Title: The Sheaf-Theoretic Structure of Non-Locality and Contextuality

Date: Nov 01, 2011 03:30 PM

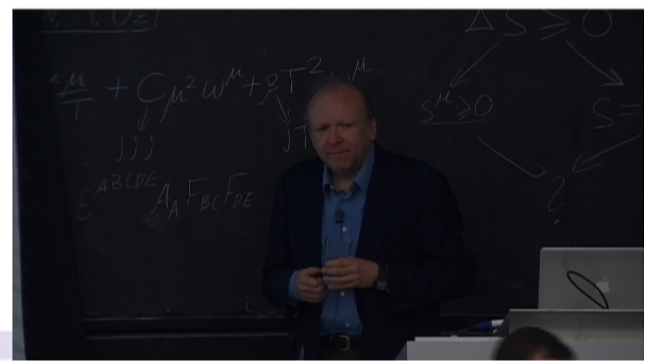
URL: http://pirsa.org/11110108

Abstract: We use the mathematical language of sheaf theory to give a unified treatment of non-locality and contextuality, which generalizes the familiar probability tables used in non-locality theory to cover Kochen-Specker configurations and more. We show that contextuality, and non-locality as a special case, correspond exactly to *obstructions to the existence of global sections*.

We describe a linear algebraic approach to computing these obstructions, which allows a systematic treatment of arguments for non-locality and contextuality. A general correspondence is shown between the existence of local hidden-variable realizations using negative probabilities, and no-signalling. Maximal non-locality is generalized to maximal contextuality, and characterized in purely qualitative terms, as the non-existence of global sections in the support. Some ongoing work with Shane Mansfield and Rui Soares Barbosa is described, which identifies *cohomological obstructions* to the existence of global sections, opening the possibility of applying the powerful methods of cohomology to non-locality and contextuality.

The Sheaf-Theoretic Structure Of Non-Locality and Contextuality

Samson Abramsky
Joint work with Adam Brandenburger



Overview

- The usual probability tables of non-locality theory ('Bell-type scenarios') are generalized to **measurement covers**. These include Kochen-Specker configurations, and more. This provides a setting for a fully unified treatment of contextuality and non-locality.

Overview

- The usual probability tables of non-locality theory ('Bell-type scenarios') are generalized to **measurement covers**. These include Kochen-Specker configurations, and more. This provides a setting for a fully unified treatment of contextuality and non-locality.
- We use the mathematical language of **sheaf theory**. We show that non-locality and contextuality can be characterized precisely in terms of the existence of **obstructions to global sections**.

Overview

- The usual probability tables of non-locality theory ('Bell-type scenarios') are generalized to **measurement covers**. These include Kochen-Specker configurations, and more. This provides a setting for a fully unified treatment of contextuality and non-locality.
- We use the mathematical language of **sheaf theory**. We show that non-locality and contextuality can be characterized precisely in terms of the existence of **obstructions to global sections**.
- Sheaf theory is exactly about functorial variation over contexts; it provides a general 'logic of contextuality'. Has been used this way, e.g. in CS. Opens the possibility of links between study of non-locality and contextuality in Quantum Foundations, and other fields.

Overview

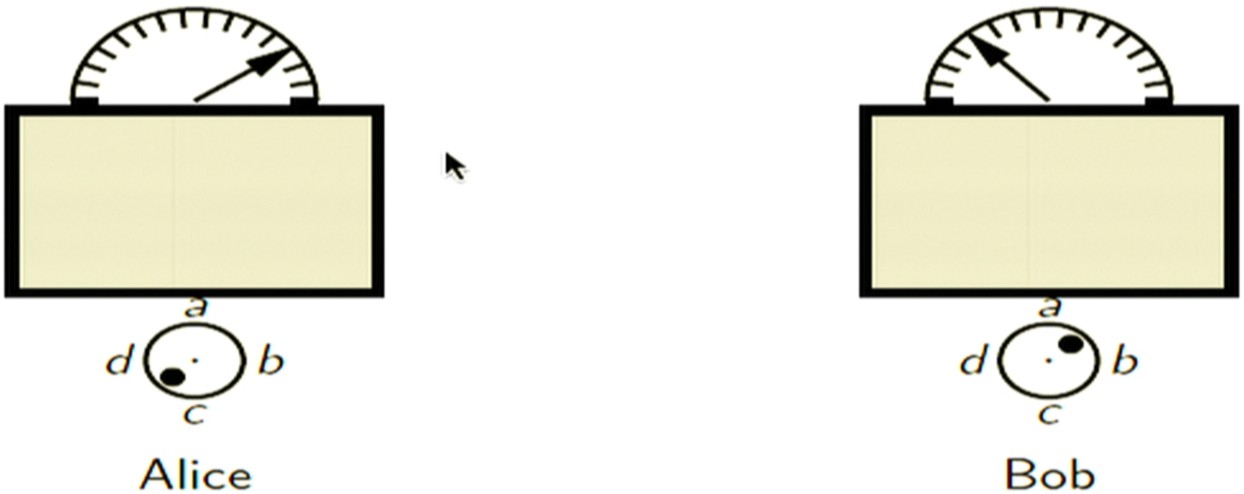
- The usual probability tables of non-locality theory ('Bell-type scenarios') are generalized to **measurement covers**. These include Kochen-Specker configurations, and more. This provides a setting for a fully unified treatment of contextuality and non-locality.
- We use the mathematical language of **sheaf theory**. We show that non-locality and contextuality can be characterized precisely in terms of the existence of **obstructions to global sections**.
- Sheaf theory is exactly about functorial variation over contexts; it provides a general 'logic of contextuality'. Has been used this way, e.g. in CS. Opens the possibility of links between study of non-locality and contextuality in Quantum Foundations, and other fields.
- Contrast with 'generalized probability theories'. We use classical probability, encapsulated in the distribution functor/monad; contextuality arises from **functorial variation over contexts**.

Overview

- The usual probability tables of non-locality theory ('Bell-type scenarios') are generalized to **measurement covers**. These include Kochen-Specker configurations, and more. This provides a setting for a fully unified treatment of contextuality and non-locality.
- We use the mathematical language of **sheaf theory**. We show that non-locality and contextuality can be characterized precisely in terms of the existence of **obstructions to global sections**.
- Sheaf theory is exactly about functorial variation over contexts; it provides a general 'logic of contextuality'. Has been used this way, e.g. in CS. Opens the possibility of links between study of non-locality and contextuality in Quantum Foundations, and other fields.
- Contrast with 'generalized probability theories'. We use classical probability, encapsulated in the distribution functor/monad; contextuality arises from **functorial variation over contexts**.

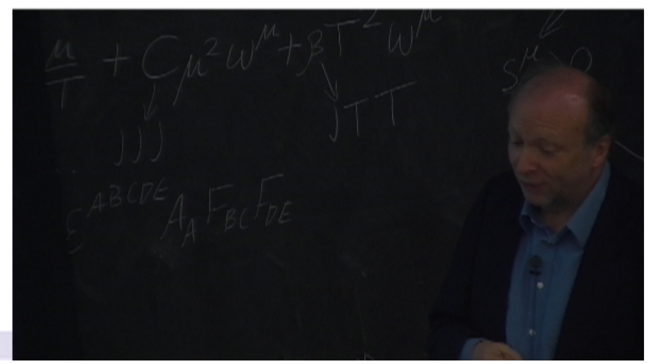
S. Abramsky and A. Brandenburger, The Sheaf-Theoretic Structure of Non-Locality and Contextuality. Available at `arXiv:1102.0264`. To appear in *New Journal of Physics*.

The Basic Scenario



A Probabilistic Model Of An Experiment

A	B	(0, 0)	(1, 0)	(0, 1)	(1, 1)
a	b	1/2	0	0	1/2
a'	b	3/8	1/8	1/8	3/8
a	b'	3/8	1/8	1/8	3/8
a'	b'	1/8	3/8	3/8	1/8

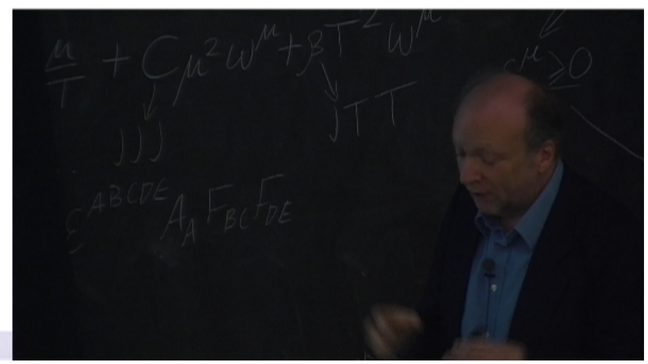


A Probabilistic Model Of An Experiment

A	B	(0,0)	(1,0)	(0,1)	(1,1)
a	b	1/2	0	0	1/2
a'	b	3/8	1/8	1/8	3/8
a	b'	3/8	1/8	1/8	3/8
a'	b'	1/8	3/8	3/8	1/8

The **measurement contexts** are

$$\{a, b\}, \quad \{a', b\}, \quad \{a, b'\}, \quad \{a', b'\}.$$



A Probabilistic Model Of An Experiment

A	B	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
a	b	$1/2$	0	0	$1/2$
a'	b	$3/8$	$1/8$	$1/8$	$3/8$
a	b'	$3/8$	$1/8$	$1/8$	$3/8$
a'	b'	$1/8$	$3/8$	$3/8$	$1/8$

The **measurement contexts** are

$$\{a, b\}, \quad \{a', b\}, \quad \{a, b'\}, \quad \{a', b'\}.$$

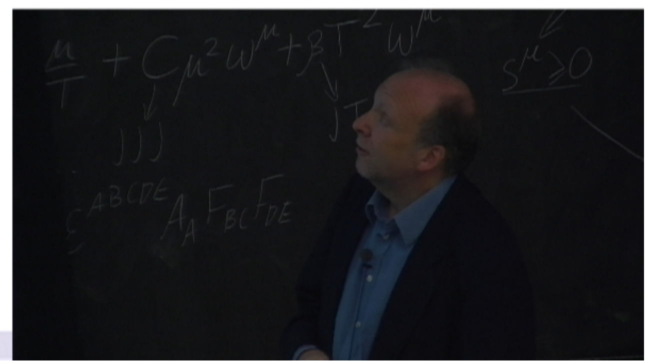
Each measurement has possible outcomes 0 or 1. The matrix entry at row (a', b) and column $(0, 1)$ indicates the **event**

$$\{a' \mapsto 0, \ b \mapsto 1\}.$$

The Presheaf of Distributions

We fix a set of measurements X, and a set of outcomes O.

For each set of measurements $U \subseteq X$, we define $\mathcal{D}_R\mathcal{E}(U)$ to be the set of probability distributions on events $s : U \to O$. Such an event specifies that outcome $s(m)$ occurs for each measurement $m \in U$.



The Presheaf of Distributions

We fix a set of measurements X, and a set of outcomes O.

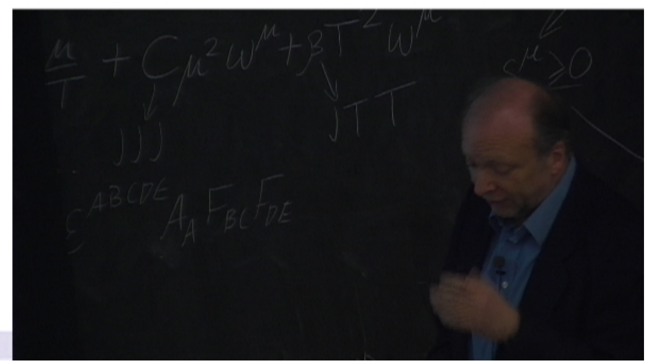
For each set of measurements $U \subseteq X$, we define $\mathcal{D}_R\mathcal{E}(U)$ to be the set of probability distributions on events $s : U \to O$. Such an event specifies that outcome $s(m)$ occurs for each measurement $m \in U$.

Given $U \subseteq U'$, we have an operation of **restriction**:

$$\mathcal{D}_R\mathcal{E}(U') \longrightarrow \mathcal{D}_R\mathcal{E}(U) :: d \mapsto d|U,$$

where for each $s \in \mathcal{E}(U)$:

$$d|U(s) \ := \sum_{s' \in \mathcal{E}(U'), s'|U = s} d(s').$$



The Presheaf of Distributions

We fix a set of measurements X, and a set of outcomes O.

For each set of measurements $U \subseteq X$, we define $\mathcal{D}_R\mathcal{E}(U)$ to be the set of probability distributions on events $s : U \rightarrow O$. Such an event specifies that outcome $s(m)$ occurs for each measurement $m \in U$.

Given $U \subseteq U'$, we have an operation of **restriction**:

$$\mathcal{D}_R\mathcal{E}(U') \longrightarrow \mathcal{D}_R\mathcal{E}(U) :: d \mapsto d|U,$$

where for each $s \in \mathcal{E}(U)$:

$$d|U(s) \ := \sum_{s' \in \mathcal{E}(U'), s'|U = s} d(s').$$

Thus $d|U$ is the **marginal** of the distribution d, which assigns to each section s in the smaller context U the sum of the weights of all sections s' in the larger context which restrict to s.

The Presheaf of Distributions

We fix a set of measurements X, and a set of outcomes O.

For each set of measurements $U \subseteq X$, we define $\mathcal{D}_R\mathcal{E}(U)$ to be the set of probability distributions on events $s : U \to O$. Such an event specifies that outcome $s(m)$ occurs for each measurement $m \in U$.

Given $U \subseteq U'$, we have an operation of **restriction**:

$$\mathcal{D}_R\mathcal{E}(U') \longrightarrow \mathcal{D}_R\mathcal{E}(U) :: d \mapsto d|U,$$

where for each $s \in \mathcal{E}(U)$:

$$d|U(s) \;:=\; \sum_{s' \in \mathcal{E}(U'), s'|U = s} d(s').$$

Thus $d|U$ is the **marginal** of the distribution d, which assigns to each section s in the smaller context U the sum of the weights of all sections s' in the larger context which restrict to s.

Mathematical notes: (i) This is functorial, hence defines a presheaf.
(ii) Composed from the sheaf $\mathcal{E}(U) := O^U$ and the distributions monad $\mathcal{D}_R$.
(iii) We can vary R.

Empirical Models: Reconstructing Probability Tables

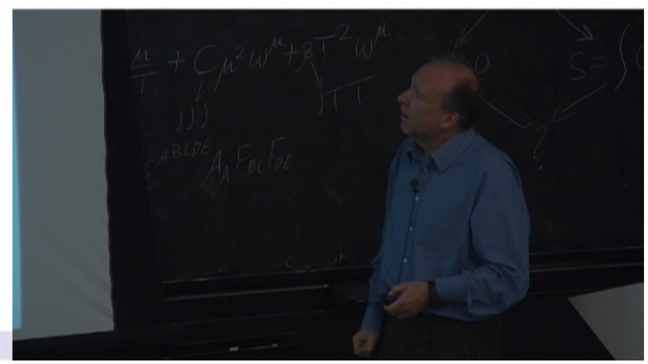
Corresponding to the choices of measurements by agents, or more generally to the idea that it may not be possible to perform all measurements together, we consider a **cover** $\mathcal{M}$: a family of subsets of X which covers X, $\bigcup \mathcal{M} = X$.

Empirical Models: Reconstructing Probability Tables

Corresponding to the choices of measurements by agents, or more generally to the idea that it may not be possible to perform all measurements together, we consider a **cover** $\mathcal{M}$: a family of subsets of X which covers X, $\bigcup \mathcal{M} = X$.

The sets $C \in \mathcal{M}$ are the **measurement contexts**; the sets of measurements which can be performed together.

These are the sets which index the rows of a generalized probability table.

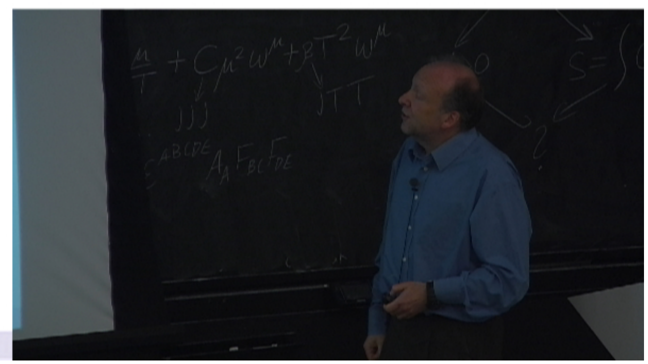


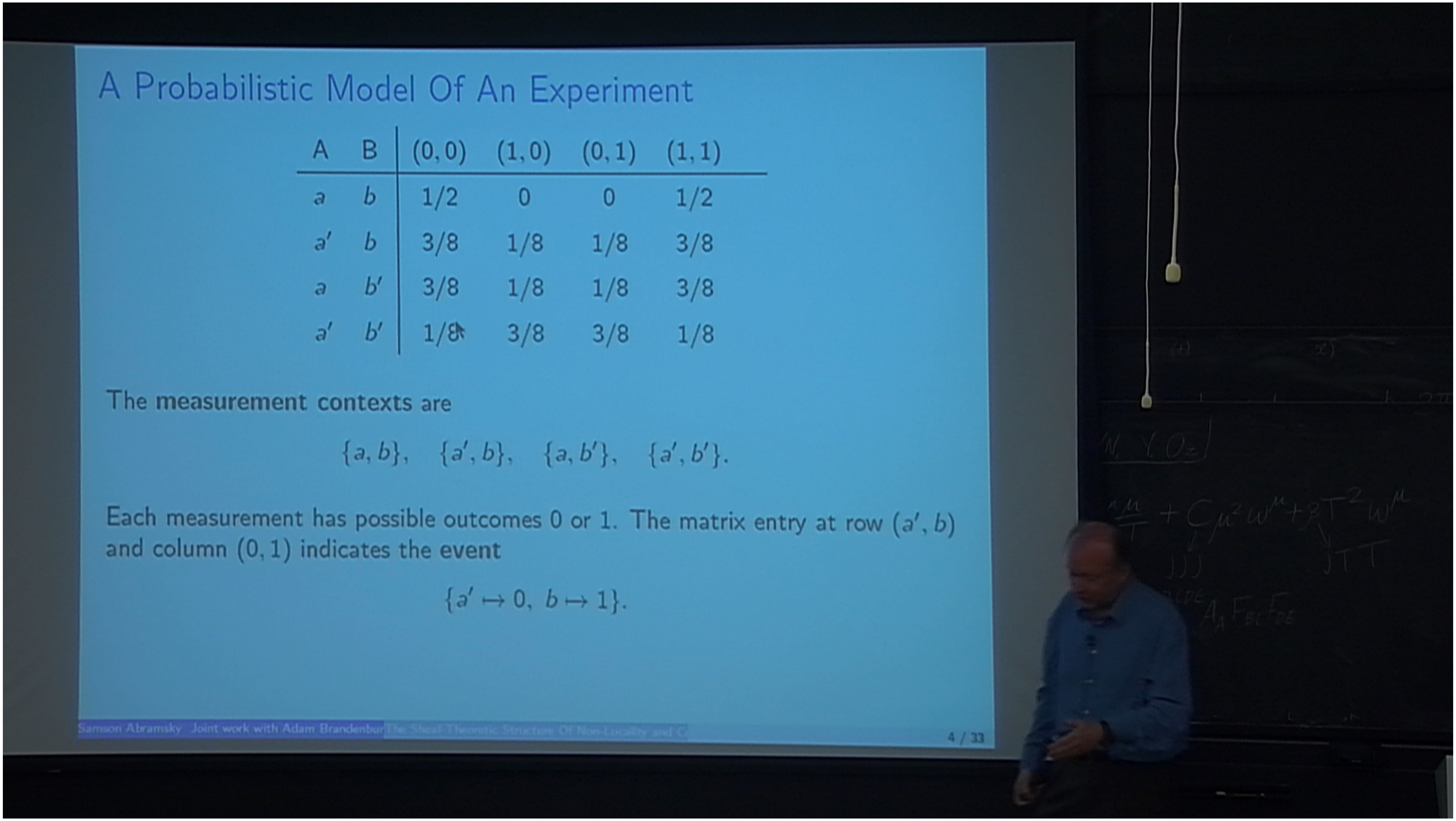
Empirical Models: Reconstructing Probability Tables

Corresponding to the choices of measurements by agents, or more generally to the idea that it may not be possible to perform all measurements together, we consider a **cover** $\mathcal{M}$: a family of subsets of X which covers X, $\bigcup \mathcal{M} = X$.

The sets $C \in \mathcal{M}$ are the **measurement contexts**; the sets of measurements which can be performed together.

These are the sets which index the rows of a generalized probability table.




A Probabilistic Model Of An Experiment
A B (0,0) (1,0) (0,1) (1,1)
a b 1/2 0 0 1/2
a' b 3/8 1/8 1/8 3/8
a b' 3/8 1/8 1/8 3/8
a' b' 1/8 3/8 3/8 1/8
The measurement contexts are
{a, b}, {a', b}, {a, b'}, {a', b'}.
Each measurement has possible outcomes 0 or 1. The matrix entry at row (a', b) and column (0, 1) indicates the event
{a' ↦ 0, b ↦ 1}.
Samson Abramsky Joint work with Adam Brandenbur The Sheaf-Theoretic Structure Of Non-Locality and C
4 / 33

Empirical Models: Reconstructing Probability Tables

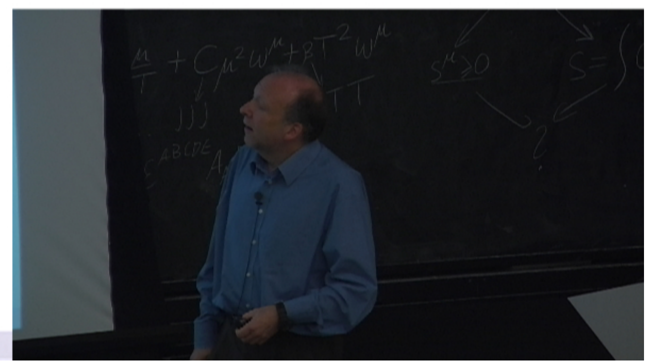
Corresponding to the choices of measurements by agents, or more generally to the idea that it may not be possible to perform all measurements together, we consider a **cover** $\mathcal{M}$: a family of subsets of X which covers X, $\bigcup \mathcal{M} = X$.

The sets $C \in \mathcal{M}$ are the **measurement contexts**; the sets of measurements which can be performed together.

These are the sets which index the rows of a generalized probability table.

Covers are general: they include both the usual 'Bell scenarios', and Kochen-Specker type constructions.

An **empirical model** for $\mathcal{M}$ is a family $\{e_C\}_{C \in \mathcal{M}}$, $e_C \in \mathcal{D}_R\mathcal{E}(C)$.



Compatibility And No-Signalling

We shall consider models $\{e_C \mid C \in \mathcal{M}\}$ which are **compatible** in the sense of agreeing on overlaps: for all $C, C' \in \mathcal{M}$,

$$e_C | C \cap C' = e_{C'} | C \cap C'.$$

Global Sections

We are given an empirical model $\{e_C\}_{C \in \mathcal{M}}$.

Question: does there exist a **global section** for this family?

Global Sections

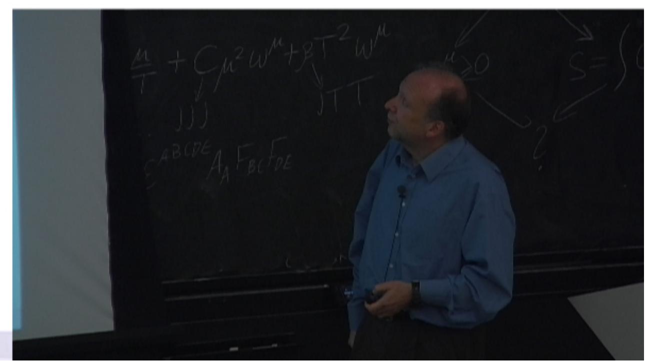
We are given an empirical model $\{e_C\}_{C \in \mathcal{M}}$.

Question: does there exist a **global section** for this family?

I.e. $d \in \mathcal{D}_R\mathcal{E}(X)$ such that, for all $C \in \mathcal{M}$

$$d|C = e_C.$$

A distribution, defined on **all** measurements, which marginalizes to yield the empirically observed probabilities?



Global Sections

We are given an empirical model $\{e_C\}_{C \in \mathcal{M}}$.

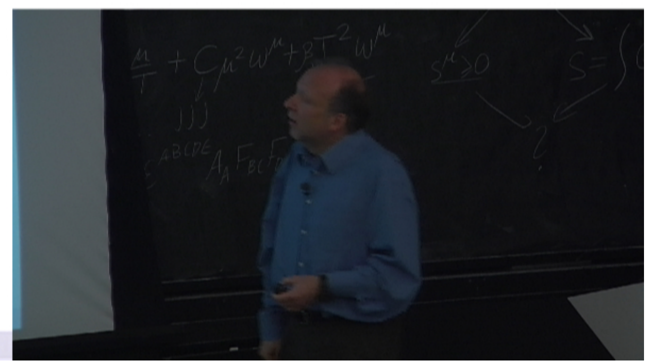
Question: does there exist a **global section** for this family?

I.e. $d \in \mathcal{D}_R\mathcal{E}(X)$ such that, for all $C \in \mathcal{M}$

$$d|C = e_C.$$

A distribution, defined on **all** measurements, which marginalizes to yield the empirically observed probabilities?

Note that $s \in \mathcal{E}(X) = O^X$ specifies an outcome for every measurement simultaneously, independent of the measurement context.



Global Sections

We are given an empirical model $\{e_C\}_{C \in \mathcal{M}}$.

Question: does there exist a **global section** for this family?

I.e. $d \in \mathcal{D}_R\mathcal{E}(X)$ such that, for all $C \in \mathcal{M}$

$$d|C = e_C.$$

A distribution, defined on **all** measurements, which marginalizes to yield the empirically observed probabilities?

Note that $s \in \mathcal{E}(X) = O^X$ specifies an outcome for every measurement simultaneously, independent of the measurement context.
For every context C, it restricts to yield $s|C$.

Thus it can be seen as a **deterministic hidden variable**.

If d is a global section for the model $\{e_C\}$, we recover the predictions of the model by **averaging over the values of these hidden variables**:

$$e_C(s) \;=\; d|C(s) \;=\; \sum_{s' \in \mathcal{E}(X), s'|C = s} d(s') \;=\; \sum_{s' \in \mathcal{E}(X)} \delta_{s'|C}(s) \cdot d(s').$$

Global Sections

We are given an empirical model $\{e_C\}_{C \in \mathcal{M}}$.

Question: does there exist a **global section** for this family?

I.e. $d \in \mathcal{D}_R\mathcal{E}(X)$ such that, for all $C \in \mathcal{M}$

$$d|C = e_C.$$

A distribution, defined on **all** measurements, which marginalizes to yield the empirically observed probabilities?

Note that $s \in \mathcal{E}(X) = O^X$ specifies an outcome for every measurement simultaneously, independent of the measurement context.
For every context C, it restricts to yield $s|C$.

Thus it can be seen as a **deterministic hidden variable**.

If d is a global section for the model $\{e_C\}$, we recover the predictions of the model by **averaging over the values of these hidden variables**:

$$e_C(s) = d|C(s) = \sum_{s' \in \mathcal{E}(X), s'|C = s} d(s') = \sum_{s' \in \mathcal{E}(X)} \delta_{s'|C}(s) \cdot d(s').$$

Global Sections Subsume Hidden-Variable Theories

Note also that this is a **local** model:

$$\delta_s|C(s') \;=\; \prod_{x \in C} \delta_{s|x}(s'|x).$$

The joint probabilities determined by s factor as a product of the probabilities assigned to the individual measurements, independent of the context in which they appear. This subsumes **Bell locality** .

So a global section **is** a deterministic local hidden-variable model.

The general result is as follows:

Theorem

Any factorizable (i.e. local) hidden-variable model defines a global section.

Global Sections Subsume Hidden-Variable Theories

Note also that this is a **local** model:

$$\delta_s | C(s') \;=\; \prod_{x \in C} \delta_{s|x}(s'|x).$$

The joint probabilities determined by s factor as a product of the probabilities assigned to the individual measurements, independent of the context in which they appear. This subsumes **Bell locality** .

So a global section **is** a deterministic local hidden-variable model.

The general result is as follows:

Theorem

Any factorizable (i.e. local) hidden-variable model defines a global section.

So:

existence of a local hidden-variable model for a given empirical model
IFF
empirical model has a global section

Global Sections Subsume Hidden-Variable Theories

Note also that this is a **local** model:

$$\delta_s | C(s') \;=\; \prod_{x \in C} \delta_{s|x}(s'|x).$$

The joint probabilities determined by s factor as a product of the probabilities assigned to the individual measurements, independent of the context in which they appear. This subsumes **Bell locality** .

So a global section **is** a deterministic local hidden-variable model.

The general result is as follows:

Theorem

Any factorizable (i.e. local) hidden-variable model defines a global section.

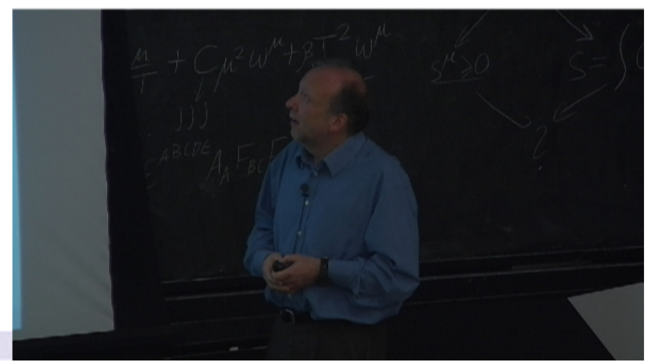
Hence:

No such h.v. model exists (the empirical model is **non-local/contextual**)
IFF
there is an **obstruction to the existence of a global section**

Existence of Global Sections

Linear algebraic method.

Define system of linear equations $\mathbf{M}\mathbf{x} = \mathbf{v}$.



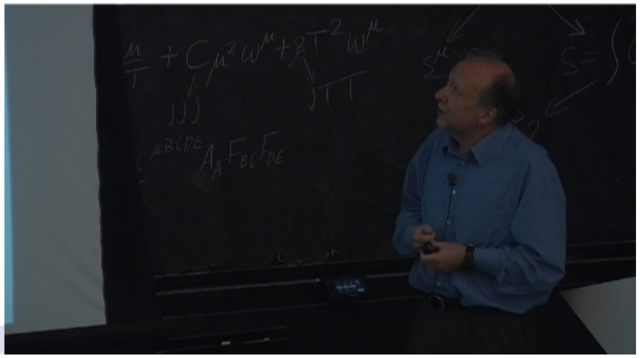
Existence of Global Sections

Linear algebraic method.

Define system of linear equations $\mathbf{Mx} = \mathbf{v}$.

$$\text{Solutions} \quad \longleftrightarrow \quad \text{Global sections}$$

Incidence matrix $\mathbf{M}$ (0/1 entries). Depends only on $\mathcal{M}$ and O.



Existence of Global Sections

Linear algebraic method.

Define system of linear equations $\mathbf{Mx} = \mathbf{v}$.

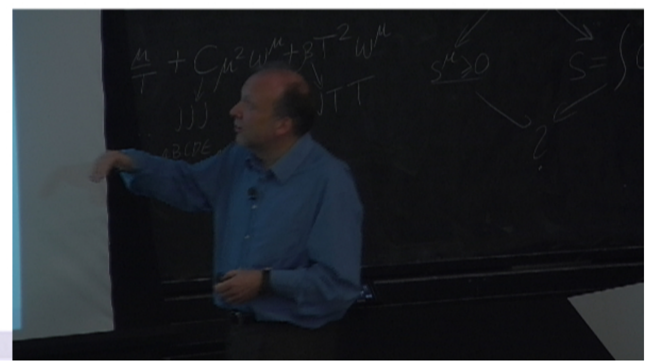
$$\text{Solutions} \quad \longleftrightarrow \quad \text{Global sections}$$

Incidence matrix $\mathbf{M}$ (0/1 entries). Depends only on $\mathcal{M}$ and O.

Enumerate $\coprod_{C \in \mathcal{M}} O^C$ as $s_1, \ldots, s_p$.

Enumerate O^X as $t_1, \ldots, t_q$.

$$\mathbf{M}[i,j] = 1 \iff t_j | C = s_i \quad (s_i \in \mathcal{E}(C)).$$



Existence of Global Sections

Linear algebraic method.

Define system of linear equations $\mathbf{Mx} = \mathbf{v}$.

$$\text{Solutions} \quad \longleftrightarrow \quad \text{Global sections}$$

Incidence matrix $\mathbf{M}$ (0/1 entries). Depends only on $\mathcal{M}$ and O.

Enumerate $\coprod_{C \in \mathcal{M}} O^C$ as $s_1, \ldots, s_p$.

Enumerate O^X as $t_1, \ldots, t_q$.

$$\mathbf{M}[i,j] = 1 \iff t_j|C = s_i \quad (s_i \in \mathcal{E}(C)).$$

Conceptually, boolean matrix representation of the map

$$\mathcal{E}(X) \longrightarrow \prod_{C \in \mathcal{M}} \mathcal{E}(C) :: s \mapsto (s|C)_{C \in \mathcal{M}}.$$

Existence of Global Sections

Linear algebraic method.

Define system of linear equations $\mathbf{M}\mathbf{x} = \mathbf{v}$.

$$\text{Solutions} \quad \longleftrightarrow \quad \text{Global sections}$$

Incidence matrix $\mathbf{M}$ (0/1 entries). Depends only on $\mathcal{M}$ and O.

Enumerate $\coprod_{C \in \mathcal{M}} O^C$ as $s_1, \ldots, s_p$.

Enumerate O^X as $t_1, \ldots, t_q$.

$$\mathbf{M}[i, j] = 1 \iff t_j | C = s_i \quad (s_i \in \mathcal{E}(C)).$$

Conceptually, boolean matrix representation of the map

$$\mathcal{E}(X) \longrightarrow \prod_{C \in \mathcal{M}} \mathcal{E}(C) :: s \mapsto (s|C)_{C \in \mathcal{M}}.$$

Bell scenarios (n, k, l): matrix is $(kl)^n \times l^{kn}$.

The $(2,2,2)$ Incidence Matrix

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

This matrix has rank 9.

The $(2, 2, 2)$ Incidence Matrix

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

This matrix has rank 9.

In general, the matrix for $(n, 2, 2)$ has rank 3^n. This is a special case of a much more general result we will describe later.

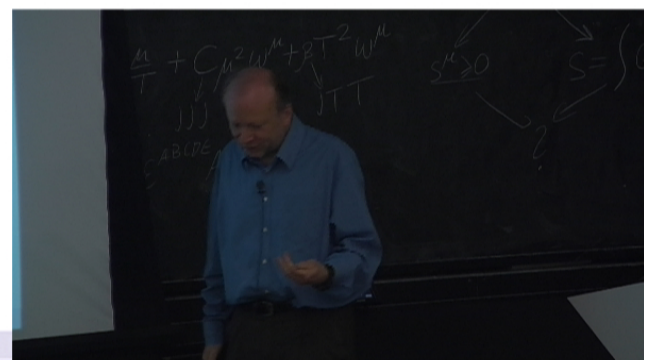
The Linear System

A model e determines a vector $\mathbf{v} = [e(s_1), \ldots, e(s_p)]$.

Solve

$$\mathbf{M}\mathbf{x} = \mathbf{v}$$

for $\mathbf{x}$ over the semiring R.



The Linear System

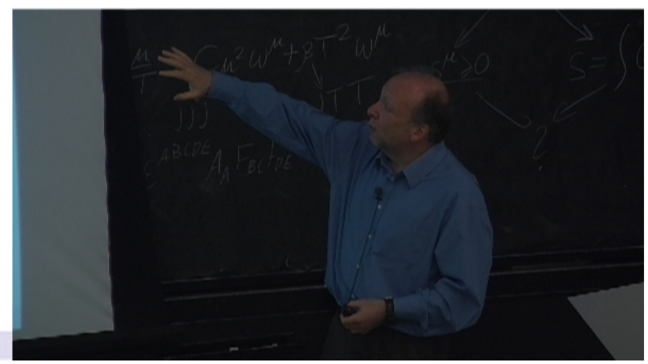
A model e determines a vector $\mathbf{v} = [e(s_1), \ldots, e(s_p)]$.

Solve

$$\mathbf{M}\mathbf{x} = \mathbf{v}$$

for $\mathbf{x}$ over the semiring R.

The solution yields weights in R for the global assignments in O^X; *i.e.* a distribution in $\mathcal{D}_R\mathcal{E}(X)$.



The Linear System

A model e determines a vector $\mathbf{v} = [e(s_1), \ldots, e(s_p)]$.

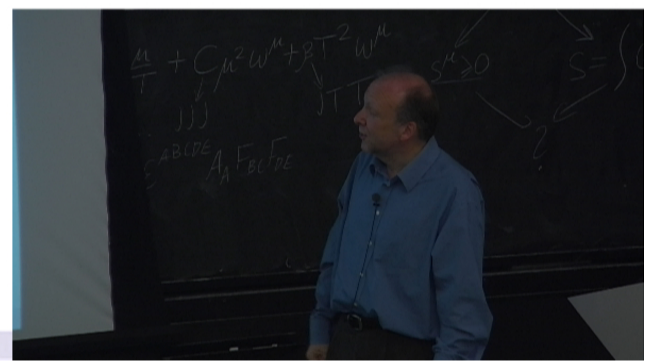
Solve

$$\mathbf{M}\mathbf{x} = \mathbf{v}$$

for $\mathbf{x}$ over the semiring R.

The solution yields weights in R for the global assignments in O^X; *i.e.* a distribution in $\mathcal{D}_R\mathcal{E}(X)$.

The equations enforce the constraints that this distribution marginalizes to yield the probabilities of the empirical model.



The Linear System

A model e determines a vector $\mathbf{v} = [e(s_1), \ldots, e(s_p)]$.

Solve

$$\mathbf{M}\mathbf{x} = \mathbf{v}$$

for $\mathbf{x}$ over the semiring R.

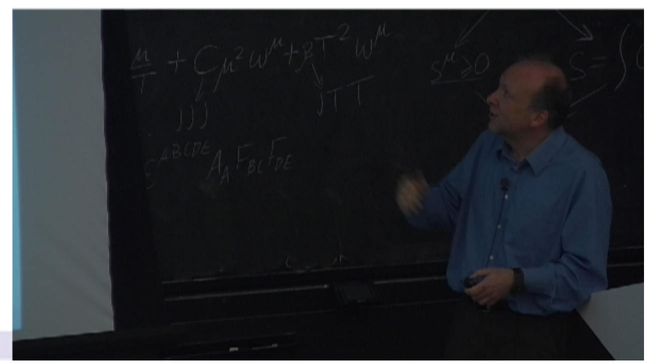
The solution yields weights in R for the global assignments in O^X; *i.e.* a distribution in $\mathcal{D}_R\mathcal{E}(X)$.

The equations enforce the constraints that this distribution marginalizes to yield the probabilities of the empirical model.

Hence solutions correspond exactly to global sections — which as we have seen, correspond exactly to local hidden-variable realizations!

The Bell Model

	(0,0)	(1,0)	(0,1)	(1,1)
(a, b)	1/2	0	0	1/2
(a', b)	3/8	1/8	1/8	3/8
(a, b')	3/8	1/8	1/8	3/8
(a', b')	1/8	3/8	3/8	1/8



The Bell Model

	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
(a, b)	$1/2$	0	0	$1/2$
(a', b)	$3/8$	$1/8$	$1/8$	$3/8$
(a, b')	$3/8$	$1/8$	$1/8$	$3/8$
(a', b')	$1/8$	$3/8$	$3/8$	$1/8$

Solutions in the non-negative reals: this corresponds to solving the linear system over $\mathbb{R}$, subject to the constraint that $\mathbf{x} \geq \mathbf{0}$ (linear programming problem).

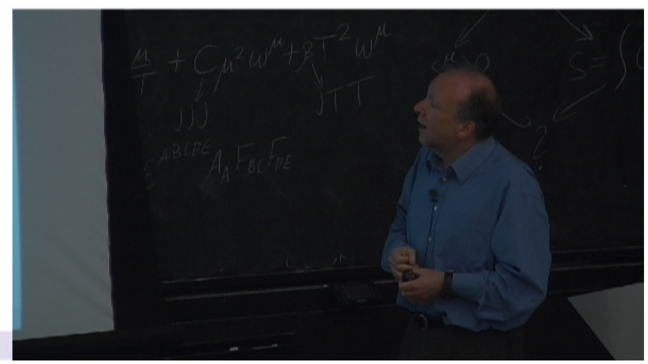
Bell's Theorem

Proposition

The Bell model has no global section.

Proof We focus on 4 out of the 16 equations, corresponding to rows 1, 6, 11 and 13 of the incidence matrix. We write X_i rather than $\mathbf{x}[i]$.

$$
\begin{array}{ccccccccc}
X_1 & + & X_2 & + & X_3 & + & X_4 & = & 1/2 \\
X_2 & + & X_4 & + & X_6 & + & X_8 & = & 1/8 \\
X_3 & + & X_4 & + & X_{11} & + & X_{12} & = & 1/8 \\
X_1 & + & X_5 & + & X_9 & + & X_{13} & = & 1/8
\end{array}
$$



The Bell Model

	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
(a, b)	$1/2$	0	0	$1/2$
(a', b)	$3/8$	$1/8$	$1/8$	$3/8$
(a, b')	$3/8$	$1/8$	$1/8$	$3/8$
(a', b')	$1/8$	$3/8$	$3/8$	$1/8$

Solutions in the non-negative reals: this corresponds to solving the linear system over $\mathbb{R}$, subject to the constraint that $\mathbf{x} \geq \mathbf{0}$ (linear programming problem).

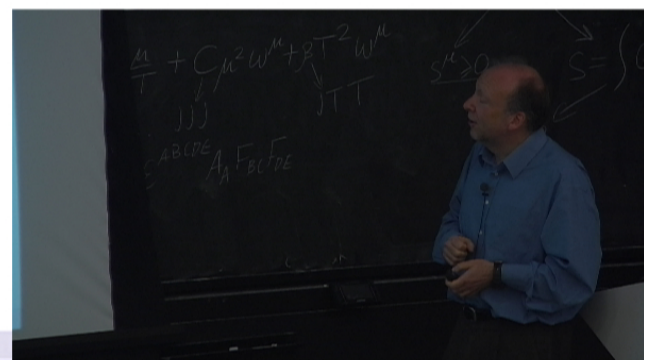
Bell's Theorem

Proposition

The Bell model has no global section.

Proof We focus on 4 out of the 16 equations, corresponding to rows 1, 6, 11 and 13 of the incidence matrix. We write X_i rather than $\mathbf{x}[i]$.

$$
\begin{array}{ccccccccc}
X_1 & + & X_2 & + & X_3 & + & X_4 & = & 1/2 \\
X_2 & + & X_4 & + & X_6 & + & X_8 & = & 1/8 \\
X_3 & + & X_4 & + & X_{11} & + & X_{12} & = & 1/8 \\
X_1 & + & X_5 & + & X_9 & + & X_{13} & = & 1/8
\end{array}
$$



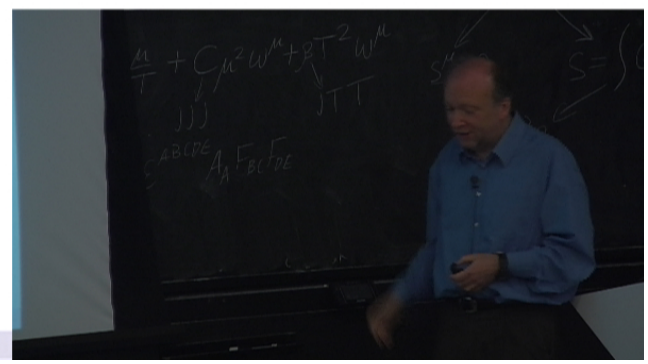
Bell's Theorem

Proposition

The Bell model has no global section.

Proof We focus on 4 out of the 16 equations, corresponding to rows 1, 6, 11 and 13 of the incidence matrix. We write X_i rather than $\mathbf{x}[i]$.

$$
\begin{aligned}
X_1 &+ X_2 + X_3 + X_4 = 1/2 \\
X_2 &+ X_4 + X_6 + X_8 = 1/8 \\
X_3 &+ X_4 + X_{11} + X_{12} = 1/8 \\
X_1 &+ X_5 + X_9 + X_{13} = 1/8
\end{aligned}
$$



Bell's Theorem

Proposition

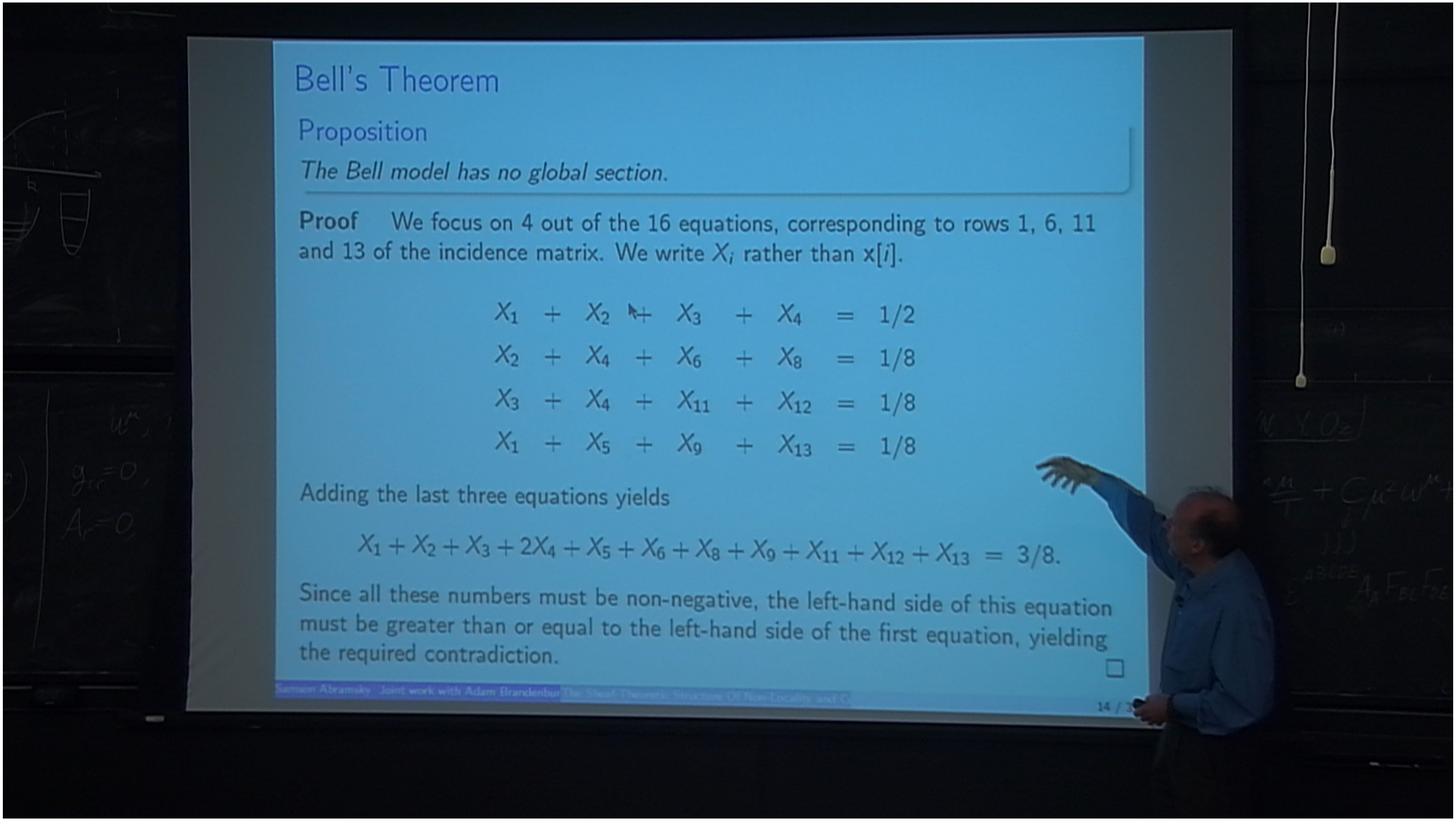
The Bell model has no global section.

Proof We focus on 4 out of the 16 equations, corresponding to rows 1, 6, 11 and 13 of the incidence matrix. We write X_i rather than $\mathbf{x}[i]$.

$$\begin{array}{ccccccccc} X_1 & + & X_2 & + & X_3 & + & X_4 & = & 1/2 \\ X_2 & + & X_4 & + & X_6 & + & X_8 & = & 1/8 \\ X_3 & + & X_4 & + & X_{11} & + & X_{12} & = & 1/8 \\ X_1 & + & X_5 & + & X_9 & + & X_{13} & = & 1/8 \end{array}$$

Adding the last three equations yields

$$X_1 + X_2 + X_3 + 2X_4 + X_5 + X_6 + X_8 + X_9 + X_{11} + X_{12} + X_{13} \ = \ 3/8.$$


Bell's Theorem
Proposition
The Bell model has no global section.
Proof We focus on 4 out of the 16 equations, corresponding to rows 1, 6, 11 and 13 of the incidence matrix. We write X_i rather than x[i].
$X_1 + X_2 + X_3 + X_4 = 1/2$
$X_2 + X_4 + X_6 + X_8 = 1/8$
$X_3 + X_4 + X_{11} + X_{12} = 1/8$
$X_1 + X_5 + X_9 + X_{13} = 1/8$
Adding the last three equations yields
$X_1 + X_2 + X_3 + 2X_4 + X_5 + X_6 + X_8 + X_9 + X_{11} + X_{12} + X_{13} = 3/8.$
Since all these numbers must be non-negative, the left-hand side of this equation must be greater than or equal to the left-hand side of the first equation, yielding the required contradiction.
Samson Abramsky Joint work with Adam Brandenbur

The Hardy Model

We consider the possibilistic version of the Hardy model, specified by the following table.

	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
(a,b)	1	1	1	1
(a',b)	0	1	1	1
(a,b')	0	1	1	1
(a',b')	1	1	1	0

The Hardy Model

We consider the possibilistic version of the Hardy model, specified by the following table.

	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
(a, b)	1	1	1	1
(a', b)	0	1	1	1
(a, b')	0	1	1	1
(a', b')	1	1	1	0

This is obtained from a standard probabilistic Hardy model by replacing all positive entries by 1; thus it can be interpreted as the **support** of the probabilistic model.

The Hardy Model

We consider the possibilistic version of the Hardy model, specified by the following table.

	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
(a,b)	1	1	1	1
(a',b)	0	1	1	1
(a,b')	0	1	1	1
(a',b')	1	1	1	0

This is obtained from a standard probabilistic Hardy model by replacing all positive entries by 1; thus it can be interpreted as the **support** of the probabilistic model.

Now we are interested in solutions over the **boolean semiring**, *i.e.* a boolean satisfiability problem. E.g. the equation specified by the first row of the incidence matrix gives the clause

$$X_1 \vee X_2 \vee X_3 \vee X_4$$

while the fifth yields the formula

$$\neg X_1 \wedge \neg X_3 \wedge \neg X_5 \wedge \neg X_7.$$

The 'Hardy paradox'

A solution is an assignment of boolean values to the variables which simultaneously satisfies all these formulas. Again, it is easy to see by a direct argument that no such assignment exists.

Proposition

The possibilistic Hardy model has no global section over the booleans.

Proof We focus on the four formulas corresponding to rows 1, 5, 9 and 16 of the incidence matrix:

$$
\begin{array}{ccccccc}
X_1 & \vee & X_2 & \vee & X_3 & \vee & X_4 \\
\neg X_1 & \wedge & \neg X_3 & \wedge & \neg X_5 & \wedge & \neg X_7 \\
\neg X_1 & \wedge & \neg X_2 & \wedge & \neg X_9 & \wedge & \neg X_{10} \\
\neg X_4 & \wedge & \neg X_8 & \wedge & \neg X_{12} & \wedge & \neg X_{16}
\end{array}
$$

The Hardy Model

We consider the possibilistic version of the Hardy model, specified by the following table.

	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
(a,b)	1	1	1	1
(a',b)	0	1	1	1
(a,b')	0	1	1	1
(a',b')	1	1	1	0

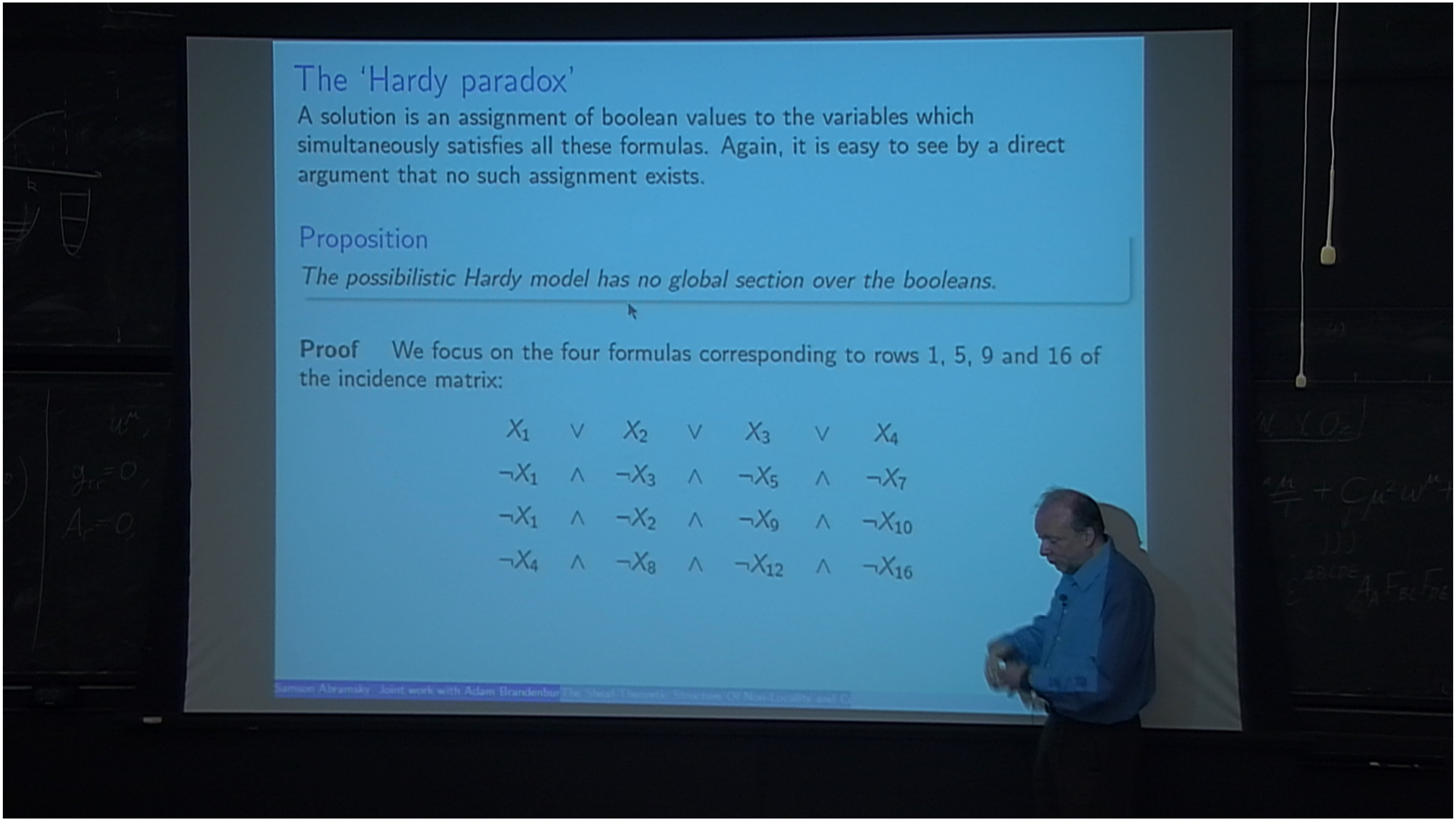
This is obtained from a standard probabilistic Hardy model by replacing all positive entries by 1; thus it can be interpreted as the **support** of the probabilistic model.

Now we are interested in solutions over the **boolean semiring**, *i.e.* a boolean satisfiability problem. E.g. the equation specified by the first row of the incidence matrix gives the clause

$$X_1 \vee X_2 \vee X_3 \vee X_4$$

while the fifth yields the formula

$$\neg X_1 \wedge \neg X_3 \wedge \neg X_5 \wedge \neg X_7.$$


The 'Hardy paradox'
A solution is an assignment of boolean values to the variables which simultaneously satisfies all these formulas. Again, it is easy to see by a direct argument that no such assignment exists.
Proposition
The possibilistic Hardy model has no global section over the booleans.
Proof We focus on the four formulas corresponding to rows 1, 5, 9 and 16 of the incidence matrix:
$X_1 \vee X_2 \vee X_3 \vee X_4$
$\neg X_1 \wedge \neg X_3 \wedge \neg X_5 \wedge \neg X_7$
$\neg X_1 \wedge \neg X_2 \wedge \neg X_9 \wedge \neg X_{10}$
$\neg X_4 \wedge \neg X_8 \wedge \neg X_{12} \wedge \neg X_{16}$
Samson Abramsky Joint work with Adam Brandenbur

The 'Hardy paradox'

A solution is an assignment of boolean values to the variables which simultaneously satisfies all these formulas. Again, it is easy to see by a direct argument that no such assignment exists.

Proposition

The possibilistic Hardy model has no global section over the booleans.

Proof We focus on the four formulas corresponding to rows 1, 5, 9 and 16 of the incidence matrix:

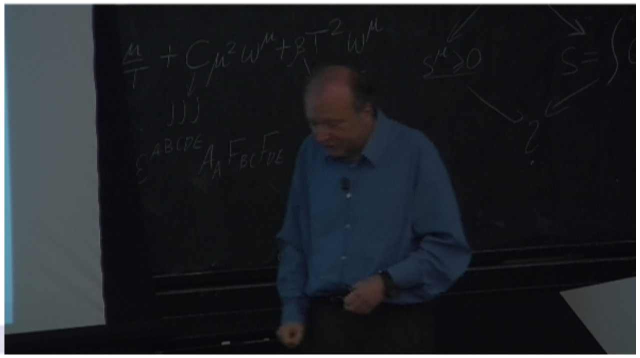
$$\begin{array}{ccccccc} X_1 & \vee & X_2 & \vee & X_3 & \vee & X_4 \\ \neg X_1 & \wedge & \neg X_3 & \wedge & \neg X_5 & \wedge & \neg X_7 \\ \neg X_1 & \wedge & \neg X_2 & \wedge & \neg X_9 & \wedge & \neg X_{10} \\ \neg X_4 & \wedge & \neg X_8 & \wedge & \neg X_{12} & \wedge & \neg X_{16} \end{array}$$

Since every disjunct in the first formula appears as a negated conjunct in one of the other three formulas, there is no satisfying assignment. □

Boolean obstructions are stronger than probabilistic ones

Proposition

Let $\mathbf{v}$ *be the vector over* $\mathbb{R}_{\geq 0}$ *for a probabilistic model,* $\mathbf{v}_b$ *the boolean vector obtained by replacing non-zero elements of* $\mathbf{v}$ *by* 1. *If* $\mathbf{M}\mathbf{x} = \mathbf{v}$ *has a solution over* $\mathbb{R}_{\geq 0}$, *then* $\mathbf{M}\mathbf{x} = \mathbf{v}_b$ *has a solution over the booleans.*



Boolean obstructions are stronger than probabilistic ones

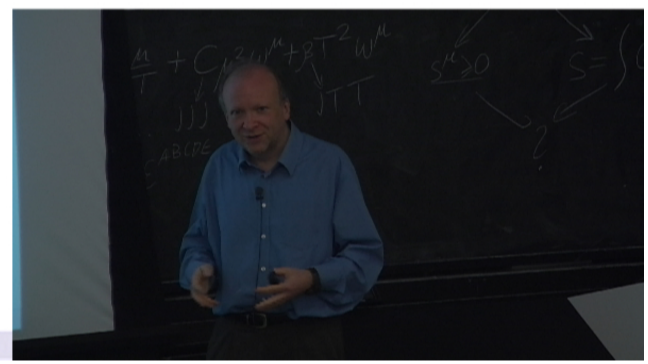
Proposition

Let $\mathbf{v}$ *be the vector over* $\mathbb{R}_{\geq 0}$ *for a probabilistic model,* $\mathbf{v}_b$ *the boolean vector obtained by replacing non-zero elements of* $\mathbf{v}$ *by* 1. *If* $\mathbf{Mx} = \mathbf{v}$ *has a solution over* $\mathbb{R}_{\geq 0}$, *then* $\mathbf{Mx} = \mathbf{v}_b$ *has a solution over the booleans.*

Proof Simply because

$$0 \mapsto 0, \qquad r > 0 \mapsto 1$$

is a semiring homomorphism. □



Boolean obstructions are stronger than probabilistic ones

Proposition

Let $\mathbf{v}$ *be the vector over* $\mathbb{R}_{\geq 0}$ *for a probabilistic model,* $\mathbf{v}_b$ *the boolean vector obtained by replacing non-zero elements of* $\mathbf{v}$ *by* 1*. If* $\mathbf{Mx} = \mathbf{v}$ *has a solution over* $\mathbb{R}_{\geq 0}$*, then* $\mathbf{Mx} = \mathbf{v}_b$ *has a solution over the booleans.*

Proof Simply because

$$0 \mapsto 0, \qquad r > 0 \mapsto 1$$

is a semiring homomorphism. □

So:

$$\text{non-existence of solution over booleans} \\ \Rightarrow \\ \text{non-existence of solution over } \mathbb{R}_{\geq 0}$$

Boolean obstructions are stronger than probabilistic ones

Proposition

Let $\mathbf{v}$ *be the vector over* $\mathbb{R}_{\geq 0}$ *for a probabilistic model,* $\mathbf{v}_b$ *the boolean vector obtained by replacing non-zero elements of* $\mathbf{v}$ *by* 1*. If* $\mathbf{M}\mathbf{x} = \mathbf{v}$ *has a solution over* $\mathbb{R}_{\geq 0}$*, then* $\mathbf{M}\mathbf{x} = \mathbf{v}_b$ *has a solution over the booleans.*

Proof Simply because

$$0 \mapsto 0, \qquad r > 0 \mapsto 1$$

is a semiring homomorphism. □

So:

$$\text{non-existence of solution over booleans}$$
$$\Rightarrow$$
$$\text{non-existence of solution over } \mathbb{R}_{\geq 0}$$

Bell: no solution over $\mathbb{R}_{\geq 0}$; solution over the booleans.

Boolean obstructions are stronger than probabilistic ones

Proposition

Let $\mathbf{v}$ *be the vector over* $\mathbb{R}_{\geq 0}$ *for a probabilistic model,* $\mathbf{v}_b$ *the boolean vector obtained by replacing non-zero elements of* $\mathbf{v}$ *by* 1*. If* $\mathbf{Mx} = \mathbf{v}$ *has a solution over* $\mathbb{R}_{\geq 0}$*, then* $\mathbf{Mx} = \mathbf{v}_b$ *has a solution over the booleans.*

Proof Simply because

$$0 \mapsto 0, \qquad r > 0 \mapsto 1$$

is a semiring homomorphism. □

So:

$$\text{non-existence of solution over booleans} \\ \Rightarrow \\ \text{non-existence of solution over } \mathbb{R}_{\geq 0}$$

Bell: no solution over $\mathbb{R}_{\geq 0}$; solution over the booleans.
Hardy: no solution over the booleans.

Conclusion: Bell < Hardy.

Negative Probabilities And No-Signalling

Distributions over $\mathbb{R}$: signed measures ('negative probabilities').
Wigner, Dirac, Feynman, Sudarshan, . . .

Negative Probabilities And No-Signalling

Distributions over $\mathbb{R}$: signed measures ('negative probabilities').
Wigner, Dirac, Feynman, Sudarshan, . . .

Feynman:

> *The only difference between a probabilistic classical world and the equations of the quantum world is that somehow or other it appears as if the probabilities would have to go negative . . .*

Negative Probabilities And No-Signalling

Distributions over $\mathbb{R}$: signed measures ('negative probabilities').
Wigner, Dirac, Feynman, Sudarshan, ...

Feynman:

> *The only difference between a probabilistic classical world and the equations of the quantum world is that somehow or other it appears as if the probabilities would have to go negative ...*

Theorem

Probabilistic models have local hidden-variable realizations with negative probabilities if and only if they satisfy no-signalling.

Linear Span Theorem

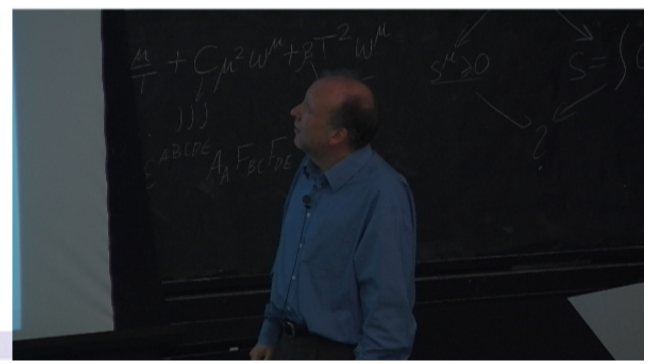
The fact that all probabilistic models have such global sections over signed measures is a consequence of the following:

Theorem

The linear subspace generated by the local models over an arbitrary measurement cover $\mathcal{M}$ coincides with that generated by the no-signalling models. Their common dimension — and the rank of the incidence matrix — is

$$D \;:=\; \sum_{U \in \mathcal{U}} (l-1)^{|U|}$$

where $l = |O|$ and $\mathcal{U}$ is the abstract simplicial complex generated by $\mathcal{M}$.



Linear Span Theorem

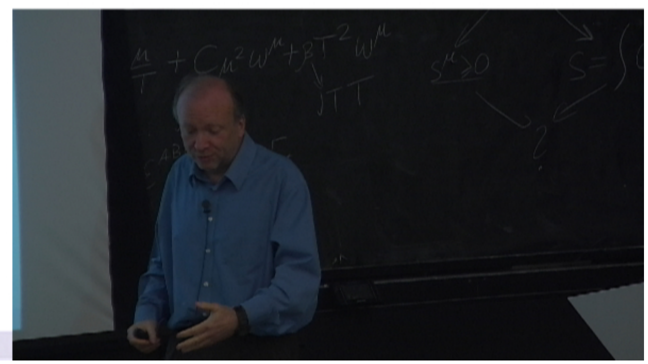
The fact that all probabilistic models have such global sections over signed measures is a consequence of the following:

Theorem

The linear subspace generated by the local models over an arbitrary measurement cover $\mathcal{M}$ coincides with that generated by the no-signalling models. Their common dimension — and the rank of the incidence matrix — is

$$D := \sum_{U \in \mathcal{U}} (l-1)^{|U|}$$

where $l = |O|$ and $\mathcal{U}$ is the abstract simplicial complex generated by $\mathcal{M}$.



Linear Span Theorem

The fact that all probabilistic models have such global sections over signed measures is a consequence of the following:

Theorem

The linear subspace generated by the local models over an arbitrary measurement cover $\mathcal{M}$ coincides with that generated by the no-signalling models. Their common dimension — and the rank of the incidence matrix — is

$$D := \sum_{U \in \mathcal{U}} (l-1)^{|U|}$$

where $l = |O|$ and $\mathcal{U}$ is the abstract simplicial complex generated by $\mathcal{M}$.

Since the local models are included in the no-signalling models, this is proved by showing that every compatible model is determined by linear equations in D variables; while there are D linearly independent local models.

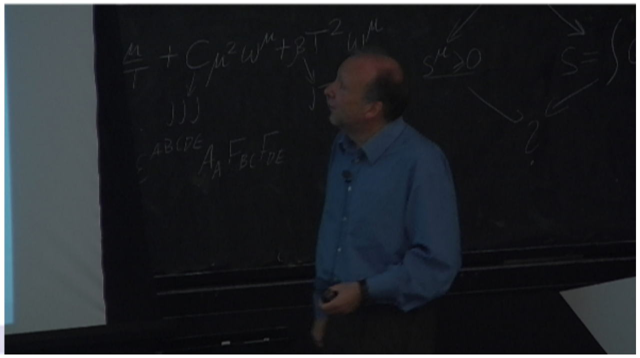
As a special case, we derive a formula for the dimension for Bell-type (n, k, l)-scenarios:

$$D = (k \cdot (l-1) + 1)^n.$$

Example: PR Boxes have global sections over $\mathbb{R}$

The 'Popescu-Rohrlich box':

	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
(a,b)	$1/2$	0	0	$1/2$
(a',b)	$1/2$	0	0	$1/2$
(a,b')	$1/2$	0	0	$1/2$
(a',b')	0	$1/2$	$1/2$	0



Example: PR Boxes have global sections over $\mathbb{R}$

The 'Popescu-Rohrlich box':

	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
(a, b)	$1/2$	0	0	$1/2$
(a', b)	$1/2$	0	0	$1/2$
(a, b')	$1/2$	0	0	$1/2$
(a', b')	0	$1/2$	$1/2$	0

The PR boxes exhibit super-quantum correlations, and cannot be realized in quantum mechanics.

Example solution for PR Box:

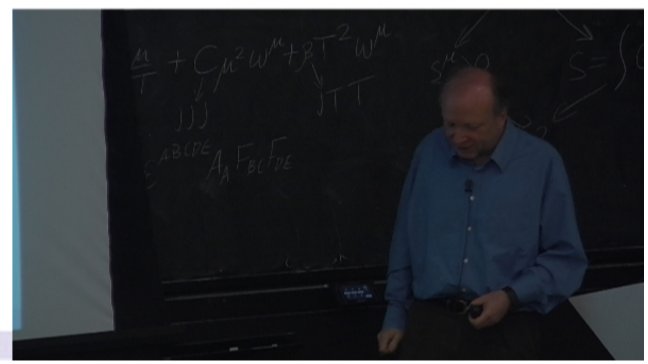
$$[1/2, 0, 0, 0, -1/2, 0, 1/2, 0, -1/2, 1/2, 0, 0, 1/2, 0, 0, 0].$$

Strong Contextuality

Given an empirical model e, we define the set

$$S_e \ := \ \{s \in \mathcal{E}(X) : \forall C \in \mathcal{M}.\, s|C \in \mathsf{supp}(e_C)\}.$$

A consequence of the extendability of e is that S_e is non-empty.



Strong Contextuality

Given an empirical model e, we define the set

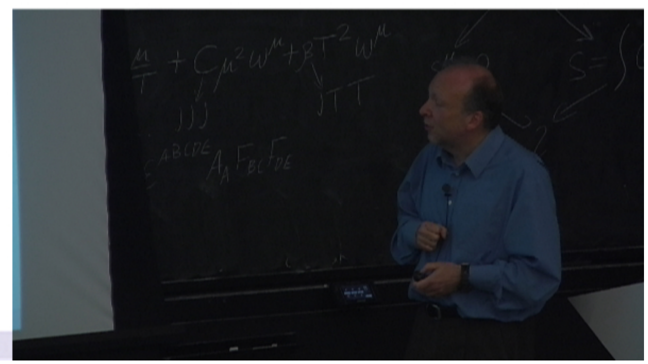
$$S_e \;:=\; \{s \in \mathcal{E}(X) : \forall C \in \mathcal{M}.\, s|C \in \mathsf{supp}(e_C)\}.$$

A consequence of the extendability of e is that S_e is non-empty.

We say that the model e is **strongly contextual** if this set S_e is *empty*. Thus strong non-contextuality implies non-extendability.

In fact, it is strictly stronger. The Hardy model, which as we saw in the previous section is possibilistically non-extendable, is *not* strongly contextual. The Bell model similarly fails to be strongly contextual.

The question now arises: are there models arising from quantum mechanics which are strongly contextual in this sense?



Strong Contextuality

Given an empirical model e, we define the set

$$S_e \; := \; \{s \in \mathcal{E}(X) : \forall C \in \mathcal{M}.\, s|C \in \mathsf{supp}(e_C)\}.$$

A consequence of the extendability of e is that S_e is non-empty.

We say that the model e is **strongly contextual** if this set S_e is *empty*. Thus strong non-contextuality implies non-extendability.

In fact, it is strictly stronger. The Hardy model, which as we saw in the previous section is possibilistically non-extendable, is *not* strongly contextual. The Bell model similarly fails to be strongly contextual.

The question now arises: are there models arising from quantum mechanics which are strongly contextual in this sense?

We shall now show that the well-known GHZ models, of type $(n, 2, 2)$ for all $n > 2$, are strongly contextual. This will establish a strict hierarchy

$$\text{Bell} < \text{Hardy} < \text{GHZ}$$

of increasing strengths of obstructions to non-contextual behaviour for these salient models.

GHZ Models

The GHZ model of type $(n, 2, 2)$ can be specified as follows. We label the two measurements at each part as $X^{(i)}$ and $Y^{(i)}$, and the outcomes as 0 and 1.

For each maximal context C, every s in the support of the model satisfies the following conditions:

- If the number of Y measurements in C is a multiple of 4, the number of 1's in the outcomes specified by s is even.
- If the number of Y measurements is $4k + 2$, the number of 1's in the outcomes is odd.

GHZ Models

The GHZ model of type $(n, 2, 2)$ can be specified as follows. We label the two measurements at each part as $X^{(i)}$ and $Y^{(i)}$, and the outcomes as 0 and 1.

For each maximal context C, every s in the support of the model satisfies the following conditions:

- If the number of Y measurements in C is a multiple of 4, the number of 1's in the outcomes specified by s is even.
- If the number of Y measurements is $4k + 2$, the number of 1's in the outcomes is odd.

NB: a model with these properties can be realized in quantum mechanics.

GHZ Models Are Strongly Contextual

We consider the case where $n = 4k$. Assume for a contradiction that we have a global section.

If we take Y measurements at every part, the number of R outcomes under the assignment has a parity P. Replacing any two Y's by X's changes the residue class mod 4 of the number of Y's, and hence must result in the opposite parity for the number of R outcomes under the assignment.

Thus for any $Y^{(i)}$, $Y^{(j)}$ assigned the **same** value, if we substitute X's in those positions they must receive **different** values. Similarly, for any $Y^{(i)}$, $Y^{(j)}$ assigned different values, the corresponding $X^{(i)}$, $X^{(j)}$ must receive the same value.

Suppose not all $Y^{(i)}$ are assigned the same value. Then for some i, j, k, $Y^{(i)}$ is assigned the same value as $Y^{(j)}$, and $Y^{(j)}$ is assigned a different value to $Y^{(k)}$. Thus $Y^{(i)}$ is also assigned a different value to $Y^{(k)}$. Then $X^{(i)}$ is assigned the same value as $X^{(k)}$, and $X^{(j)}$ is assigned the same value as $X^{(k)}$. By transitivity, $X^{(i)}$ is assigned the same value as $X^{(j)}$, yielding a contradiction.

The remaining cases are where all Y's receive the same value. Then any pair of X's must receive different values. But taking any 3 X's, this yields a contradiction, since there are only two values, so some pair must receive the same value.

GHZ Models Are Strongly Contextual

We consider the case where $n = 4k$. Assume for a contradiction that we have a global section.

If we take Y measurements at every part, the number of R outcomes under the assignment has a parity P. Replacing any two Y's by X's changes the residue class mod 4 of the number of Y's, and hence must result in the opposite parity for the number of R outcomes under the assignment.

Thus for any $Y^{(i)}$, $Y^{(j)}$ assigned the **same** value, if we substitute X's in those positions they must receive **different** values. Similarly, for any $Y^{(i)}$, $Y^{(j)}$ assigned different values, the corresponding $X^{(i)}$, $X^{(j)}$ must receive the same value.

Suppose not all $Y^{(i)}$ are assigned the same value. Then for some i, j, k, $Y^{(i)}$ is assigned the same value as $Y^{(j)}$, and $Y^{(j)}$ is assigned a different value to $Y^{(k)}$. Thus $Y^{(i)}$ is also assigned a different value to $Y^{(k)}$. Then $X^{(i)}$ is assigned the same value as $X^{(k)}$, and $X^{(j)}$ is assigned the same value as $X^{(k)}$. By transitivity, $X^{(i)}$ is assigned the same value as $X^{(j)}$, yielding a contradiction.

The remaining cases are where all Y's receive the same value. Then any pair of X's must receive different values. But taking any 3 X's, this yields a contradiction, since there are only two values, so some pair must receive the same value.

Strong Contextuality and Maximal Contextuality

Strong contextuality is defined in a simple 'qualitative' fashion. It is equivalent to a notion which can be defined in quantitative terms, and has been studied in this form in the special case of Bell-type scenarios

We consider convex decompositions

$$e = \lambda L + (1 - \lambda)q, \qquad 0 \leq \lambda \leq 1, \tag{1}$$

where L is a local model, and q a no-signalling model.

We define the **non-contextual fraction** of e to be the supremum over all λ appearing in such convex decompositions (1).

Quantitative Contextuality

We can consider the followed 'relaxed' version of the linear programming problem for contextuality:

(LP1) Maximize $\mathbf{1} \cdot \mathbf{x}$, subject to the constraints $\mathbf{M}\mathbf{x} \leq \mathbf{v}$ and $\mathbf{x} \geq \mathbf{0}$.

Proposition

The values that $\mathbf{1} \cdot \mathbf{x}^*$ can take, for any $\mathbf{M}$ and $\mathbf{v}$, lie in the unit interval. Moreover:

$$\mathbf{1} \cdot \mathbf{x}^* = 1 \iff \mathbf{M}\mathbf{x}^* = \mathbf{v}.$$

Thus the distance of $\mathbf{1} \cdot \mathbf{x}^*$ from 1 quantifies 'how contextual' the model is.

Proposition

The following are equivalent:

1. $\mathbf{1} \cdot \mathbf{x}^* = \mathbf{y}^* \cdot \mathbf{v} = 0$.
2. *The model is strongly contextual.*

Cohomology of Non-Locality and Contextuality

Joint work with Shane Mansfield and Rui Soares Barbosa.
Paper in Proceedings of QPL 2011.

Cohomology of Non-Locality and Contextuality

Joint work with Shane Mansfield and Rui Soares Barbosa.
Paper in Proceedings of QPL 2011.

The basic idea: to view non-locality and contextuality as **cohomological obstructions to global sections**.

- Given an empirical model e on a cover $\mathcal{U}$, we define an **abelian presheaf** $\mathcal{F} := F_{\mathbb{Z}} S_e$, the free abelian group functor applied to the support presheaf of the model.
- We work with the Čech cohomology groups $\check{H}^q(\mathcal{U}, \mathcal{F})$ for this presheaf.

Cohomology of Non-Locality and Contextuality

Joint work with Shane Mansfield and Rui Soares Barbosa.
Paper in Proceedings of QPL 2011.

The basic idea: to view non-locality and contextuality as **cohomological obstructions to global sections**.

- Given an empirical model e on a cover $\mathcal{U}$, we define an **abelian presheaf** $\mathcal{F} := F_{\mathbb{Z}} S_e$, the free abelian group functor applied to the support presheaf of the model.
- We work with the Čech cohomology groups $\check{H}^q(\mathcal{U}, \mathcal{F})$ for this presheaf.
- To each $s \in S_e(C)$, we associate an element $\gamma(s) \in \check{H}^1(\mathcal{U}, \mathcal{F}_{\tilde{C}})$ of a cohomology group, which can be regarded as an obstruction to s having an extension within the support of e to a global section. In particular, the existence of such an extension implies that the obstruction vanishes. Thus the non-vanishing of the obstruction provides a **cohomological witness** for contextuality and strong contextuality.
- We show for many examples, including GHZ, PR boxes, various Kochen-Specker constructions, the Peres-Mermin square etc. that this obstruction does indeed not vanish for any section, yielding witnesses for strong contextuality.

Important Equivalence

The following are equivalent:

1. The cohomology obstruction vanishes: $\gamma(s_1) = 0$
2. There is a family $\{r_i \in \mathcal{F}(C_i)\}$ with $s_1 = r_1$, and for all i, j:

$$r_i | C_i \cap C_j = r_j | C_i \cap C_j$$

Sufficient Condition for Non-Locality/Contextuality

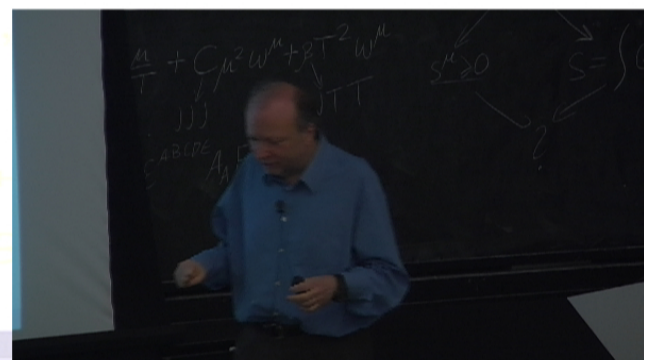
- e is local/ non-contextual $\rightarrow$ obstruction vanishes for every section in the support
- e is **not** strongly contextual $\rightarrow$ obstruction vanishes for some section in the support

The Hardy Model

Support of the Hardy Model

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(A,B)	s_1	s_2	s_3	s_4
(A,B')	0	s_6	s_7	s_8
(A',B)	0	s_{10}	s_{11}	s_{12}
(A',B')	s_{13}	s_{14}	s_{15}	0

- Label non-zero sections



The Hardy Model

Support of the Hardy Model

	(0,0)	(0,1)	(1,0)	(1,1)
(A, B)	s_1	s_2	s_3	s_4
(A, B')	0	s_6	s_7	s_8
(A', B)	0	s_{10}	s_{11}	s_{12}
(A', B')	s_{13}	s_{14}	s_{15}	0

- Label non-zero sections
- Compatible family of $\mathbb{Z}$-linear combinations of sections:

$$r_1 = s_1, \quad r_2 = s_6 + s_7 - s_8, \quad r_3 = s_{11}, \quad r_4 = s_{15}$$

- One can check that

$$\begin{aligned} r_2|A &= 1\cdot(A \mapsto 0) + 1\cdot(A \mapsto 1) - 1\cdot(A \mapsto 1) &= r_1|A, \\ r_2|B' &= 1\cdot(B' \mapsto 1) + 1\cdot(B' \mapsto 0) - 1\cdot(B' \mapsto 1) &= r_4|B' \end{aligned}$$

The Hardy Model

Support of the Hardy Model

	(0,0)	(0,1)	(1,0)	(1,1)
(A, B)	s_1	s_2	s_3	s_4
(A, B')	0	s_6	s_7	s_8
(A', B)	0	s_{10}	s_{11}	s_{12}
(A', B')	s_{13}	s_{14}	s_{15}	0

- Label non-zero sections
- Compatible family of $\mathbb{Z}$-linear combinations of sections:

$$r_1 = s_1, \quad r_2 = s_6 + s_7 - s_8, \quad r_3 = s_{11}, \quad r_4 = s_{15}$$

- One can check that

$$\begin{aligned} r_2|A &= 1 \cdot (A \mapsto 0) + 1 \cdot (A \mapsto 1) - 1 \cdot (A \mapsto 1) &= r_1|A, \\ r_2|B' &= 1 \cdot (B' \mapsto 1) + 1 \cdot (B' \mapsto 0) - 1 \cdot (B' \mapsto 1) &= r_4|B' \end{aligned}$$

The Hardy Model

- $\gamma(s_1)$ vanishes!
- This example illustrates that false positives do arise
- Cohomological prescription does not pick up on the non-locality of the Hardy model

Kochen-Specker-type Models

- In a Kochen-Specker problem, we wish to assign the outcome 1 to a single measurement in each context
- So sections in the support are the ones with exactly one 1

Kochen-Specker-type Models

- In a Kochen-Specker problem, we wish to assign the outcome 1 to a single measurement in each context
- So sections in the support are the ones with exactly one 1
- E.g. 18-vector K-S model

	1000	0100	0010	0001
ABCD	*a*	*b*	*c*	*d*
AEFG	*a*	*e*	*f*	*g*
HICJ	*h*	*i*	*c*	*j*
HKGL	*h*	*k*	*g*	*l*
BEMN	*b*	*e*	*m*	*n*
IKNO	*i*	*k*	*n*	*o*
PQDJ	*p*	*q*	*d*	*j*
PRFL	*p*	*r*	*f*	*l*
QRMO	*q*	*r*	*m*	*o*

Kochen-Specker-type Models

- In a Kochen-Specker problem, we wish to assign the outcome 1 to a single measurement in each context
- So sections in the support are the ones with exactly one 1
- E.g. 18-vector K-S model

$$
\begin{aligned}
b + c + d &= e + f + g \\
a + b + d &= h + i + j \\
a + c + d &= e + m + n \\
a + b + c &= p + q + j \\
a + f + g &= b + m + n \\
a + e + f &= h + k + l \\
a + e + g &= p + r + l \\
i + c + j &= k + g + l \\
h + c + j &= k + n + o \\
h + i + c &= p + q + d \\
h + g + l &= i + n + o \\
h + k + g &= p + r + f \\
b + e + n &= q + r + o \\
b + e + m &= i + k + o \\
i + k + n &= q + r + m \\
q + d + j &= r + f + l \\
p + d + j &= r + m + o \\
p + f + l &= q + m + o
\end{aligned}
$$

A Class of KS-type Models

Proposition (Abramsky-Brandenburger)

A necessary condition for Kochen-Specker-type models to have a global section is:

$$\gcd\{d_m \mid m \in X\} \quad \mid \quad |\mathcal{U}|,$$

where $d_m := |\{C \in \mathcal{U} \mid m \in C\}|$

Corollary

All models that do not satisfy the above condition are therefore strongly contextual

A Class of KS-type Models

Proposition (AMB)

If $\gamma(s)$ vanishes for some section s in the support of a connected Kochen-Specker-type model, then the GCD condition holds for that model

Corollary

The vanishing of the cohomological obstruction is a complete invariant for the non-locality/contextuality of any connected KS-type model that violates the GCD condition

A Class of KS-type Models

Proposition (Abramsky-Brandenburger)

A necessary condition for Kochen-Specker-type models to have a global section is:

$$\mathrm{gcd}\{d_m \mid m \in X\} \quad | \quad |\mathcal{U}|,$$

where $d_m := |\{C \in \mathcal{U} \mid m \in C\}|$

Corollary

All models that do not satisfy the above condition are therefore strongly contextual

Kochen-Specker-type Models

- In a Kochen-Specker problem, we wish to assign the outcome 1 to a single measurement in each context
- So sections in the support are the ones with exactly one 1
- E.g. 18-vector K-S model

	1000	0100	0010	0001
$ABCD$	a	b	c	d
$AEFG$	a	e	f	g
$HICJ$	h	i	c	j
$HKGL$	h	k	g	l
$BEMN$	b	e	m	n
$IKNO$	i	k	n	o
$PQDJ$	p	q	d	j
$PRFL$	p	r	f	l
$QRMO$	q	r	m	o

Limitations and Further Directions

- In general, the cohomological condition for contextuality is sufficient, but not necessary

Conjecture

Under suitable assumptions of symmetry and connectedness, the cohomology obstruction is a complete invariant for strong contextuality

- We have been computing the obstructions by brute force enumeration
- We would like to use the machinery of homological algebra and exact sequences to obtain more conceptual and general results

Limitations and Further Directions

- In general, the cohomological condition for contextuality is sufficient, but not necessary

Conjecture

Under suitable assumptions of symmetry and connectedness, the cohomology obstruction is a complete invariant for strong contextuality

- We have been computing the obstructions by brute force enumeration
- We would like to use the machinery of homological algebra and exact sequences to obtain more conceptual and general results