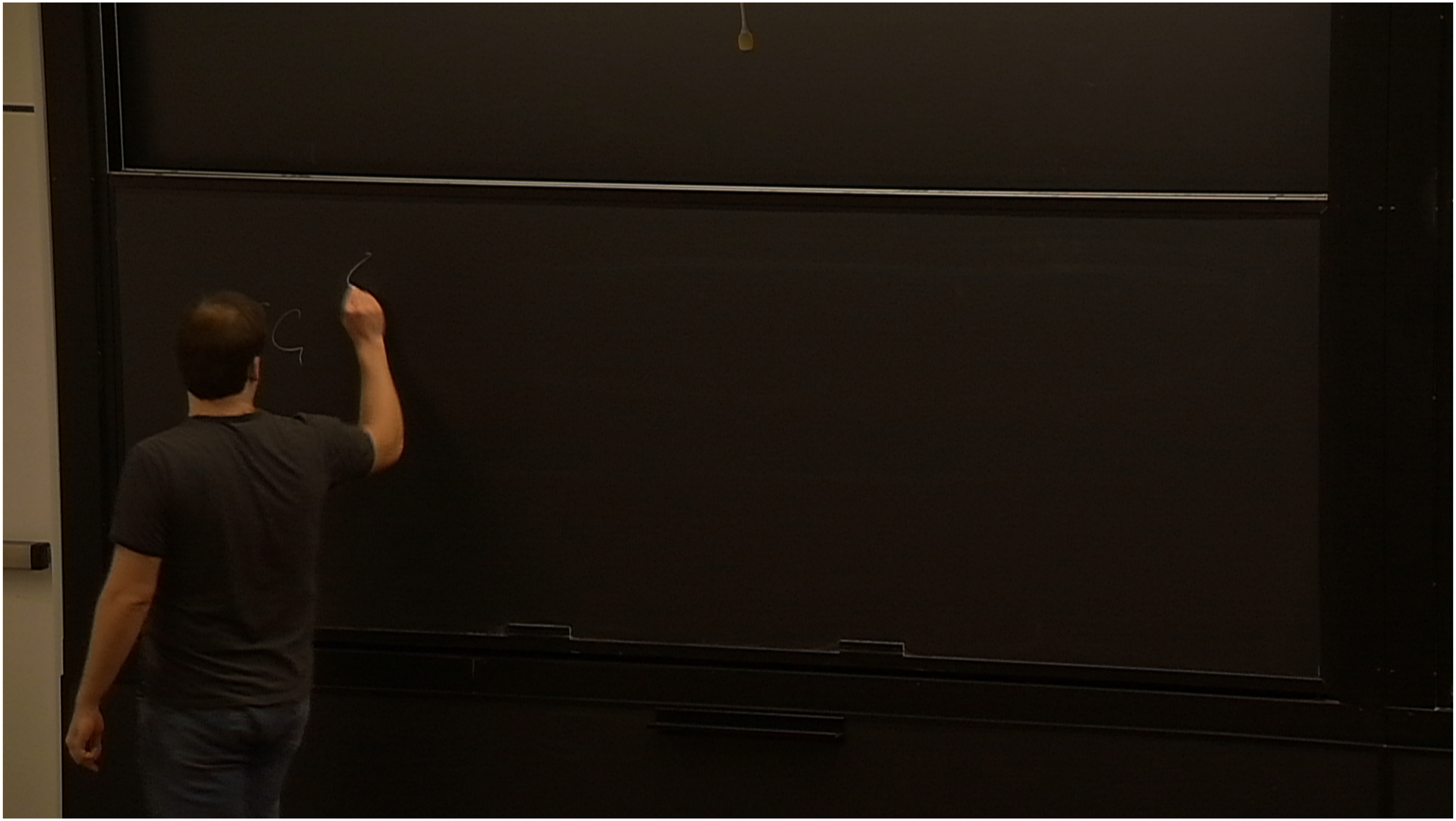
Title: Non-commutative Geometry via Toposes

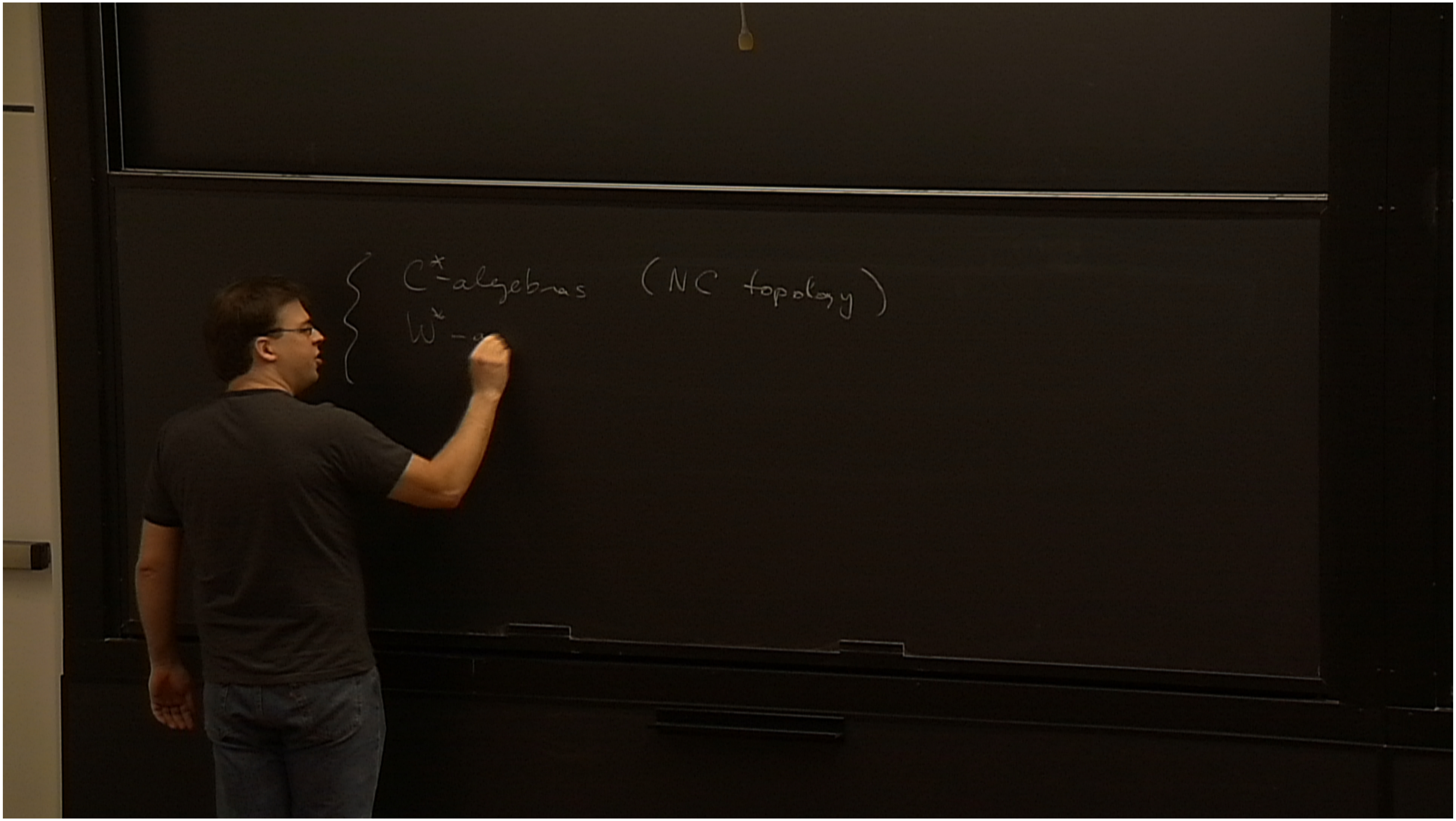
Date: Nov 24, 2011 10:30 AM

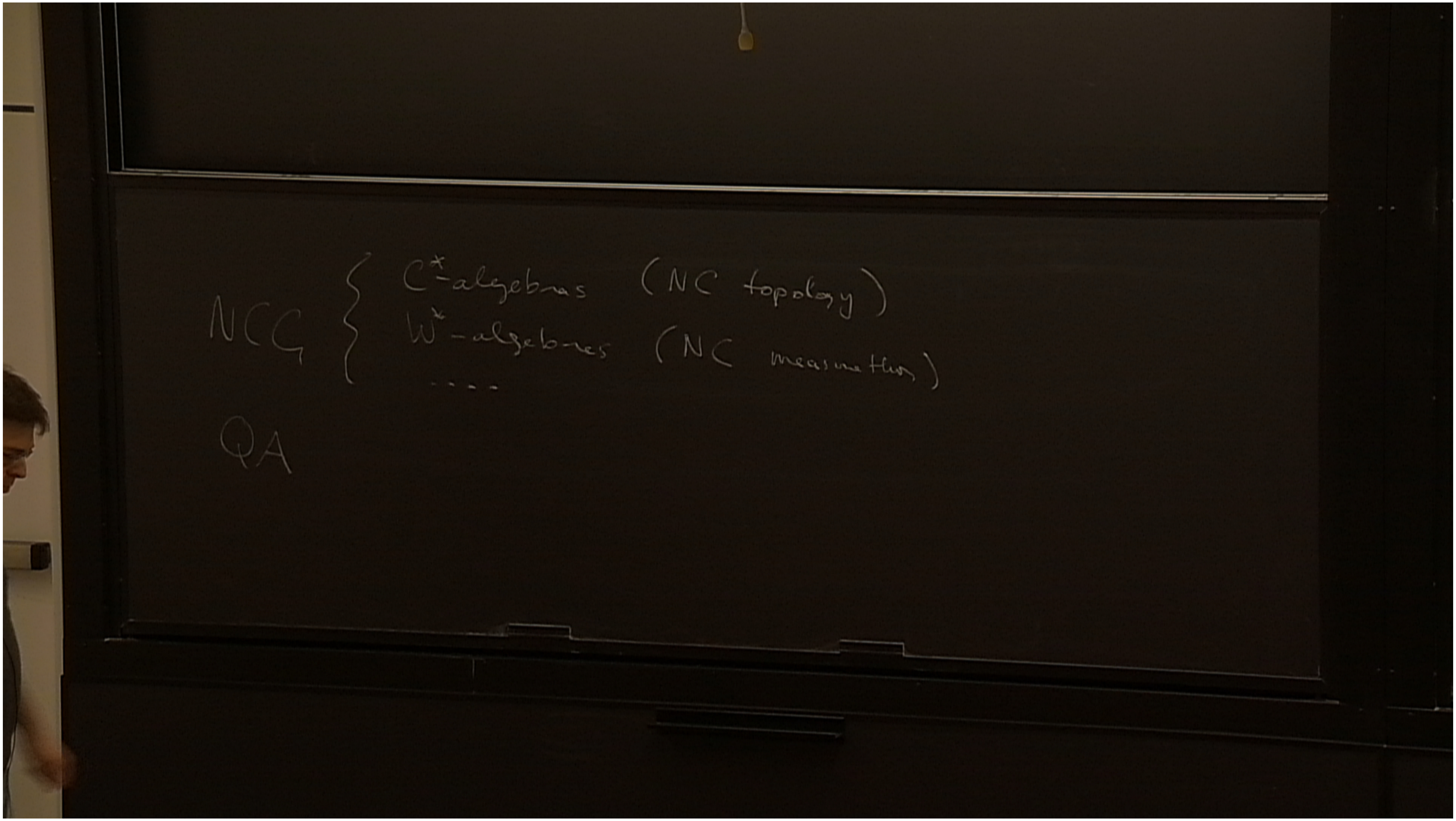
URL: http://pirsa.org/11110124

Abstract: It is my contention that non-commutative geometry is really "ordinary geometry" carried out in a non-commutative logic. I will sketch a specific project, relating groupoid C*-algebras to toposes, by means of which I hope to detect the nature of this non-commutative logic.




NCG
QA


C*-algebras (NC topology)
W*-


NCG
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W*-algebras (NC measure theory)
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QA

NCG { C^*-algebras (NC topology)
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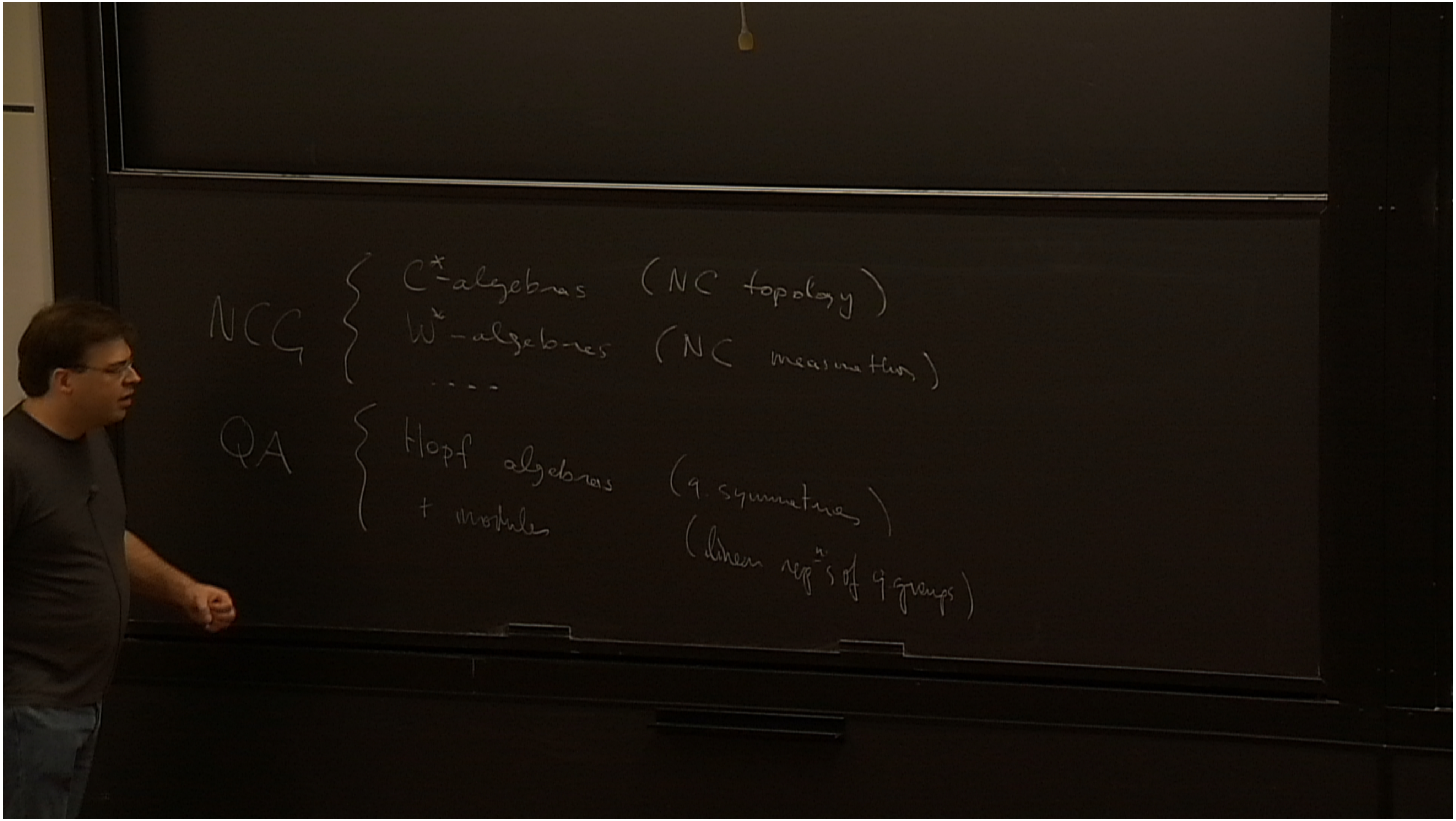
....

QA { Hopf algebras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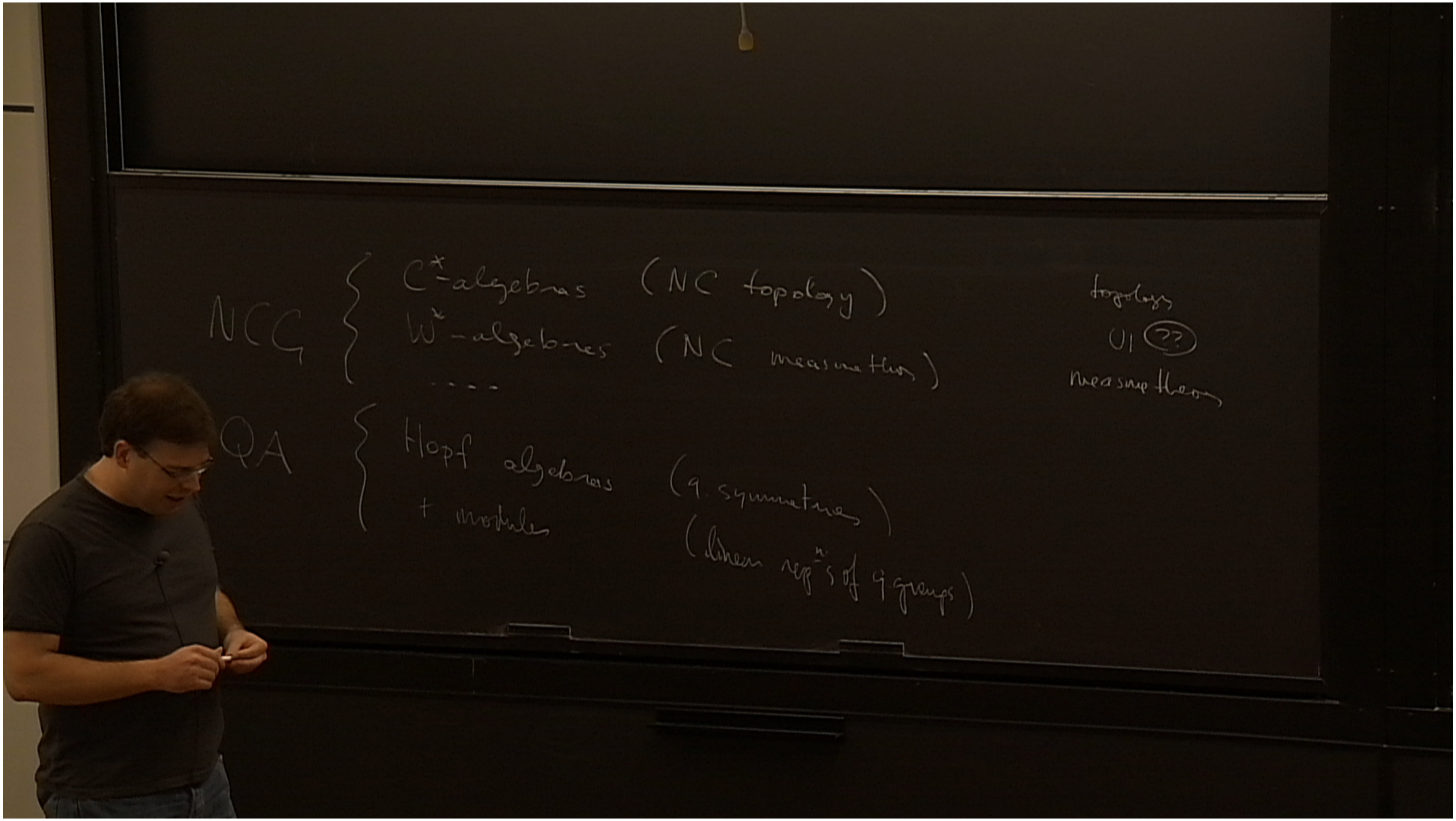
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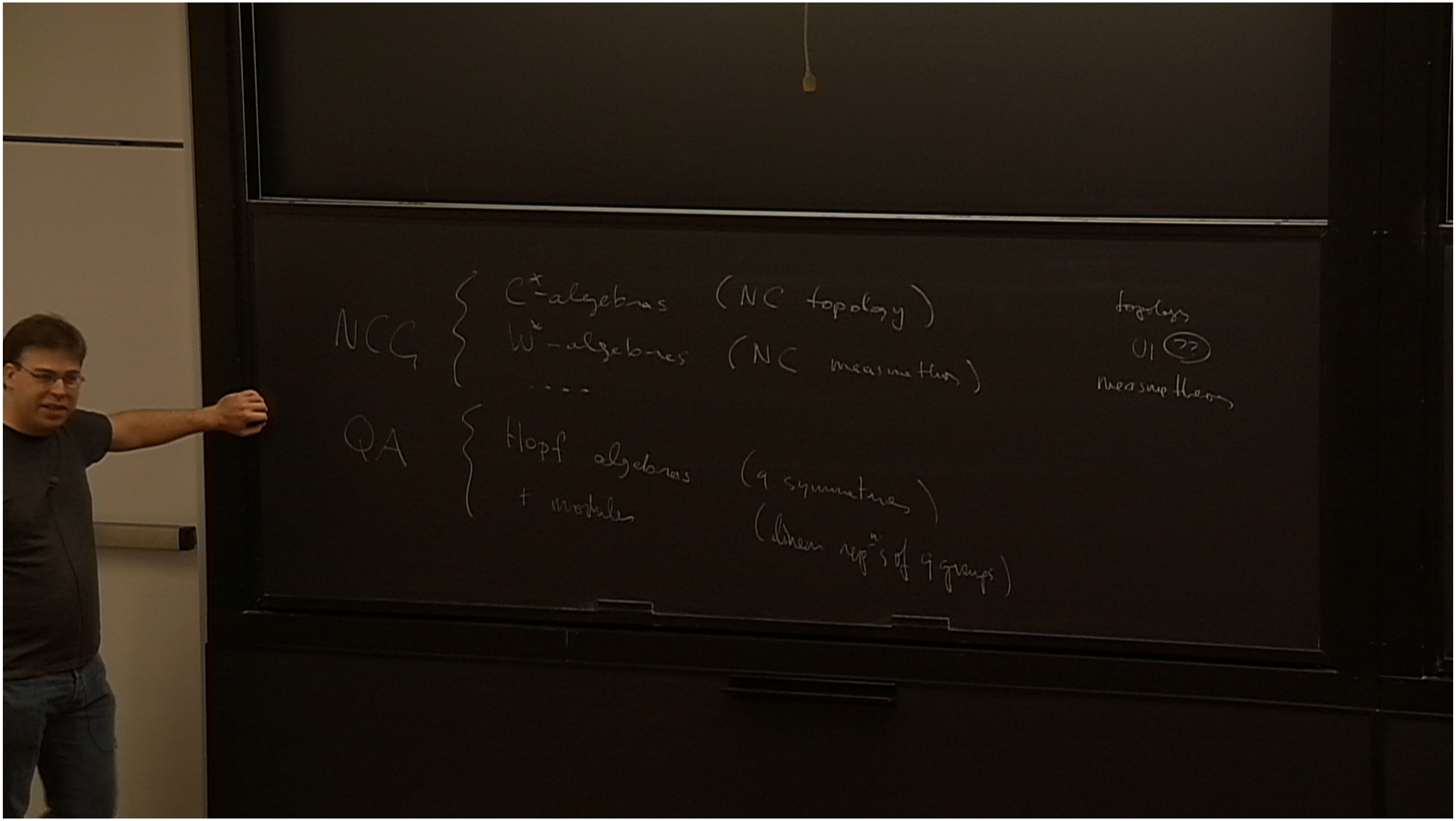
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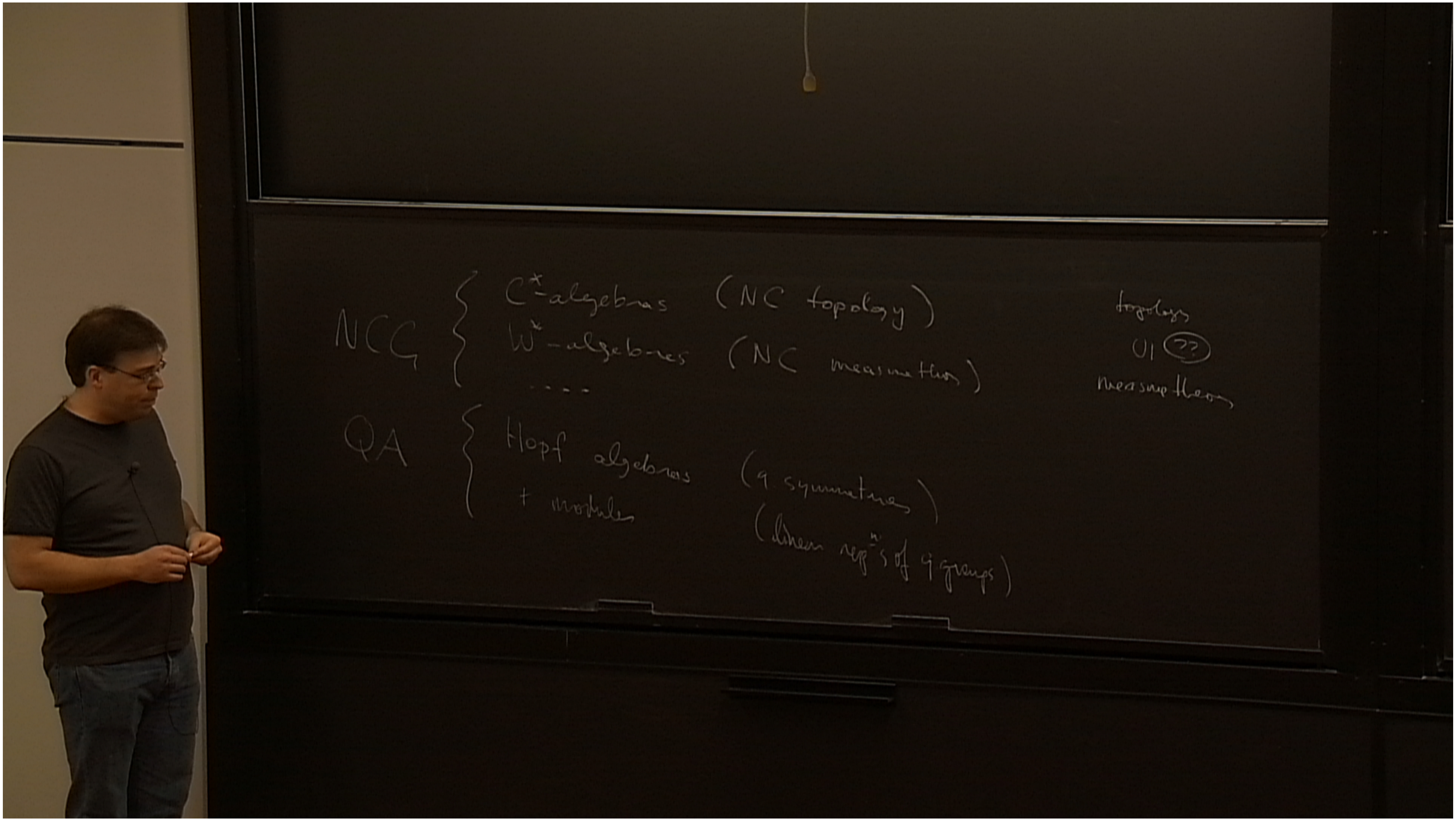
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measure theory

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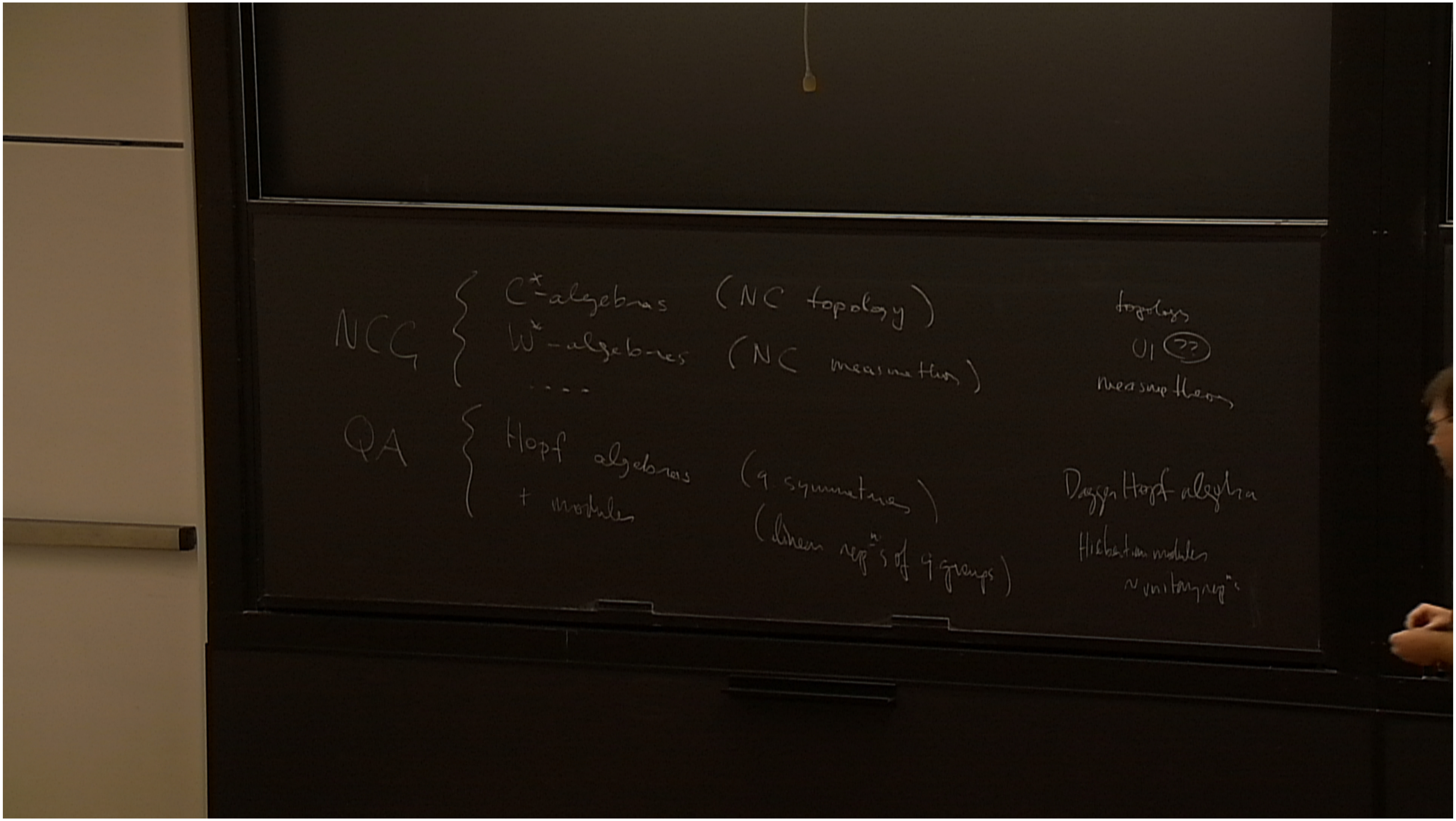
Dagger Hopf algebra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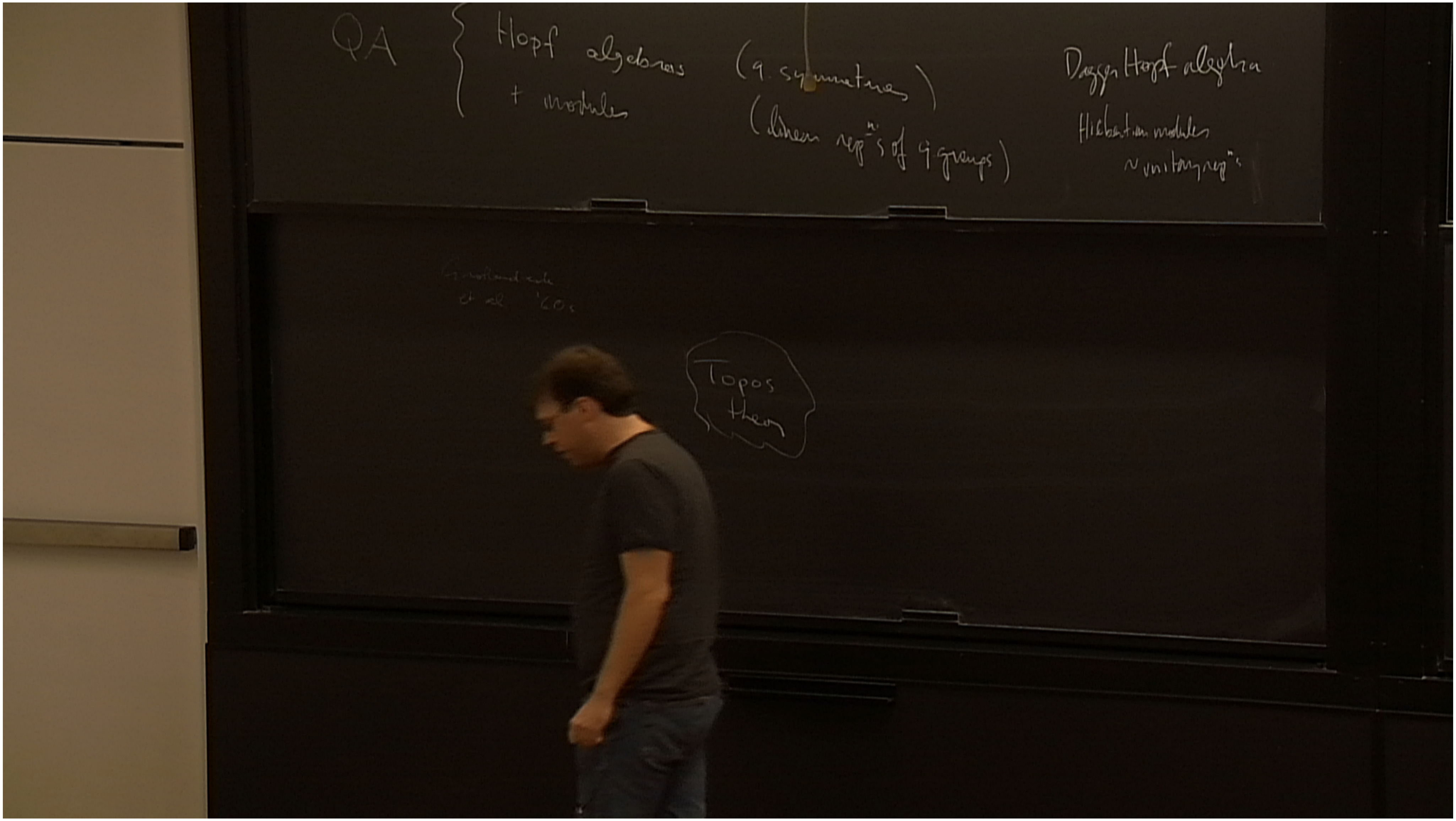
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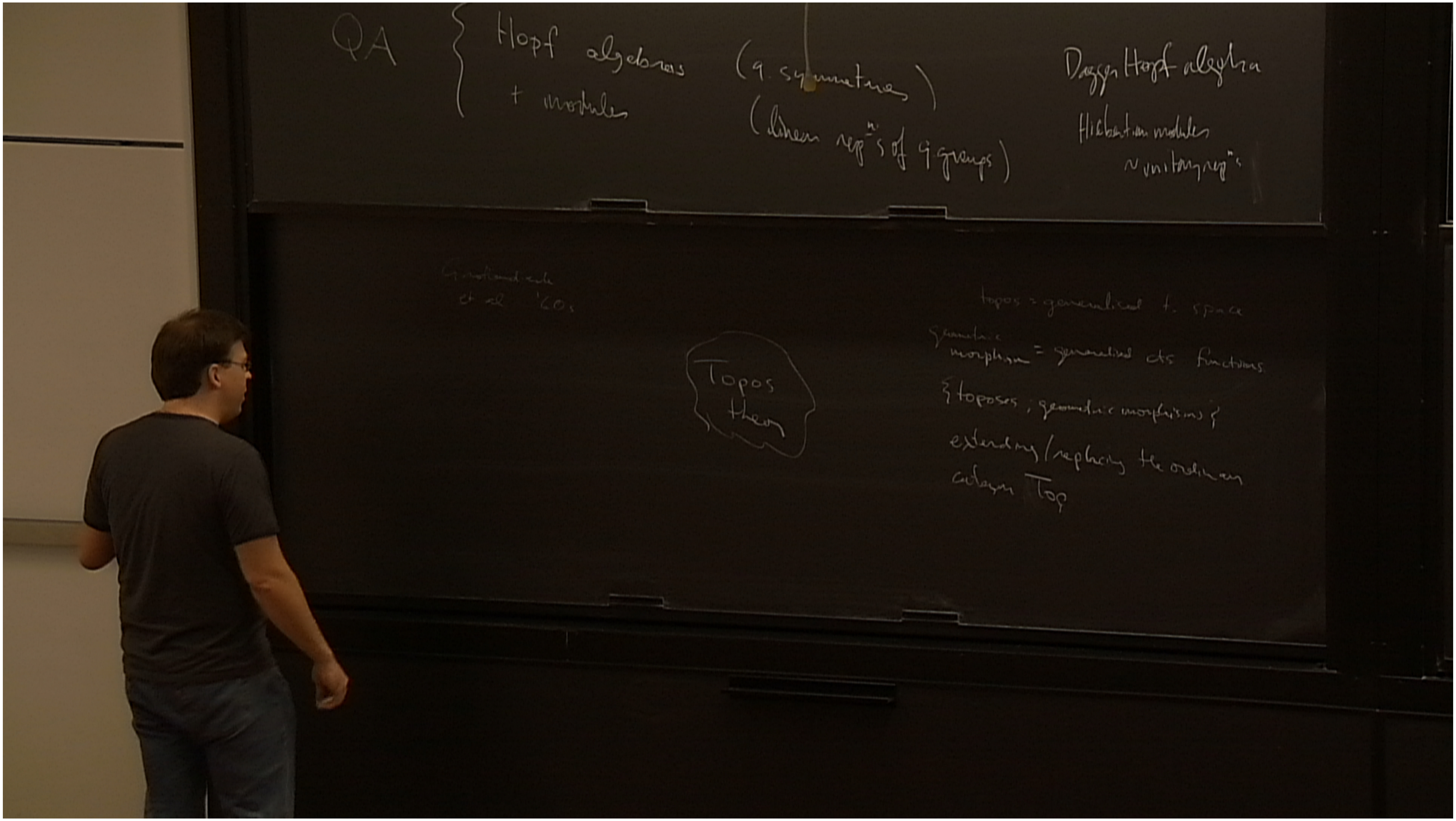
Topos theory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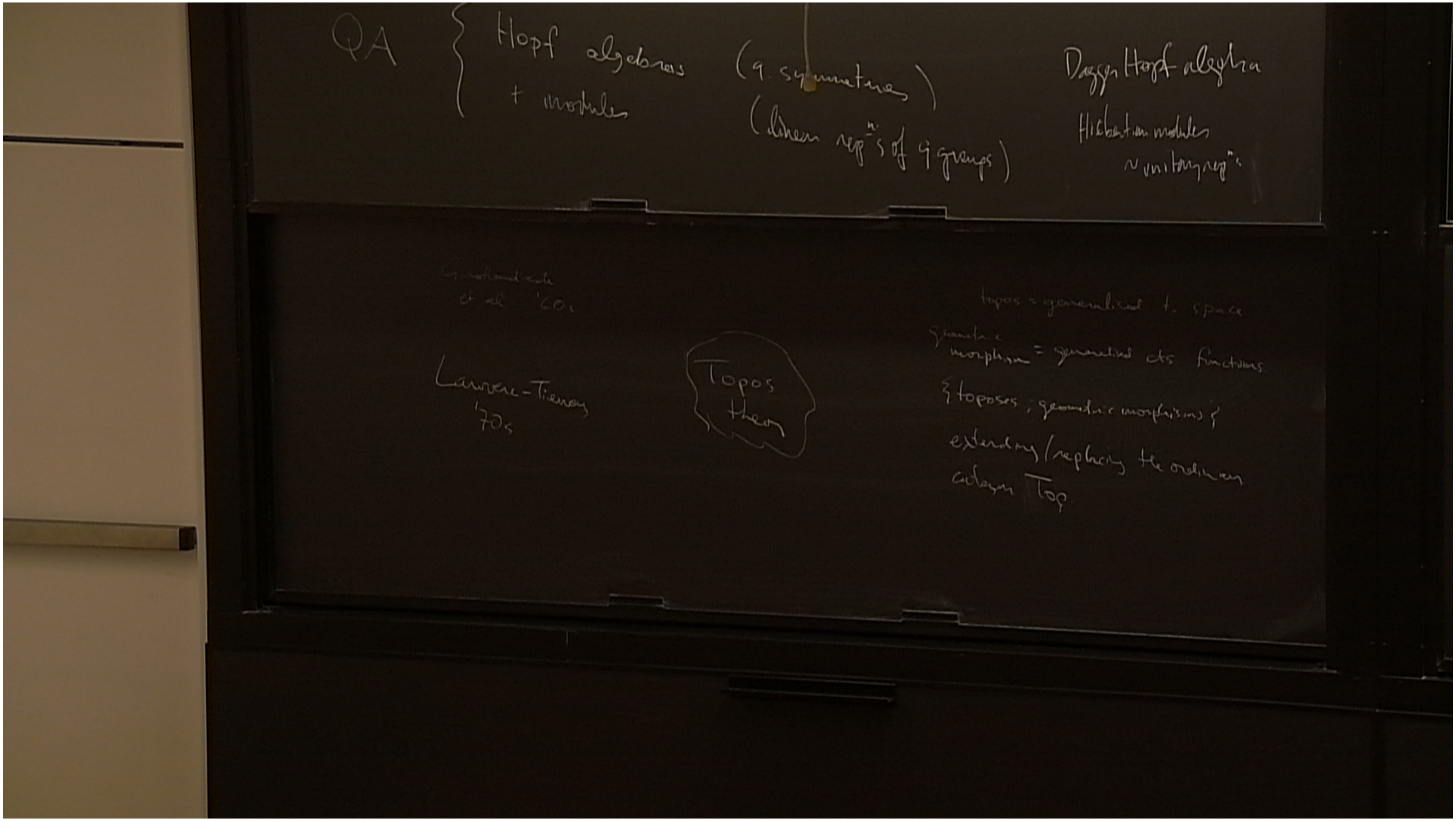
Topos theory
topos = generalized t. space
morphism = generalized cts functions
{toposes, geometric morphisms}
extending/replacing the ordinary category Top



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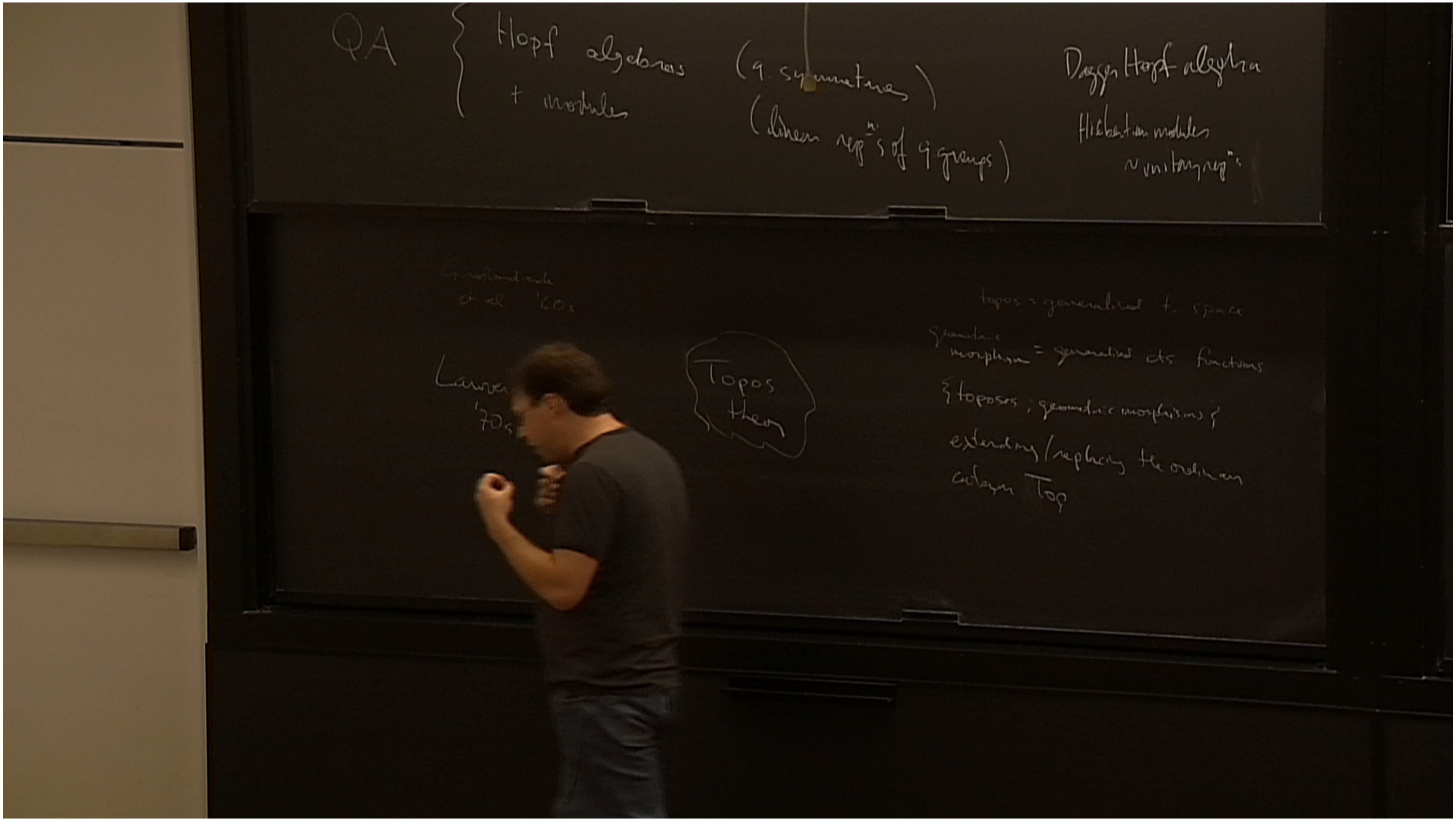
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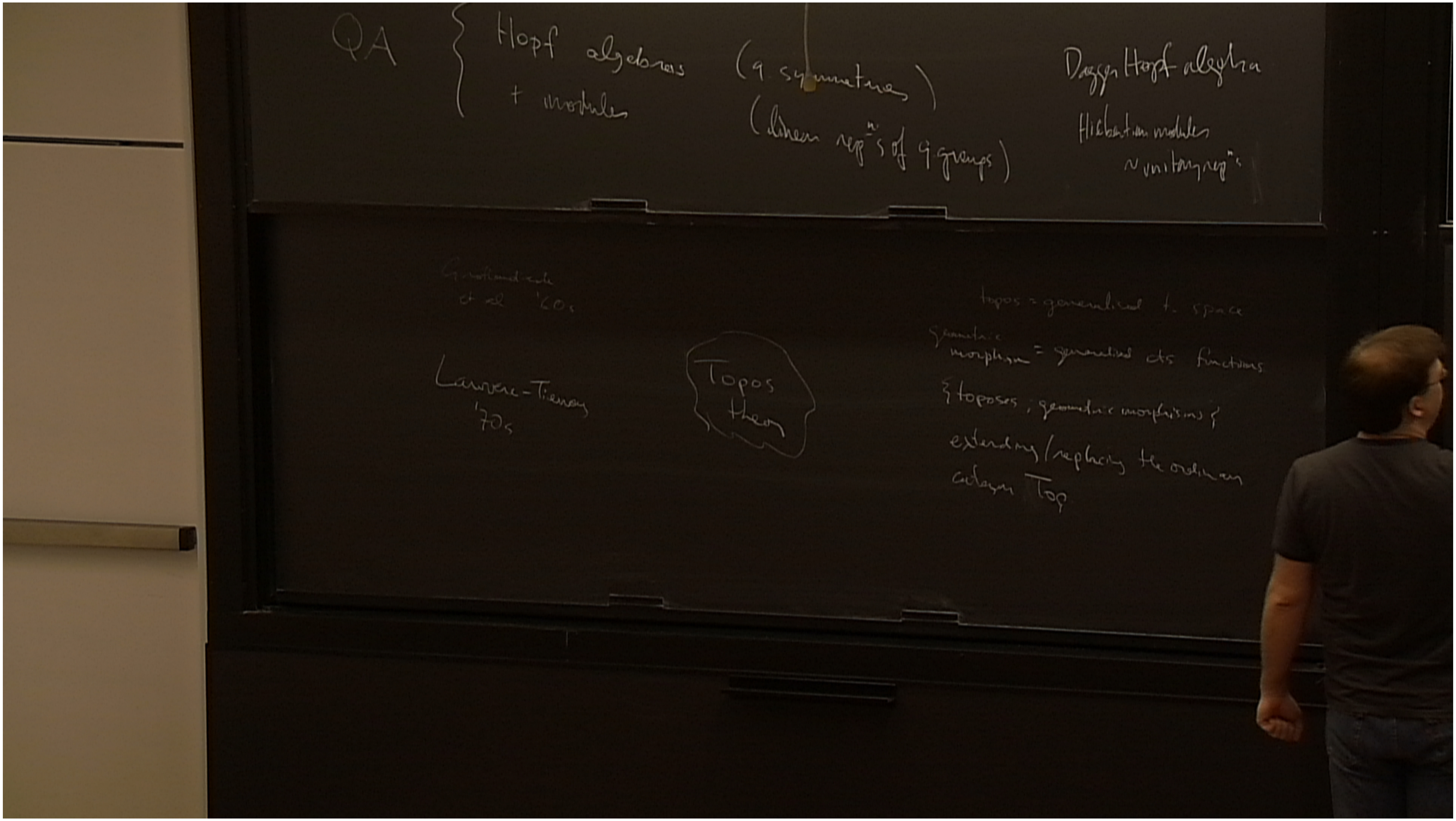
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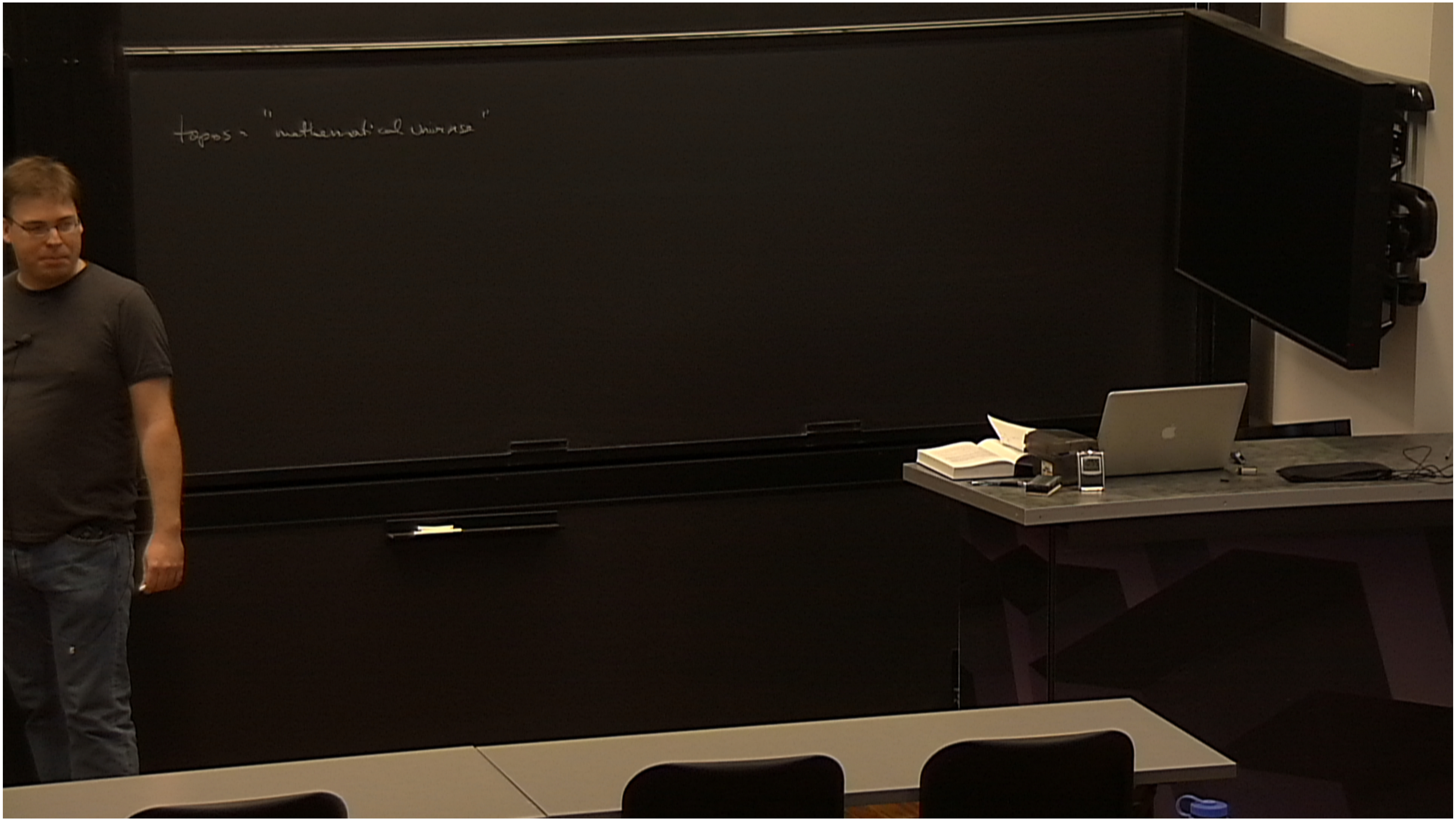
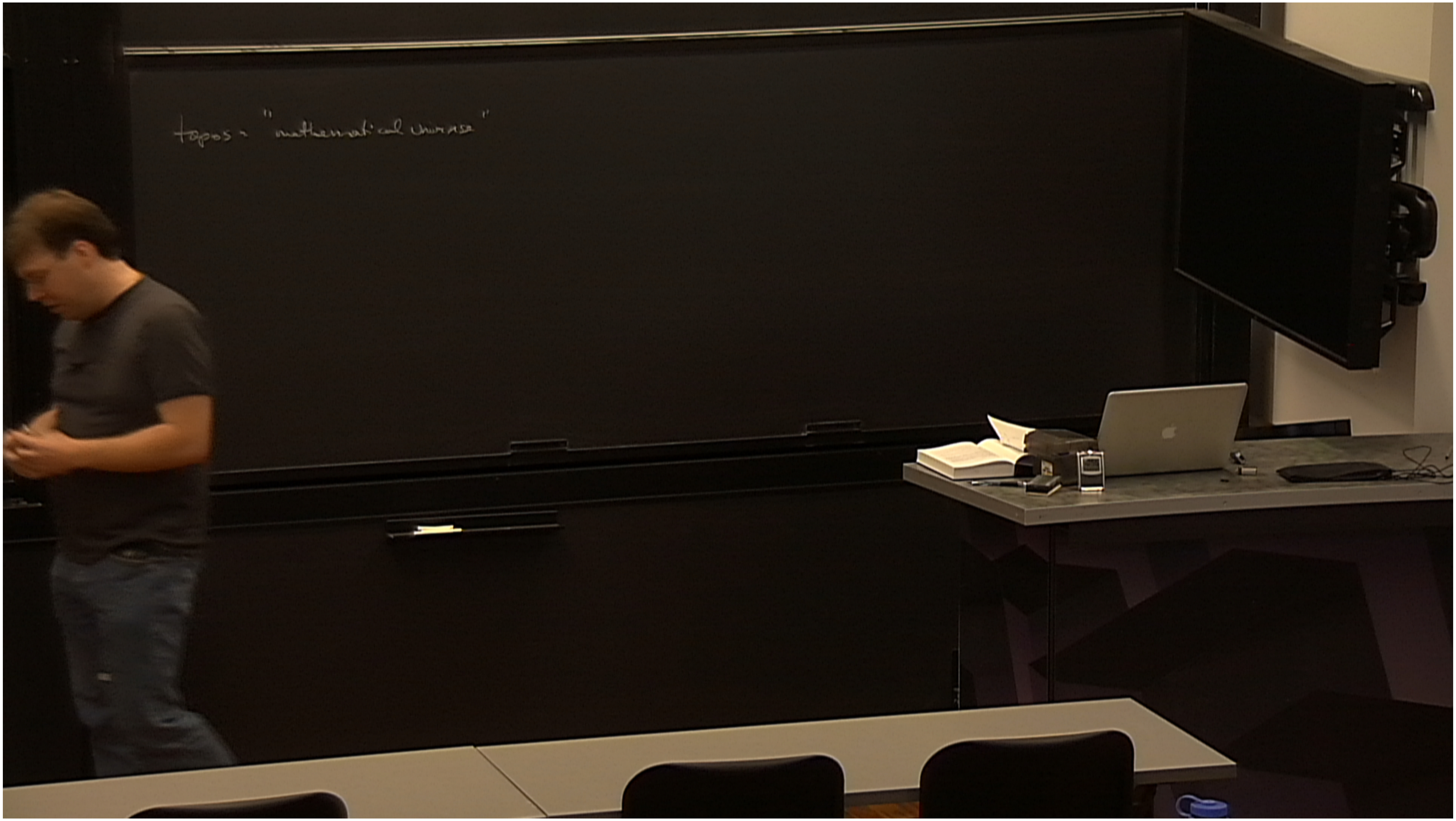

topos = "mathematical universe"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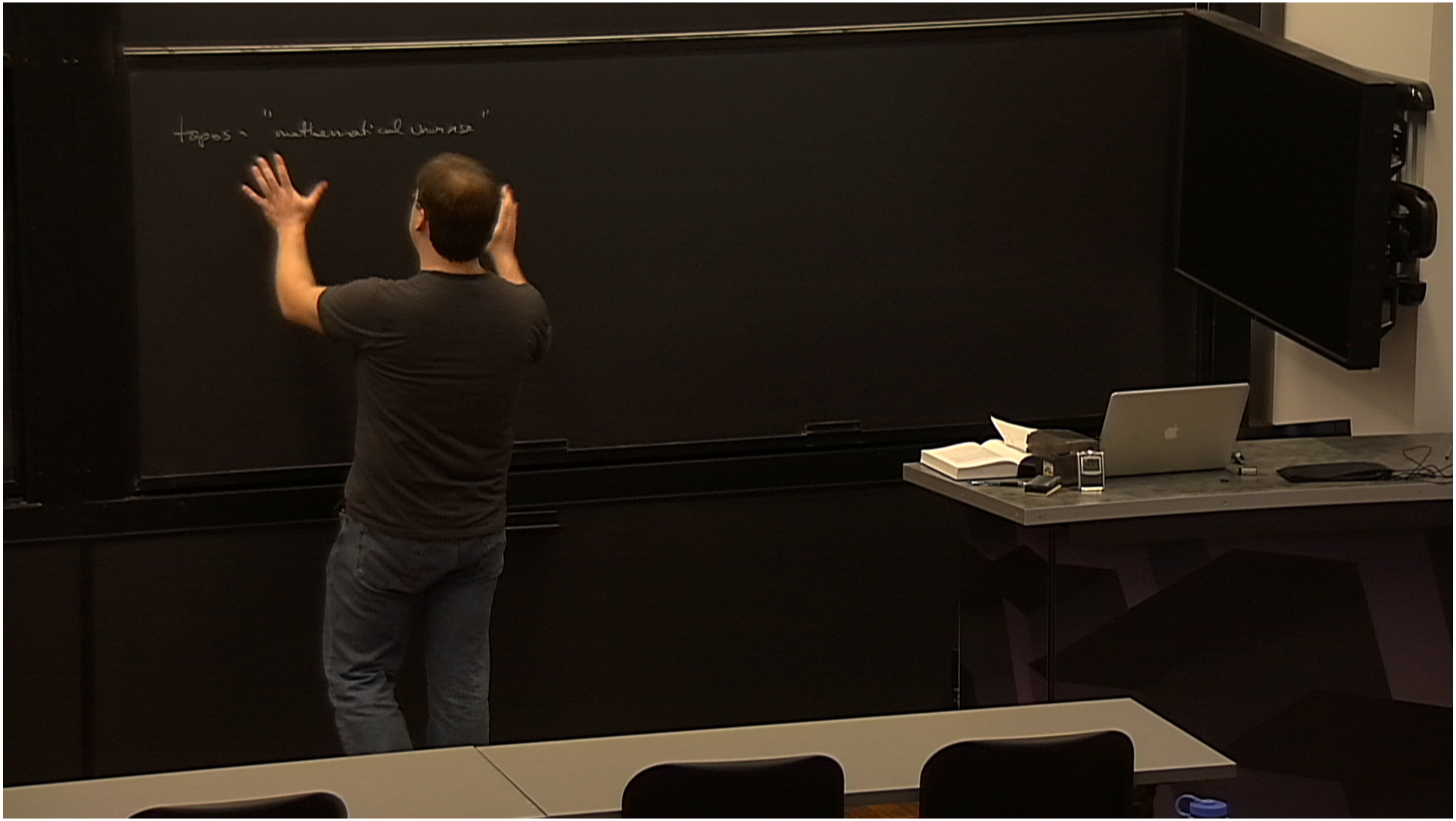
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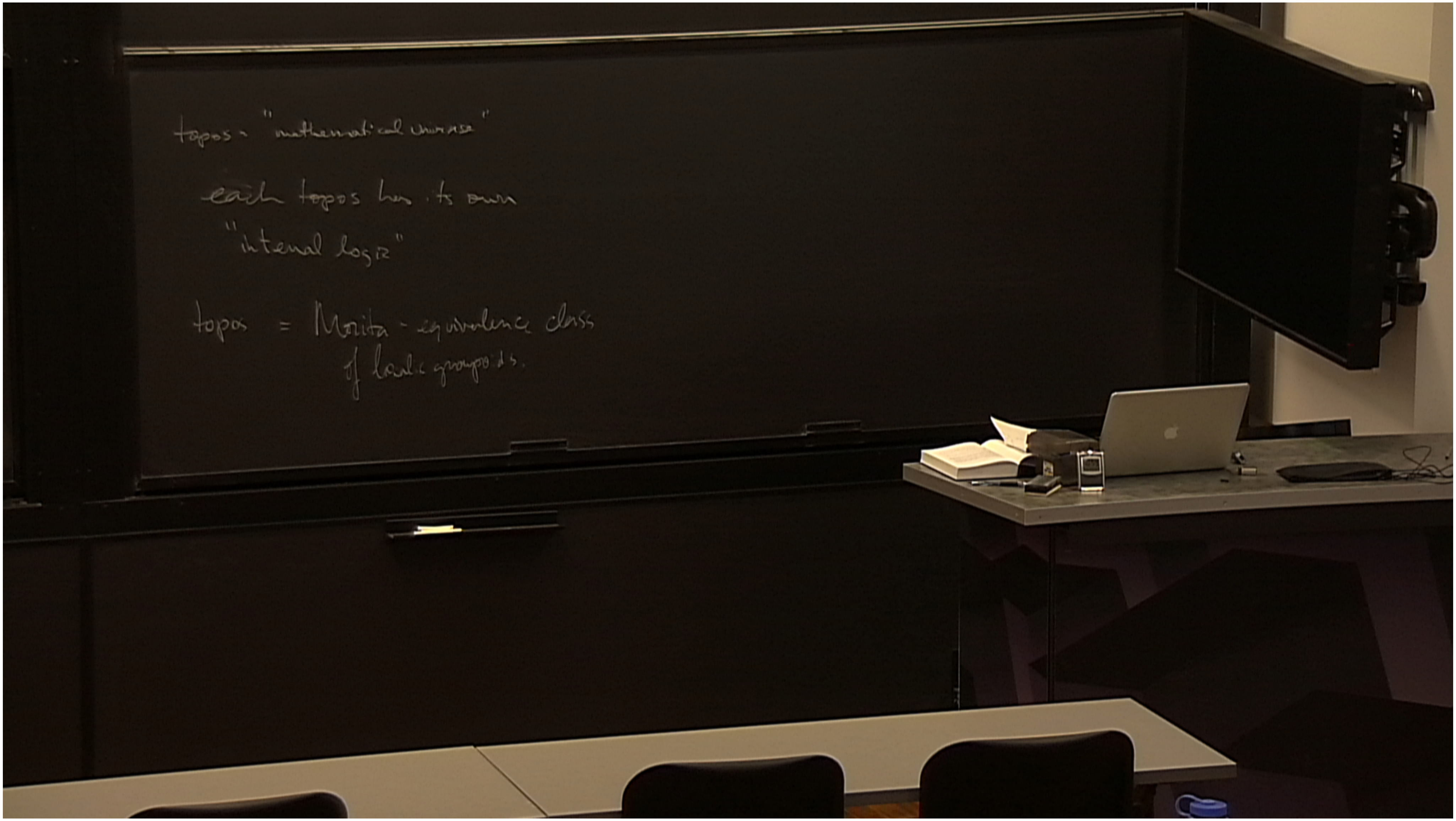
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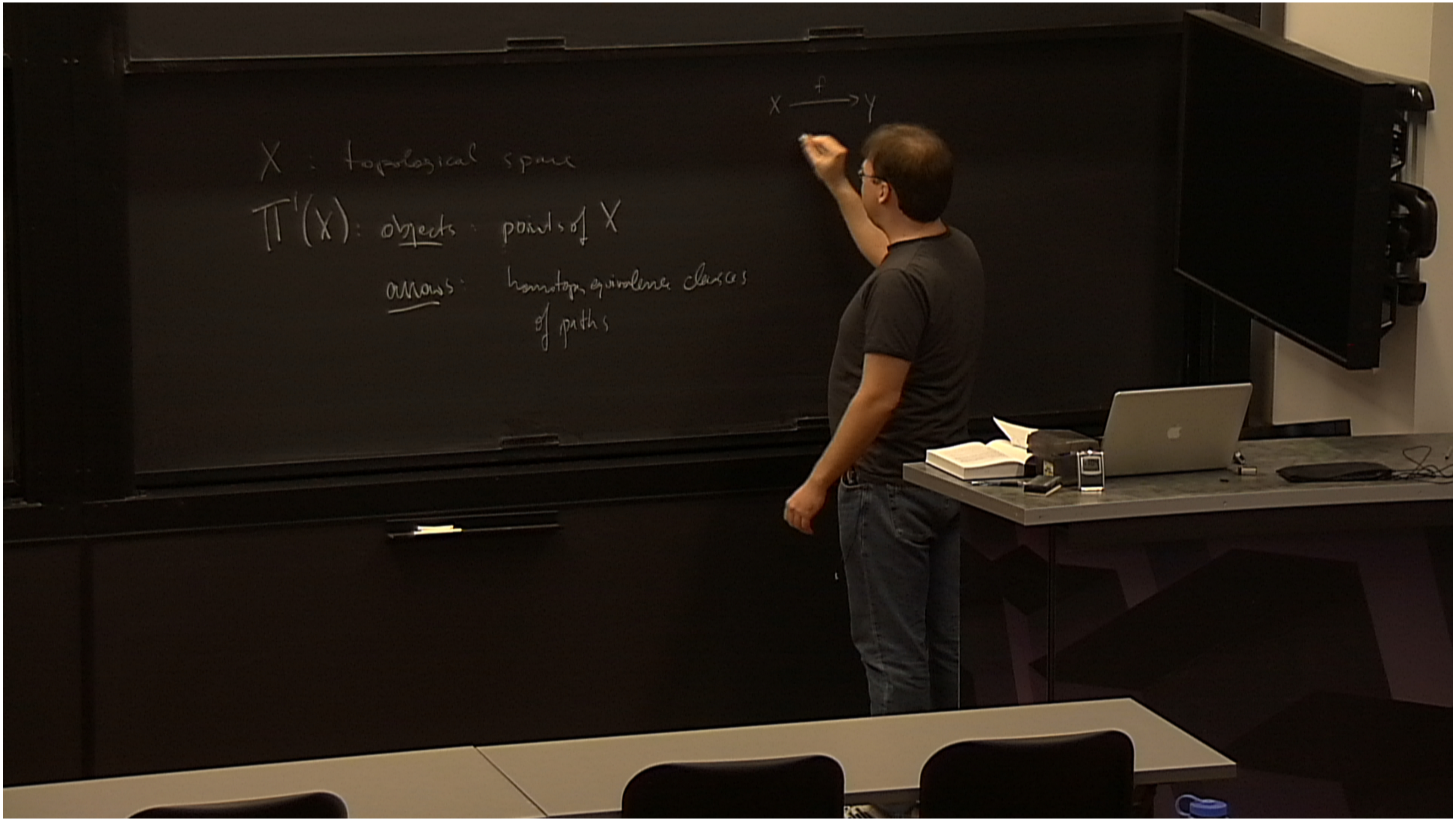
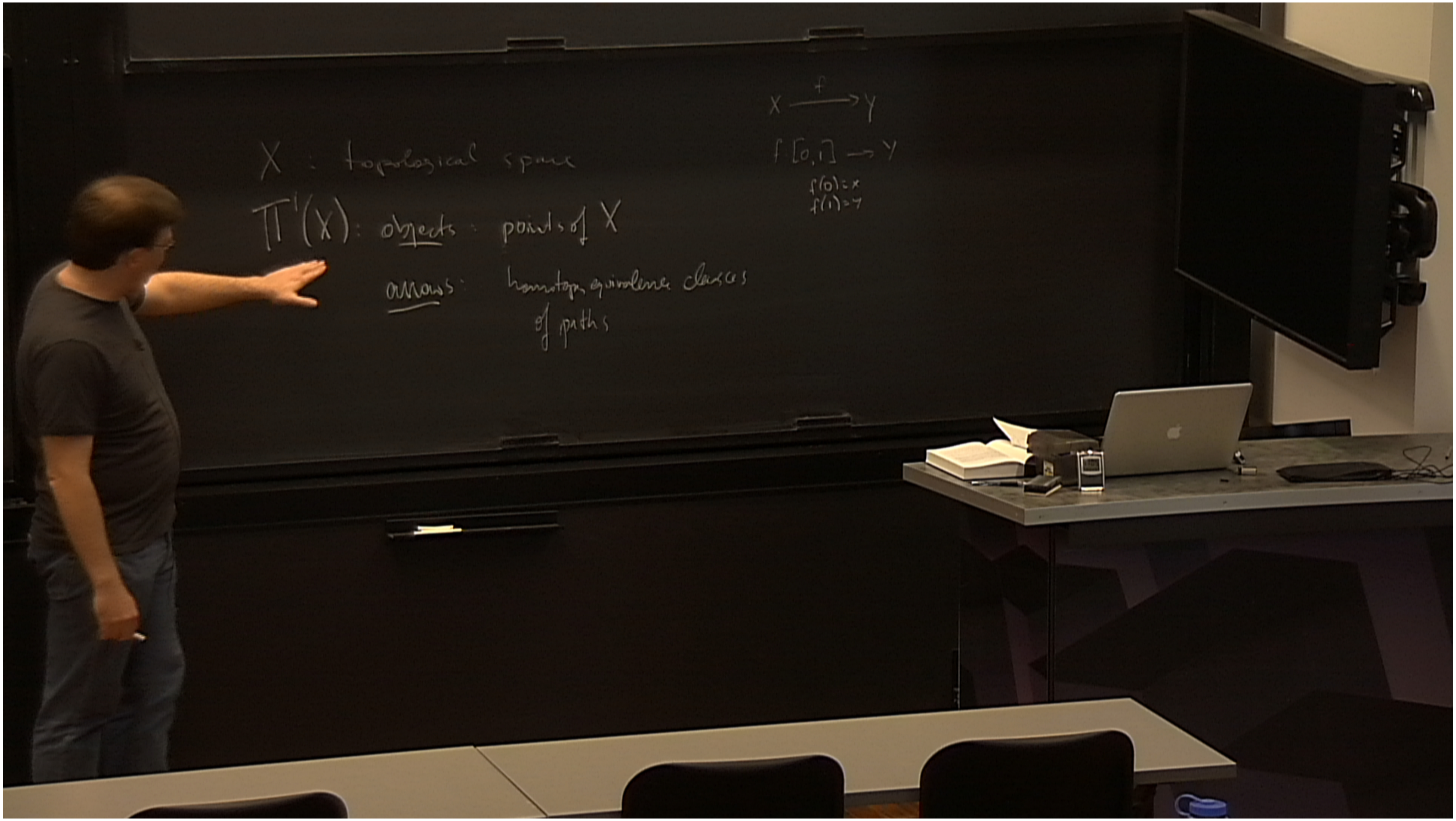

X —f→ Y
X : topological space
Π¹(X): objects : points of X
arrows: homotopy equivalence classes
of paths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X —f→ Y
f: [0,1] → Y
f(0)=x
f(1)=y


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arrows
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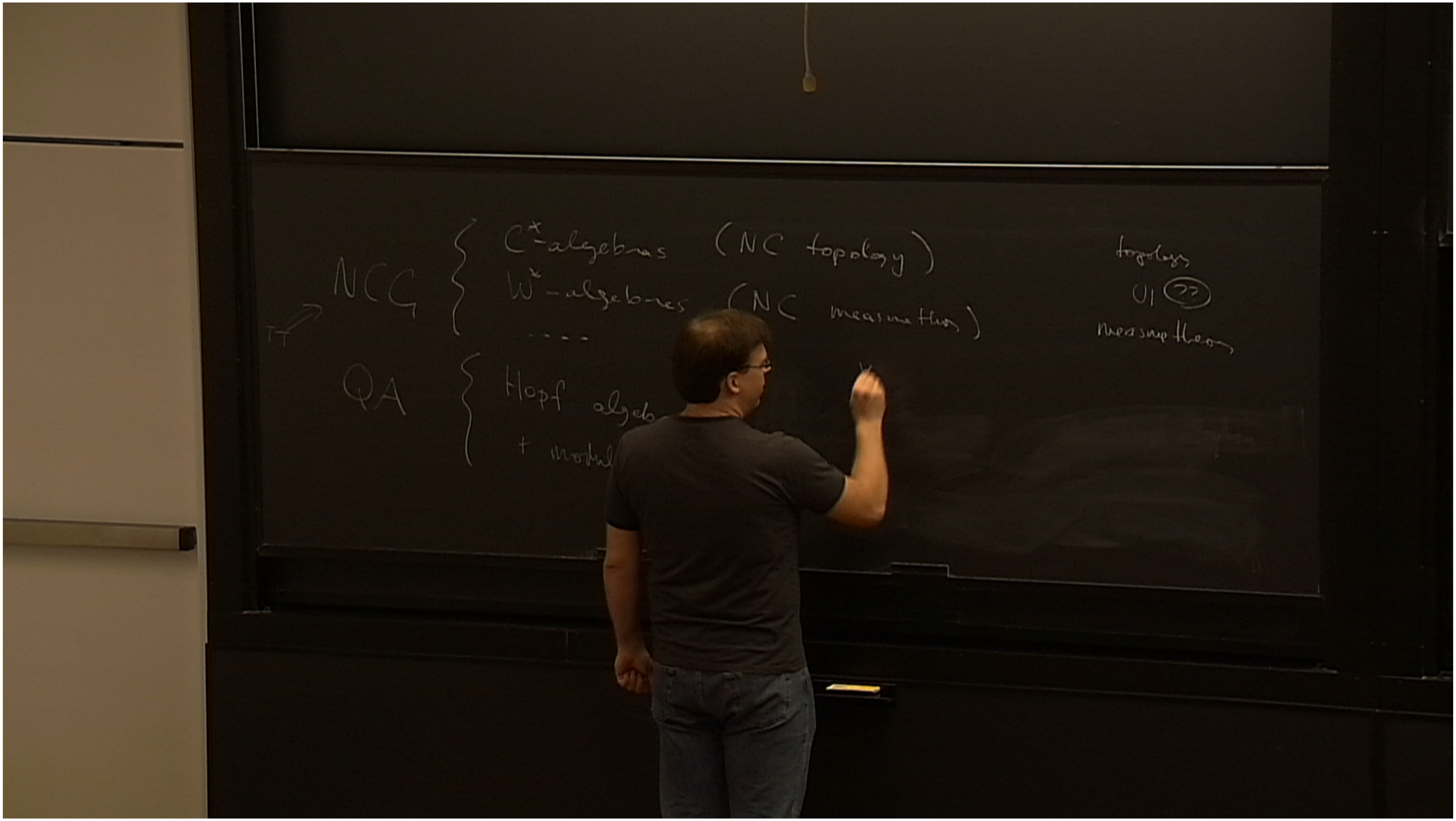
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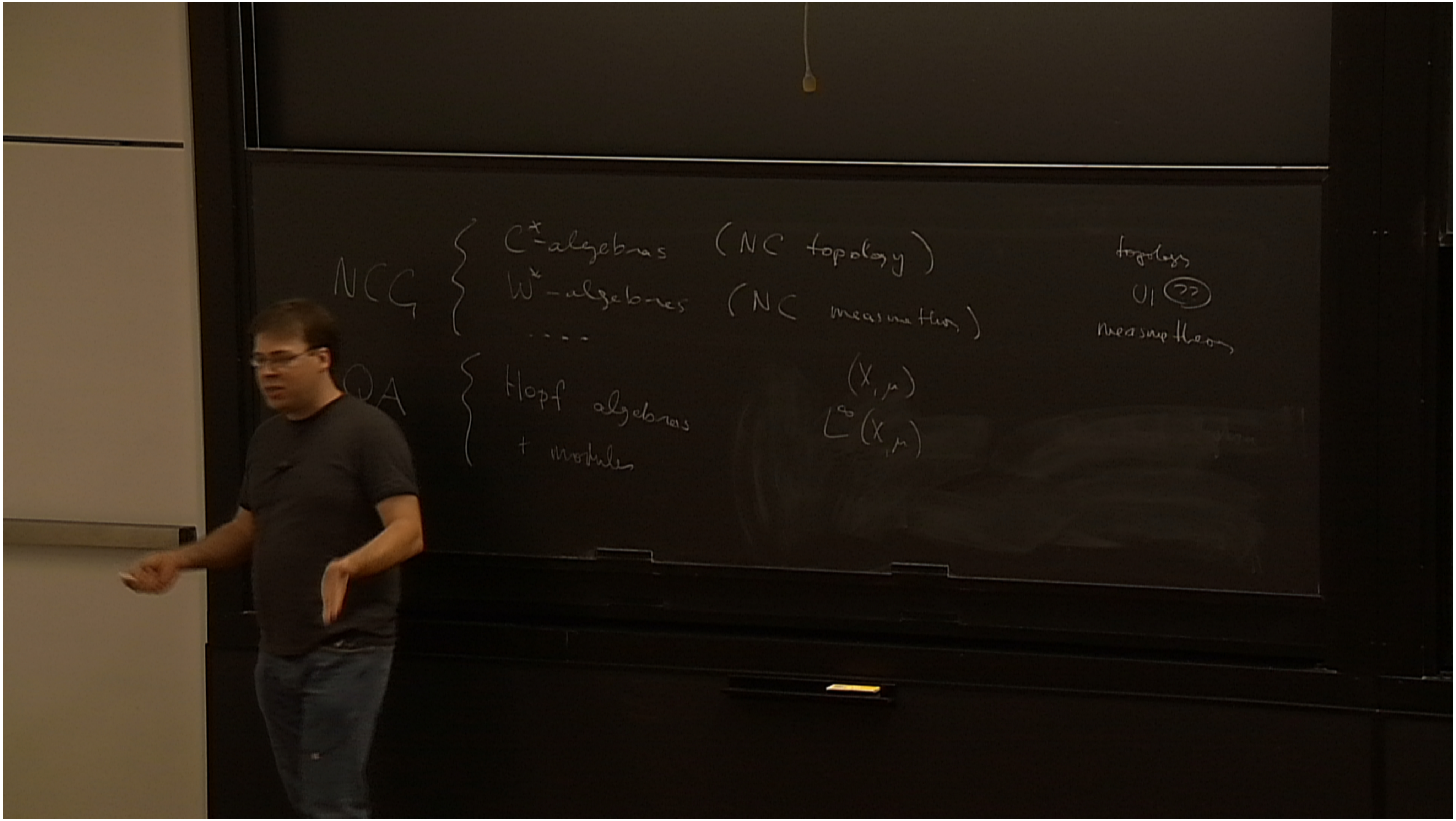
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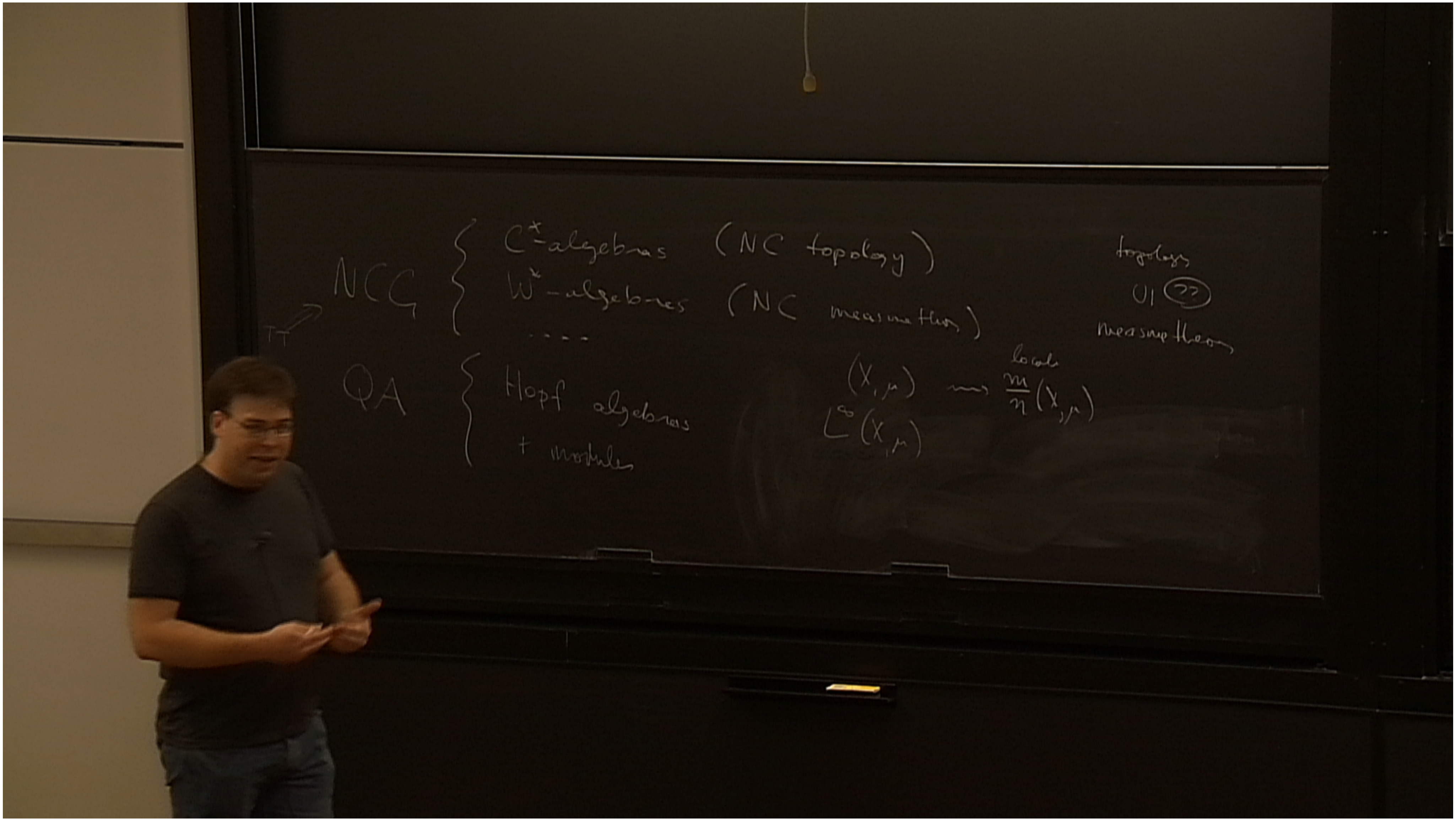
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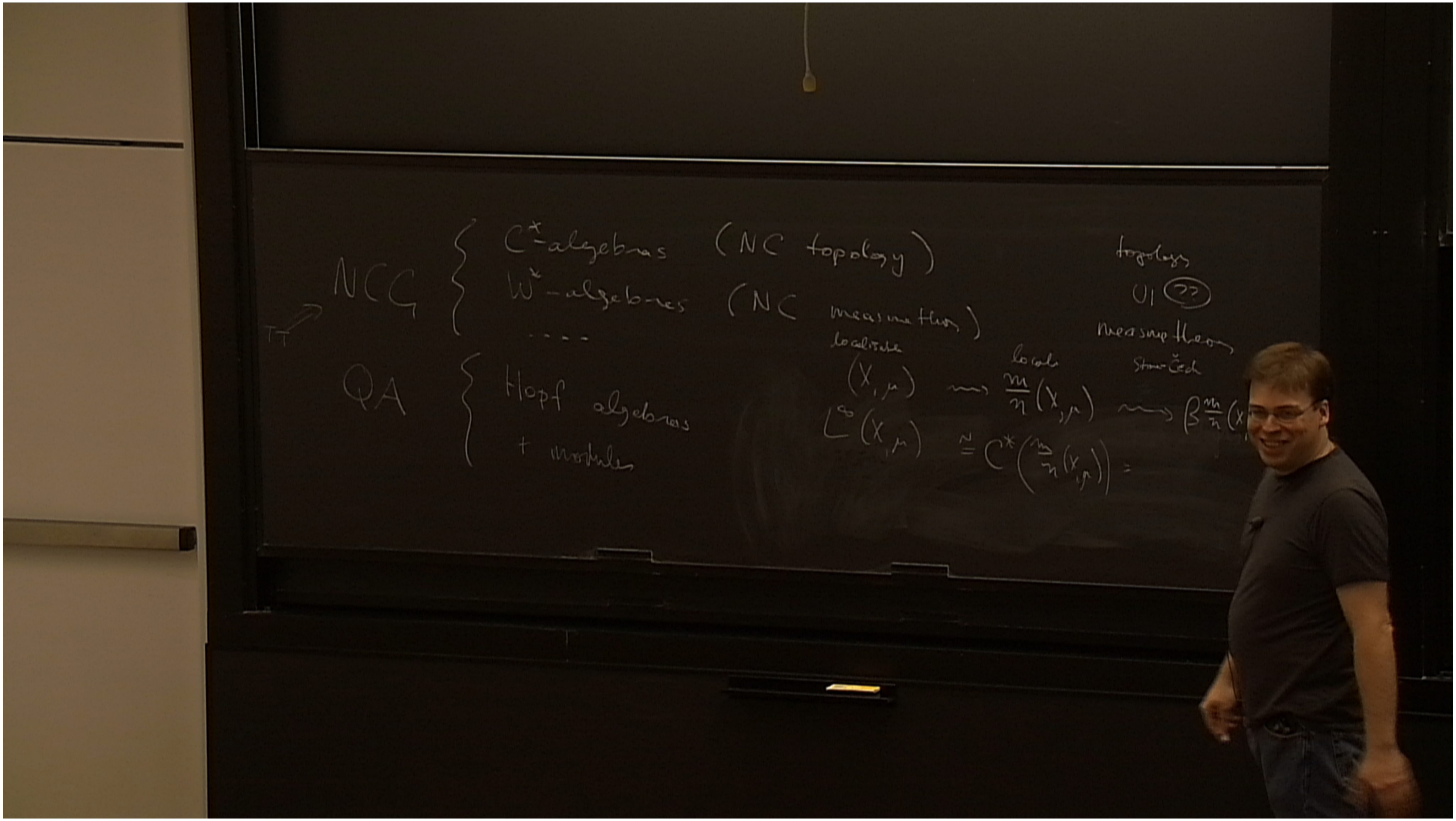
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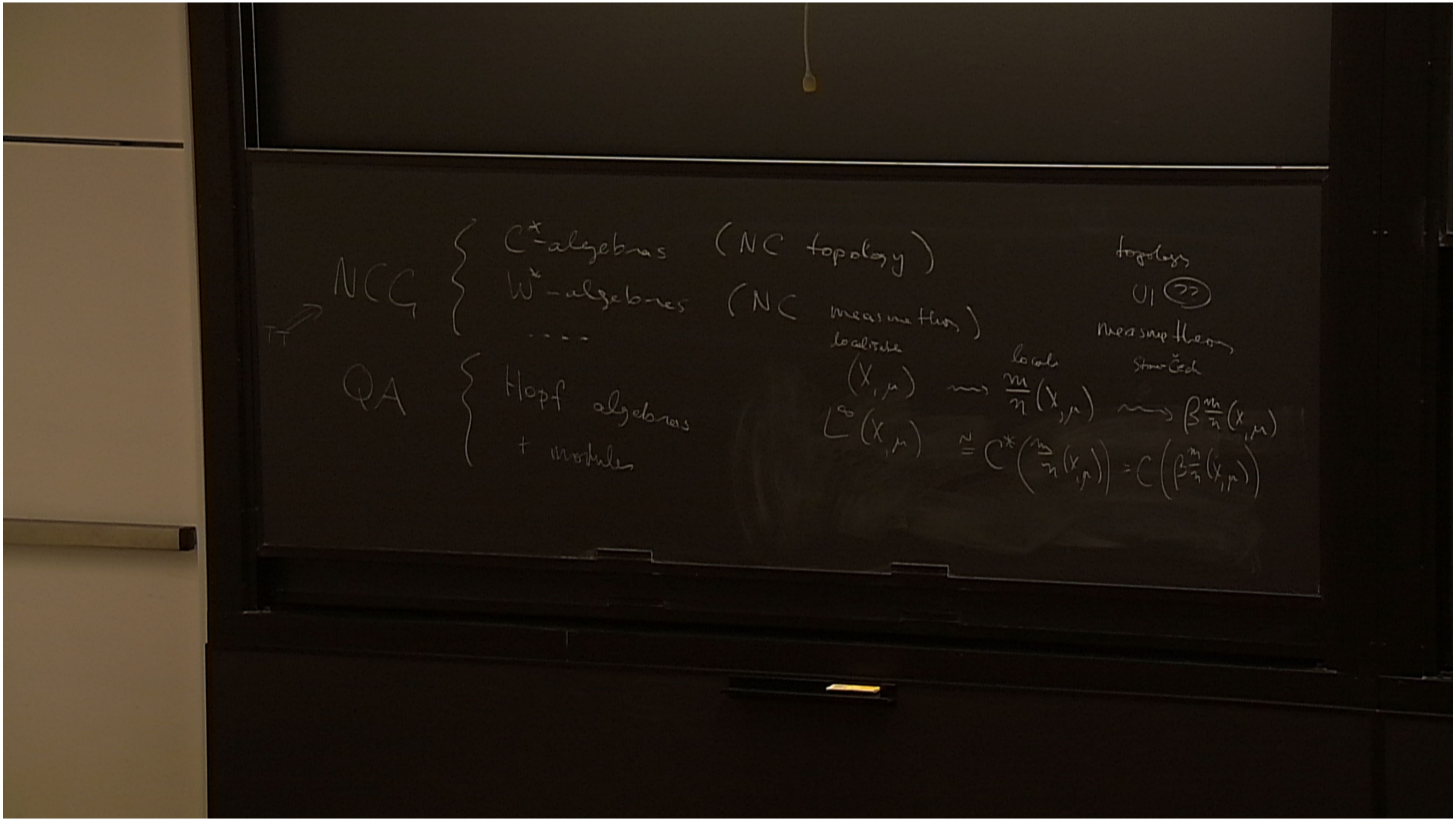
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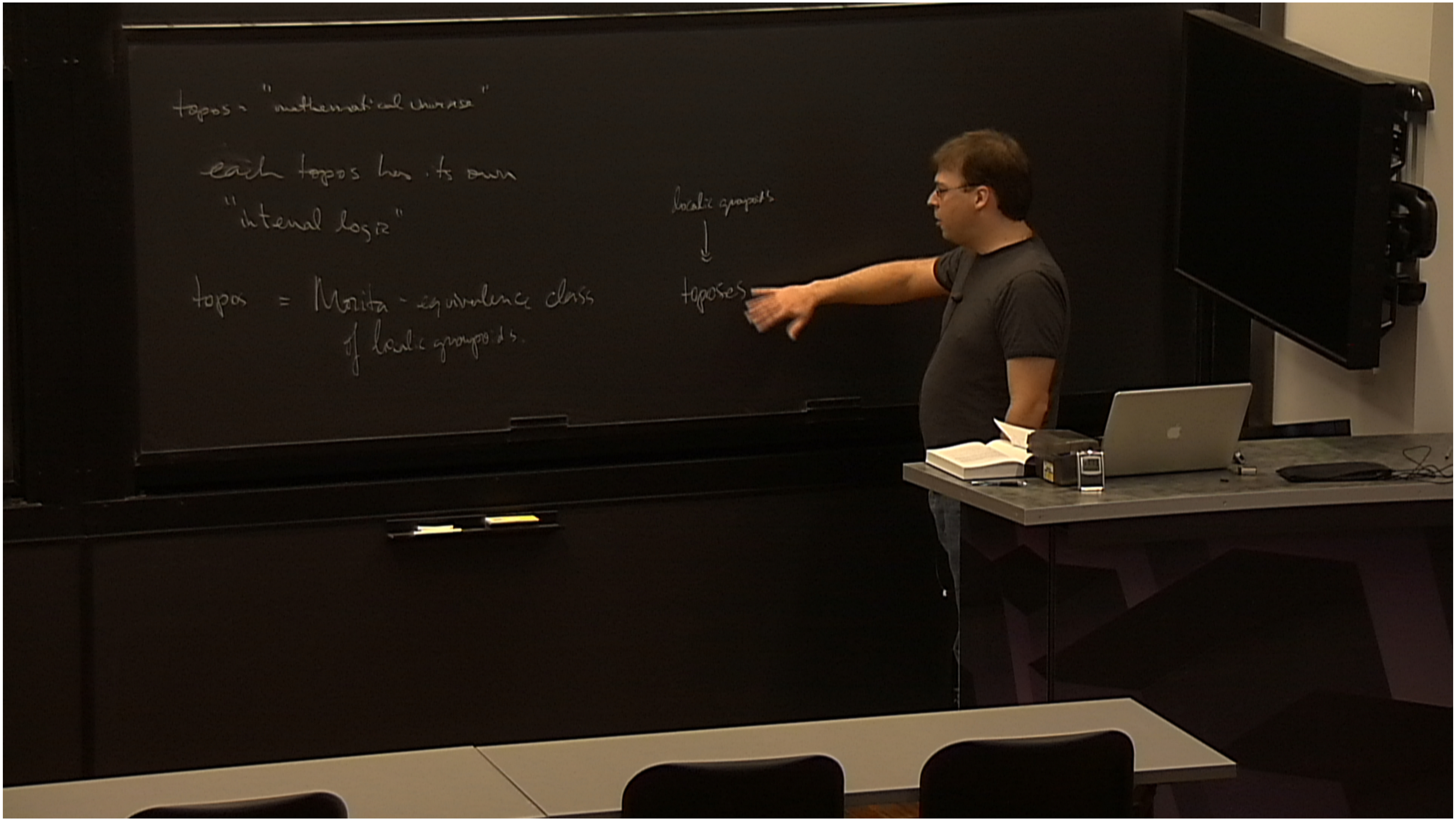
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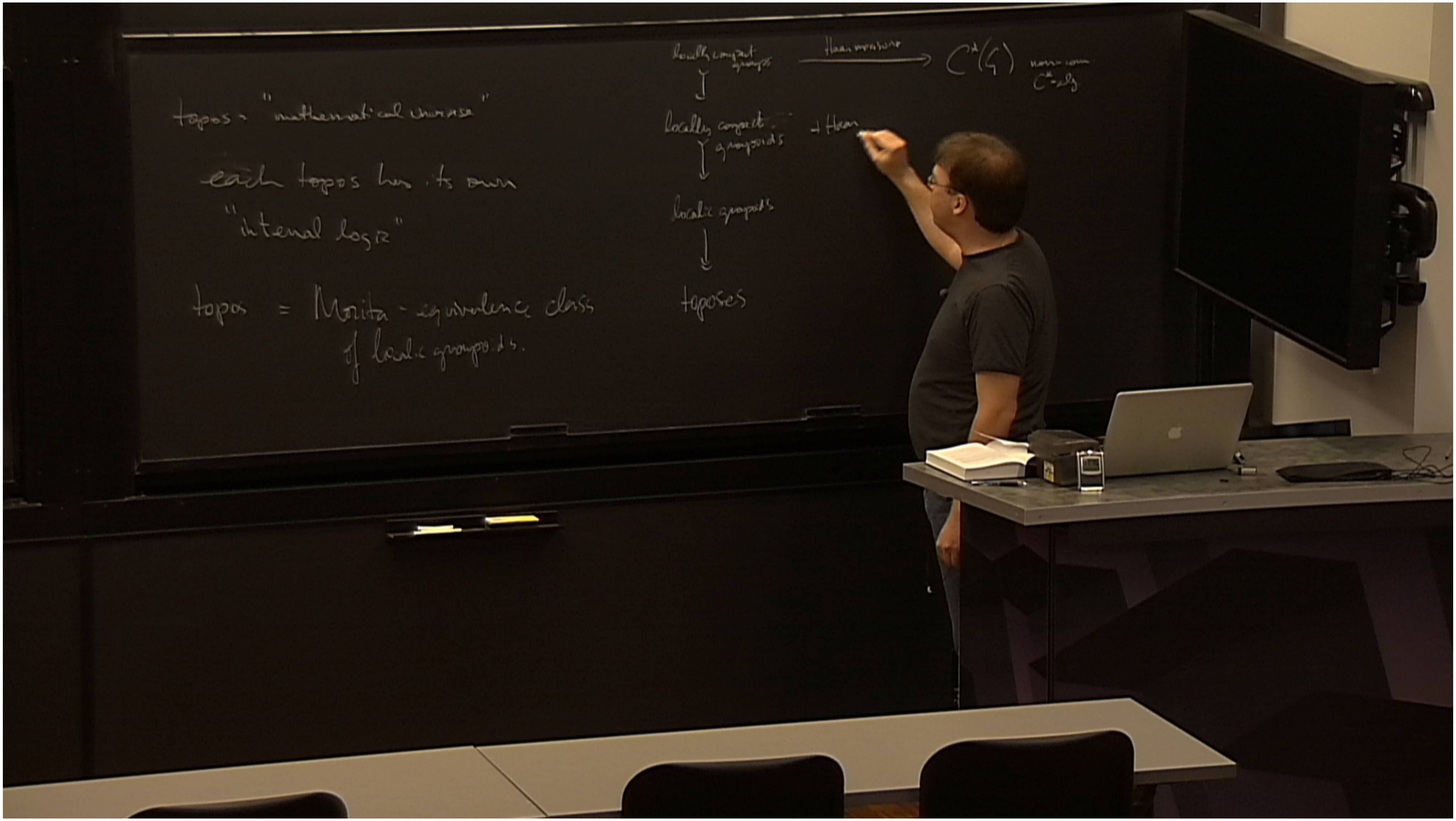
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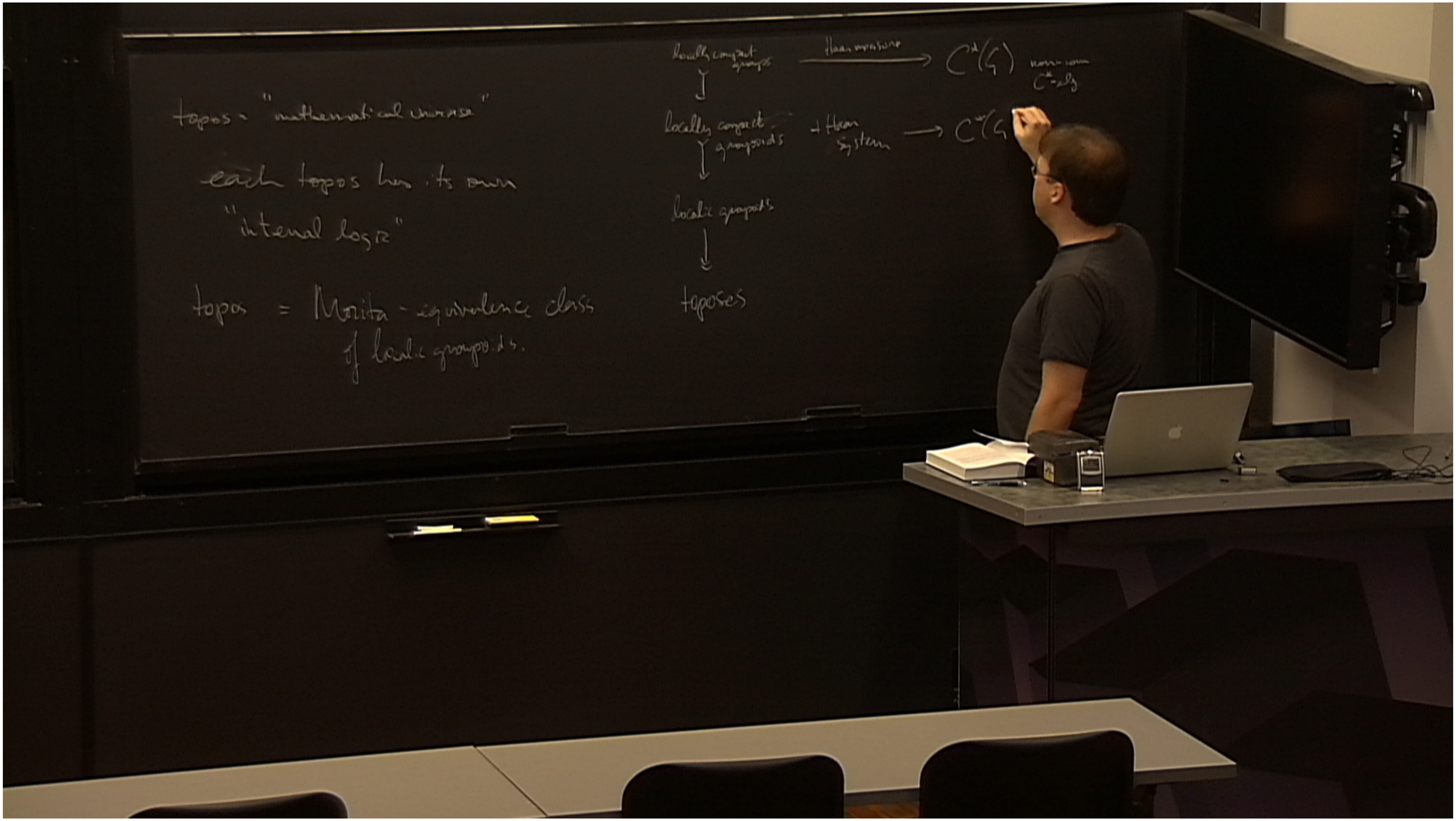
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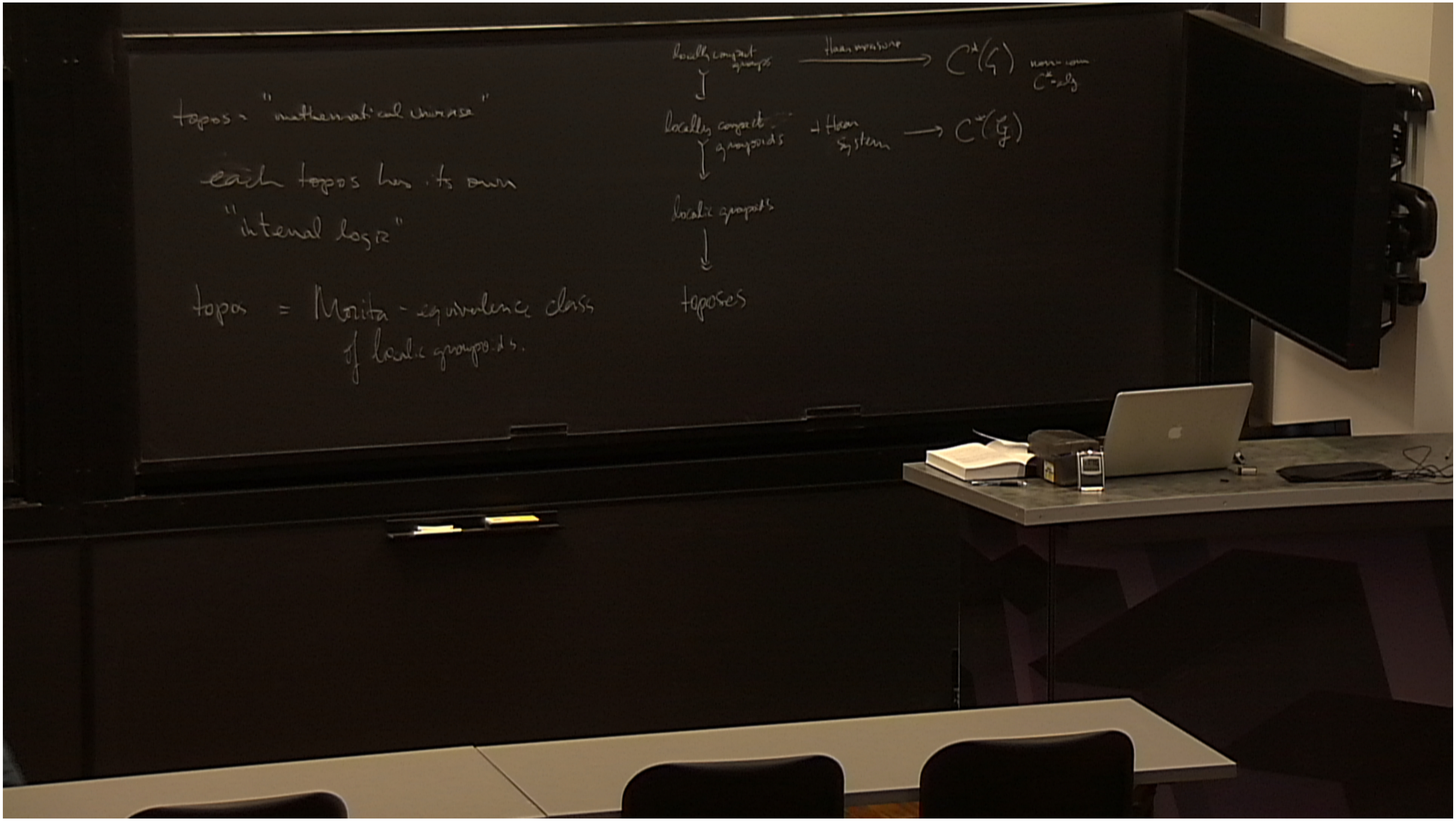
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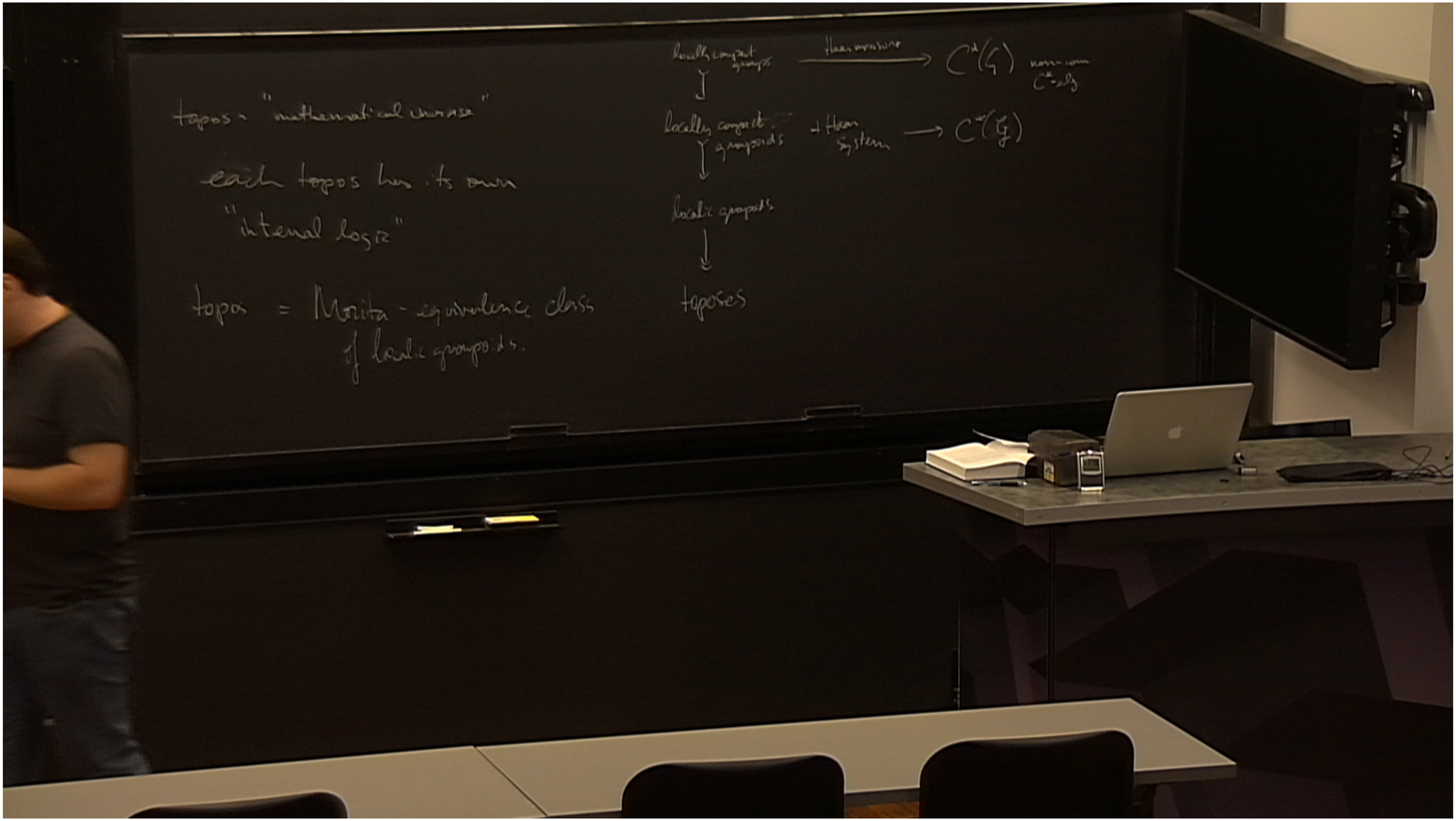
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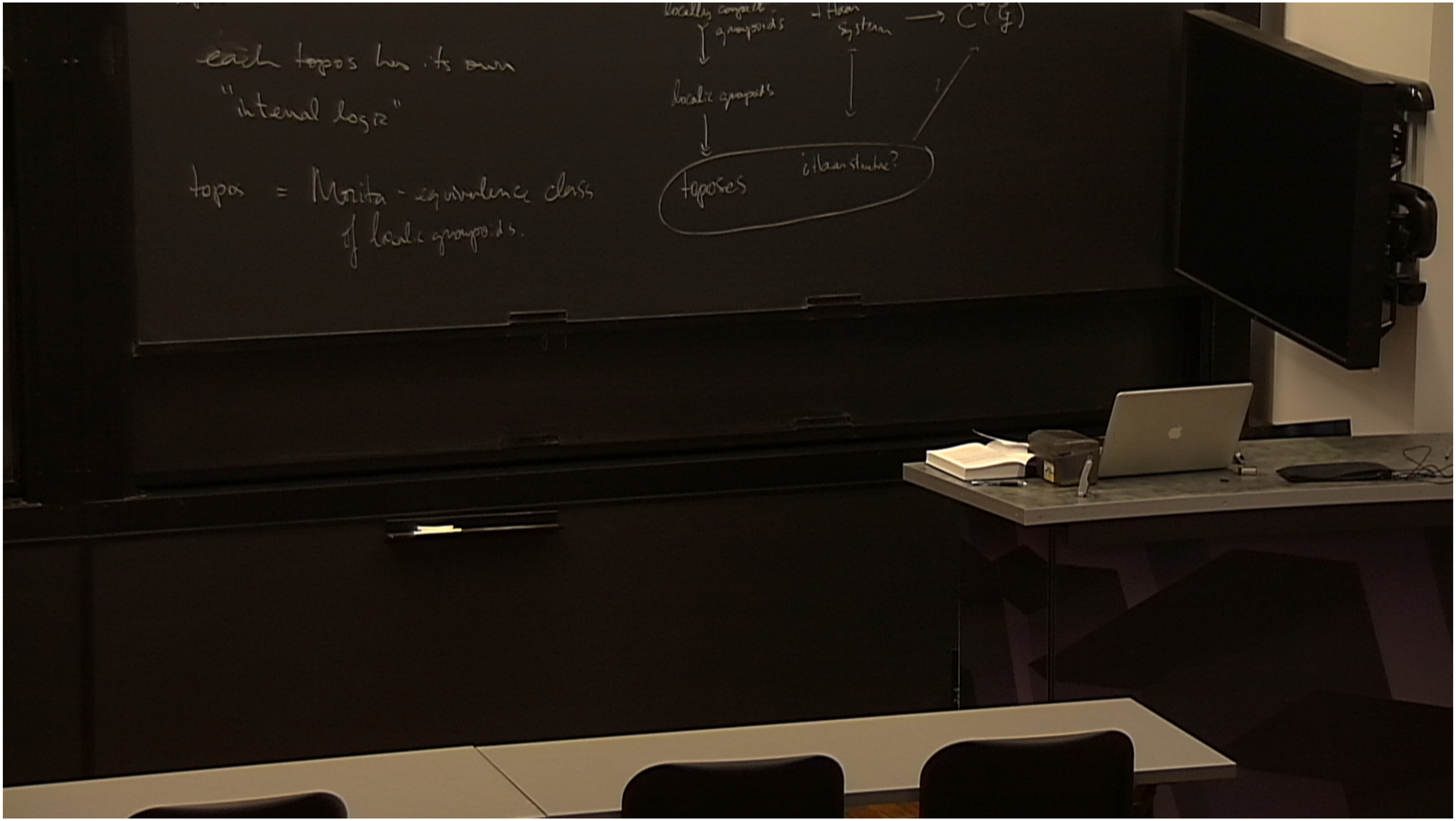
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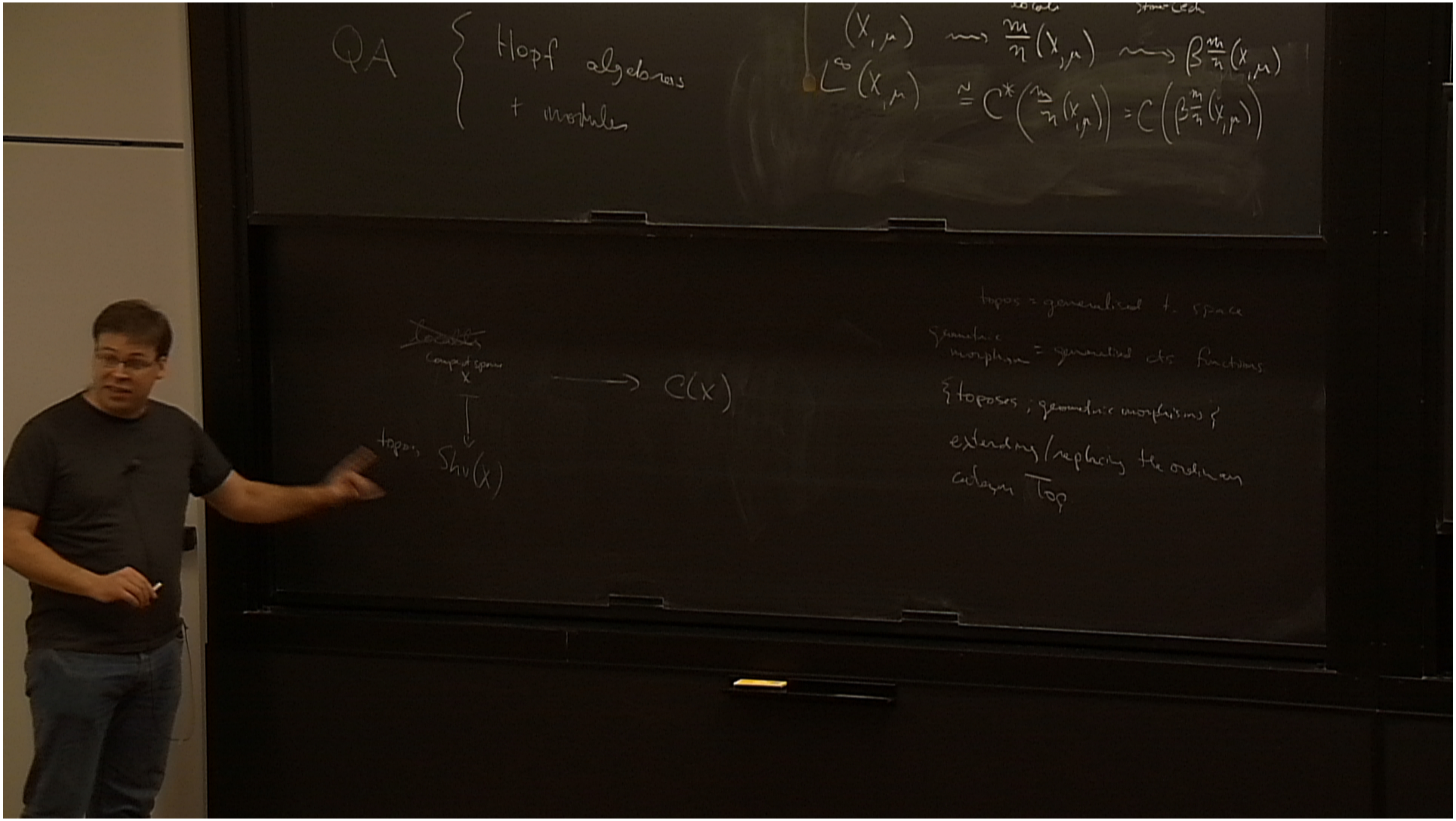
¿Haar structure?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Hopf algebras
+ modules
$(X,\mu) \rightsquigarrow \frac{m}{n}(X,\mu) \rightsquigarrow \beta\frac{m}{n}(X,\mu)$
$L^\infty(X,\mu) \cong C^*\left(\frac{m}{n}(X,\mu)\right) = C\left(\beta\frac{m}{n}(X,\mu)\right)$
~~locales~~
compact space
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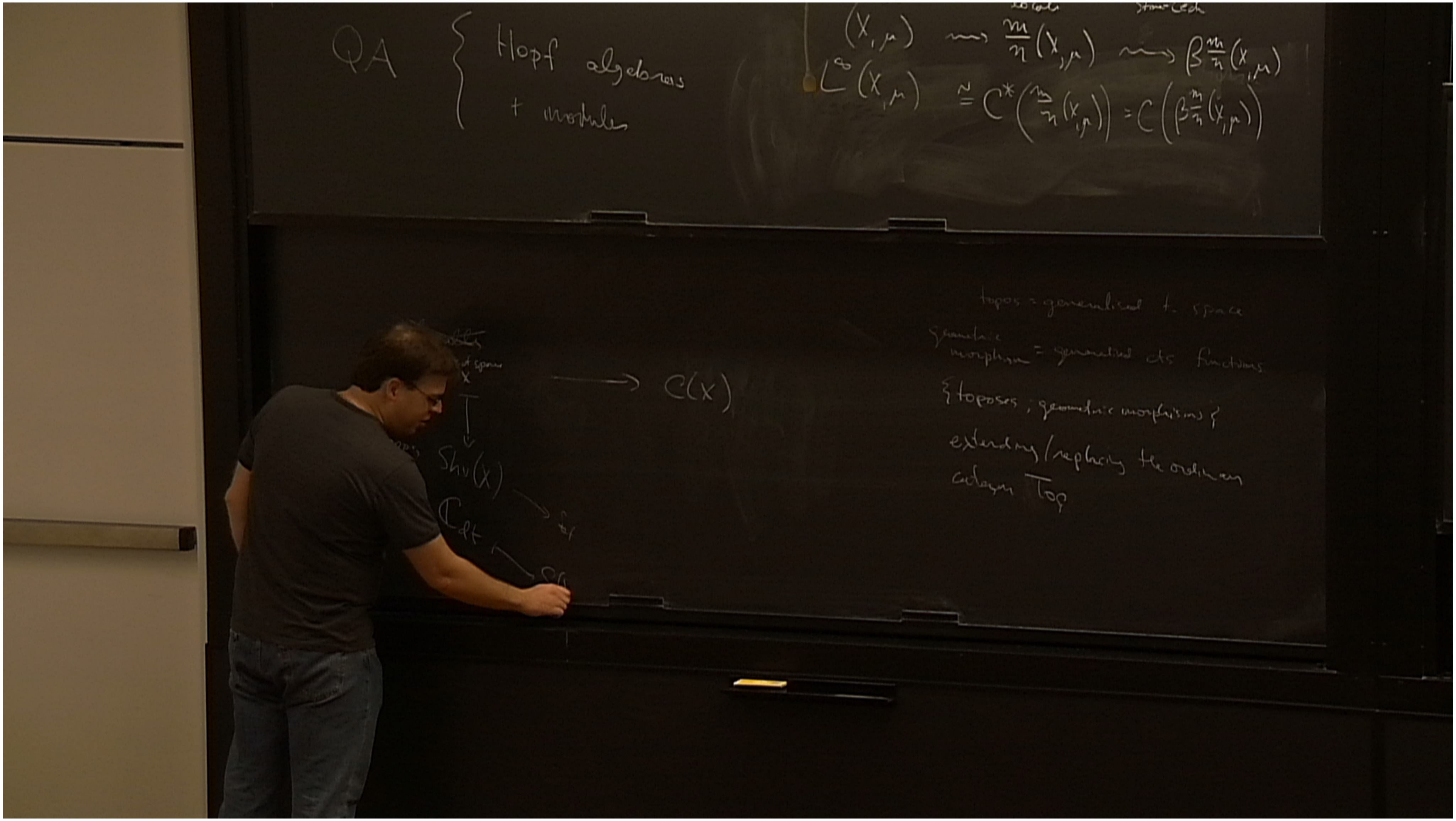
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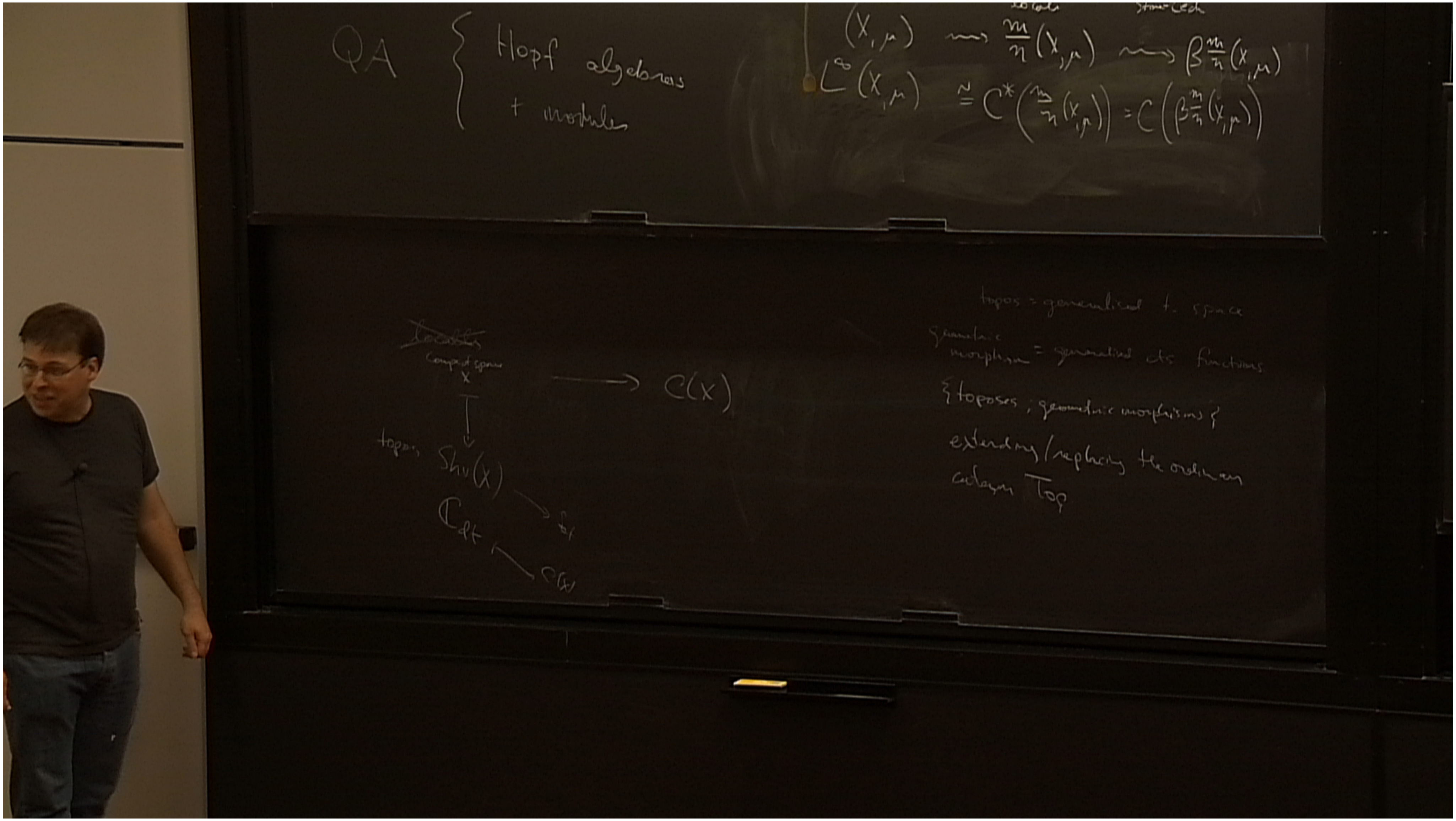
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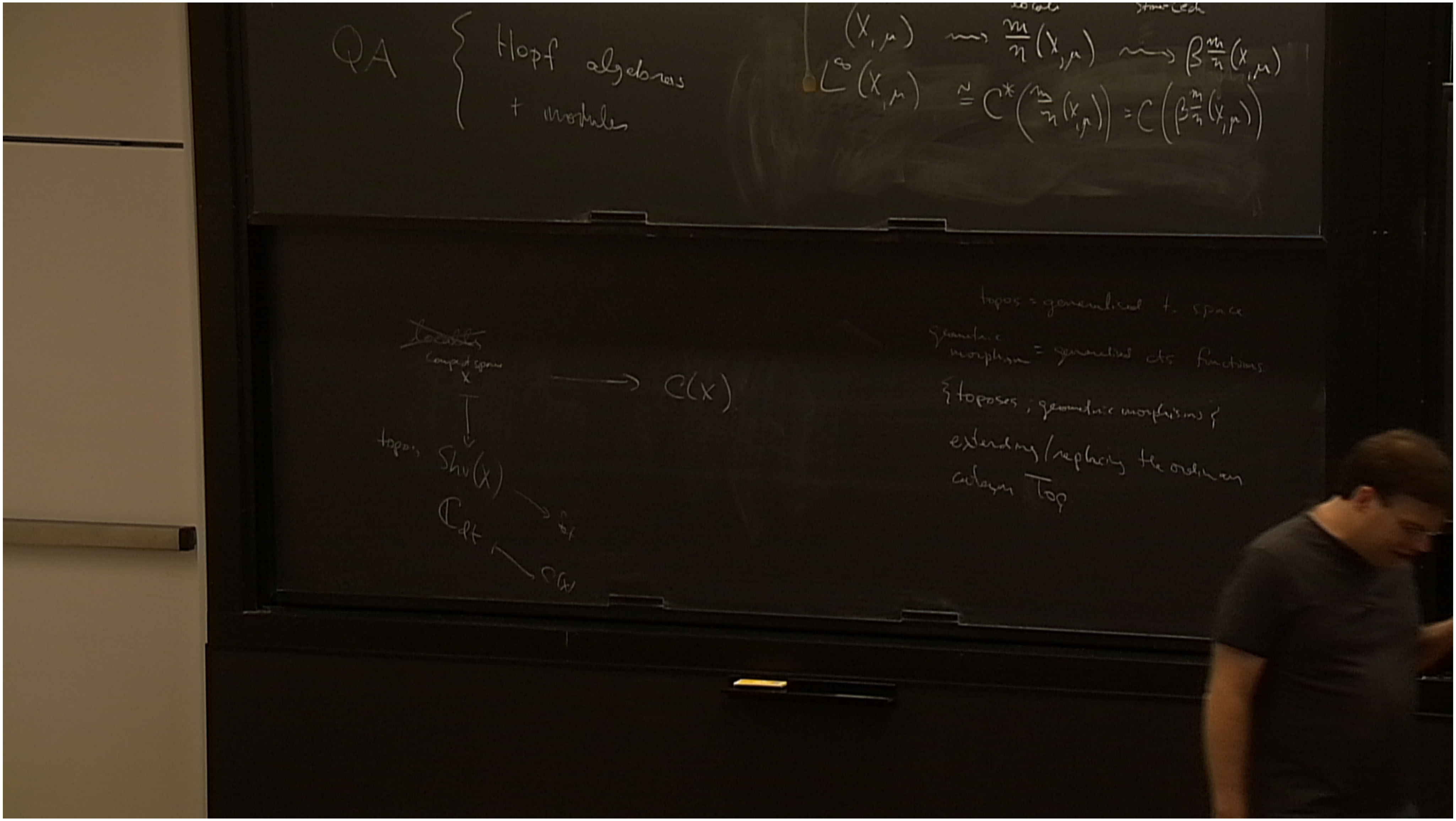
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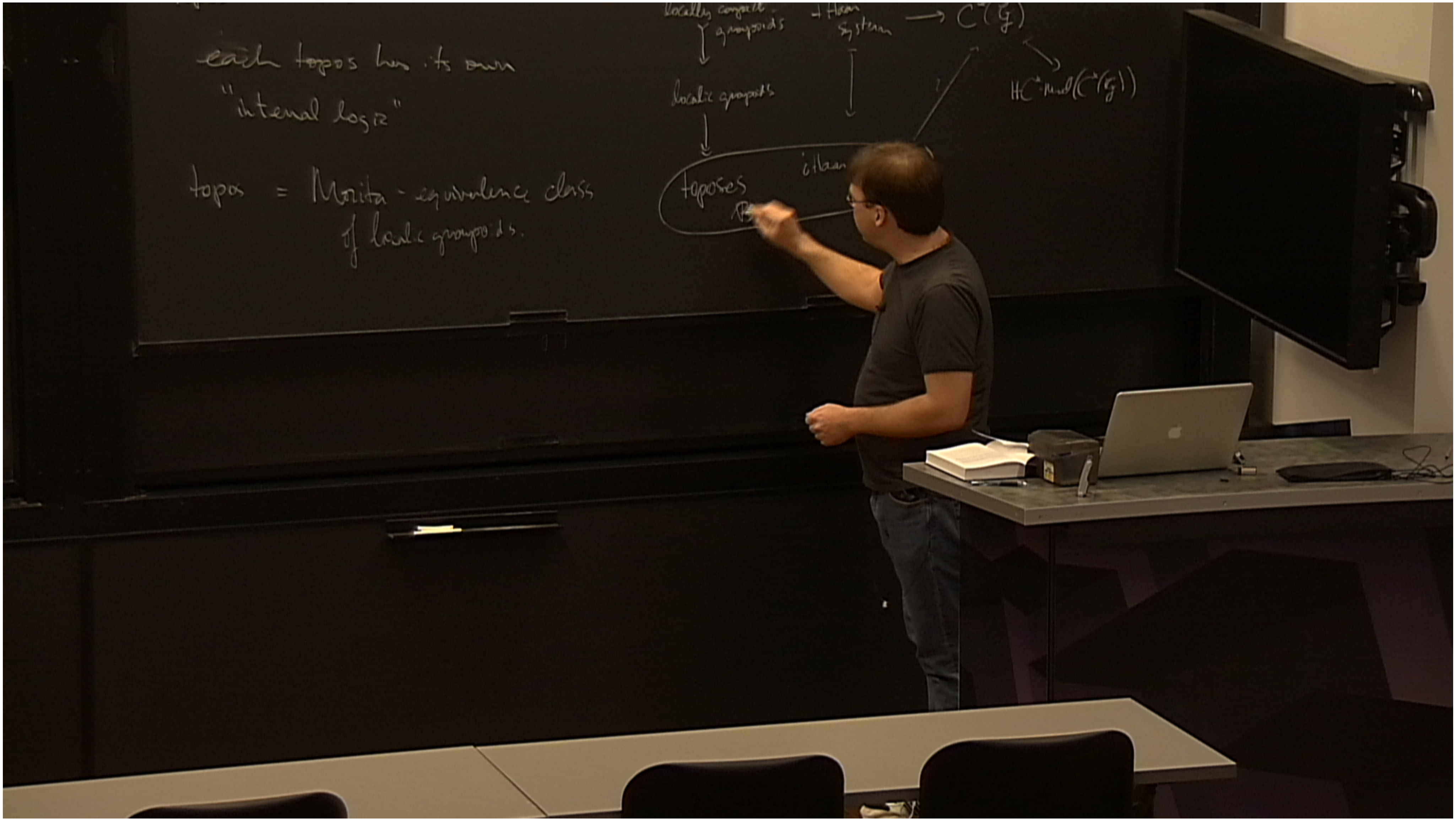
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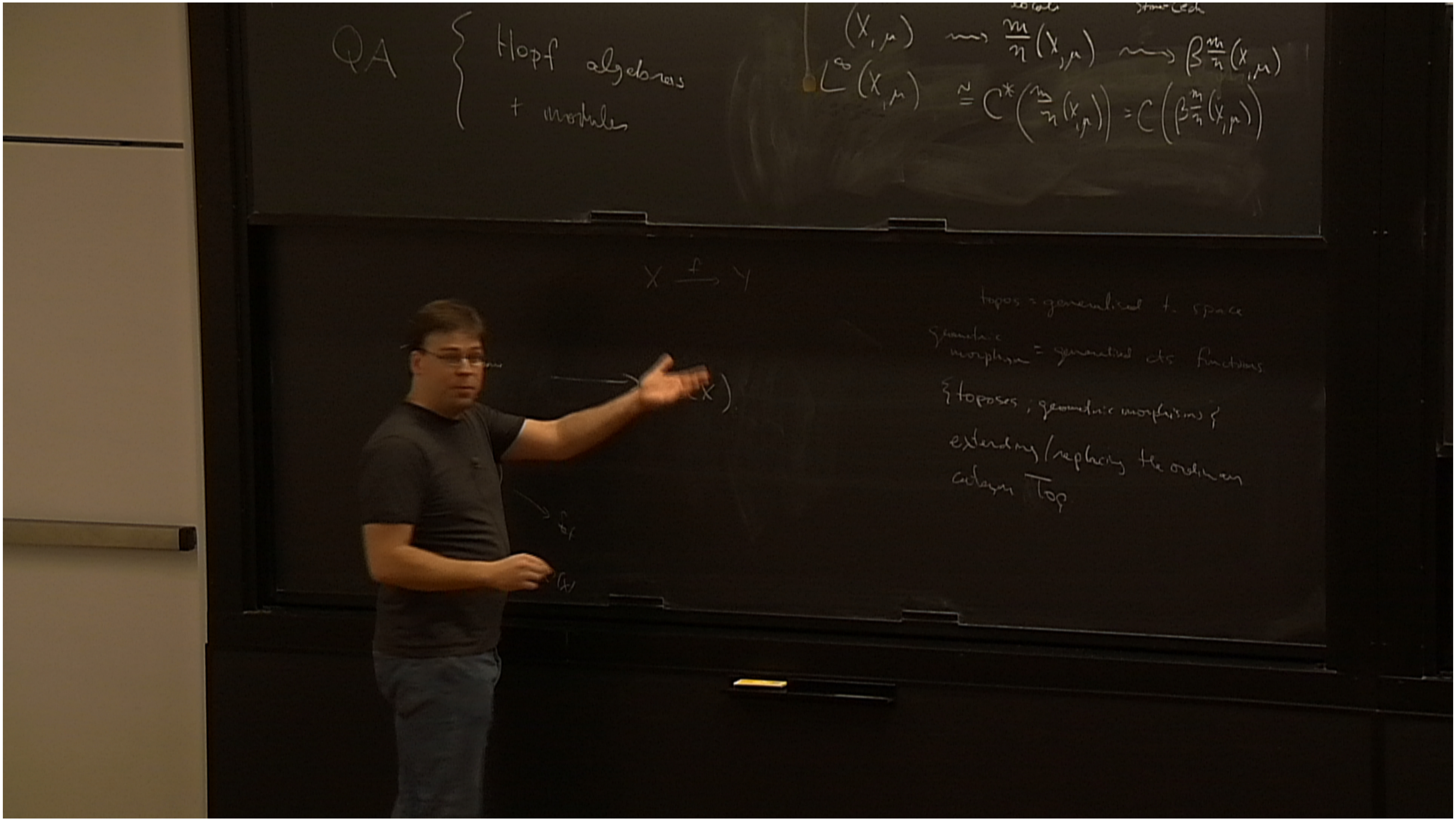
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