Title: On Renormalization Group Flows in Diverse Dimensions

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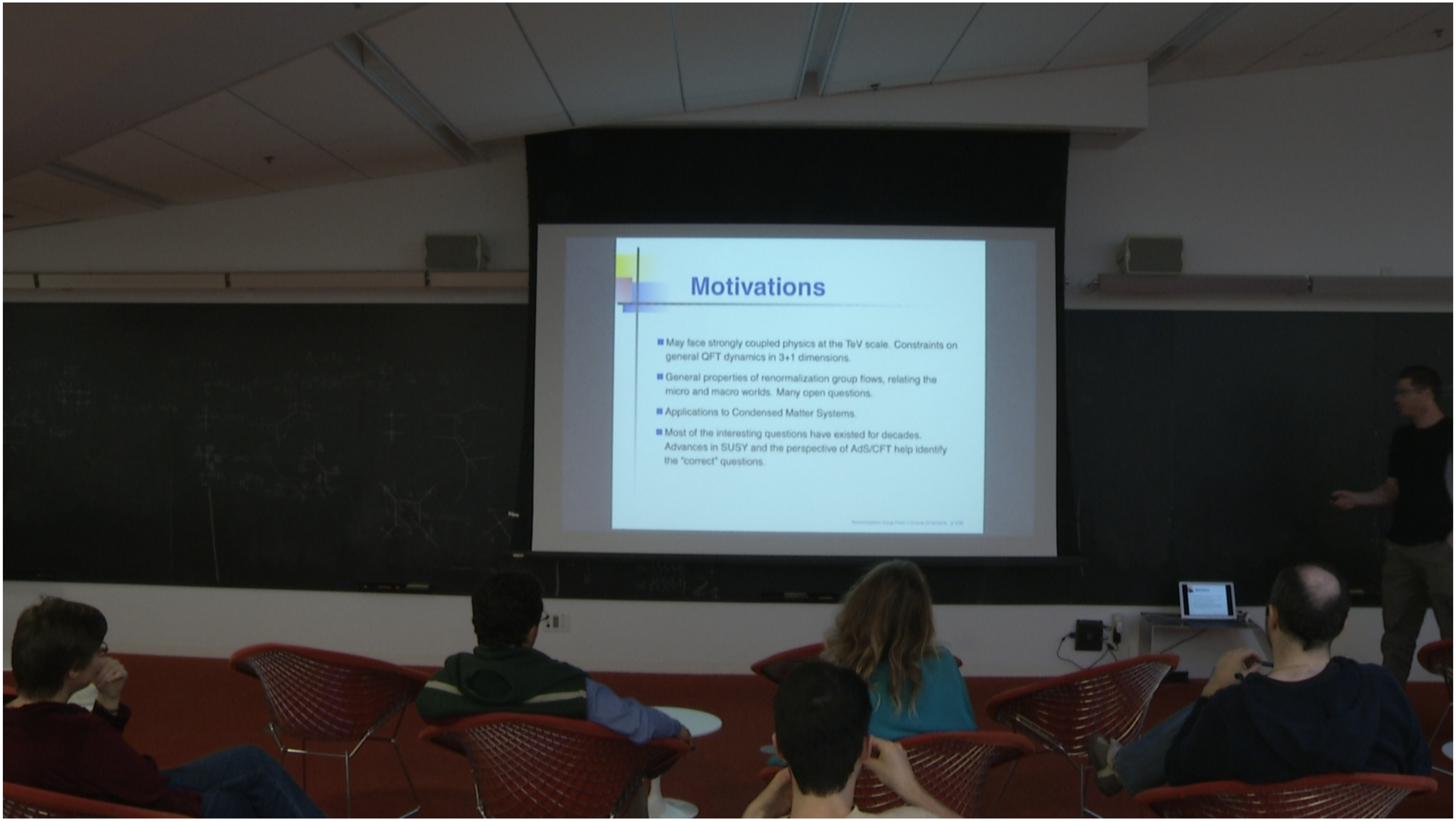
Abstract: TBA


Motivations
May face strongly coupled physics at the TeV scale. Constraints on general QFT dynamics in 3+1 dimensions.
General properties of renormalization group flows, relating the micro and macro worlds. Many open questions.
Applications to Condensed Matter Systems.
Most of the interesting questions have existed for decades. Advances in SUSY and the perspective of AdS/CFT help identify the "correct" questions.

Introduction

Is there a function

$$DOF : \mathcal{M} \to \mathbb{R}\,, \quad \mathcal{M} = \{all\ CFTs\}$$

such that if $CFT_{UV} \to CFT_{IR}$ by some relevant deformation then

$$DOF(CFT_{UV}) > DOF(CFT_{IR})$$

One can also try to establish a monotonic function interpolating between $DOF(CFT_{UV})$ and $DOF(CFT_{IR})$.

Introduction

Applications of DOF (if exists)

- If solutions to 't Hooft's anomaly matching conditions are too complicated and require many fields, a contradiction with $DOF(CFT_{UV}) > DOF(CFT_{IR})$ can establish symmetry breaking.
- If the breaking of some symmetry entails too many Nambu-Goldstone bosons, a contradiction with $DOF(CFT_{UV}) > DOF(CFT_{IR})$ implies no symmetry breaking.
- The space of CFTs can be foliated and the flow has a fixed direction. No cycles exist in the space of theories. If $CFT_A \to CFT_B$ by a relevant operator, no relevant operator can cause $CFT_B \to CFT_A$.
- Applications to Condensed Matter systems.

Introduction

- 0+1: a promising proposal exists (the "g-theorem," by Affleck and Ludwig), proof only in special cases by Friedan et al.
- 1+1: Solved by Zamolodchikov, his answer: $DOF(CFT) = c$, where c is the central charge of the two-dimensional conformal theory. In the end of the talk we will give a new proof of this.
- 2+1: Very inspiring proposal due to Myers and Sinha. Further elaborated on by Jafferis,Klebanov,Pufu,Safdi. Evidence consists of some $\mathcal{N} = 2$ models, perturbative fixed points, holography.
- 3+1: Cardy (followed by Osborn et al.) proposed $DOF(CFT) = a$, where a is the Euler trace anomaly. Evidence consists of BZ fixed points, SUSY, holography. Today we will develop some new tools in QFT to prove this.

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Introduction

For a free (thus conformal) theory in four space-time dimensions

$$a = \frac{1}{90(8\pi)^2}\left(\#_{scalars} + \frac{11}{2}\#_{Weyl\ fermions} + 62\#_{gauge\ fields}\right)$$

It is some measure of the number of degrees of freedom. One can also prove $a > 0$ for every CFT (see also Hofman,Maldacena ; Kulaxizi,Parnachev).

Notably, other intuitive measures of the number of degrees of freedom, like the free energy, do not work in general (Appelquist,Cohen,Schmaltz).

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Outline

- Conformal symmetry breaking and trace anomalies
- The Little a-Theorem
- General RG flows and the a-theorem
- Examples
- A new proof of Zamolodchikov's theorem
- Open questions

Conformal Symmetry Breaking

Consider a spontaneously broken CFT. The Nambu-Goldstone theorem tells us there is a massless particle, the dilaton $\tau(x)$.

We follow the rules of nonlinear Lagrangians. Introduce a space-time BACKGROUND metric $g_{\mu\nu}(x)$ and demand that the theory is invariant under diff$\times$Weyl transformations

$$g_{\mu\nu} \longrightarrow e^{2\sigma} g_{\mu\nu} \, , \qquad \tau \longrightarrow \tau + \sigma$$

Diffeomorphisms act as usual, with the dilaton a space-time scalar.

Useful to denote $\hat{g} = e^{-2\tau} g_{\mu\nu}$, since it transforms as a metric under diffeomorphisms and is Weyl invariant.

Conformal Symmetry Breaking

Most general theory up to two derivatives is

$$f^2 \int d^4x \sqrt{-\det \hat{g}} \left(\Lambda + \frac{1}{6} \hat{R} \right)$$

where $\hat{R} = \hat{g}^{\mu\nu} R_{\mu\nu}[\hat{g}]$. f is the "decay constant."

If the dilaton really comes from spontaneously broken conformal symmetry the vacuum degeneracy cannot be lifted and hence $\Lambda = 0$.

Let us go back to $g_{\mu\nu} = \eta_{\mu\nu}$. We get

$$S = f^2 \int d^4x e^{-2\tau} (\partial\tau)^2$$

We see that this is an ordinary free massless particle. The equation of motion is

$$\Box \tau = (\partial\tau)^2$$

Conformal Symmetry Breaking

Consider terms in the effective action with more derivatives. With four derivatives, one has three coefficients

$$\int d^4x \sqrt{-\hat{g}} \left(\kappa_1 \hat{R}^2 + \kappa_2 \hat{R}^2_{\mu\nu} + \kappa_3 \hat{R}^2_{\mu\nu\rho\sigma} \right)$$

There is a better basis

$$E_4 = R^2_{\mu\nu\rho\sigma} - 4R^2_{\mu\nu} + R^2 \,, \qquad W^2_{\mu\nu\rho\sigma} = R^2_{\mu\nu\rho\sigma} - 2R^2_{\mu\nu} + \frac{1}{3}R^2$$

Thus, we can also write

$$\int d^4x \sqrt{-\hat{g}} \left(\kappa_1' \hat{R}^2 + \kappa_2' \hat{E}_4 + \kappa_3' \hat{W}^2_{\mu\nu\rho\sigma} \right)$$

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$$\int d^4x\sqrt{-\hat{g}}\left(\kappa_1'\hat{R}^2 + \kappa_2'\hat{E}_4 + \kappa_3'\hat{W}^2_{\mu\nu\rho\sigma}\right)$$

We see that κ_2' term is a total derivative. If we set $g_{\mu\nu} = \eta_{\mu\nu}$, then $\hat{g}_{\mu\nu} = e^{-2\tau}\eta_{\mu\nu}$ is conformal to the flat metric and hence also the κ_3' term does not play any role.

So only κ_1' matters in flat space. A straightforward calculation yields

$$\int d^4x\sqrt{-\hat{g}}\hat{R}^2\Big|_{g_{\mu\nu}=\eta_{\mu\nu}} = 36\int d^4x\left(\Box\,\tau - (\partial\tau)^2\right)^2$$

But the combination $\Box\,\tau - (\partial\tau)^2 = 0$ by the two-derivative theory eom.

Thus, the diff×Weyl invariant terms in the Lagrangian do not yield a genuine tree-level four-derivative interaction.

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Conformal Symmetry Breaking

Better said, there is no $s^2 + t^2 + u^2$ term in the low momentum expansion of the scattering amplitude of four dilatons.

We will later explain that such a contribution must be nonzero by unitarity so something must save the day.

Trace Anomalies

So far we only examined functionals of $g_{\mu\nu}, \tau$ which are diff$\times$Weyl invariant. But all physical theories have trace anomalies.

The most general anomalous functional satisfying the WZ consistency conditions is

$$\delta_\sigma S_{anomaly} = \int d^4x\sqrt{-g}\sigma\left(cW^2_{\mu\nu\rho\sigma} - aE_4\right)$$

We need to solve for $S_{anomaly}$.

We solve for $S_{anomaly}$ iteratively, by first replacing σ on the right-hand side above with τ and then keep fixing.

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$$S_{anomaly} = -a \int d^4x\sqrt{-g}\left(\tau E_4 + 4\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right)\partial_\mu\tau\partial_\nu\tau - 4(\partial\tau)^2\Box\,\tau \right.$$

$$\left. +2(\partial\tau)^4\right) + c\int d^4x\sqrt{-g}\tau W^2_{\mu\nu\rho\sigma}$$

There is a self-interactions of the dilaton even in flat space due to the "non-Abelian" nature of the a-anomaly.

This is reminiscent of the Wess-Zumino term in pions physics that leads to the universal $KK \to \pi\pi\pi$ decay.

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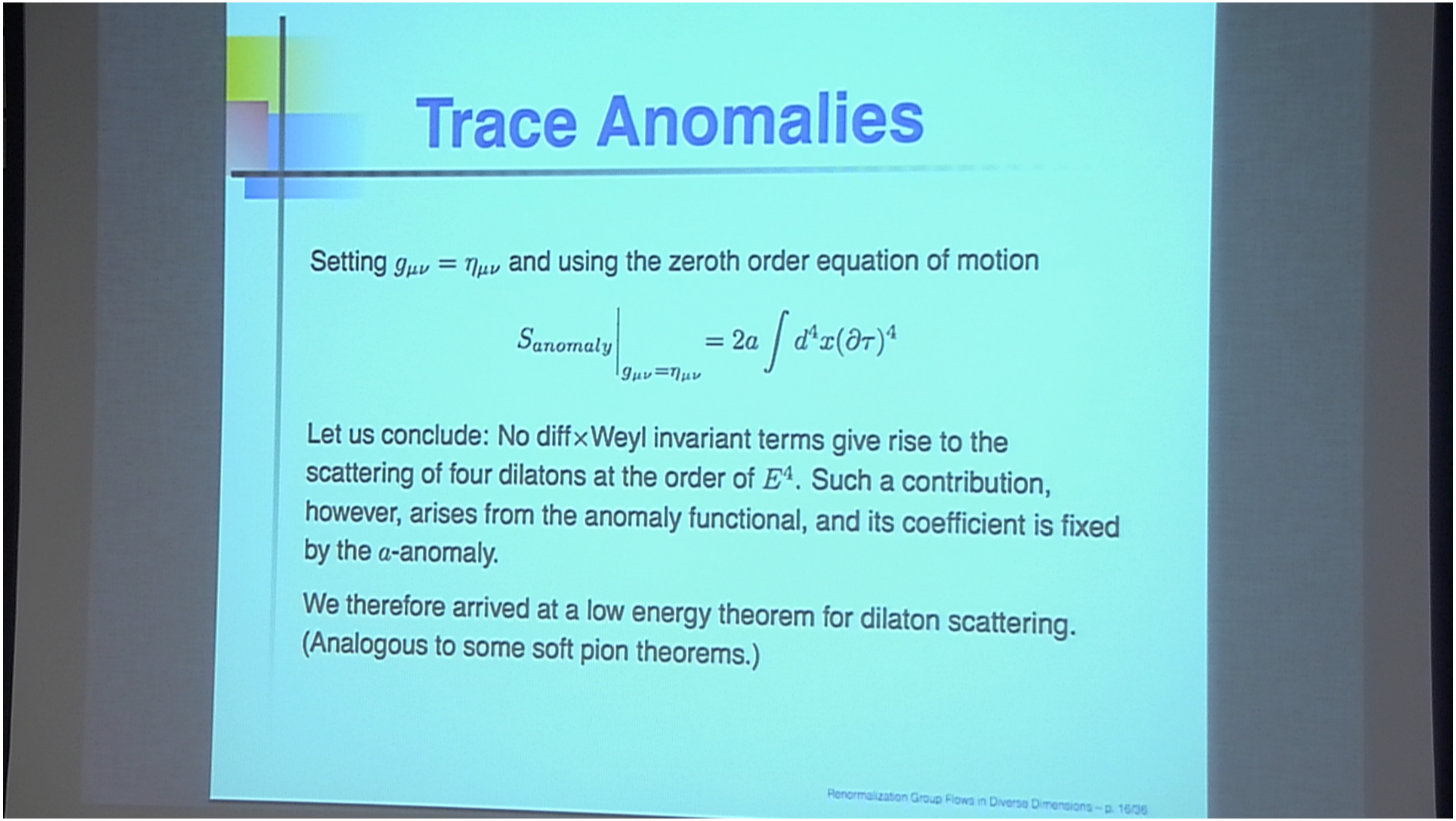
Trace Anomalies

Setting $g_{\mu\nu} = \eta_{\mu\nu}$ and using the zeroth order equation of motion

$$S_{anomaly}\Big|_{g_{\mu\nu}=\eta_{\mu\nu}} = 2a\int d^4x(\partial\tau)^4$$

Let us conclude: No diff$\times$Weyl invariant terms give rise to the scattering of four dilatons at the order of E^4. Such a contribution, however, arises from the anomaly functional, and its coefficient is fixed by the a-anomaly.

We therefore arrived at a low energy theorem for dilaton scattering. (Analogous to some soft pion theorems.)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Trace Anomalies

Our first iteration

$$S_{anomaly} = \int d^4x \sqrt{-g}\,\tau \left(cW^2_{\mu\nu\rho\sigma} - aE_4\right) + \cdots .$$

The variation of this includes the anomaly but this is not the whole answer because E_4 is not Weyl invariant.

Thus, the c-anomaly is “Abelian” and the a-anomaly “non-Abelian.” The a-anomaly is therefore quite distinct algebraically from the c-anomaly!

The final answer for $S_{anomaly}$ is:

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Digression: Unitarity

Consider any massless spin 0 particle. Its low energy $2 \to 2$ scattering is given by

$$\mathcal{A}(\tau\tau \to \tau\tau) = \alpha(s^2 + t^2 + u^2) + \mathcal{O}(E^6)$$

In the forward limit we set $t = 0$ and get

$$\mathcal{A}(\tau\tau \to \tau\tau) = 2\alpha s^2 + \mathcal{O}(s^3)$$

Looking at $\mathcal{A}/s^2$ it has a branch cut and a pole at the origin. Closing the contour we get

$$\alpha = \frac{1}{\pi}\int_{s'>0} ds' \frac{\sigma(\tau\tau \to X)}{s'^2}$$

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Such unitarity constraints have been used in many contexts, e.g. chiral Lagrangians (Pham and Truong), W and Z bosons (e.g. Distler, Grinstein, Porto, Rothstein), higher fermion interactions (Adams, Jenkins, O'Connell), SUSY (Dine, Festuccia, ZK).

Also an important interpretation of such analyticity constraints was given by Adams, Arkani-Hamed, Dubovsky, Nicolis, Rattazzi. They found applications for theories of modified gravity.

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The Little a-Theorem

Consider a conformal field theory, CFT$_{UV}$ (with anomalies a_{UV}, c_{UV}), which has a moduli space of vacua $\{VAC\}$.

Pick a conformal symmetry-breaking vacuum. The dilaton must be massless. In addition to it, there may be a nontrivial IR CFT, denoted CFT$_{IR}$ (with anomalies a_{IR}, c_{IR}).

Anomalies in the UV and IR must agree by the usual logic of 't Hooft (emphasized in Schwimmer, Theisen).

This does not mean the a- and c-anomalies of CFT$_{UV,IR}$ match, rather, that the difference must be compensated for by the dilaton.

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The Little a-Theorem

The matching of the total anomaly forces the effective action in the IR to take the form

$$S_{IR}[g_{\mu\nu}] = \mathrm{CFT}_{IR}[g_{\mu\nu}] + \frac{1}{6}f^2\int d^4x\sqrt{-\hat{g}}\hat{R} + \frac{\kappa}{36}\int d^4x\sqrt{-\hat{g}}\hat{R}^2$$

$$+\kappa'\int d^4x\sqrt{-\hat{g}}\hat{W}^2_{\mu\nu\rho\sigma}$$

$$-(a_{UV}-a'_{IR})\int d^4x\sqrt{-g}\left(\tau E_4+4\left(R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R\right)\partial_\mu\tau\partial_\nu\tau-4(\partial\tau)^2\Box\tau+2(\partial\tau)^4\right)$$

$$+(c_{UV}-c'_{IR})\int d^4x\sqrt{-g}\tau W^2_{\mu\nu\rho\sigma}+\cdots$$

Here $a'_{IR} = a_{IR} + a_{scalar}$, $c'_{IR} = c_{IR} + c_{scalar}$. The shift is due to the usual quantum anomaly of the dilaton.

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The Little a-Theorem

We now examine the effective action with a flat space-time metric $g_{\mu\nu} = \eta_{\mu\nu}$. The result is

$$S_{IR} = \mathrm{CFT}_{IR}$$

$$+ \int d^4x \left(f^2 e^{-2\tau} (\partial\tau)^2 + \kappa \left(\Box\,\tau - (\partial\tau)^2\right)^2 + (a_{UV} - a'_{IR}) \left(4(\partial\tau)^2 \Box\,\tau - 2(\partial\tau)^4\right.\right.$$

We see that the difference between the a-anomalies $a_{UV} - a'_{IR}$ appears in front of some specific four-derivative and thus

$$\mathcal{A}(s,t) = \frac{a_{UV} - a'_{IR}}{f^4} \left(s^2 + t^2 + u^2\right) + \ldots$$

Thus,

$$a_{UV} - a'_{IR} = \frac{f^4}{\pi} \int_{s'>0} ds' \frac{\sigma(\tau\tau \to X)}{s'^2} > 0$$

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We now examine the effective action with a flat space-time metric $g_{\mu\nu} = \eta_{\mu\nu}$. The result is

$$S_{IR} = \mathrm{CFT}_{IR}$$

$$+ \int d^4x \left(f^2 e^{-2\tau} (\partial\tau)^2 + \kappa \left(\Box\,\tau - (\partial\tau)^2\right)^2 + (a_{UV} - a'_{IR}) \left(4(\partial\tau)^2 \Box\,\tau - 2(\partial\tau)^4\right.\right.$$

We see that the difference between the a-anomalies $a_{UV} - a'_{IR}$ appears in front of some specific four-derivative and thus

$$\mathcal{A}(s,t) = \frac{a_{UV} - a'_{IR}}{f^4} \left(s^2 + t^2 + u^2\right) + \ldots$$

Thus,

$$a_{UV} - a'_{IR} = \frac{f^4}{\pi} \int_{s'>0} ds' \frac{\sigma(\tau\tau \to X)}{s'^2} > 0$$

The Little a-Theorem

This result applies to many highly nontrivial flows that can be investigated (such moduli space are ubiquitous in SUSY). In fact, a “counterexample” (recently revisited by Gaiotto,Seiberg,Tachikawa) was supposed to utilize such a mechanism.

Let us now consider general massive RG flows.

General RG Flows

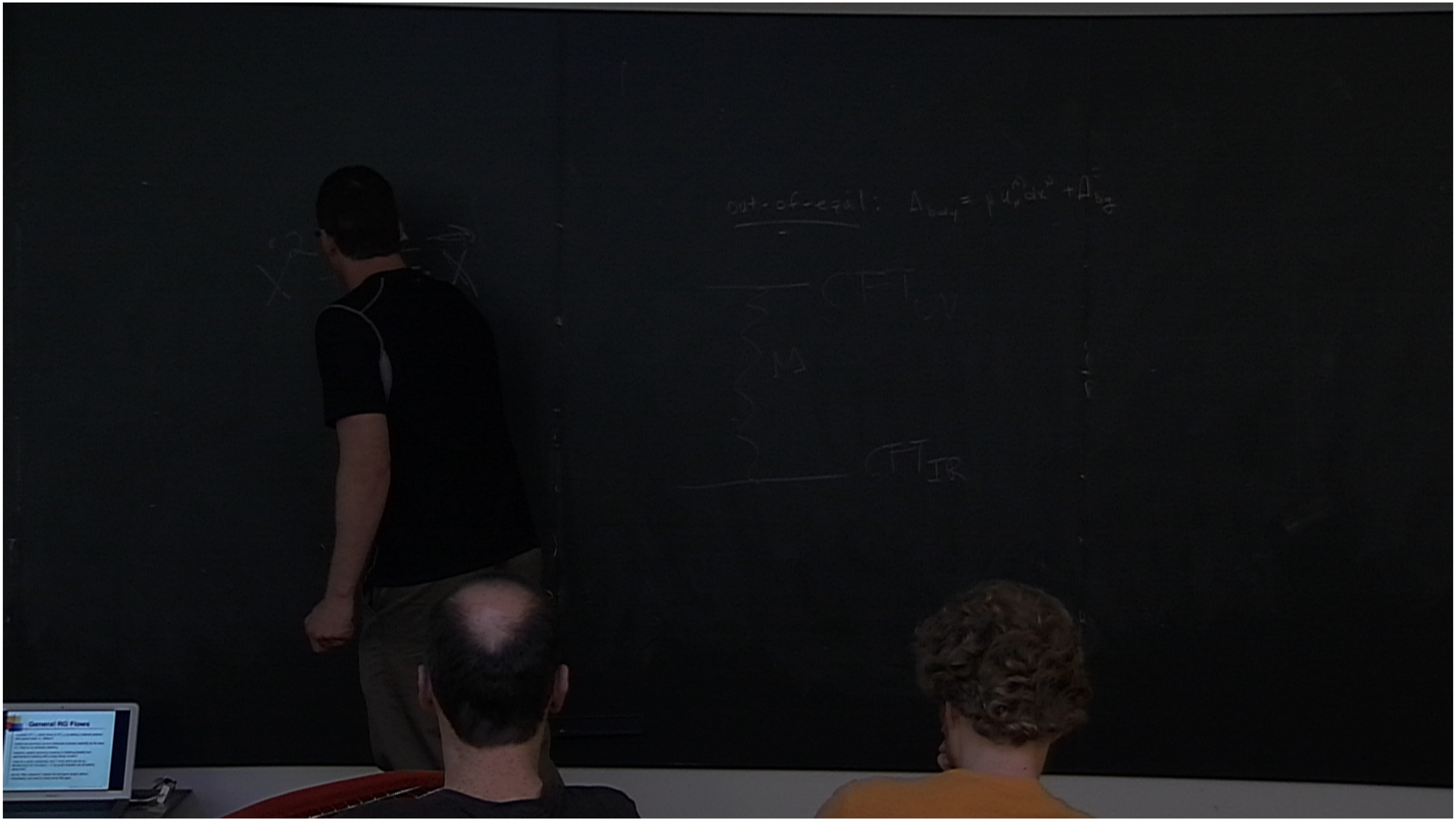
Consider CFT$_{UV}$ which flows to CFT$_{IR}$ by adding a relevant operator with typical mass M. (Wilson)

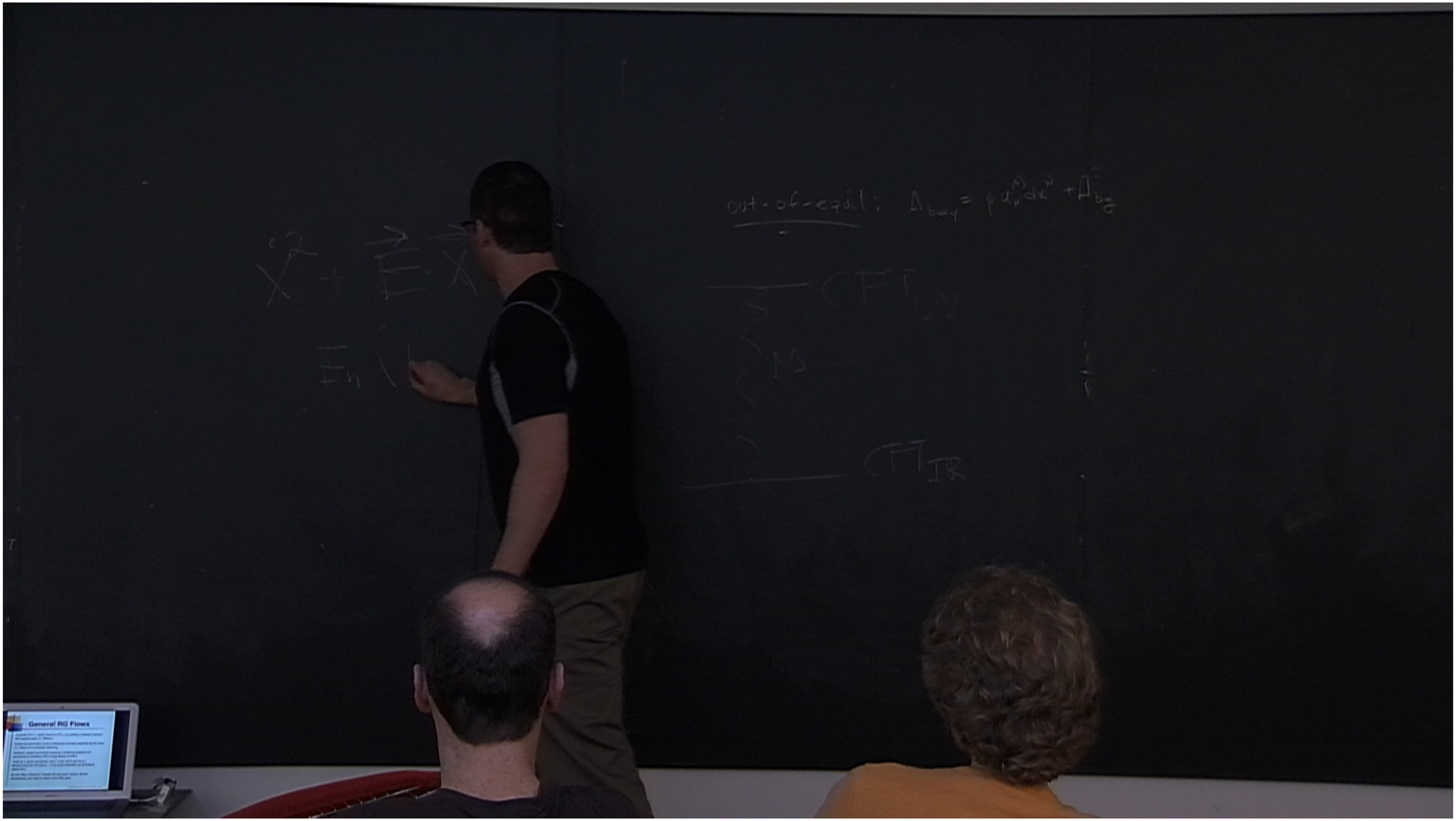
Conformal symmetry at short distances is broken explicitly by the mass M. There is no anomaly matching.

However, explicit symmetry breaking is indistinguishable from spontaneous breaking with a large decay constant.

THIS IS A VERY GENERAL FACT THAT APPLIES IN ALL BRANCHES OF PHYSICS – IT IS ALSO KNOWN AS SPURION ANALYSIS.

So the “little a-theorem” implies the strongest version almost immediately, just need to close some little gaps.


out-of-equil:
CFT UV
CFT IR
General RG Flows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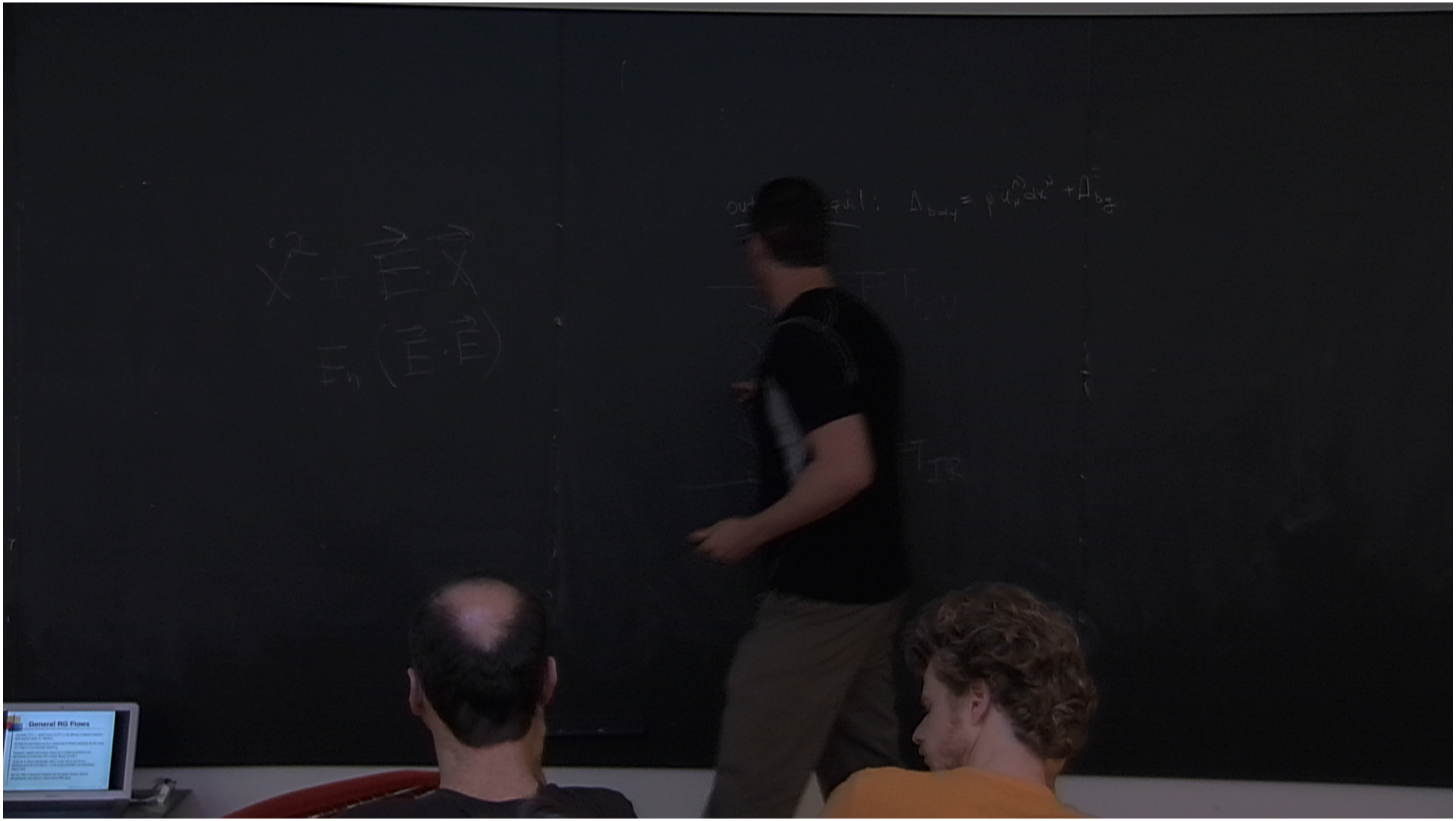
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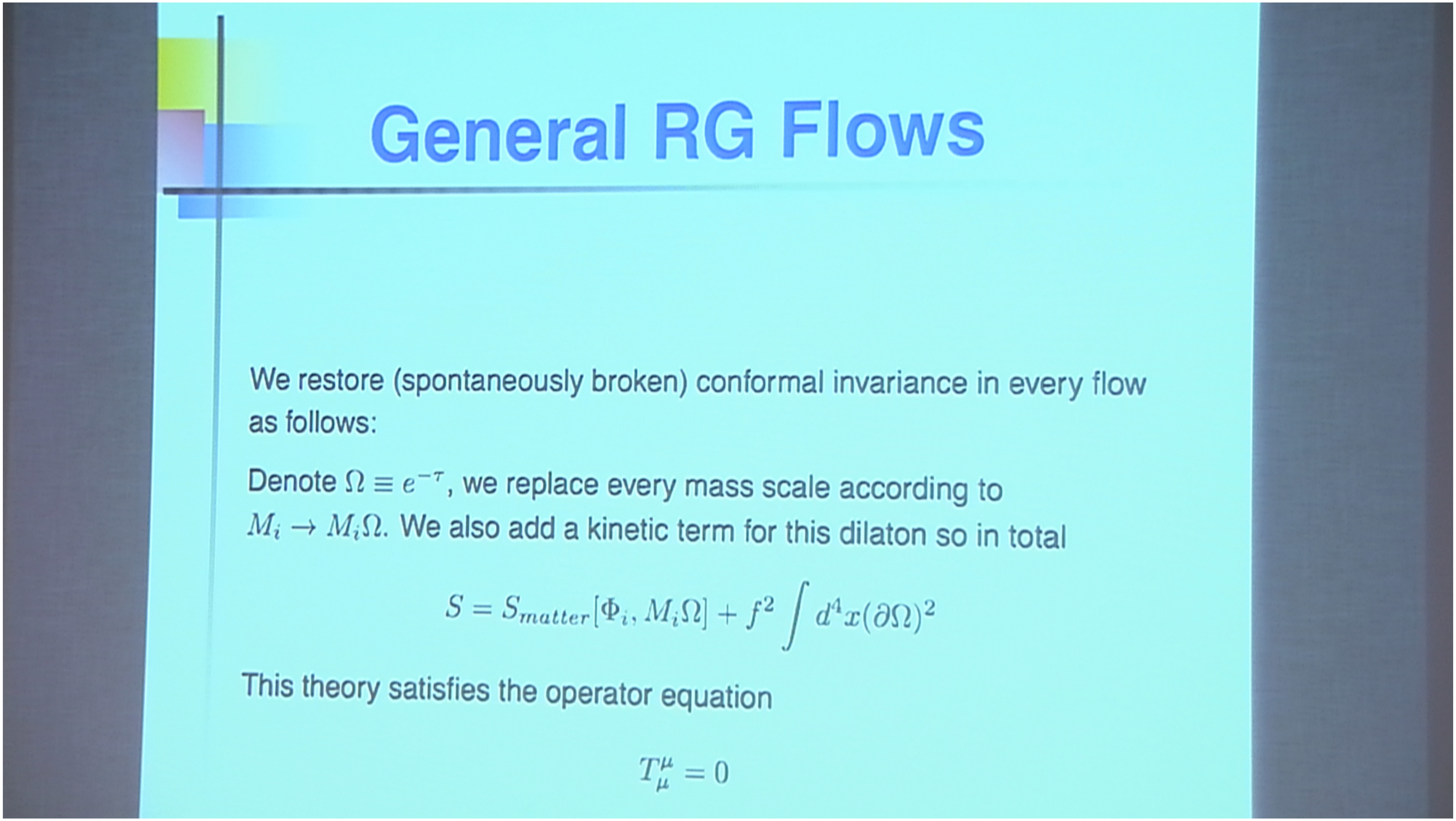
General RG Flows
We restore (spontaneously broken) conformal invariance in every flow as follows:
Denote $\Omega \equiv e^{-\tau}$, we replace every mass scale according to $M_i \to M_i\Omega$. We also add a kinetic term for this dilaton so in total
$$S = S_{matter}[\Phi_i, M_i\Omega] + f^2 \int d^4x (\partial\Omega)^2$$
This theory satisfies the operator equation
$$T^\mu_\mu = 0$$

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General RG Flows

Since f appears as the coefficient of the kinetic term of the dilaton, we see that the physical dilaton fluctuations couple to matter fields by inverse powers of f and thus if we take

$$M_i \ll f$$

the coupling between the dilaton and the matter sector is arbitrarily weak.

So the original flow is unperturbed.

Now anomaly matching applies; the dilaton carries the difference between the anomalies of the UV and IR theories.

We proceed as before, the IR action is:

General RG Flows

$$S_{IR}[g_{\mu\nu}] = \mathrm{CFT}_{IR}[g_{\mu\nu}] + \frac{1}{6}f^2 \int d^4x\sqrt{-\hat{g}}\hat{R} + \frac{\kappa}{36}\int d^4x\sqrt{-\hat{g}}\hat{R}^2$$

$$+\kappa' \int d^4x\sqrt{-\hat{g}}\hat{W}^2_{\mu\nu\rho\sigma}$$

$$-(a_{UV}-a_{IR})\int d^4x\sqrt{-g}\left(\tau E_4 + 4\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right)\partial_\mu\tau\partial_\nu\tau - 4(\partial\tau)^2\Box\,\tau + 2(\partial\tau)^4\right.$$

$$+(c_{UV} - c_{IR})\int d^4x\sqrt{-g}\tau W^2_{\mu\nu\rho\sigma} + \cdots$$

Thus,

$$a_{UV} - a_{IR} = \frac{f^4}{\pi}\int_{s'>0} ds' \frac{\sigma(\tau\tau \to X)}{s'^2} > 0$$

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Examples

Consider a free massive scalar

$$S = \frac{1}{2}\int d^4x \left((\partial\Phi)^2 - M^2\Phi^2\right)$$

It flows from the CFT of a single massless particle to the empty theory. We can render this flow spontaneously conformally broken via the dilaton

$$S = \frac{1}{2}\int d^4x \left((\partial\Phi)^2 - M^2 e^{-2\tau}\Phi^2\right)$$

We can now perform the path integral over Φ and obtain an effective action for the dilaton.

This is done by expanding the τ dependent $\det(\Box + M^2 e^{-2\tau})$ in derivatives of τ.

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Examples

There are many nontrivial cancelations, and the answer in fact eventually matches the answer allowed by our analysis of conformal invariance

$$\kappa\left(\Box\tau - (\partial\tau)^2\right)^2 + 4a(\partial\tau)^2\Box\tau - 2a(\partial\tau)^4$$

This agreement depends on infinitely many consistency checks, for instance, because the heat kernel above terminates at four τs.

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Examples

We also demonstrate the procedure by studying a general perturbative flow. Consider a CFT with an operator $\mathcal{O}$ of dimension $\Delta = 4 - \epsilon$. Take $\epsilon \ll 1$. Then deforming

$$\delta S = \lambda \mu^{\epsilon} \mathcal{O}$$

we have $\beta_\lambda = -\epsilon\lambda + C\lambda^2 + \ldots$ Thus

$$\lambda_* = \epsilon / C$$

is a nearby nontrivial CFT.

The massive flow evolving from $\lambda = 0$ to λ_* can be rendered conformal by writing

$$\delta S = \lambda(e^{\tau}(x)\mu)\mu^{\epsilon}\mathcal{O}$$

Examples

Integrating out from the scale μ to a lower scale we find that one generates at four derivatives the effective action

$$\delta\kappa\left(\Box\tau-(\partial\tau)^2\right)^2+4\delta a(\partial\tau)^2\Box\tau-2\delta a(\partial\tau)^4 .$$

This again depends on non-trivial consistency checks.

One finds $\delta a=-Vol(S^3)\beta_\lambda^2\delta\log\mu$ which can be integrated to give

$$\Delta a=-Vol(S^3)\int_0^{\lambda_*}\beta_\lambda d\lambda$$

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Zamolodchikov's Theorem

Here we explain how to prove the C-theorem in two dimensions in the spirit of our ideas.

As before, consider the dilation coupled to some background metric.

One can write many diff×Weyl invariants, but with two derivatives there is only one candidate $\int \sqrt{g}\hat{R}$, which is a total derivative. So there is no diff×Weyl invariant term with two derivatives.

We must invoke anomalies to find such a term. Indeed, there is the WZ action for trace anomalies also in two dimensions

$$S_{WZ} = \int \sqrt{g} \left(\tau R + (\partial \tau)^2 \right)$$

This is the sole source for two-derivative terms in the action.

Zamolodchikov's Theorem

We again couple τ as an auxiliary field that restores conformal invariance. The coupling as before takes the form τT^{μ}_{μ}.

Now that the theory breaks conformal invariance only spontaneously we invoke anomaly matching to find that the coefficient of the WZ term must be

$$S_{WZ} = \Delta c \int \sqrt{g} \left(\tau R + (\partial \tau)^2 \right) \ .$$

We hence find that

$$\Delta c = \frac{\partial^2}{\partial q \partial q} \langle T^{\mu}_{\mu}(q) T^{\mu}_{\mu}(-q) \rangle \Big|_{q=0} \ .$$

Zamolodchikov's Theorem

Fourier transforming this to position space we find

$$\Delta c = \int x^2 \langle T^\mu_\mu(x) T^\mu_\mu(0) \rangle d^2x > 0 \ .$$

This agree with the known results about the C function.

Note that here we do not need a scattering matrix argument to establish positivity, it follows directly from reflection positivity in Euclidean field theories. In other words, it is just the statement that the auxiliary field acquires a positive definite kinetic term.

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Open Questions

- The "non-Abelian" structure of the Euler anomaly is important. This leads to the universal WZ-like $2 \to 2$ scattering. It would be interesting to understand better the algebraic (cohomological) structure of this phenomenon.
- We have constructed a monotonic decreasing function that interpolates between a_{UV} and a_{IR}. However, we have not said anything about gradient flow. This should be interesting to address.
- The dilaton is clearly an auxiliary object, it is a bookkeeping device for four-point functions of the EM tensor. The fact that the a-anomaly appears in four point functions is the key for positivity.

Open Questions

- Interesting to understand the effective action of the dilaton on the moduli space of $\mathcal{N} = 4$ Yang-Mills theory and to compare with expectations from strong coupling.
- These WZ-like dilaton self-interaction may have an interesting manifestation in holography. Consider conformal symmetry breaking in AdS spaces and try to identify the (perhaps geometric) reason for the universality of the coefficient of this interaction.
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