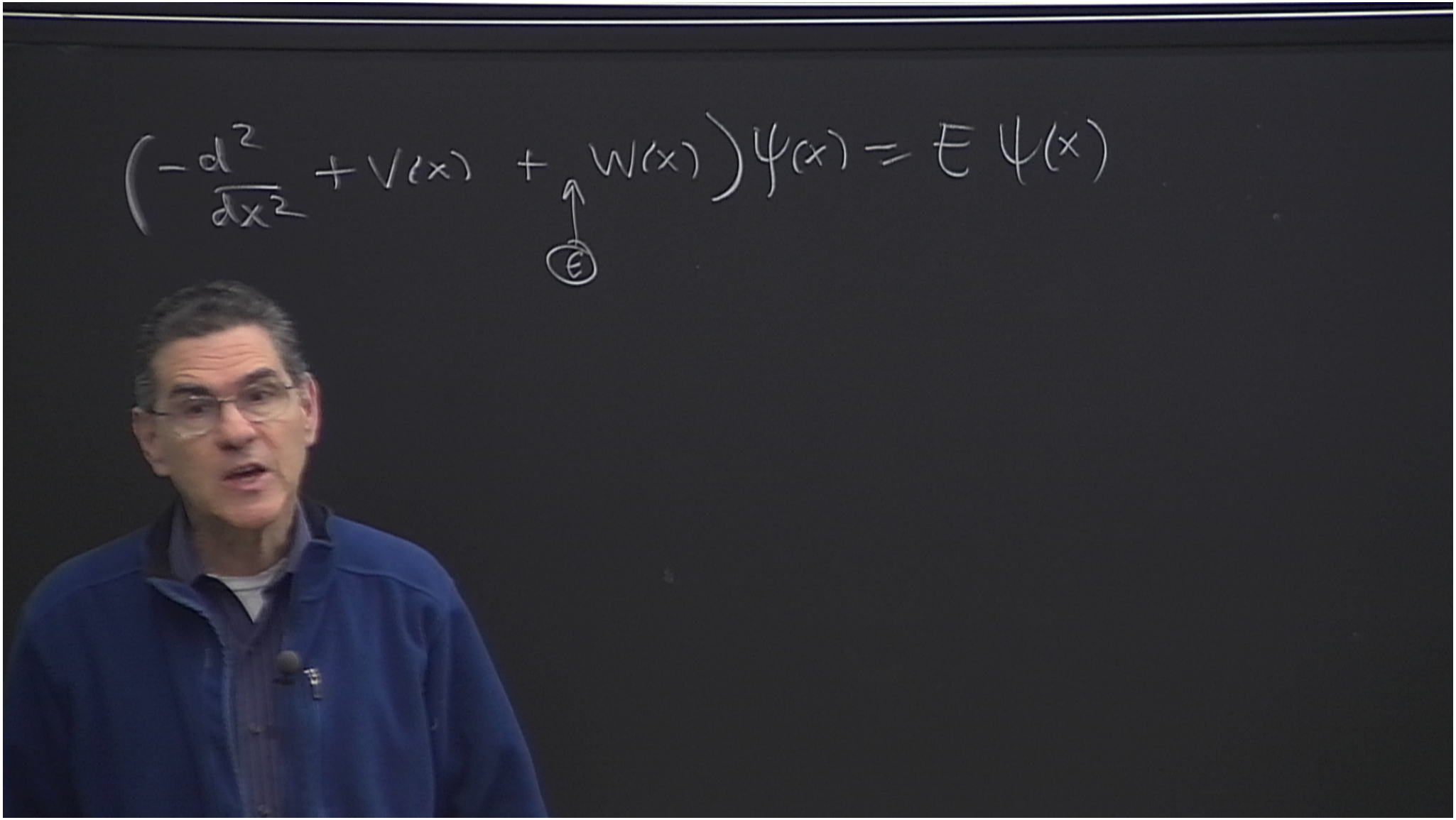
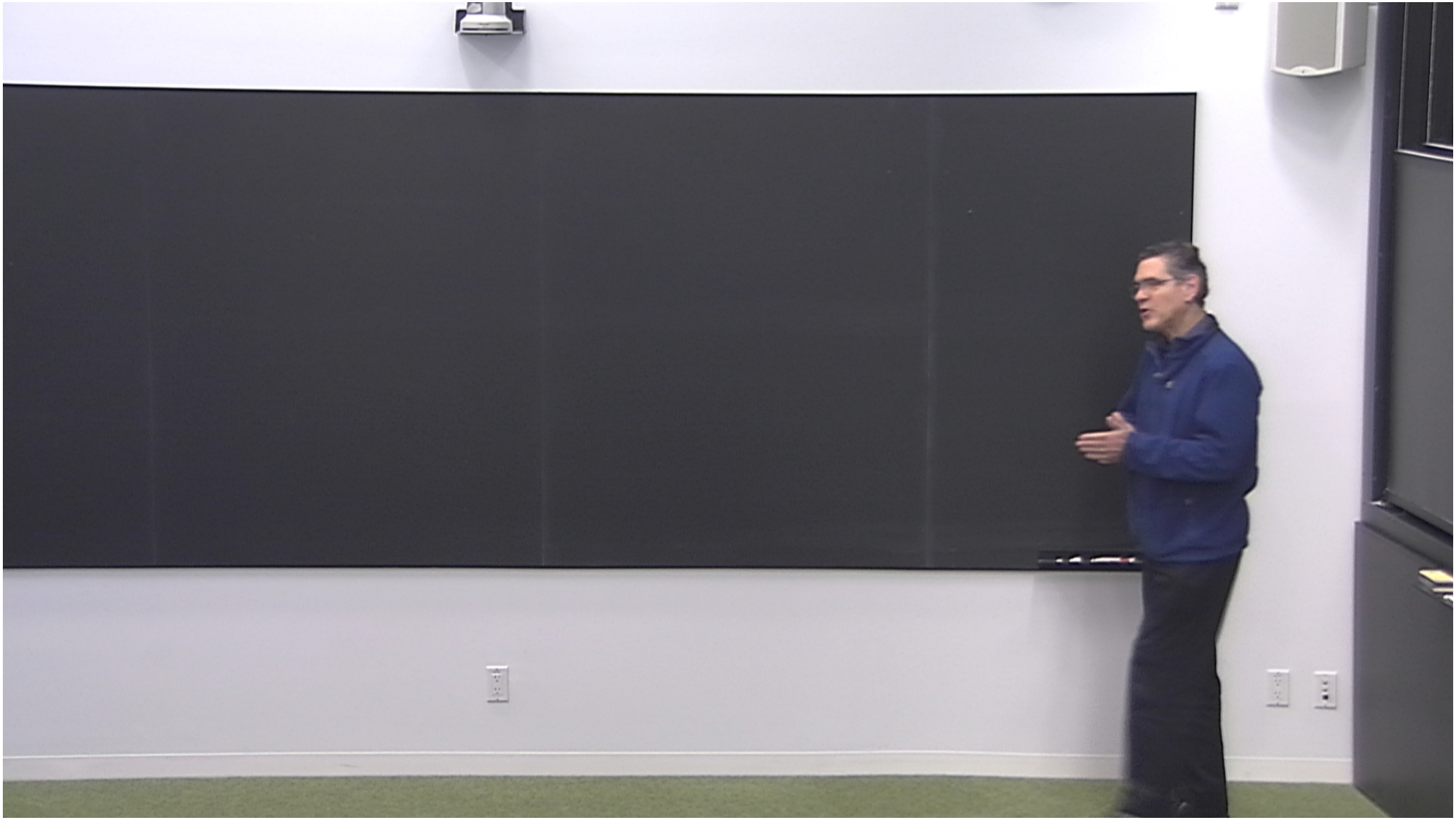
Title: Mathematical Physics - Lecture 12

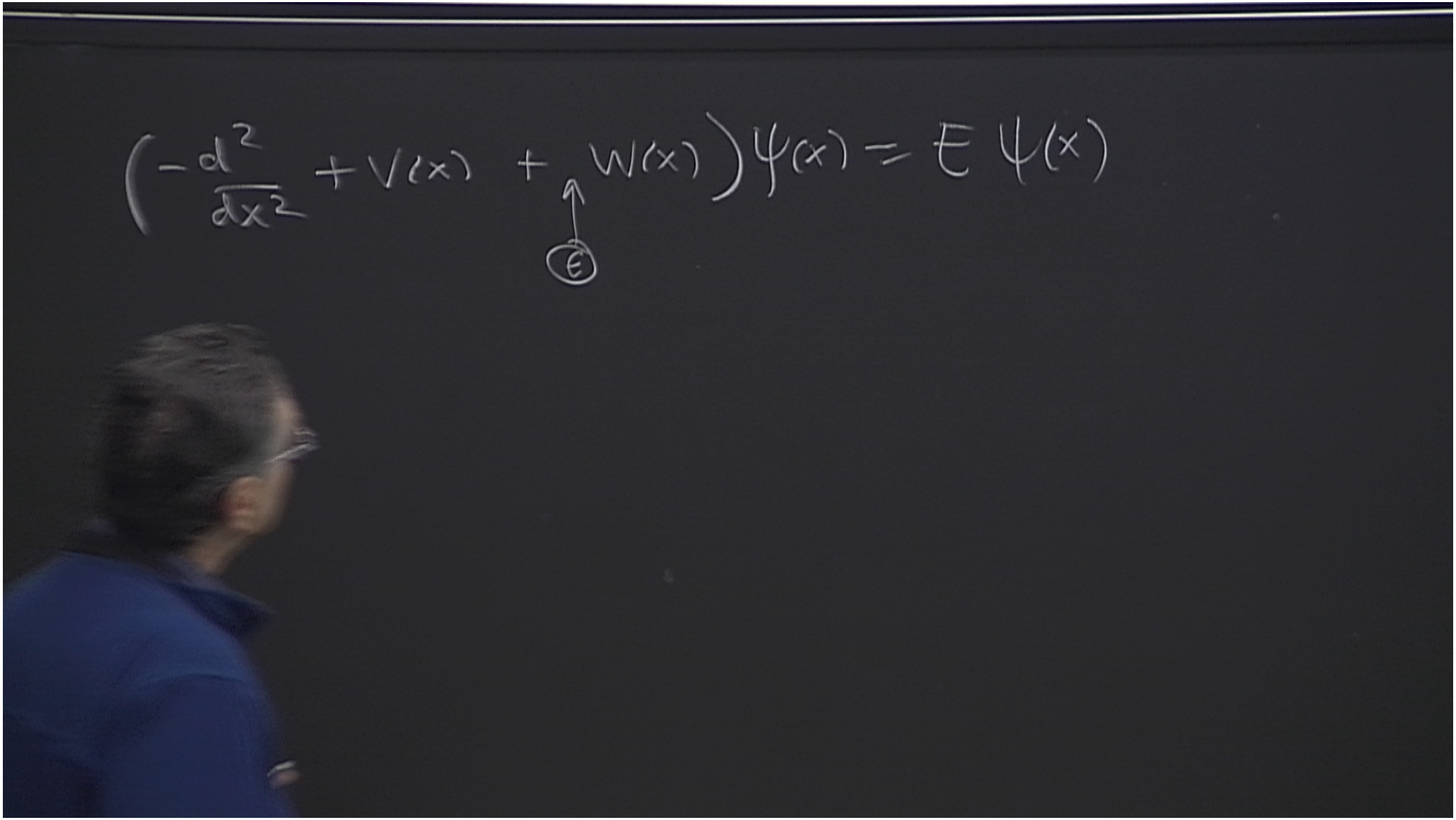
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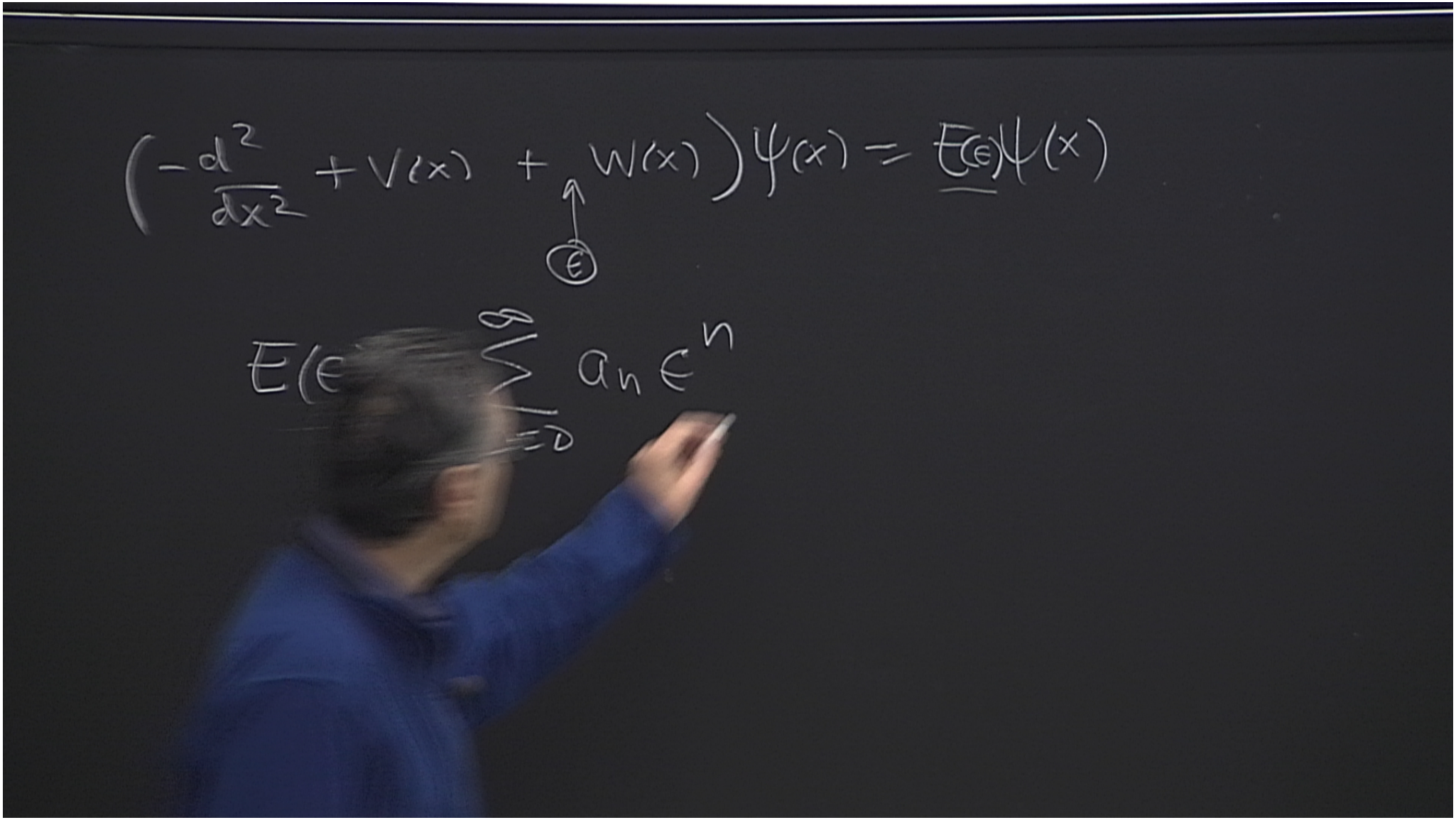
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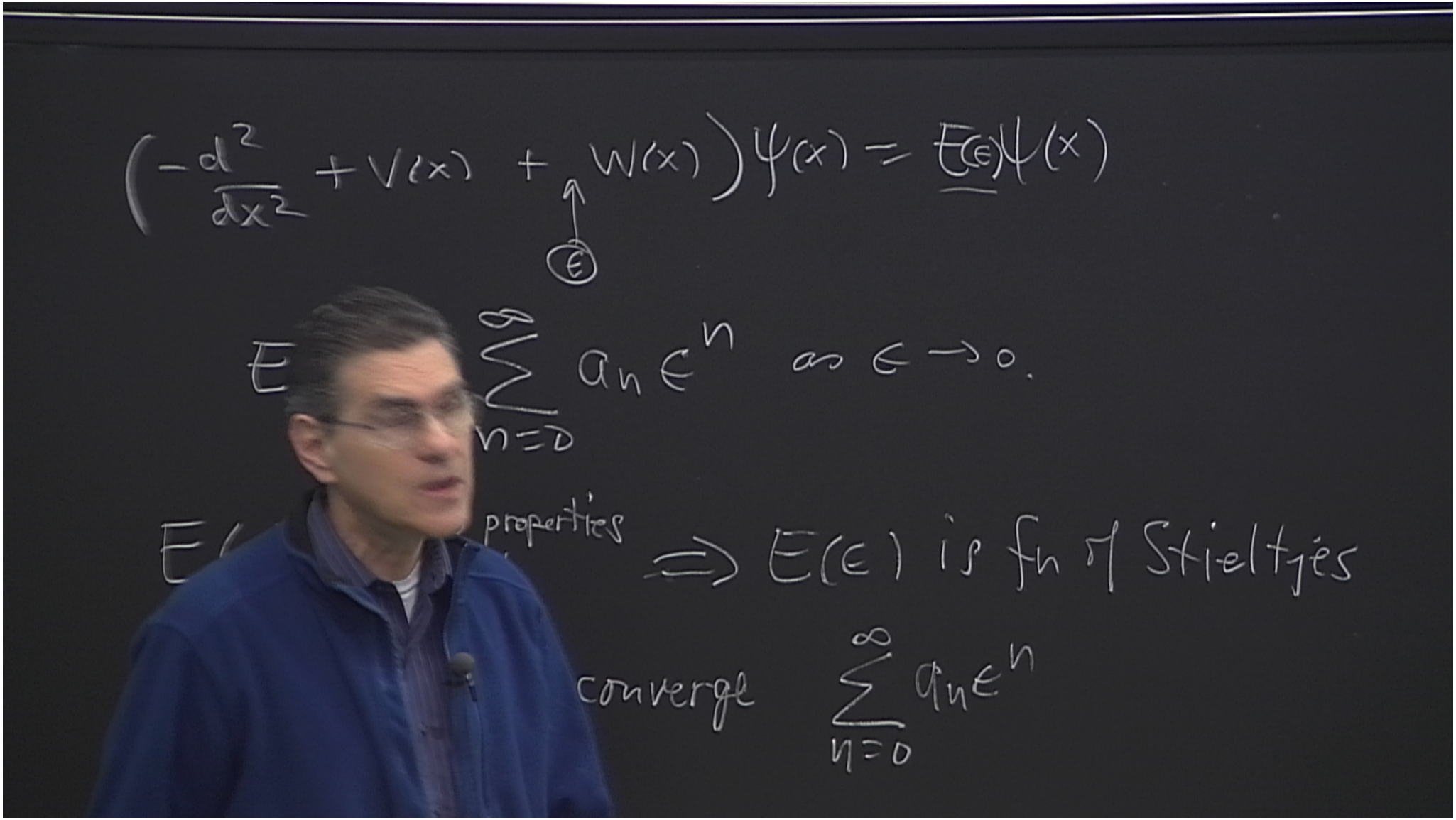
Abstract:

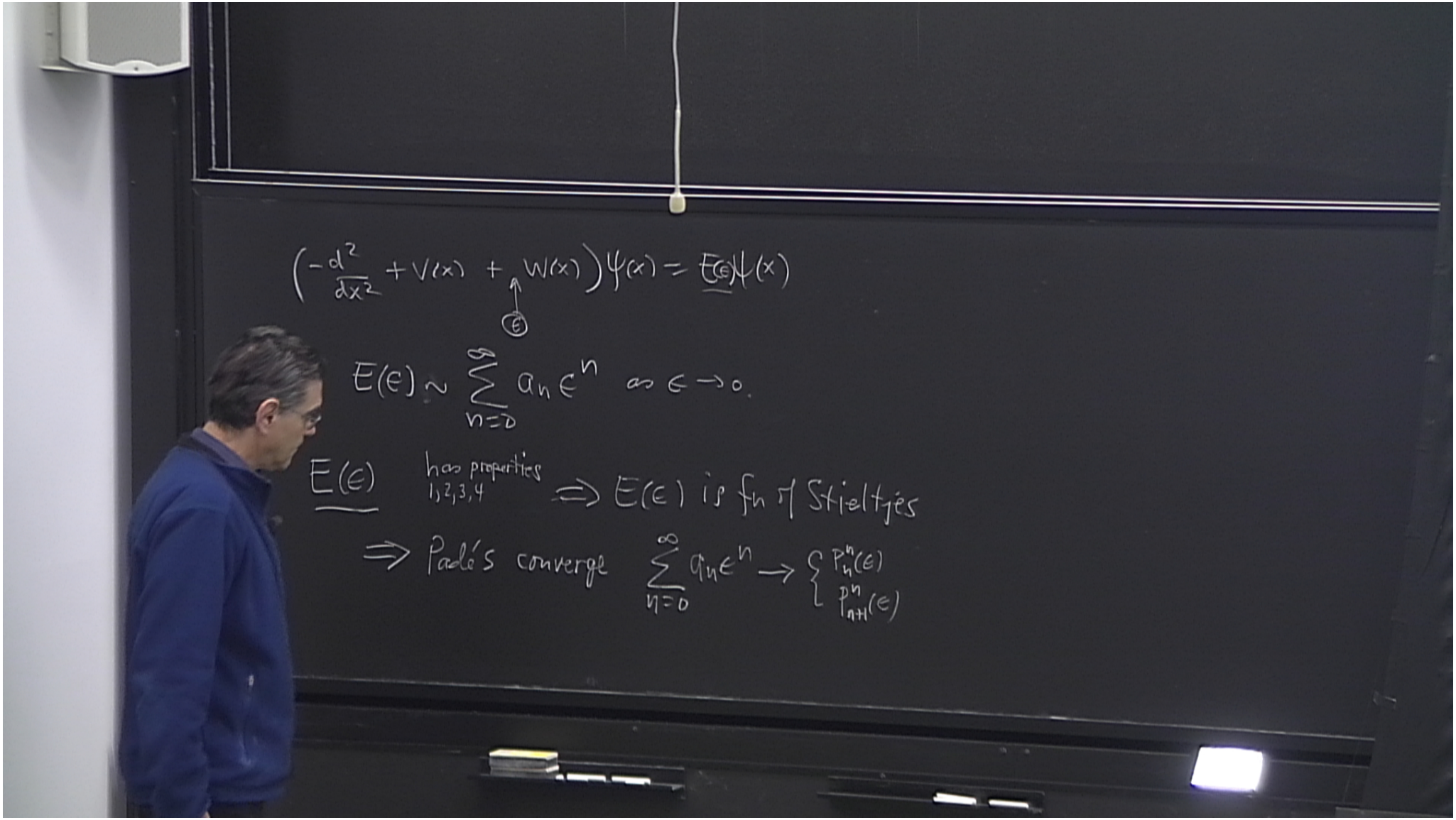

$\left(-\frac{d^2}{dx^2} + V(x) + W(x)\right)\psi(x) = E\,\psi(x)$
E

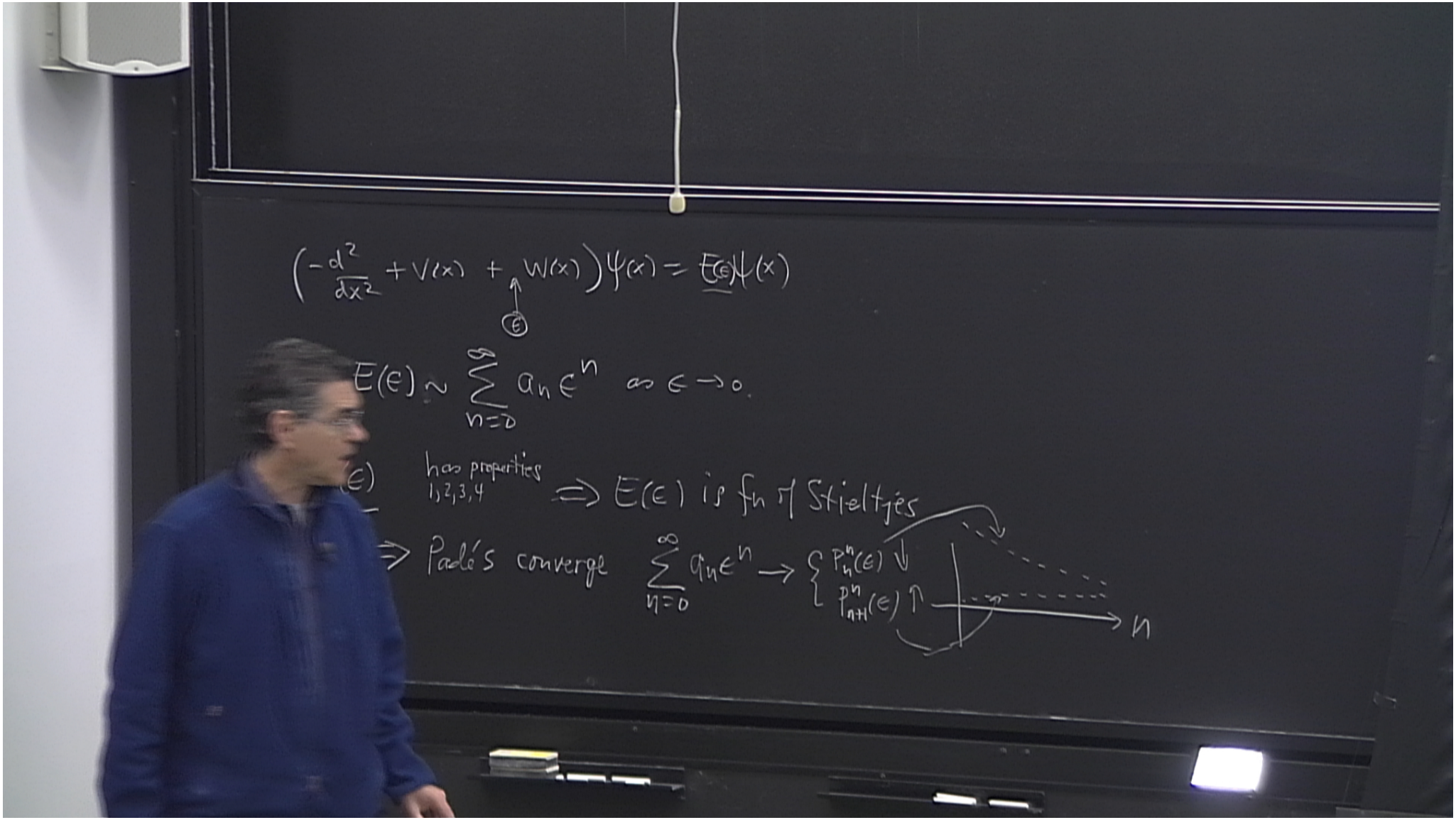



$\left(-\frac{d^2}{dx^2} + V(x) + W(x)\right)\psi(x) = E\,\psi(x)$
ϵ


$\left(-\frac{d^2}{dx^2} + V(x) + W(x)\right)\psi(x) = E(\epsilon)\psi(x)$
ϵ
$E(\epsilon) = \sum_{n=0} a_n \epsilon^n$


$\left(-\frac{d^2}{dx^2} + V(x) + W(x)\right)\psi(x) = E(\epsilon)\psi(x)$
ϵ
$E \sim \sum_{n=0}^{\infty} a_n \epsilon^n$ as $\epsilon \to 0$.
properties
$\Rightarrow E(\epsilon)$ is fn of Stieltjes
converge $\sum_{n=0}^{\infty} a_n \epsilon^n$


$\left(-\frac{d^2}{dx^2} + V(x) + W(x)\right)\psi(x) = E(\epsilon)\psi(x)$
ϵ
$E(\epsilon) \sim \sum_{n=0}^{\infty} a_n \epsilon^n$ as $\epsilon \to 0$.
$E(\epsilon)$ has properties 1,2,3,4 $\Rightarrow$ $E(\epsilon)$ is fn of Stieltjes
$\Rightarrow$ Padés converge $\sum_{n=0}^{\infty} a_n \epsilon^n \to \{ P^n_n(\epsilon), P^n_{n+1}(\epsilon)$


$\left(-\frac{d^2}{dx^2} + V(x) + W(x)\right)\psi(x) = E(\epsilon)\psi(x)$
ϵ
$E(\epsilon) \sim \sum_{n=0}^{\infty} a_n \epsilon^n$ as $\epsilon \to 0$.
(ϵ) has properties 1,2,3,4 $\Rightarrow$ $E(\epsilon)$ is fn of Stieltjes
$\Rightarrow$ Padés converge $\sum_{n=0}^{\infty} a_n \epsilon^n \to \{ P^n_n(\epsilon) \downarrow$ $P^n_{n+1}(\epsilon) \uparrow$
n

$W(x))\psi(x) = E(\epsilon)\psi(x)$

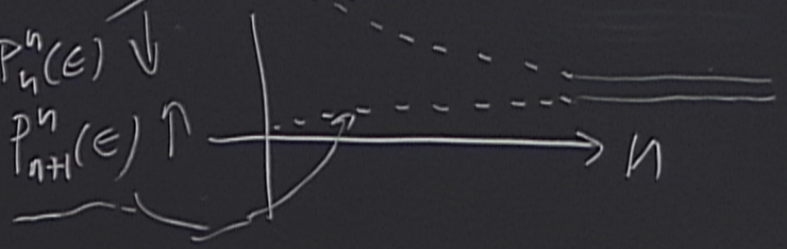
ϵ

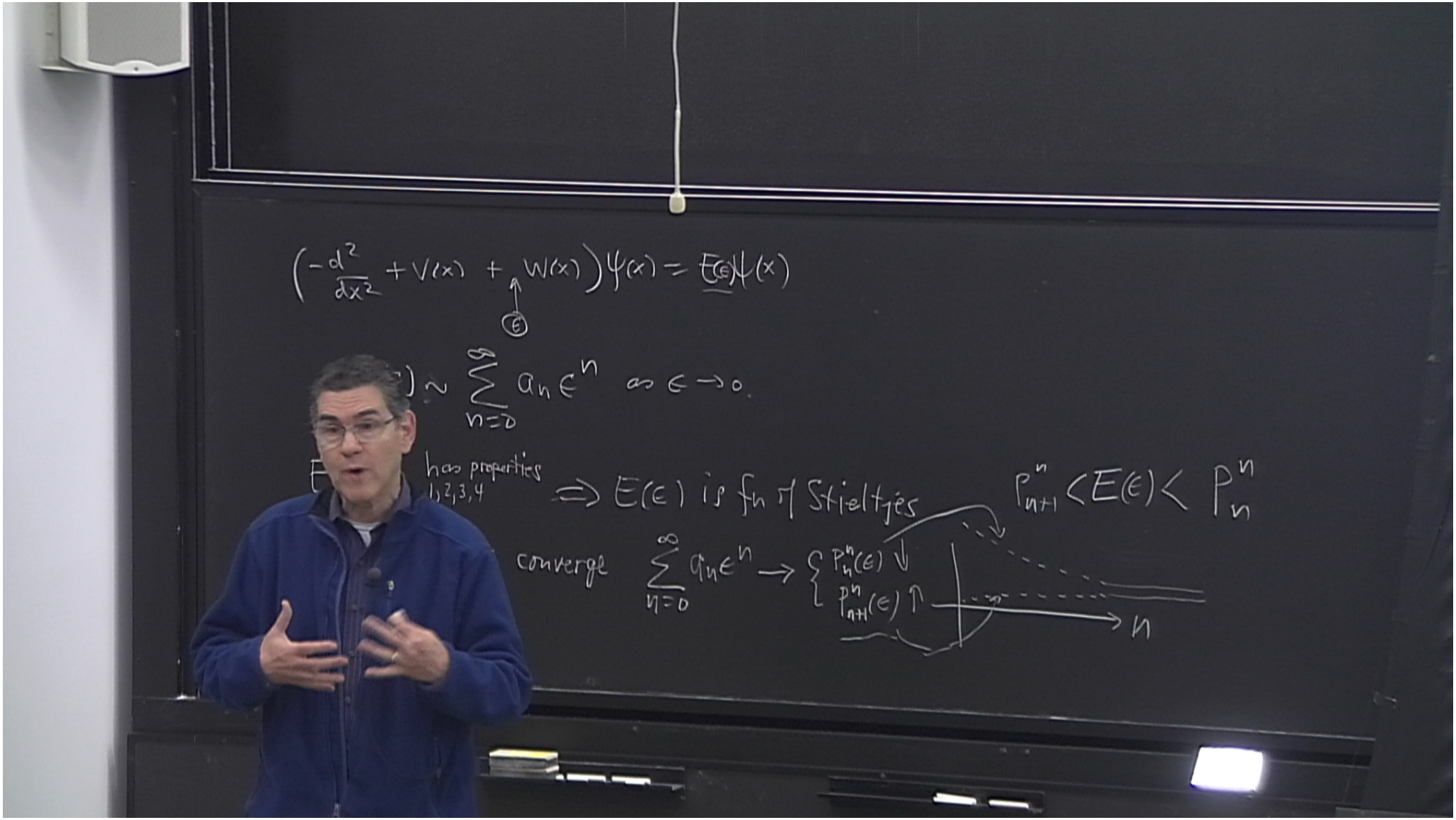
$a_n \epsilon^n$ as $\epsilon \to 0$.

$\Rightarrow E(\epsilon)$ is fn of Stieltjes

converge $\sum_{n=0}^{\infty} a_n \epsilon^n \to$ { $P_n^n(\epsilon) \downarrow$, $P_{n+1}^n(\epsilon) \uparrow$

$P_{n+1}^n < E(\epsilon) < P_n^n$




$\left(-\frac{d^2}{dx^2} + V(x) + W(x)\right)\psi(x) = E(\epsilon)\psi(x)$
ϵ
$\sim \sum_{n=0}^{\infty} a_n \epsilon^n$ as $\epsilon \to 0$.
has properties 1,2,3,4
$\Rightarrow E(\epsilon)$ is fn of Stieltjes
$P_{n+1}^n < E(\epsilon) < P_n^n$
converge
$\sum_{n=0}^{\infty} a_n \epsilon^n \to \{ P_n^n(\epsilon) \downarrow$
$P_{n+1}^n(\epsilon) \uparrow$
n

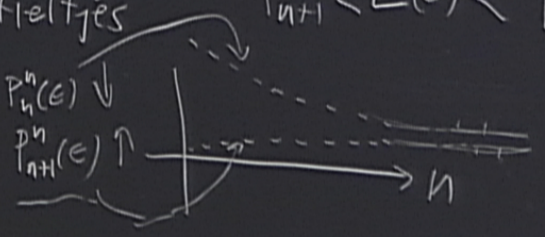
$$\left(-\frac{d^2}{dx^2} + V(x) + W(x)\right)\psi(x) = E(\epsilon)\psi(x)$$

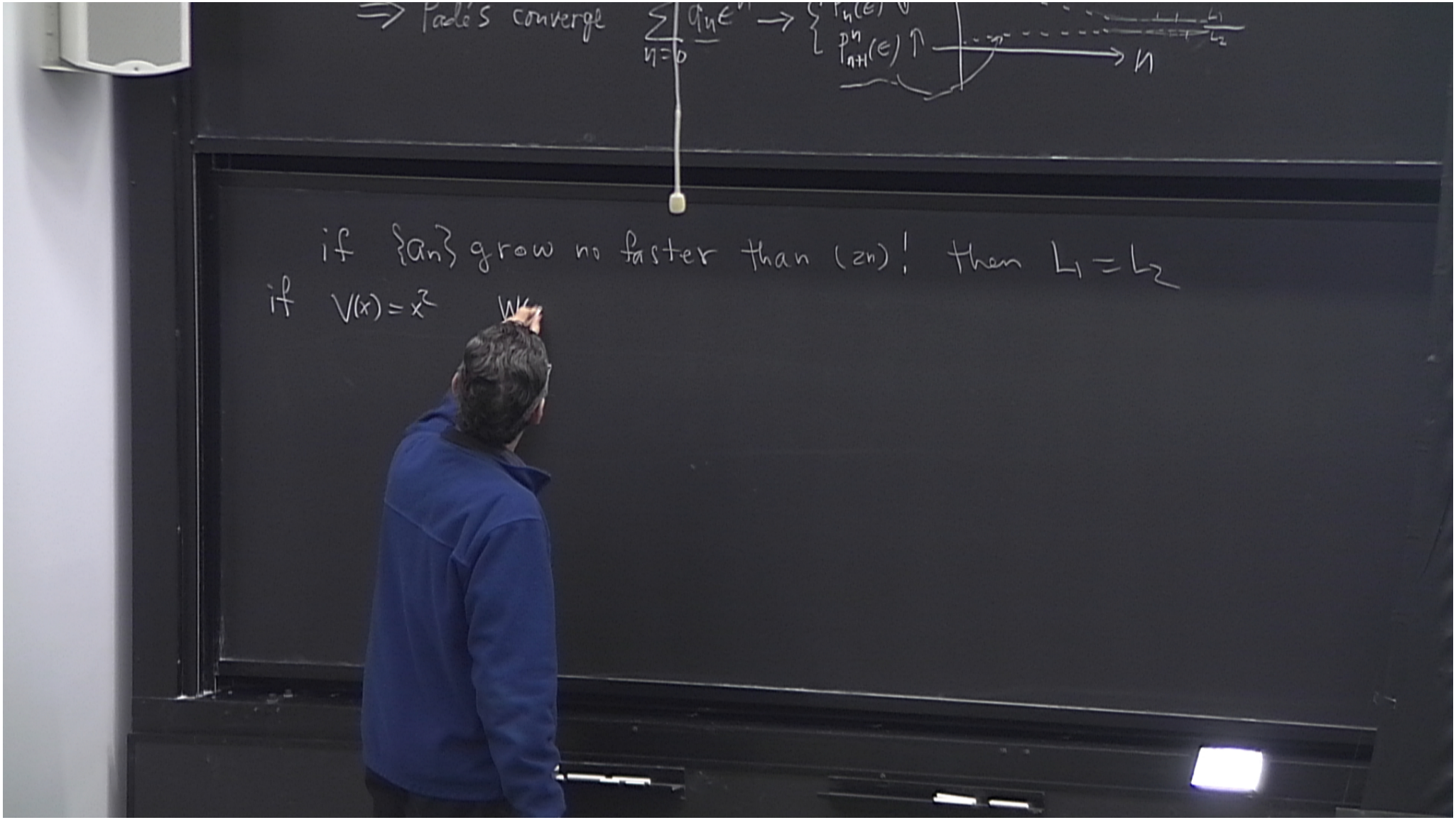
ϵ

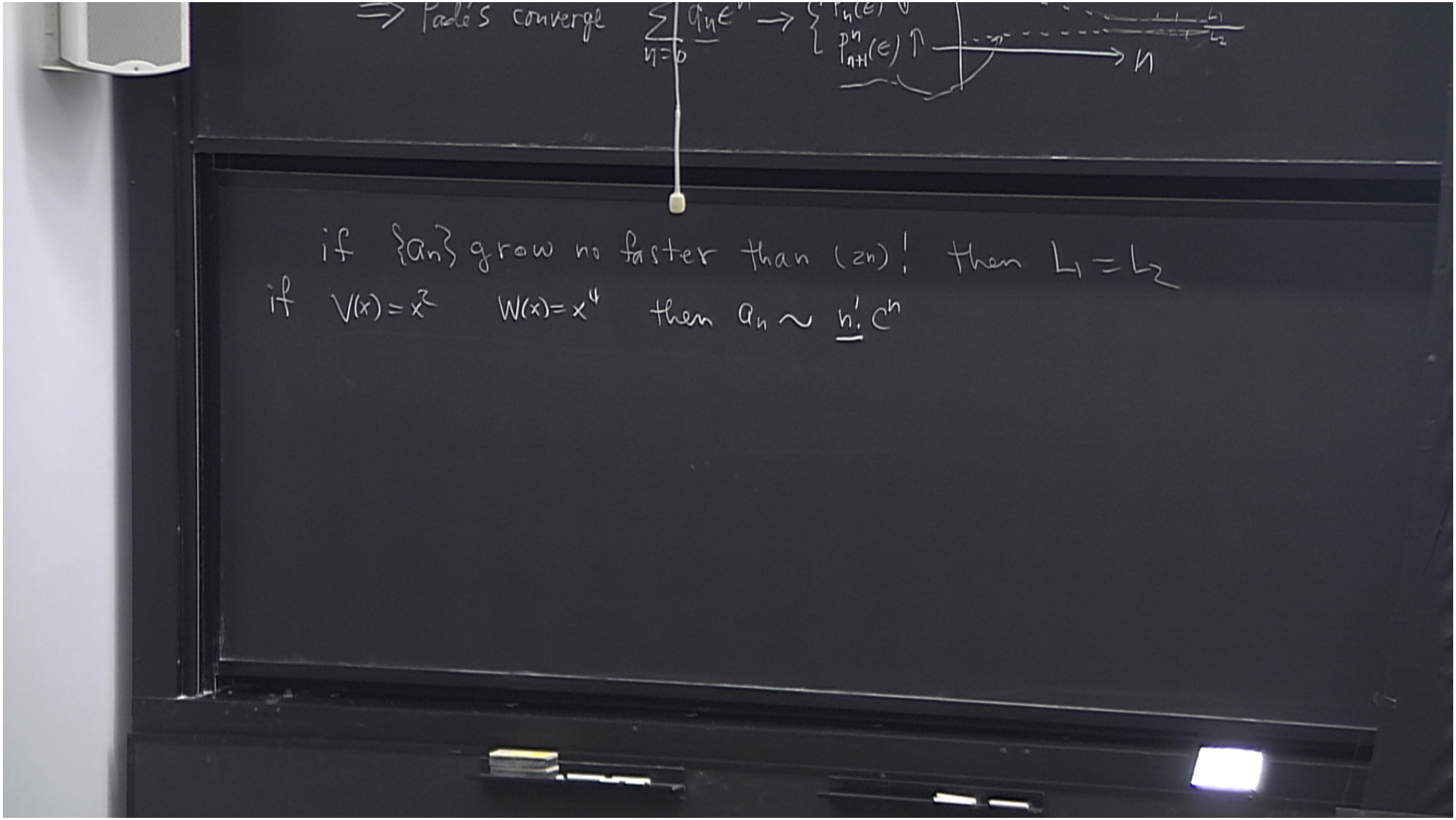
$$E(\epsilon) \sim \sum_{n=0}^{\infty} a_n \epsilon^n \quad \text{as } \epsilon \to 0.$$

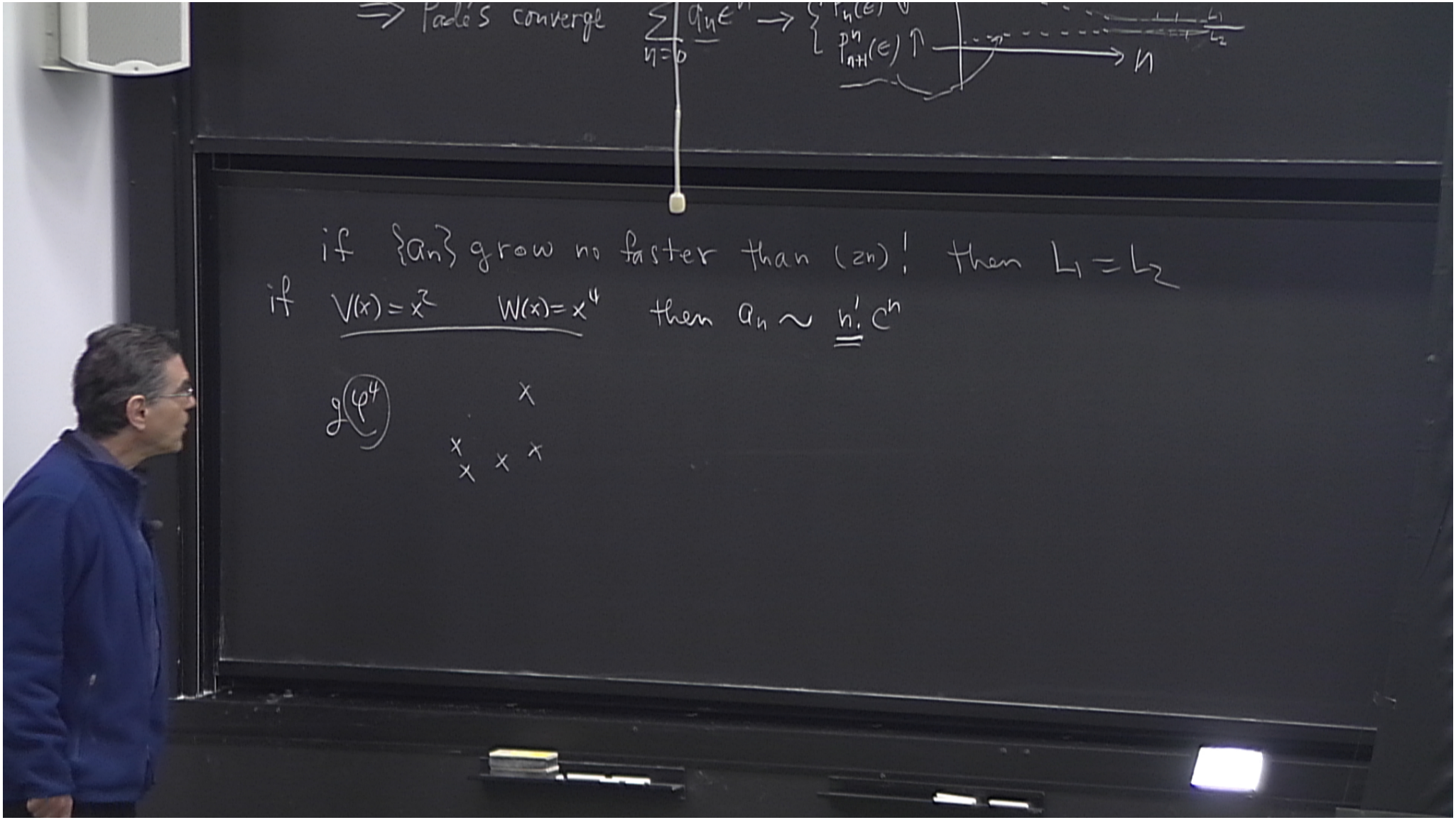
$E(\epsilon)$ has properties 1,2,3,4 $\Rightarrow$ $E(\epsilon)$ is fn of Stieltjes $\quad P^n_{n+1} < E(\epsilon) < P^n_n$

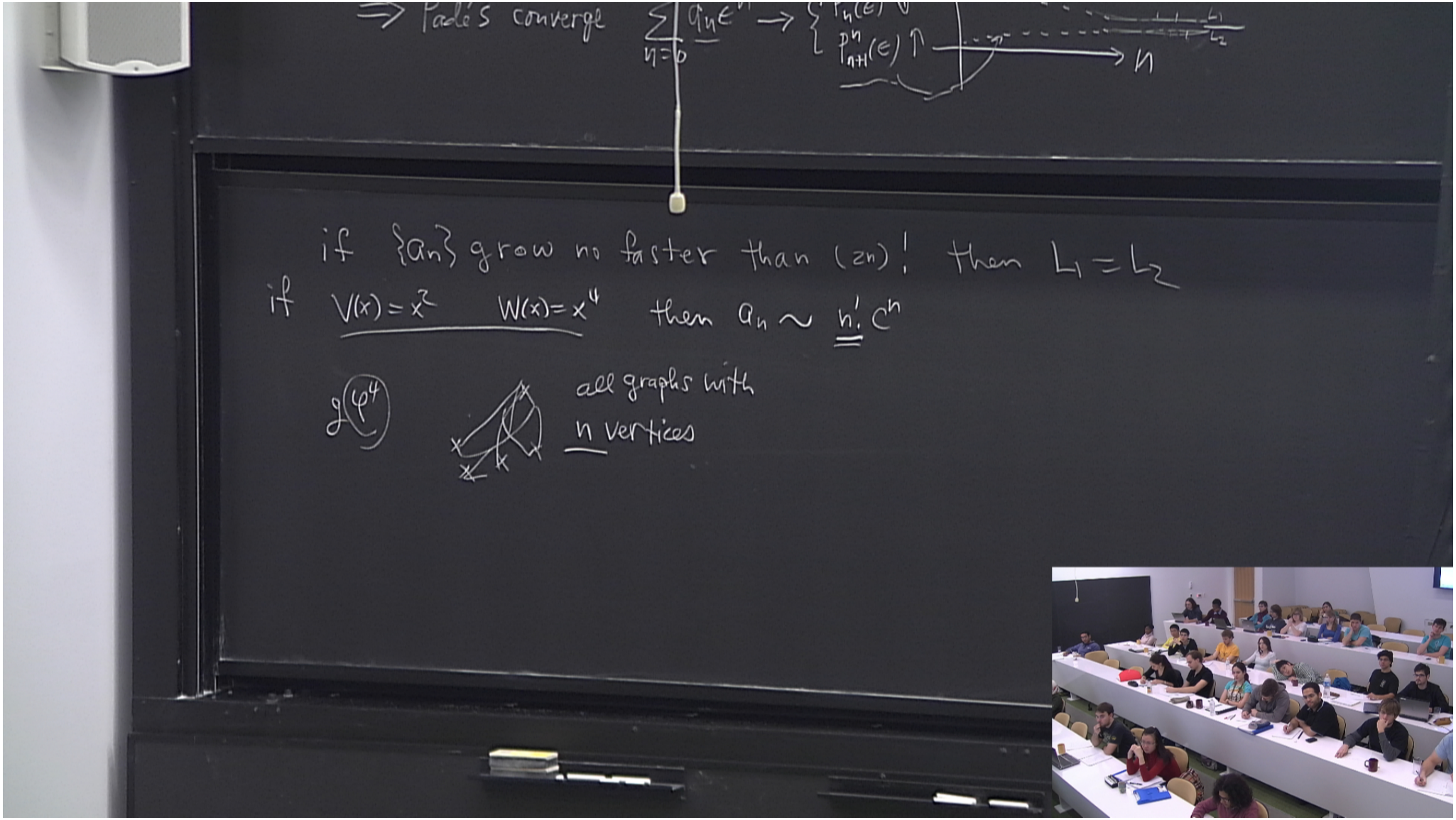
$\Rightarrow$ Padés converge $\sum_{n=0}^{\infty} a_n \epsilon^n \to \begin{cases} P^n_n(\epsilon) \downarrow \\ P^n_{n+1}(\epsilon) \uparrow \end{cases}$



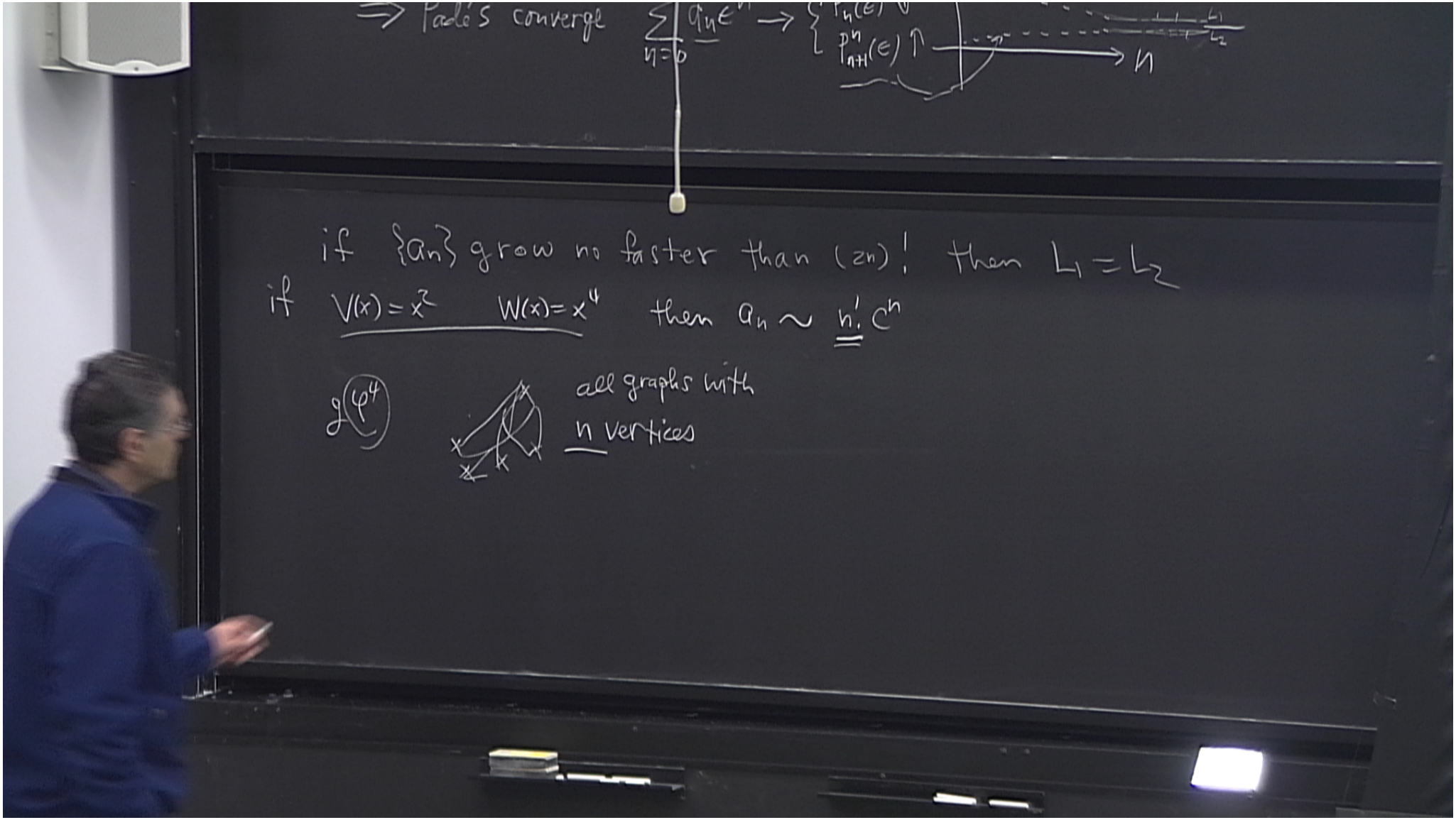

⇒ Padé's converge
$\sum_{n=0}$ $a_n \epsilon^n$ →
$P^n_n(\epsilon)$
$P^n_{n+1}(\epsilon)$
L_1
L_2
n
if $\{a_n\}$ grow no faster than $(2n)!$ then $L_1 = L_2$
if $V(x) = x^2$

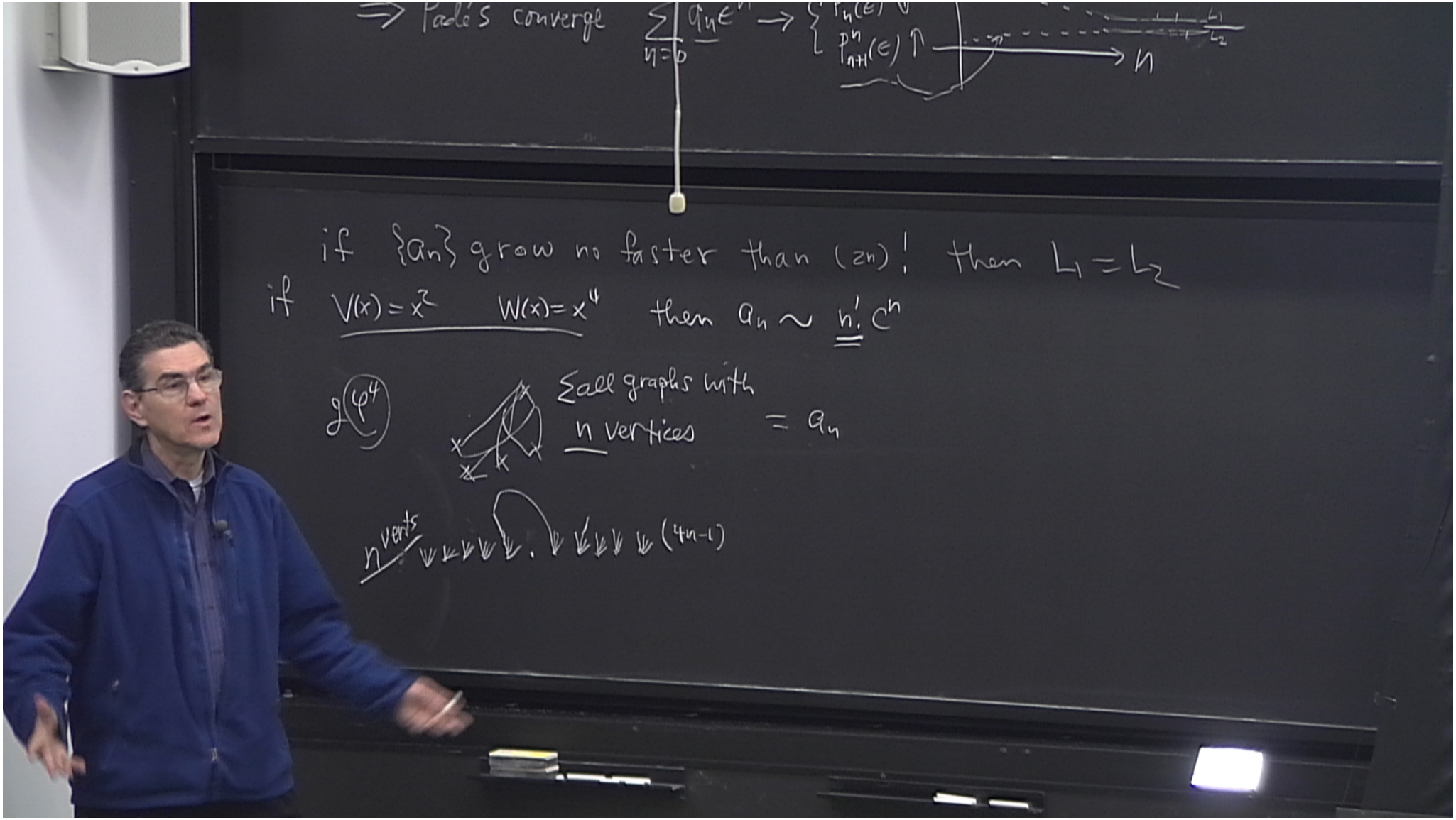

⇒ Padé's converge
$\sum_{n=0}$ $a_n \epsilon^n$ →
$P^n_{n+1}(\epsilon)$
L_1
L_2
n
if $\{a_n\}$ grow no faster than $(2n)!$ then $L_1 = L_2$
if $V(x) = x^2$ $W(x) = x^4$ then $a_n \sim n!\, c^n$

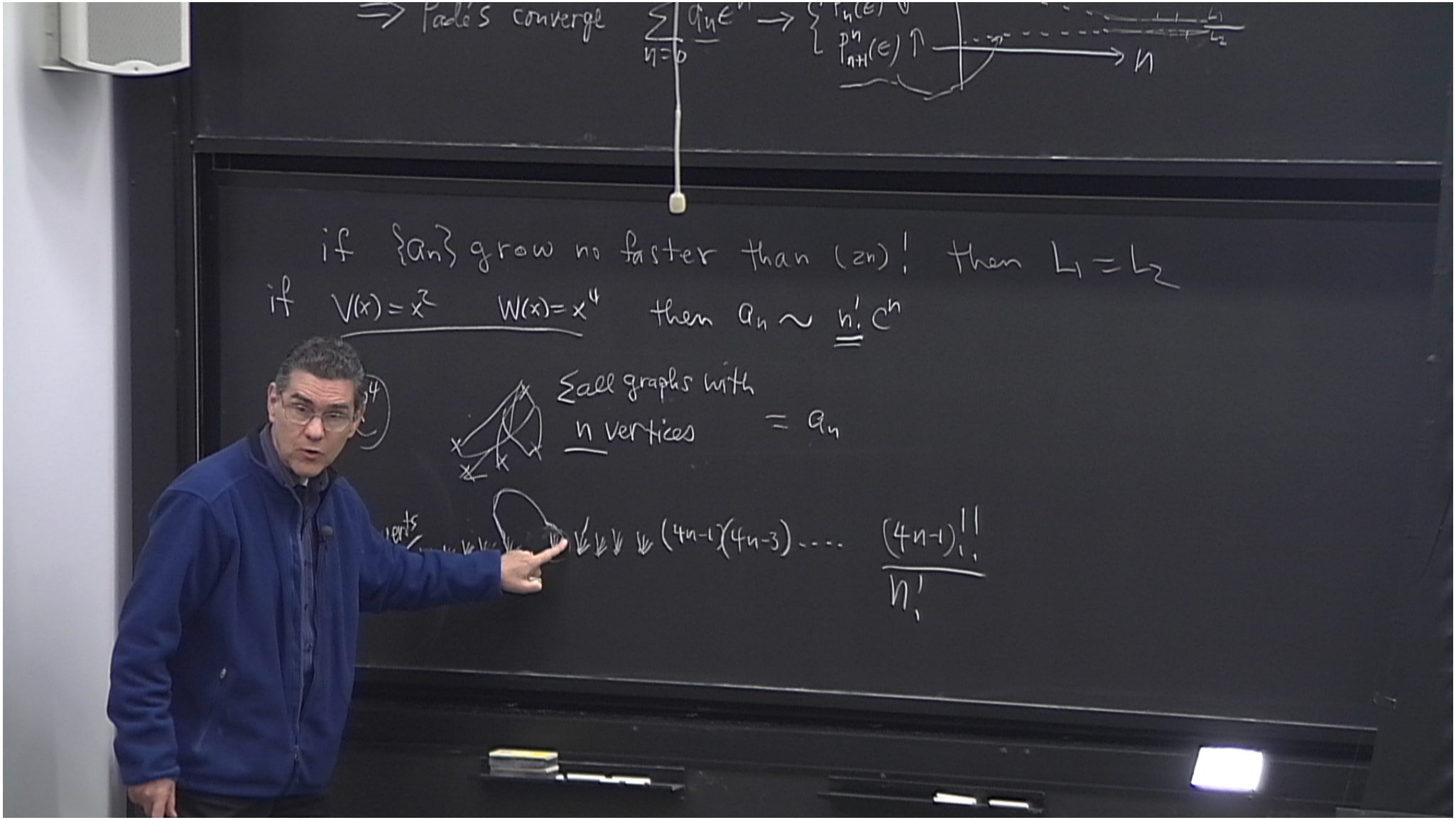

⇒ Padé's converge
$\sum_{n=0}$ $a_n \epsilon^n$ →
$P^n_n(\epsilon)$
$P^n_{n+1}(\epsilon)$
L_1
L_2
n
if $\{a_n\}$ grow no faster than $(2n)!$ then $L_1 = L_2$
if $V(x) = x^2$ $W(x) = x^4$ then $a_n \sim n!\, c^n$
$g(\varphi^4)$


⇒ Padé's converge
$\sum_{n=0}$ $a_n \epsilon^n$ →
$P^n_{n+1}(\epsilon)$
if $\{a_n\}$ grow no faster than $(2n)!$ then $L_1 = L_2$
if $V(x) = x^2$ $W(x) = x^4$ then $a_n \sim n!\, C^n$
$g(\varphi^4)$
all graphs with
n vertices


⇒ Pólya converge
if
then
if
then


⇒ Padé's converge
$\sum_{n=0}$ $a_n \epsilon^n$ →
$P^n_{n+1}(\epsilon)$
L_1
L_2
n
if $\{a_n\}$ grow no faster than $(2n)!$ then $L_1 = L_2$
if $V(x) = x^2$ $W(x) = x^4$ then $a_n \sim n!\, c^n$
$g(\varphi^4)$
all graphs with
n vertices


⇒ Padé's converge
$\sum_{n=0} a_n \epsilon^n$
$P^n_{n+1}(\epsilon)$
if $\{a_n\}$ grow no faster than $(zn)!$ then $L_1 = L_2$
if $V(x) = x^2$ $W(x) = x^4$ then $a_n \sim n!\, c^n$
$g(\varphi^4)$
Σ all graphs with n vertices $= a_n$
n verts
$(4n-1)$

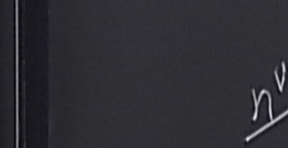

⇒ Padé's converge
$\sum_{n=0} a_n \epsilon^n \to$
$P^n_{n+1}(\epsilon)$
n
if $\{a_n\}$ grow no faster than $(2n)!$ then $L_1 = L_2$
if $V(x) = x^2$ $W(x) = x^4$ then $a_n \sim n!\, c^n$
Σ all graphs with n vertices $= a_n$
$(4n-1)(4n-3)\cdots$
$\frac{(4n-1)!!}{n!}$

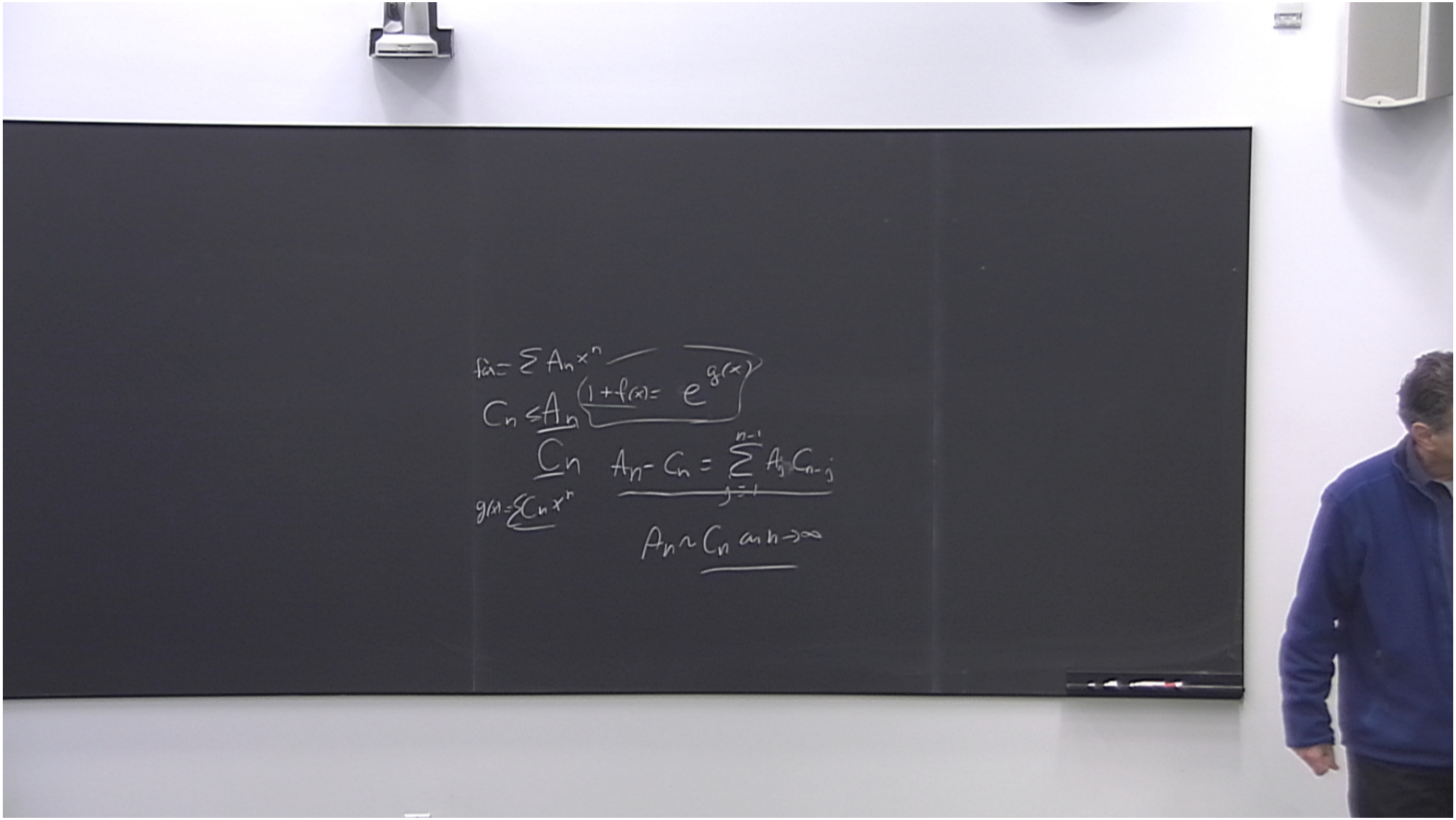
$\Rightarrow$ Padé's converge $\sum_{n=0} a_n \epsilon^n \rightarrow$ $P^n_{n+1}(\epsilon)$ $\uparrow$ n

if $\{a_n\}$ grow no faster than $(zn)!$ then $L_1 = L_2$

if $V(x) = x^2$ $W(x) = x^4$ then $a_n \sim n!\, c^n$

$g(\varphi^4)$  Σ all graphs with n vertices $= a_n$

n verts  $(4n-1)(4n-3)\cdots \dfrac{(4n-1)!!}{n!\,(24)^n}$


$f(x) = \sum A_n x^n$
$1 + f(x) = e^{g(x)}$
$C_n \le A_n$
C_n
$A_n - C_n = \sum_{j=1}^{n-1} A_j C_{n-j}$
$g(x) = \sum C_n x^n$
$A_n \sim C_n$ as $n \to \infty$

grow no faster than $(2n)!$ then $L_1 = L_2$

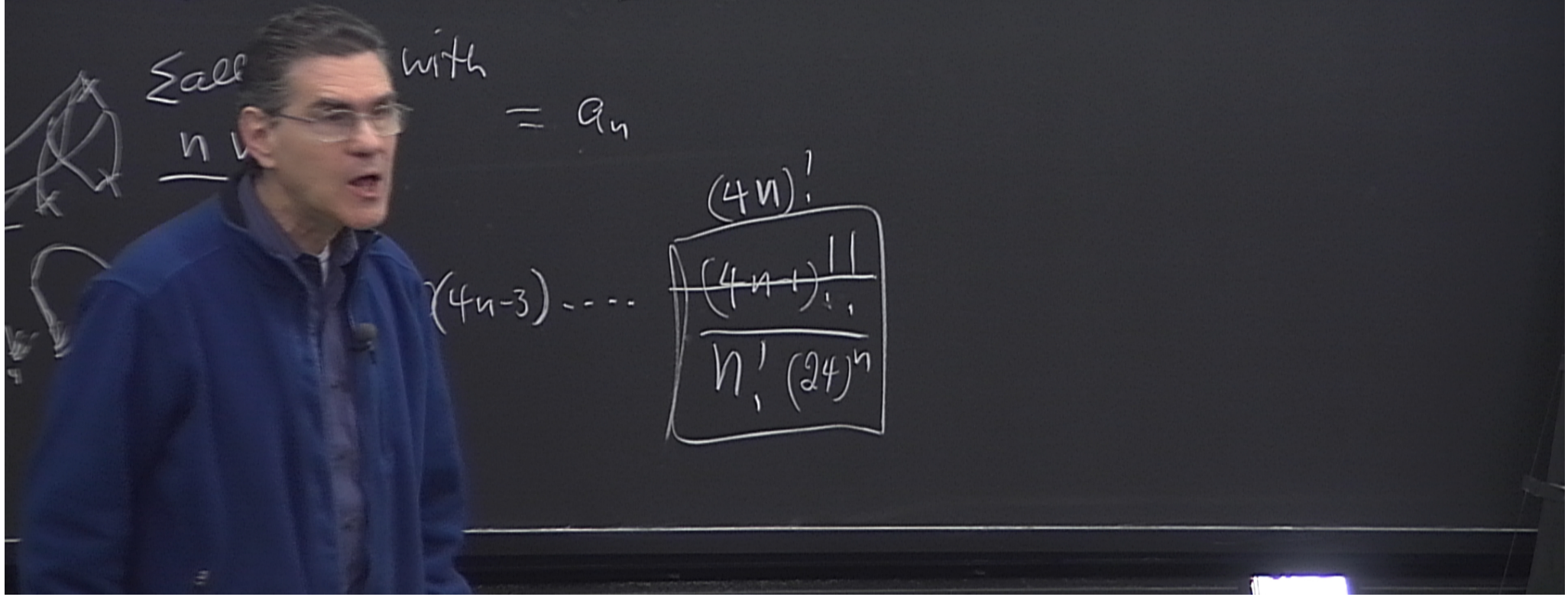
$W(x) = x^4$ then $a_n \sim n!\, C^n$

Each ... with

$= a_n$

$(4n)!$

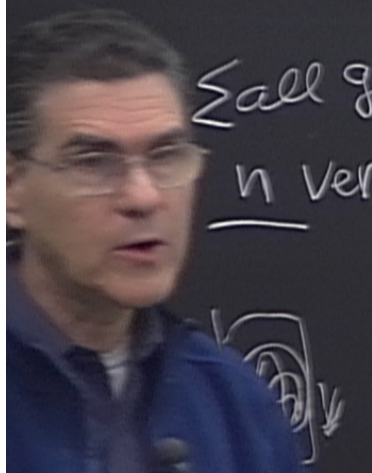
$(4n-3)\cdots$ $\dfrac{(4n-1)!!}{n!\,(24)^n}$



grow no faster than $(2n)!$ then $L_1 = L_2$

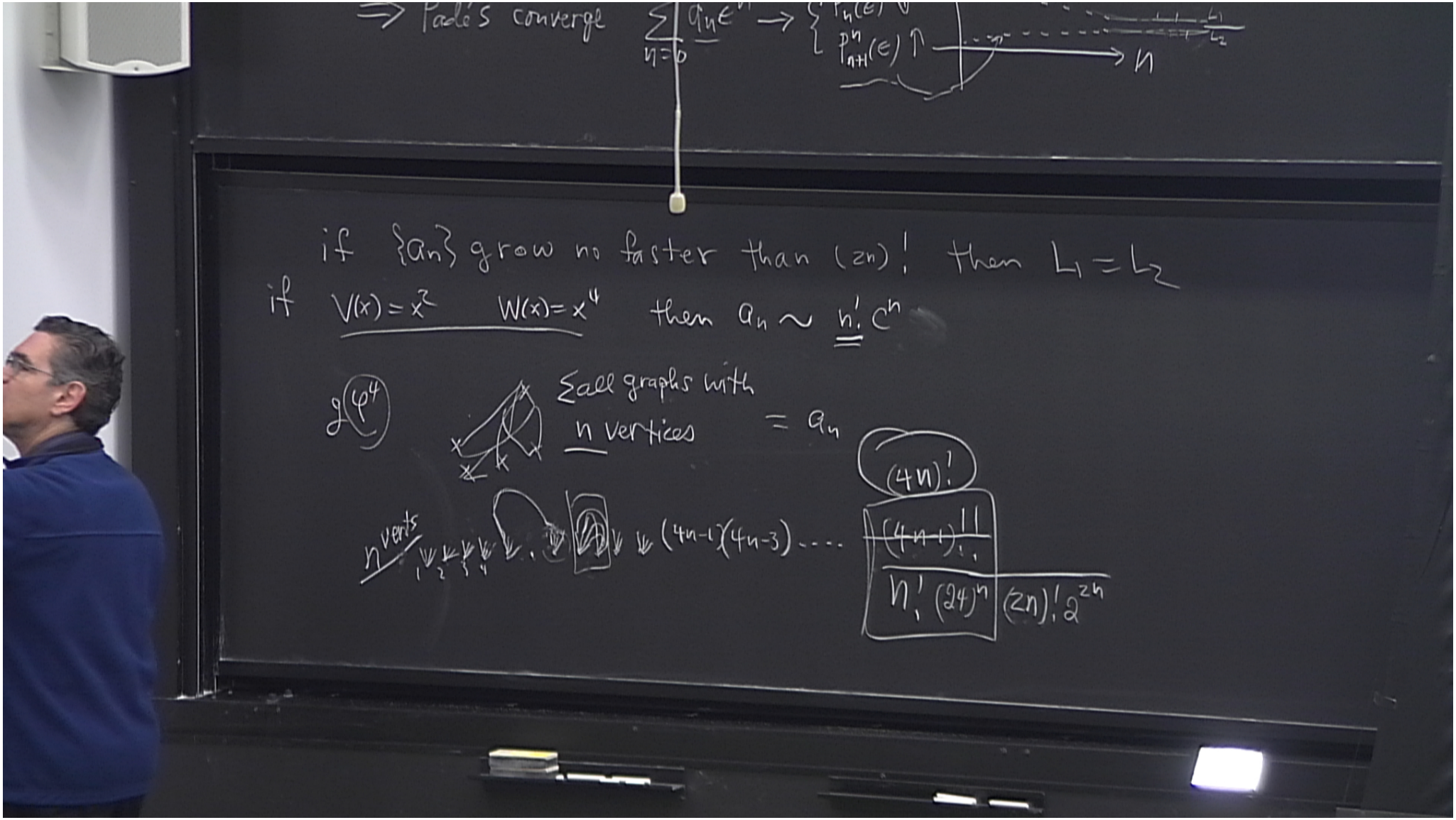
$W(x) = x^4$ then $a_n \sim n!\, C^n$

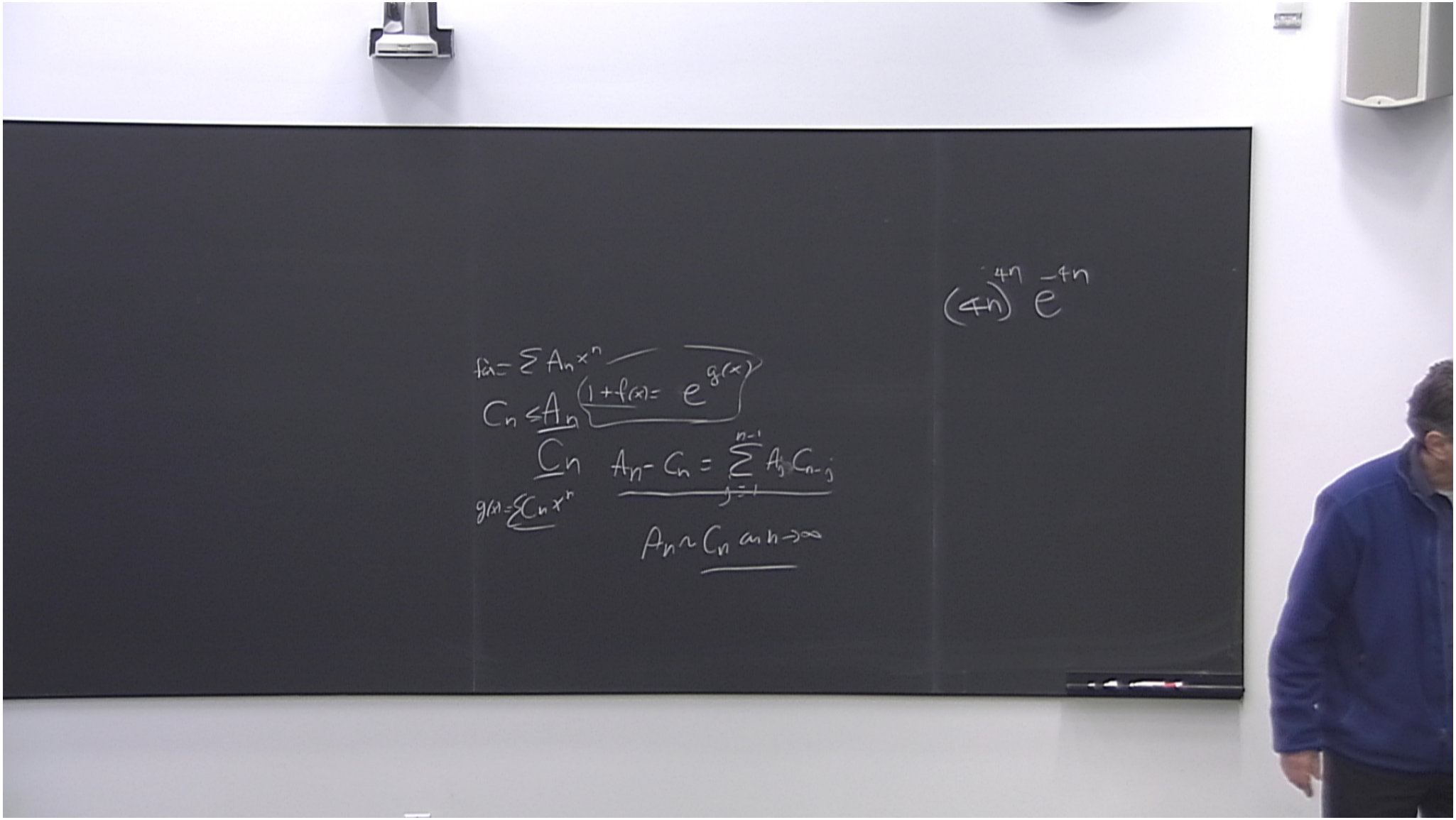
Σ all graphs with n vertices $= a_n$



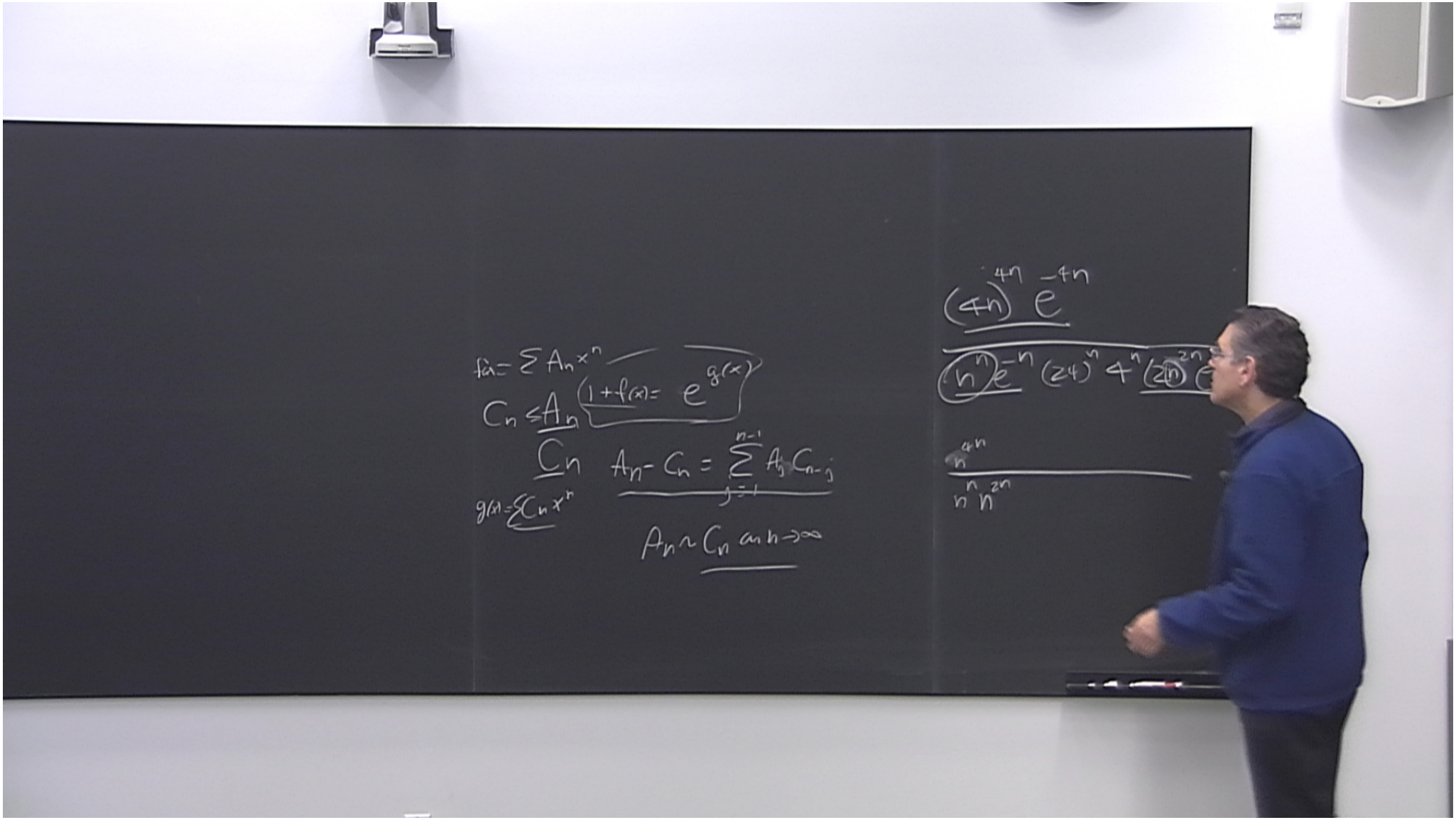
$(4n)!$ / $(4n-1)!!$ / $n!\,(24)^n$ $(2n)!\,2^{2n}$

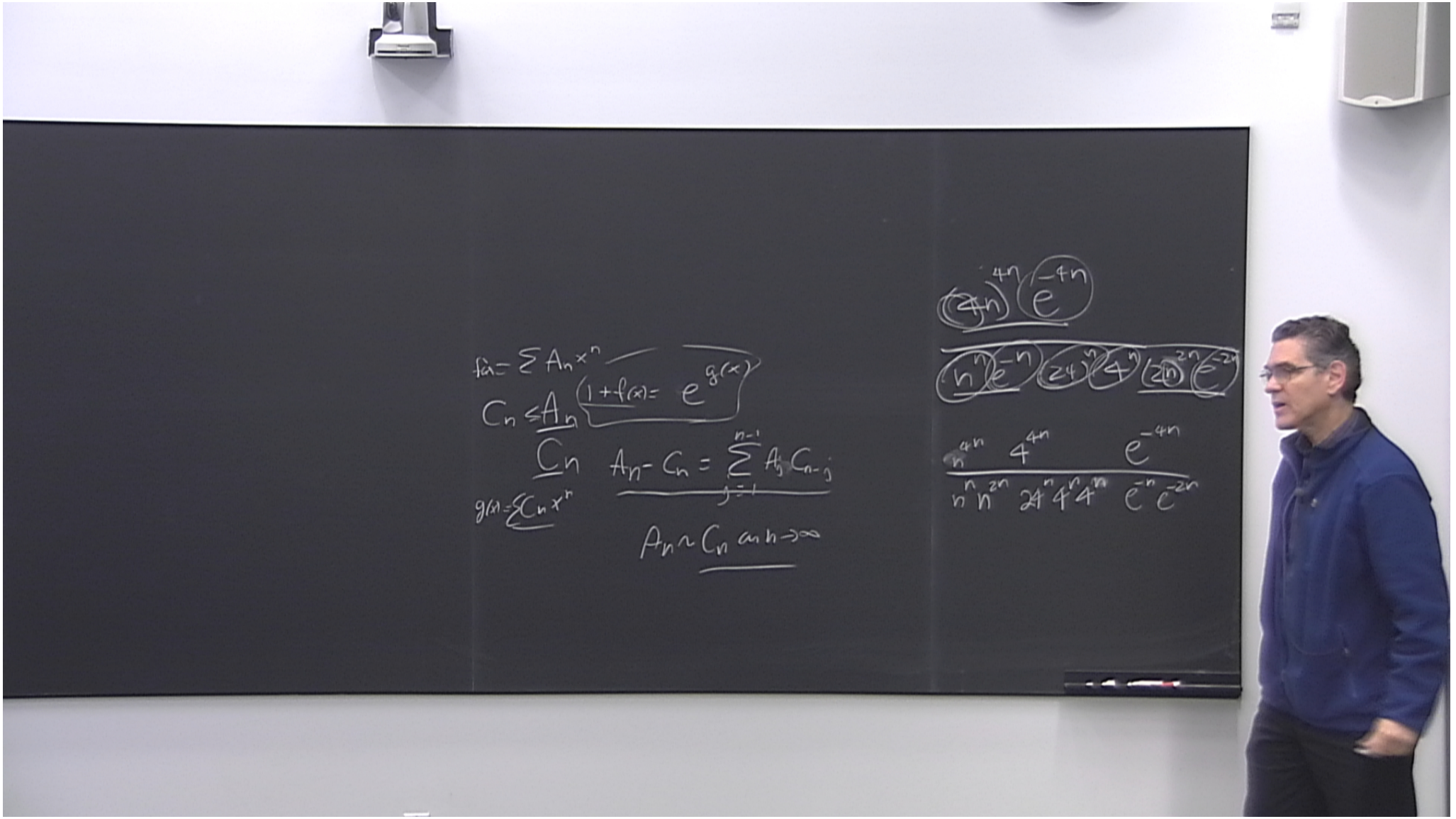
$(4n-1)(4n-3)\cdots$

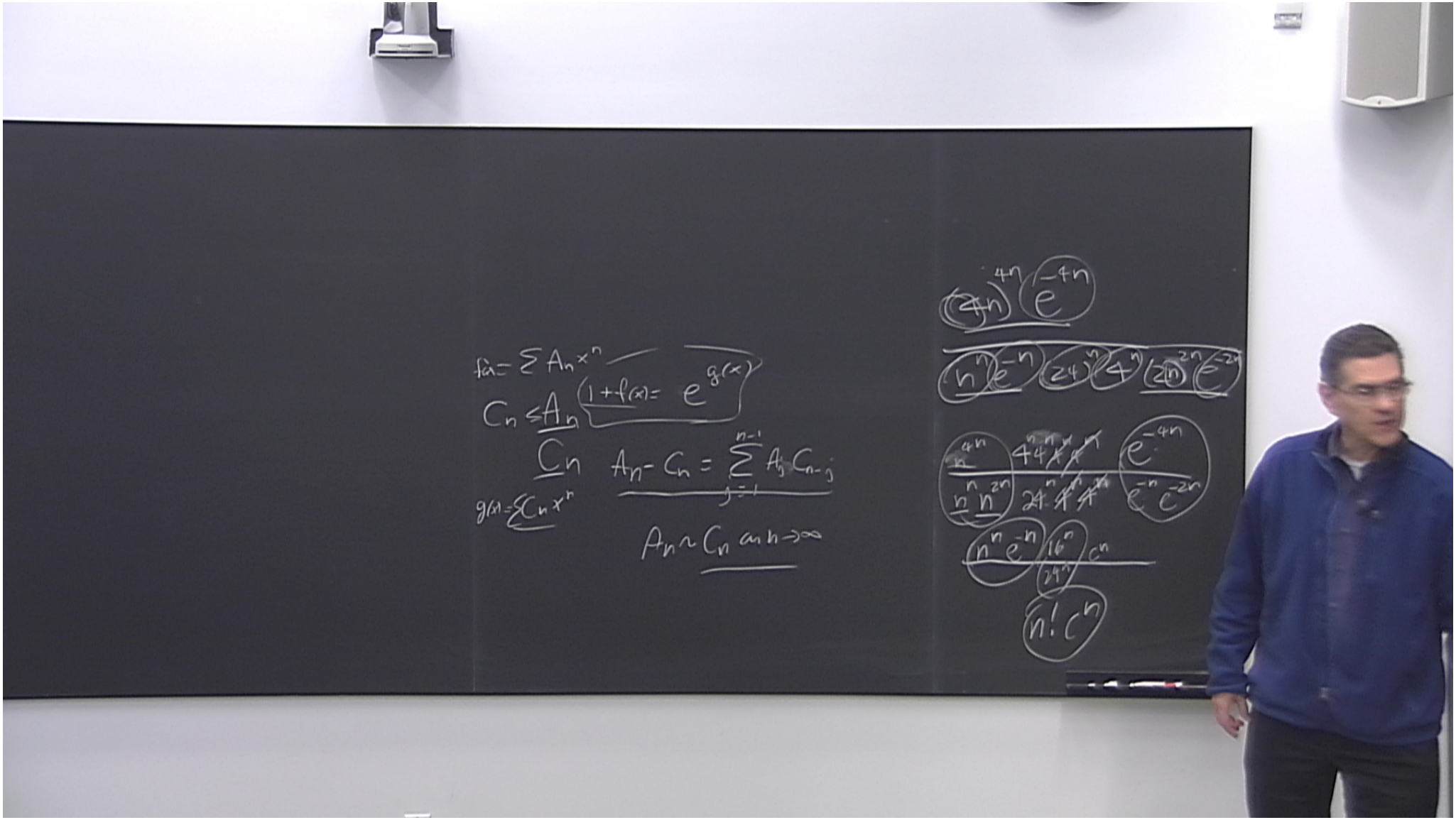

⇒ Padé's converge
$\sum_{n=0} a_n \epsilon^n$
$P^n_{n+1}(\epsilon)$
L_1
L_2
n
if $\{a_n\}$ grow no faster than $(2n)!$ then $L_1 = L_2$
if $V(x) = x^2$ $W(x) = x^4$ then $a_n \sim n!\, c^n$
$g(\varphi^4$
Σ all graphs with n vertices $= a_n$
n verts
$(4n-1)(4n-3)\ldots$
$(4n)!$
$\frac{(4n-1)!!}{n!\,(24)^n}$
$(2n)!\,2^{2n}$


$f(x) = \sum A_n x^n$
$1 + f(x) = e^{g(x)}$
$C_n \le A_n$
C_n
$A_n - C_n = \sum_{j=1}^{n-1} A_j C_{n-j}$
$g(x) = \sum C_n x^n$
$A_n \sim C_n$ as $n \to \infty$
$(4n)^{4n} e^{-4n}$


$E(\epsilon) \sim \sum_{n=0}^{\infty} a_n \epsilon^n$ as $\epsilon \to 0$.
$E(\epsilon)$ has properties 1,2,3,4 $\Rightarrow$ $E(\epsilon)$ is fn of Stieltjes
$\Rightarrow$ Padé's converge
$P^n_{n+1} < E(\epsilon) < P$
$\sum_{n=0}^{\infty} a_n \epsilon^n \to \{ P^n_n(\epsilon) \downarrow , P^n_{n+1}(\epsilon) \uparrow$
if $\{a_n\}$ grow no faster than $(2n)!$ then $L_1 = L_2$
if $V(x) = x^2$ $W(x) = x^4$ then $a_n \sim n!\, c^n$
Σ all graphs with n vertices $= a_n$
$(4n)!$
$(4n-1)!!$
$n!$ $(2^4)^n$ $(2n)!$ 2^{2n}
$f(x) = \sum A_n x^n$
$C_n \le A_n$
$1 + f(x) = e^{g(x)}$
C_n $A_n - C_n = \sum_{j=1}^{n-1} A_j C_{n-j}$
$g(x) = \sum C_n x^n$
$A_n \sim C_n$ as $n \to \infty$
$(4n)^{4n} e^{-4n}$
$n^n e^{-n} (2^4)^n 4^n (2n)^{2n} e^{-2n}$


$f(x) = \sum A_n x^n$
$1 + f(x) = e^{g(x)}$
$C_n \le A_n$
C_n
$A_n - C_n = \sum_{j=1}^{n-1} A_j C_{n-j}$
$g(x) = \sum C_n x^n$
$A_n \sim C_n$ as $n \to \infty$
$(4n)^{4n} e^{-4n}$
n^{4n}
$n^n n^{2n}$


$f(x) = \sum A_n x^n$
$C_n \le A_n$
$1 + f(x) = e^{g(x)}$
C_n
$A_n - C_n = \sum_{j=1}^{n-1} A_j C_{n-j}$
$g(x) = \sum C_n x^n$
$A_n \sim C_n$ as $n \to \infty$
$(4n)^{4n} e^{-4n}$
$n^{4n} \quad 4^{4n} \quad e^{-4n}$
$n^n n^{2n} \; 2^{4n} 4^n 4^n \; e^{-n} e^{-2n}$

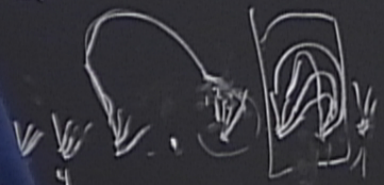

$f(x) = \sum A_n x^n$
$1 + f(x) = e^{g(x)}$
$C_n \le A_n$
C_n
$A_n - C_n = \sum_{j=1}^{n-1} A_j C_{n-j}$
$g(x) = \sum C_n x^n$
$A_n \sim C_n$ as $n \to \infty$
$(4n)^{4n} e^{-4n}$
$n^n e^{-n}$ $(2^4)^n$ 4^n $(2n)^{2n} e^{-2n}$
$n^n n^{2n}$ $2^n 4^n$ e^{-4n} $e^{-n} e^{-2n}$
$n^n e^{-n}$ 16^n c^n
2^{4n}
$n! c^n$

if $\{a_n\}$ grow no faster than $(2n)!$ then $L_1 = L_2$

If $V(x) = x^2$ $W(x) = x^4$ then $a_n \sim n!\, c^n$

Σ all graphs with n vertices $= a_n$





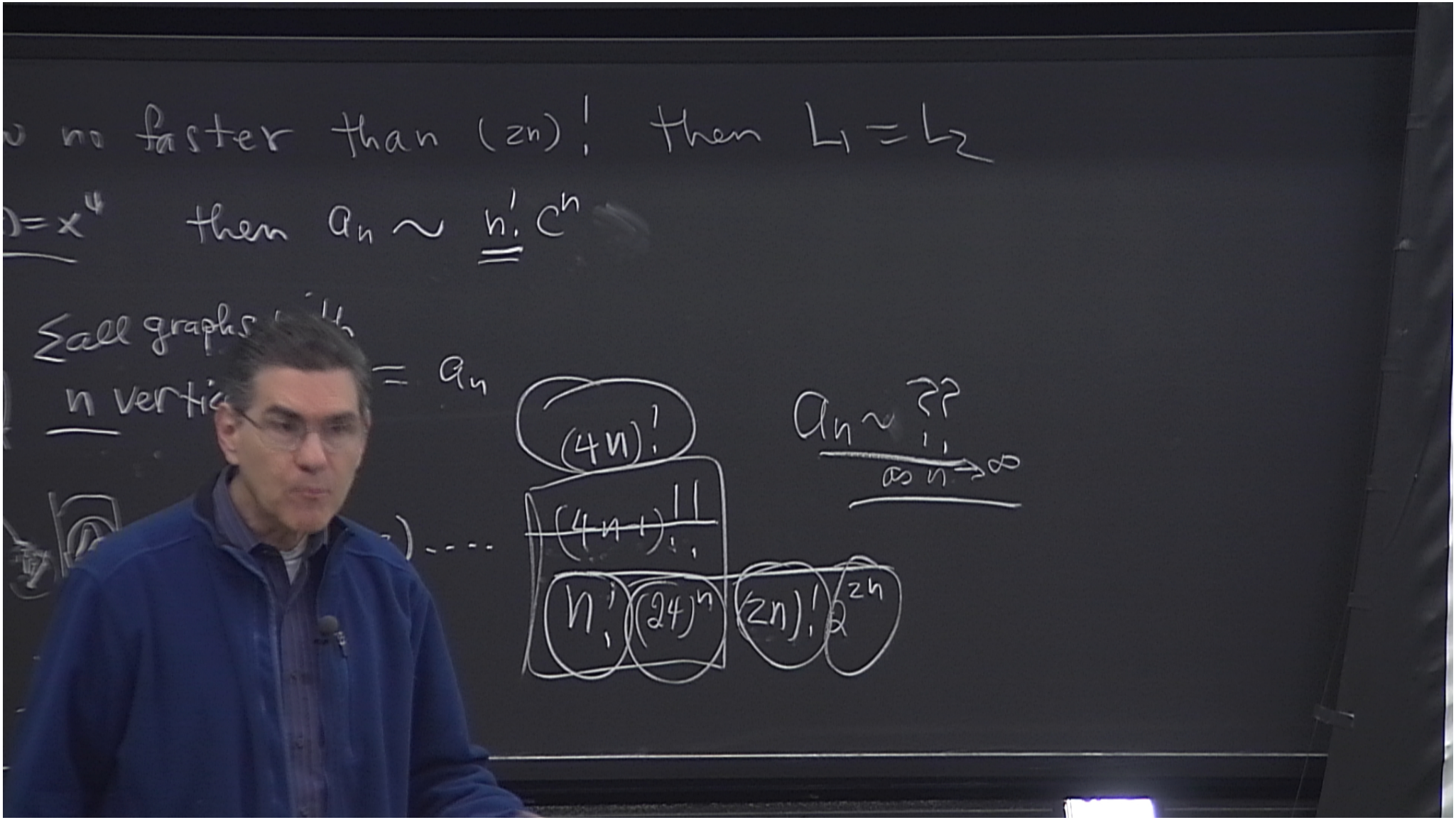
$(4n-1)(4n-3)\cdots$

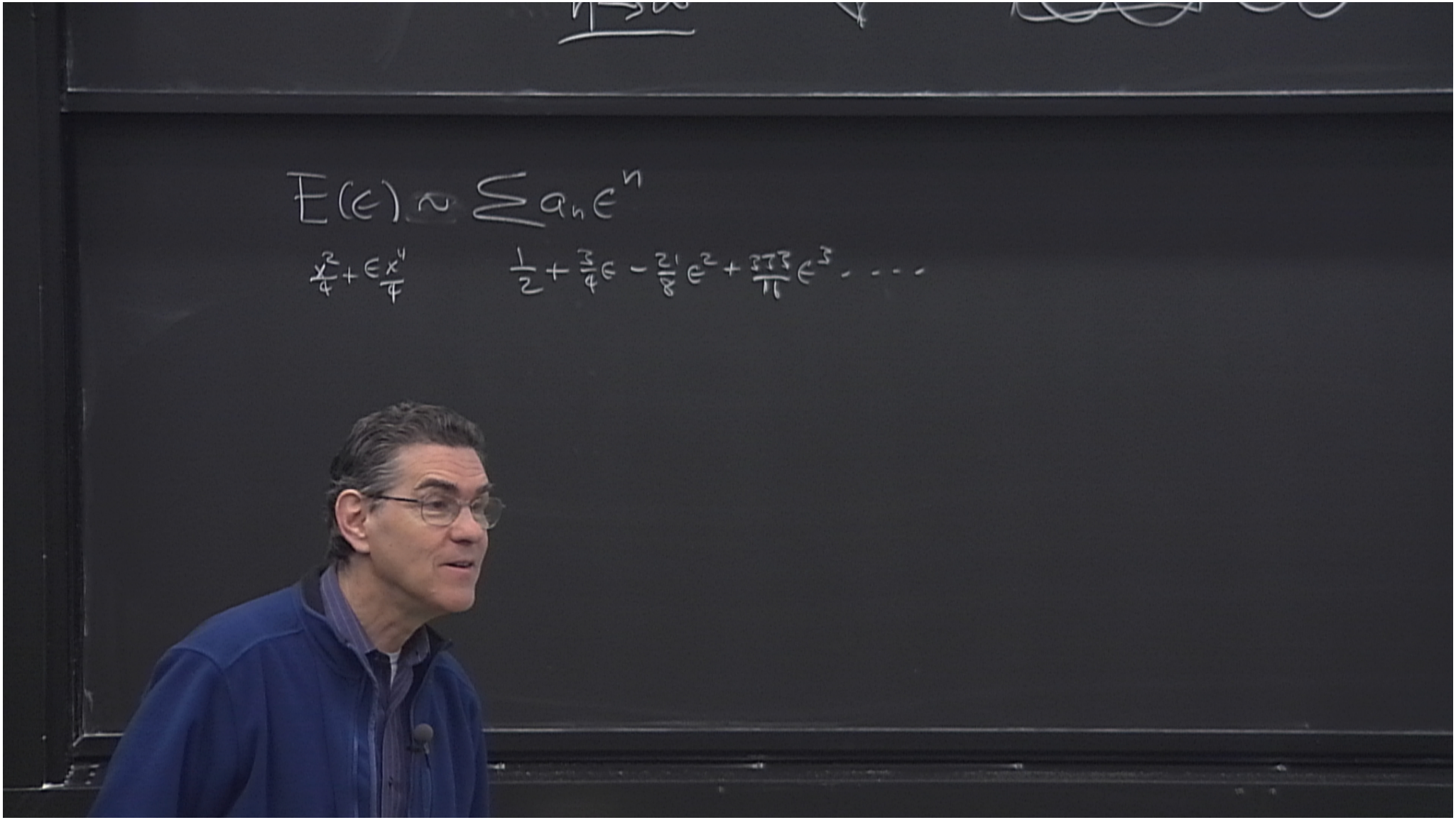
$(4n)!$

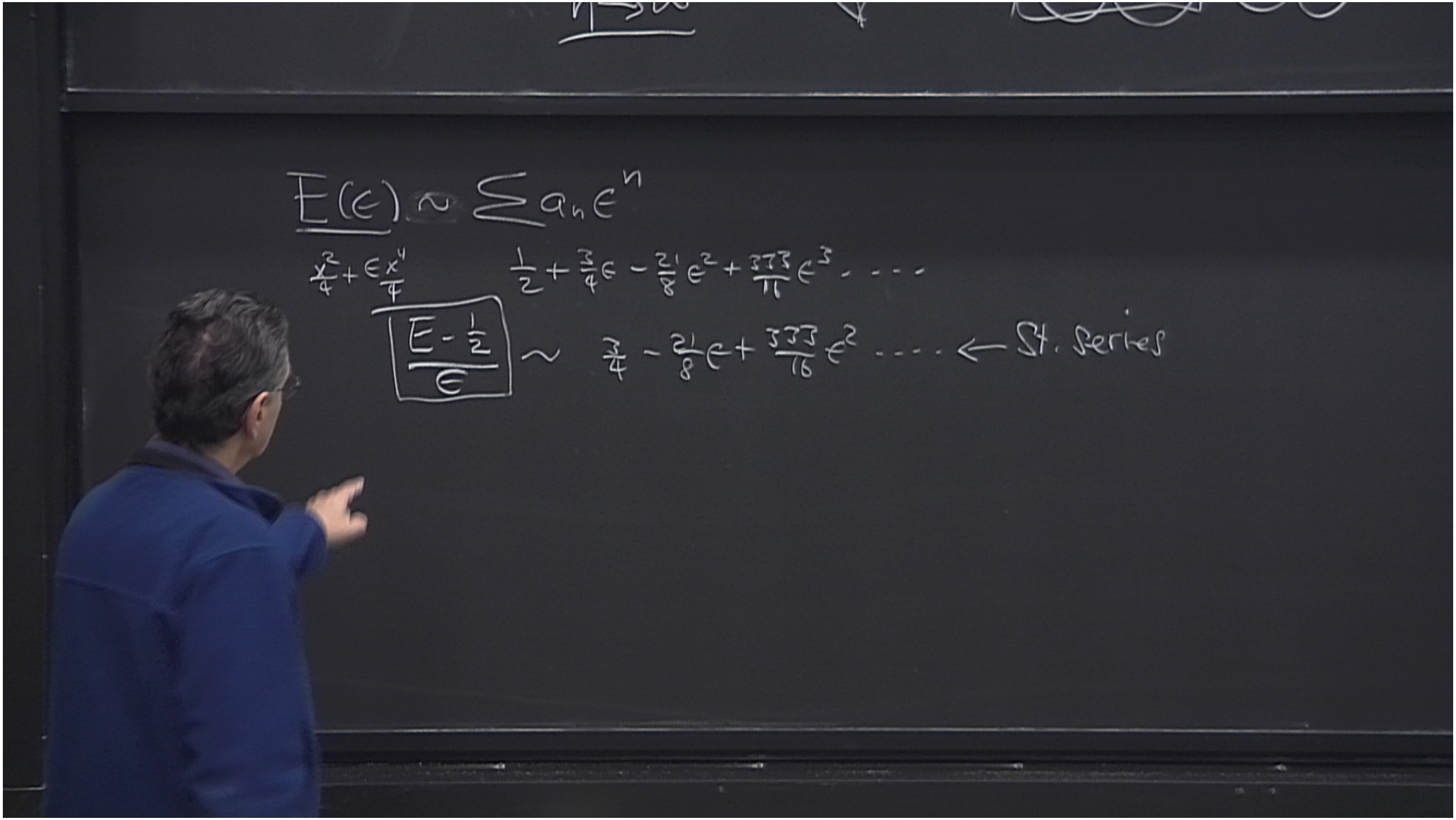
$(4n+1)!!$

$n!$ $(24)^n$ $(2n)!$ 2^{2n}


E(ε) ~ Σ a_n ε^n as ε → 0
E(ε) has properties 1,2,3,4 ⇒ E(ε) is fn of Stieltjes
⇒ Padé's converge
if {a_n} grow no faster than (2n)! then L1 = L2
if V(x) = x^2 W(x) = x^4 then a_n ~ n! c^n
Σ all graphs with n vertices = a_n
n → ∞
(4n)!
n!
(2n)!


no faster than (2n)! then $L_1 = L_2$
$= x^4$ then $a_n \sim n!\, c^n$
$= a_n$
$a_n \sim ??$
∞ $n \to \infty$
(4n)!
(4n+1)!!
n! (24)^n (2n)! 2^{2n}

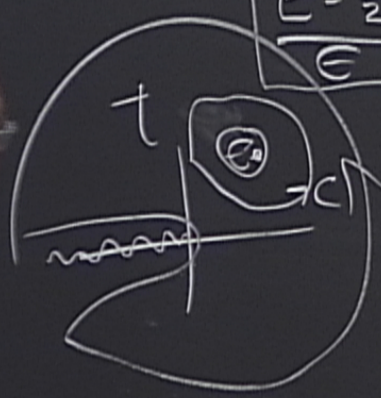

$E(\epsilon) \sim \sum a_n \epsilon^n$
$\frac{x^2}{4} + \epsilon \frac{x^4}{4}$
$\frac{1}{2} + \frac{3}{4}\epsilon - \frac{21}{8}\epsilon^2 + \frac{333}{11}\epsilon^3 \ldots$


$E(\epsilon) \sim \sum a_n \epsilon^n$
$\frac{x^2}{4} + \epsilon \frac{x^4}{4}$
$\frac{1}{2} + \frac{3}{4}\epsilon - \frac{21}{8}\epsilon^2 + \frac{333}{16}\epsilon^3 \ldots$
$\frac{E - \frac{1}{2}}{\epsilon} \sim \frac{3}{4} - \frac{21}{8}\epsilon + \frac{333}{16}\epsilon^2 \ldots$ ← St. series

$E(\epsilon) \sim \sum a_n \epsilon^n$

$\frac{x^3}{4} + \epsilon \frac{x^4}{4}$ $\quad \frac{1}{2} + \frac{3}{4}\epsilon - \frac{21}{8}\epsilon^2 + \frac{333}{16}\epsilon^3 - \cdots$

$\frac{E - \frac{1}{2}}{\epsilon} \sim \frac{3}{4} - \frac{21}{8}\epsilon + \frac{333}{16}\epsilon^2 \cdots \leftarrow$ St. series

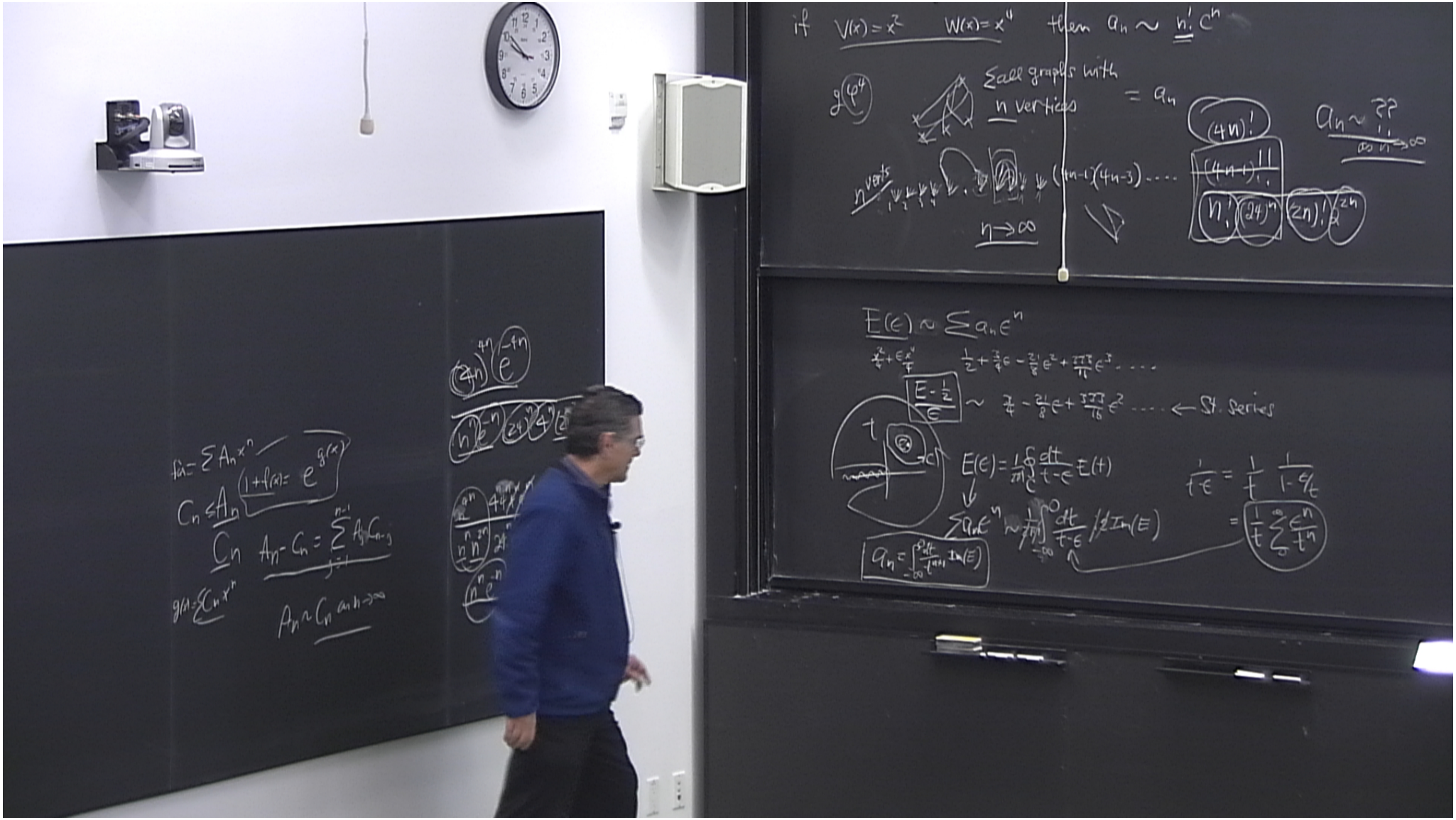


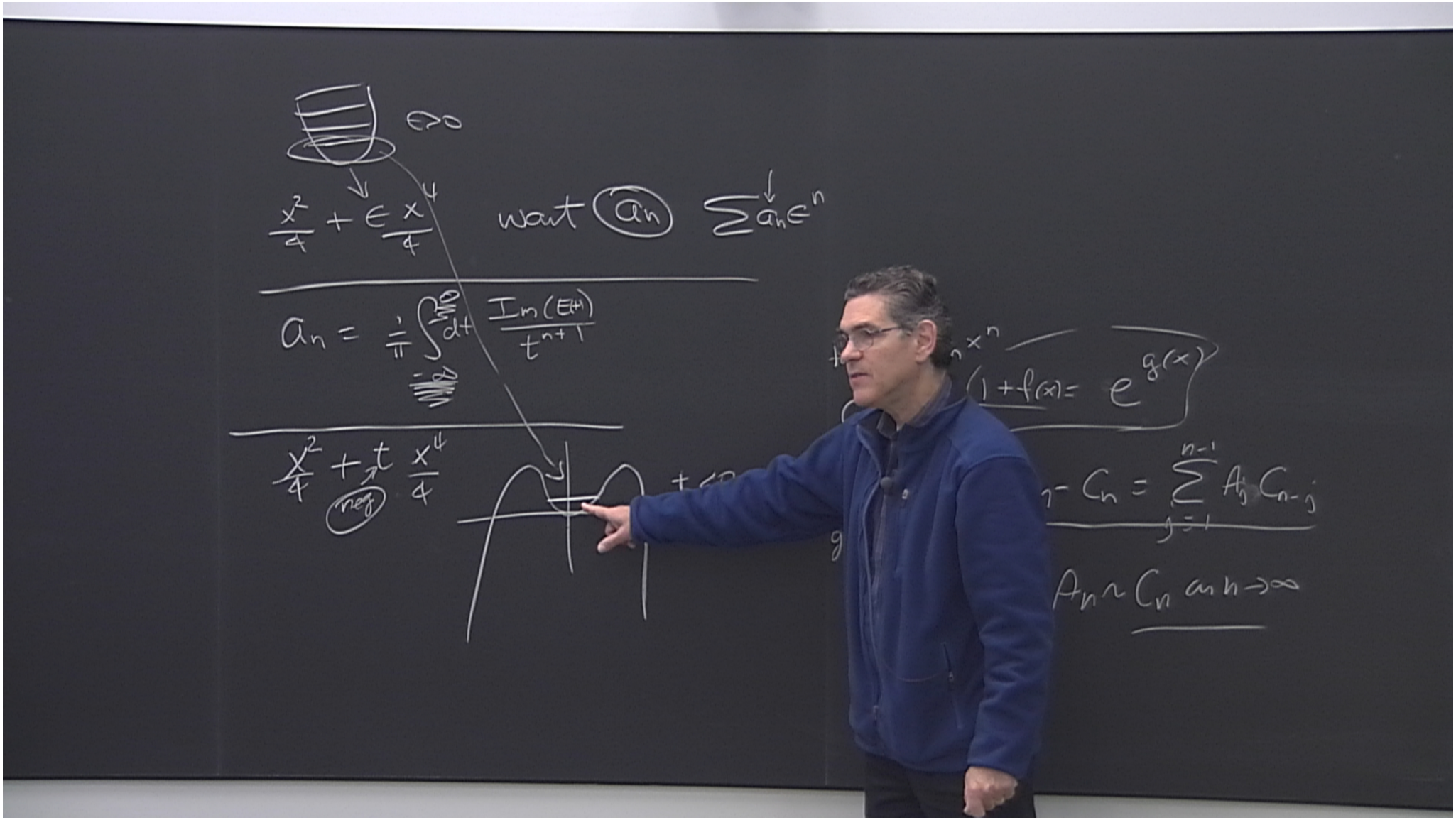
$E(\epsilon) = \frac{1}{2\pi i}\oint_C \frac{dt}{t-\epsilon} E(t)$

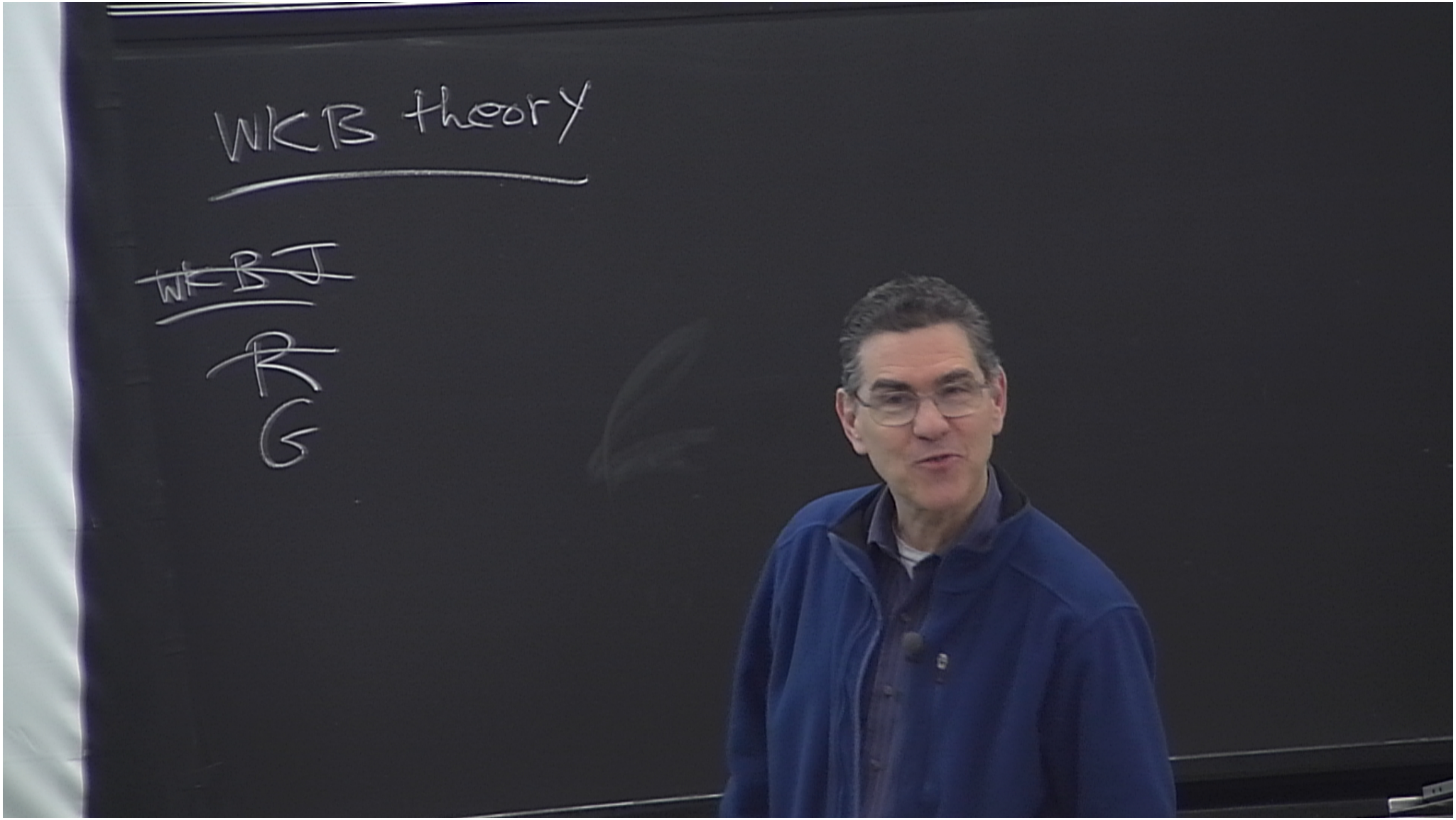
$= \frac{1}{2\pi i}\int_{-\infty}^{0} \frac{dt}{t-\epsilon} \cdot 2\,\mathrm{Im}(E)$

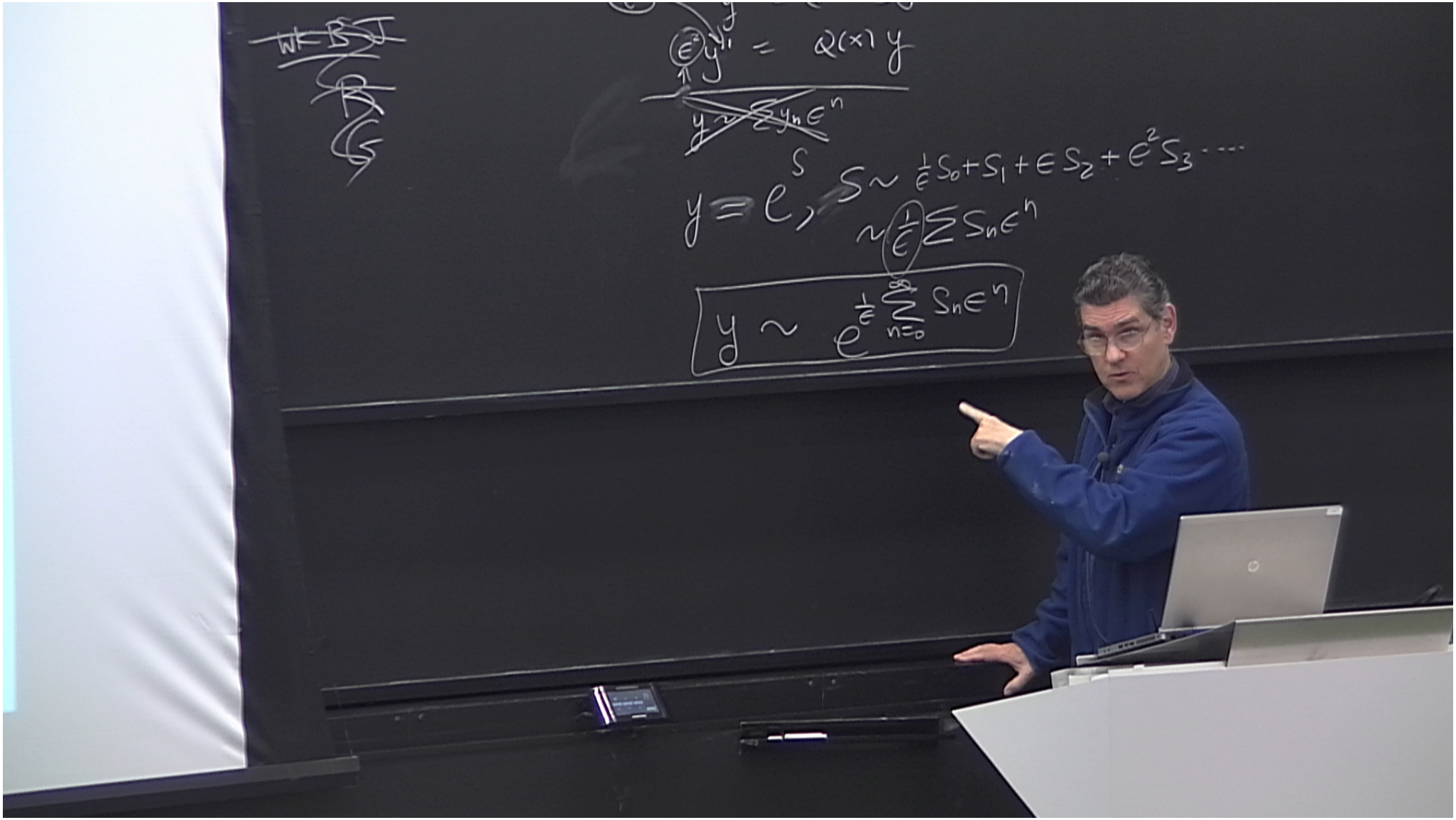
$\frac{1}{t-\epsilon} = \frac{1}{t}\frac{1}{1-\epsilon/t}$

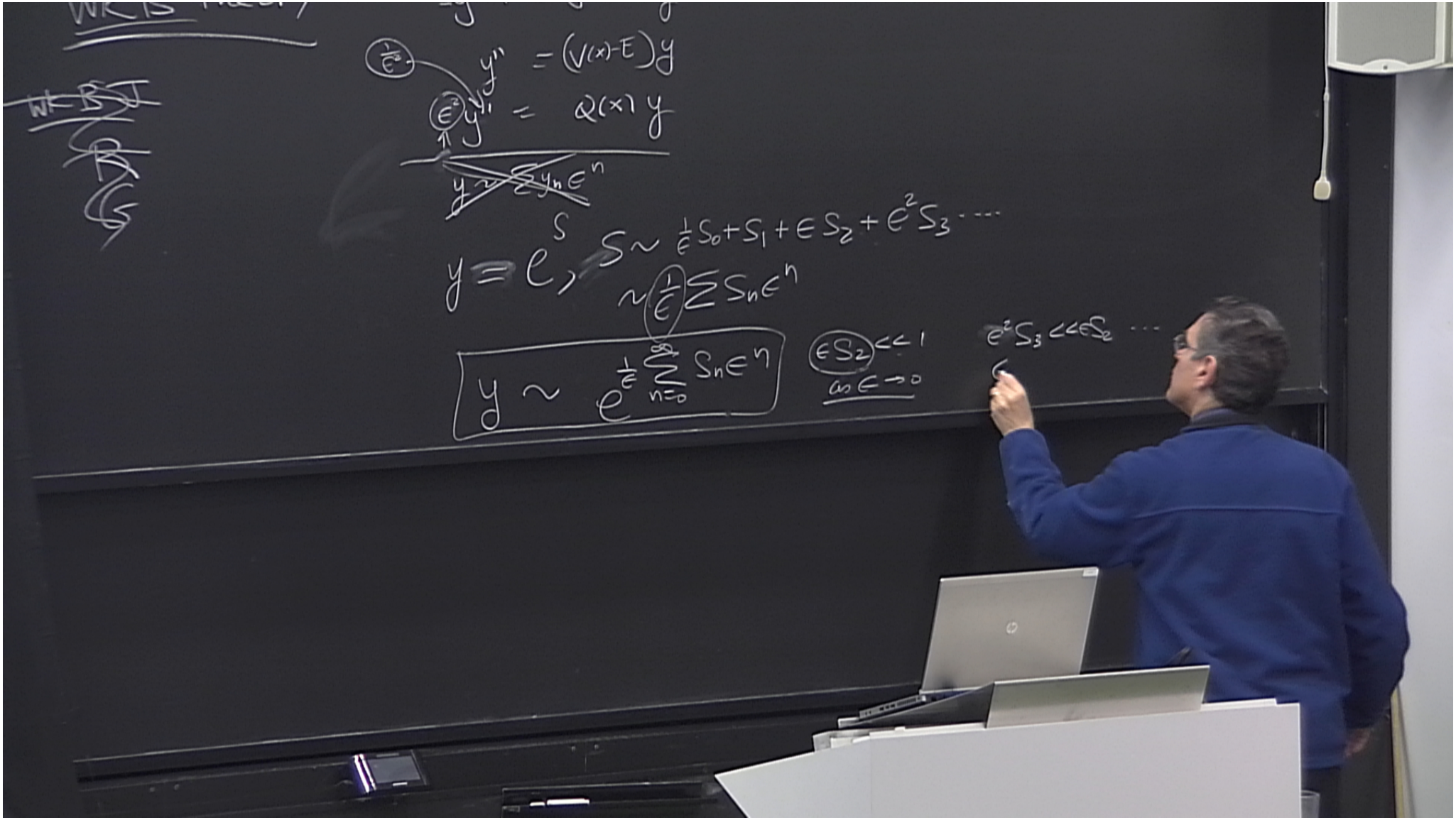
$= \frac{1}{t}\sum_0^\infty \frac{\epsilon^n}{t^n}$

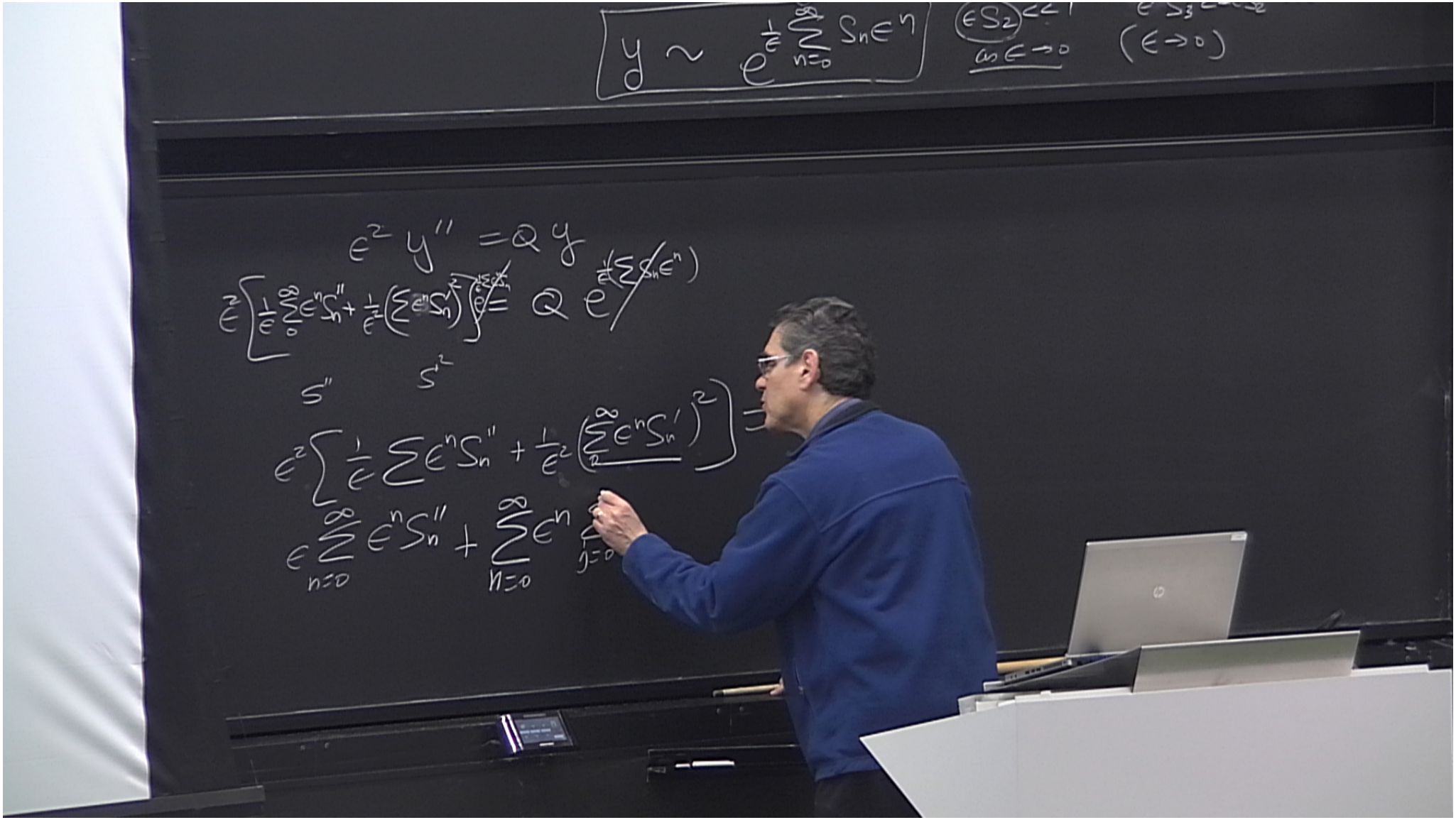

if $V(x)=x^2$ $W(x)=x^4$ then $a_n \sim n!\,c^n$
$g(\varphi^4)$
Σ all graphs with n vertices $= a_n$
n verts
$n \to \infty$
$(4n)!$
$(4n-1)!!$
$n!\,(24)^n$
$(2n)!\,2^{2n}$
$a_n \sim ??$
$n \to \infty$
$E(\epsilon) \sim \Sigma a_n \epsilon^n$
$\frac{1}{2} + \frac{3}{4}\epsilon - \frac{21}{8}\epsilon^2 + \frac{231}{16}\epsilon^3 - \dots$
$\frac{E-\frac{1}{2}}{\epsilon} \sim \frac{3}{4} - \frac{21}{8}\epsilon + \frac{231}{16}\epsilon^2 - \dots$ ← St. Series
$E(\epsilon) = \frac{1}{2\pi i}\oint \frac{dt}{t-\epsilon} E(t)$
$\frac{1}{t-\epsilon} = \frac{1}{t}\frac{1}{1-\frac{\epsilon}{t}}$
$= \frac{1}{t}\sum_0^\infty \frac{\epsilon^n}{t^n}$
$\Sigma a_n \epsilon^n \sim \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{dt}{t-\epsilon}\, \frac{1}{2}\mathrm{Im}(E)$
$a_n = \int_{-\infty}^{\infty} \frac{dt}{t^{n+1}}\, \mathrm{Im}(E)$
$f(x) = \Sigma A_n x^n$
$1 + f(x) = e^{g(x)}$
$C_n \le A_n$
$A_n - C_n = \sum_{j=1}^{n-1} A_j C_{n-j}$
$g(x) = \Sigma C_n x^n$
$A_n \sim C_n$ as $n \to \infty$

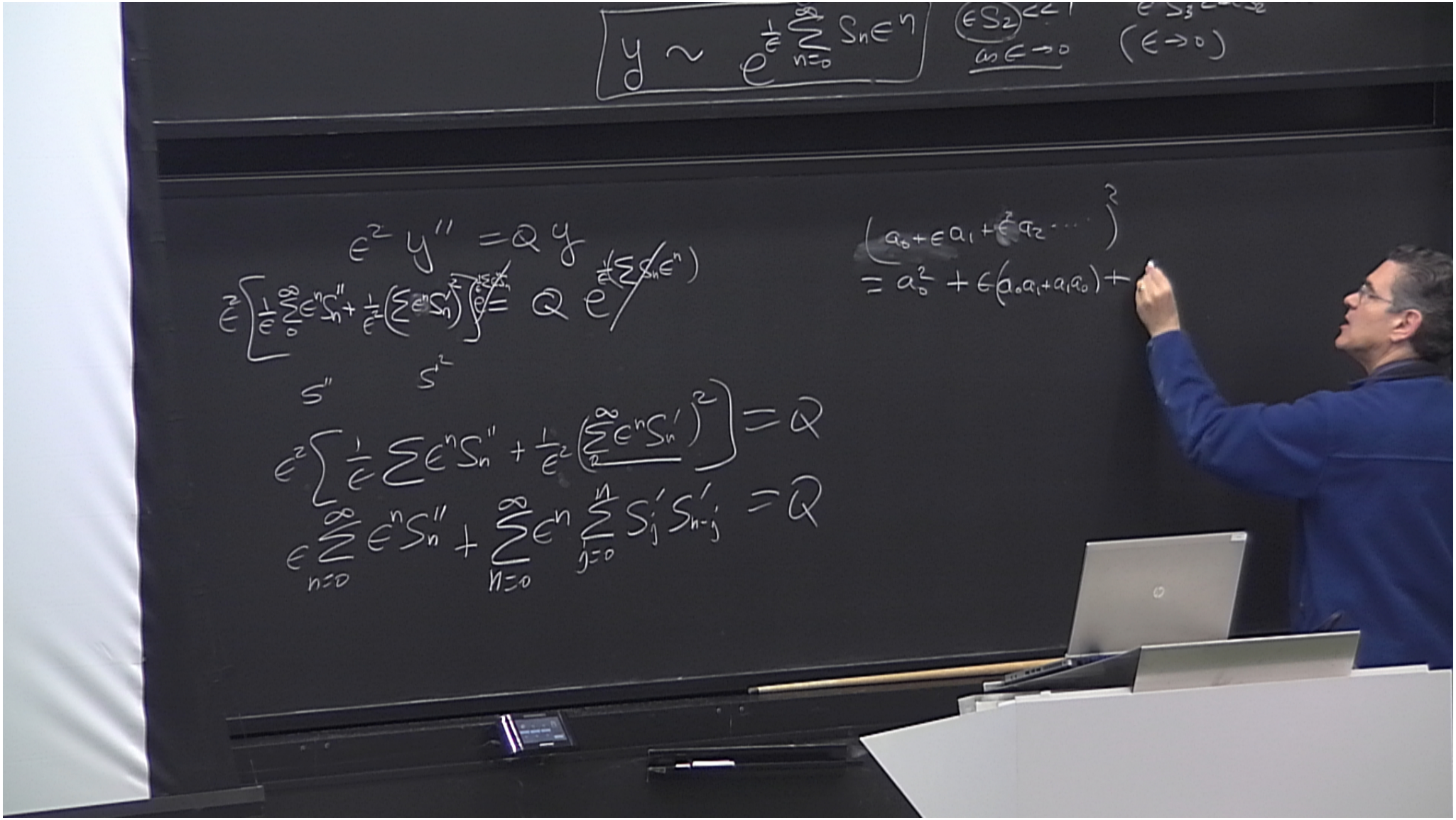

$\epsilon > 0$
$\frac{x^2}{4} + \epsilon \frac{x}{4}$
want a_n $\sum a_n \epsilon^n$
$a_n = \frac{1}{\pi} \int \frac{\mathrm{Im}(E+i)}{t^{n+1}}$
$\frac{x^2}{4} + t \frac{x}{4}$
neg
$1 + f(x) = e^{g(x)}$
$C_n = \sum_{j=1}^{n-1} A_j C_{n-j}$
$A_n \sim C_n$ as $n \to \infty$

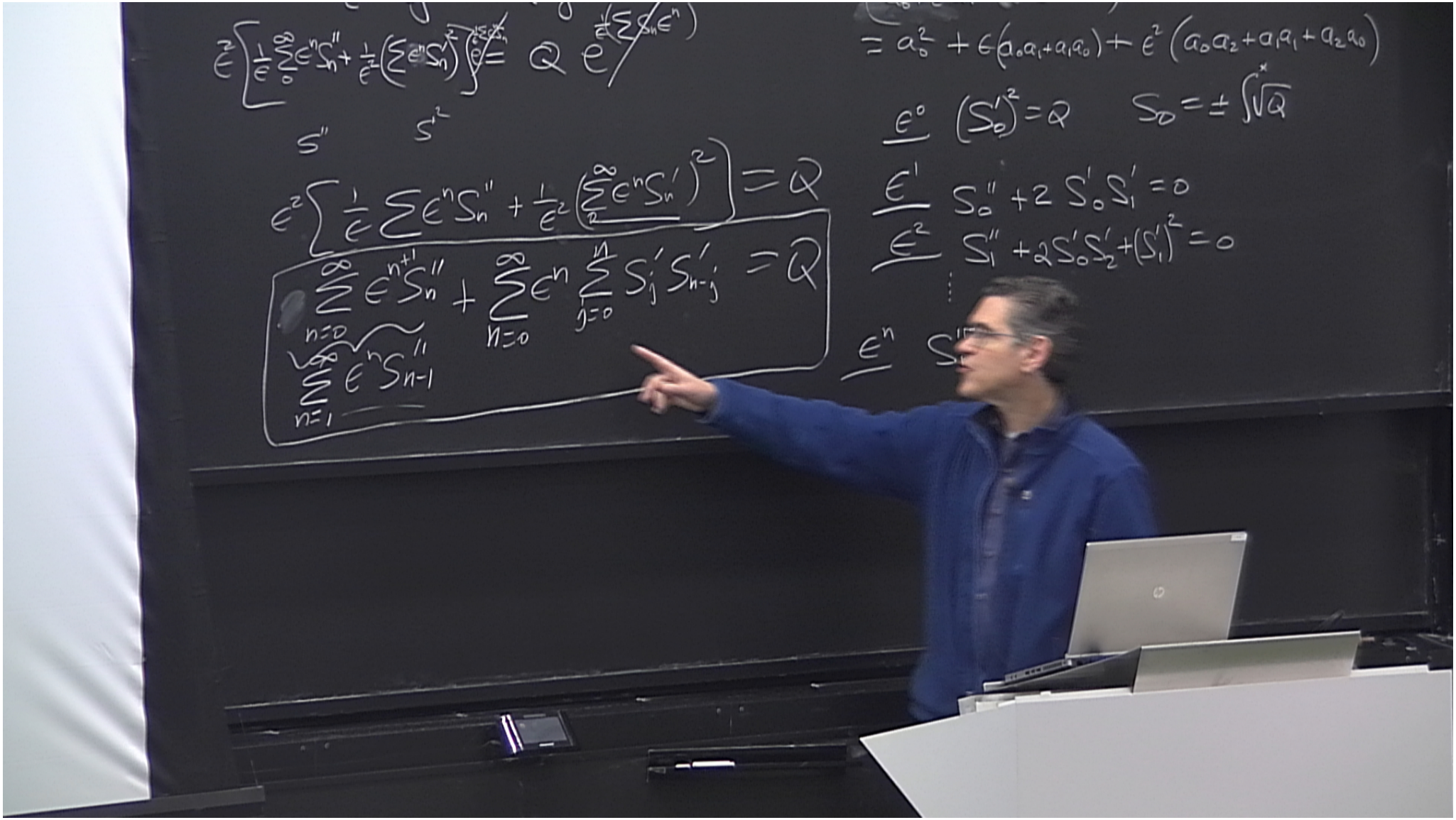

WKB theory
WKBJ
R
G

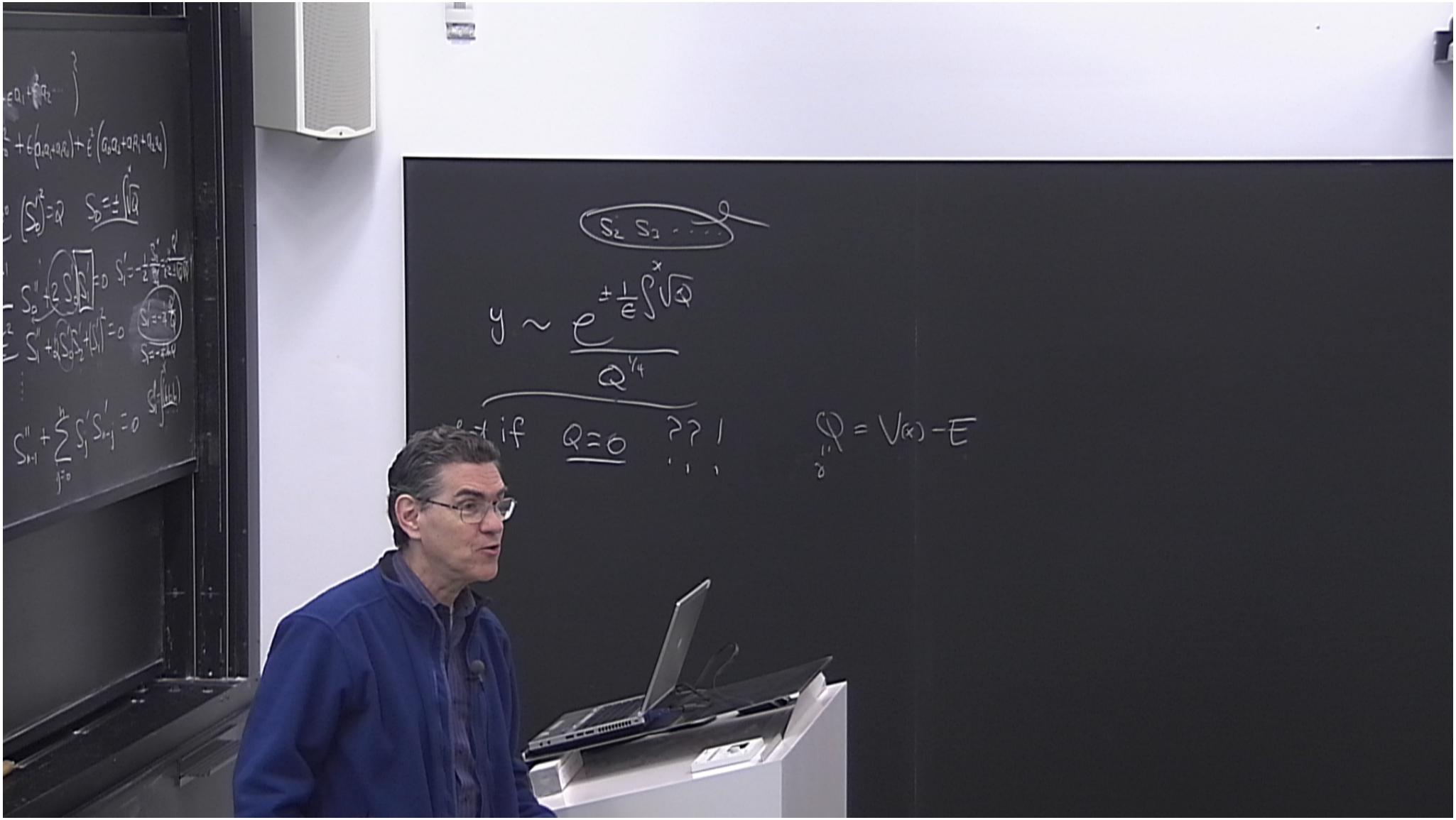

WKBJ
$\epsilon^2 y'' = Q(x)\, y$
$y = e^{S}, \quad S \sim \frac{1}{\epsilon} S_0 + S_1 + \epsilon S_2 + \epsilon^2 S_3 \cdots$
$\sim \frac{1}{\epsilon} \sum S_n \epsilon^n$
$y \sim e^{\frac{1}{\epsilon} \sum_{n=0}^{\infty} S_n \epsilon^n}$


WKB
WKB
$\epsilon^2 y'' = (V(x)-E)y$
$y'' = Q(x)\,y$
$y = e^{S}$, $S \sim \frac{1}{\epsilon}S_0 + S_1 + \epsilon S_2 + \epsilon^2 S_3 \cdots$
$\sim \frac{1}{\epsilon}\sum S_n \epsilon^n$
$y \sim e^{\frac{1}{\epsilon}\sum_{n=0}^{\infty} S_n \epsilon^n}$
$\epsilon S_2 \ll 1$
as $\epsilon \to 0$
$\epsilon^2 S_3 \ll \epsilon S_2 \cdots$


$y \sim e^{\frac{1}{\epsilon}\sum_{n=0}^{\infty} S_n \epsilon^n}$
$\epsilon S_2 \ll 1$
$\omega\epsilon \to 0$
$(\epsilon \to 0)$
$\epsilon^2 y'' = Q y$
$\epsilon^2\left[\frac{1}{\epsilon}\sum_0^\infty \epsilon^n S_n'' + \frac{1}{\epsilon^2}\left(\sum \epsilon^n S_n'\right)^2\right] = Q e^{\frac{1}{\epsilon}\sum S_n \epsilon^n}$
S''
S'^2
$\epsilon^2\left[\frac{1}{\epsilon}\sum \epsilon^n S_n'' + \frac{1}{\epsilon^2}\left(\sum_0^\infty \epsilon^n S_n'\right)^2\right] =$
$\epsilon \sum_{n=0}^{\infty} \epsilon^n S_n'' + \sum_{n=0}^{\infty} \epsilon^n$


$y \sim e^{\frac{1}{\epsilon}\sum_{n=0}^{\infty} S_n \epsilon^n}$
$\epsilon S_2 \ll 1$
$\epsilon \to 0$
$\epsilon^2 y'' = Q y$
$\epsilon^2 \left[\frac{1}{\epsilon} \sum_0^\infty \epsilon^n S_n'' + \frac{1}{\epsilon^2} \left(\sum \epsilon^n S_n' \right)^2 \right] = Q$
S'' S'^2
$\epsilon^2 \left[\frac{1}{\epsilon} \sum \epsilon^n S_n'' + \frac{1}{\epsilon^2} \left(\sum_0^\infty \epsilon^n S_n' \right)^2 \right] = Q$
$\epsilon \sum_{n=0}^{\infty} \epsilon^n S_n'' + \sum_{n=0}^{\infty} \epsilon^n \sum_{j=0}^{n} S_j' S_{n-j}' = Q$
$(a_0 + \epsilon a_1 + \epsilon^2 a_2 \cdots)^2$
$= a_0^2 + \epsilon (a_0 a_1 + a_1 a_0) +$


$\epsilon^2\left[\frac{1}{\epsilon}\sum_0^\infty \epsilon^n S_n'' + \frac{1}{\epsilon^2}\left(\sum \epsilon^n S_n'\right)^2\right] = Q$
$\sum_{n=0}^{\infty} \epsilon^{n+1} S_n'' + \sum_{n=0}^{\infty} \epsilon^n \sum_{j=0}^{n} S_j' S_{n-j}' = Q$
$\sum_{n=1}^{\infty} \epsilon^n S_{n-1}''$
$= a_0^2 + \epsilon(a_0 a_1 + a_1 a_0) + \epsilon^2(a_0 a_2 + a_1 a_1 + a_2 a_0)$
$\epsilon^0 \quad (S_0')^2 = Q \qquad S_0 = \pm\int^x \sqrt{Q}$
$\epsilon^1 \quad S_0'' + 2 S_0' S_1' = 0$
$\epsilon^2 \quad S_1'' + 2 S_0' S_2' + (S_1')^2 = 0$


$S_2\ S_1 \cdots$
$y \sim \frac{e^{\pm\frac{1}{\epsilon}\int^x \sqrt{Q}}}{Q^{1/4}}$
if $Q=0$??!
$\frac{\hbar^2}{2m}\psi'' = V(x) - E$