

Title: Mathematical Physics - Lecture 9

Date: Dec 01, 2011 09:00 AM

URL: <http://pirsa.org/11120001>

Abstract:

$$x^3 y'' = y$$

y behave? near $x=0$

$$x^3 y'' = y$$

y behave? near $x=0$

$$y'' = e^{-x} y$$

$$x^3 y'' = y$$

y behave? near $x=0$

$$y'' = \underline{e^{-x}} y$$

$x=0$ is a Reg pt.

$$y(x) = \sum a_n x^n$$

$$x^3 y'' = y$$

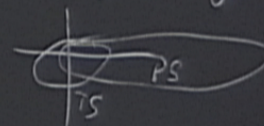
y behave? near $x=0$

$$y'' = \underline{e^{-x}} y$$

$x=0$ is a Reg pt.

→ (1) $y(x) = \sum a_n x^n$ Local

→ (2) $y(x) = \sum_{n=0}^{\infty} e^n y_n$ Global
 $y'' = e^{-x} y$



$$x^3 y'' = y$$

y behave? near $x=0$

$$y \sim e^{\frac{2}{\sqrt{x}}} x^{3/4} \sum_{n=0}^{\infty} a_n x^{\frac{n}{2}}$$

$\uparrow$
 $a_0 = 1$
 $a_1 = -\frac{3}{16}$
 $\vdots$

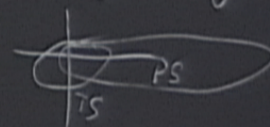
$$a_n \approx n!$$

$$y'' = e^{-x} y$$

$x=0$ is a Reg pt.

(1) $y(x) = \sum a_n x^n$ Local

(2) $y(x) = \sum_{n=0}^{\infty} e^{-x} y_n$ Global
 $y'' = e^{-x} y$



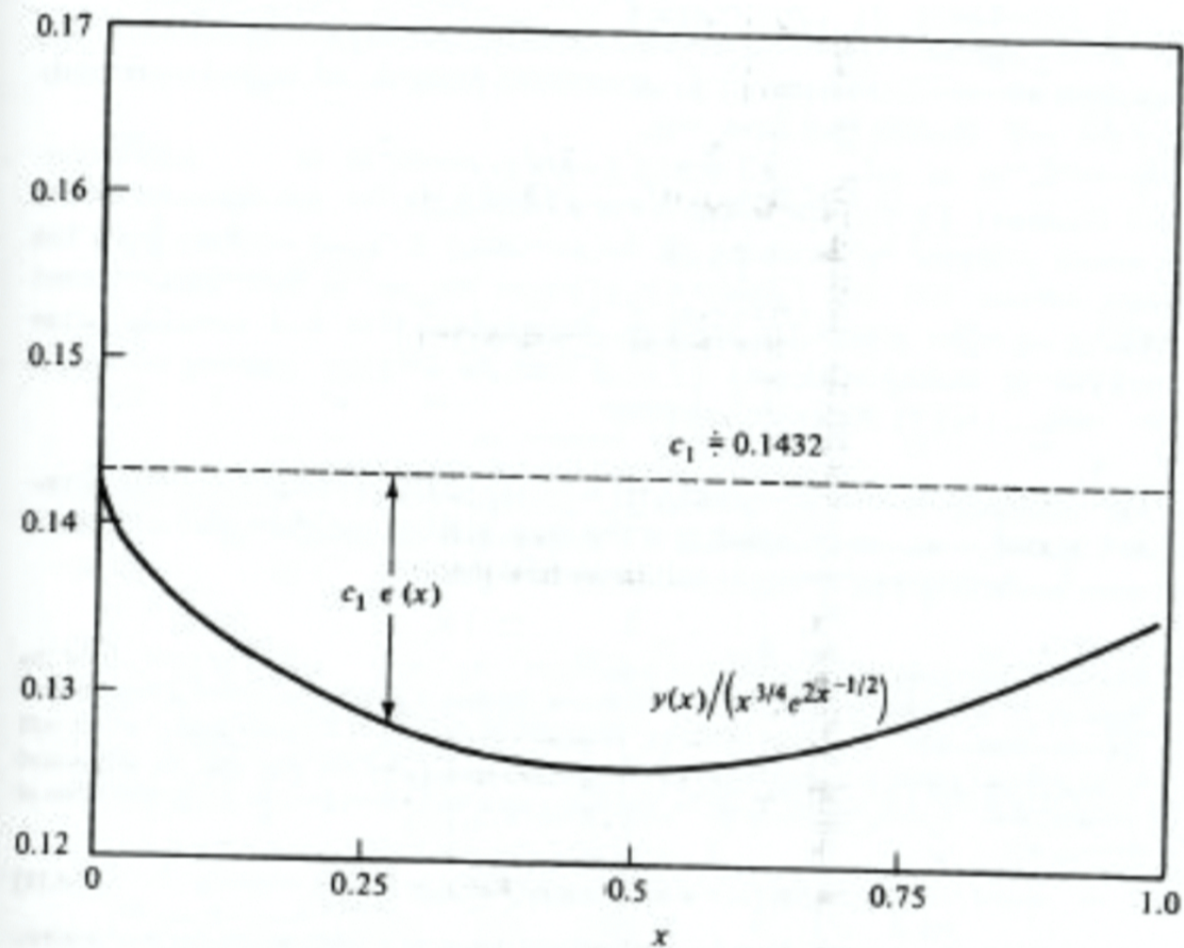


Figure 3.2 A plot of the ratio of $y(x)$, the numerical solution of (3.4.1) with $y(1) = 1$ and $y'(1) = 0$, to $x^{3/4} \exp(2x^{-1/2})$ for $0 \leq x \leq 1$. Asymptotic analysis [see (3.4.17)] shows that this ratio approaches a constant c_1 . For these initial conditions, $c_1 \approx 0.1432$. The difference between the plotted ratio and c_1 is $c_1 \epsilon(x)$.

$$x^3 y'' = y$$

y behave? near $x=0$

$$y(x) \sim \left(e^{\frac{2}{\sqrt{x}}} x^{3/4} \right) \sum_{n=0}^{\infty} a_n x^{\frac{n}{2}} \quad \text{as } x \rightarrow 0$$

$\uparrow$
 $a_0 = 1$
 $a_1 = -\frac{3}{16}$
 $\vdots$

$a_n \sim n!$

$$y'' = e^{-x} y$$

$x=0$ is a Reg pt.

(1) $y(x) = \sum a_n x^n$ Local

(2) $y(x) = \sum_{n=0}^{\infty} e^n y_n$ Global
 $y'' = e^{-x} y$

$\frac{PS}{15}$

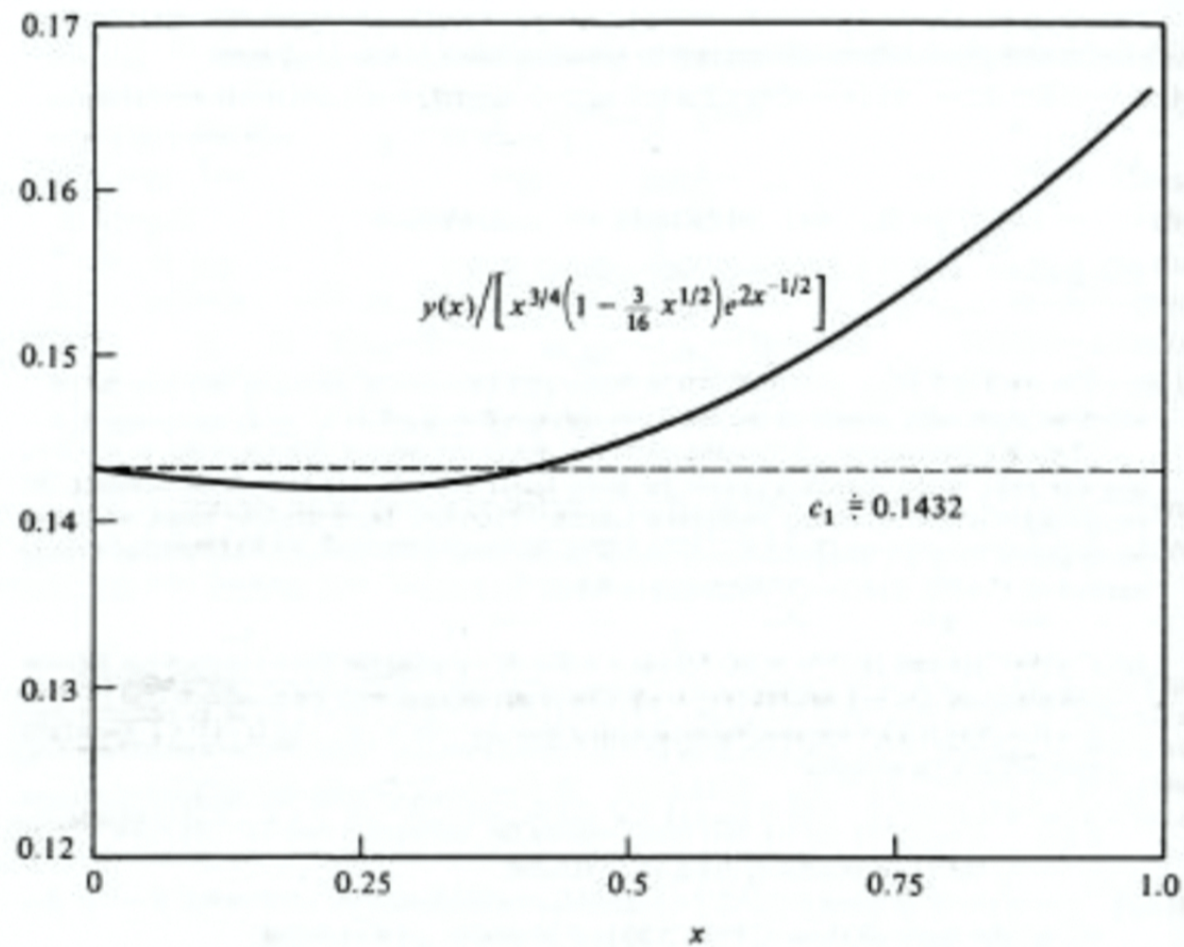


Figure 3.3 A plot of the ratio $y(x)$, the exact solution to (3.4.1) with $y(1) = 1$ and $y'(1) = 0$, to $x^{3/4}(1 - \frac{3}{16}x^{1/2}) \exp(2x^{-1/2})$ for $0 \leq x \leq 1$. A comparison of the results plotted in this figure with those of Fig. 3.2 shows that including the term $-\frac{3}{16}x^{1/2}$ in $w(x)$ improves the estimate of the numerical solution $y(x)$ as $x \rightarrow 0+$.

$$f(x) \sim \sum_{n=0}^{\infty} a_n x^n \quad \text{as } x \rightarrow 0$$

$$\text{for all } N \quad \left[f(x) - \sum_{n=0}^N a_n x^n \sim a_{N+1} x^{N+1} \right] \text{ as } x \rightarrow 0$$

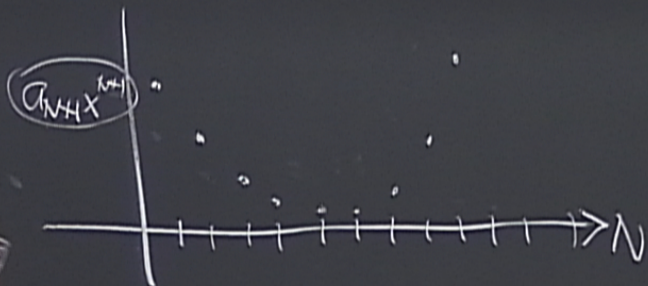
$$y'' = e^{-x} y$$

$$\frac{PS}{TS}$$

Optimal

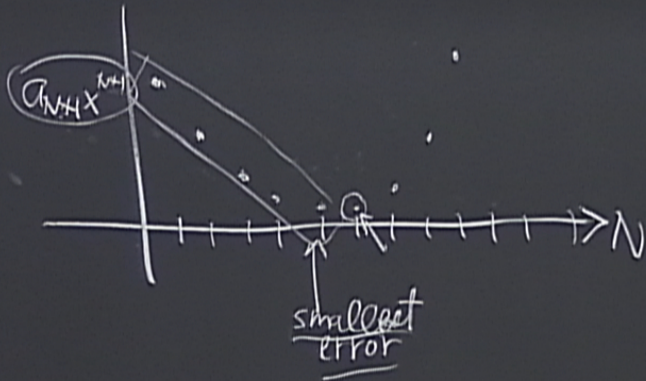
$$\text{for all } N \left| f(x) - \sum_{n=0}^N a_n x^n \sim a_{N+1} x^{N+1} \right| \text{ as } x \rightarrow 0$$

Optimal asymp approx



for all N $\left| f(x) - \sum_{n=0}^N a_n x^n \right| \sim \frac{a_{N+1} x^{N+1}}{\alpha} \text{ as } x \rightarrow 0$
 measure of error $\sim \frac{1}{N+1}$

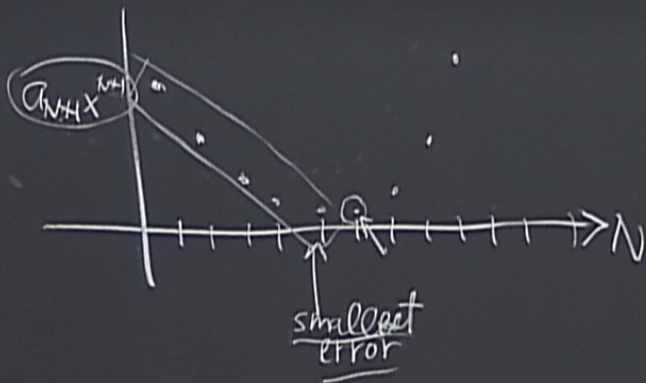
Optimal asymp approx



$$\sum a_n \alpha^n \quad \alpha = \frac{1}{137}$$

for all N $\left| f(x) - \sum_{n=0}^N a_n x^n \right| \sim \frac{a_{N+1} x^{N+1}}{1} \text{ as } x \rightarrow 0$
 measure of error $\sim \frac{1}{N!}$

Optimal asymp approx

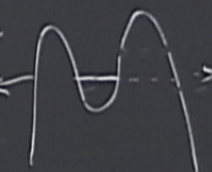


$$\sum a_n x^n \quad \alpha = \frac{1}{137}$$

$$\frac{m^2}{2} \phi^2 + g \phi^4$$

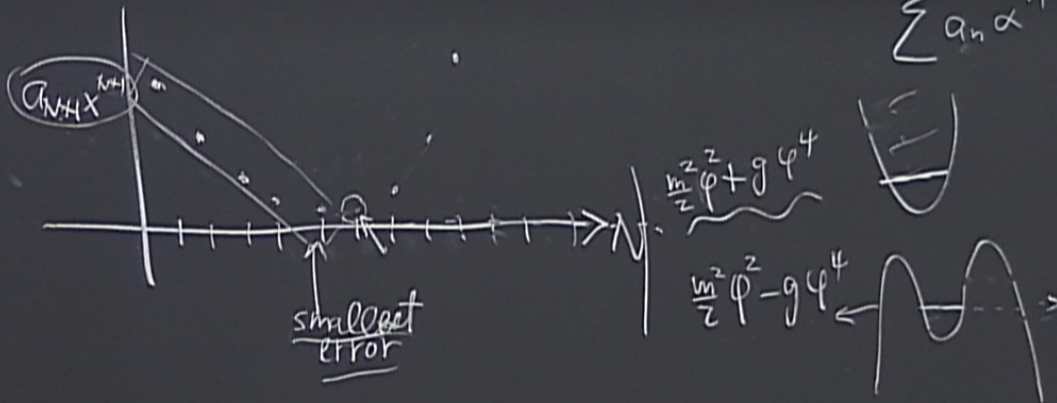


$$\frac{m^2}{2} \phi^2 - g \phi^4$$



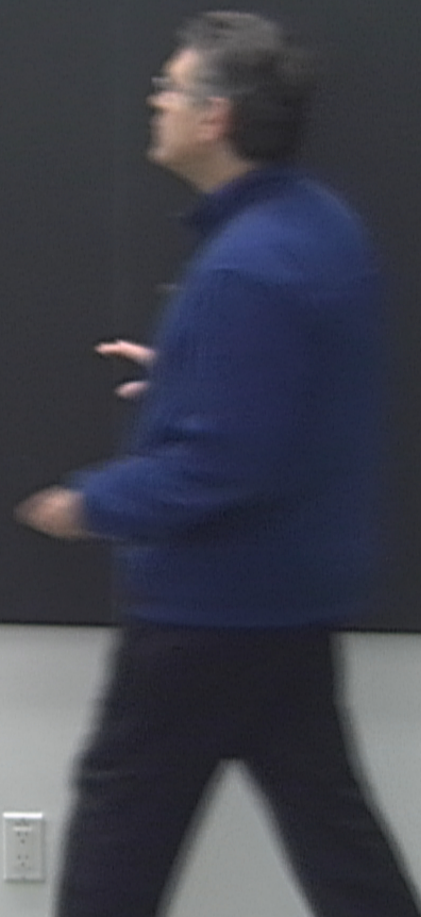
for all N $\left| f(x) - \sum_{n=0}^N a_n x^n \right| \sim \frac{a_{N+1} x^{N+1}}{1} \text{ as } x \rightarrow 0$
 measure of error

Optimal asymp approx



$\sum a_n x^n \quad \alpha = \frac{1}{137}$

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2}$$



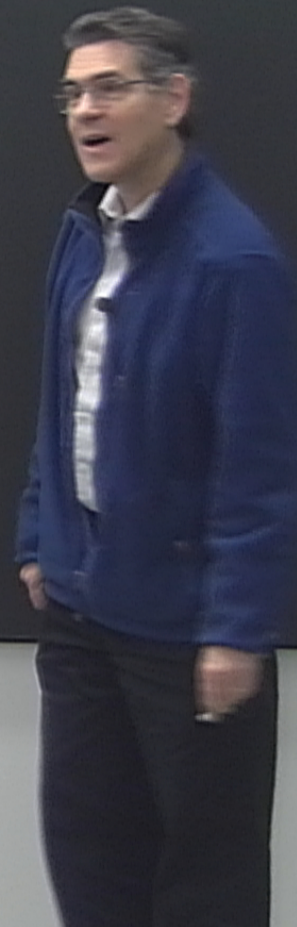
$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2}$$

how does $f(x)$ behave as $x \rightarrow \infty$?

$$f'(x) = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!(n-1)!}$$

$$x f'(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!(n-1)!}$$

$$(x f')' = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!n!}$$



$f(x)$ behave as $x \rightarrow \infty$?

$$\frac{x^{n-1}}{n!(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!n!} = f(x)$$

$n \rightarrow n+1$

$= f$

$x = \infty$ is an irreg sing pt

$$f(x) = e^{S(x)}$$

$$x(S'' + (S')^2) + S' = 1$$

$$S'' \ll (S')^2 \text{ as } x \rightarrow \infty$$

$$x(S')^2 + S' \sim 1 \text{ as } x \rightarrow \infty$$

ISP at $x = \infty$

$$f \approx e^{ax^p} \quad p > 0$$

$f(x)$ behave as $x \rightarrow \infty$?

$$\frac{x^{n-1}}{n!(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!n!} = f(x)$$

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$$x(S')^2 \sim 1 \quad S' \sim 1$$

ISP at $x = \infty$

$$f \approx e^{ax^p} \quad p > 0$$

$f(x)$ behave as $x \rightarrow \infty$?

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$= f$

$x = \infty$ is an irreg sing pt

$$f(x) = e^{S(x)}$$

$$x(S'' + (S')^2) + S' = 1$$

$S'' \ll (S')^2$ as $x \rightarrow \infty$.

$$x(S')^2 + S' \sim 1 \text{ as } x \rightarrow \infty.$$

$$\frac{x(S')^2}{S' + 1}$$

1/6

$$\begin{aligned} x(S')^2 &\sim -S' \\ xS' &\sim -1 \\ S' &\sim -\frac{1}{x} \end{aligned}$$

ISP at $x = \infty$

$$f \sim e^{ax^p} \quad p > 0$$

$$x = \frac{1}{t}$$

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2} \approx e^{+2\sqrt{x}}$$

how does $f(x)$ behave as $x \rightarrow \infty$?

$$f'(x) = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!(n-1)!}$$

$$x f'(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!(n-1)!}$$

$$(x f')' = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!(n-1)!} = \sum_{n \rightarrow n+1}^{\infty} \frac{x^n}{n!n!} = f(x)$$

$$\boxed{x f'' + f' = f}$$

$x = \infty$ is an irreg st

$$f(x) = e^{S(x)}$$

$$x(S'' + (S')^2) + S' = 1$$

$$S'' \ll (S')^2 \text{ as } x \rightarrow \infty$$

$$x(S')^2 + S' \sim 1$$

$$\frac{x(S')^2}{S'} + 1 \sim \frac{1}{S'}$$

$$\frac{x(S')^2}{S'} \sim -S' \sim -\frac{1}{x}$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2} \approx e^{+2\sqrt{x}}$$

How does $f(x)$ behave as $x \rightarrow \infty$?

$$f(x) = \sum_{n=0}^{\infty} \frac{x^{n-1}}{n!(n-1)!}$$

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!(n-1)!}$$

$$f'(x) = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!n!} = f(x)$$

$$x f'' + f' = f$$

$x = \infty$ is an irreg sing pt

$$f(x) = e^{S(x)}$$

$$x(S'' + (S')^2) + S' = 1$$

$$S'' \ll (S')^2 \text{ as } x \rightarrow \infty$$

$$x(S')^2 + S' \sim 1 \text{ as } x \rightarrow \infty$$

$$\cancel{x(S')^2} \quad S' \sim \frac{1}{x}$$

$$\begin{aligned} x(S')^2 &\sim -S' \\ xS' &\sim -1 \\ S' &\sim -\frac{1}{x} \end{aligned}$$

ISP at $x = \infty$

$$f \sim e^{ax^p} \quad p > 0$$

(S')

$$x(S')^2 \sim 1$$

$$S'^2 \sim \frac{1}{x}$$

$$S' \sim \pm \frac{1}{\sqrt{x}}$$

$$\begin{aligned} S &\sim \pm 2\sqrt{x} \text{ as } x \rightarrow \infty \\ f &\sim e^{\pm 2\sqrt{x}} \end{aligned}$$

error

$$\text{Let } S = 2\sqrt{x} + C(x)$$

$$S' = \frac{1}{\sqrt{x}} + C'$$

$$S'' = -\frac{1}{2x^{3/2}} + C''$$

$$C(x) \ll \sqrt{x} \quad \text{as } x \rightarrow \infty$$

$$C' \ll \frac{1}{\sqrt{x}} \quad "$$

$$C'' \ll \frac{1}{x^{3/2}} \quad "$$

$$x \left[-\frac{1}{2x^{3/2}} + C'' + \frac{1}{x} + C'^2 + \frac{2}{\sqrt{x}} C' \right] + \frac{1}{\sqrt{x}} + C' = 1$$

error

$$\text{Let } S = 2\sqrt{x} + C(x)$$

$$S' = \frac{1}{\sqrt{x}} + C'$$

$$S'' = -\frac{1}{2x^{3/2}} + C''$$

$$C(x) \ll \sqrt{x} \quad \text{as } x \rightarrow \infty$$

$$C' \ll \frac{1}{\sqrt{x}} \quad "$$

$$C'' \ll \frac{1}{x^{3/2}} \quad "$$

$$-\frac{1}{2x^{3/2}} + C'' + \cancel{C'^2} + \frac{2}{\sqrt{x}} C' + \frac{1}{\sqrt{x}} + C' = \cancel{0}$$

$$-\frac{1}{2x^{3/2}} + C'' + C'^2 + \frac{2}{\sqrt{x}} C' + \frac{1}{x^{3/2}} + \frac{C'}{x} = 0$$

error

$$\text{Let } S = 2\sqrt{x} + C(x)$$

$$S' = \frac{1}{\sqrt{x}} + C'$$

$$S'' = -\frac{1}{2x^{3/2}} + C''$$

$$C(x) \ll \sqrt{x} \quad \text{as } x \rightarrow \infty$$

$$C' \ll \frac{1}{\sqrt{x}} \quad "$$

$$C'' \ll \frac{1}{x^{3/2}} \quad "$$

$$x \left[-\frac{1}{2x^{3/2}} + C'' + \cancel{C'^2} + \frac{2}{\sqrt{x}} C' \right] + \frac{1}{\sqrt{x}} + C' = \cancel{0}$$

$$-\frac{1}{2x^{3/2}} + \underbrace{C''}_{\text{error}} + \underbrace{C'^2}_{\text{error}} + \frac{2}{\sqrt{x}} C' + \frac{1}{\sqrt{x}} + \frac{C'}{x} = 0$$

error

$$\text{Let } S = 2\sqrt{x} + C(x)$$

$$S' = \frac{1}{\sqrt{x}} + C'$$

$$S'' = -\frac{1}{2x^{3/2}} + C''$$

$$C(x) \ll \sqrt{x} \quad \text{as } x \rightarrow \infty$$

$$C' \ll \frac{1}{\sqrt{x}} \quad "$$

$$C'' \ll \frac{1}{x^{3/2}} \quad "$$

$$\frac{1}{2x^{3/2}} + C'' + \cancel{C'^2} + \frac{2}{\sqrt{x}} C' + \frac{1}{\sqrt{x}} + C' = \cancel{0}$$

$$\cancel{-\frac{1}{2x^{3/2}}} + \underbrace{C''}_{\text{error}} + \underbrace{C'^2}_{\text{error}} + \frac{2}{\sqrt{x}} C' + \frac{1}{2x^{3/2}} + \underbrace{\frac{C'}{x}}_{\text{error}} = 0$$

error

$$\text{Let } S = 2\sqrt{x} + C(x)$$

$$S' = \frac{1}{\sqrt{x}} + C'$$

$$S'' = -\frac{1}{2x^{3/2}} + C''$$

$$C(x) \ll \sqrt{x} \quad \text{as } x \rightarrow \infty$$

$$C' \ll \frac{1}{\sqrt{x}} \quad "$$

$$C'' \ll \frac{1}{x^{3/2}} \quad "$$

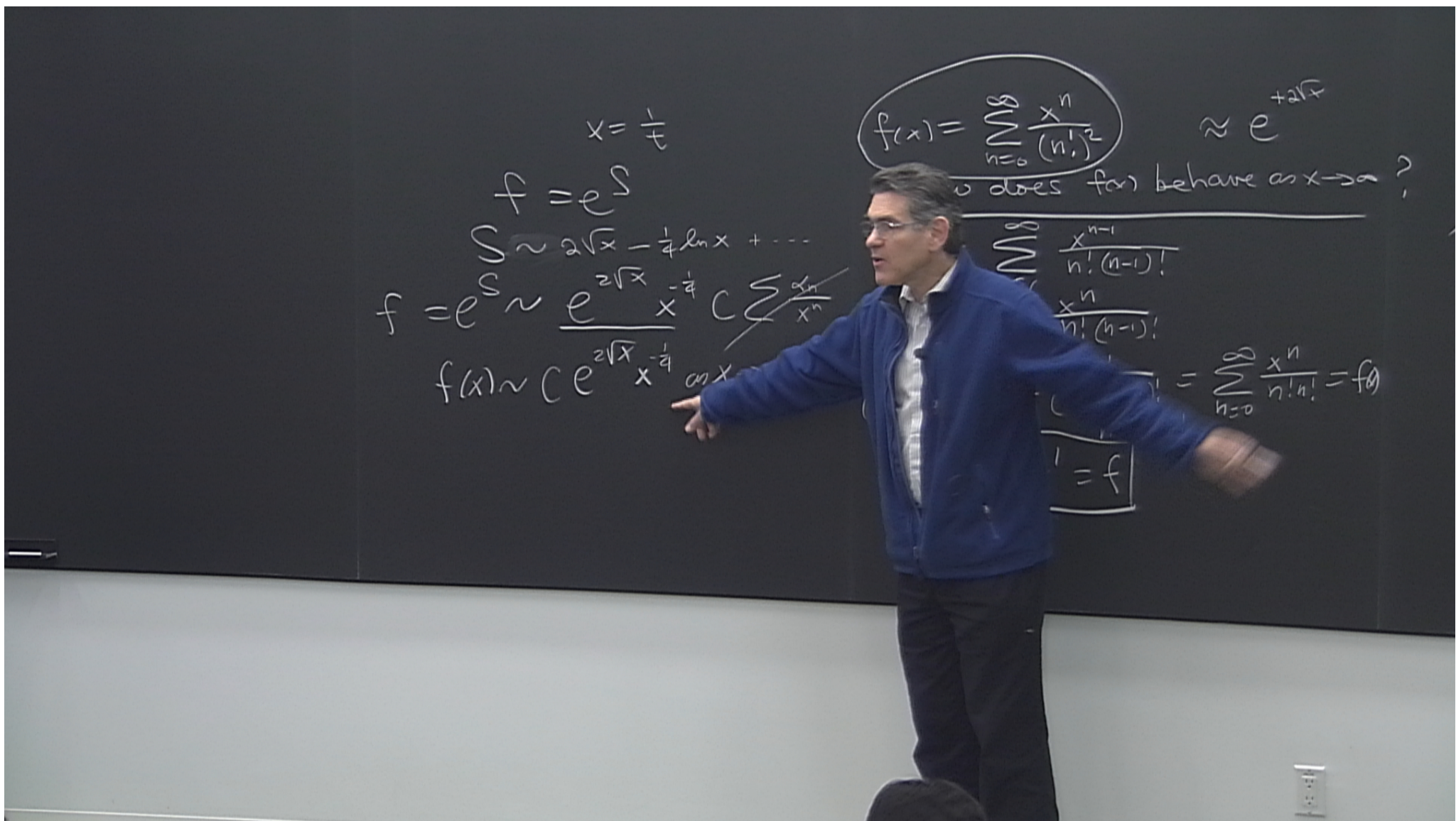
$$x \left[-\frac{1}{2x^{3/2}} + C'' + \cancel{x} + C'^2 + \frac{2}{\sqrt{x}} C' \right] + \frac{1}{\sqrt{x}} + C' = \cancel{x}$$

$$-\cancel{\frac{1}{2x^{3/2}}} + \underbrace{C''}_{\text{neg}} + \underbrace{C'^2}_{\text{neg}} + \frac{2}{\sqrt{x}} C' + \frac{1}{2x^{3/2}} + \underbrace{\frac{C'}{x}}_{\text{neg}} = 0$$

$$\frac{2}{\sqrt{x}} C' \sim -\frac{1}{2x^{3/2}}$$

$$C' \sim -\frac{1}{4x}$$

$$C \sim -\frac{1}{4} \ln x$$



$$x = \frac{1}{t}$$

$$f = e^S$$

$$S \sim 2\sqrt{x} - \frac{1}{4} \ln x + \dots$$

$$f = e^S \sim e^{2\sqrt{x}} x^{-\frac{1}{4}} C \sum \frac{x^n}{x^n}$$

$$f(x) \sim C e^{2\sqrt{x}} x^{-\frac{1}{4}} \text{ as } x \rightarrow \infty$$

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2}$$

$$\approx e^{+2\sqrt{x}}$$

how does $f(x)$ behave as $x \rightarrow \infty$?

$$f'(x) = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!(n-1)!}$$

$$x f'(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!(n-1)!}$$

$$(x f')' = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!n!} = f(x)$$

$$x f'' + f' = f$$

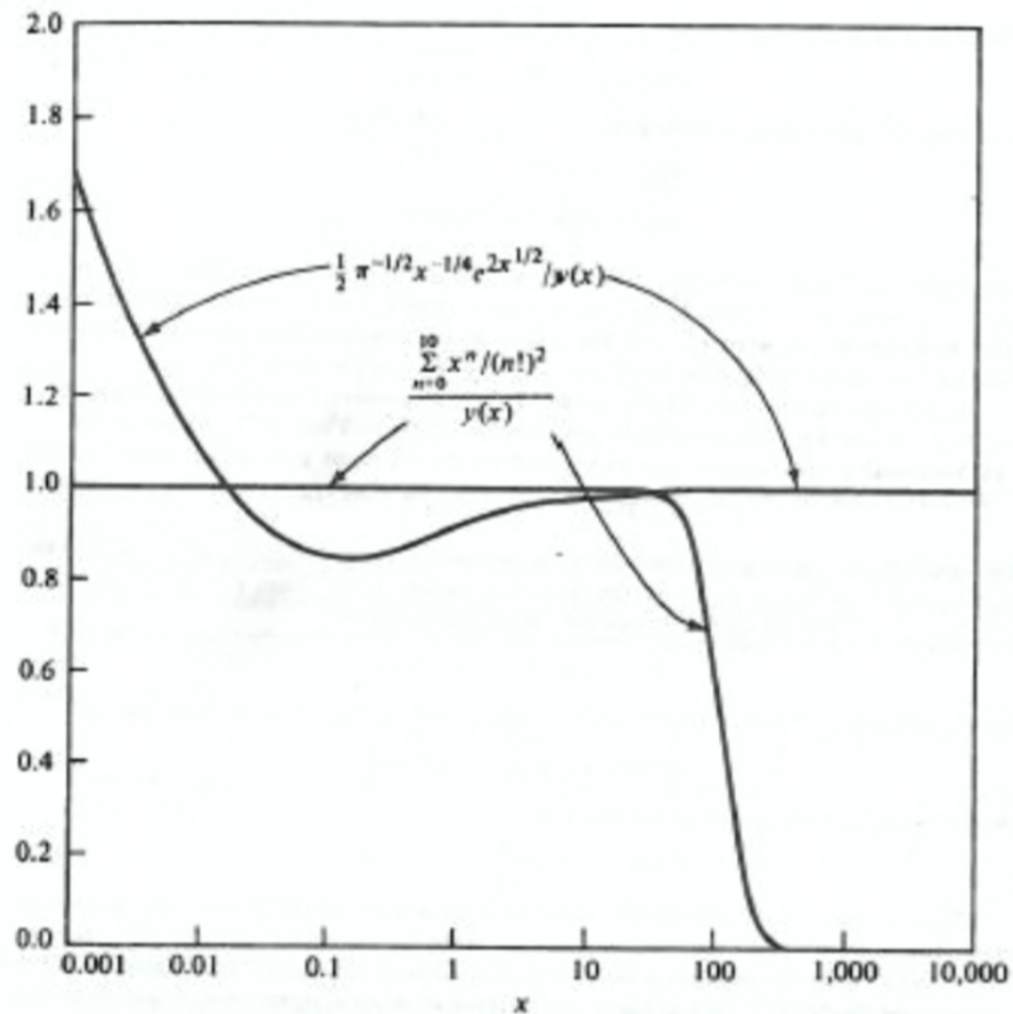


Figure 3.4 A comparison of the leading asymptotic approximation and an 11-term Taylor series approximation to the function $y(x)$ defined by the sum (3.5.1). The ratios of these approximations to $y(x)$ are plotted for $0.001 \leq x \leq 10,000$.

$f(x)$ exact.

$$(1) \quad \sum_{n=0}^{10} \frac{x^n}{(n!)^2} / f(x)_{\text{exact.}}$$

$$(2) \quad e^{2\sqrt{x} - \frac{1}{4}} / f(x)_{\text{exact.}}$$

if
good

$$\sum_{n=0}^{\infty} \frac{1}{(n!)} \frac{f(x)}{f(x)_{\text{exact}}}$$

$$e^{2\sqrt{x} - \frac{2}{3}} / f(x)_{\text{exact}}$$

$$y'' = xy$$

$$\text{Ai}(x) \leftarrow$$

$$\text{Bi}(x) \leftarrow$$

Airy

$$\begin{aligned} & \text{as } x \rightarrow \infty \\ & y'' = Q(x)y \\ & \text{as } x \rightarrow \infty, Q(x) \rightarrow \infty \\ & y(x) \sim k_{\pm} Q^{-1/4} e^{\pm \int \sqrt{Q}} \end{aligned}$$

$$y(x) \sim k_{\pm} \frac{e^{\pm \frac{2}{3} x^{3/2}}}{x^{1/4}}$$

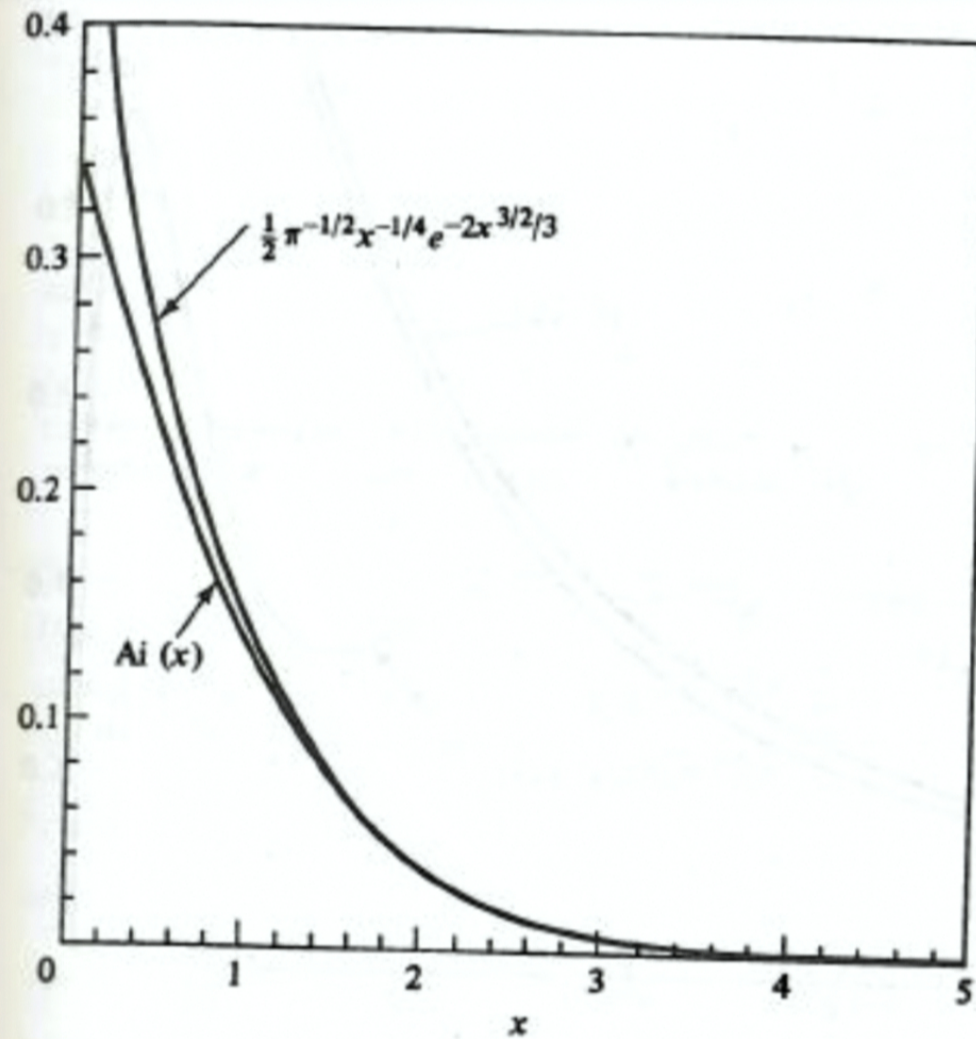
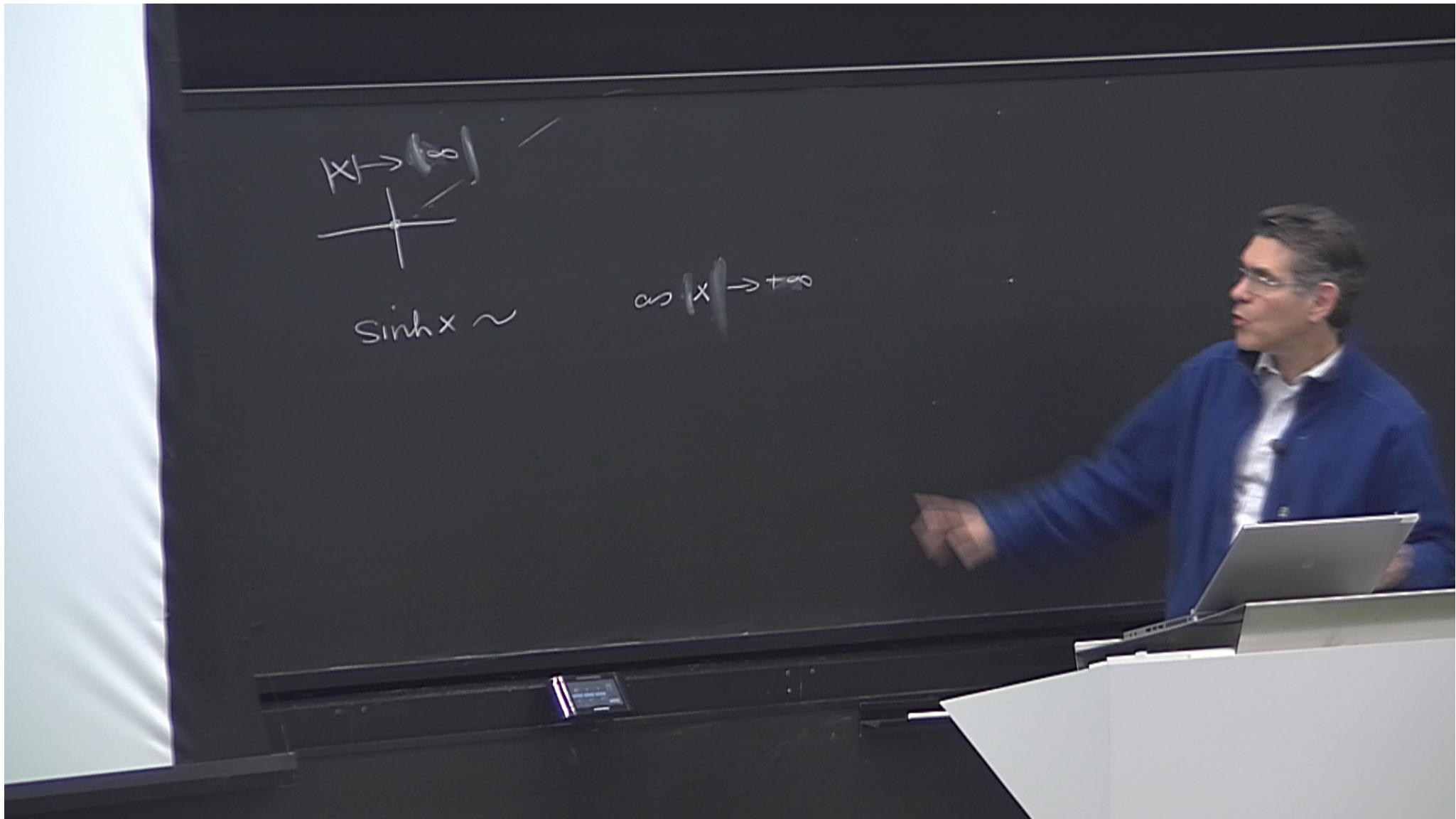
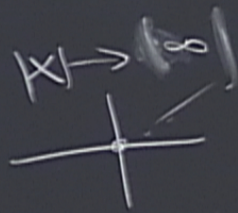


Figure 3.9 A plot of the Airy function $\text{Ai}(x)$ and its leading behavior (3.5.17a) for $0 \leq x \leq 5$.

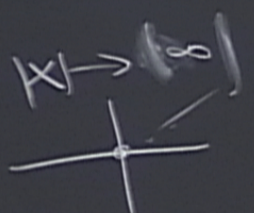
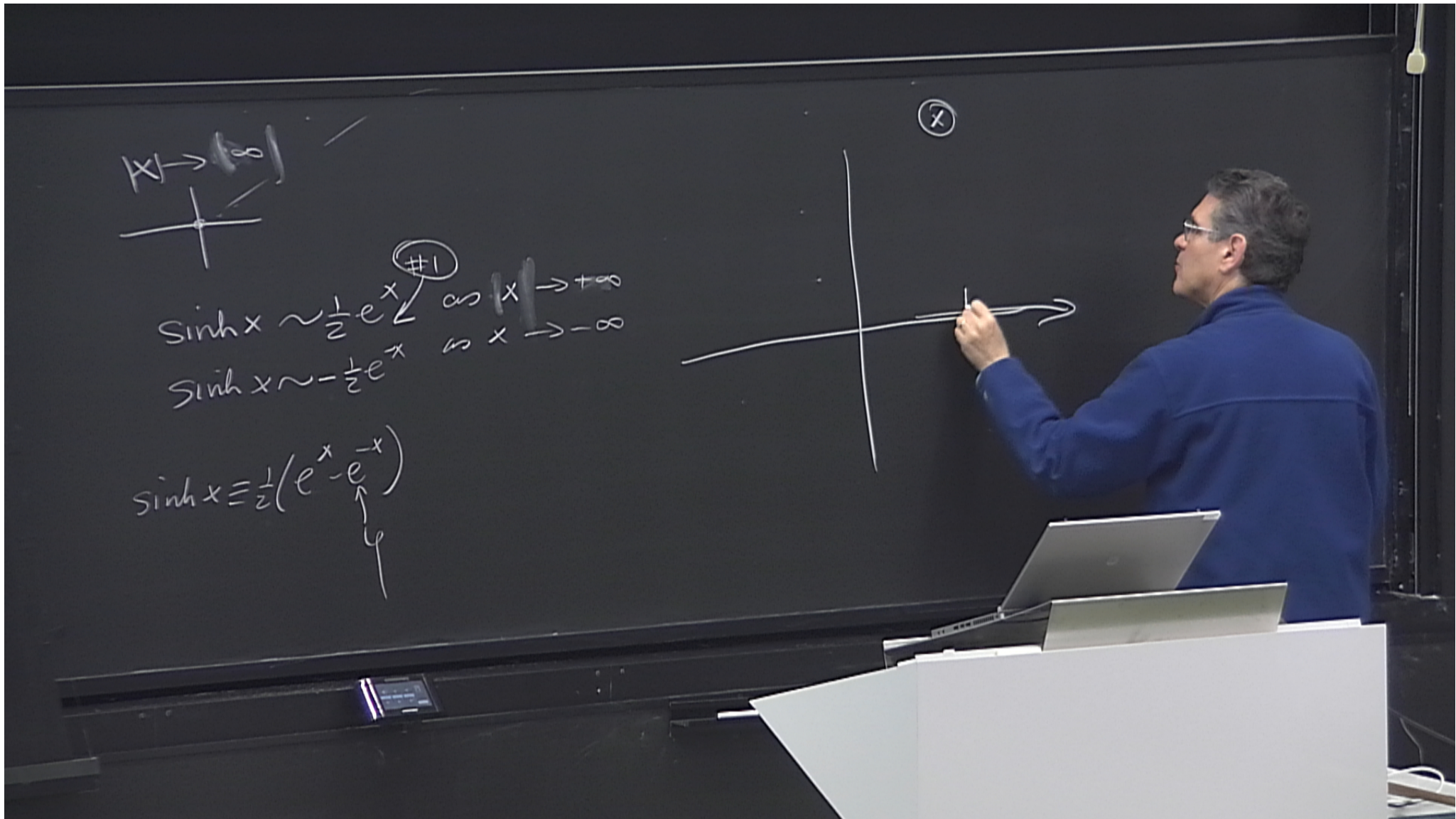




$$\sinh x \sim$$

$$\cos x \rightarrow +\infty$$

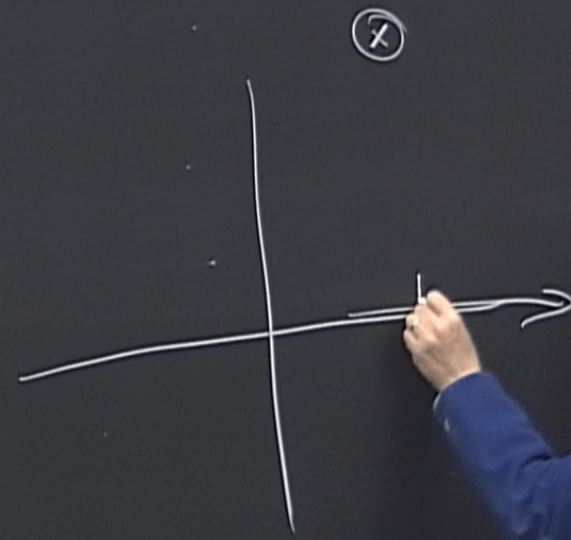
$$\sinh x \equiv \frac{1}{2}(e^x - e^{-x})$$

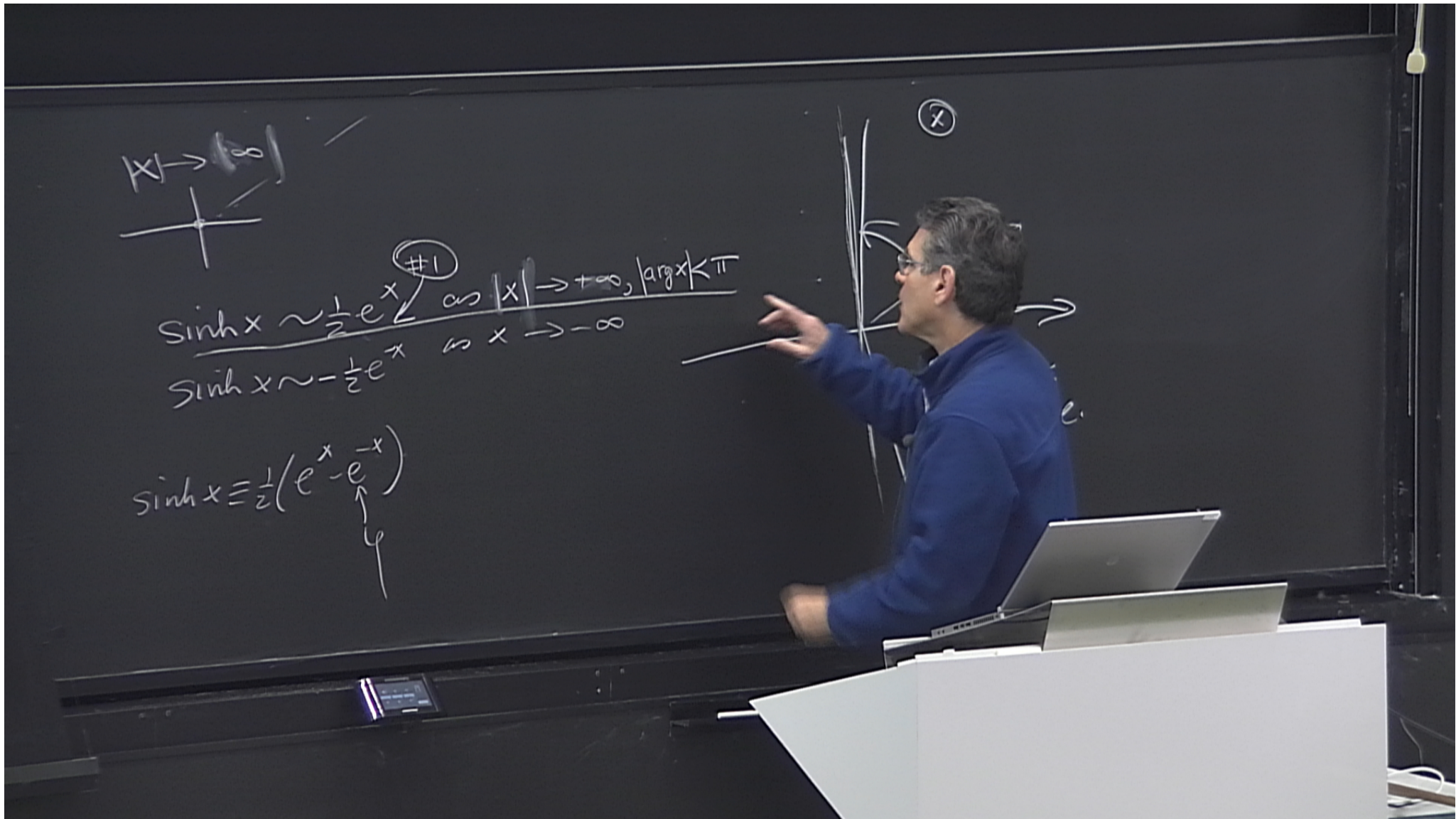


$|x| \rightarrow \infty$

$\sinh x \sim \frac{1}{2}e^x$ as $x \rightarrow +\infty$
 $\sinh x \sim -\frac{1}{2}e^{-x}$ as $x \rightarrow -\infty$

$$\sinh x \equiv \frac{1}{2}(e^x - e^{-x})$$





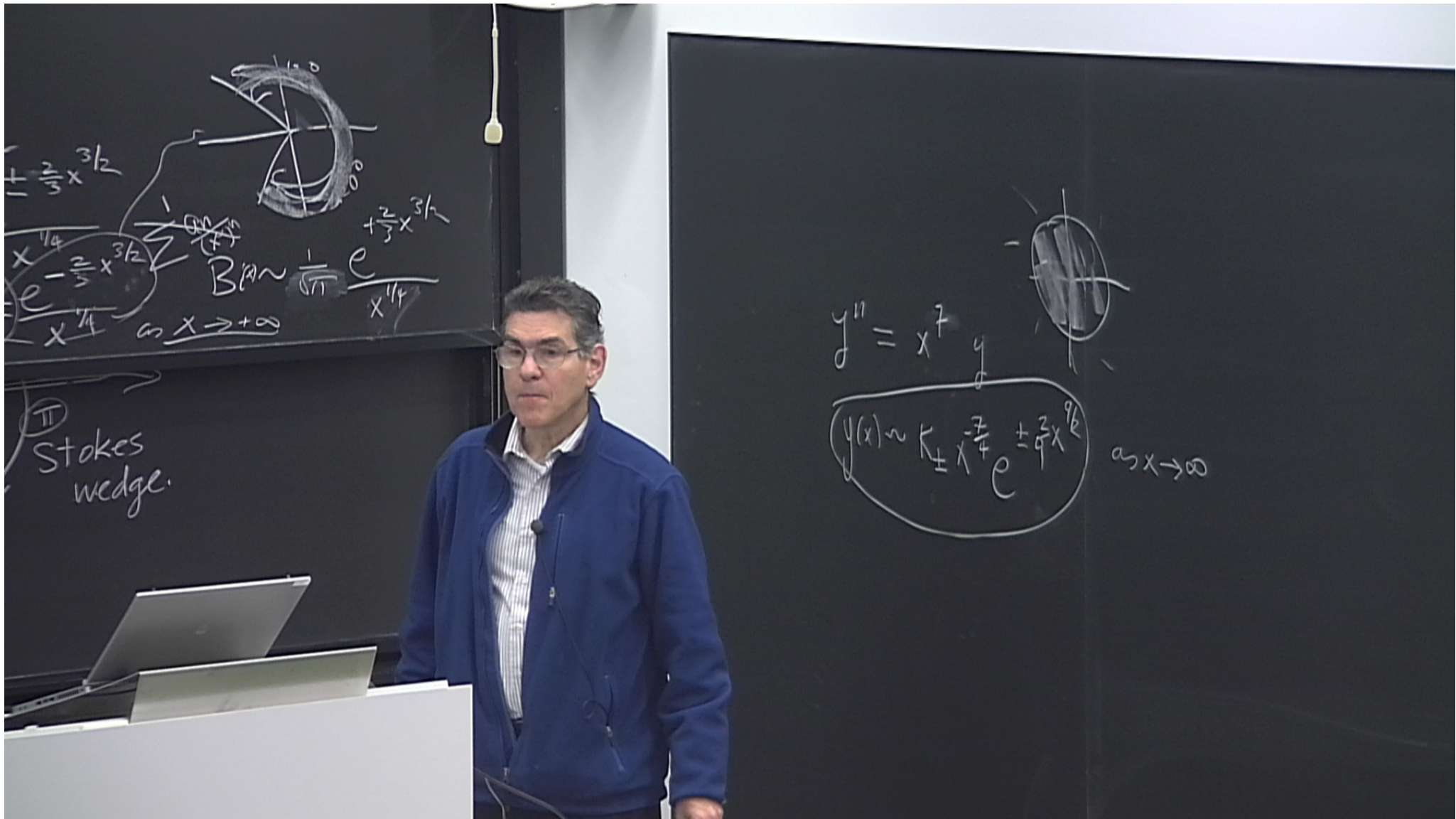
$\sum_{n=0}^{\infty} \frac{x^n}{(n!)^2} / f(x)_{\text{exact.}}$
 $e^{2\sqrt{x} - \frac{2}{3}x} / f(x)_{\text{exact.}}$

$y'' = xy$ Airy
 $\text{Ai}(x) \leftarrow$
 $\text{Bi}(x) \leftarrow$

$y'' = Q(x)y$
 $\text{as } x \rightarrow \infty, Q(x) \rightarrow \infty$
 $y(x) \sim k_{\pm} Q^{-1/4} e^{\pm \int \sqrt{Q} dx}$

$y(x) \sim k_{\pm} e^{\pm \frac{2}{3}x^{3/2}}$
 $\text{Ai}(x) \sim \frac{1}{2\sqrt{\pi}} e^{-\frac{2}{3}x^{3/2}} x^{1/4}$
 $\text{Bi}(x) \sim \frac{1}{\sqrt{\pi}} e^{+\frac{2}{3}x^{3/2}} x^{1/4}$

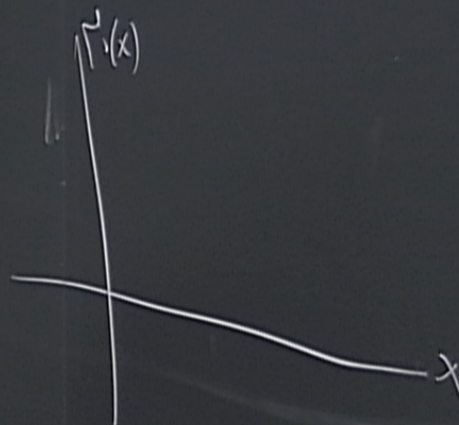
Stokes wedge.
 $\frac{\pi}{2} < \arg x < \frac{3\pi}{2}$

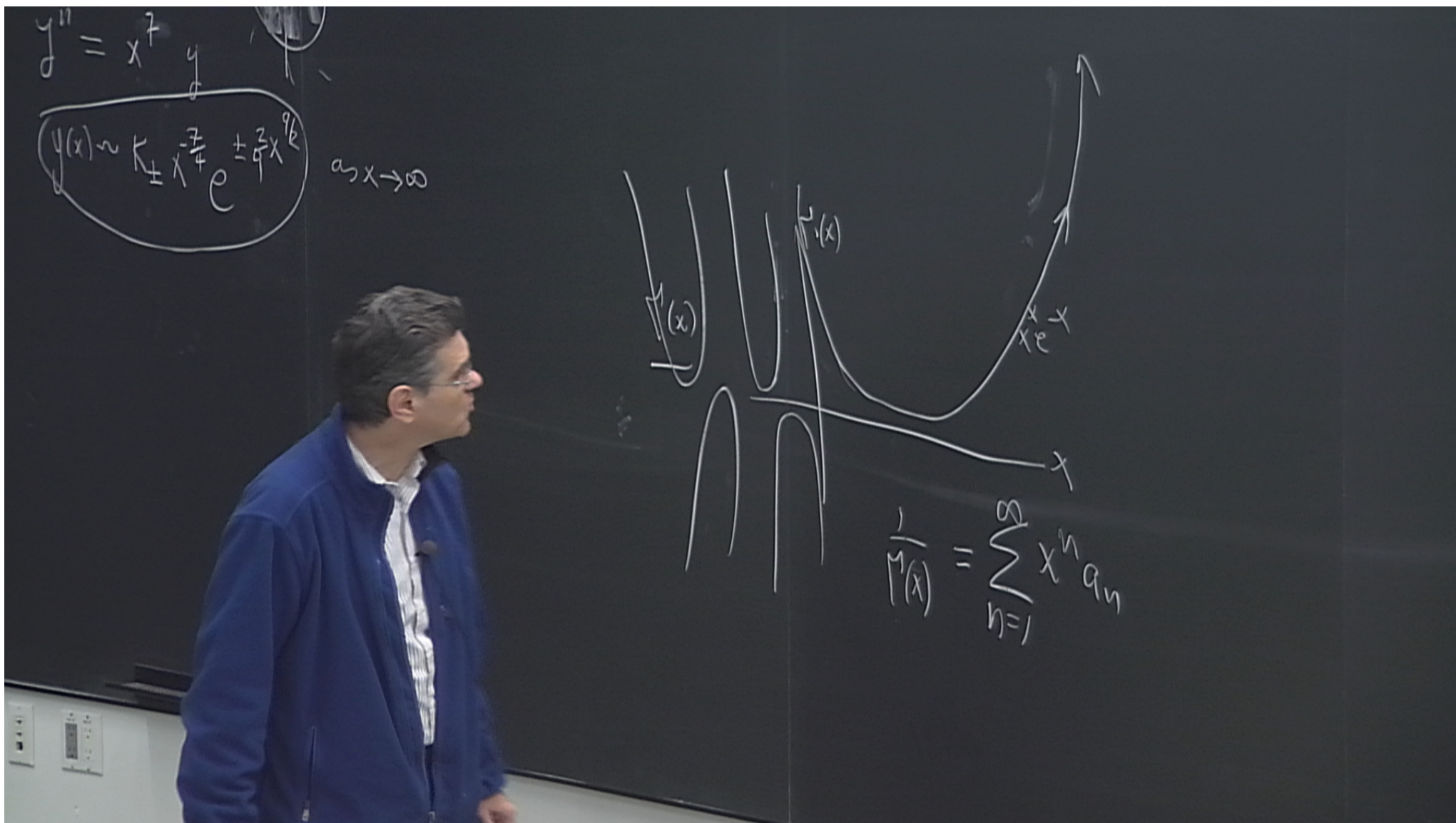


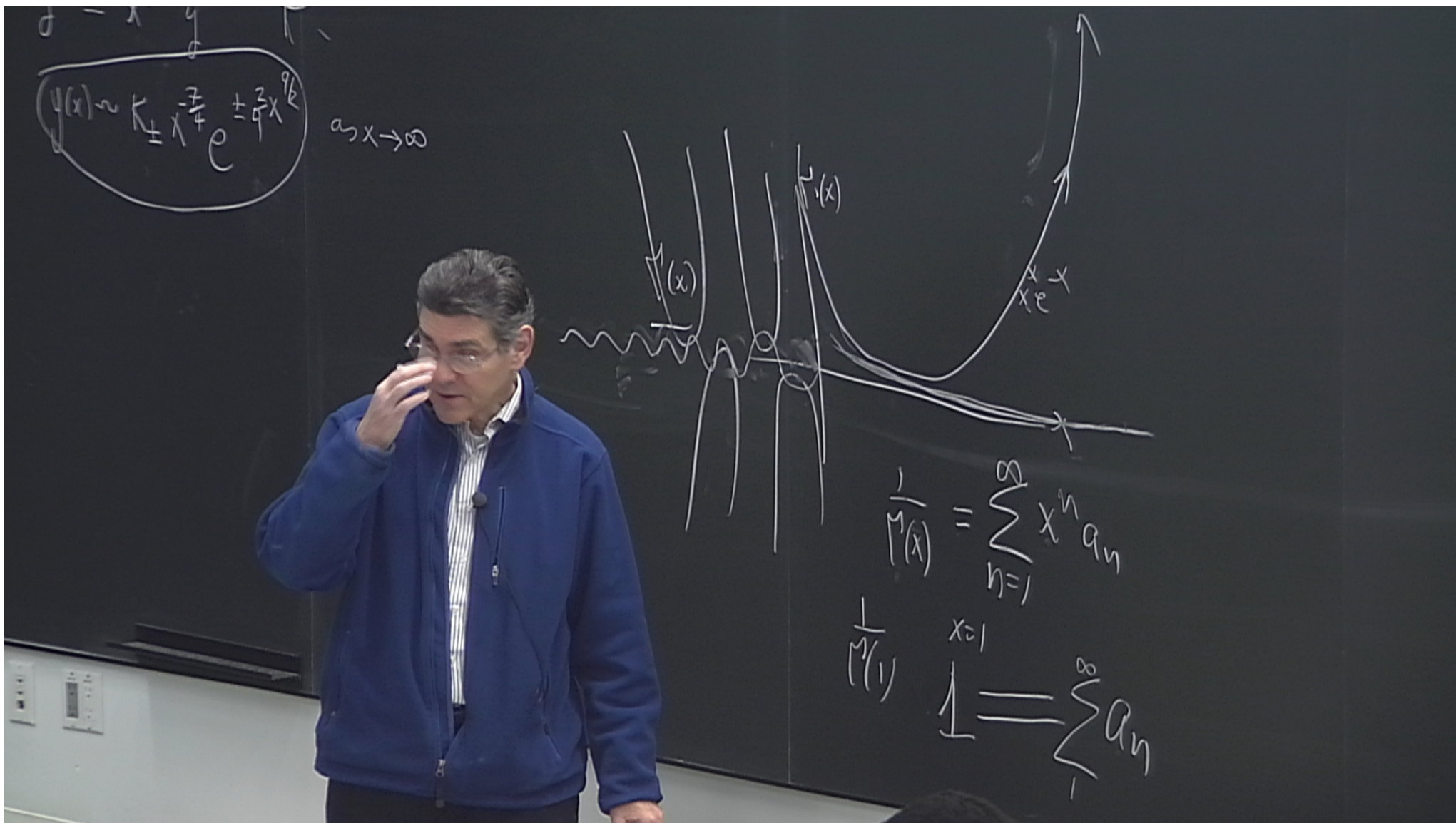
$$y'' = x^7 y$$

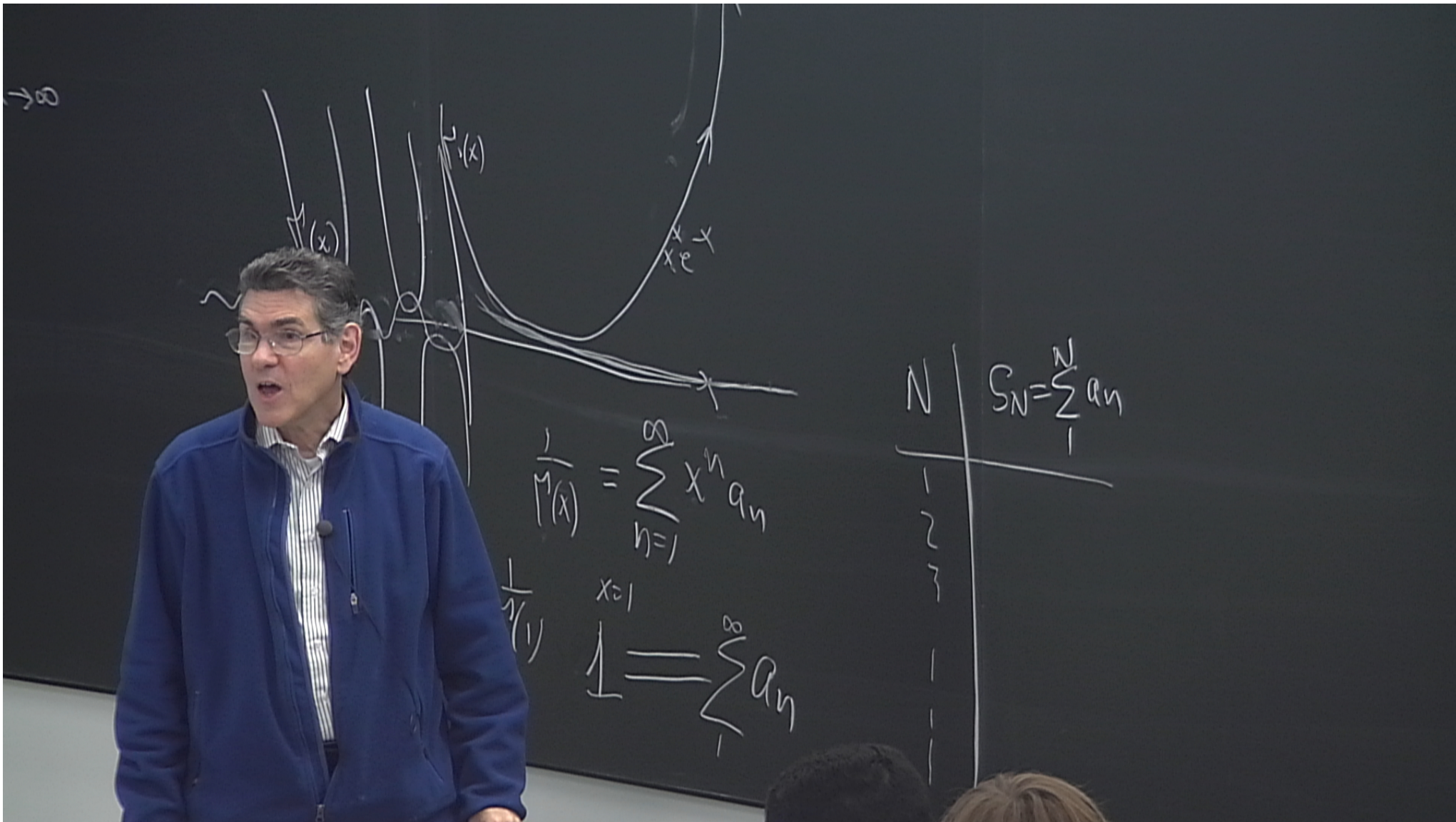
$$y(x) \sim K_{\pm} x^{-\frac{7}{4}} e^{\pm \frac{2}{9} x^{\frac{9}{2}}} \text{ as } x \rightarrow \infty$$

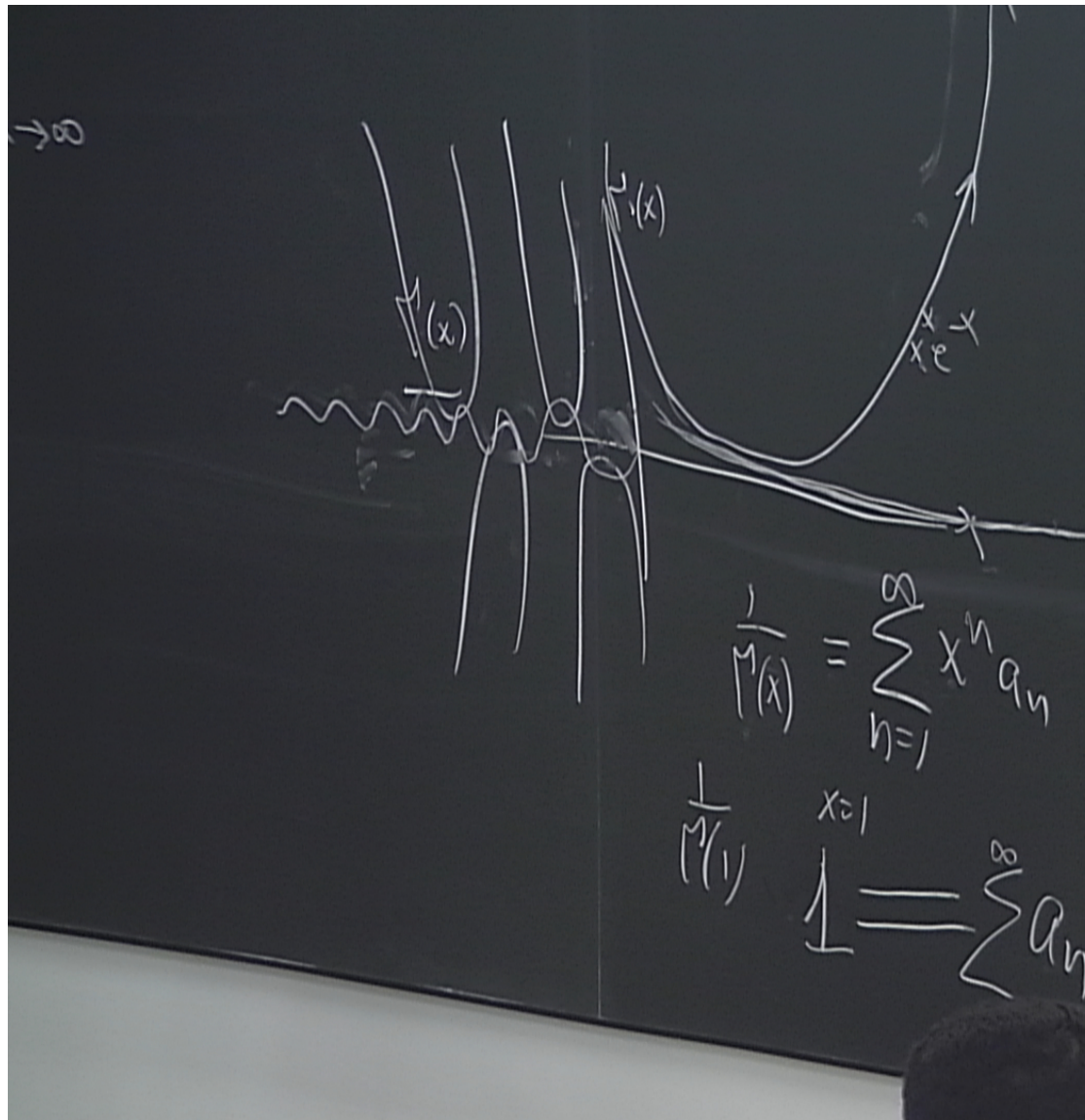
$$\tilde{V}(x)$$





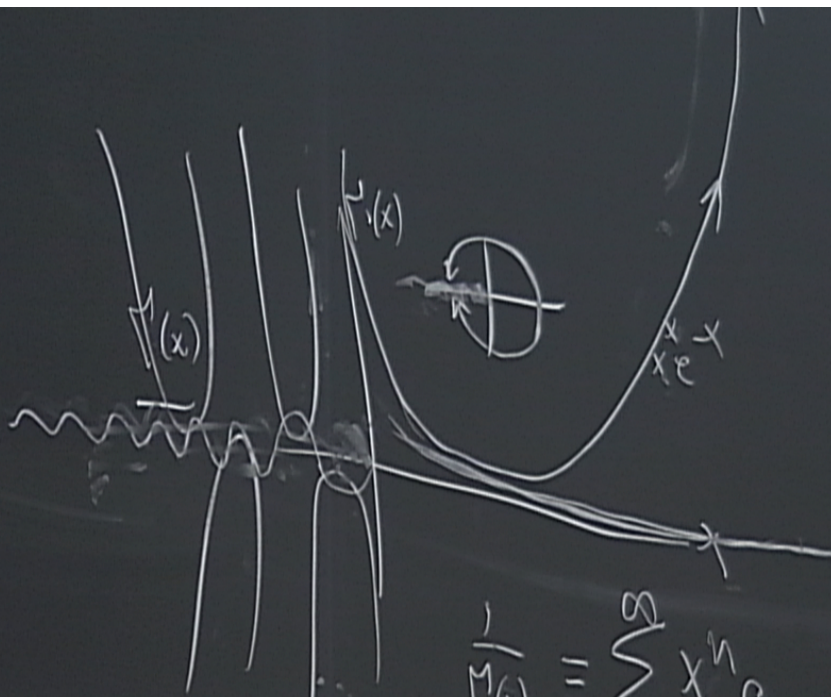






N	$S_N = \sum_{n=1}^N a_n$
1	.7
2	-5
3	4
...	...

$$x \pm x^{\frac{7}{4}} e^{\pm \frac{2}{9} x^{\frac{9}{4}}} \quad \text{as } x \rightarrow \infty$$



$$\frac{1}{f(x)} = \frac{1}{f}$$

$$\frac{1}{f(x)} = \sum_{n=1}^{\infty} x^n a_n$$

$$\frac{1}{f(2)}$$

$$1 = \sum_{n=1}^{\infty} a_n (2)^n$$

N	$S_N = \sum_{i=1}^N a_i$
1	.7
2	-5
3	4
...	...