Title: TBA

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Abstract: TBA

A covariant inner product

- A natural object in a spinor space is $\sigma^{\alpha A'A} = (\delta^{A'A}, \sigma^{iA'A})$

Representations of the Lorentz group

- Lorentz transformations are represented by $\Lambda_A^{\ B}\psi_B$
- These are representations of $SL(2, \mathbb{C})$
- The $\Lambda_A^{\ B}$ (with $I^{A'A} = \delta^{A'A}$) are however not unitary i.e.

$$\langle \Lambda\psi | \Lambda\chi \rangle \neq \langle \psi | \chi \rangle$$

Problem

- We would like a 2D representation of the Lorentz group acting on $\mathcal{H}$:
  There are no faithful finite-dimensional unitary representations of the Lorentz group [Wigner].

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A COVARIANT INNER PRODUCT

- A natural object in a spinor space is $\sigma^{\alpha A'A} = (\delta^{A'A}, \sigma^{iA'A})$

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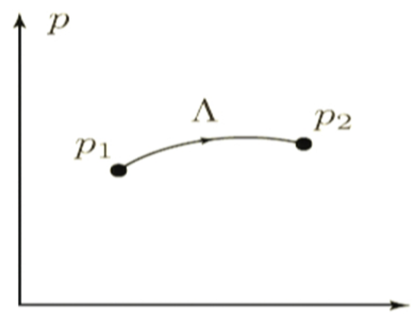
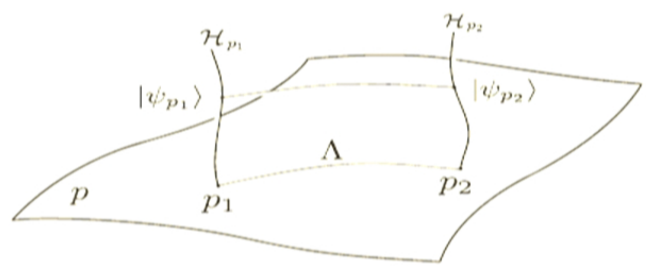
- Given that we have the inner product $I_u^{A'A} \Rightarrow \mathcal{H}_p$

$$\langle \Lambda\phi | \Lambda\chi \rangle = \langle \phi | \chi \rangle$$

- We regain unitarity of Lorentz transformations!
- So how have we stepped around Wigners Theorem?
  - Hilbert spaces are labelled by momentum: $\mathcal{H}_p$
  - $\Lambda : \mathcal{H}_{p_1} \mapsto \mathcal{H}_{p_2} \Rightarrow$ This is not a representation!





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Wigner representation

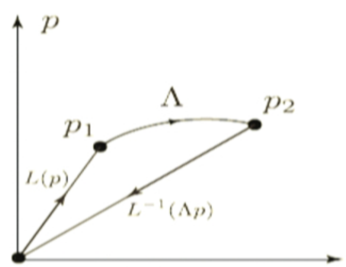
- An alternative (∞-dim) representation is due to Wigner:

$$\hat{P}^{\mu} \left| p, \sigma \right\rangle = p^{\mu} \left| p, \sigma \right\rangle$$

- A general Lorentz transformations acts according to:

$$\mathcal{U}(\Lambda) \left| p, \sigma \right\rangle = \sum_{\sigma} D_{\sigma,\sigma'} \left| \Lambda p, \sigma' \right\rangle$$

- $D_{\sigma,\sigma'}(\Lambda, p) = L^{-1}(\Lambda p) \Lambda L(p)$ is called a Wigner rotation



	Spinor	Wigner
$\mathcal{H}$	$\mathcal{H}_p$	$\mathbb{C}^2 \otimes L^2(\mathbb{R}^4)$
Λ	Λ_A^B	$D_{\sigma,\sigma'}(\Lambda, p)$

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Outline

Unitary representation of spin
- Spinor rotation
- Unitarity and identifying an inner product
- Wigner representation

Relativistic Stern-Gerlach measurement
- Measurement formalism
- Physical model of Stern-Gerlach
- Dirac equation in the WKB limit

Relativistic spin operators

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Covariant Hermitian operators

- Given the inner product structure $I_u^{A'A}$ the representation of Hermitian operators must be modified.
- Starting with the definition of a Hermitian operator

$$\langle\chi|A\psi\rangle - \langle A\chi|\psi\rangle = u_\alpha \sigma^{\alpha A'A} A_A^{\ B} \psi_B \bar{\chi}_{A'} - u_\alpha \sigma^{\alpha A'A} \bar{A}_{A'}^{\ B'} \bar{\chi}_{B'} \psi_A = 0$$

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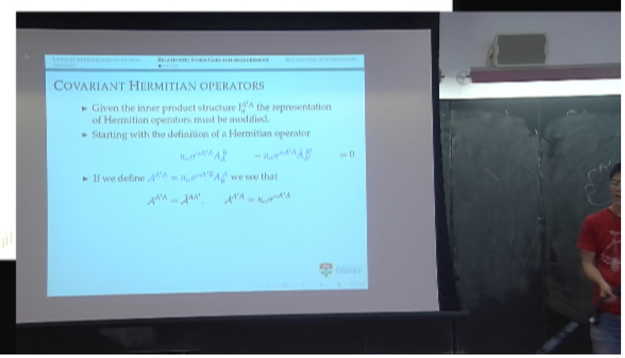
COVARIANT HERMITIAN OPERATORS

- Given the inner product structure $I_u^{A'A}$ the representation of Hermitian operators must be modified.
- Starting with the definition of a Hermitian operator

$$u_\alpha \sigma^{\alpha A'A} A_A^{\ B} \qquad - u_\alpha \sigma^{\alpha A'A} \bar{A}_{A'}^{\ B'} \qquad = 0$$

- If we define $\mathcal{A}^{A'A} = u_\alpha \sigma^{\alpha A'B} A_B^{\ A}$ we see that

$$\mathcal{A}^{A'A} = \bar{\mathcal{A}}^{AA'}, \qquad \mathcal{A}^{A'A} = n_\alpha \sigma^{\alpha A'A}$$



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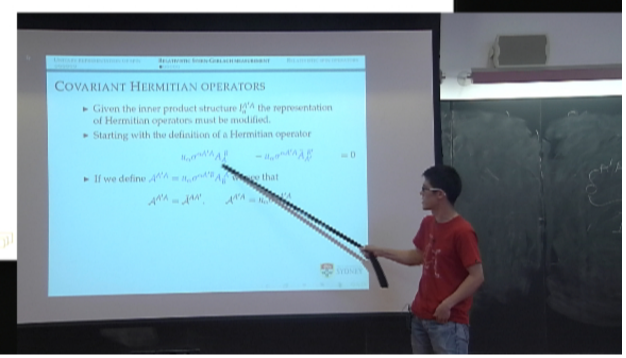
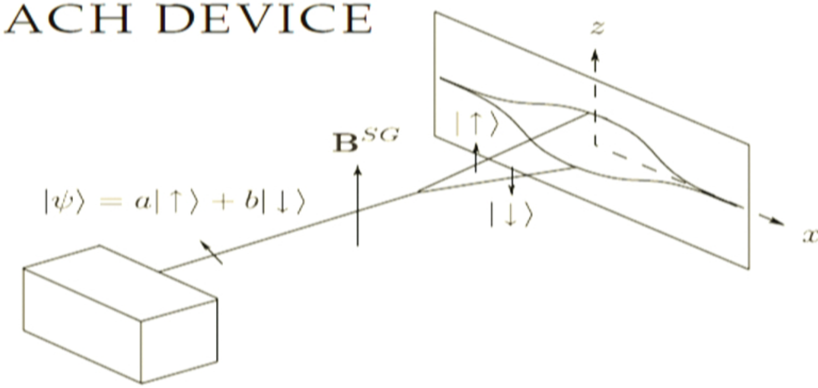
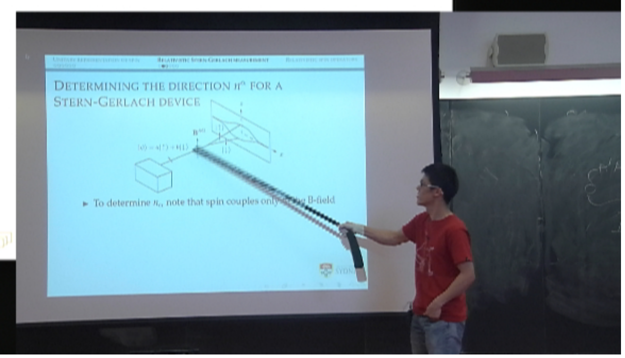


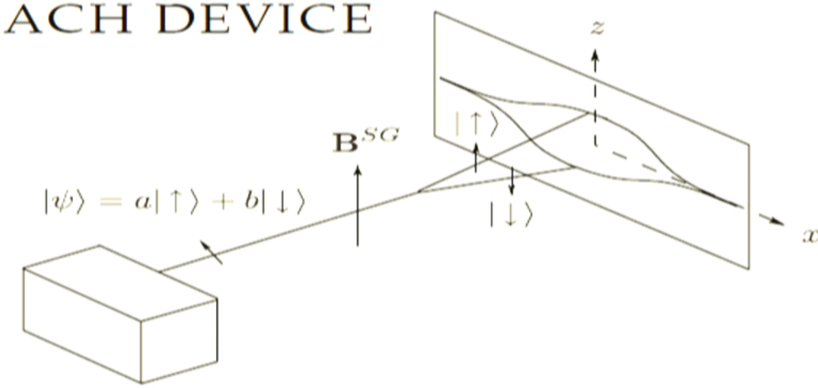
Determining the direction n^α for a Stern-Gerlach device



- To determine n_α note that spin couples only to the B-field



DETERMINING THE DIRECTION n^α FOR A STERN-GERLACH DEVICE



- To determine n_α note that spin couples only to the B-field
- The EM field of a SG device is expressed as

$$F_{\alpha\beta} = -\epsilon_{\alpha\beta\gamma\delta} v^\gamma B^\delta_{\text{SG}}$$

v^γ is the velocity; B^δ_{SG} the magnetic field of SG device.

- in the rest frame $v^\gamma \overset{**}{=} (1, 0, 0, 0)$ and $B^\delta_{\text{SG}} \overset{**}{=} (0, B^i)$

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Determining the direction n^α for a Stern-Gerlach device

- The magnetic fields in the two frames are related by

$$B_\beta^{\mathrm{RF}} = (v^\alpha u_\alpha) B_\beta^{\mathrm{SG}} - (B_\alpha^{\mathrm{SG}} u^\alpha) v_\beta$$

- The measurement operator for a Stern-Gerlach measurement is given by

$$\boxed{\hat{\mathcal{S}} = -2i u_\alpha B_\beta^{\mathrm{RF}} \hat{L}^{\alpha\beta} \stackrel{*}{=} B_i^{\mathrm{RF}} \hat{\sigma}^i}$$

where the direction n^α being measured corresponds to B_β^{RF}

- the eigenstates and eigenvalues are defined as

$$\hat{\mathcal{S}}\,|\psi^\pm\rangle = \pm|B^{RF}|\,|\psi^\pm\rangle$$

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DETERMINING THE DIRECTION n^α FOR A STERN-GERLACH DEVICE

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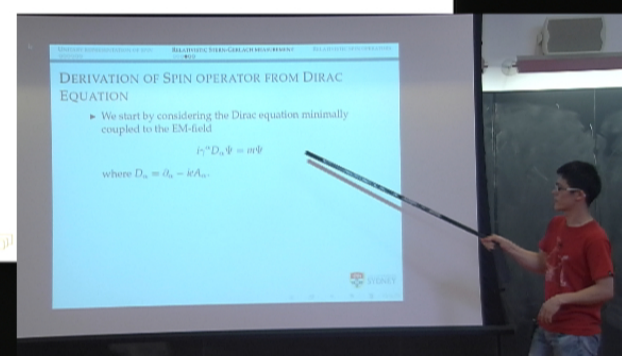
DERIVATION OF SPIN OPERATOR FROM DIRAC EQUATION

- We start by considering the Dirac equation minimally coupled to the EM-field

$$i\gamma^\alpha D_\alpha \Psi = m\Psi$$

where $D_\alpha = \partial_\alpha - ieA_\alpha$.



Dirac equation in the WKB limit

- The starting point is to assume a solution of the form:

$$\phi_A(x) = a\varphi_A^+(x)e^{i\theta^+(x)/\varepsilon} + b\varphi_A^-(x)e^{i\theta^-(x)/\varepsilon}$$

- The van der Waerden equation in the WKB limit is:

$$\Box\varphi_A^\pm - eF_{\alpha\beta}L^{\alpha\beta}{}_A{}^B\varphi_B^\pm + \frac{i}{\varepsilon}(2k^\alpha\partial_\alpha\varphi_A^\pm + \varphi_A^\pm\partial_\alpha k^\alpha) - \frac{1}{\varepsilon^2}k_\alpha k^\alpha\varphi_A^\pm + m^2\varphi_A^\pm = 0 \quad (1)$$

where $k_\alpha \equiv \partial_\alpha\theta - \varepsilon e A_\alpha$

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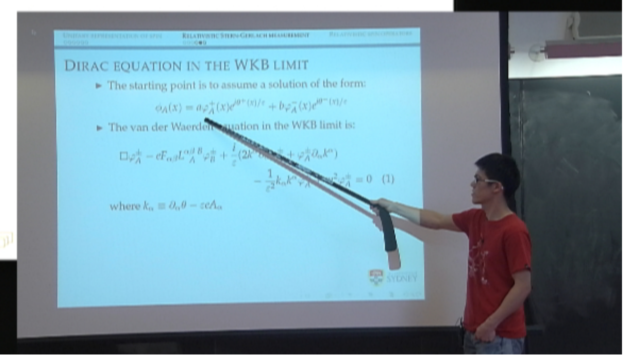

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where $k_\alpha \equiv \partial_\alpha\theta - \varepsilon e A_\alpha$

- In the limit $\varepsilon \longrightarrow 0$ the different orders in ε decouple and we approximate (1) as:

$$\begin{aligned} 0 &= 2k^\alpha\partial_\alpha\varphi_A^\pm + \varphi_A^\pm\partial_\alpha k^\alpha \\ 0 &= -k_\alpha k^\alpha\varphi_A^\pm + m^2\varphi_A^\pm - \varepsilon e F_{\alpha\beta}L^{\alpha\beta}{}_A{}^B\varphi_B^\pm \end{aligned}$$

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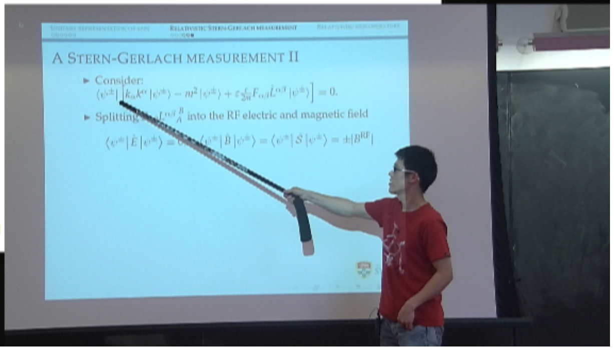
A Stern-Gerlach measurement II

- Consider:
  $\langle\psi^{\pm}|\left[k_\alpha k^\alpha|\psi^{\pm}\rangle - m^2|\psi^{\pm}\rangle + \varepsilon\frac{e}{2m}F_{\alpha\beta}\hat{L}^{\alpha\beta}|\psi^{\pm}\rangle\right] = 0.$
- Splitting $F_{\alpha\beta}L^{\alpha\beta}{}_A^B$ into the RF electric and magnetic field

$$\langle\psi^{\pm}|\,\hat{E}\,|\psi^{\pm}\rangle = 0, \quad \langle\psi^{\pm}|\,\hat{B}\,|\psi^{\pm}\rangle = \langle\psi^{\pm}|\,\hat{\mathcal{S}}\,|\psi^{\pm}\rangle = \pm|B^{\mathrm{RF}}|$$



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- Splitting $F_{\alpha\beta}L^{\alpha\beta}{}_A{}^B$ into the RF electric and magnetic field

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- Substituting and taking the gradient, we get

$$k^\alpha \partial_\alpha k_\beta + ek^\alpha F_{\alpha\beta} \pm \frac{e}{2m}\partial_\beta(B^{\mathrm{RF}}) = 0$$

- The relativistic Spin operator for a SG measurement is:

$$\boxed{\hat{\mathcal{S}} = -2iu_\alpha B^{\mathrm{RF}}_\beta \hat{L}^{\alpha\beta} \stackrel{*}{=} B^{\mathrm{RF}}_i \hat{\sigma}^i}$$

with $B^{\mathrm{RF}}_\beta = (v^\alpha u_\alpha)B^{\mathrm{SG}}_\beta - (B^{\mathrm{SG}}_\alpha u^\alpha)v_\beta$

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RELATIVISTIC SPIN OPERATORS I

In the frame of the S-G ($v_\alpha \overset{**}{=} (1, 0, 0, 0)$) a Spin-observable $\hat{S}_i$ should satisfy the following properties:

1. It should reduce to the rest frame spin observable i.e. the Pauli matrices $\hat{\sigma}^i$;
2. It should be a three vector in the Stern-Gerlach frame:

$$\left[\epsilon_{iln}\hat{L}^{ln}, \hat{S}_j\right] \overset{**}{=} i\epsilon_{ij}{}^{k}\hat{S}_k$$

3. And it should satisfy the spin commutation relations

$$\left[\hat{S}_i, \hat{S}_j\right] \overset{**}{=} i\epsilon_{ij}{}^{k}\hat{S}_k$$

- The first two conditions imply:

$$\hat{S}_i \overset{**}{=} \frac{a}{m}\left[\hat{W}_i - b\hat{W}_0 p_i\right] \text{ where } \hat{W}^\alpha = -\frac{1}{2}\epsilon^{\alpha}{}_{\beta\gamma\delta}p^\beta \hat{L}^{\gamma\delta}$$

$\hat{W}^\alpha$ is the Pauli-Lubanski vector

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Relativistic Spin operators I

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RELATION TO OTHER SPIN OPERATORS I

- Recall

$$\hat{\mathcal{S}} = -2iu_\alpha b^{\mathrm{RF}}_\beta \hat{L}^{\alpha\beta}$$

where $b^{\mathrm{RF}}_\beta = \frac{(v^\alpha u_\alpha)B^{\mathrm{SG}}_\beta - (B^{\mathrm{SG}}_\alpha u^\alpha)v_\beta}{|B^{\mathrm{RF}}|}$

- in the SG frame this can be rearranged as:

$$\hat{\mathcal{S}} \overset{**}{=} b^i_{\mathrm{SG}} \hat{S}_i \overset{**}{=} b^i_{\mathrm{SG}} \frac{\gamma}{m}\left[\hat{W}_i - \frac{p_i}{p_0}\hat{W}_0\right]$$

where $a = \gamma$ and $b = 1/p_0$; note that $\hat{\mathcal{S}} \overset{*}{=} b^i_{\mathrm{RF}}\hat{\sigma}_i$.

- Comparing with the other operators

$$\hat{S}^{(B)}_i \overset{**}{=} \frac{1}{m}\left[\hat{W}_i - \frac{p_i}{m+p_0}\hat{W}_0\right] \quad , \qquad \hat{S}^{(C)}_i \overset{**}{=} \frac{1}{p_0}\hat{W}_i$$

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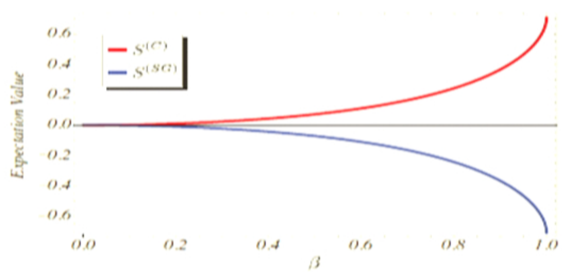
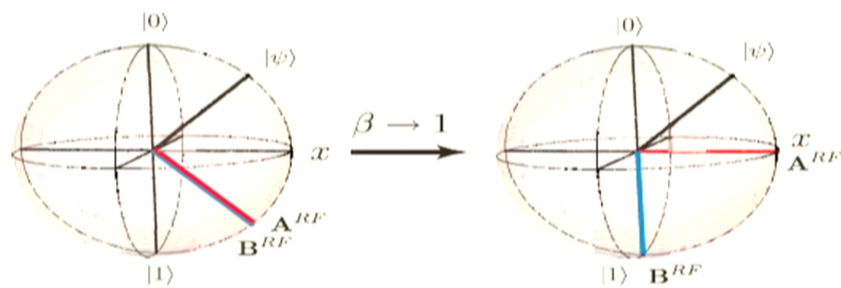
Relation to other spin operators II

- Preparing the state:

$$|\psi\rangle = \cos(\pi/8)\,|0\rangle + \sin(\pi/8)\,|1\rangle$$

- Then perform a SG measurement where $B^{\alpha}_{SG} \stackrel{**}{=} (0, 1, 0, -1)$
- The expectation values give.





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