Title: On Four Point Functions of 1/2 BPS operators in the AdS/CFT Correspondence

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Abstract: In this talk I will provide evidence supporting the Dolan/Nirschl/Osborn conjecture for the precise form of the amplitude of four-point functions of 1/2-BPS operators in N=4 SYM theory at strong coupling and in the large N limit. I will also discuss the methods that allowed the evaluation of amplitudes involving operators of arbitrary conformal dimension.

In a **conformal field theory** the full **dynamical information** is contained in

the spectrum of **conformal weights**

the coefficients of three point functions of primary operators **(OPE coefficients)**

$\mathcal{N} = 4$

Integrability

Quantitative understanding of spectrum of anomalous dimensions. Unclear how will it enter computations of 3p coefficients

The **simplest** operators to discuss are the **chiral primary operators**.

$$[0, p, 0] \qquad SU(4) \qquad \Delta = p$$

They are related to **Kaluza-Klein** modes in the expansion of **supergravity** fields on the sphere.

Two- and Three-point functions are fixed by conformal symmetry, apart from an **overall constant**.

Normalisation constant computed using SUGRA is identical to **large N free field** theory result.

Four-point functions have a non-trivial dependence on the coupling, though they are constrained by

Non-renormalisation theorems
Ward Identities

All cases computed via SUGRA so far can be given an **OPE interpretation**.

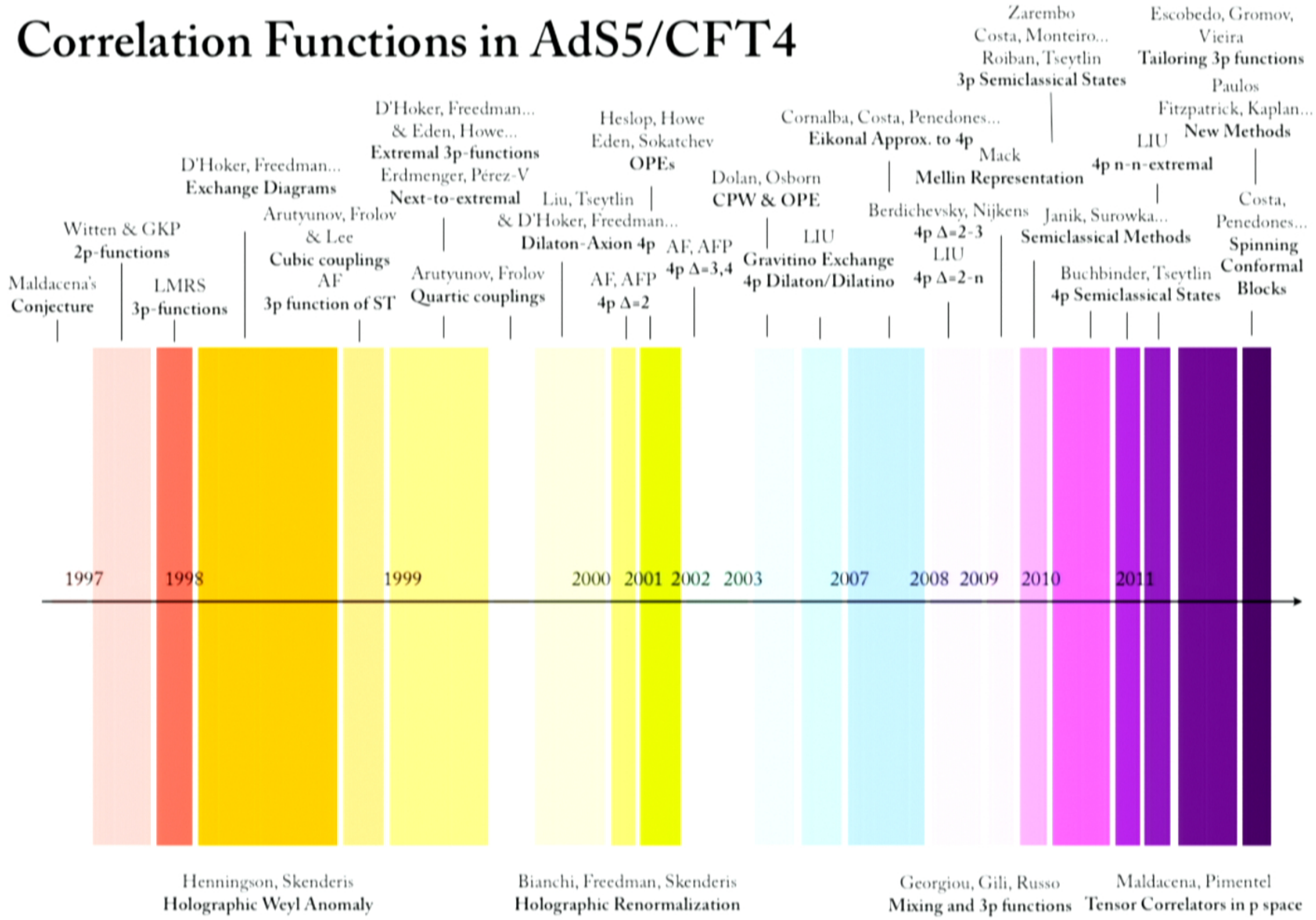
> All the **power–singular terms** in the direct channel limit **exactly match** the corresponding contributions to the OPE of the operator dual to the exchanged bulk field and of its conformal descendents - *e.g.* graviton / stress-energy tensor

But calculations are in general **quite cumbersome** to perform.

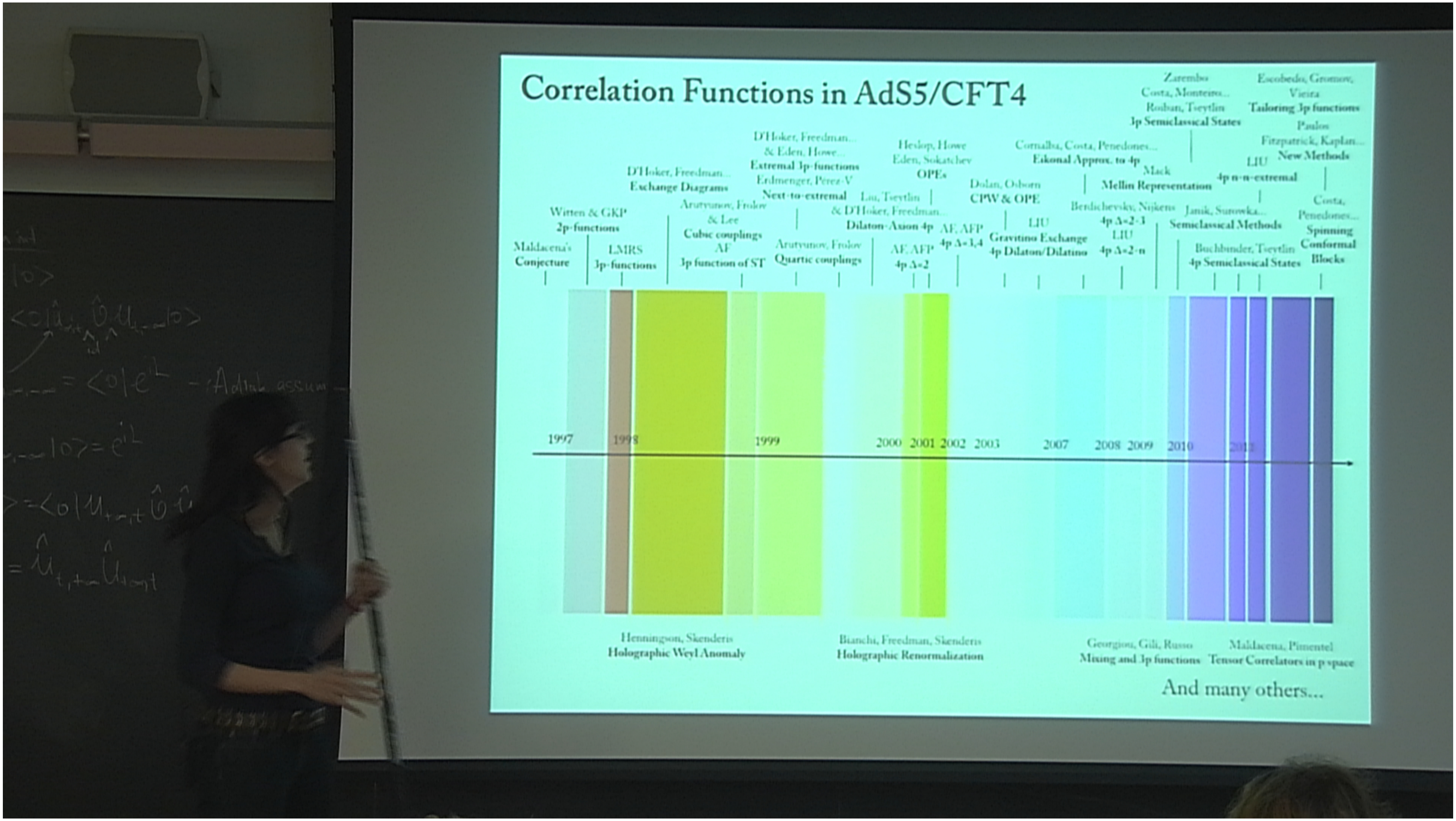
and the community moved on to greener pastures.

But let me give you an overview of the story of **correlation functions in the AdS/CFT correspondence so far**, and because **infographics and data visualization** are all the rage now...

Correlation Functions in AdS5/CFT4



And many others...


Correlation Functions in AdS5/CFT4
Maldacena's Conjecture
Witten & GKP 2p-functions
LMRS 3p-functions
D'Hoker, Freedman... Exchange Diagrams
Arutyunov, Frolov & Lee Cubic couplings
AF 3p function of ST
D'Hoker, Freedman... & Eden, Howe... Extremal 3p-functions
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Escobedo, Gromov, Vieira Tailoring 3p functions
Paulos Fitzpatrick, Kaplan... New Methods
Costa, Penedones... Spinning Conformal Blocks
1997 1998 1999 2000 2001 2002 2003 2007 2008 2009 2010 2011
Henningson, Skenderis Holographic Weyl Anomaly
Bianchi, Freedman, Skenderis Holographic Renormalization
Georgiou, Gili, Russo Mixing and 3p functions
Maldacena, Pimentel Tensor Correlators in p-space
And many others...

But **what goes around comes back around.**

More recently, there has been a **renewed interest** in computing correlation functions.

Going **beyond** operators dual to supergravity fields

Semiclassical Methods

Janik, et. al.

Introduced formalism for using **semiclassical methods** to evaluate correlation functions of operators dual to **classical spinning strings**

Zarembo, Costa, et.al. Roiban and Tseytlin

Developed methods to evaluate **three-point functions** for the case in which two operators are dual to **semiclassical states** and the other is dual to a **SUGRA mode**.

Buchbinder and Tseytlin

Evaluated **four-point functions** for the case in which two operators are dual to **semiclassical states** and the others are dual to **SUGRA modes**.

Finding a **new formalism** for evaluating these quantities.

Mellin representation

Mach. Penedones

Change of basis from coordinate space by **Mellin transform** leads to simplifications.
"making the physics of CFT correlation functions simple and transparent"

Paulos. Fitzpatrick, Kaplan, et.al.

Gave simple **diagrammatic rules** for the construction of Mellin amplitudes corresponding to **tree-level Witten diagrams**

Possibility of evaluating **higher point diagrams** and hence higher point correlation functions

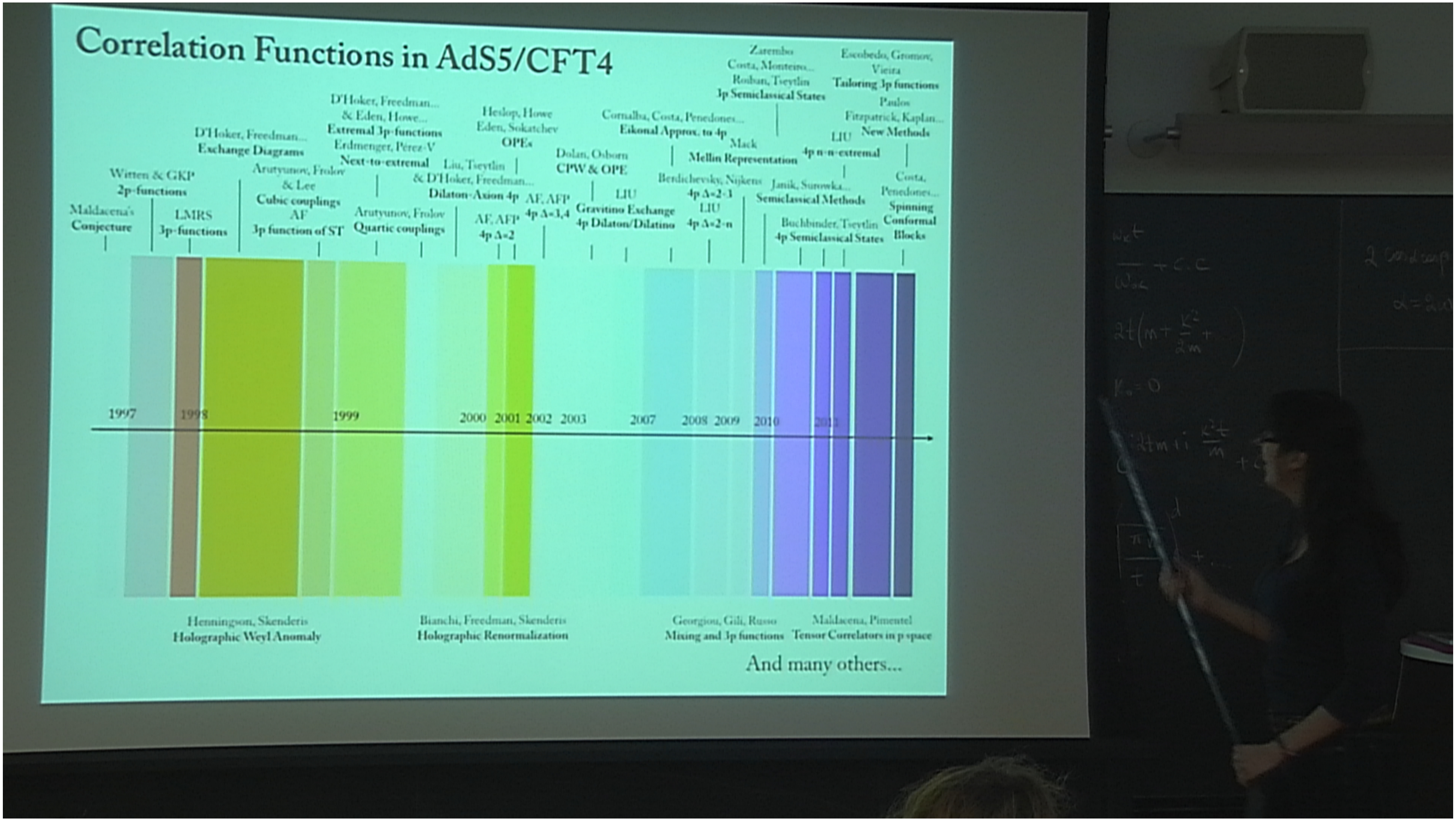
Unfortunately, computing correlation functions holographically is more than evaluating Witten diagrams and requires the determination of the **effective lagrangian** (but for cases computed in AdS supergravity, it is possible to determine the Mellin transform)

3

Trying to use the power of **integrability** for going beyond 2p-functions.

Escobedo, Gromov, Sever, Vieira

Computation of planar three-point functions - structure constants - using the **underlying exactly solvable structures** of these theories.



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Independently of how the previous efforts might change the playing field, there are still **questions to be answered** when referring to specific processes in AdS supergravity.

In particular, there was an outstanding conjecture...

Dolan, Nirschl and Osborn

Gave an **expression for the four-point function** of 1/2-BPS operators belonging to $[0,p,0]$ representation of $SU(4)$ in $\mathcal{N}$=4 superconformal theories **at strong coupling and large *N*** with $p << N$

Observation

Large *N* SUGRA AdS correlation functions reduce to a sum of contact interactions

$$D_{\Delta_1,\cdots,\Delta_n}(\vec{x}_1,\cdots,\vec{x}_n) = \frac{1}{\pi^{d/2}}\int \frac{d^{d+1}z}{z_0^{d+1}} \prod_{i=1}^{n}\left(\frac{z_0}{z_0^2+(\vec{z}-\vec{x})^2}\right)^{\Delta_i}$$

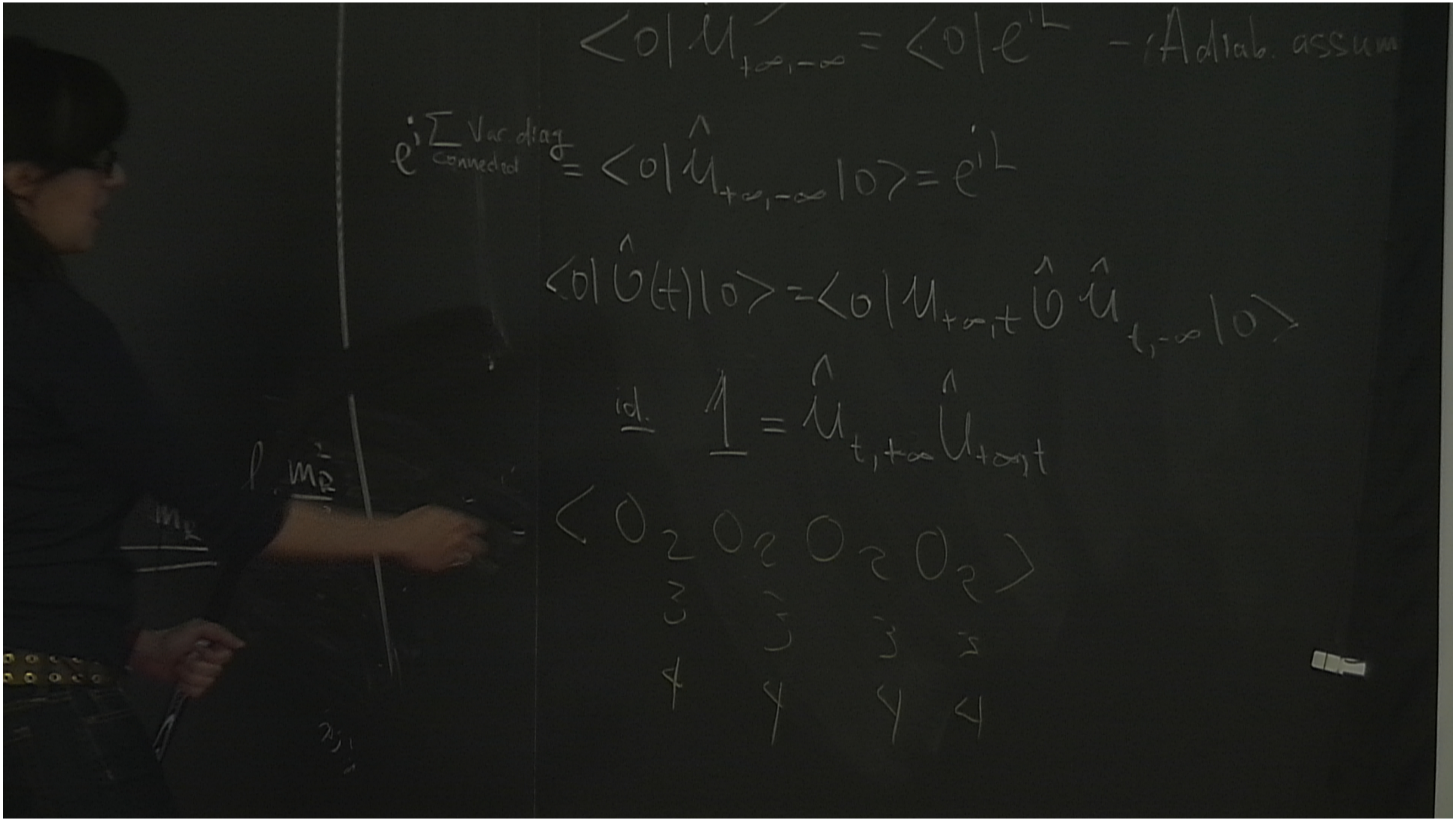
$D_{\Delta_1\Delta_2\Delta_3\Delta_4} =$

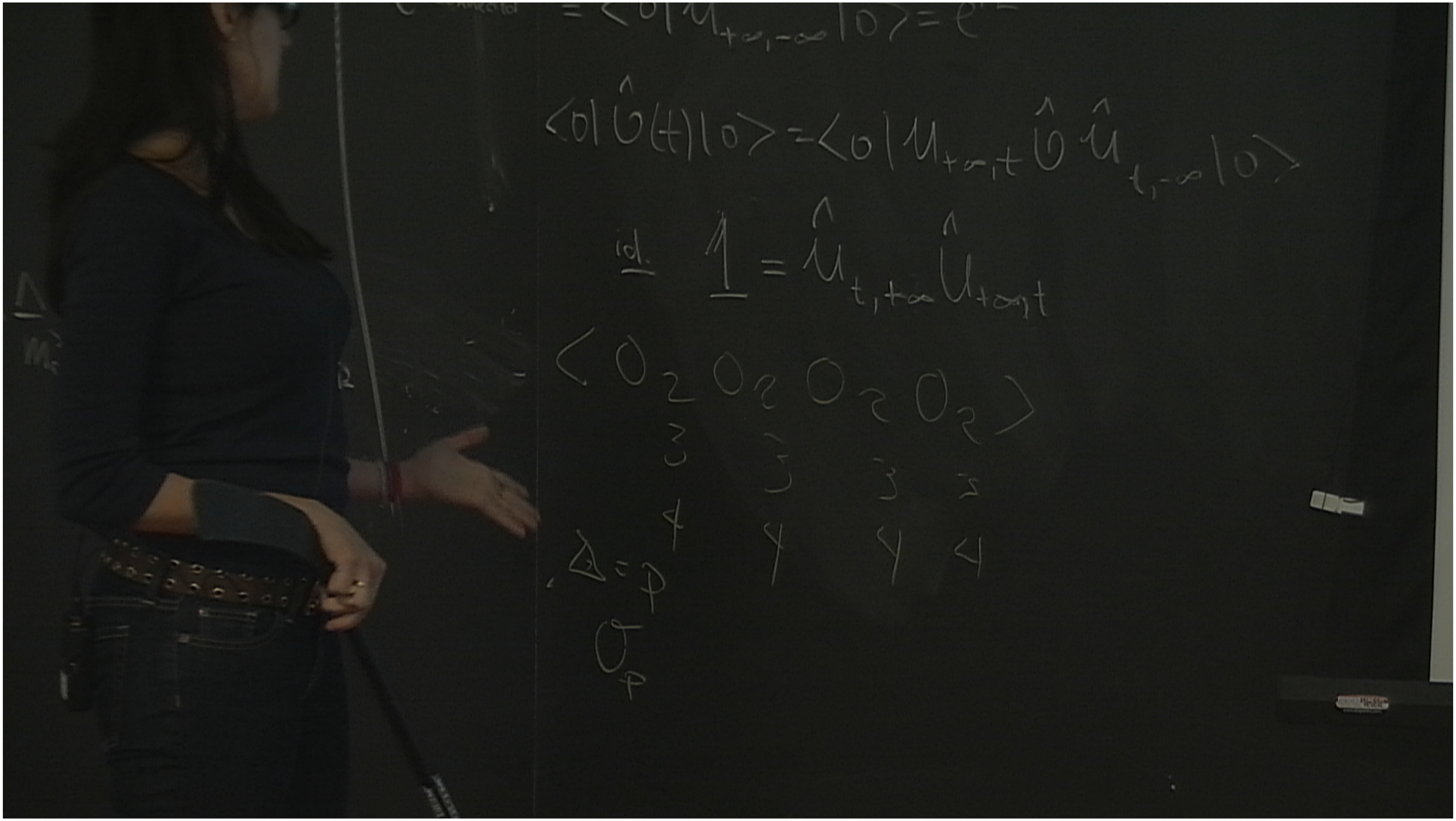
For n=3, the integral reduces to the **standard form** for the three-point function.

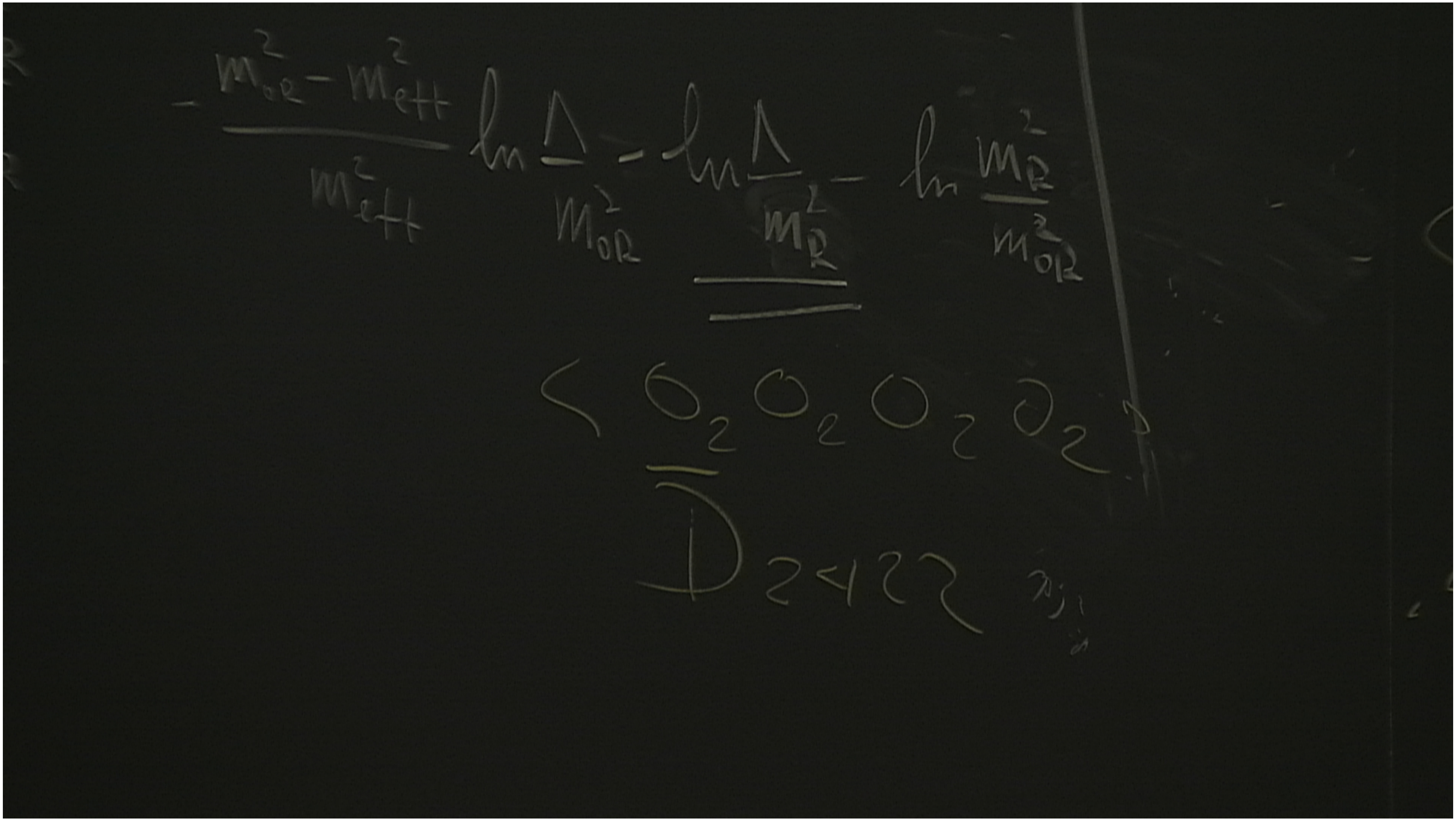
For n=4, the integral can be expressed in terms of yet another function, independent of d, of two conformal invariants u, and v. Namely, $\bar{D}_{\Delta_1,\Delta_2,\Delta_3,\Delta_4}(u,v)$

$$D_{\Delta_1,\Delta_2,\Delta_3,\Delta_4}(\vec{x}_1,\vec{x}_2,\vec{x}_3,\vec{x}_4) = \frac{\pi^{\frac{d}{2}}}{2}\frac{\Gamma(\Sigma-\frac{d}{2})}{\prod_{i=1}^{4}\Gamma(\Delta_i)}\frac{(x_{14}^2)^{\Sigma-\Delta_1-\Delta_4}(x_{34}^2)^{\Sigma-\Delta_3-\Delta_4}}{(x_{13}^2)^{\Sigma-\Delta_4}(x_{24}^2)^{\Sigma-\Delta_2}}\bar{D}_{\Delta_1,\Delta_2,\Delta_3,\Delta_4}(u,v)$$

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$$


⟨0| U_{+∞,−∞} = ⟨0| e^{iL} − Adiab. assum
e^{i Σ Var. diag connected} = ⟨0| Û_{+∞,−∞} |0⟩ = e^{iL}
⟨0| Ô(t) |0⟩ = ⟨0| U_{+∞,t} Ô Û_{t,−∞} |0⟩
id. 1 = Û_{t,+∞} Û_{+∞,t}
⟨O_2 O_2 O_2 O_2⟩


$\langle 0|\hat{O}(t)|0\rangle = \langle 0|U_{+\infty,t}\,\hat{O}\,\hat{U}_{t,-\infty}|0\rangle$
id. $\mathbb{1} = \hat{U}_{t,+\infty}\hat{U}_{+\infty,t}$
$\langle O_2\, O_2\, O_2\, O_2 \rangle$
3 3 3 3
4 4 4 4
$\Delta = p$
O_p



All known SUGRA results reduced to **a sum of $\overline{\mathrm{D}}$-functions,** of the form

$$\bar{D}_{i,p+2,j,k}(u,v) \qquad \text{for} \qquad i,j,l \leq p$$

These have a **series expansion** in powers of u, $1-v$, in which terms of the form $\log u$ are also present. Log terms are interpreted as arising from the leading term in the $1/N$ expansion of the **anomalous dimensions of long multiplets**.

Long Multiplets $\mathcal{A}^{\Delta}_{nm,l}$

Superconformal symmetry

Anomalous dimensions arise only for long multiplets where the lowest dimension operator belongs to $[n-m, 2m, n-m]$, scale Δ and spin l.

OPE analysis / Ward identities demand that long multiplets may **only be present** for $m \leq n \leq p-2$, and **anomalous dimensions** are obtained only for multiplets with twist $\Delta - l \geq 2p$.

3

Unitarity

Unitarity bound in superconformal representation theory only requires $\Delta - l \geq 2n+2$

So long multiplets with twist $\Delta - l < 2p$ **must be absent** from the OPE of two CPOs in the large N limit.

Short multiplets $\mathcal{B}_{nm}$ and Semi-Short multiplets $\mathcal{C}_{nm,l}$

Contributions **without anomalous dimensions** correspond to operators in short and semi-short multiplets, w/ lowest dimension operator belonging to $[n-m,2m,n-m]$, scale $\Delta=n$ and spin $l=0$ or $\Delta-l=n+2$ and spin l.

$$\mathcal{A}^{2n+l+2}_{nm,l} \sim \mathcal{C}_{nm,l} \oplus \mathcal{C}_{n+1m,l-1} \oplus \cdots \qquad 0 \leq m \leq n$$

$$C_{nm,-1} \sim \mathcal{B}_{n+1m} \qquad n > m$$

(Decomposition of long multiplets at unitarity threshold)

Only **such short or semi-short multiplets** contribute to the OPE for twist $\Delta-l<2p$. This is, **long multiplets** which decompose in semi-short multiplets are the only ones contributing to the OPE.

4 Observation from Sugra results for $p=2$

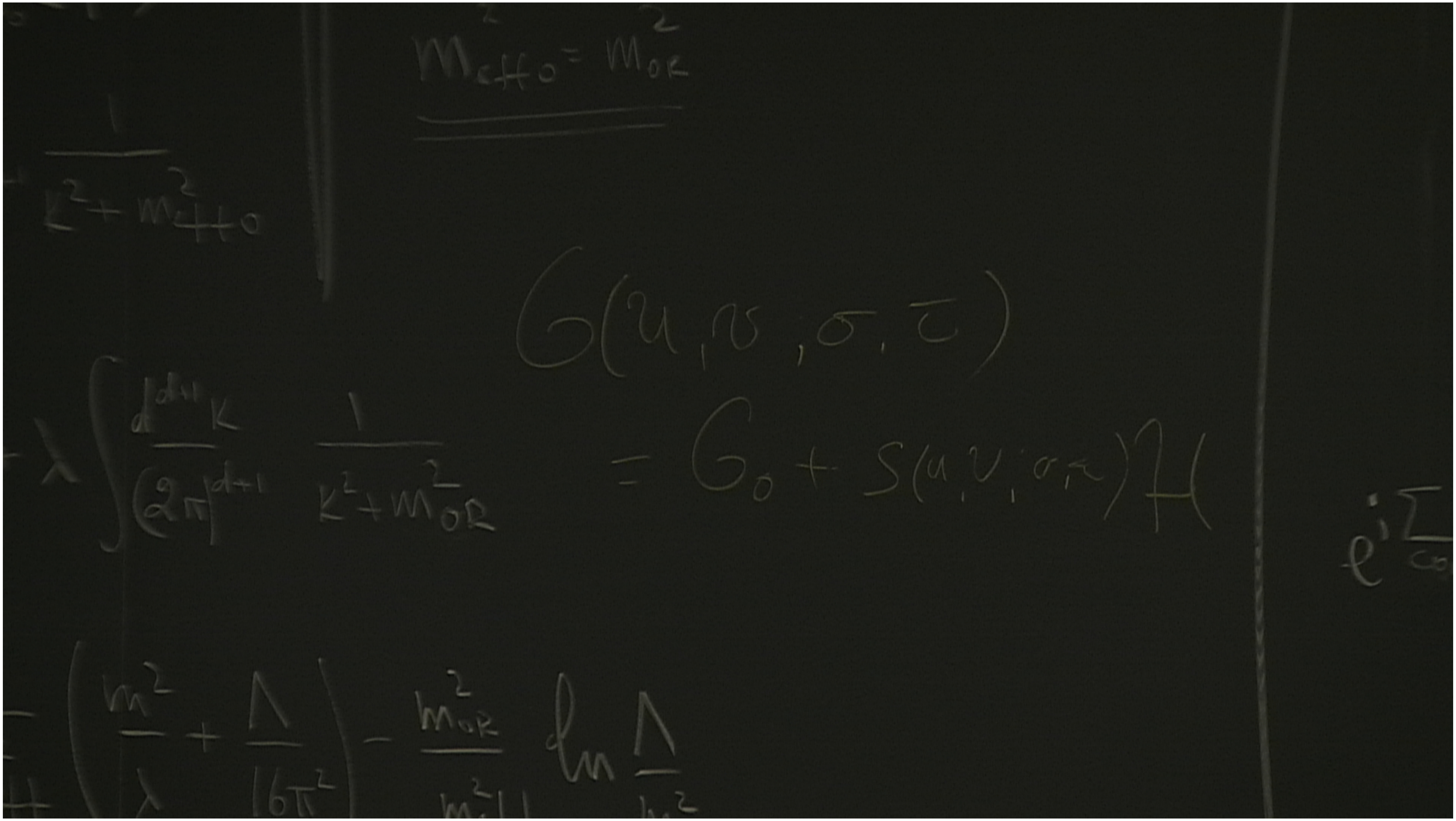
The only **twist two** singlet operator necessary in the OPE for $p=2$, is when $l=2$. This is, $\Delta=4$ corresponding to the stress-energy tensor. All other leading twist two singlet operators belonging to long multiplets were absent for any l.

(In free field theory, twist 2 singlet operators are present for any l. e.g. Konishi scalar, $l=0$)

The disappearance of **twist 2 operators** belonging to long multiplets at strong coupling requires a **non-trivial cancellation** between the **free field** contributions at order $O(1/N^2)$ and the leading non $\log u$ terms from the dynamical D-functions.

Hence the OPE for large N at strong coupling **only has contributions** from multi-trace operators with anomalous dimensions suppressed by powers of $1/N^2$

This fact was used as a **guiding principle** to conjecture the form of the dynamical piece of the amplitude at large N and strong coupling


G(u, v, σ, τ)
= G₀ + S(u, v, σ, τ) H(

Direct evaluation of the amplitude

$$\langle \mathcal{O}_2 \mathcal{O}_2 \mathcal{O}_n \mathcal{O}_n \rangle$$

in AdS supergravity, provided **further evidence** for the DNO conjecture.

Additionally, this result yielded some interesting puzzles regarding its connection to the four-point function of operators dual to two classical strings and two supergravity modes.

Ideally, we would like to evaluate the **most general four-point function of CPOs of arbitrary conformal weight**, but this is a step too far with current techniques. We settle for:

$$\langle \mathcal{O}_{k+2} \mathcal{O}_{k+2} \mathcal{O}_{n-k} \mathcal{O}_{n+k} \rangle$$

Which is a **next-next-extremal process** as:

$$\Delta_1 + \Delta_2 + \Delta_3 - \Delta_4 = 4$$

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where the t's are **six dimensional complex null vectors.** One can then write down the four-point function as

$$\langle \mathcal{O}_{k+2}(\vec{x}_1, t_1)\mathcal{O}_{k+2}(\vec{x}_2, t_2)\mathcal{O}_{n-k}(\vec{x}_3, t_3)\mathcal{O}_{n+k}(\vec{x}_4, t_4)\rangle$$
$$= \left(\frac{t_1\cdot t_2}{|\vec{x}_{12}|^2}\right)^2 \left(\frac{t_1\cdot t_4}{|\vec{x}_{14}|^2}\right)^k \left(\frac{t_2\cdot t_4}{|\vec{x}_{24}|^2}\right)^k \left(\frac{t_3\cdot t_4}{|\vec{x}_{34}|^2}\right)^{n-k} \mathcal{G}(u,v;\sigma,\tau)$$

where we introduced *SU(4)* invariants

$$\sigma = \frac{t_1\cdot t_3\, t_2\cdot t_4}{t_1\cdot t_2\, t_3\cdot t_4} \qquad \tau = \frac{t_1\cdot t_4\, t_2\cdot t_3}{t_1\cdot t_2\, t_3\cdot t_4}$$

so the correlation function is a polynomial in these quantities.

$$\mathcal{G}(u,v;\sigma,\tau) = a(u,v) + b_1(u,v)u\sigma + b_2(u,v)\frac{u}{v}\tau + c_1(u,v)u^2\sigma^2 + c_2(u,v)\frac{u^2}{v^2}\tau^2 + d(u,v)\frac{u^2}{v}\sigma\tau$$

Crossing symmetry reduces the number of coefficient functions to 4 since under exchange 1 - 2

$$\mathcal{G}(u,v;\sigma,\tau) = \mathcal{G}(\tfrac{u}{v},\tfrac{1}{v};\tau,\sigma)$$

When $n=2k+2$ there is an **additional symmetry** reducing the number of coefficient functions further to 2.

Ward Identities and dynamical considerations force the function to split into two distinct pieces:

$$\mathcal{G}(u,v;\sigma,\tau)=\mathcal{G}_0(u,v;\sigma,\tau)+s(u,v;\sigma,\tau)\mathcal{H}_I(u,v;\sigma,\tau)$$

The first corresponds to the contribution coming from **Free Fields.** The second contains all the **non-trivial dynamics**. Here

$$s(u,v;\sigma,\tau)=v+\sigma^2uv+\tau^2u+\sigma v(v-u-1)+\tau(1-u-v)+\sigma\tau(u-v-1)$$

We are interested in

Showing that the same **structure** is respected in SUGRA.

Comparing the form of H with the one **conjectured** by DNO.

3 Comparing the **free field contribution** as read off from the SUGRA amplitude with the results obtained by **direct computation** in YM.

4 Analysing **connections** to results obtained by Buchbinder & Tseytlin in the limit in which n becomes large.

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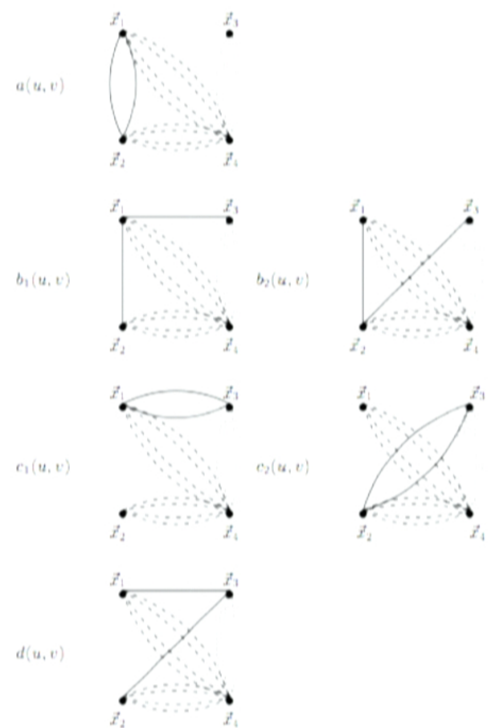
Free Field Theory at Large N

Using the **propagator basis** there are six diagrams to consider



Introduce a basis for $U(N)$, and the basic two-point function of adjoint scalars $\{T_a\}, a = 1 \cdots N^2$

$$[T_a, T_b] = i f_{abc} T_c \qquad \mathrm{tr}(T_a T_b) = \frac{1}{2}\delta_{ab}$$

$$\mathrm{tr}(T_a A)\mathrm{tr}(T_a B) = \frac{1}{2}\mathrm{tr}(AB) \qquad T_a T_a = \frac{1}{2} N \mathbb{I}$$

$$\langle X_a X_b \rangle = 2\delta_{ab}$$

We evaluate the **two-point** function of CPOs

$$\begin{aligned} \langle \mathrm{tr}(X^p)\mathrm{tr}(X^p)\rangle &= 2^p p!\, \mathrm{tr}(T_{(a_1} \cdots T_{a_p)})\, \mathrm{tr}(T_{(a_1} \cdots T_{a_p)}) \\ &\simeq 2^p p\, \mathrm{tr}(T_{a_1} \cdots T_{a_p})\, \mathrm{tr}(T_{a_p} \cdots T_{a_1}) \\ &= 2^{p-1} p\, \mathrm{tr}(T_{a_1} \cdots T_{a_{p-1}} T_{a_{p-1}} \cdots T_{a_1}) = pN^p \end{aligned}$$

and the **three-point** function of equal weight CPOs

$$\langle \mathrm{tr}(X^{p_1})\mathrm{tr}(X^{p_2})\mathrm{tr}(X^{p_3})\rangle = p_1 p_2 p_3\, N^{\frac{1}{2}(p_1+p_2+p_3)-1}$$

Four-point functions need to be evaluated taking care of the number of ways in which one can contract the operators

$$\langle \mathrm{tr}(X^{k+2})\mathrm{tr}(X^{k+2})\mathrm{tr}(X^{n-k})\mathrm{tr}(X^{n+k})\rangle_a = N^{n+k}2k(k+2)^2(n-k)(n+k)$$
$$\langle \mathrm{tr}(X^{k+2})\mathrm{tr}(X^{k+2})\mathrm{tr}(X^{n-k})\mathrm{tr}(X^{n+k})\rangle_{b_1,b_2} = N^{n+k}(k+2)^2(n-k)(n+k)(k+1)$$
$$\langle \mathrm{tr}(X^{k+2})\mathrm{tr}(X^{k+2})\mathrm{tr}(X^{n-k})\mathrm{tr}(X^{n+k})\rangle_{c_1,c_2} = N^{n+k}(k+2)^2(n-k)(n+k)(n-2)$$
$$\langle \mathrm{tr}(X^{k+2})\mathrm{tr}(X^{k+2})\mathrm{tr}(X^{n-k})\mathrm{tr}(X^{n+k})\rangle_{d_1} = N^{n+k}(k+2)^2(n-k)(n+k)(n-k-1)$$

By normalising the two-point function so it has unit coefficient, the **large N free field result for the correlation function** reads:

$$\mathcal{G}_0(u,v;\sigma,\tau) = \frac{1}{N^2}\sqrt{(k+2)^2(n-k)(n+k)}\Big\{2k + (k+1)\Big(\sigma u + \tau\frac{u}{v}\Big) + (n-2)\Big(\sigma^2 + \tau^2\frac{u^2}{v^2}\Big) + (n-k-1)\sigma\tau\frac{u^2}{v}\Big\}$$

Notice that in the limit in which $n=2k+2$, the result becomes:

$$\mathcal{G}_0(u,v;\sigma,\tau) = \frac{1}{N^2}\sqrt{(k+2)^3(3k+2)}\Big\{2k\Big(1+\sigma^2+\tau^2\frac{u^2}{v^2}\Big) + (k+1)\Big(\sigma u + \tau\frac{u}{v} + \sigma\tau\frac{u^2}{v}\Big)\Big\}$$

Evaluating **correlation functions** in **AdS supergravity**

Write down 5d **effective lagrangian**

All couplings available in the literature.
Just need to do perturbation theory.

Evaluate **Witten Diagrams**

Some formulae in literature, remaining processes easily calculable.

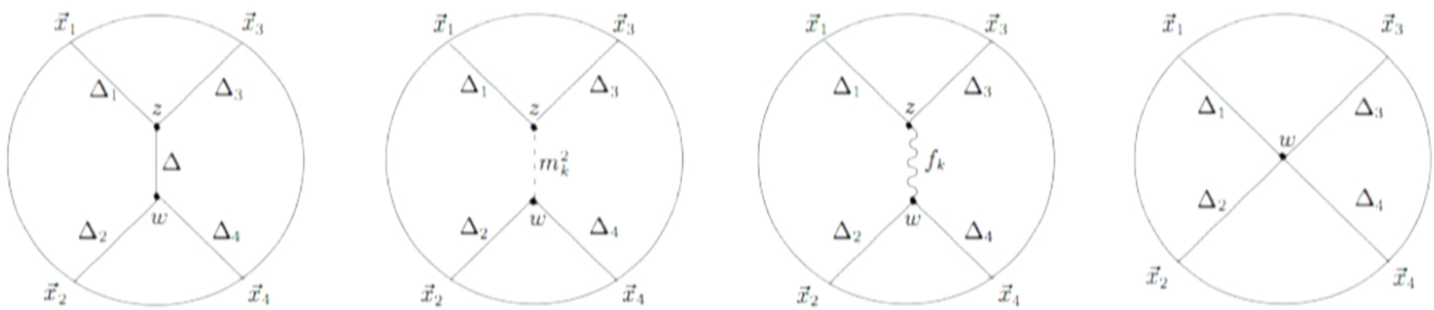
3 Evaluate **effective vertices** and **quartic couplings**

New techniques had to be introduced to carry out integrations over the sphere and to show that the four-derivative quartic couplings vanished.

4 **Simplify and rewrite** in terms of D-functions

Quite painful but possibly made easier if expressed in Mellin space.

Witten Diagrams



	s-channel	*t*-channel
	[0,2k+2,0]	[0,n,0]
Scalars	$m^2 = (2k+2)(2k-2)$	$m^2 = n(n-4)$
	[1,2k,1]	[1,n-2,1]
Vectors	$m^2_{2k+1} = 4k(k+1)$	$m^2_{n-1} = n(n-2)$
	[0,2k,0]	[0,n-2,0]
Tensors	$f(2k) = 4k(k+2)$	$f(n-2) = (n-2)(n+2)$

plus a **contact diagram**

The **five-dimensional lagrangian** reads:

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4$$

Represent the solutions to the equations of motion in the form:

$$s_p = s_p^0 + \tilde{s}_p \qquad A_\mu = A_\mu^0 + \tilde{A}_\mu \qquad \phi_{\mu\nu} = \phi_{\mu\nu}^0 + \tilde{\phi}_{\mu\nu}$$

where the fields with the "0" superscript are **solutions to the linearised equations** with fixed boundary conditions and the tilded fields represent the **fields in the bulk** with vanishing boundary conditions.

Express the tilded fields as integrals on the bulk, by introducing corresponding Green's functions.

2 Evaluate the **on-shell value** of the action.

We still need to evaluate the effective couplings coming from integrals over the five-sphere.

Integrals in S^5

How do these expressions relate to the standard techniques in the literature? The typical **scalar cubic coupling** has the form

$$a_{I_1 I_2 I_3} \equiv \omega_5^{3/2} \int_{S^5} Y_{k_1}^{I_1} Y_{k_2}^{I_2} Y_{k_3}^{I_3} \quad = A_{123}(k_1, k_2, k_3) \langle C^{I_1}_{[0,k_1,0]} C^{I_2}_{[0,k_2,0]} C^{I_3}_{[0,k_3,0]} \rangle$$

where

$$\langle C^{I_1}_{[0,k_1,0]} C^{I_2}_{[0,k_2,0]} C^{I}_{[0,k_3,0]} \rangle = C^{I_1}_{i_1 \dots i_{\alpha_2} j_1 \dots j_{\alpha_3}} C^{I_2}_{j_1 \dots j_{\alpha_3} l_1 \dots l_{\alpha_1}} C^{I}_{l_1 \dots l_{\alpha_1} i_1 \dots i_{\alpha_2}}$$

and the *C's* form a basis of symmetric traceless tensors in *SO(6)*. Generic exchange integrals contain effective couplings of the form:

$$\langle C^1_{p_1} C^2_{p_2} C^5_{[0,k_5,0]} \rangle \langle C^3_{p_3} C^4_{p_4} C^5_{[0,k_5,0]} \rangle$$

Contracting them give raise to Kronecker deltas, so the higher the representation, the **more complicated** it gets. The original SUGRA induced 4p functions that were calculated, employed the formula explicitly.

$$C^I_{i_1 \dots i_n} C^I_{j_1 \dots j_n} = \sum_{k=0}^{[\frac{n}{2}]} \theta_k \sum_{(l_{2k-1} \dots l_{2k})} \delta_{i_{l_1} i_{l_2}} \dots \delta_{i_{l_1} i_{l_2}} \dots \delta_{i_{l_{2k-1}} i_{l_{2k}}} \delta^{(n-2k)}_{i_1 \dots \hat{i}_{l_1} \dots \hat{i}_{l_{2k}} \dots i_{l_n}, (j_{2k+1} \dots j_n} \delta_{j_1 j_2} \dots \delta_{j_{2k-1} j_{2k})}$$

For generic representations it was clear one needed to do **something different**.

In fact, these effective couplings coming from the sphere can be expressed as eigenfunctions of the Casimir operator:

$$L^2 = \frac{1}{2} L_{ab} L_{ab} \qquad\qquad L^2 Y(\sigma, \tau) = -2C Y(\sigma, \tau)$$

Up to a normalisation constant, each function Y can be identified with an irreducible representations of , so *Ynm* corresponds to the representation with Dynkin labels

Determination of the **normalisation constant** for the 22pp case was done by explicit calculation with lower p cases *(p=1,2,34)*. For the k+2k+2n-k n+k this was not enough.

Define spherical harmonics transforming in the *[0,k,0]* representation of *SU(4)*

$$Y_k^I = z(k) C^I_{i_1 \ldots i_k} \xi^{i_1} \ldots \xi^{i_k}$$

$$\xi \in S^5$$

Given that the basis of totally symmetric tensors is orthonormal, we can fix *z(k)*

$$\int_{S^5} Y_k^{I_1} Y_k^{I_2} = \omega_5 \delta^{I_1 I_2}.$$

so

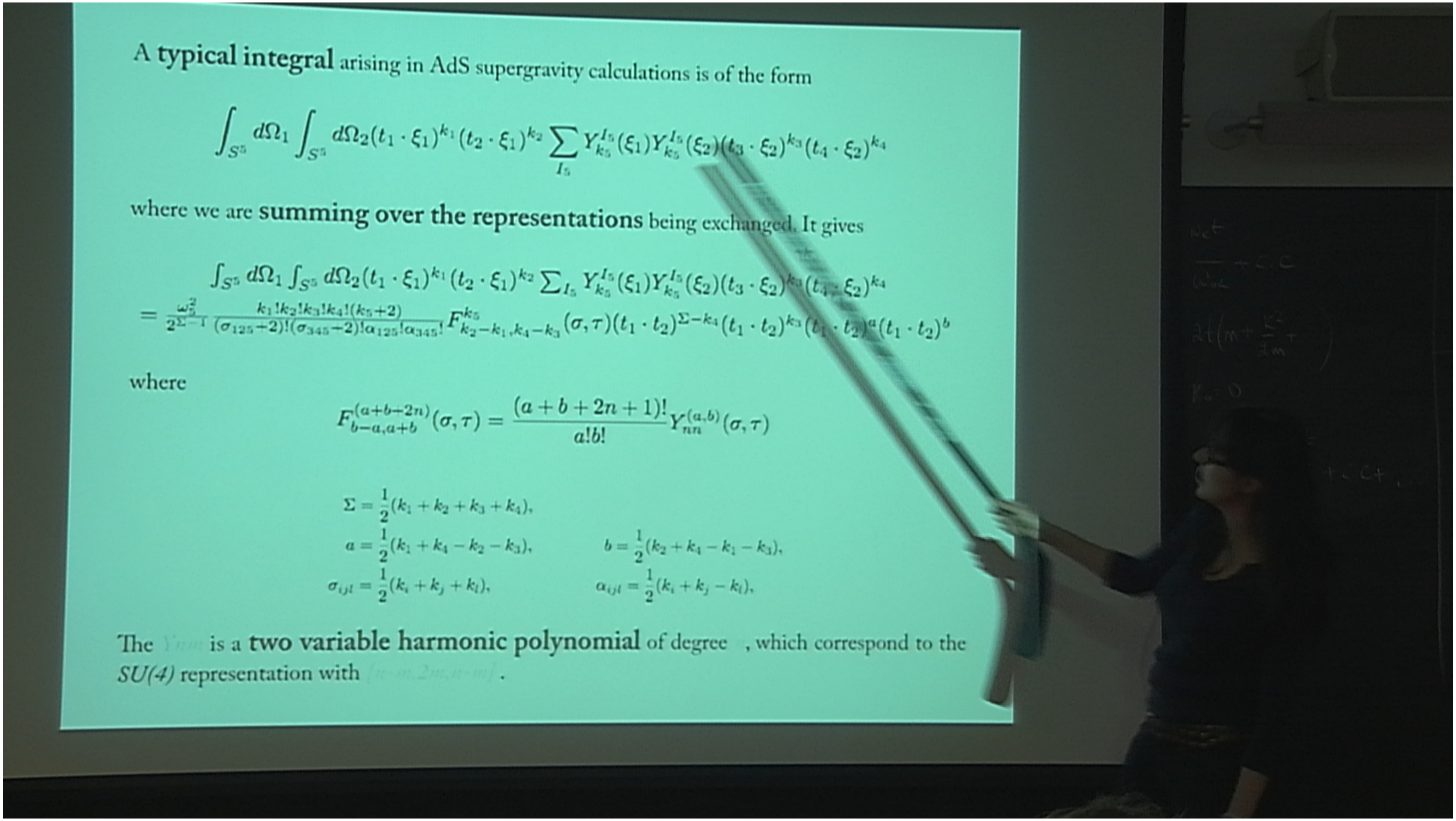
$$z(k) = \sqrt{2^{k-1}(k+1)(k+2)}.$$

One can prove that:

$$\int_{S^5} (t_1 \cdot \xi)^k (t_2 \cdot \xi)^k = \frac{\omega_5}{2^{k-1}(k+1)(k+2)} (t_1 \cdot t_2)^k = \frac{\omega_5}{(z(k))^2} (t_1 \cdot t_2)^k$$

so it is possible to establish the result

$$Y_k^{(k)} \to z(k)(t \cdot \xi)^k$$


A typical integral arising in AdS supergravity calculations is of the form
$\int_{S^5} d\Omega_1 \int_{S^5} d\Omega_2 (t_1 \cdot \xi_1)^{k_1} (t_2 \cdot \xi_1)^{k_2} \sum_{I_5} Y^{I_5}_{k_5}(\xi_1) Y^{I_5}_{k_5}(\xi_2)(t_3 \cdot \xi_2)^{k_3}(t_4 \cdot \xi_2)^{k_4}$
where we are summing over the representations being exchanged. It gives
$\int_{S^5} d\Omega_1 \int_{S^5} d\Omega_2 (t_1 \cdot \xi_1)^{k_1} (t_2 \cdot \xi_1)^{k_2} \sum_{I_5} Y^{I_5}_{k_5}(\xi_1) Y^{I_5}_{k_5}(\xi_2)(t_3 \cdot \xi_2)^{k_3}(t_4 \cdot \xi_2)^{k_4}$
$= \frac{\omega_5^2}{2^{\Sigma-1}} \frac{k_1!k_2!k_3!k_4!(k_5+2)}{(\sigma_{125}+2)!(\sigma_{345}-2)!\alpha_{125}!\alpha_{345}!} F^{k_5}_{k_2-k_1,k_4-k_3}(\sigma,\tau)(t_1 \cdot t_2)^{\Sigma-k_4}(t_1 \cdot t_2)^{k_3}(t_1 \cdot t_2)^{a}(t_1 \cdot t_2)^{b}$
where
$F^{(a+b-2n)}_{b-a,a+b}(\sigma,\tau) = \frac{(a+b+2n+1)!}{a!b!} Y^{(a,b)}_{nn}(\sigma,\tau)$
$\Sigma = \frac{1}{2}(k_1+k_2+k_3+k_4),$
$a = \frac{1}{2}(k_1+k_4-k_2-k_3),$
$b = \frac{1}{2}(k_2+k_4-k_1-k_3),$
$\sigma_{ijl} = \frac{1}{2}(k_i+k_j+k_l),$
$\alpha_{ijl} = \frac{1}{2}(k_i+k_j-k_l),$
The is a two variable harmonic polynomial of degree , which correspond to the SU(4) representation with .

A **typical integral** arising in AdS supergravity calculations is of the form

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where we are **summing over the representations** being exchanged. It gives

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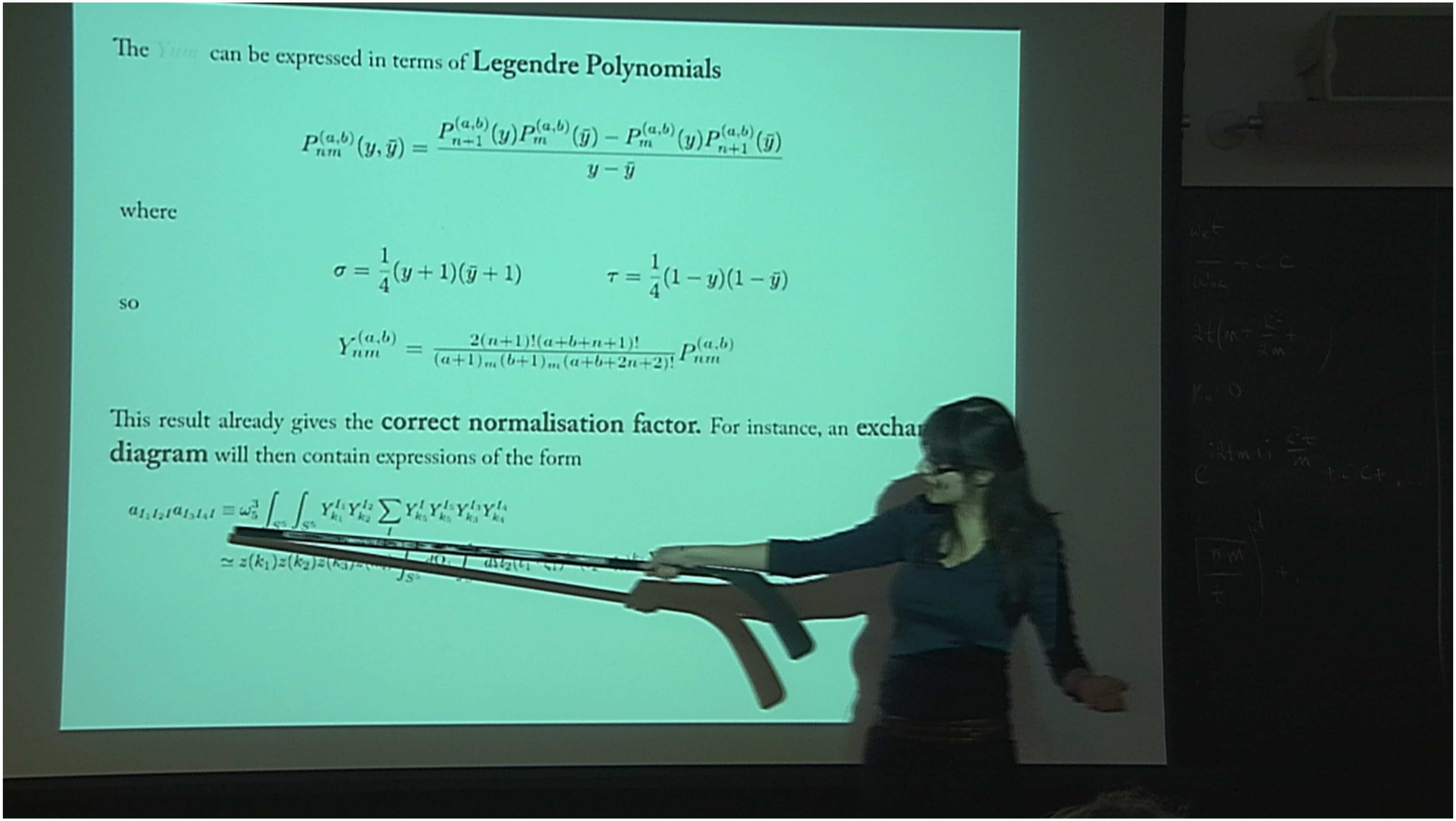
$$= \frac{\omega_5^2}{2^{\Sigma-1}} \frac{k_1! k_2! k_3! k_4! (k_5+2)}{(\sigma_{125}+2)!(\sigma_{345}+2)! \alpha_{125}! \alpha_{345}!} F^{k_5}_{k_2-k_1, k_4-k_3}(\sigma,\tau)(t_1 \cdot t_2)^{\Sigma-k_4}(t_1 \cdot t_2)^{k_3}(t_1 \cdot t_2)^a (t_1 \cdot t_2)^b$$

where

$$F^{(a+b+2n)}_{b-a,a+b}(\sigma,\tau) = \frac{(a+b+2n+1)!}{a!b!} Y^{(a,b)}_{nn}(\sigma,\tau)$$

$$\Sigma = \frac{1}{2}(k_1 + k_2 + k_3 + k_4),$$
$$a = \frac{1}{2}(k_1 + k_4 - k_2 - k_3), \qquad b = \frac{1}{2}(k_2 + k_4 - k_1 - k_3),$$
$$\sigma_{ijl} = \frac{1}{2}(k_i + k_j + k_l), \qquad \alpha_{ijl} = \frac{1}{2}(k_i + k_j - k_l),$$

The Y_{nm} is a **two variable harmonic polynomial** of degree n, which correspond to the *SU(4)* representation with $[n-m, 2m, n-m]$.


The Y_{nm} can be expressed in terms of **Legendre Polynomials**
$$P^{(a,b)}_{nm}(y,\bar{y}) = \frac{P^{(a,b)}_{n+1}(y)P^{(a,b)}_{m}(\bar{y}) - P^{(a,b)}_{m}(y)P^{(a,b)}_{n+1}(\bar{y})}{y-\bar{y}}$$
where
$$\sigma = \frac{1}{4}(y+1)(\bar{y}+1) \qquad \tau = \frac{1}{4}(1-y)(1-\bar{y})$$
so
$$Y^{(a,b)}_{nm} = \frac{2(n+1)!(a+b+n+1)!}{(a+1)_m(b+1)_m(a+b+2n+2)!}P^{(a,b)}_{nm}$$
This result already gives the **correct normalisation factor.** For instance, an **excha**
diagram will then contain expressions of the form
$$a_{l_1 l_2 l} a_{l_3 l_4 l} = \omega_5^3 \int_{S^5}\int_{S^5} Y^{l_1}_{k_1} Y^{l_2}_{k_2} \sum_l Y^{l}_{k_5} Y^{l_5}_{k_5} Y^{l_3}_{k_3} Y^{l_4}_{k_4}$$
$\simeq z(k_1)z(k_2)z(k_3)$

The Y_{nm} can be expressed in terms of **Legendre Polynomials**

$$P_{nm}^{(a,b)}(y,\bar{y}) = \frac{P_{n+1}^{(a,b)}(y)P_m^{(a,b)}(\bar{y}) - P_m^{(a,b)}(y)P_{n+1}^{(a,b)}(\bar{y})}{y-\bar{y}}$$

where

$$\sigma = \frac{1}{4}(y+1)(\bar{y}+1) \qquad \tau = \frac{1}{4}(1-y)(1-\bar{y})$$

so

$$Y_{nm}^{(a,b)} = \frac{2(n+1)!(a+b+n+1)!}{(a+1)_m(b+1)_m(a+b+2n+2)!}P_{nm}^{(a,b)}$$

This result already gives the **correct normalisation factor.** For instance, an **exchange diagram** will then contain expressions of the form

$$\begin{aligned} a_{I_1 I_2 I} a_{I_3 I_4 I} &\equiv \omega_5^3 \int_{S^5}\int_{S^5} Y_{k_1}^{I_1} Y_{k_2}^{I_2} \sum_I Y_{k_5}^{I} Y_{k_5}^{I_5} Y_{k_3}^{I_3} Y_{k_4}^{I_4} \\ &\simeq z(k_1)z(k_2)z(k_3)z(k_4)\int_{S^5} d\Omega_1 \int_{S^5} d\Omega_2 (t_1\cdot\xi_1)^{k_1}(t_2\cdot\xi_1)^{k_2}\sum_I Y_{k_5}^I(\xi_1)Y_{k_5}^I(\xi_2)(t_3\cdot\xi_2)^{k_3}(t_4\cdot\xi_2)^{k_4} \end{aligned}$$

but using the formulas before, we can immediately express these in terms of classical polynomials in σ, τ

$$\begin{aligned}
&\langle C^{I_1}_{[0,k_1,0]} C^{I_2}_{[0,k_2,0]} C^{I}_{[0,k_5,0]} \rangle \langle C^{I_3}_{[0,k_3,0]} C^{I_4}_{[0,k_4,0]} C^{I}_{[0,k_5,0]} \rangle \\
&\to \frac{(t_1 \cdot t_2)^{\Sigma - k_4} (t_1 \cdot t_2)^{k_3} (t_1 \cdot t_2)^{a} (t_1 \cdot t_2)^{b}}{A_{125}(k_1, k_2, k_5) A_{345}(k_3, k_4, k_5)} \frac{z(k_1) z(k_2) z(k_3) z(k_4)}{2^{\Sigma - 1}} \frac{k_1! k_2! k_3! k_4! (k_5 + 2)}{(\sigma_{125} + 2)! (\sigma_{345} + 2)! \alpha_{125}! \alpha_{345}!} \frac{(k_5 + 1)!}{a! b!} Y^{(a,b)}_{nn} \times \\
&= \frac{2^{\sigma_{125} - 1} 2^{\sigma_{345} - 1} \alpha_{251}! \alpha_{512}! \alpha_{453}! \alpha_{534}!}{(k_5!)^2 z(k_5)^2} \frac{1}{2^{\Sigma - 1}} \frac{(k_5 + 2)!}{a! b!} Y^{(a,b)}_{nn} (t_1 \cdot t_2)^{\Sigma - k_4} (t_1 \cdot t_2)^{k_3} (t_1 \cdot t_2)^{a} (t_1 \cdot t_2)^{b} \\
&= \frac{\alpha_{251}! \alpha_{512}! \alpha_{453}! \alpha_{534}!}{k_5!} \frac{1}{a! b!} Y^{(a,b)}_{nn} (t_1 \cdot t_2)^{\Sigma - k_4} (t_1 \cdot t_2)^{k_3} (t_1 \cdot t_2)^{a} (t_1 \cdot t_2)^{b}
\end{aligned}$$

Quartic interactions also contain expressions of the form $a_{125} a_{345}$ and in fact, they are the hardest to evaluate.

One first shows that the **four-derivative couplings** vanish, as they should so the lagrangian is of the sigma-model type.

$$\begin{aligned}
\tilde{\mathcal{L}}_4^{(2)} = \left[3 \left(m^2_{k+2} + \frac{m^2_{n-k} + m^2_{n+k}}{2} - 4 \right) \Sigma^{1234} + \frac{1}{2} \tilde{B}_1^{1234} \right] &\left[s^1_{k+2} \nabla_\mu s^2_{k+2} s^3_{n-k} \nabla^\mu s^4_{n+k} \right. \\
&\left. + s^1_{k+2} \nabla_\mu s^2_{k+2} s^3_{n+k} \nabla^\mu s^4_{n-k} \right]
\end{aligned}$$

$$\begin{aligned}
\tilde{\mathcal{L}}_4^{(0)} = \frac{1}{2} \Big[C_1^{1234} - \frac{1}{2} (m^2_{n+k} + m^2_{n-k} + 2 m^2_{k+2}) B_2^{1234} &+ \Sigma^{1234} (m^2_{k+2} + \frac{1}{2} m^2_{n+k} + \frac{1}{2} m^2_{n-k} - 4) \\
&\times (m^2_{k+2} + \frac{1}{2} m^2_{n+k} + \frac{1}{2} m^2_{n-k}) \Big] s^1_{k+2} s^2_{k+2} s^3_{n-k} s^4_{n+k}
\end{aligned}$$

Results from Supergravity

$$\langle O_{k+2}(\vec{x}_1)O_{k+2}(\vec{x}_2)O_{n-k}(\vec{x}_3)O(\vec{x}_4)\rangle = \frac{(2\pi)^8}{2N^4}\sqrt{\frac{\Gamma(k)^2\Gamma(n-k-2)\Gamma(n+k-2)}{k^2(n-k-2)(n+k-2)\Gamma(k+2)^2\Gamma(n-k)\Gamma(n+k)}} \times \frac{\delta}{\delta s_{k+2}(\vec{x}_1)}\frac{\delta}{\delta s_{k+2}(\vec{x}_2)}\frac{\delta}{\delta s_{n-k}(\vec{x}_3)}\frac{\delta}{\delta s_{n+k}(\vec{x}_4)}(-S)$$

and we substitute the on-shell value of the action. In general, the result is extremely messy and is written in terms of **sums D-functions**, σ, τ . Normalisation is such that the **two-point function has unit coefficient**. The amplitude as a whole can be shown to respect crossing symmetry and is consistent with conformal symmetry.

e.g. Coefficient of order 0

$$\begin{aligned}\tilde{a}(u,v) = &\frac{2}{(2k+3)}(k+1)^2(2k+1)u\bar{D}_{k+1\ k+1\ n-k\ n+k}\\ &+\frac{u}{(2k+3)}\left((k+1)\left(v\bar{D}_{k+1\ k+3\ n-k+1\ n+k+1}+\bar{D}_{k+3\ k+1\ n-k+1\ n+k+1}\right)\right.\\ &\left.-(k+2)(1+v-u)\bar{D}_{k+2\ k+2\ n-k+1\ n+k+1}\right)\\ &-\left(\frac{(k+2)(n-k-2)}{(2k+3)}+(k+1)\right)2u^2\bar{D}_{k+2\ k+2\ n-k\ n+k}\end{aligned}$$

but one can show, that every coefficient function can be reduced to a **single D-function**.

$$\tilde{a}(u,v) = 2u\bar{D}_{k+2\ k+2\ n-k\ n+k+2}$$

When reducing to **D-functions**, one needs to be mindful of cases in which one of the conformal weights becomes zero:

$$\bar{D}_{\Delta_1\Delta_2\Delta_3+1\Delta_4+1} + u\bar{D}_{\Delta_1+1\Delta_2+1\Delta_3\Delta_4} + v\bar{D}_{\Delta_1\Delta_2+1\Delta_3+1\Delta_4}\big|_{\Delta_1+\Delta_2+\Delta_3=\Delta_4} = \Gamma(\Delta_1)\Gamma(\Delta_2)\Gamma(\Delta_3)$$

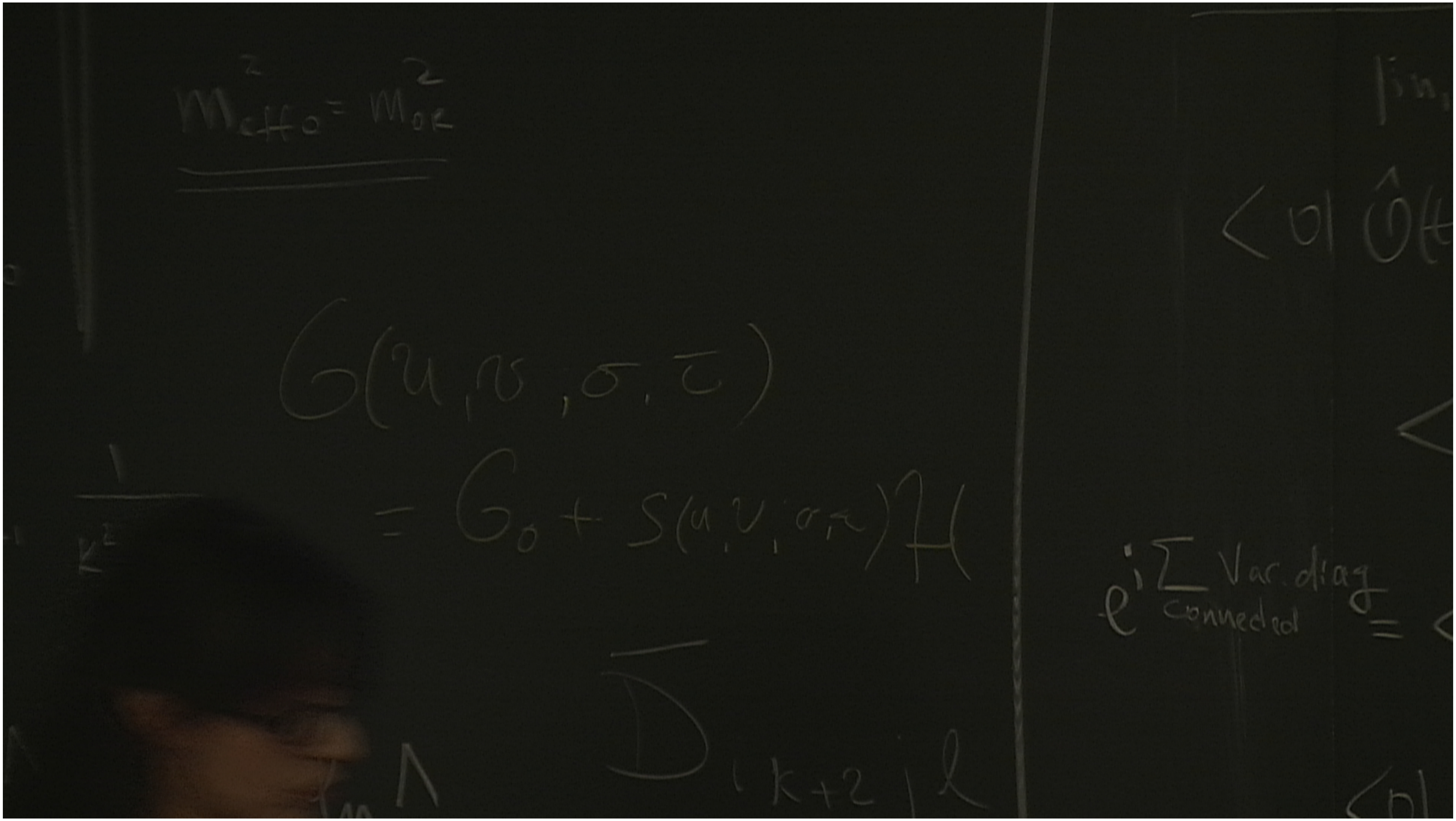
This fact leads to **finite pieces** which will determine the **"free field" contribution** to the correlation function as read from supergravity.

Final **Supergravity** amplitude:

$$\mathcal{G}(u,v;\sigma,\tau) = \frac{\sqrt{(k+2)^2(n-k)(n+k)}}{N^2}\Big\{(k+1)\left(\sigma u + \tau\frac{u}{v}\right) + (n-k-1)\sigma\tau\frac{u^2}{v} - \frac{1}{\Gamma(k+1)^2\Gamma(n-k-1)}s(u,v;\sigma,\tau)u^{n-k}v^k\bar{D}_{n-k\;\,n+k+2\;\,k+2\;\,k+2}(u,v)\Big\}$$

Free Field amplitude:

$$\mathcal{G}_0(u,v;\sigma,\tau) = \frac{1}{N^2}\sqrt{(k+2)^2(n-k)(n+k)}\Big\{2k + (k+1)\Big(\sigma u + \tau\frac{u}{v}\Big) + (n-2)\Big(\sigma^2 + \tau^2\frac{u^2}{v^2}\Big) + (n-k-1)\sigma\tau\frac{u^2}{v}\Big\}$$


Var. diag
Connected

Dynamical piece of the amplitude:

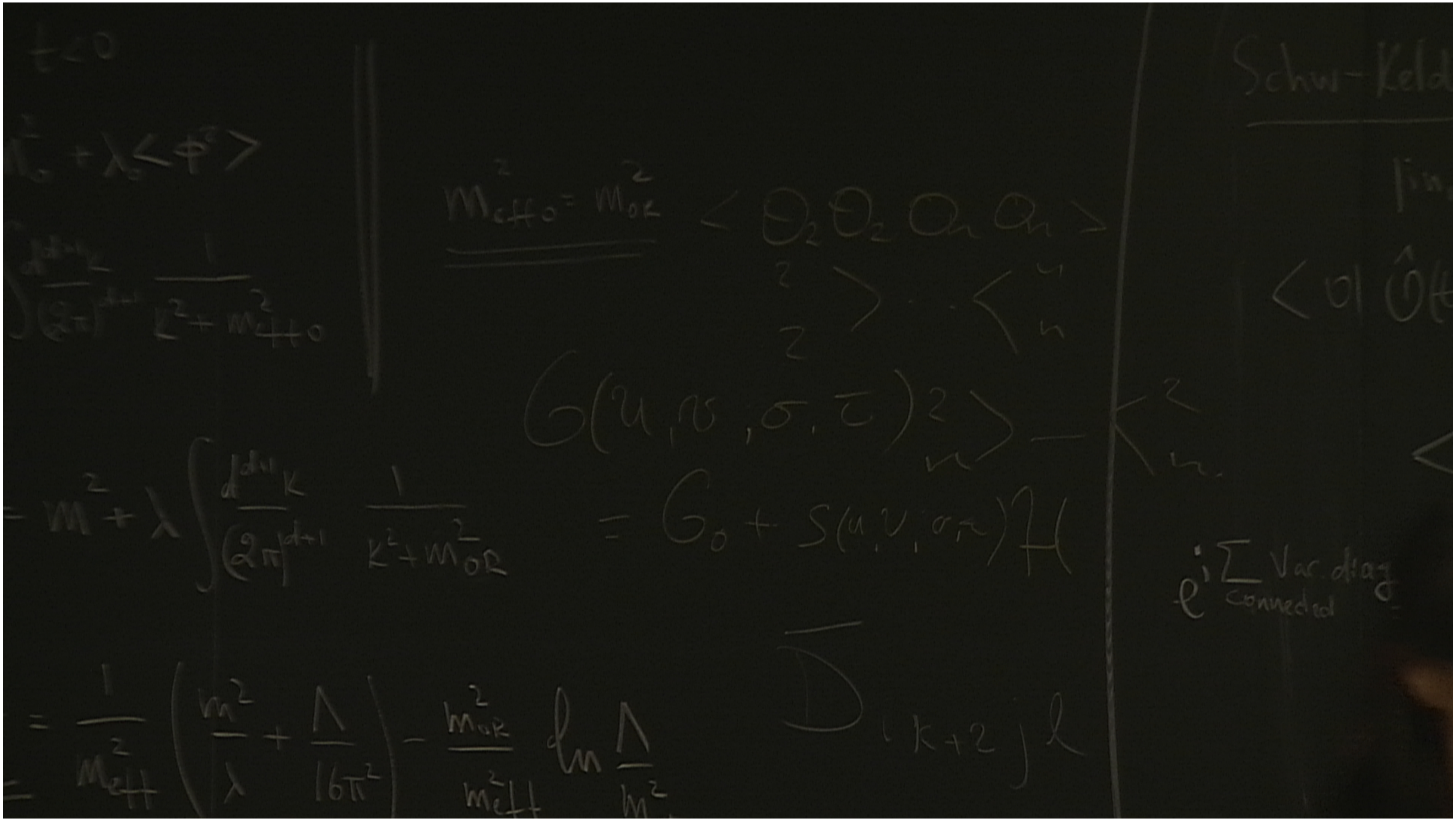
$$\mathcal{H}_I(u,v) = \frac{\sqrt{(k+2)^2(n-k)(n+k)}}{N^2\Gamma(k+1)^2\Gamma(n-k-1)} u^{n-k} v^k \bar{D}_{n-k\ n+k+2\ k+2\ k+2}(u,v)$$

This result **supports the DNO conjecture** and the known field-theoretic arguments for the **partially non-renormalised** form of four-point amplitudes of CPOs.

In fact one could then conjecture that the **most general next-next-to extremal process** will have the form (at strong coupling, large N)

$$\mathcal{G}(u,v;\sigma,\tau) = \frac{\sqrt{p_1p_2p_3p_4}}{N^2}\Big\{(p_1-1)\sigma u + (p_2-1)\tau\frac{u}{v} + (p_3-1)\sigma\tau\frac{u^2}{v} - \frac{1}{\Gamma(p_1-1)\Gamma(p_2-1)\Gamma(p_3-1)} s(u,v;\sigma,\tau) u^{p_3} v^{p_1-2} \bar{D}_{p_3\ p_4+2\ p_1\ p_2}(u,v)\Big\}$$

which in turns suggests that the **DNO conjecture can be generalised** and the generic four-point amplitude of CPOs with different conformal weights, can be given in closed form.


Schw-Keld
Var diag
connected

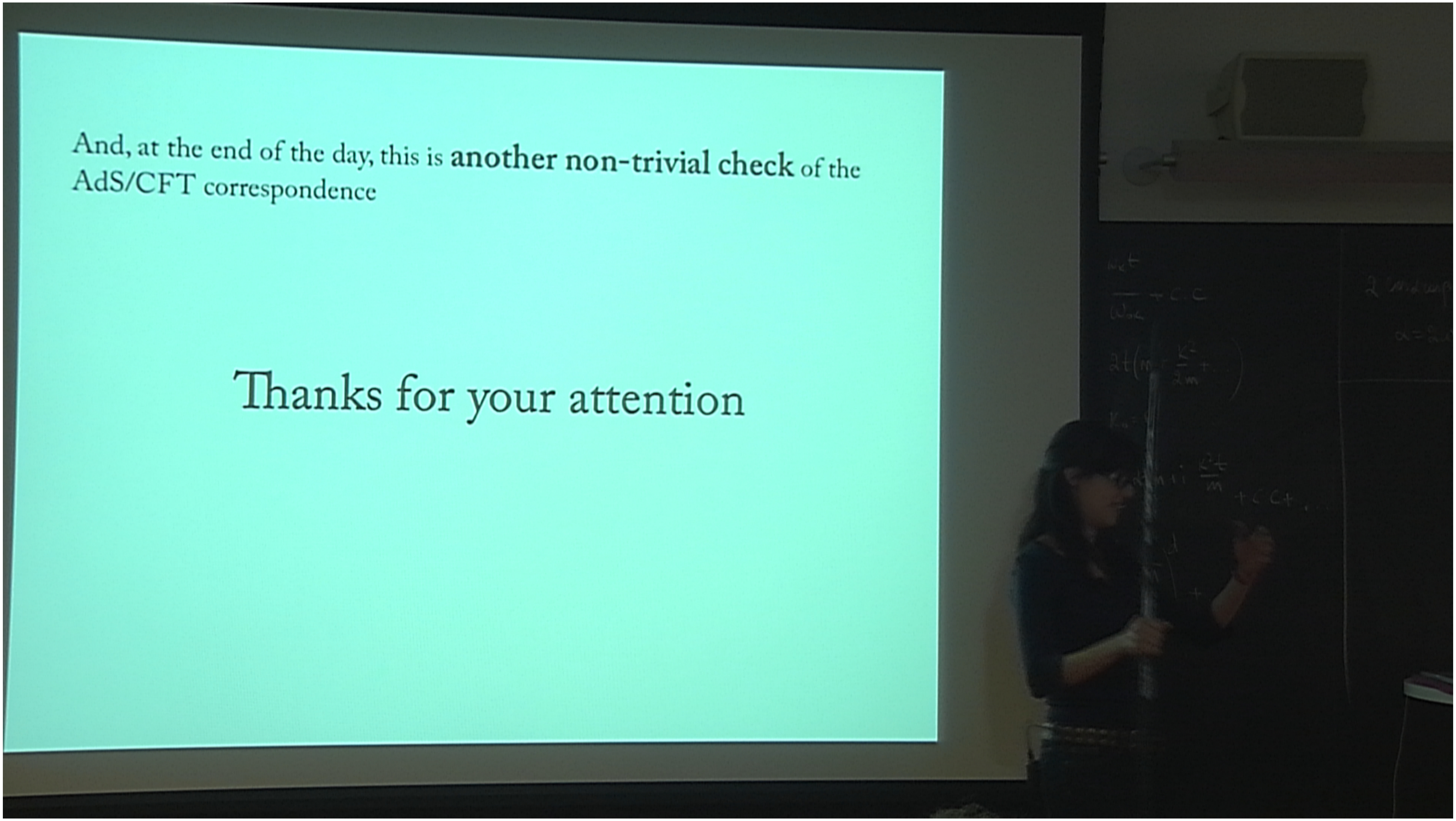
Summary & Conclusions

We have given further evidence for the DNO conjecture (and its generalisation) which specifies that the dynamical piece of the **strongly coupled large N four-point amplitude of CPOs** in N=4 SYM theory is determined by specific combinations of D-functions.

In particular, in the next-next-to extremal case, the dynamical contribution is determined by a single **single D-function.** The result is consistent with superconformal symmetry.

The DNO conjecture **could be generalised** to determine the most general form of the dynamical piece of the four-point amplitude for generic CPOs.

Maybe worth looking at more examples making use of recent developments (?)


And, at the end of the day, this is **another non-trivial check** of the AdS/CFT correspondence
Thanks for your attention