Title: Relative Locality, Kappa-PoincarÃ© and the Relativity Principle

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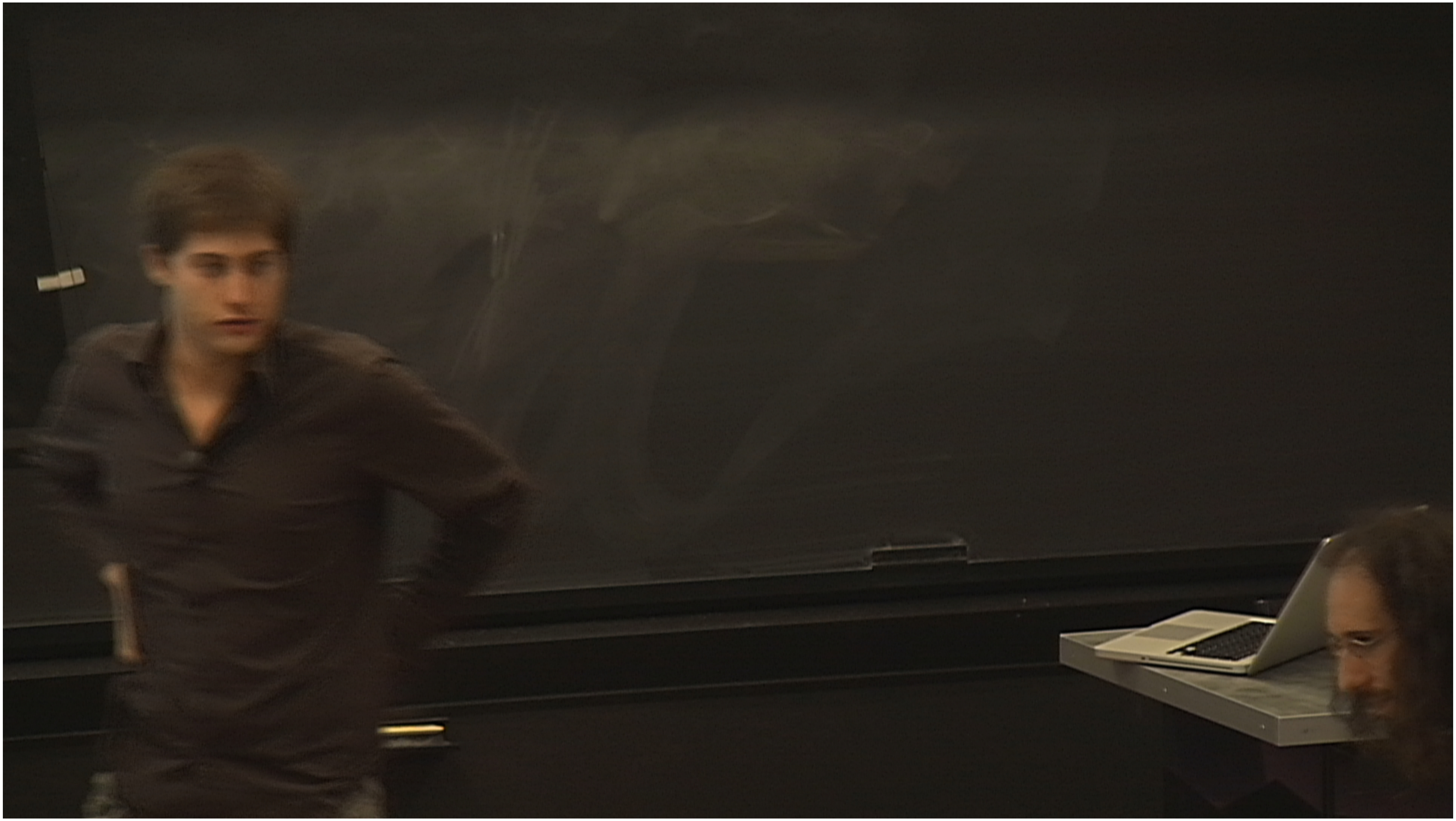
Abstract: I briefly introduce the recently introduced idea of relativity of locality, which is a consequence of a non-flat geometry of momentum space. Momentum space can acquire nontrivial geometrical properties due to quantum gravity effects. I study the relation of this framework with noncommutative geometry, and the Quantum Group approach to noncommutative spaces. In particular I'm interested in kappa-PoincarÃ©, which is a Quantum Group that, as shown by Freidel and Livine, in the 1+1D case emerges as the symmetry of effective field theory coupled with quantum gravity, once that the gravitational degrees of freedom are integrated out. I'm interested in particular in the Lorentz covariance of this model which is present, but is realized in a nontrivial way. If I still have time, I'll then speak about an under-course general study of the Lorentz covariance of Relative Locality models.



$\hbar \to 0$ LIMIT OF QUANTUM GRAVITY

It can be taken in two complementary ways, depending on G and $M_p = \sqrt{\frac{c\hbar}{G}}$

General Relativity	**Relative Locality**
$G = \text{const}, \quad M_p \to 0$	$G \to 0, \quad M_p = \text{const}$
Particle Phase Space = $\Gamma^*(\mathscr{M})$	Particle Phase Space = $\Gamma^*(\mathscr{P})$
($\mathscr{M}$ curved, $\mathscr{P} = T^*(\mathscr{M})$ linear)	($\mathscr{M} = T^*(\mathscr{P})$ linear, $\mathscr{P}$ curved)

(Born's reciprocity principle recovered)

Amelino-Camelia, Freidel, Kowalski-Glikman, Smolin (AFKS)
[arXiv:1101.0931]

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Relativistic dynamics

Free particle

$$S_{free} = \int ds \left[x^a \dot{p}_a - N \left(d^2(p,o) - m^2 \right) \right]$$

N is a Lagrange multiplier

Poisson brackets: $\{x^a, p_b\} = \delta^a_b$

Equations of motion

$$\dot{p}_a = 0 \qquad \dot{x}_a = N \frac{\partial d^2(p,o)}{\partial p_a} \qquad d^2(p,o) = m^2$$

In SR $\quad d^2(p,o) = \eta^{ab} p_a p_b \quad \Rightarrow \quad \dot{x}_a = 2N p_a$

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INTERACTIONS

The composition law $\oplus$ enters into the conservation law. For example, for a three-particle process:

$$p \oplus q = k \quad \Rightarrow \quad \mathcal{K}_a = ((p_1 \oplus p_2) \ominus p_3)_a = 0$$

we can enforce this with a Lagrange multiplier

$$S_{int} = \int ds\, \delta(s - s_0)\, z^a\, \mathcal{K}_a(s) = z^a\, \mathcal{K}_a(s_0)$$

we choose an arbitrary s_0 to label the interaction (we have reparametrization invariance). Then in the free action we have to distinguish between incoming and outgoing particles, and introduce a boundary at s_0:

$$S_{free} = \int_{-\infty}^{s_0} ds \left[x_1^a\, \dot{p}_a - z_1\,(d^2(p_1, o) - m_1^2) + x_2^a\, \dot{q}_a - z_2\,(d^2(p_2, o) - m_2^2)\right]$$

$$+ \int_{s_0}^{\infty} ds \left[x_3^a\, \dot{k}_a - z_3\,(d^2(p_3, o) - m_3^2)\right]$$

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$$\delta S_{int} = \delta z^a\, \mathcal{K}_a(s_0) \pm x_j^a(s_0)\,\delta p_a^j(s_0) + z^a \frac{\delta \mathcal{K}_a}{\delta p_b^j}(s_0)\,\delta p_b^j$$

$$x_j^b(s_0) = \pm z^a \frac{\delta \mathcal{K}_a}{\delta p_b^j}(s_0)$$

In SR same interaction point for all particles (absolute locality):

$$(p_1 \oplus p_2) \ominus p_3 = p_1 + p_2 - p_3 \qquad \Rightarrow \qquad x_j^b(s_0) = z^a \quad \forall j$$

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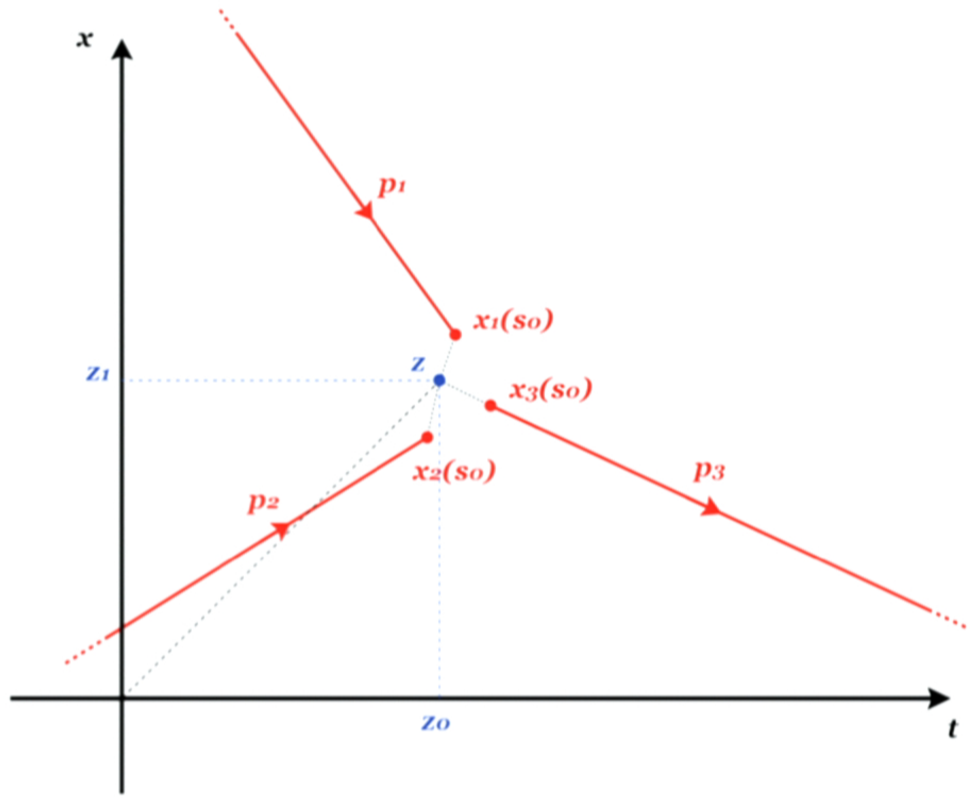
x
p1
x1(s0)
z
z1
x3(s0)
x2(s0)
p3
p2
z0
t

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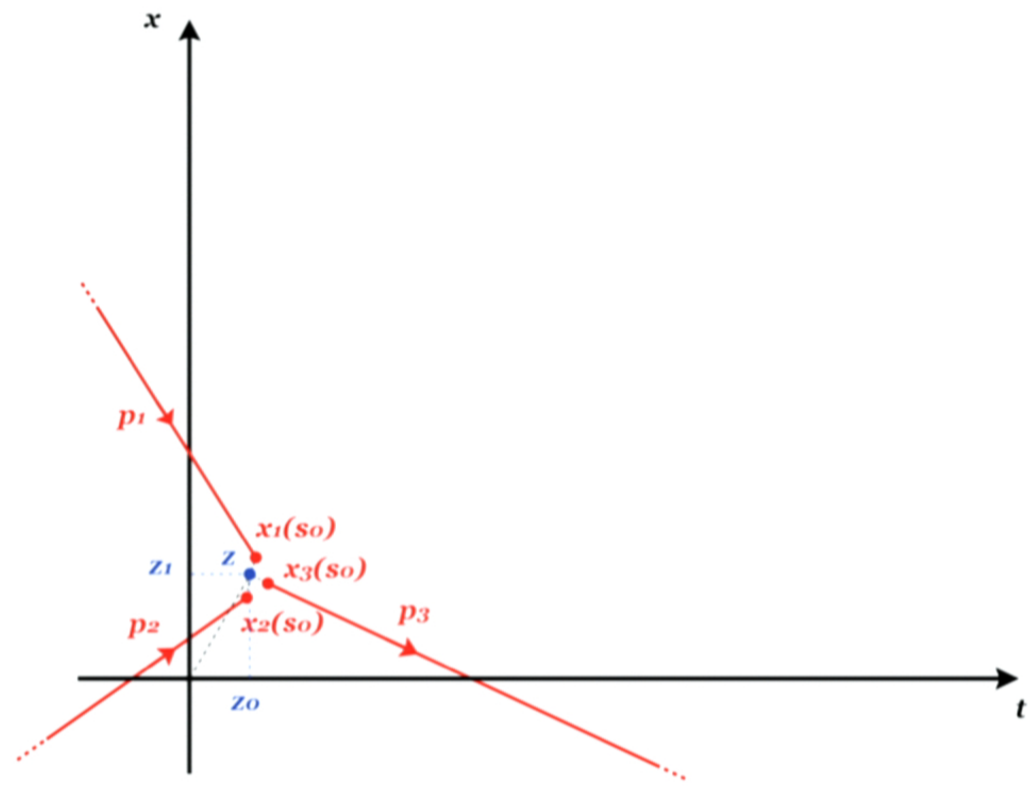

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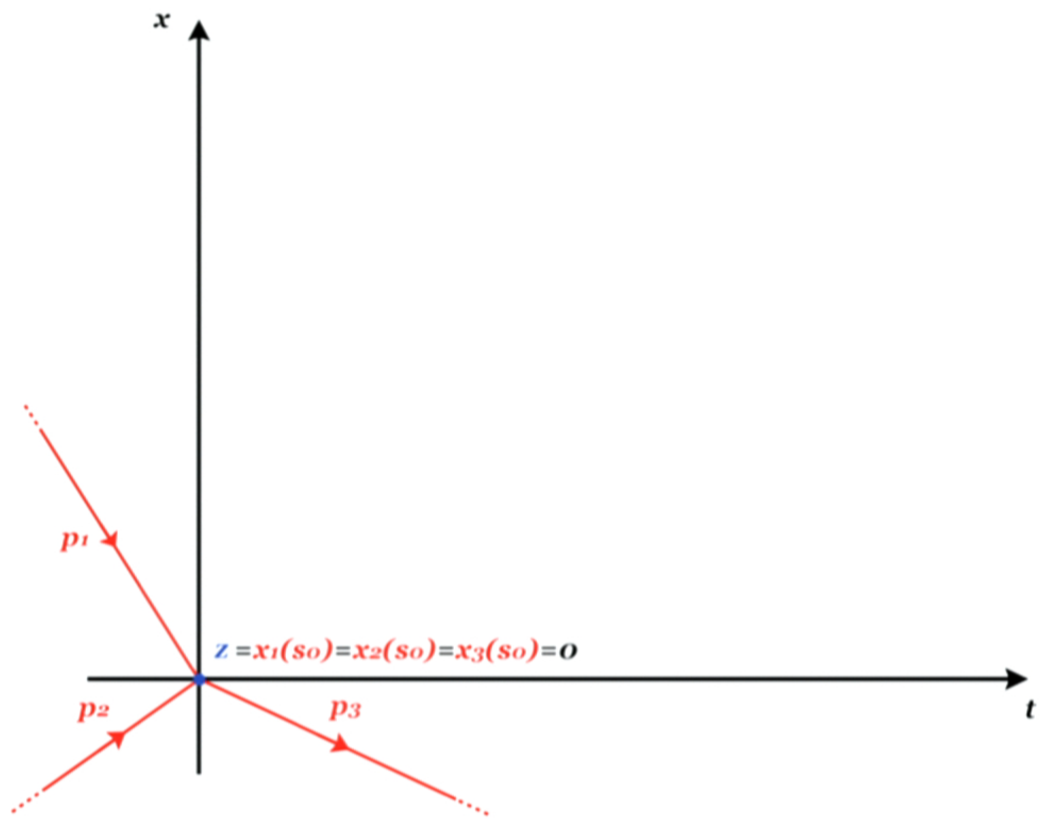

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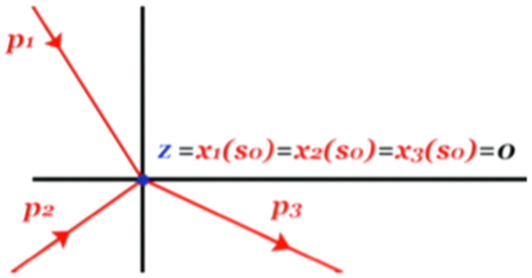
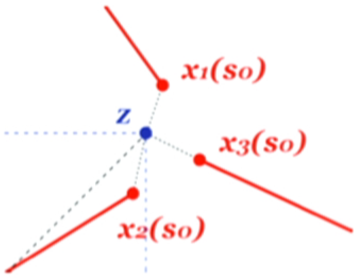
Relativity of locality

Closeby events are always local



Far events appear nonlocal



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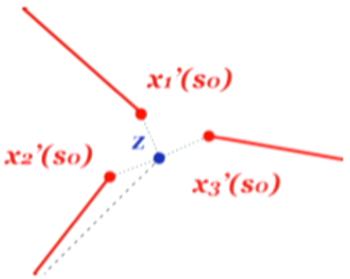

$$p_a \to p'_a, \quad x'^a(s) = x^b(s)\,\frac{\partial p_b}{\partial p'_a}$$

$$\Rightarrow x^a(\tilde{s}) = 0 \to x'^a(\tilde{s}) = 0$$

...wordlines are not diffeo-invariant. The fact that they cross the origin is.

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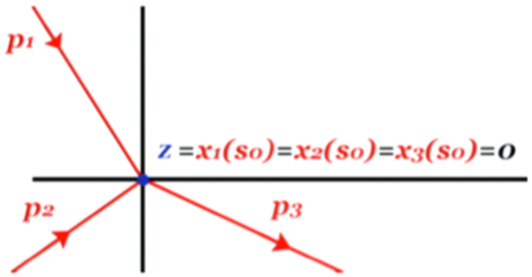

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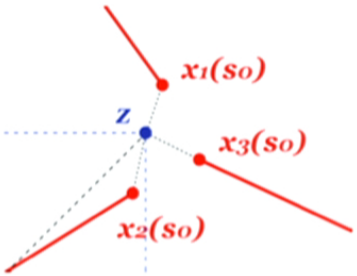

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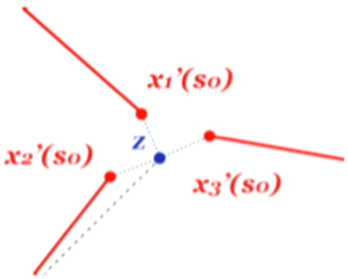

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Relative Locality: the present status

Phenomenology

- **GRBs**
  [AFKS] [Freidel-Smolin]
- **Dual Gravitational Lensing**
  [Freidel-Smolin]
- **Atom Interferometry**
  [AFKS]
  [FM+Rome group, PRL 103]

Theoretical issues

- **Soccer-Ball Problem**
  [AFKS]
- **Translations**
  [AFKS] [Kowalski+Rome group]
- **BH Information Paradox**
  [Smolin]
- **Lorentz Invariance**
  [FM+Gubitosi] [Amelino-Camelia]
  [FM+Carmona+Cortes]

Relative Locality in κ-Poincaré

[arXiv:1106.5710]

with Giulia Gubitosi (UC Berkeley)

Quantum Groups (Hopf algebras)

Generalizations of Lie groups to describe symmetries of *noncommutative spaces*

If G is a Lie group and $\mathbb{C}(G)$ is the algebra of complex-valued functions over G

$$(f \cdot g)(g_1) = f(g_1)g(g_1) = (g \cdot f)(g_1)$$

$$\Delta : \mathbb{C}(G) \to \mathbb{C}(G) \otimes \mathbb{C}(G) \sim \mathbb{C}(G \times G) \quad \text{coproduct} \quad \Delta(f)(g_1, g_2) = f(g_1 \cdot g_2)$$

$$S : \mathbb{C}(G) \to \mathbb{C}(G) \quad \text{antipode} \quad S(f)(g) = f(g^{-1})$$

$$\varepsilon : \mathbb{C}(G) \to \mathbb{C} \quad \text{counit} \quad \varepsilon(f) = f(1_G)$$

$\mathbb{C}(G)$ can be generalized to a noncommutative algebra $\mathscr{A}$

κ-Poincaré Quantum Group

Coalgebraic structures are the same as the Poincaré group:

$$\Delta(\Lambda^\mu{}_\nu) = \Lambda^\mu{}_\rho \otimes \Lambda^\rho{}_\nu\,, \qquad \Delta(a^\mu) = \Lambda^\mu{}_\nu \otimes a^\nu + a^\mu \otimes \mathbb{1}\,,$$

$$S(\Lambda^\mu{}_\nu) = (\Lambda^{-1})^\mu{}_\nu\,, \qquad S(a^\mu) = -(\Lambda^{-1})^\mu{}_\nu a^\nu\,,$$

$$\varepsilon(\Lambda^\mu{}_\nu) = \delta^\mu{}_\nu\,, \qquad \varepsilon(a^\mu) = 0\,.$$

algebra $\mathbb{C}(G)$ deformed into a noncommutative algebra $\mathbb{C}_\kappa(G)$:

$$[a_j, a_0] = \frac{i\hbar}{\kappa} a_j, \qquad [a_j, a_k] = 0, \qquad [\Lambda^\mu{}_\nu, \Lambda^\rho{}_\sigma] = 0$$

$$[a^\mu, \Lambda^\rho{}_\sigma] = \frac{i\hbar}{\kappa}\left[(\Lambda^\rho{}_0 - \delta^\rho{}_0)\Lambda^\mu{}_\sigma + \left(\Lambda^0{}_\sigma - \delta^0{}_\sigma\right)\eta^{\rho\mu}\right]$$

Zakrzewski [J. Phys. A27 (1994)]

κ-MINKOWSKI NONCOMMUTATIVE SPACETIME

Quotienting the κ-Poincaré group by its Lorentz Hopf-subalgebra

$$\mathbb{C}_\kappa(G)/\mathbb{C}(SO(3,1)) = \mathscr{A}$$

we obtain the noncommutative κ-Minkowski algebra,

$$[x_j, x_0] = \frac{i\hbar}{\kappa} x_j, \qquad \Delta x_\mu = x_\mu \otimes \mathbb{1} + \mathbb{1} \otimes x_\mu, \qquad S(x_\mu) = -x_\mu, \qquad \varepsilon(x_\mu) = 0$$

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QUANTUM GROUPS (HOPF ALGEBRAS)

Hopf algebras are self-dual structures: the dual of a Quantum Group, a Quantum Algebra, is still an Hopf algebra.

In the case of a Lie group G written as an Hopf algebra $\mathbb{C}(G)$, its dual is the Universal Enveloping Algebra $U(G^*)$ of the Lie algebra G^*:

$$(f \cdot g)(g_1) = f(g_1)g(g_1) = (g \cdot f)(g_1) \qquad \Leftrightarrow \qquad \Delta(t_i) = t_i \otimes 1 + 1 \otimes t_i$$

$$\Delta(f)(g_1, g_2) = f(g_1 \cdot g_2) \qquad \Leftrightarrow \qquad [t_i, t_j] = f_{ij}{}^k\, t_k$$

$$S(f)(g) = f(g^{-1}) \qquad \Leftrightarrow \qquad S(t_i) = -t_i$$

$$\varepsilon(f) = f(1_G) \qquad \Leftrightarrow \qquad \varepsilon(t_i) = 0$$

κ-Poincaré algebra (1+1D)

Taking the dual of κ-Poincaré group, we obtain its Quantum Algebra:

$$[P,E]=0\,,\qquad [N,P]=\frac{\kappa}{2}\left(1-e^{-2E/\kappa}\right)-\frac{1}{2\kappa}P^2\,,\qquad [N,E]=P\,,$$

$$\Delta E = E\otimes \mathbb{1} + \mathbb{1}\otimes E\,,\quad \Delta P = P\otimes \mathbb{1} + e^{-E/\kappa}\otimes P\,,$$
$$\Delta N = N\otimes \mathbb{1} + e^{-E/\kappa}\otimes N\,,$$

$$S(E)=-E\,,\quad S(P)=-e^{E/\kappa}P\,,\quad S(N)=-e^{E/\kappa}N\,,$$
$$\varepsilon(E)=\varepsilon(P)=\varepsilon(N)=0\,.$$

Lukierski [Phys. Lett. B264 (1991)]

κ-POINCARÉ AND 2+1D QUANTUM GRAVITY

Freidel and Livine [PRL 96 (2006)] showed that quantum gravity amplitudes for particles coupled to 2+1D gravity

$$Z = \int Dg \int D\phi \; e^{iS(g,\phi)} e^{iS_{GR}(\phi)}$$

integrating out the gravitational degrees of freedom

$$Z = \int D\phi \; e^{iS_{eff}(\phi)}$$

are related to the amplitudes of a noncommutative field theory

$$S_{eff} = \frac{1}{2}\int \left(\partial_\mu\phi \star \partial^\mu\phi + \frac{\sin^2 m\kappa}{\kappa^2}\phi\star\phi\right) + \frac{\lambda}{3!}\int \phi\star\phi\star\phi$$

where $\phi \in \mathscr{A} = \mathbb{C}_\kappa(G)/\mathbb{C}(SO(3,1))$ and $\star$ is a realization of the product of $\mathscr{A}$. This field theory is invariant under the κ-Poincaré algebra.

κ-Poincaré algebra (1+1D)

E and P close a commutative Hopf-subalgebra:

$$[P,E] = 0 \,,$$

$$\Delta E = E \otimes \mathbb{1} + \mathbb{1} \otimes E \,, \quad \Delta P = P \otimes \mathbb{1} + e^{-E/\kappa} \otimes P \,,$$

$$S(E) = -E \,, \quad S(P) = -e^{E/\kappa} P \,,$$

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we interpret it as the algebra of functions over a momentum space $\mathscr{P}$

κ-POINCARÉ ALGEBRA (1+1D)

- E and P are "coordinate functions": $P^\mu(p) = \mu$-th coordinates of point p.

- Diffeos $P^\mu \to f^\mu(P)$ correspond to change of basis in the enveloping algebra

- The coproduct & antipode define a composition law $\Delta(P^\mu)(p,q) = P^\mu(p \oplus q)$ and its inverse $S(P^\mu)(p) = P^\mu(\ominus p)$

- The counit give the coordinates of the origin $\varepsilon(P^\mu) = P^\mu(o)$. In the "bi-crossproduct" basis these coordinates are $\{0,0\}$.

- The "coassociativity" axiom of Hopf algebras $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$ imply associativity of composition rule ($\Rightarrow$ flat connection) (want curved connections? try with Hopf algebroids)

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LET'S FIND THE METRIC...

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these can be integrated wrt ξ. The solution is

$$E' = E + \kappa \log\left[\left(\cosh\xi/2 + \frac{p}{\kappa}\sinh\xi/2\right)^2 - e^{-2E/\kappa}\sinh^2\xi/2\right]$$

$$p' = \kappa\frac{\left(\mathrm{ch}\,\xi/2 + \frac{p}{\kappa}\mathrm{sh}\,\xi/2\right)\left(\mathrm{sh}\,\xi/2 + \frac{p}{\kappa}\mathrm{ch}\,\xi/2\right) - e^{-2E/\kappa}\mathrm{ch}\,\xi/2\ \mathrm{sh}\,\xi/2}{\left(\mathrm{ch}\,\xi/2 + \frac{p}{\kappa}\mathrm{sh}\,\xi/2\right)^2 - e^{-2E/\kappa}\sinh^2\xi/2}$$

[Rossano-Bruno, Amelino-Camelia, Kowalski-Glikman, PLB 522 (2001)]

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de Sitter metric in comoving coordinates

$$ds^2 = dE^2 - e^{2E/\kappa}dp^2$$

this metric is invariant under the Lorentz transformations shown above:

$$ds'^2 = dE'^2 - e^{2E'/\kappa}dp'^2 = ds^2$$

consider this coordinate transformation:

$$\eta_0 = \kappa \sinh(E/\kappa) + e^{E/\kappa}P^2/2\kappa$$

$$\eta_1 = e^{E/\kappa}P$$

the algebra in this basis turn into Poincaré algebra

$$[\eta_0, \eta_1] = 0 \,, \qquad [N, \eta_0] = \eta_1 \,, \qquad [N, \eta_1] = \eta_0$$

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DE SITTER METRIC IN EMBEDDING COORDINATES

the coordinate transformation is not invertible:

$$E_{\pm} = \kappa \log\left(\frac{\eta_0 \pm \sqrt{\kappa^2 + \eta_0^2 - \eta_1^2}}{\kappa}\right) \, , \quad P_{\pm} = \frac{\kappa\, \eta_1}{\eta_0 \pm \sqrt{\kappa^2 + \eta_0^2 - \eta_1^2}} \, .$$

then the coalgebra in these coordinates won't close

$$\Delta \eta_1 = \eta_1 \otimes e^{E/\kappa} + 1 \otimes \eta_1$$

$$\Delta \eta_0 = \eta_0 \otimes e^{E/\kappa} + e^{-E/\kappa} \otimes \eta_0 + \frac{1}{\kappa} e^{-E/\kappa} \eta_1 \otimes \eta_1$$

but extending the algebra with another generator

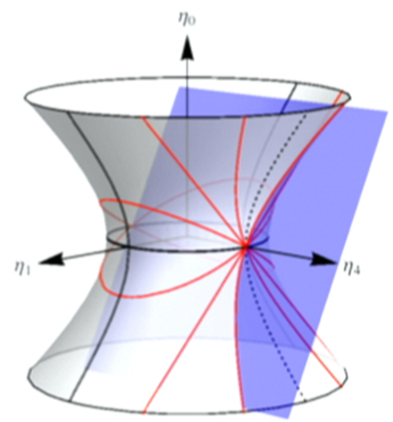
$$\eta_4 = \kappa \cosh(E/\kappa) - e^{E/\kappa} P^2 / 2\kappa$$

the change of basis results invertible:

$$E = \kappa \log\left(\frac{\eta_0 + \eta_4}{\kappa}\right) \, , \quad P = \frac{\kappa\, \eta_1}{\eta_0 + \eta_4} \, ,$$

Geodesics and dispersion curves

de Sitter geodesics passing through the origin o are given by the intersections with the plans through the η_4 axis



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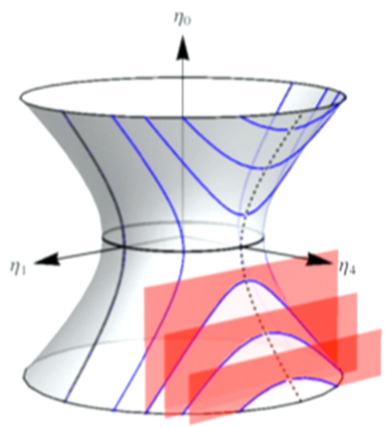
Geodesics and dispersion curves

curves at fixed metric distance from the origin (dispersion curves) are given by the intersections with plans orthogonal to η_4



Dispersion relation

The equation relating the distance from o to the value η_4 for these planes is:

$$\eta_4 = \kappa \cosh(d/\kappa)$$

this gives us the dispersion relation in embedding coordinates:

$$d(p,o) = \kappa \operatorname{arccosh}(\eta_4/\kappa) = \kappa \operatorname{arccosh}\left(\cosh(E/\kappa) - e^{-E/\kappa} P^2/2\kappa^2\right) = m$$

and therefore the dispersion relation in comoving coordinates:

$$\cosh(E/\kappa) - e^{-E/\kappa} P^2/2\kappa = \cosh(m/\kappa)$$

$\Longrightarrow$ we got that the mass is the rest energy $(E(P=0) = m)$

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Connection

In comoving coordinates the composition law is

$$(p \oplus q)_0 = p_0 + q_0, \qquad (p \oplus q)_1 = p_1 + e^{-p_0/\kappa} q_1$$

"translating" the relation $\Gamma^{\mu\nu}_{\rho}(o) = -\frac{\partial}{\partial p_\mu}\frac{\partial}{\partial q_\nu}(p \oplus q)\Big|_{p=q=o}$ we can calculate the connection everywhere:

$$\Gamma^{\mu\nu}_{\rho}(k) = -\frac{\partial}{\partial p_\mu}\frac{\partial}{\partial q_\nu}(k \oplus ((\ominus k \oplus p) \oplus (\ominus k \oplus q)))\Big|_{p=q=k}$$

In κ-Poincaré $\quad \Gamma^{\mu\nu}_{\rho}(k) = \frac{1}{\kappa}\delta^{\mu}{}_{0}\delta^{\nu}{}_{1}\delta^{1}{}_{\rho}$

Connection

The κ-Poincaré connection possess:

- Torsion: $T_{\rho}^{\mu\nu} = \frac{1}{\kappa}\delta^{[\mu}{}_0\delta^{\nu]}{}_1\delta^1{}_\rho$
- Nonmetricity: $\nabla^\rho g^{\mu\nu} = \frac{1}{\kappa}(2\delta^\mu{}_1\delta^\nu{}_1\delta^\rho{}_0 + \delta^\mu{}_0\delta^\nu{}_1\delta^\rho{}_1 + \delta^\mu{}_1\delta^\nu{}_0\delta^\rho{}_1)\, e^{2E/\kappa}$
- Zero curvature: $R_\mu^{\nu\rho\sigma} = 0$ (the composition law is associative)

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RELATIVITY PRINCIPLE

The deformed Lorentz transformations shown above:

$$E' = E + \kappa \log\left[\left(\cosh\xi/2 + \frac{p}{\kappa}\sinh\xi/2\right)^2 - e^{-2E/\kappa}\sinh^2\xi/2\right]$$

$$p' = \kappa\frac{(\text{ch}\,\xi/2 + \frac{p}{\kappa}\text{sh}\,\xi/2)\,(\text{sh}\,\xi/2 + \frac{p}{\kappa}\text{ch}\,\xi/2) - e^{-2E/\kappa}\text{ch}\,\xi/2\ \text{sh}\,\xi/2}{(\text{ch}\,\xi/2 + \frac{p}{\kappa}\text{sh}\,\xi/2)^2 - e^{-2E/\kappa}\sinh^2\xi/2}$$

leave the dispersion relation invariant (they map dispersion curves into themselves):

$$\cosh(E'/\kappa) - e^{-E'/\kappa}p'^2/2\kappa = \cosh(E/\kappa) - e^{-E/\kappa}p^2/2\kappa = \cosh(m/\kappa)$$

but what about the composition law $p \oplus q = \{p_0 + q_0, p_1 + e^{-p_0/\kappa}q_1\}$?

RELATIVITY PRINCIPLE

It turns out that it's not invariant in a naïve sense (it's not a $\oplus$-homomorphism):

$$p' \oplus q' \neq (p \oplus q)'$$

this is due to the nontrivial coproduct of the boost generator:

$$\Delta N = N \otimes \mathbb{1} + e^{-E/\kappa} \otimes N \,,$$

which introduces a "back-reaction" $e^{-E/\kappa}$ of the momenta on the boosts

Majid [hep-th/0604130] exponentialized this back-reaction, turning it into an action of momenta on Lorentz group elements, labelled by their rapidity ξ:

$$\xi \triangleleft p = 2\,\mathrm{arcsinh}\left(\frac{e^{-p_0/\kappa} \sinh \frac{\xi}{2}}{\sqrt{\left(\cosh \frac{\xi}{2} + \frac{p_1}{\kappa} \sinh \frac{\xi}{2}\right)^2 - e^{-2p_0/\kappa} \sinh^2 \frac{\xi}{2}}} \right)$$

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RELATIVITY PRINCIPLE

Then, writing the boosts as a one-parameter family of automorphisms of momentum space: $\Lambda : \mathbb{R} \times \mathscr{P} \to \mathscr{P}$, parameterized by the rapidity ξ, we get:

$$\Lambda(\xi, p \oplus q) = \Lambda(\xi, p) \oplus \Lambda(\xi \triangleleft p, q)$$

"$\triangleleft$" satisfies some compatibility relations with the coalgebra:

- $(\xi \triangleleft q) \triangleleft p = \xi \triangleleft (q \oplus p)$ ($\triangleright$ is a *right* action)

  $$\Rightarrow \quad \Lambda(\xi, p \oplus q \oplus k) = \Lambda(\xi, p) \oplus \Lambda(\xi \triangleleft p, q) \oplus \Lambda(\xi \triangleleft (p \oplus q), k)$$

- $\Lambda(\xi \triangleleft p, S(p)) = S(\Lambda(\xi, p))$

we can parametrize everything in terms of the rapidity of every single particle:

$$\Lambda(\xi' \triangleleft S(p), p \oplus q) = \Lambda(\xi' \triangleleft S(p), p) \oplus \Lambda(\xi', q)$$

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Relativity principle

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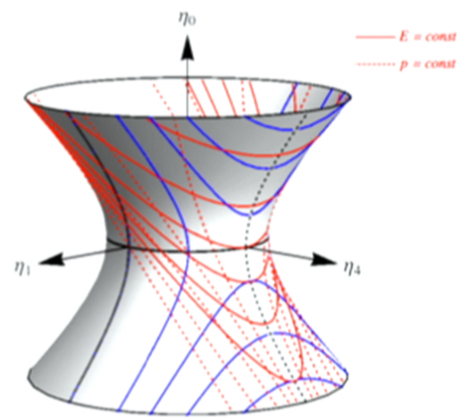
RELATIVITY PRINCIPLE

CAUTION: if the momentum p is outside of the upper light-cone $d(p,o) \geq 0$, there exist a finite critical boost ξ_c which makes $|p_0| \to \infty$. Conversely, for all ξ, there exists a critical curve on momentum space for which $\xi \triangleleft |p_c| \to \infty$

This behavior is due to the comoving coordinate system (coordinate singularity):



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RELATIVITY PRINCIPLE, REPRISE

can we establish which momentum spaces preserve Lorentz invariance?

[to appear]

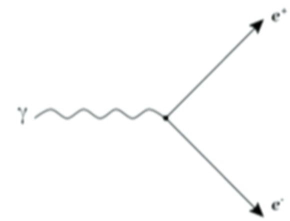
With J. Carmona & J. Cortes (Zaragoza)

RELATIVITY PRINCIPLE, REPRISE

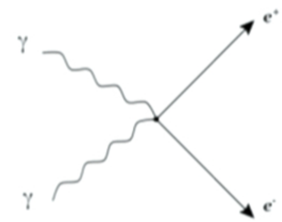
Amelino-Camelia recently studied in some generality the problem of the equivalence between inertial observers in RL [arXiv:1110.5081]

He introduced two golden rules:

"no in-vacuo photon decays"



"no switch-off threshold for photoproduction"



that a momentum space has to satisfy in oder to preserve the relativity principle.

RELATIVITY PRINCIPLE, REPRISE

Amelino-Camelia recently studied in some generality the problem of the equivalence between inertial observers in RL [arXiv:1110.5081]

He introduced two golden rules:

"no in-vacuo photon decays"	The process $\gamma \to e^+ e^-$ is never allowed, no matter E_γ
"no switch-off threshold for photoproduction"	No matter how small the energy of a photon E_γ, there always is a sufficiently hard photon $E_{\gamma'}$ that permits the process $\gamma \gamma' \to e^+ e^-$

that a momentum space has to satisfy in oder to preserve the relativity principle.

RELATIVITY PRINCIPLE, REPRISE

Working at first order in M_p^{-1}, he introduces a (not fully general) ansatz for the dispersion relation and the composition law:

$$p_0^2 - p_1^2 + \frac{1}{M_p}\left(\alpha_1\, p_0\, p_1^2 + \alpha_2\, p_0^3\right) = m^2$$

$$(k \oplus p)_0 = k_0 + p_0 + \frac{1}{M_p}(\beta_1\, k_1\, p_1 + \beta_2\, k_0 p_0)$$

$$(k \oplus p)_1 = k_1 + p_1 + \frac{1}{M_p}(\gamma_1\, k_0\, p_1 + \gamma_2\, k_1\, p_0)$$

He founds that both "no in-vacuo photon decays" and "no switch-off threshold for photoproduction" constraints are satisfies if and only if

$$\boxed{\alpha_1 + \alpha_2 + \beta_1 + \beta_2 - \gamma_1 - \gamma_2 = 0}$$

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RELATIVITY PRINCIPLE, REPRISE

He conjectures that the "golden rule" $\alpha_1 + \alpha_2 + \beta_1 + \beta_2 - \gamma_1 - \gamma_2 = 0$ is a necessary and sufficient condition for a momentum-space to be relativistic.

We studied the problem in full generality (assuming only undeformed rotational invariance and polynomial expressions in the momenta):

$$p_0^2 - p_1^2 + \frac{1}{M_p}\left(\alpha_1 p_0 p_1^2 + \alpha_2 p_0^3\right) = m^2$$

$$(k \oplus p)_0 = k_0 + p_0 + \frac{1}{M_p}\left(\beta_1 k_1 p_1 + \beta_2 k_0 p_0 + \beta_3 k_0^2 + \beta_4 p_0^2 + \beta_5 k_1^2 + \beta_6 p_1^2\right)$$

$$(k \oplus p)_1 = k_1 + p_1 + \frac{1}{M_p}(\gamma_1 k_0 p_1 + \gamma_2 k_1 p_0 + \gamma_3 k_1 p_0 + \gamma_4 k_1 k_0)$$

(I show only th 1+1D case, for simplicity)

Relativity principle, reprise

The novelty of our approach consists in only demanding the existence of a consistent set of deformed Lorentz transformations:

$$\Lambda[\xi, p] = \begin{pmatrix} p_0 \\ p_1 \end{pmatrix} + \xi \begin{pmatrix} N, p_0 \\ N, p_1 \end{pmatrix} = \begin{pmatrix} p_0 \\ p_1 \end{pmatrix} + \xi \begin{pmatrix} p_1 + \frac{1}{M_p} \lambda_1 p_1 p_0 \\ p_0 + \frac{1}{M_p} \left(\lambda_2 p_1^2 + \lambda_3 p_0^2 \right) \end{pmatrix}$$

with the most generic back-reaction of momenta on rapidity:

$$\Lambda[\xi, p \oplus q] = \Lambda \left[\xi + \frac{\xi}{M_p} (\eta_1 q_0 + \eta_2 p_0), p \right] \oplus \Lambda \left[\xi + \frac{\xi}{M_p} (\eta_3 p_0 + \eta_4 q_0), q \right]$$

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RELATIVITY PRINCIPLE, REPRISE

Consistency conditions:

1. Invariance of the dispersion relation $C(p) = p_0^2 - p_1^2 + \frac{1}{M_p}\left(\alpha_1 p_0 p_1^2 + \alpha_2 p_0^3\right)$

$$C(\Lambda[\xi, p]) = C(p)$$

2. Closure of a Lorentz algebra (in the 3+1D case)

$$[N_j, N_k] = \varepsilon_{jkl} R_l$$

3. Invariance of the composition law

$$\Lambda[\xi, p \oplus q] = \Lambda\left[\xi + \frac{\xi}{M_p}(\eta_1 q_0 + \eta_2 p_0), p\right] \oplus \Lambda\left[\xi + \frac{\xi}{M_p}(\eta_3 p_0 + \eta_4 q_0), q\right]$$

RELATIVITY PRINCIPLE, REPRISE

We find that the 19 parameters $\alpha_j, \beta_j, \gamma_j, \lambda_j$ and η_j are not independent:

$$\lambda_1 + 2\lambda_2 + 2\lambda_3 = 0$$

$$\alpha_1 = 2\lambda_2 \qquad \alpha_2 = -\lambda_1 - 2\lambda_2$$

$$\gamma_1 = -\eta_1 \qquad \gamma_2 = -\eta_3 \qquad \gamma_3 = -\eta_2 \qquad \gamma_4 = -\eta_4$$

$$\beta_1 = \gamma_1 + \gamma_2 - \alpha_1 \qquad \beta_2 = -\alpha_2 \qquad \beta_3 = \beta_4 = 0 \qquad \beta_5 = \gamma_3 \qquad \beta_6 = \gamma_4$$

these 13 relations allow to express $\alpha_j, \beta_j, \gamma_j$ in function of the 6 independent λ_j and η_j. we have 6 degrees of freedom

RELATIVITY PRINCIPLE, REPRISE

If the above relations are satisfied, we have (up to order M_p^{-2}) a relativistically invariant system with two invariant scales (c and M_p)

We tested our rules with Amelino-camelia's two golden rules:

THEY ARE AUTOMATICALLY SATISFIED.

that is, no in-vacuo photon decays are allowed, and no there is no switch-off threshold for photoproduction.

But that's not the end of the story...

RELATIVITY PRINCIPLE, REPRISE

Amelino-Camelia's rules are necessary, but NOT sufficient for a theory to be relativistic!

That is, there are choices of the $\alpha_j, \beta_j, \gamma_j, \lambda_j$ and η_j parameters that satisfy A-C's golden rules, but don't satisfy our constraints.

This can be easily seen in the simpler case considered by Amelino-Camelia,

$$p_0^2 - p_1^2 + \frac{1}{M_p}\left(\alpha_1 \, p_0 \, p_1^2 + \alpha_2 \, p_0^3\right) = m^2$$

$$(k \oplus p)_0 = k_0 + p_0 + \frac{1}{M_p}(\beta_1 k_1 p_1 + \beta_2 k_0 p_0)$$

$$(k \oplus p)_1 = k_1 + p_1 + \frac{1}{M_p}(\gamma_1 k_0 p_1 + \gamma_2 k_1 p_0)$$

which, to us, corresponds to the choice $\eta_2 = \eta_4 = 0$

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RELATIVITY PRINCIPLE, REPRISE

In this case, imposing Amelino-Camelia's golden rule:

$$\alpha_1 + \alpha_2 + \beta_1 + \beta_2 - \gamma_1 - \gamma_2 = 0$$

we find that, despite no in-vacuo photon decay and no switch-off threshold for photoproduction are allowed, we still don't have a relativistic theory unless:

$$\beta_2 = -\alpha_2$$

we conjecture that there are OTHER GOLDEN RULES (a number of forbidden processes) that, together with A-C's two, ensure that the relativity principle holds.

CONCLUSIONS

- Why should we care about that? because it's measurable (GRBs...)
- Quantum Groups fit very naturally into the RL scheme
- Where did noncommutative spaces end? turn $\hbar$ on
- Lorenz transformations, initially expected to be more trivial than translations, hid a surprise: different particles transform with different rapidities
- Up to first order in M_p^{-1} we have the most general Lorentz-invariant case. Described by 6 independent parameters
- Two golden rules are insufficient to specify it: how many are enough?

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