

Title: Mathematical Physics - Lecture 11

Date: Dec 05, 2011 09:00 AM

URL: <http://pirsa.org/11120005>

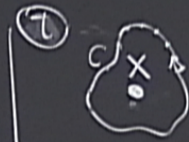
Abstract:

④ $-f(x)$ is Herglotz $\leftarrow$ $\begin{matrix} \text{Stieltjes} \\ \text{in UHP } (\operatorname{Im} x > 0), \operatorname{Im}[f(x)] > 0 \\ \text{in LHP } (\operatorname{Im} x < 0), \operatorname{Im}[-f(x)] < 0 \end{matrix}$

if $f(x)$ satisfies
 ① ② ③ ④ $\Rightarrow f$ is Stieltjes!

Cauchy

$$f(x) = \frac{1}{z\pi i}$$



Stieltjes fns. $f(x)$

$$f(x) \equiv \int_0^{\infty} \frac{dt p(t)}{1+xt}$$

$p(t) \geq 0$ for all $t \geq 0$
and $a_n = \int_0^{\infty} dt p(t) t^n$ exists

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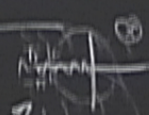
$p(t) \geq 0$ for all $t \geq 0$
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① $f(x)$ is anal in cut- x plane. ~~non-anal~~ ^② (anal means $f'(x)$ exists as a complex deriv.)

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- ① $f(x)$ is anal in cut- x plane.  (anal means $f'(x)$ exists as a complex deriv.)
- ② $f(x) \rightarrow 0$ as $|x| \rightarrow \infty$ in 
- ③ $f(x) \sim \sum_{n=0}^{\infty} (-1)^n a_n x^{-n}$ as $|x| \rightarrow \infty$, $-\pi < \arg x < \pi$
Stieltjes series
- ④

$$s(x) = \int_0^\infty (1+xt)^{-1} dt$$

$p(t) \geq 0$ for all $t \geq 0$
and $a_n = \int_0^\infty t^n p(t) dt$ exists

- ① $f(x)$ is anal in cut- x plane. ~~normal~~ [⊗] (anal means $f'(x)$ exists as a complex deriv.)
- ② $f(x) \rightarrow 0$ as $|x| \rightarrow \infty$ in $\nearrow$
- ③ $f(x) \sim \sum_{n=0}^{\infty} (-1)^n a_n x^n$ as $|x| \rightarrow 0$, $-\pi < \arg x < \pi$
- ④ $-f(x)$ is Herglotz $\leftarrow$ ^{strictly series} in UHP ($\text{Im} x > 0$), $\text{Im}[f(x)] > 0$
in LHP ($\text{Im} x < 0$), $\text{Im}[-f(x)] < 0$

ts

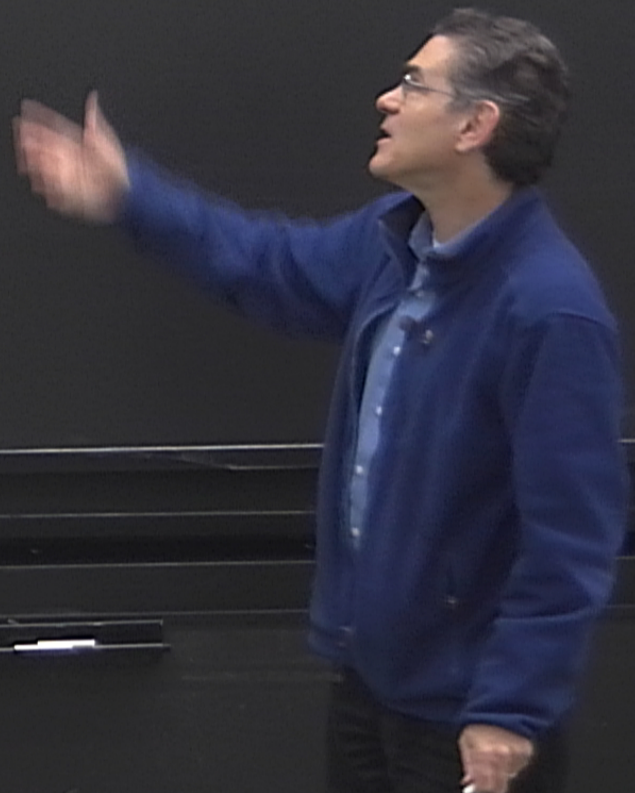
⊕ (anal means $f'(x)$ exists as a complex deriv.)

$$-\pi < \arg \lambda \leq \pi$$

$$(\operatorname{Im} x > 0), \operatorname{Im}[f(x)] > 0$$

$$(\operatorname{Im} x < 0), \operatorname{Im}[-f(x)] < 0$$

①
$$f'(x) = - \int_0^{\infty} \frac{p(t) dt}{(1+xt)^2}$$



means $f'(x)$ exists
complex deriv.

$\operatorname{Re} f(x) > 0$
 $\operatorname{Im} f(x) < 0$

$$\textcircled{1} \quad f'(x) = - \int_0^{\infty} \frac{p(t) dt}{(1+xt)^2}$$

$$\textcircled{2} \quad x = a + ib$$

$$f(x) = \int_0^{\infty} \frac{dt p(t)}{1 + (a+ib)t} = \frac{1+a-i}{1+a-i} \int_0^{\infty} \frac{dt p(t)}{1+a-i+it}$$

$$= \int_0^{\infty} \frac{dt p(t) (1+a-ibt)}{(1+at)^2 + b^2 t^2}$$

$$\operatorname{Im} f(x) = -b \int_0^{\infty} \frac{dt p(t) t}{(1+at)^2 + b^2 t^2}$$

$$\textcircled{1} \quad f'(x) = - \int_0^{\infty} \frac{p(t) dt}{(1+xt)^2}$$

$$\textcircled{2} \quad x = a + ib$$

$$f(x) = \int_0^{\infty} \frac{dt p(t)}{1 + (a+ib)t} \frac{1+a-t-ibt}{1+a-t-ibt}$$

$$= \int_0^{\infty} \frac{dt p(t) (1+a-t-ibt)}{(1+at)^2 + b^2 t^2}$$

$$\text{Im} f(x) = -b \int_0^{\infty} \frac{dt p(t) t}{(1+at)^2 + b^2 t^2}$$

$$\text{Im}(f(x)) \propto$$

Herglotz is powerful!

(entire means anal all finite $\mathbb{C} \times \mathbb{X}$)

ex: $\text{poly}, e^x, \sin x, \sinh x, \dots, Ai(x), \dots$

if $f(x)$ is entire & Herglotz then...

Herglotz is powerful!

(entire means anal all finite cxX)

ex: poly, e^x , $\sin x$, $\sinh x$, ..., $Ai(x)$, ...

if $f(x)$ is entire & Herglotz then...

entire $f(x) = \sum_{n=0}^{\infty} a_n x^n$ (all x)

$$x = re^{i\theta}$$
$$f(x) = \sum_{n=0}^{\infty} a_n r^n e^{in\theta}$$

Herglotz is powerful!

(entire means anal all finite $\mathbb{C} \times \mathbb{X}$)

ex: $\text{poly}, e^x, \sin x, \sinh x, \dots, \text{Ai}(x), \dots$

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entire $f(x) = \sum_{n=0}^{\infty} a_n x^n$ (all x)

$x = re^{i\theta}$
 $f(x) = \sum_{n=0}^{\infty} a_n r^n \underbrace{e^{in\theta}}_{\cos n\theta + i \sin n\theta}$

$\text{Im} f(x) = \sum_{n=0}^{\infty} a_n r^n \sin(n\theta)$

Hergl $\Rightarrow$

8

is powerful!

anal all finite (x, X)

$x, \sin x, \dots, A(x), \dots$

& Herglotz then...

(all x)

$$\frac{e^{in\theta}}{\cos n\theta + i \sin n\theta}$$

$$n \sin(n\theta)$$

$$\operatorname{Im} X = \operatorname{Im} r e^{i\theta} = r \sin \theta$$

Hergl $\Rightarrow$

Sign of

$$\sum_{n=0}^{\infty} a_n r^n \sin(n\theta)$$

Same as
Sign of

$$\sin \theta$$

is powerful!
 anal all finite (x, X)

$x, \sin x, \dots, A(x), \dots$
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$$e^{in\theta} = \cos n\theta + i \sin n\theta$$

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Hergl $\Rightarrow$

Sign of

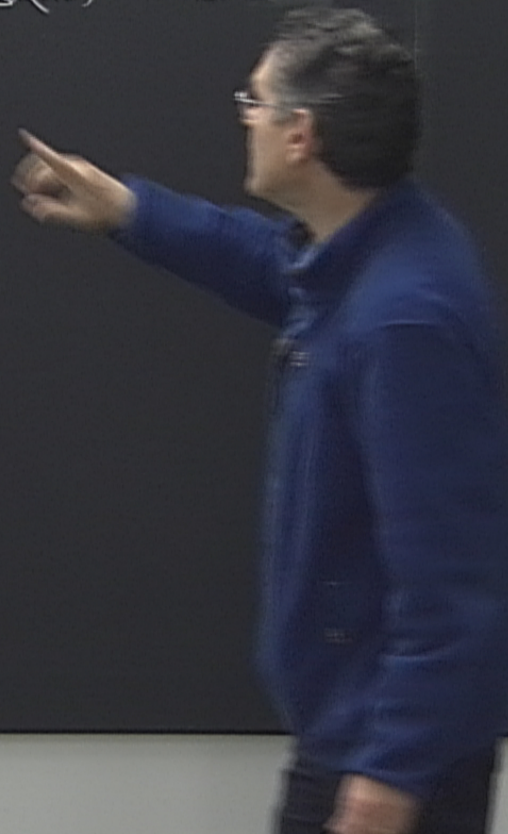
$$\sum_{n=0}^{\infty} a_n r^n \sin(n\theta)$$

Same as
 Sign of

$$\sin \theta$$

$$n \sin \theta \geq \sin(n\theta)$$

for $0 \leq \theta \leq \pi$



is powerful!

anal all finite αx)

$x, \sin x, \dots, A(x), \dots$

& Herglotz then...

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$$\frac{e^{in\theta}}{\cos n\theta + i \sin n\theta}$$

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Same as
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$$n \sin \theta \geq \sin(n\theta) \text{ for } 0 \leq \theta \leq \pi$$

$n \sin \theta \pm \sin(n\theta)$ same sign as

is powerful!

anal all finite (x) X)

x, $\sin x, \dots, A(x), \dots$

& Herglotz then...

(all x)

$$\frac{e^{in\theta}}{\cos n\theta + i \sin n\theta}$$

$$n \sin(n\theta)$$

$$mX = \text{Im} re^{i\theta} = r \sin \theta$$

Hergl $\Rightarrow$

Sign of

$$\sum_{n=0}^{\infty} a_n r^n \sin(n\theta)$$

Same as
Sign of

$$\sin \theta \pm \sin(n\theta)$$

$$n \sin \theta \geq \sin(n\theta) \text{ for } 0 \leq \theta \leq \pi$$

$$n \sin \theta \pm \sin(n\theta) \text{ same sign as } \sin \theta$$

$$\int_0^\pi d\theta \sin(n\theta) \sin(m\theta) = \begin{cases} 0 & n \neq m \\ \frac{\pi}{2} & n = m \end{cases}$$

$$n \sin \theta \pm \sin(n\theta), \int_0^\pi d\theta$$

is powerful!

anal all finite αx

$x, \sin x, \dots, A(x), \dots$

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(all x)

$$\frac{e^{in\theta}}{\cos n\theta + i \sin n\theta}$$

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$$m x = \operatorname{Im} r e^{i\theta} = r \sin \theta$$

Hergl $\Rightarrow$

Sign of

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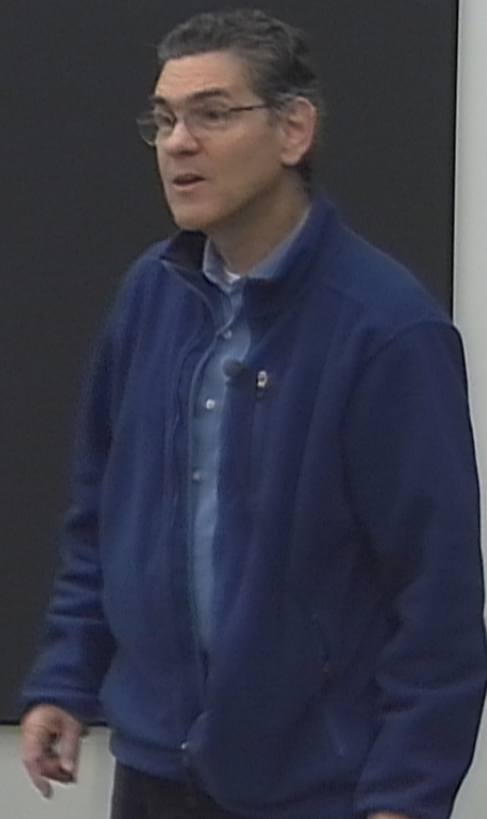
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$$n \sin \theta \pm \sin(n\theta), \int_0^\pi d\theta$$

$$\frac{\pi}{2} a_1 r \pm \frac{\pi}{2} a_n r^n \text{ same sign } \left(n \frac{\pi}{2} \right) +$$

$$a_n = 0 \text{ for } n > 1 \quad \left(\begin{matrix} // \\ // \end{matrix} \right)$$



is powerful!

anal all finite (x)

$x, \sin x, \dots, A(x), \dots$

& Herglotz then...

(all x)

$$e^{in\theta} = \cos n\theta + i \sin n\theta$$

$$n \sin(n\theta)$$

$$mx = \text{Im} re^{i\theta} = r \sin \theta$$

Hergl $\Rightarrow$

Sign of

$$\sum_{n=0}^{\infty} a_n r^n \sin(n\theta)$$

Same as

$$\sin \theta \pm \sin(n\theta)$$

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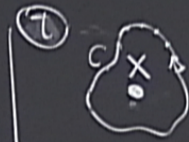
$$f(x) = a + bx$$

④ $-f(x)$ is Herglotz $\leftarrow$ $\begin{matrix} \text{Stieltjes} \\ \text{in UHP } (\operatorname{Im} x > 0), \operatorname{Im}[f(x)] > 0 \\ \text{in LHP } (\operatorname{Im} x < 0), \operatorname{Im}[-f(x)] < 0 \end{matrix}$

if $f(x)$ satisfies
 ① ② ③ ④ $\Rightarrow f$ is Stieltjes!

Cauchy

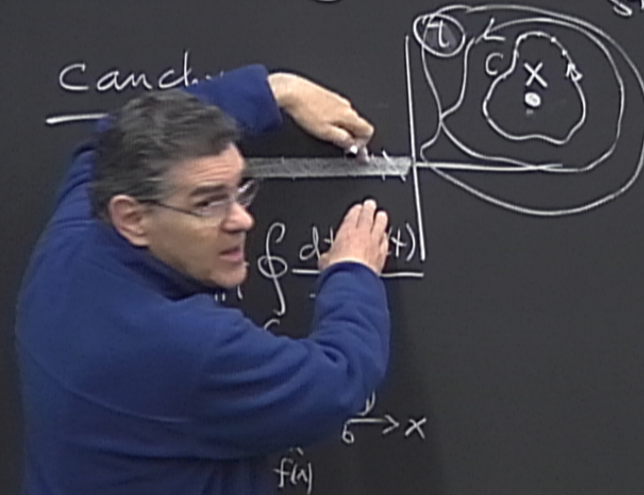
$$f(x) = \frac{1}{z\pi i}$$



④ $-f(x)$ is Herglotz $\Leftarrow$ $\begin{matrix} \text{in UHP (Im } x > 0), \text{Im}[f(x)] > 0 \\ \text{in LHP (Im } x < 0), \text{Im}[-f(x)] < 0 \end{matrix}$

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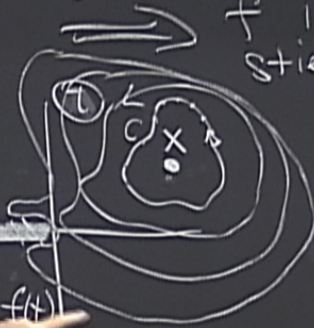
④ $-f(x)$ is Herglotz $\leftarrow$ $\begin{matrix} \text{stieljes} \\ \text{in UHP (Im } x > 0), \text{Im}[f(x)] \geq 0 \\ \text{in LHP (Im } x < 0), \text{Im}[-f(x)] < 0 \end{matrix}$

if $f(x)$ satisfies
 ① ② ③ ④ $\Rightarrow f$ is stieljes!

Cauchy

$$f(x) = \frac{1}{2\pi i} \oint \frac{f(t) dt}{t - x}$$

$$\frac{f(x)}{x} \xrightarrow{x \rightarrow \infty} 0$$



④ $-f(x)$ is Herglotz $\leftarrow$ $\begin{matrix} \text{stetig} \\ \text{in UHP (Im } x > 0), \text{Im}[f(x)] > 0 \\ \text{in LHP (Im } x < 0), \text{Im}[-f(x)] < 0 \end{matrix}$

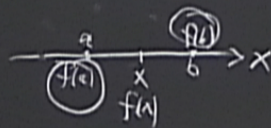
if $f(x)$ satisfies

① ② ③ ④

f is stetig!

Cauchy

$$f(x) = \frac{1}{2\pi i} \oint_C \frac{dt f(t)}{t-x}$$



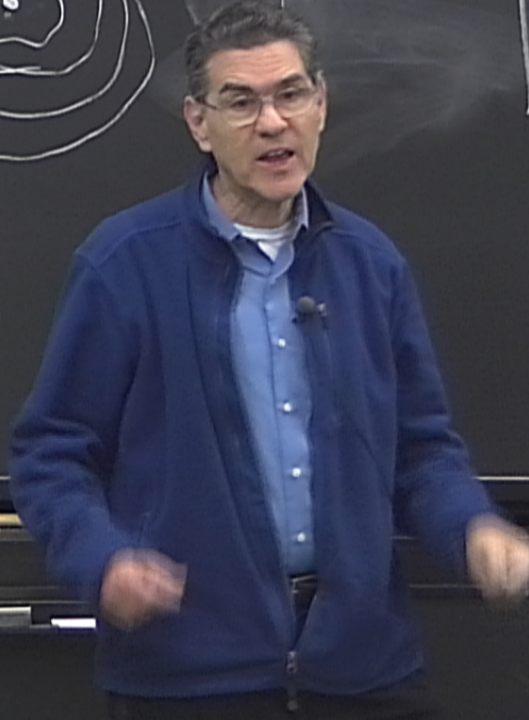
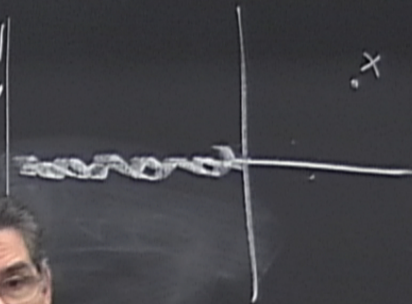
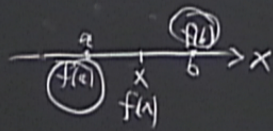
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f is stieljes !

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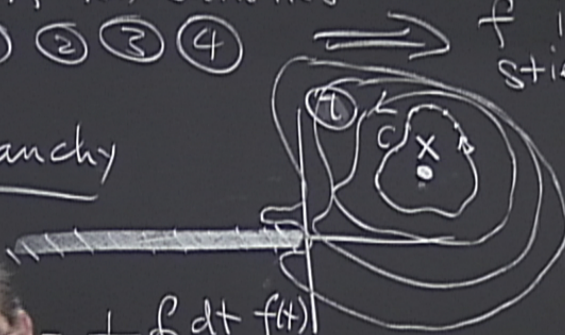
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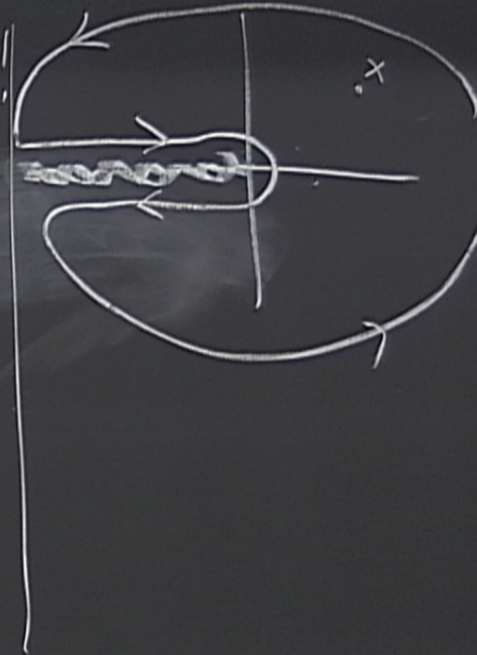
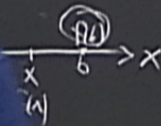
① ② ③ ④

Cauchy

f is Stieltjes



$$= \frac{1}{2\pi i} \oint_C \frac{dt f(t)}{t-x}$$



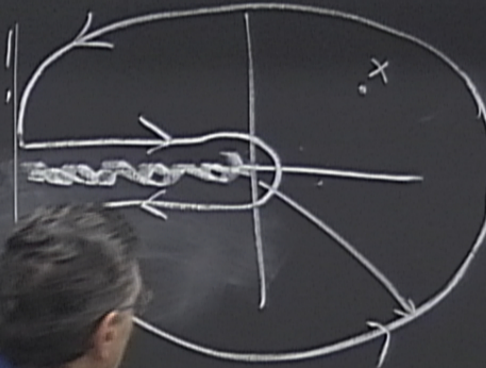
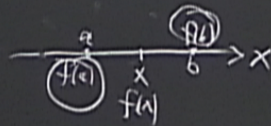
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if $f(x)$ satisfies
① ② ③ ④

f is
stetiges

Cauchy

$$f(x) = \frac{1}{2\pi i} \oint_C \frac{dt f(t)}{t-x}$$



④ $-f(x)$ is Herglotz $\leftarrow$ $\begin{matrix} \text{Stieltjes} \\ \text{in UHP } (\text{Im } x > 0), \text{Im}[f(x)] > 0 \\ \text{in LHP } (\text{Im } x < 0), \text{Im}[-f(x)] < 0 \end{matrix}$

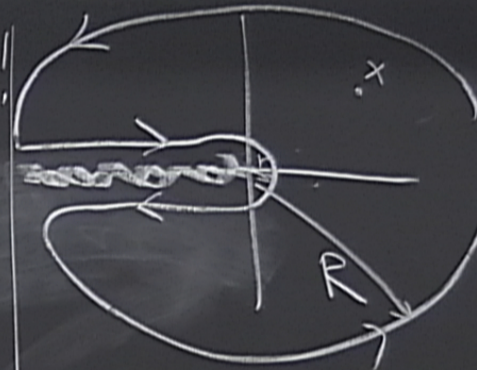
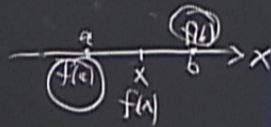
if $f(x)$ satisfies

① ② ③ ④

f is Stieltjes

Cauchy

$$f(x) = \frac{1}{2\pi i} \oint_C \frac{dt f(t)}{t-x}$$



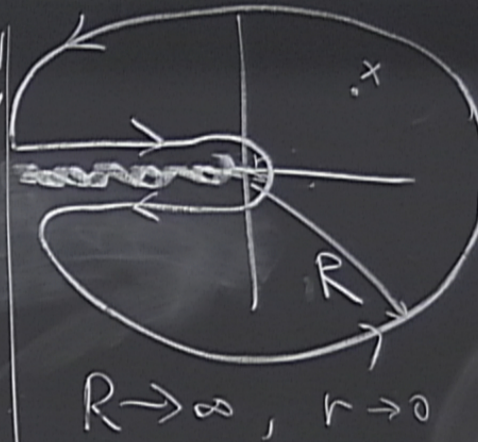
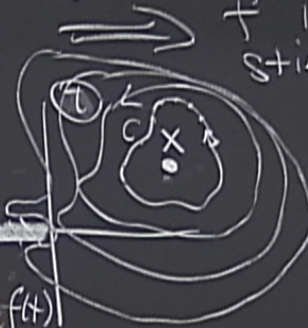
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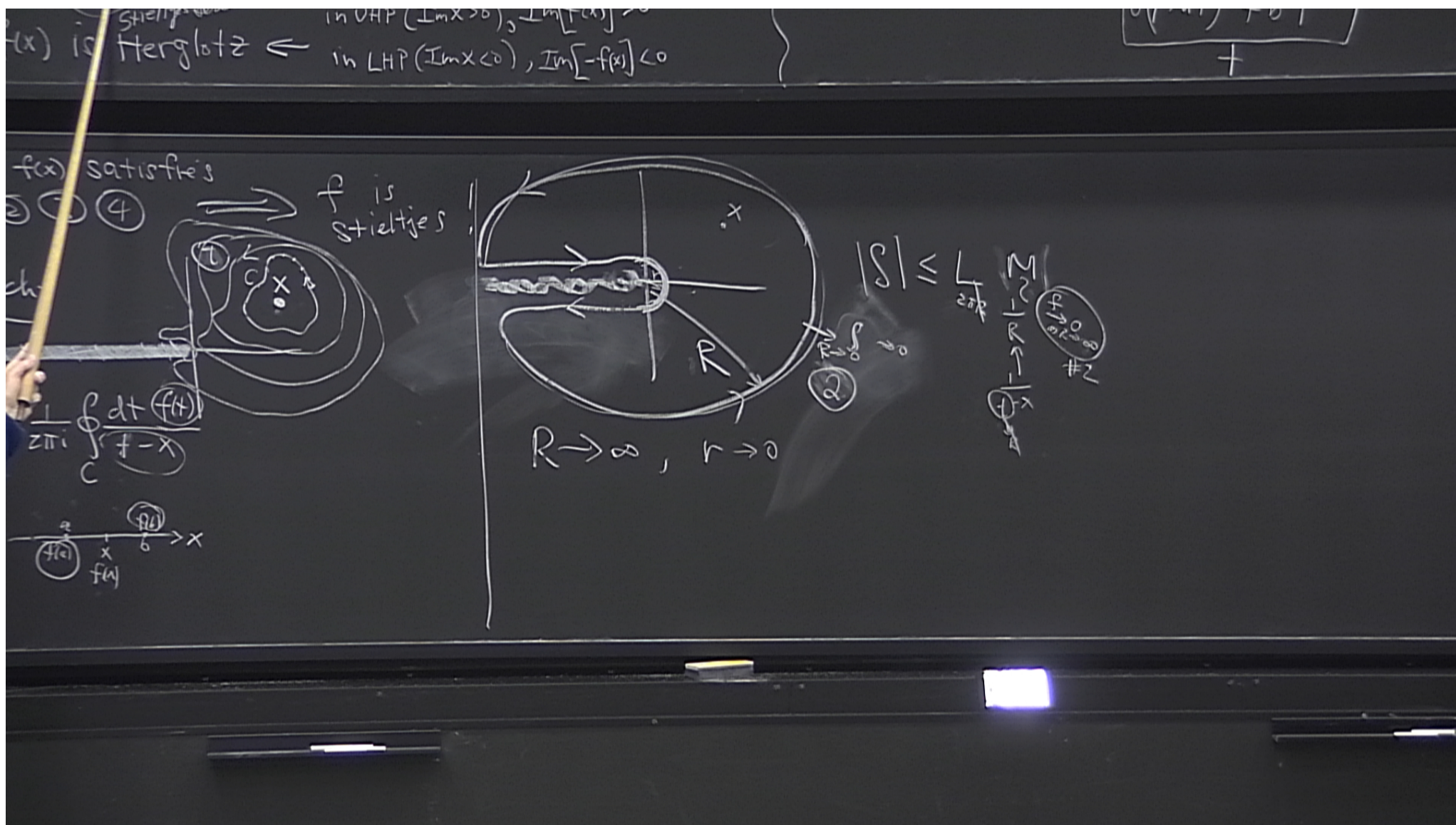
if $f(x)$ satisfies
① ② ③ ④

f is
stieljes

canchy

$$f(x) = \frac{f(t)}{x}$$





$\text{in LHP } (\text{Im } x < 0), \text{Im}[f(x)] < 0$

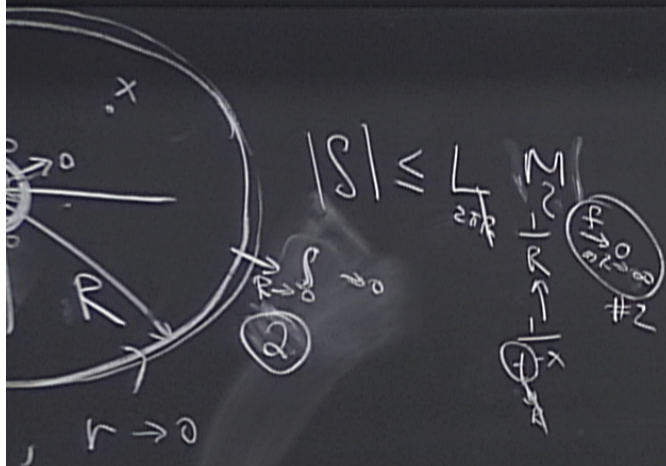
satisfies f is Stieltjes

$R \rightarrow \infty, r \rightarrow 0$

$$f(x) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{dt}{t-x} [f(t+i\eta) - f(t-i\eta)]$$

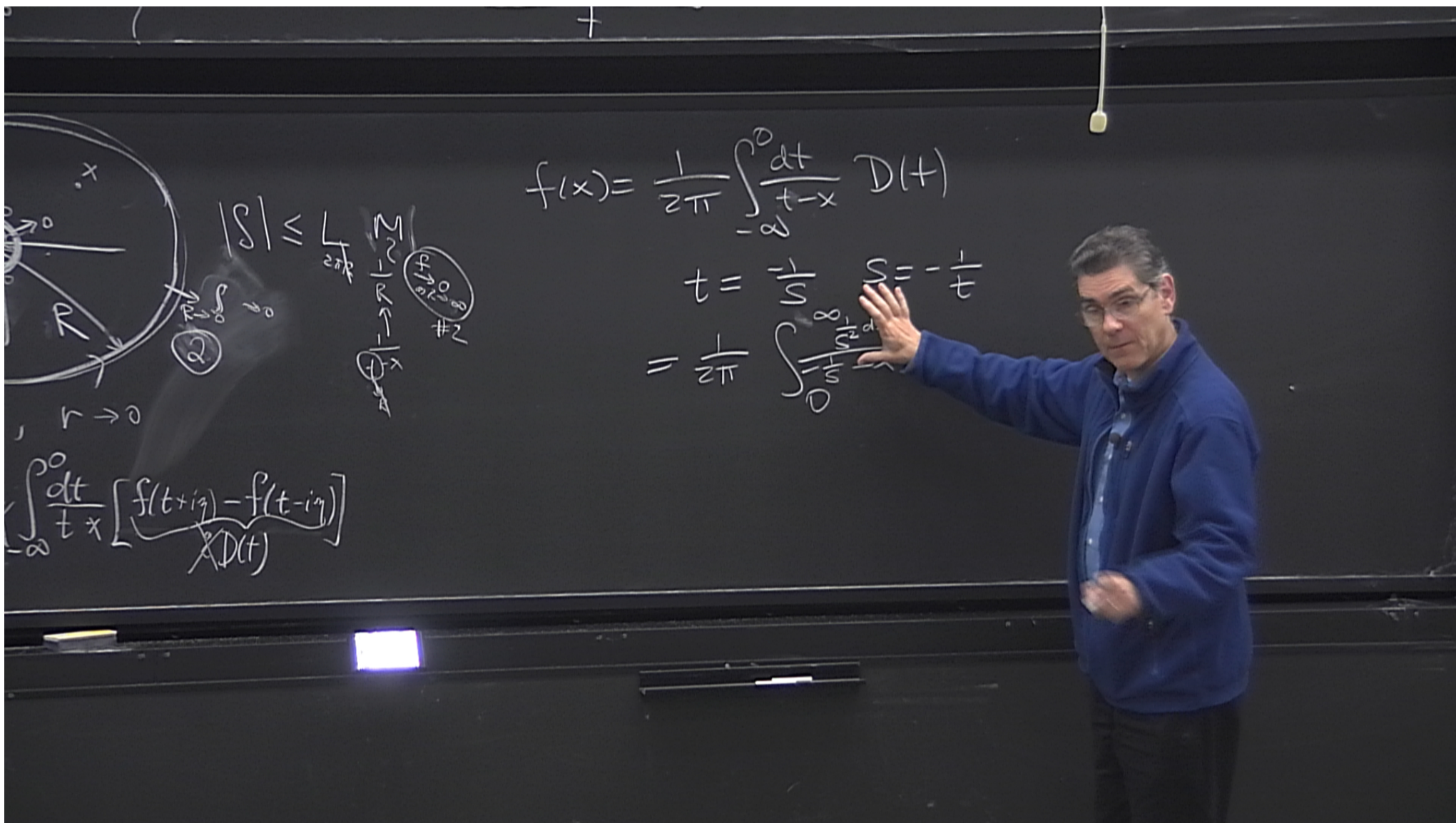
$|S| \leq \frac{L}{2\pi} \int_{-\infty}^{\infty} \frac{dt}{t-x}$

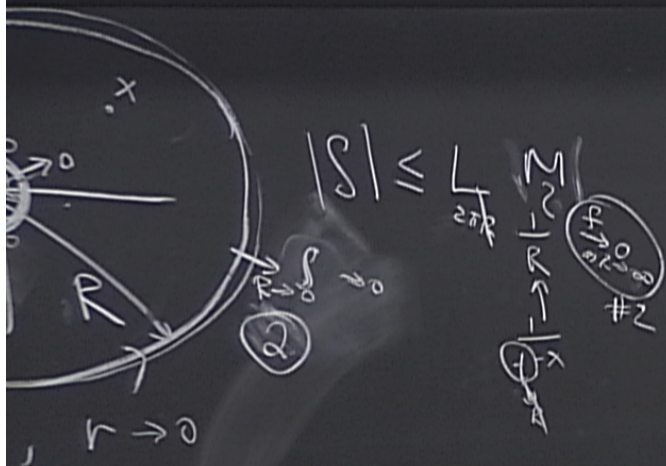
$\#2$



$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dt}{t-x} D(t)$$

$$\int_{-\infty}^{\infty} \frac{dt}{t-x} \left[\underbrace{f(t+i\eta) - f(t-i\eta)}_{D(t)} \right]$$





$$\int_{-\infty}^{\infty} \frac{dt}{t} \left[\underbrace{f(t+i\eta) - f(t-i\eta)}_{\times D(t)} \right]$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dt}{t-x} D(t)$$

$$t = -\frac{1}{s} \quad s = -\frac{1}{t}$$

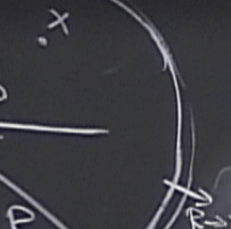
$$= \frac{1}{2\pi} \int_0^{\infty} \frac{s^{\frac{1}{2}} ds}{-\frac{1}{s} - x} D\left(-\frac{1}{s}\right)$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dt}{t-x} D(t)$$

$$t = -\frac{1}{s} \quad s = -\frac{1}{t}$$

$$= \frac{1}{2\pi} \int_{-\frac{1}{s}}^{\frac{1}{s}} \frac{ds}{-x} D\left(\frac{1}{s}\right)$$

$$= \int_{-\infty}^{\infty} \frac{ds}{1 + xs} \left(\frac{-D(-\frac{1}{s})}{2\pi s} \right)$$


 $|S| \leq \frac{L}{2\pi R}$
 $\frac{1}{2\pi R} \int_{-\infty}^{\infty} \frac{f(t+i\eta) - f(t-i\eta)}{t} dt$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dt}{t-x} D(t)$$

$$f(x) = \int_0^{\infty} \frac{ds}{1+xs} p(s)$$

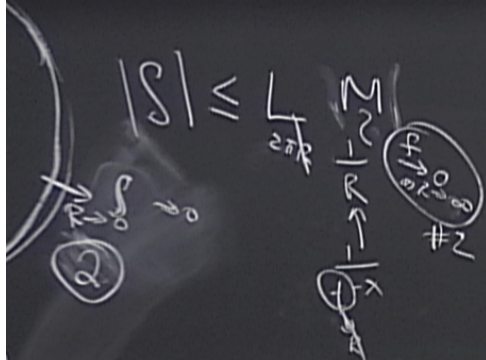
$$t = -\frac{1}{s} \quad s = -\frac{1}{t}$$

$$= \frac{1}{2\pi} \int_0^{\infty} \frac{s^{\frac{1}{2}} ds}{\frac{1}{s} - x} D\left(\frac{-1}{s}\right)$$

$$= \int_{\sigma} \frac{ds}{1+xs} \quad \left(\frac{-D(-\frac{1}{s})}{2\pi s} \right)$$

[illegible]

$$\frac{f(t+i\eta) - f(t-i\eta)}{2D(t)}$$



$$\frac{f(t+i\eta) - f(t-i\eta)}{2\eta}$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dt}{t-x} D(t)$$

$$t = -\frac{1}{s} \quad s = -\frac{1}{t}$$

$$= \frac{1}{2\pi} \int_{-\frac{1}{s}-x}^{\frac{1}{s}} \frac{ds}{s^2} D\left(-\frac{1}{s}\right)$$

$$= \int_0^{\infty} \frac{ds}{1+xs} \left(\frac{-D(-\frac{1}{s})}{2\pi s} \right) \leftarrow p(s)$$

$$f(x) = \int_0^{\infty} \frac{ds}{1+xs} p(s)$$

more to do!

$$\begin{array}{c} a \\ \textcircled{f(a)} \\ x \\ f(x) \\ b \end{array} \rightarrow x$$

$$f(x) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{dt}{t} \times \left[\underbrace{f(t+i\eta) - f(t-i\eta)}_{\times D(t)} \right]$$

Stieltjes stated:

$$\begin{array}{c} f(x) \\ \uparrow \\ \text{St. fn} \end{array} \sim \sum_{n=0}^{\infty} a_n x^n \xrightarrow{\text{Padé}} \begin{cases} P_n^n(x) \rightarrow \text{converge} \\ P_{n+1}^n(x) \rightarrow c \end{cases} \quad \text{as } n \rightarrow \infty.$$

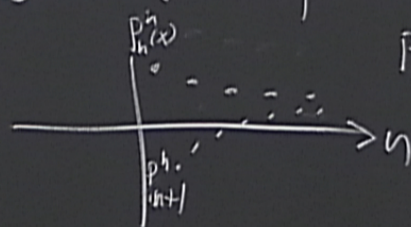
$$\begin{array}{c} a \\ \textcircled{f(a)} \end{array} \quad \begin{array}{c} x \\ f(x) \end{array} \quad \begin{array}{c} b \\ \textcircled{f(b)} \end{array} \rightarrow x$$

$$f(x) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{dt}{t-x} \left[\underbrace{f(t+ig) - f(t-ig)}_{\cancel{X} D(t)} \right]$$

Stieltjes showed:

$$\sum_{n=0}^{\infty} a_n x^n \xrightarrow{\text{Padé}} \begin{cases} P_n^n(x) \rightarrow \text{converges in int.} \\ P_{n+1}^n(x) \rightarrow \text{converges in int.} \end{cases} \text{ as } n \rightarrow \infty.$$

Also, for real x



$P_n^n(x) \downarrow$ monotone.

$$P_{n+1}^n(x) < P_m^m(x) \text{ for all } n, m.$$

$$-f(t-i\eta)] \\ (D(t))$$

$$= \int_0^\infty \frac{ds}{1+xs} \left(\frac{-D(-s)}{\pi s} \right)$$

$\rightarrow \infty$.

in CUT PLANE

$f(x) \downarrow$ monotone.

$$P_{n+1}^n(x) < P_n^m(x) \text{ for all } n, m.$$

#5! if the moment problem has a unique solution then $L_1 = L_2 = f(x)$ in $\mathbb{C}_x$ plane Padé's conv to $f(x)$

$$-f(t-i\eta)] \\ (D(t))$$

$$= \int_0^\infty \frac{ds}{1+xs} \left(\frac{-D(-s)}{\pi s} \right)$$

$\rightarrow \infty$.

in CUT PLANE

$p(x) \downarrow$ monotone.

$$p_{n+1}^n(x) < p_{n+1}^m(x) \text{ for all } n, m.$$

#5! if the moment problem has a unique solution then $L_1 = L_2 = f(x)$ in cx plane Padé's conv to $f(x)$

$$a_n = \int_0^\infty p(t) dt t^n$$

given for $n=0,1,2,\dots,\infty$

$$\text{find } p(t) \\ \text{s.t. } p(t) \geq 0$$

$$a_n = (8n+7)!$$

$$n=0,1,\dots$$

find $p(t)$

$$p(t) = \frac{e^{-t^{1/8}}}{8} + a e^{-t^{1/4}} \sin t^{1/4}$$

$$-0.14761 < a < 1.584$$

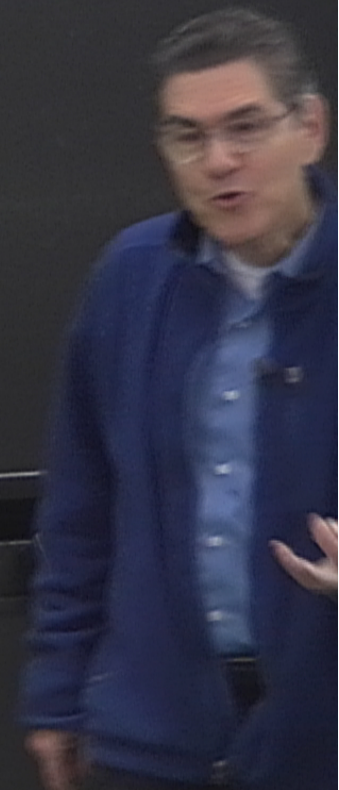
$$a_n = (8n+7)!$$

$$n=0,1,\dots$$

find $p(t)$

$$p(t) = \frac{e^{-t^{1/8}}}{8} + a e^{-t^{1/4}} \frac{\sin t^{1/4}}{8} \geq 0$$

$$\left[-0.14761\dots < a < 1.21584\dots \right]$$



Carleman condition
if a_n grow no faster than $(2n)!$
then solⁿ is unique,
(to moment prob)

$$t^{1/4} \geq 0$$

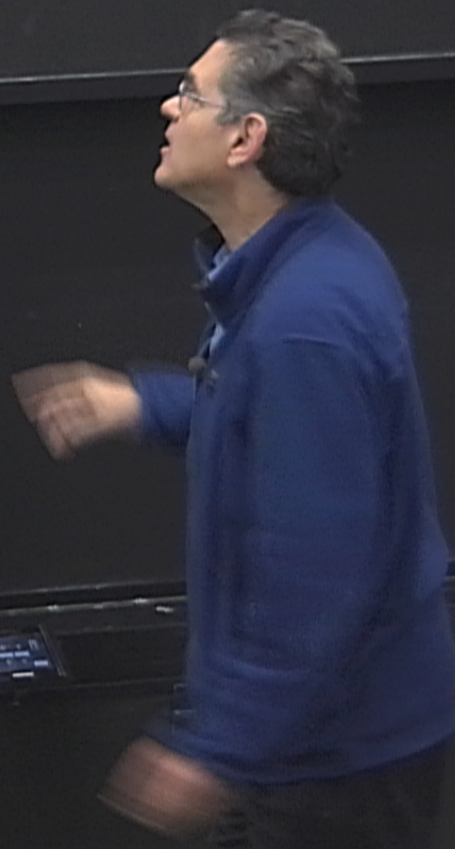
a $-84. \dots$

$$\left[\frac{\sin t^{1/4}}{t} > 1.21584 \right] \geq 0$$

$$\left(-\frac{d^2}{dx^2} + V(x) + \epsilon W(x) \right) \psi = \underline{E(\epsilon)} \psi$$

$$\left[-\psi'' + \frac{x^2}{4} \psi + \epsilon x^4 \psi = E \psi \right]$$

$$E(\epsilon) \sim \sum a_n \epsilon^n$$



Can show

$E(e)$ is and in cut plane

$\sum a_n e^n$ is asymp in cut plane as $e \rightarrow 0$

etc.

a_n grow no faster than $(2n)!$
Then soln is unique,
 (to moment prob)

$$\left(-\frac{d^2}{dx^2} + V(x) + \epsilon W(x) \right) \psi = E(\epsilon) \psi$$

$$\left[-\psi'' + \left(\frac{x^2}{2} \right) \psi + \epsilon x^4 \psi = E(\epsilon) \psi \right]$$

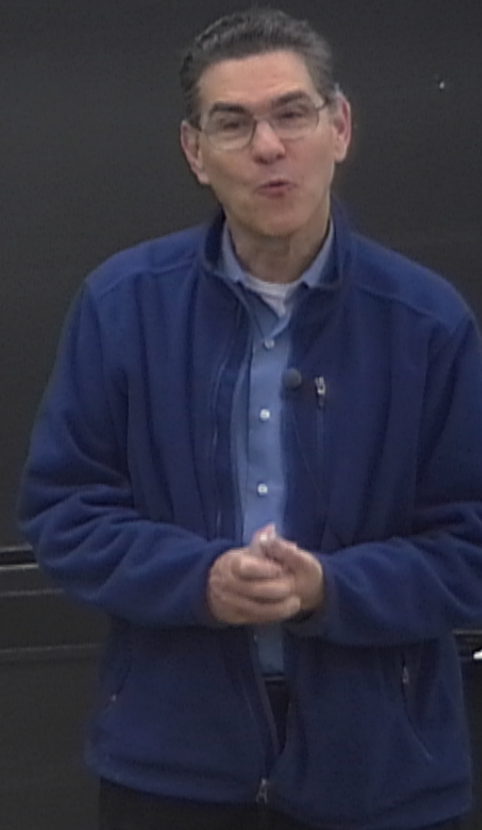
$$E(\epsilon) \sim \sum a_n \epsilon^n \quad (\epsilon \rightarrow 0)$$

$a^* a$ real
 +

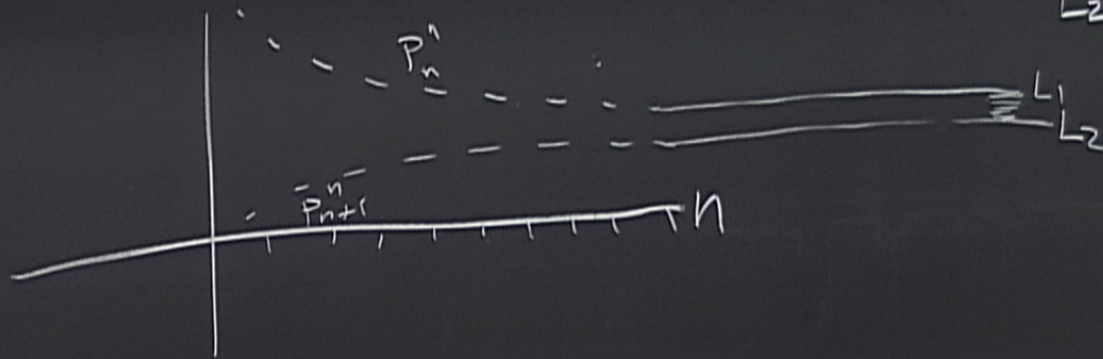
$$\begin{aligned} & \int_{-\infty}^{\infty} dx \left(\psi^*(x) \left(-\psi''(x) + \left(\frac{x^2}{2} \right) \psi(x) + \epsilon x^4 \psi(x) \right) - \frac{1}{2} \left(\psi'(x) \right)^2 \right) \\ & + \int_{-\infty}^{\infty} dx V(x) \psi^* \psi + \epsilon \int_{-\infty}^{\infty} dx W(x) \psi^* \psi = E \int_{-\infty}^{\infty} dx \psi^* \psi \end{aligned}$$

$$\frac{Im}{\oplus} Im E = \oplus Im E$$

can show all 4 properties
 $E(e)$ is and in cut plane
 $\sum a_n e^n$ is a smp in cut plane $\Leftrightarrow$
etc.

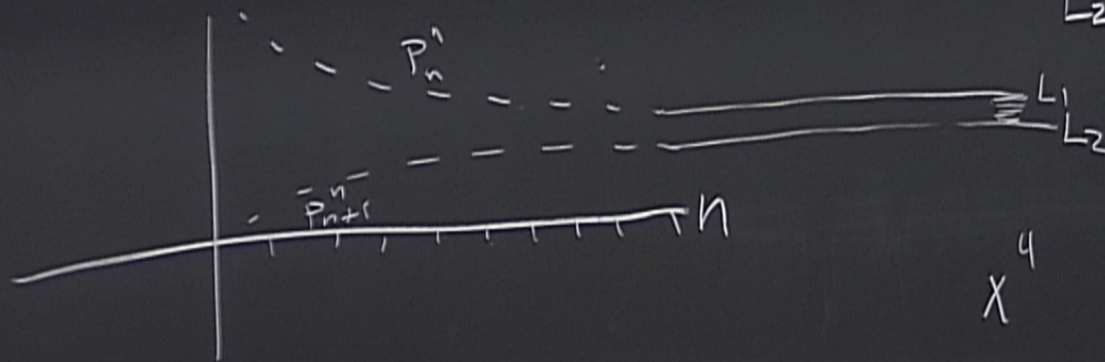


always true that $P_n^n \downarrow > P_{n+1}^n \uparrow$



$$L_2 \leq \lim_{n \rightarrow \infty} P_n^n \leq L_1$$

always true that $P_n^{\downarrow} > P_{n+1}^{\uparrow}$



$$L_2 \leq \text{exact answer} \leq L_1$$

$$x^4$$

WKB $\frac{1}{\sqrt{x^4}} e^{-\int \sqrt{x^4}}$
 $\sim \frac{1}{x} e^{-\frac{2}{3}x^{3/2}}$