Title: Dualities and Dimensional Reduction in Topological Quantum Order and Processing of Quantum Information

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Abstract: Dualities in physics are well known for their conceptual depth and quantitative predictive power in contexts where perturbation theory is unreliable. They are also remarkable for the staggering arrange of physical problems that exploit them, ranging from the study of confinement and unconventional phases in statistical mechanics and field theory to the unification of the string theory landscape.
In this talk I will present a new, completely general approach to dualities that affords a systematic theory of quantum dualities and incorporates classical dualities as well into one unified framework. This new algebraic approach is remarkably successful in extending powerful duality techniques to the context of topologically quantum ordered systems and quantum information processing, and affords a compelling foundation for a general theory of exact dimensional reduction or holographic correspondences. Many systems however display only approximate dimensional reduction, and so I will present general inequalities -some of them based on entanglement- linking quantum systems of different spatial dimensionality. These inequalities provide bounds on the expectation values of observables and correlators that can enforce an effective dimensional reduction. In closing I will discuss some implications for (topological) quantum memories.

Dualities and Dimensional Reduction
in Topological Quantum Order and Processing of Quantum Information

Emilio Cobanera

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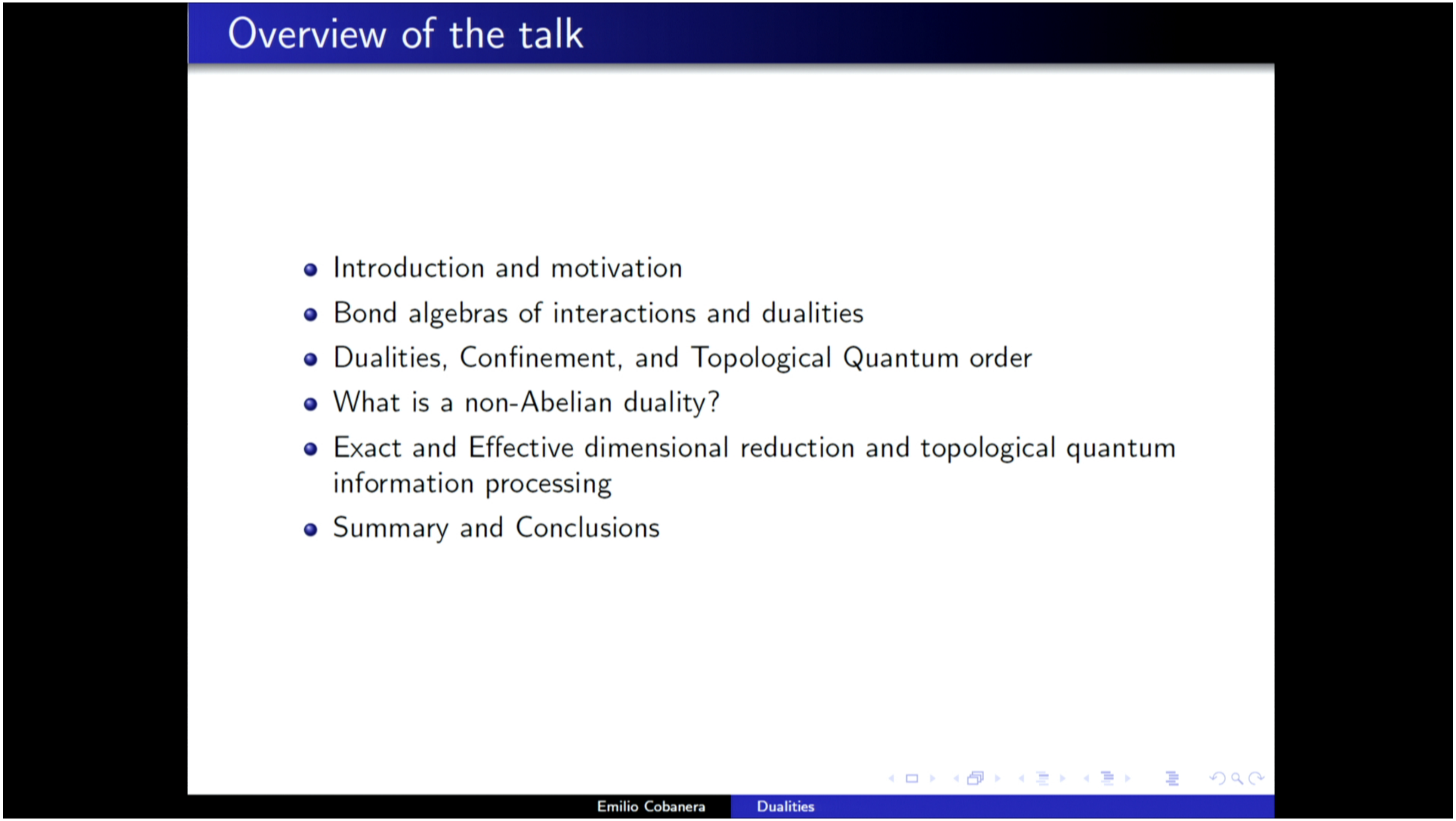
Perimeter Institute, November 15, 2011

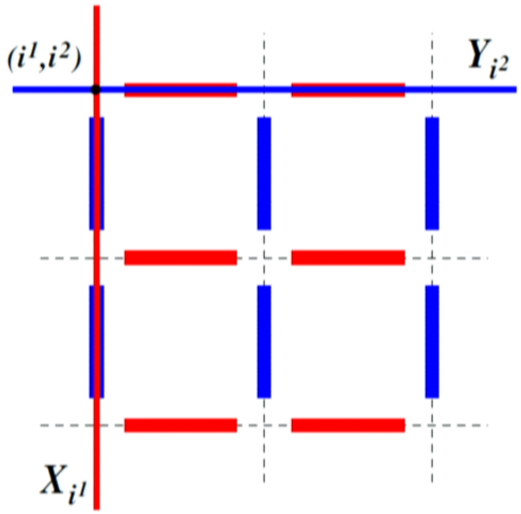

Overview of the talk
Introduction and motivation
Bond algebras of interactions and dualities
Dualities, Confinement, and Topological Quantum order
What is a non-Abelian duality?
Exact and Effective dimensional reduction and topological quantum information processing
Summary and Conclusions
Emilio Cobanera
Dualities

A Symmetry Principle for Dimensional Reduction and TQO

- What is the link between TQO and dimensional reduction?
- Models of TQO tipically display d-**dimensional gauge-like symmetries**, that combined with dimensional-reduction techniques can yield important information about phase diagrams.

$$H_{\mathrm{POC}} = -\sum_{\mathbf{r}} (J_1 \; \sigma^x_{\mathbf{r}} \sigma^x_{\mathbf{r}+\mathbf{e}_1} + J_2 \; \sigma^y_{\mathbf{r}} \sigma^y_{\mathbf{r}+\mathbf{e}_2})$$

$$X_{i^1} = \prod_{i^2} \sigma^x_{i^1, i^2} \qquad Y_{i^2} = \prod_{i^1} \sigma^y_{i^1, i^2}$$



- d gauge-like symmetries have been proposed to be the symmetry principle underlying both TQO and dimensional reduction. (Cobanera et. al. arXiv:1110.2179v1 [cond-mat.stat-mech])

Introduction: Interactions are More Important than Elementary Degrees of Freedom I

"Instead of using the analogy of heat, a fluid, the properties of which are entirely at our disposal, is assumed as the vehicle of mathematical reasoning ... The mathematical ideas obtained from the fluid are then applied to various parts of electrical science." - Maxwell (1855)

Any Periodic Motion	Mass on a Spring
Electrodynamics	Hydrodynamics
Thermodynamics	Black Hole Physics

Why are analogies so pervasive?

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Why are analogies so pervasive?

Because they emphasise the structure of physical laws over elementary degrees of freedom.

Interactions are more important than elementary degrees of freedom III: Bonds

Model Building in Quantum Mechanics

$$\textbf{EDFs} \;\Rightarrow\; \textbf{basic interactions } \{h_\Gamma\}_\Gamma \Rightarrow$$

$$\Rightarrow\; H = \sum_\Gamma \lambda_\Gamma h_\Gamma \;\Rightarrow\; \textbf{Emergent EDFs}$$

The **BONDS** h_Γ are the "atomic constituents" of the Hamiltonian.

Interactions are more important than elementary degrees of freedom III: Bonds

Model Building in Quantum Mechanics

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The **BONDS** h_Γ are the "atomic constituents" of the Hamiltonian.

$$\sigma_i^x, \sigma_i^z \;\Rightarrow\; \{\sigma_i^x, \sigma_i^z\sigma_{i+1}^z\}_i \;\Rightarrow\; H_I = \sum_i [h\sigma_i^x + J\sigma_i^z\sigma_{i+1}^z] \;\Rightarrow\; \textbf{Kinks}$$

Bonds are SPARSE: $[h_\Gamma, h_{\Gamma'}] = 0$ **for most** Γ'

Typically a consecquence of *LOCALITY*

Bond Algebra = Algebra of Interactions I

Our Philosophy: Interactions are more important than elementary degrees of freedom.

What are the EDFs? $\leftrightarrow$ What is the algebra of the EDFs?

1. fermionic or bosonic algebra?
2. $SU(N)$ spins, "Hopf spins"? etc. etc. etc. ...

What are the interactions? $\leftrightarrow$ What is the algebra of interactions?

Definition

The **bond algebra** of $H = \sum_\Gamma \lambda_\Gamma h_\Gamma$ is the von Neumann algebra of operators $\mathcal{A}_H$ generated by the set of bonds $\{h_\Gamma\}_\Gamma$.

(Cobanera et. al., PRL 104, 020402 (2010))

Bond Algebra = Algebra of Interactions II

$$\boxed{H = \sum_{\Gamma} \lambda_\Gamma h_\Gamma}$$

LOCALITY of INTERACTIONS

BOND ALGEBRA:

$$\mathcal{A}_H = \textit{Linear Span}\ \{\mathbb{1}, h_\Gamma, h_\Gamma^\dagger, h_\Gamma h_{\Gamma'}, h_\Gamma^\dagger h_{\Gamma'}, h_{\Gamma'}^\dagger h_\Gamma, h_{\Gamma'}^\dagger h_\Gamma^\dagger, h_\Gamma h_{\Gamma'} h_{\Gamma''}, \cdots\}$$

- CLOSED under *Hermitian conjugation*
- CLOSED under *addition*
- CLOSED under *multiplication*
- $\mathcal{A}_H'' = \mathcal{A}_H$ (bicommutant theorem)

Focus on Interactions = Focus on the Bond Algebra

Bond Algebras

- **Dualities**

  1. Unified theory of quantum and classical dualities
  2. Topological degrees of freedom
  3. Symmetries of phase diagrams
  4. Gauge theories of TQO
  5. Exact dimensional reduction

  Cobanera et. al., PRL 104, 020402 (2010), Adv. Phys. 60, 679 (2011))

- **Fermionization** in more than one dimension.

  Cobanera et. al., Adv. Phys. 60, 679 (2011))

- **Exact solvability** (Lie bond algebras)

  (Nussinov and Ortiz Phys. Rev. B 79, 214440 (2009))

Bond Algebras and Dualities

When are two sets of interactions (sets of bonds) **equivalent**? When they generate **equivalent (isomorphic) bond algebras.**

$\Phi : A_{H_1} \rightarrow A_{H_2}$ one-to-one and onto

$$\Phi(\mathbb{1}) = \mathbb{1}, \qquad \Phi(\mathcal{O}^\dagger) = \Phi(\mathcal{O})^\dagger,$$
$$\Phi(\mathcal{O}_1\mathcal{O}_2) = \Phi(\mathcal{O}_1)\Phi(\mathcal{O}_2), \qquad \Phi(\mathcal{O}_1 + \lambda\mathcal{O}_2) = \Phi(\mathcal{O}_1) + \lambda\Phi(\mathcal{O}_2).$$

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Φ is a duality, and H_1 is **dual** to H_2, if $\Phi(H_1) = H_2$

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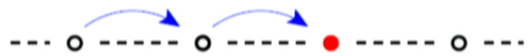
Dualities are unitary equivalences!!!

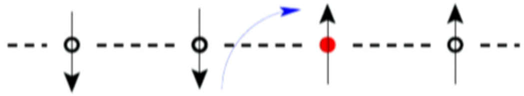
Transmutation of statistics I

$$H_{\mathrm{F}} = \sum_{i=1}^{N-1} \lambda (c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i)$$



Bonds:
$\{c_{i+1}^\dagger c_i | i = 1, \cdots, N-1\}$

$$H_{\mathrm{XY}} = \sum_{i=1}^{N-1} \lambda (\sigma_i^+ \sigma_{i+1}^- + \sigma_{i+1}^+ \sigma_i^-)$$



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Very different EDFs, but isomorphic bond algebras:

$$c_{i+1}^\dagger c_i \xrightarrow{\Phi_{\mathrm{d}}} \sigma_{i+1}^+ \sigma_i^-$$

H_{F} is **dual** (unitarily equivalent!) to H_{XY}

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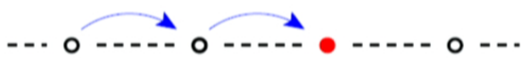

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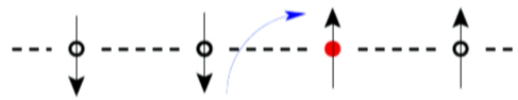

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$$\boxed{c_i \xrightarrow{\Phi_{\mathrm{d}}} ???}$$

Are the elementary Fermions in the bond algebra $\mathcal{A}_F$? NO!

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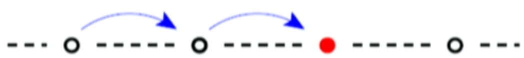

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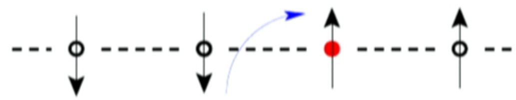

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Very different EDFs, but isomorphic bond algebras:

$c_{i+1}^\dagger c_i \xrightarrow{\Phi_\mathrm{d}} \sigma_{i+1}^+ \sigma_i^-$ $\quad$ H_F is **dual** (unitarily equivalent!) to H_XY

$c_i \xrightarrow{\Phi_\mathrm{d}} ???$ $\quad$ Are the elementary Fermions in the bond algebra $\mathcal{A}_F$? NO!

Transmutation of statistics II: Fermions as dual topological collective modes

1. **Enlarge** $\mathcal{A}_F$ by adding c_1 to its set of generators

   $$\Rightarrow \quad c_2 = [c_1, c_1^\dagger c_2], \ \cdots, \ c_N = [c_{N-1}, c_{N-1}^\dagger c_N] \ \in \mathcal{A}_F$$

2. **Extend** $\Phi_{\rm d}$ so that all algebraic relations are preserved

   $$\Rightarrow \quad c_1 \xrightarrow{\Phi_{\rm d}} \sigma_1^- . \qquad \text{Then, for } i = 2, \cdots, N$$

3. $\Phi_{\rm d}(c_i) = [\Phi_{\rm d}(c_{i-1}), \Phi_{\rm d}(c_{i-1}^\dagger c_i)] = [\Phi_{\rm d}(c_{i-1}), \sigma_{i-1}^+ \sigma_i^-]$

4. $$\boxed{c_i \xrightarrow{\Phi_{\rm d}} \prod_{j=1}^{i-1} (-\sigma_j^z)\, \sigma_i^- \equiv \hat{c}_i}$$ **JW transformation** $=$ **dual fermions**

5. Fermions and Spins $1/2$ **emerge** from the reps of one abstract bond algebra $\mathcal{A} \cong \mathcal{A}_{\rm F} \cong \mathcal{A}_{\rm XY}$ (at the endpoints of strings as $V \to \infty$).

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Dualities and Fermionization

1. EDFs with very different statistics can emerge from reps of one and the same bond algebra of interactions (toy picture of string-nets?).
2. Fermionization is a duality **in any number of dimensions**. JW transformations are derivable from mappings **local** in the bonds.
3. Dualities expand the reach of fermionization techniques. Some models that cannot be fermionized in their original presentation can be fermionized after a suitable duality **(Cobanera et. al., Adv. Phys 60, 679 (2011))**
   - Example: Two-dimensional Ising model in a transverse field

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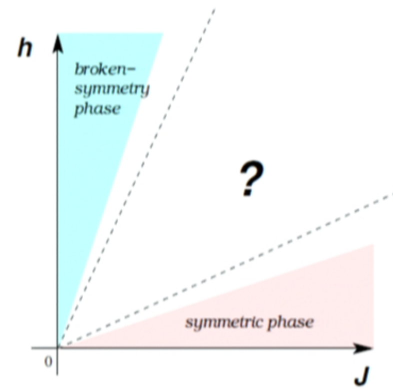
Dualities and TQO I: The $\mathbb{Z}_2$ Higgs field (Ising model)

$$H_{\mathrm{I}}[h,J] = \sum_i \left[h\sigma_i^x + J\sigma_i^z\sigma_{i+1}^z\right]$$

An infinite quantum Ising Chain



Bond	anticommutes with		Bond^2
σ_i^x	$\sigma_{i-1}^z\sigma_i^z$	$\sigma_i^z\sigma_{i+1}^z$	$\mathbb{1}$
$\sigma_i^z\sigma_{i+1}^z$	σ_i^x	σ_{i+1}^x	$\mathbb{1}$

$$\sigma_i^x \xrightarrow{\Phi_d} \sigma_i^z\sigma_{i+1}^z$$
$$\sigma_i^z\sigma_{i+1}^z \xrightarrow{\Phi_d} \sigma_{i+1}^x$$

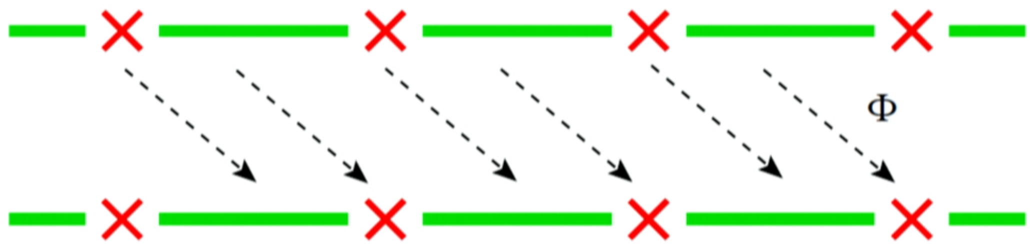
$H_{\mathrm{I}}[h,J]$ is dual (unitarily equivalent!) to $H_{\mathrm{I}}[J,h]$

$\Rightarrow E(J,h) = E(h,J) \Rightarrow \boxed{J = h}$ transition line

Self-duality and kinks

Duality Mapping

$$\sigma_i^x \xrightarrow{\Phi_d} \sigma_i^z \sigma_{i+1}^z \qquad \sigma_i^z \sigma_{i+1}^z \xrightarrow{\Phi_d} \sigma_{i+1}^x$$



A **duality** is a mapping of bonds that preserves the algebra of interactions

$$\mu_i^x \equiv \Phi_d(\sigma_i^x) = \sigma_i^z \sigma_{i+1}^z$$

$$\mu_i^z \equiv \Phi_d(\sigma_i^z) = \Phi_d(\sigma_i^z \sigma_{i+1}^z \times \sigma_{i+1}^z \sigma_{i+2}^z \times \cdots) = \sigma_{i+1}^x \sigma_{i+1}^x \sigma_{i+2}^x \cdots$$

(Fradkin and Susskind, Phys. Rev. D 17 (1978) 2637)

The $\mathbb{Z}_p$ and $U(1)$ Higgs field

- The $\mathbb{Z}_p$ Higgs field in one dimension, also known as the p-clock or vector Potts model, is **self-dual for any** p. *However*, if $p \geq 3$, the model has a **non-Abelian** group of global symmetries. Other insights from bond algebras:
  1. Explicit form of the emergent U(1) symmetry that explains BKT transition for large p
  2. New topological excitation that only exists for $p \geq 5$

- The $U(1)$ Higgs field in one dimension, also known as the quantum planar rotor model, is dual to the solid-on-solid model. **No need to invoke the Villain approximation**.

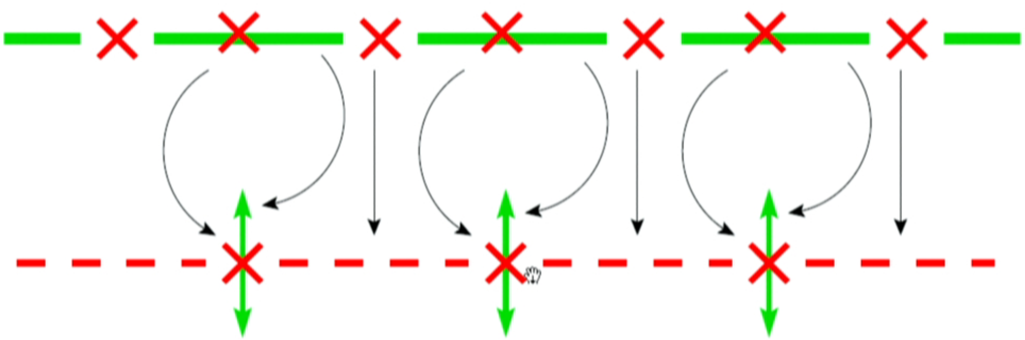
G. Ortiz, E. Cobanera, Z. Nussinov, Nuc. Phys. B 854, 780 (2012)

Dualities and TQO III

$$H^D_{\rm ETC} = \sum_i \left[J_x \sigma^x_i + h_z \sigma^z_i \sigma^z_{(i,1)} \sigma^z_{i+1} + h_x \sigma^x_{(i,1)} \right]$$

Duality Mapping:

$$\sigma^x_i \xrightarrow{\Phi_{\rm d}} \sigma^x_{(i-1,1)} \sigma^x_{(i,1)} \equiv A_i, \quad \sigma^z_i \sigma^z_{(i,1)} \sigma^z_{i+1} \xrightarrow{\Phi_{\rm d}} \sigma^z_{(i,1)} \quad \sigma^x_{(i,1)} \xrightarrow{\Phi_{\rm d}} \sigma^x_{(i,1)}$$



Have we lost degrees of freedom???

Dualities and TQO II

$$H_{\rm ETC} = \sum_i \left[h_z \sigma^z_{(i,1)} + h_x \sigma^x_{(i,1)} + J_x \sigma^x_{(i,1)} \sigma^x_{(i+1,1)} \right] \qquad \sigma^x_{(i,1)} \sigma^x_{(i+1,1)} \equiv A_{i+1}$$

Bond	anticommutes with			Bond^2
$\sigma^z_{(i,1)}$	$\sigma^x_{(i,1)}$	$\sigma^x_{(i-1,1)}\sigma^x_{(i,1)}$	$\sigma^x_{(i,1)}\sigma^x_{(i+1,1)}$	$\mathbb{1}$
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(Tupitsyn et. al., Phy. Rev. B 82, 8 (2012); two dimensions)

Dualities and TQO II

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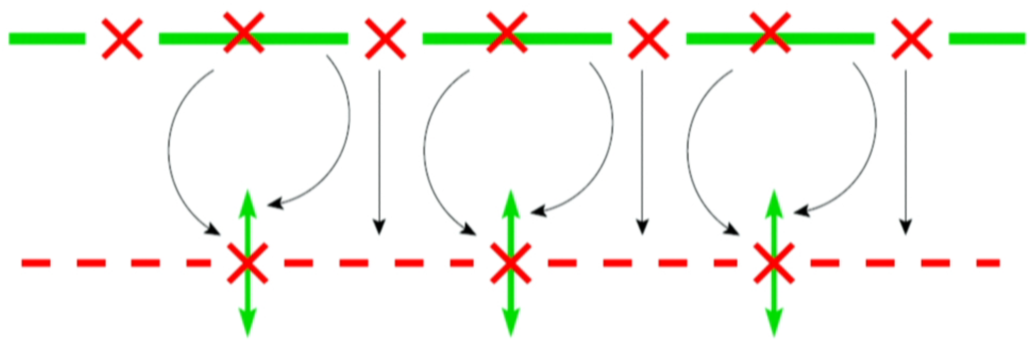

Have we lost degrees of freedom???

Dualities and Gauge Symmetries I

$$H^D_{\text{ETC}} = \sum_i \left[J_x \sigma^x_i + h_z \sigma^z_i \sigma^z_{(i,1)} \sigma^z_{i+1} + h_x \sigma^x_{(i,1)}\right]$$ $\mathbb{Z}_2$ Higgs model

(Fradkin and Shenker, Phys. Rev. D 19, 3682 (1979))

Gauge Symmetries: $\sigma^x_i A_i = \sigma^x_{(i-1,1)} \sigma^x_i \sigma^x_{(i,1)}$

A state ρ is physical if and only if $[\rho, \sigma^x_i A_i] = 0$

NOTICE: $\boxed{\sigma^x_i A_i \xrightarrow{\Phi_{\text{d}}} A_i A_i = \mathbb{1}}$

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Dualities and Gauge Symmetries I

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- Dualities and Gauge Symmetries
- Dualities and TQO

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Dualities and TQO III

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Duality Mapping:

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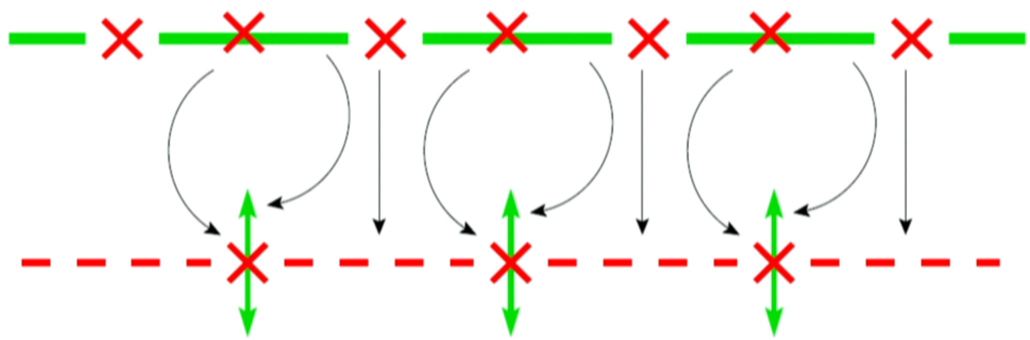
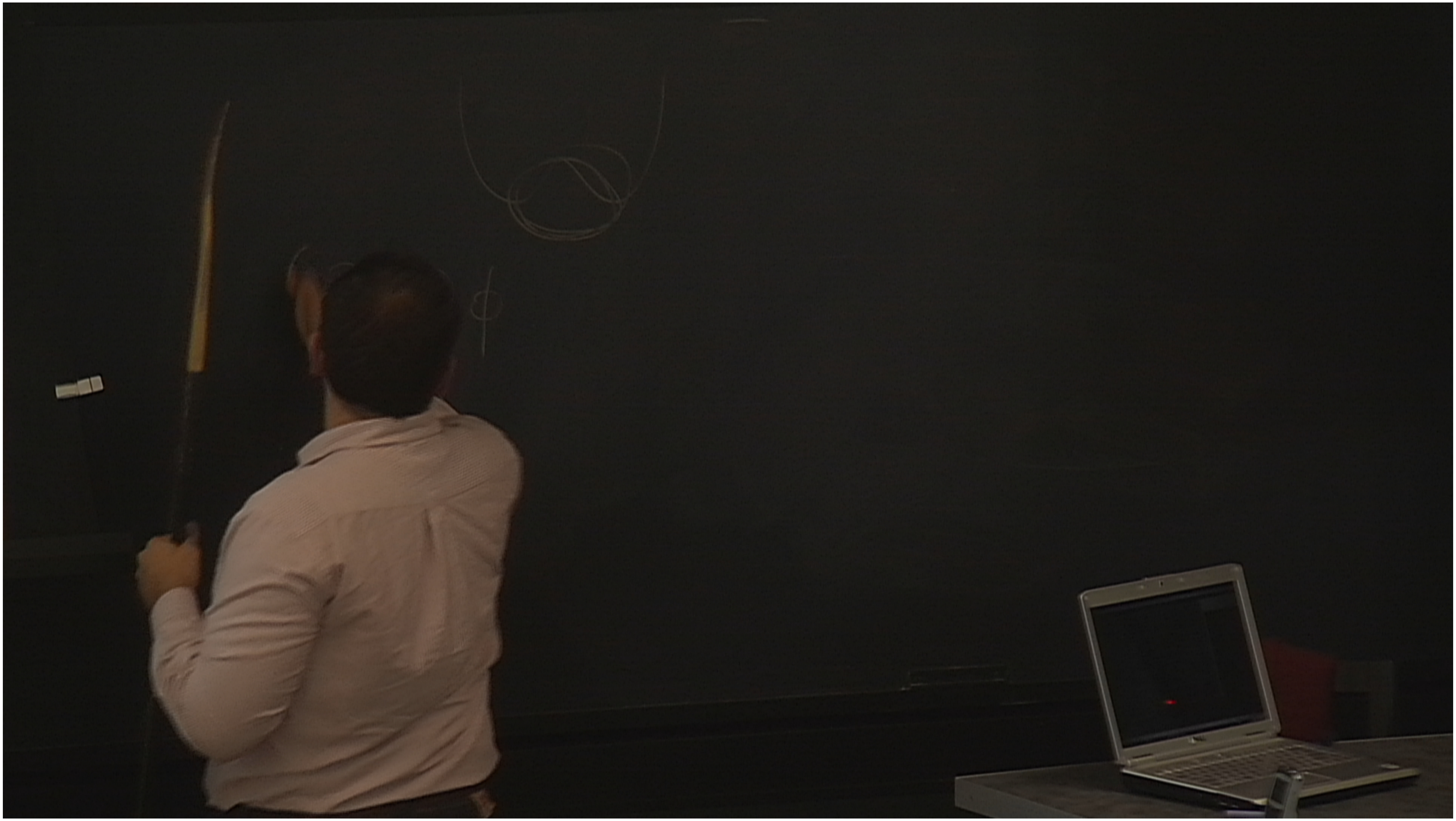


Have we lost degrees of freedom???



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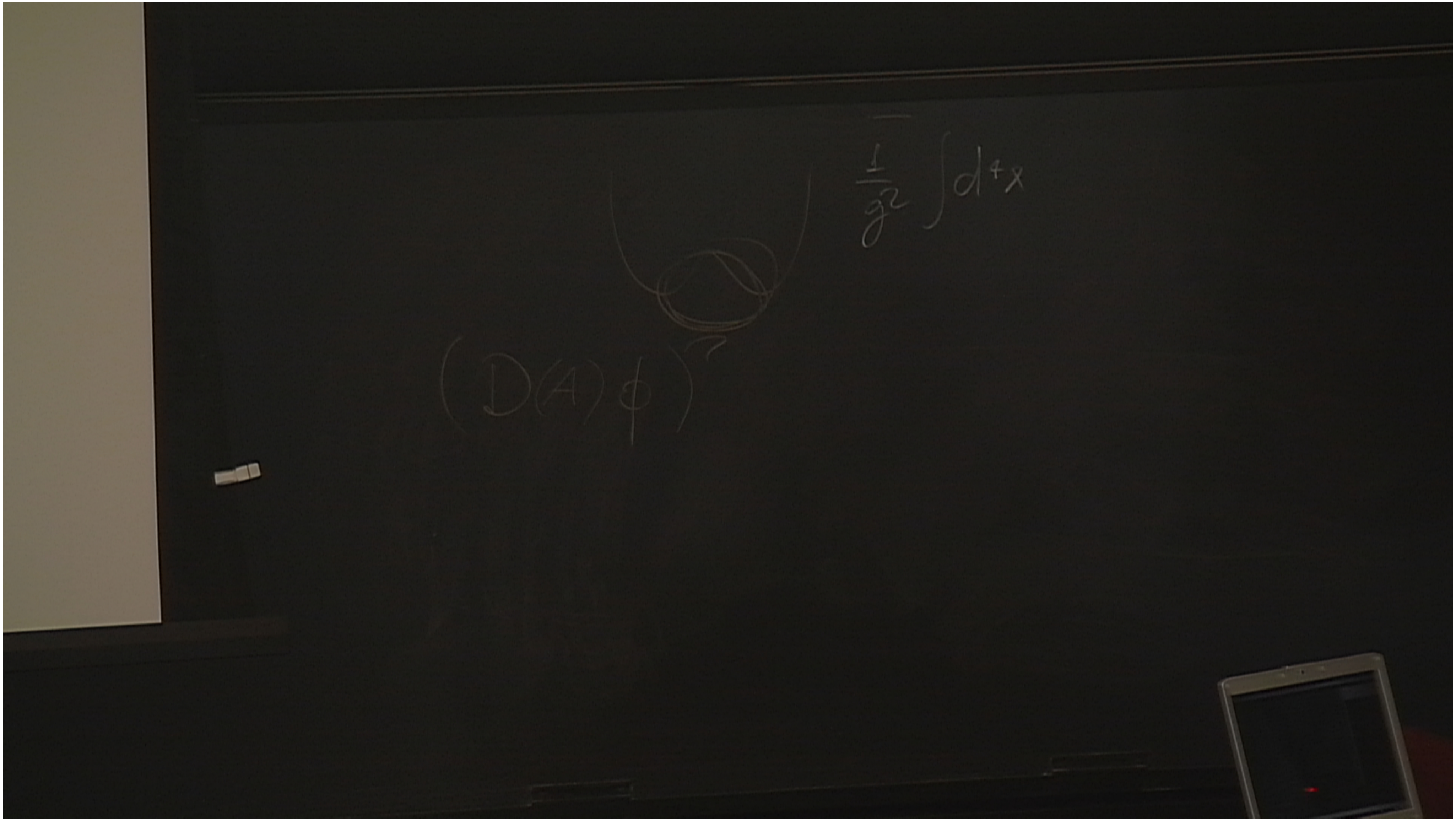
1/g^2 ∫d^4x
(D(A)φ)^2

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- the "matrix" form of $\Phi_{\rm d}$ is not given by a unitary transformation, but by a **partial isometry**. Partial isometries are mappings *between spaces of differen dimensionality* that either maps a state to the null state or preserves its norm.

Topological quantum order in the Higgs model II: Generalizations

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Emilio Cobanera Dualities

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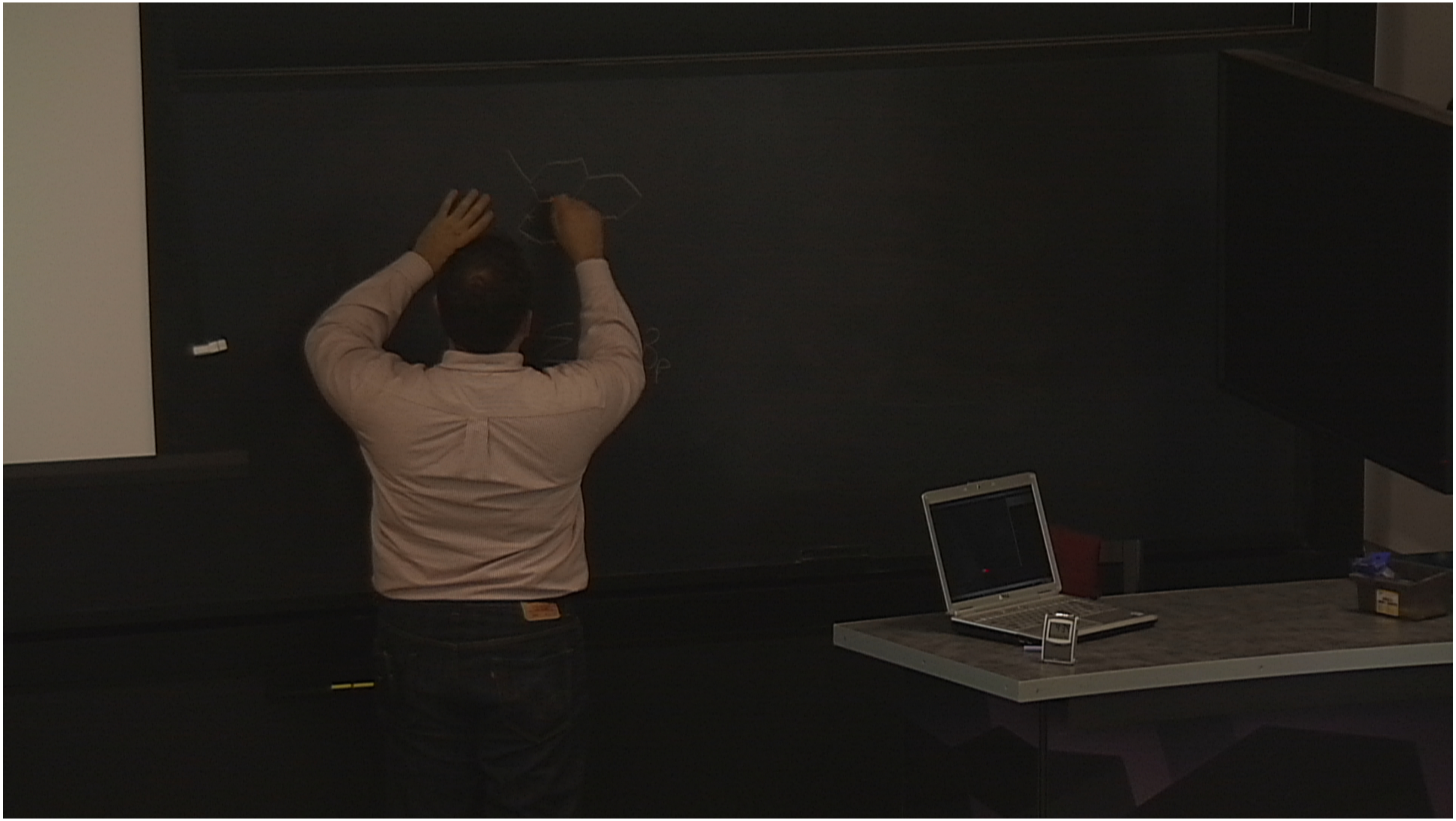
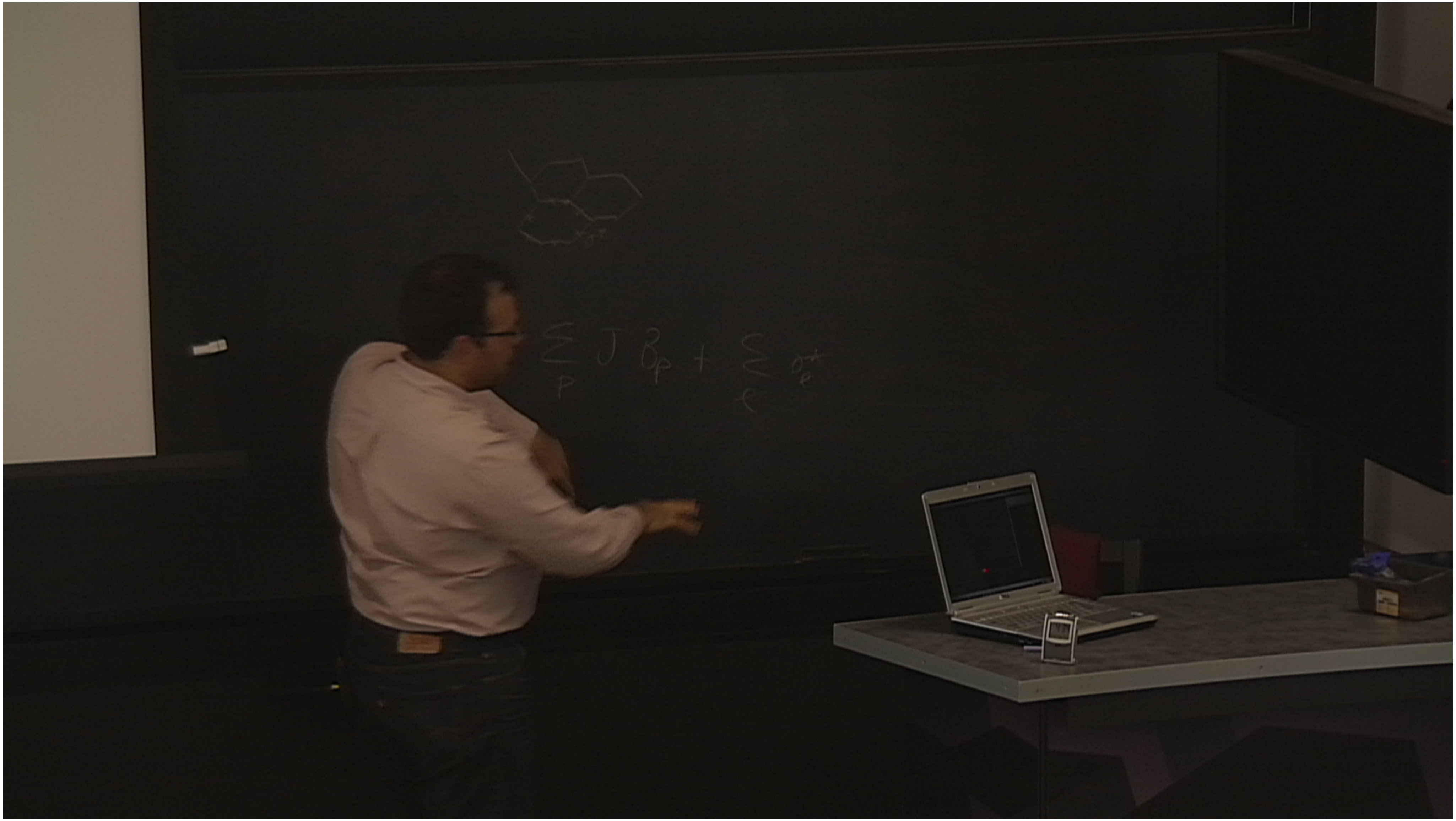
- However, we have learned already that **the character of a duality is not determined by the group of symmetries of the model**.
- We have found interesting dualities for models displaying non-Abelian symmetry groups of any complexity (E. Cobanera et. al., Adv. Phys. 60 (2011), 679; Nuc. Phys. B 854 (2012), 780), including **new dualitie for the $S = 1/2$ Heisenberg model** *in any number of dimensions*.
- We do no think these should be considered non-Abelian dualities. Presumably, a duality for an $SU(2)$ gauge field should be truly non-Abelian, but the problem requires a better characterization.





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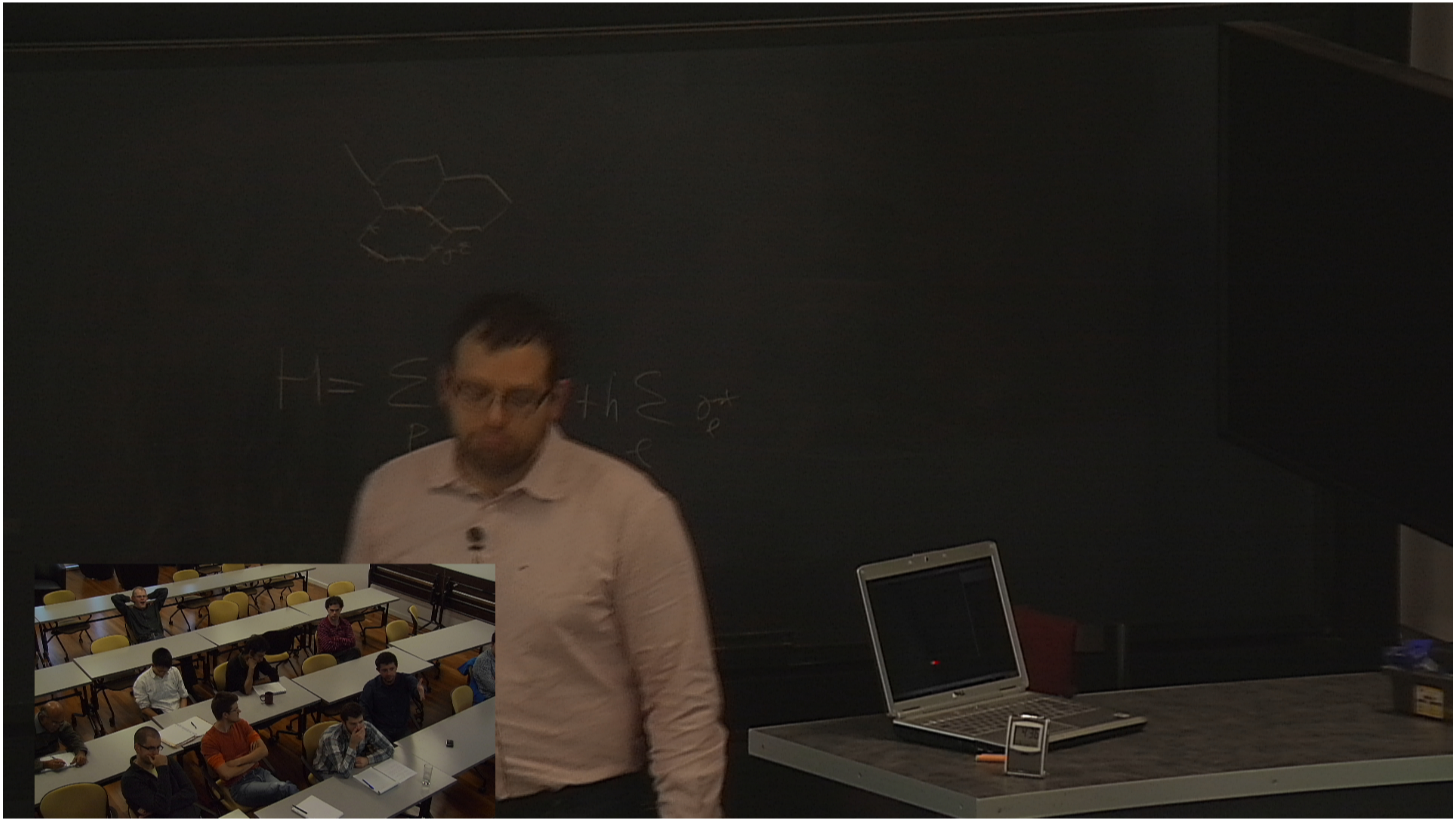
- **The Big Challenge:** What if G is **non-Abelian**?



Holographies and dualities

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  1. **Restricted dynamics from conservation laws** (sliding dynamics)
  2. **Restricted dynamics from special couplings and interactions** (layered systems)
  3. **Kaluza-Klein compactification** (string theory)
  4. **Gauge-gravity dualities** (AdS-CFT correspondence)

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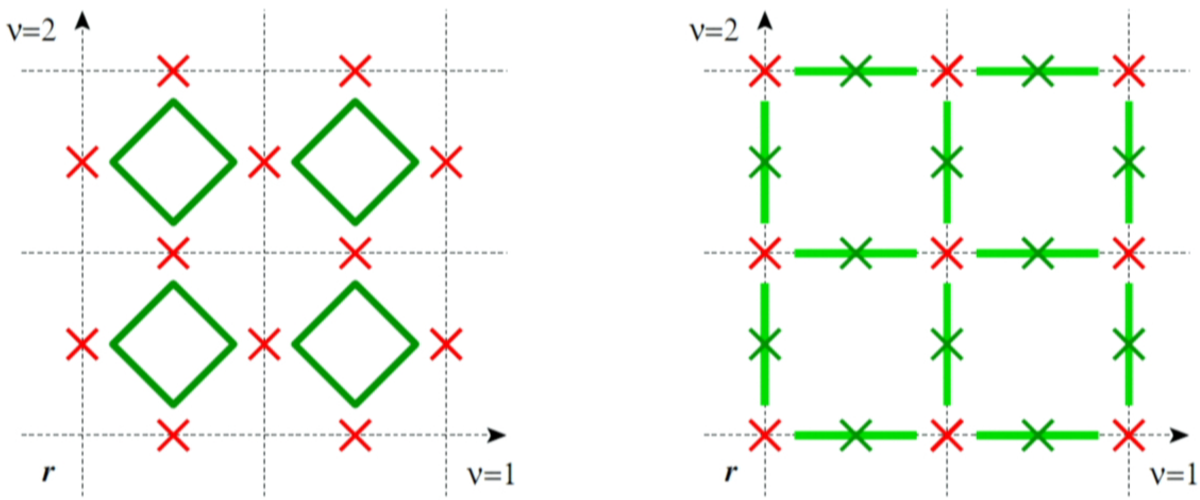
Confinement and topological quantum order: The new face of an old phase diagram

The $\mathbb{Z}_2$ Higgs model $\left(B_{(\mathbf{r},3)} \equiv \sigma^z_{(\mathbf{r},1)}\sigma^z_{(\mathbf{r}+\mathbf{e_1},2)}\sigma^z_{(\mathbf{r}+\mathbf{e_2},1)}\sigma^z_{(\mathbf{r},2)}\right)$:

$$H_{\rm AH} = \sum_{\mathbf{r}} \left(J_x\sigma^x_{\mathbf{r}} + J_z B_{(\mathbf{r},3)}\right) + \sum_{\mathbf{r}}\sum_{\nu=1,2}\left(h_z\sigma^z_{\mathbf{r}}\sigma^z_{(\mathbf{r},\nu)}\sigma^z_{\mathbf{r}+\mathbf{e}_\nu} + h_x\sigma^x_{(\mathbf{r},\nu)}\right)$$

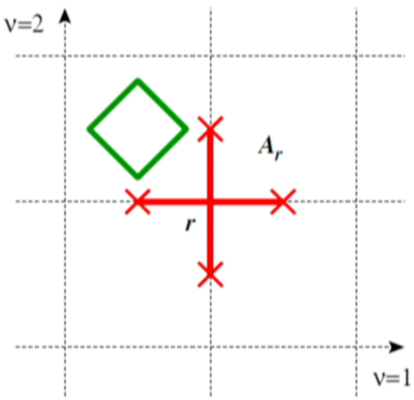


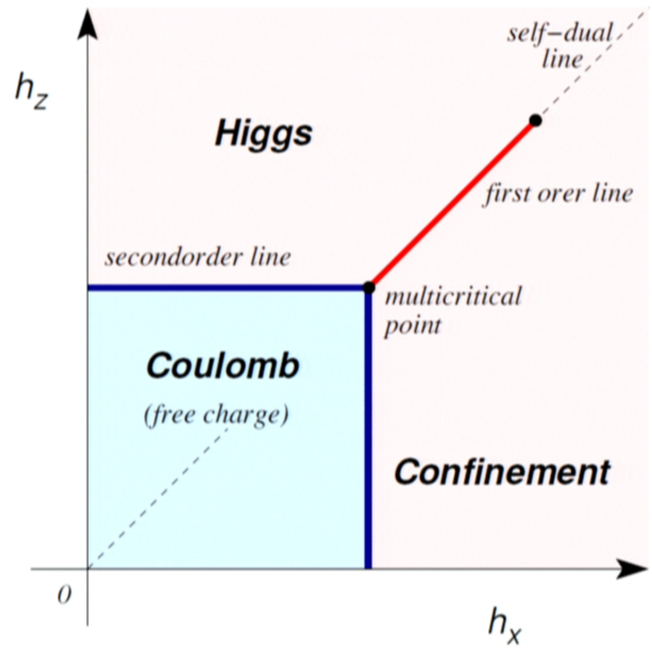
Symmetries and phase diagram of the $\mathbb{Z}_2$ Higgs model

The gauge symmetries are
$G_{\mathbf{r}} \equiv \sigma^x_r A_{\mathbf{r}}$, with

$$A_{\mathbf{r}} \equiv \sigma^x_{(\mathbf{r},1)} \sigma^x_{(\mathbf{r},2)} \sigma^x_{(\mathbf{r}-\mathbf{e}_1,1)} \sigma^x_{(\mathbf{r}-\mathbf{e}_2,2)}$$

the *star operator*.





There can be no spontaneous breakdown of gauge symmetries (Elitzur's theorem). But we can try to get rid of them to have easier access to the model's phase diagram. Dualities!