Title: Localization in 3d Gauge Theories

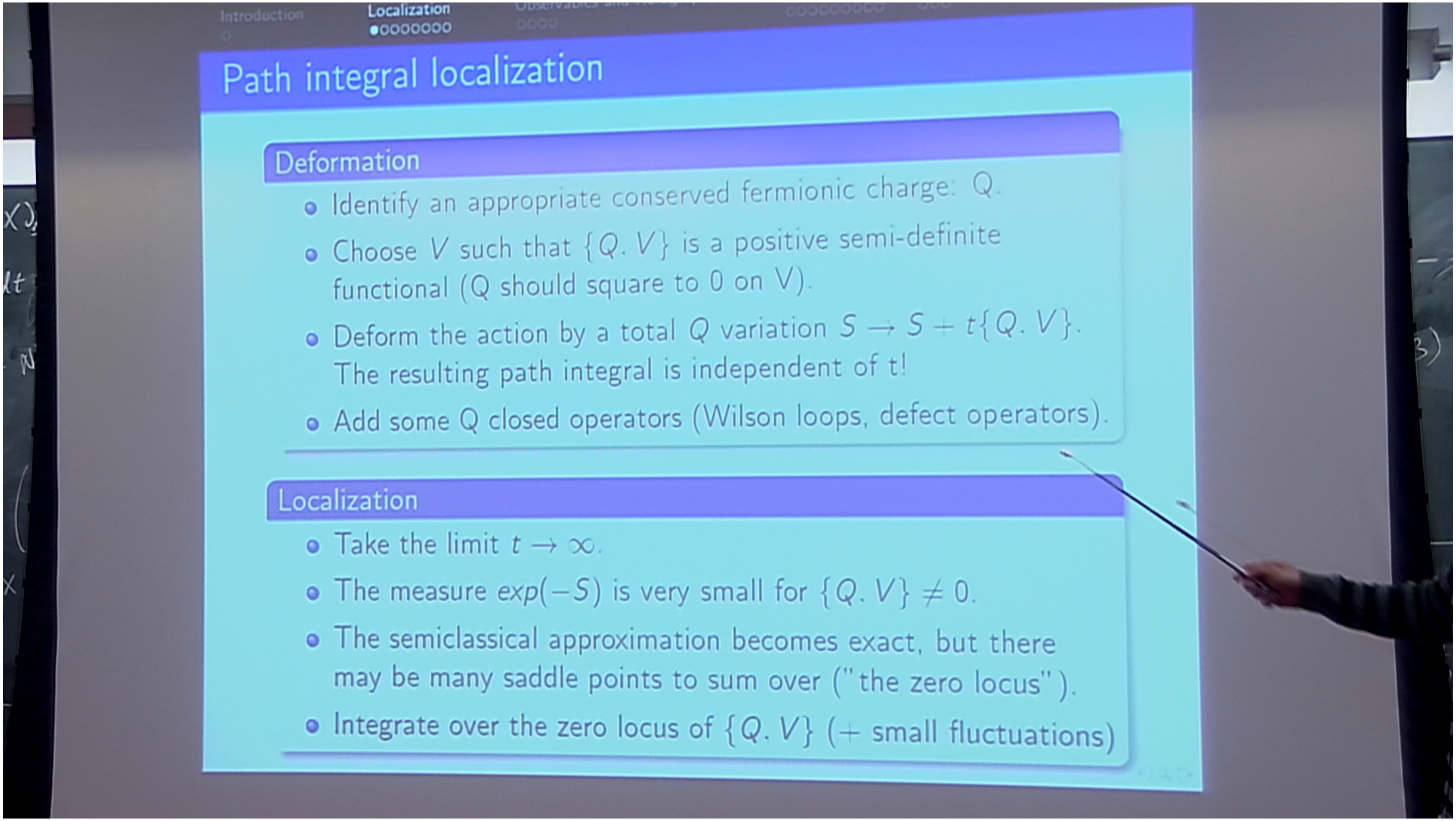
Date: Nov 24, 2011 04:00 PM

URL: http://pirsa.org/11110135

Abstract: In this talk I will give an overview of localization and some of its applications for QFTs in three dimensions. I will start by reviewing the localization procedure for N=2 supersymmetric gauge theories in three dimensions on S^3. I will then describe some of the applications to field theory dualities and to holography, and the possibility of extracting information about RG fixed points from the localized partition function.

Motivation

- Exact results in quantum field theory are hard to come by.
- Localization enables exact computations even in strongly coupled theories (IR fixed points of 3d gauge theories with a Yang-Mills term).
- Many new and interesting supersymmetric theories in $2+1$ dimensions (GW, BLG, ABJM, ABJ...).
- Lots of duality conjectures for IR fixed points.
- Holographic duals are available and comparisons can be made.
- Bonus: Euclidean partition functions knows something about the RG flow.


Introduction
Localization
Path integral localization
Deformation
Identify an appropriate conserved fermionic charge: Q.
Choose V such that {Q, V} is a positive semi-definite functional (Q should square to 0 on V).
Deform the action by a total Q variation S → S + t{Q, V}. The resulting path integral is independent of t!
Add some Q closed operators (Wilson loops, defect operators).
Localization
Take the limit t → ∞.
The measure exp(−S) is very small for {Q, V} ≠ 0.
The semiclassical approximation becomes exact, but there may be many saddle points to sum over ("the zero locus").
Integrate over the zero locus of {Q, V} (+ small fluctuations)

Supersymmetry on S^3

- We wish to compute all expectation values on S^3.
- After a conformal transformation
  1. All derivatives become covariant.
  2. Scalars with a kinetic term get a conformal mass (proportional to the Ricci scalar).
- (Covariantly) constant spinors exist only on ricci flat manifolds.
- Manifolds of constant curvature have Killing spinors satisfying $\nabla_\mu \varepsilon = \alpha \gamma_\mu \varepsilon$.
- On the three sphere $\nabla_\mu \varepsilon = \pm \frac{i}{2} \gamma_\mu \varepsilon$ (two of each).
- Actions with fermionic symmetries may be constructed using these spinors.

$\mathcal{N} = 2$ Vector multiplets

- The vector multiplet

σ, D (real)	λ_α (complex)	A_μ (real)

- Additional gauge fixing fields: c, $\bar{c}$ and b.
- The gaugino variation is (I have set the radius $r = 1$)

$$\delta\lambda = (-\frac{1}{2}\gamma^{\mu\nu}F_{\mu\nu} - D + i\gamma^{\mu}D_{\mu}\sigma - \sigma)\varepsilon$$

- We actually consider a combined supercharge

$$\tilde{Q} = Q_{\varepsilon} + Q_{BRST}. \qquad V = Tr\left(\{\tilde{Q}, \lambda\}^{\dagger}\lambda + \bar{c}\partial^{\mu}A_{\mu}\right)$$

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Vector multiplet localization

- The localizing functional is similar to a normal Yang-Mills action

$$S_Q = t\int_{\mathcal{M}} \sqrt{g}\, Tr\Big(\frac{1}{2}F^{\mu\nu}F_{\mu\nu} + D^{\mu}\sigma D_{\mu}\sigma + (D+\sigma)^2 + i\lambda^{\dagger} \not{D}\lambda$$
$$+ i[\lambda^{\dagger},\sigma]\lambda - \frac{1}{2}\lambda^{\dagger}\lambda + \nabla^{\mu}\bar{c}D_{\mu}c + b\nabla^{\mu}A_{\mu}\Big)$$

- Localizing to

$$S_Q = 0 \Leftrightarrow A_{\mu} = 0,\ \lambda = \lambda^{\dagger} = 0,\ c = 0,\ \bar{c} = 0,\ D = -\sigma = \sigma_0(const)$$

and with b unrestricted.

- Path integral reduces to

$$\int_{\sigma=const} S_{original}[\sigma = const]\,(\text{one loop determinant})$$

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Gauge sector matrix model

- The fluctuation determinant is

$$\prod_{\alpha}\prod_{l=0}^{\infty}\frac{((l+i\alpha(a))(-l-1+i\alpha(a)))^{l(l-1)}}{((l+1)^2+\alpha(a)^2)^{l(l-2)}}=\prod_{\alpha\in \text{roots}}\frac{2\sinh(\pi\alpha(a))}{(\pi\alpha(a))}$$

- The supersymmetric Chern-Simons action becomes

$$exp\left(\frac{ik}{4\pi}tr_f\int_{\mathcal{M}}\sqrt{g}(2D\sigma)\right)\to \exp(-i\pi k\, tr_f(a^2))$$

- The supersymmetric Wilson loop

$$W_{1/2}\equiv \mathcal{P}\mathrm{Tr}_R \exp[\oint (iA_\mu dx^\mu+\sigma|\dot{x}|)]\to \mathrm{Tr}_R exp(2\pi a)$$

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The Chern-Simons matrix model

The matrix integral

- The expectation value of the Wilson loop has been reduced to a matrix integral

$$\int da \underbrace{\exp(-ik\pi tr(a^2))}_{\text{classical CS term}} \underbrace{\det_{Ad} 2\sinh(\pi a)/(\pi a)}_{\text{1 loop det}} \underbrace{tr_R \exp(2\pi a)}_{\text{Wilson loop}}$$

Consistency checks

- The above matrix model was derived independently by other means for pure CS theory.
- Exact results for $U(N)$ are available and compare well with known results.
- The supersymmetric computation demands a specific "framing".

Matter fields

- Component fields and fermion transformations

ϕ,F (complex)	ψ_α (complex)

$$\delta\psi = (-i\gamma^\mu \nabla_\mu \phi - i\sigma\phi + \frac{1}{2}\phi)\varepsilon. \qquad \delta\psi^\dagger = \varepsilon^T F^\dagger$$

- The localizing term is

$$S_Q = t \int_{\mathcal{M}} \sqrt{g}\, Tr\Big[\nabla^\mu \phi^\dagger \nabla_\mu \phi + i\phi^\dagger v^\mu \nabla_\mu \phi - \phi^\dagger \sigma_0 \phi + \frac{1}{4}\phi^\dagger \phi$$

$$+ F^\dagger F + \psi^\dagger \Big(i\nabla\!\!\!/ - i\sigma_0 + \Big(\frac{1 + v\!\!\!/}{2}\Big)\Big)\psi\Big]. \qquad v_\mu \equiv \varepsilon^\dagger \gamma_\mu \varepsilon$$

- No additional zero modes arise. All fields are set to 0.

The matter determinant

A self dual representation $R \oplus R^*$ (like a hypermultiplet)

$$Z^{\text{matter}}_{1\text{ loop}} = \prod_{\rho \in \text{weights}} \frac{1}{2\cosh(\pi\rho(a))}$$

A general chiral superfield of conformal dimension Δ

$$\delta\psi = (-i\gamma^{\mu}\nabla_{\mu}\phi - i\sigma\phi + \Delta\phi)\varepsilon. \qquad \delta\psi^{\dagger} = \varepsilon^{T} F^{\dagger}$$

$$Z^{\text{matter}}_{1\text{ loop}} = \prod_{\rho \in \text{weights}} e^{\ell(1-\Delta+i(\rho(a)))}$$

where

$$\ell(z) = -z\log\left(1 - e^{2\pi i z}\right) + \frac{i}{2}\left(\pi z^2 + \frac{1}{\pi} Li_2\left(1 - e^{2\pi i z}\right)\right) - \frac{i\pi}{12}$$

is a solution to $\partial_z \ell(z) = -\pi z \cot(\pi z)$

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The topological Chern-Simons theory

- The Chern Simons partition function has been computed exactly, as were expectation values for Wilson loops (Witten).
- Comparing with the localization result we find agreement as long as we take a supersymmetric framing

$$Z_{CS} = \frac{1}{(k+N)^{N/2}} \prod_{m=1}^{N-1} \left(2 \sin \frac{\pi m}{k+N} \right)^{N-m}$$

- We can compute the expectation value for the unknot in the fundamental representation and compare

$$< W >= \frac{\frac{e^{-N\pi i}}{k}}{N} \frac{\sin\left(\frac{\pi N}{k}\right)}{\sin\left(\frac{\pi}{k}\right)}$$

A Wilson loop in ABJM

A(harony)B(ergman)J(afferis)M(aldacena)

- A superconformal $\mathcal{N}=6$ Chern-Simons matter theory.
- Gauge group $U(N)\times U(N)$ with CS levels $(k,-k)$.
- Two hypermultiplets in the $(N,\bar{N})$ representation.
- Low energy limit of $\mathcal{N}=8$ SYM (k=1) and holographically dual to M-theory on $AdS_4\times S^7/\mathbb{Z}_k$.

An $\mathcal{N}=2$ Wilson loop

- A loop operator preserving 2 real supercharges

$$W_{1/2}\equiv \mathcal{P}\mathrm{Tr}_R\exp[\oint (iA_\mu dx^\mu+\sigma|\dot{x}|)]\to \mathrm{Tr}_R exp(2\pi a)$$

- There is a 1/2 BPS (in the $\mathcal{N}=6$ sense of ABJM) version in the same cohomology class (Drukker, Trancanelli)

The ABJM matrix model

- The matrix model for ABJM involves two matrices (we rewrite with two sets of eigenvalues)

$$< W_{1/2} >= \int \Big(\prod_i e^{ik\pi(\lambda_i^2 - \hat{\lambda}_i^2)} d\lambda_i d\hat{\lambda}_i \Big) \frac{\prod_{i<j} \sinh^2(\pi(\lambda_i - \lambda_j))\sinh^2(\pi(\hat{\lambda}_i - \hat{\lambda}_j))}{\prod_{i,j} \cosh^2(\pi(\lambda_i - \hat{\lambda}_j))} \sum_i e^{2\pi\lambda_i}$$

- The expectation value matches perturbative calculations in the 't Hooft coupling $\lambda = N/k$ and for large N

$$\left\langle W_{1/2} \right\rangle = 1 + \left(\frac{5}{6} + \frac{1}{6N^2} \right) \frac{\pi^2 N^2}{k^2} - \left(\frac{1}{2} - \frac{1}{2N^2} \right) \frac{i\pi^3 N^3}{k^3} + \dots$$

Large N comparisons

Wilson loop

- Computed in the large N limit (Marino and Putrov).
- Expectation value was computed exactly in λ.
- Scales as expected from holography $< W > \approx \frac{i}{2\pi\sqrt{2\lambda}} e^{\pi\sqrt{2\lambda}}$

Partition function

- A (formerly confusing) scaling for the number of degrees of freedom for holographic M-theory duals $\#$ dof $\backsim N^{3/2}$.
- Derived from the large N limit of the matrix model partition function (Drukker, Marino, Putrov).
- Other quiver theories have $\#$ dof $\backsim N^{3/2}$ with a coefficient which matches gravity calculations on various backgrounds (Klebanov et al).

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Supersymmetric Deformations

Mass terms

Real mass terms are supersymmetric configurations for background flavor symmetry gauge fields $V_m \propto \theta\bar{\theta}m$

$$S_{mass} = -\int d^3x d^2\theta d^2\bar{\theta} \sum_{matter} (\phi^\dagger e^{2V_m}\phi + \tilde{\phi}^\dagger e^{-2V_m}\tilde{\phi})$$

in the matrix model this just shifts $\rho(a) \to \rho(a) + m$.

Fayet-Iliopoulos (FI) terms

Fayet-Iliopoulos (FI) terms for the $U(1)$ factors of the gauge group are equivalent to gauging topological symmetries $\hat{V}_{FI} \propto \theta\bar{\theta}\eta$

$$S_{FI} = Tr\int d^3x d^2\theta d^2\bar{\theta}\Sigma\hat{V}_{FI} \to e^{2\pi i\eta tr_f(a)}$$

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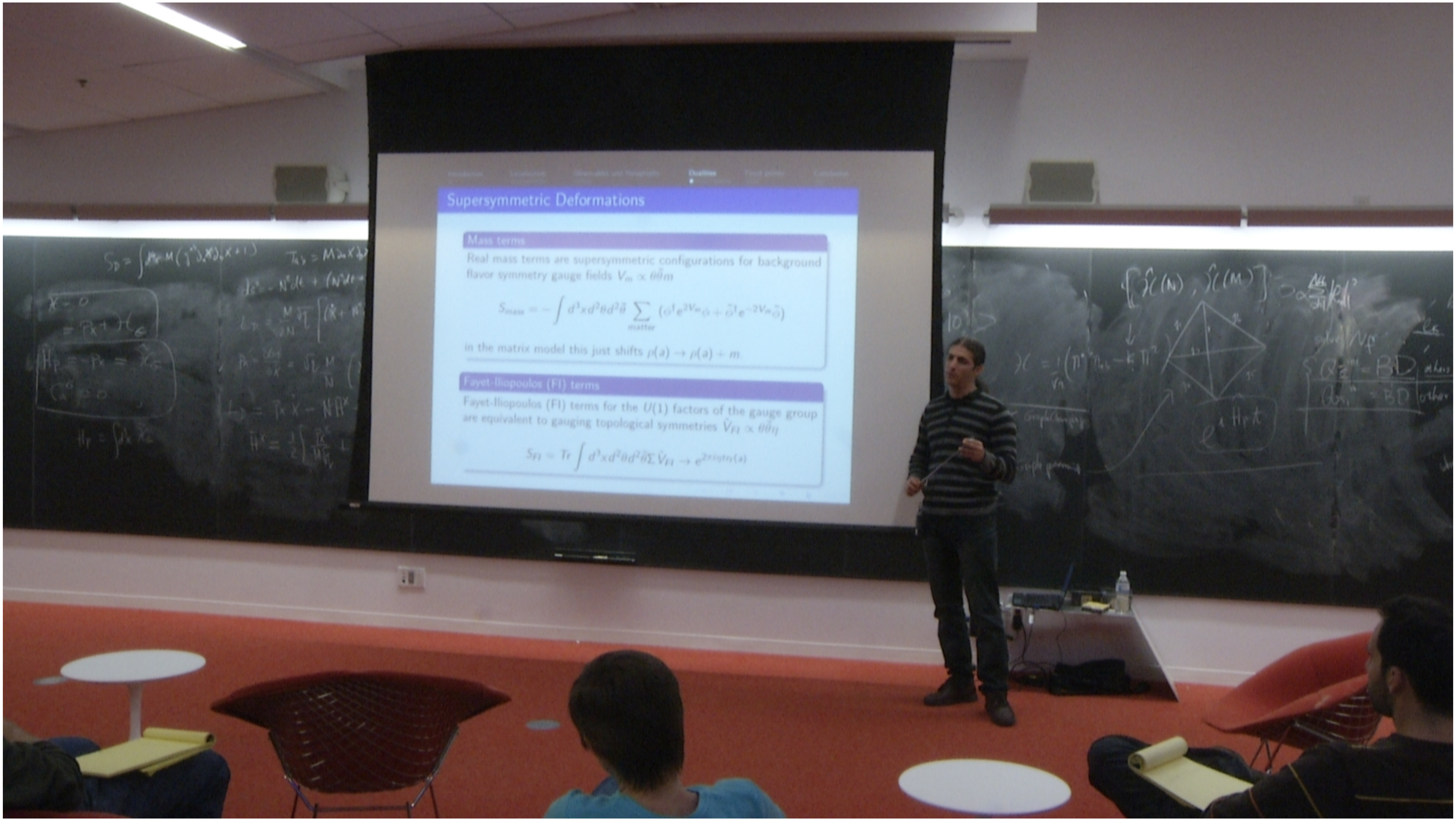
Dualities
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Mirror Symmetry

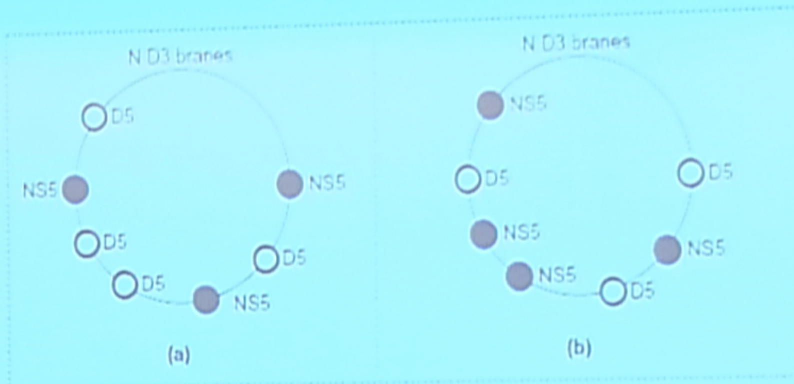

Figure: Brane construction of a 3d quiver theory and its mirror dual.

Basics of 3d mirror symmetry

- Relates the IR limit (strong coupling!) of different $\mathcal{N} = 4$ supersymmetric quiver gauge theories.
- Can be realized with S-duality in type IIB brane construction.
- Duality exchanges the Higgs and Coulomb branches of the moduli space, and real masses with FI terms.

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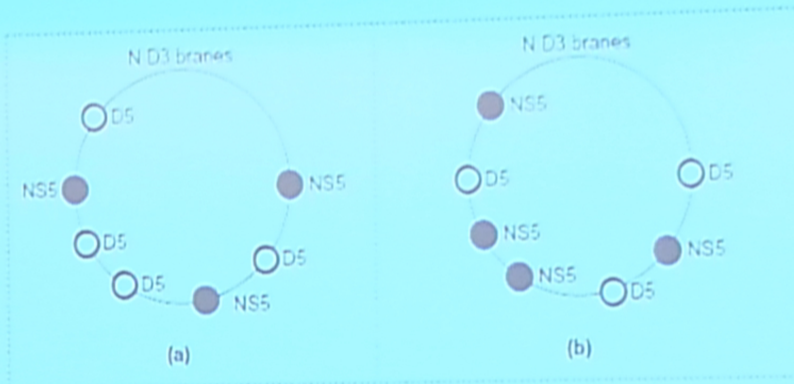

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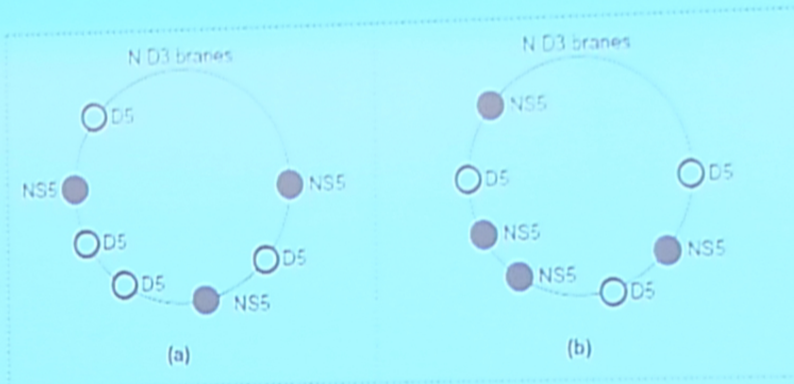

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Identities

Fourier transform

- The function $\frac{1}{\cosh(\pi x)}$ is its own Fourier transform.
- A free hypermultiplet is dual to a $U(1)$ gauge theory with a charge 1 hypermultiplet

$$\frac{1}{\cosh \pi\omega} \quad \underset{\eta \leftrightarrow \omega}{\longleftrightarrow} \quad \int d\sigma \, \frac{e^{2\pi i \sigma \eta}}{\cosh \pi\sigma}.$$

Determinant formula

- A version of the Cauchy determinant formula

$$\frac{\prod_{i<j} \sinh(x_i - x_j) \sinh(y_i - y_j)}{\prod_{i,j} \cosh(x_i - y_j)} = \sum_{\rho} (-1)^{\rho} \prod_{i} \frac{1}{\cosh(x_i - y_{\rho(i)})}$$

Transforming the partition function

- The integral can be written as a set of contributions from NS5 and D5 branes.
- Using the determinant formula and the basic Fourier transform, we can rewrite the partition function in a manifestly duality invariant way.
- Without deformations it takes the form

$$Z = \int \prod_{a=1}^{n} \frac{1}{N!} d^N\sigma_a d^N\tau_a \sum_{\rho_a} (-1)^{\rho_a} \prod_i \frac{e^{2\pi i \tau_a{}^i(\sigma_a{}^i - \sigma_{a+1}{}^{\rho_a(i)})}}{l_{\alpha_a}(\sigma_a{}^i, \tau_a{}^i)}$$

$$l_\alpha(\sigma, \tau) = \begin{cases} \cosh(\pi\sigma) & \alpha = D5 \\ \cosh(\pi\tau) & \alpha = NS5 \end{cases}$$

- The mapping of deformations takes the form expected from the brane picture.

ABJM and $\mathcal{N} = 8$ SYM

- At Chern Simons level k=1, ABJM describes coincident M2 branes in flat space. It describes the IR fixed point of $\mathcal{N} = 8$ SYM which lives on coincident D2 branes.
- We can compare the (very different looking) partition functions and find agreement

$$Z_{SYM}(\eta, \omega) = \frac{1}{N!} \int d^N\sigma \frac{\prod_{i<j} \sinh^2(\pi(\sigma_i - \sigma_j)) e^{2\pi i \eta \sum_i \sigma_i}}{\prod_{i,j} \cosh(\pi(\sigma_i - \sigma_j + \omega)) \prod_i \cosh(\pi \sigma_i)}$$

$$Z_{ABJM}(\eta, \omega) = \frac{1}{(N!)^2} \int d^N\sigma d^N\tilde{\sigma}$$

$$\frac{\prod_{i<j} \sinh^2(\pi(\sigma_i - \sigma_j)) \sinh^2(\pi(\tilde{\sigma}_i - \tilde{\sigma}_j)) e^{2\pi i \zeta \sum_i (\sigma_i + \tilde{\sigma}_i) + \pi i \sum_i (\sigma_i^2 - \tilde{\sigma}_i^2)}}{\prod_{i,j} \cosh(\pi(\sigma_i - \tilde{\sigma}_j + \xi)) \cosh(\pi(\sigma_i - \tilde{\sigma}_j - \xi))}$$

$$\eta = \xi + 2\zeta, \quad \omega = \xi - 2\zeta.$$

$\mathcal{N} = 3$ Seiberg-like dualities for a single gauge group

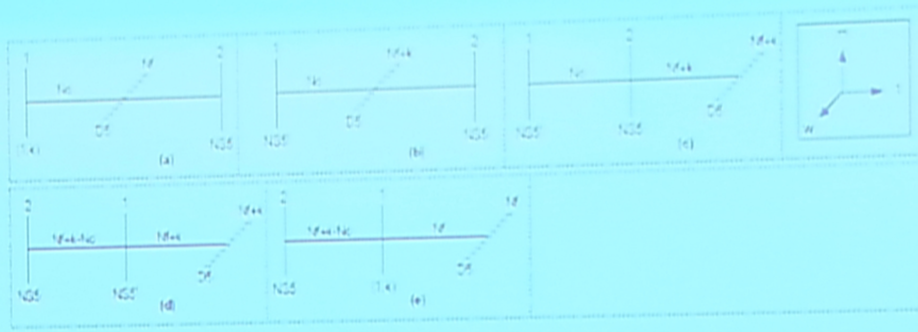

Figure: Brane moves leading to the duality of Chern Simons theories introduced by Giveon and Kutasov.

- Relates different superconformal Chern-Simons theories with $\mathcal{N} = 3$.
- Can be realized with brane moves in type IIB construction.
- Duality maps flavor symmetries to themselves and $U(N_c)_k, N_f \Leftrightarrow U(|k| + N_f - N_c)_{-k}, N_f$
- An $\mathcal{N} = 2$ version was also proposed (more complicated).

Giveon-Kutasov duality generalizes level rank duality

- The partition function of pure Chern-Simons theory ($N_f = 0$) is invariant under the exchange of level (k) and rank (N)

$$Z_{CS} = \frac{1}{(k+N)^{N/2}} \prod_{m=1}^{N-1} \left(2 \sin \frac{\pi m}{k+N} \right)^{N-m}$$

- The $N_f = 1$ partition function is a sum of unknotted Wilson loops in the pure CS theory. Duality relates these operators to Wilson loops in the dual. The reduction works in general ($N_f \to N_f - 1$) but the mapping is not known.
- Cases with a larger number of flavors were verified numerically.

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Seiberg-like dualities for quivers

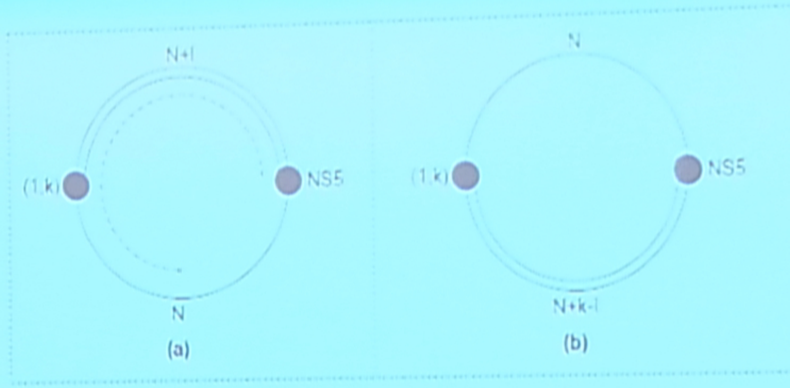

Figure: A duality between different ABJ theories. This is a nodewise version of the GK duality shown before.

- Relates superconformal Chern-Simons quiver theories with $\mathcal{N}=6$: $U(N+l)_k \times U(N)_{-k} \Leftrightarrow U(N)_k \times U(N-l-k)_{-k}$ both with two bifundamental hypers.
- Many generalizations can be found by examining the partition function.

$\mathcal{N} = 2$ dualities

- Aharony duality is the closest to 4d Seiberg duality. Implies that $U(N_c)$ Yang Mills with N_f flavors is dual to $U(N_f - N_c)$ with N_f flavors plus a $N_f \times N_f$ meson matrix (chirals).
- Addition fields are required to describe the coulomb branch. There is also a superpotential

$$W \sim \sum_i q^i \tilde{q}_j M_i^j + V_+ \tilde{V}_- + V_- \tilde{V}_-$$

- Partition functions can be shown to be equal using identities for hyperbolic gamma functions. The R-symmetry is abelian and can mix, but the comparison of partition functions is insensitive to the effect.
- The $\mathcal{N} = 2$ version of GK duality is a consequence of Aharony duality after acounting for the "parity anomaly".

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The superconformal R-symmetry

- The supermultiplet containing the EM tensor also contains, at the IR fixed point, a distinguished $U(1)_R$ symmetry. In the presence of other (flavor etc.) $U(1)$s

$$R_{IR} = R_{UV} + \sum_{\text{flavor currents}} a_i F_i$$

  Determining the correct IR R-symmetry requires additional input (besides identifying the currents).
- In 4d this is done by "a maximization". R_{IR} is the linear combination which locally maximizes the "a type" conformal anomaly as a function of trial R charges.
- The R-charge of a chiral field determines its IR conformal dimension (in 3d $\Delta = Q_R$).

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The S^3 partition function and conformal dimensions

- The localized S^3 partition function depends on Δ through the fermion variation.

$$\delta\psi = (-i\gamma^\mu \nabla_\mu \phi - i\sigma\phi - \Delta\phi)\varepsilon, \qquad \delta\psi^\dagger = \varepsilon^T F^\dagger$$

$$Z^{\text{matter}}_{1\text{ loop}} = \prod_{\rho \in \text{weights}} e^{\ell(1-\Delta+i(\rho(a)+m))}$$

- Since $\Delta = q_{R_{IR}} = q_{R_{UV}} + \sum a_i q_{F_i}$ and $m = \sum q_{F_i} m_i$, the dependence is holomorphic in $a_i + im_i$.
- The variation with respect to m_i is a one point function and therefore vanishes in the IR CFT. This implies (by holomorphicity)

$$\partial_{a_i} |Z_{S^3}| = 0$$

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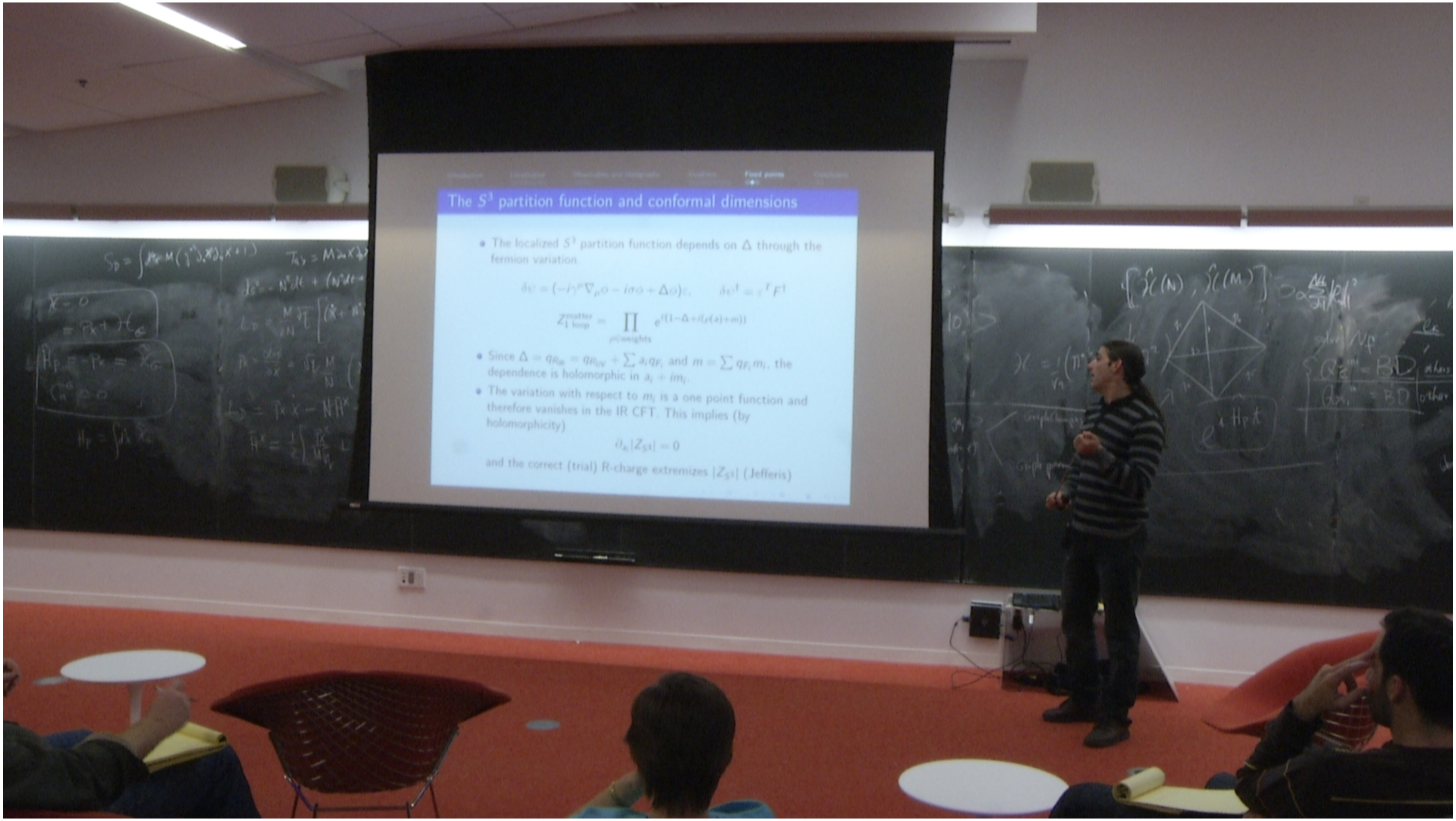
Fixed points
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The S^3 partition function and counting dof

- The free energy as for a euclidean 3d CFT has been conjectured to decrease along RG flows (Jafferis, Klebanov...)

$$F = -\log\left(|Z_{S^3}|\right)$$

- F also scales in the correct way at large N, reproducing $F \backsim N^{3/2}$ and its coefficient for $\mathcal{N} = 2$ theories.
- F is a "global" quantity, very different from the "a type" anomaly in 4d (the "a" theorem) or the conformal anomaly "c" in 2d (the "c" theorem).
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Summary

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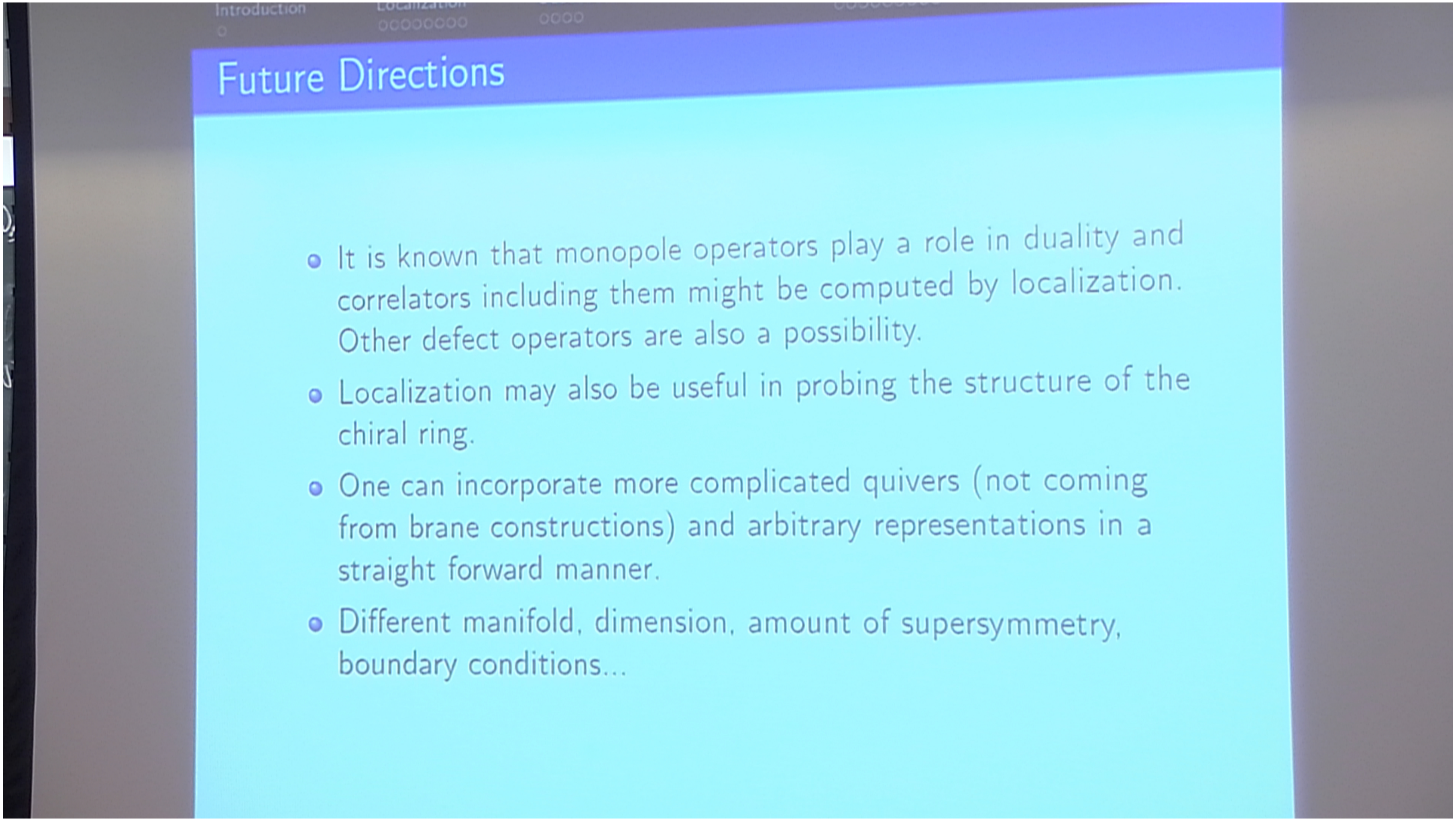
Introduction
Future Directions
It is known that monopole operators play a role in duality and correlators including them might be computed by localization. Other defect operators are also a possibility.
Localization may also be useful in probing the structure of the chiral ring.
One can incorporate more complicated quivers (not coming from brane constructions) and arbitrary representations in a straight forward manner.
Different manifold, dimension, amount of supersymmetry, boundary conditions...