Title: The mechanics of open systems and applications in effective field theory

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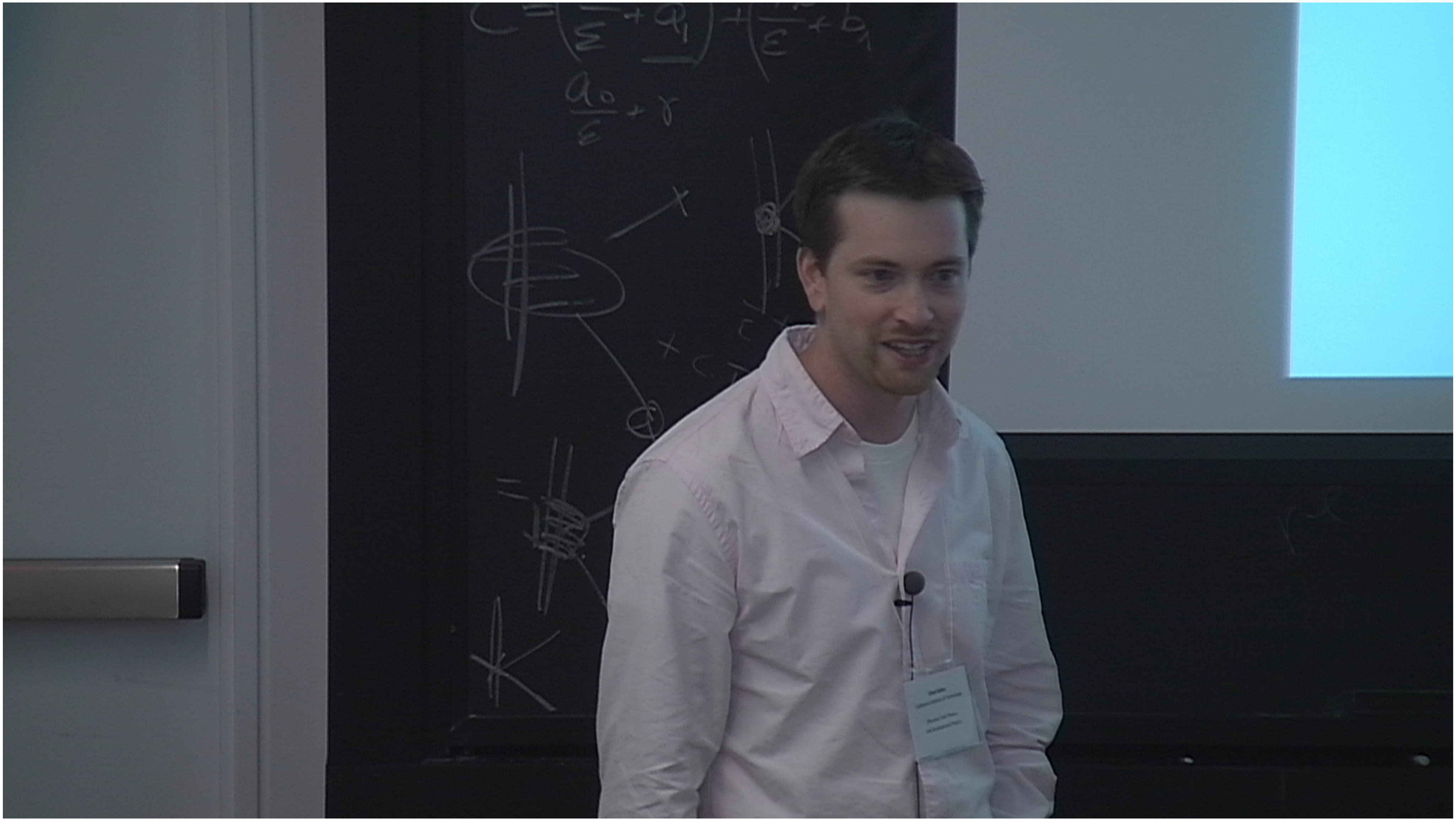
Abstract: Recent years have seen the paradigm of effective field theory (EFT) successfully applied to an increasing number of classical systems that range from the gravitational inspiral of compact binaries to hydrodynamics. Many of these systems exhibit dissipation in one form or another, such as radiation reaction or viscous fluid flow, that naturally results from the system being open. This "openness" can manifest as energy leaving the dynamical variables of interest via radiation or heat transfer, for example. As the EFT approach typically utilizes the action, and hence Hamilton's Principle of Extremal Action, it is crucial to determine how generally and consistently to accommodate dissipative effects in a variational principle. In this talk, I discuss why Hamilton's Principle fails to incorporate dissipation. I then provide a reformulation that has been used successfully to confirm well-established results as well as to provide new predictions regarding dissipative systems. I show specific examples drawn from EFT applications. Finally, I show how this reformulation of Hamilton's Principle turns out to correspond to the classical limit of quantum theories based on the so-called "in-in" or "closed-time-path" approaches.



Effective Field Theory (EFT) in a tiny nutshell

- EFT exploits a separation in scales (length, mass, energy, velocity...) to parameterize the effect of the "microphysics" on the "macrophysics."

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- EFT usually involves:

  Identifying the relevant degrees of freedom and their symmetries

  Writing most general action consistent with symmetries (i.e., derivative expansion)

What's been done with EFT: A snapshot

Potentials for non-spinning binaries thru 3PN
(Goldberger, Rothstein, Gilmore, Ross, Chu, Foffa, Sturani)

Spin-orbit & spin-spin potentials thru 4PN & 3PN, resp.
(Porto, Rothstein, Levi, Perrodin)

PN radiation reaction thru 3.5PN
(CG, Leibovich, Tiglio)

Gravitational waveform at LO
(CG, Tiglio)

Radiative moments thru 3PN
(Ross, Goldberger, Porto, Rothstein)

Tidal Love number for BH
(Smolkin, Kol)

First-order gravitational self-force
(CG, Hu)

Third-order scalar self-force
(CG)

Absorptive effects
(Rothstein, Goldberger, Porto)

Radiation reaction on extended charges
(Leibovich, Rothstein, CG)

Caged black holes
(Kol, Smolkin, Chu, Goldberger, Rothstein)

Cosmological perturbation theory
(Baumann, Nicolis, Senatore, Zaldarriaga,...)

Inflation
(Senatore, Zaldarriaga,...)

Higher dimensional BHs
(Emparan, Harmark, Niarchos, Obers)

Hydrodynamics
(Nicolis, Dubovsky, Endlich, Hui, Son,...)

Condensed matter
(Yolcu, Rothstein, Deserno)

A problem with EFT???

- Naive application of EFT to dissipative systems (e.g., radiation reaction) yields no dissipation

  - Radiation reaction in electrodynamics

$$S_{\text{eff}}[z^\mu] = -m \int d\tau + \frac{e^2}{8\pi} \int d\tau \, u^\alpha a_\alpha$$

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$$S_{\text{eff}}[z^\mu] = (\text{lower order conservative PN terms}) - \frac{G}{10} \int dt \, Q_{ij}(t) \frac{d^5 Q_{ij}(t)}{dt^5} + \cdots$$

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  - Self-force in extreme mass ratio binaries

$$S_{\rm eff}[z^\mu] = -m \int d\tau + 4\pi G m^2 \int d\tau d\tau'\, u^\alpha u^\beta \left[\frac{G^{\rm ret}_{\alpha\beta\gamma'\delta'}(z^\mu, z^{\mu'}) + G^{\rm adv}_{\alpha\beta\gamma'\delta'}(z^\mu, z^{\mu'})}{2} \right] u^{\gamma'} u^{\delta'} + \cdots$$

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- Not a problem with EFT but with the formulation of mechanics

"Mantra" of classical mechanics:
Lagrangians and Hamiltonians do not describe dissipative dynamics

Shortcomings of Hamilton's Principle

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- Formulation of Hamilton's Principle of extremal action is:

  1) Not generally consistent or applicable for initial value problems

  2) Unable to account generally for dissipative forces

  3) Unable to properly describe non-equilibrium dynamics of open systems

Shortcomings: A simple open system example (1)

- N harmonic oscillators $\{Q_n(t)\}$ coupled bi-linearly to $q(t)$

$$S[q,\{Q_n\}] = S_q[q] + \sum_{n=1}^{N} \int_{t_i}^{t_f} dt \left(\frac{M\dot{Q}_n^2}{2} - \frac{M\Omega_n^2 Q_n^2}{2} + \lambda_n q Q_n \right)$$

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- "Integrate out" $Q_n(t)$ subject to initial conditions

$$\ddot{Q}_n + \Omega_n^2 Q_n = \frac{\lambda_n}{M} q \quad \Longrightarrow \quad Q_n(t) = Q_n^{(h)}(t) + \frac{\lambda_n}{M} \int_{t_i}^{t_f} dt'\, G_{\rm ret}^{(n)}(t-t') q(t')$$

Effective action:

$$S_{\rm eff}[q] = S_q[q] + \sum_{n=1}^{N} \lambda_n \int_{t_i}^{t_f} dt\, q\, Q_n^{(h)} + \sum_{n=1}^{N} \frac{\lambda_n^2}{2M} \int_{t_i}^{t_f} dt dt'\, q(t) G_{\rm ret}^{(n)}(t-t') q(t')$$

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Shortcomings: A simple open system example (II)

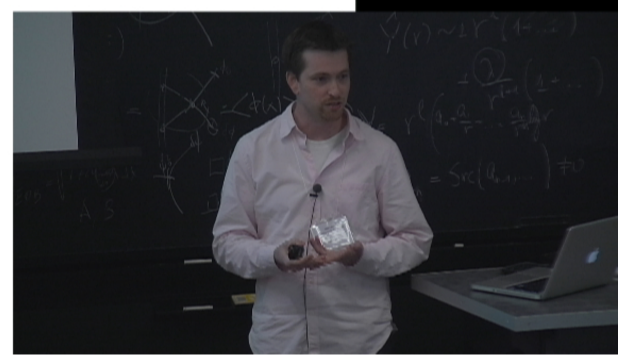
- Equation of motion for $q(t)$ is (from Hamilton's Principle of extremal action)

$$m\ddot{q} + m\omega^2 q = \sum_{n=1}^{N} \lambda_n Q_n^{(h)}(t) + \sum_{n=1}^{N} \frac{\lambda_n^2}{M} \int_{t_i}^{t_f} dt' \left[\frac{G_{\rm ret}^{(n)}(t-t') + G_{\rm adv}^{(n)}(t-t')}{2} \right] q(t')$$

- Resulting solution does not evolve causally in time

- $q(t)$ cannot be specified by initial data

$$(\text{RHS at } t = t_i) = \sum_{n=1}^{N} \lambda_n Q_{ni} + \sum_{n=1}^{N} \frac{\lambda_n^2}{2M} \int_{t_i}^{t_f} dt' \, G_{\rm adv}^{(n)}(t_i - t') q(t')$$



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and describes conservative interactions (no dissipation, damping)

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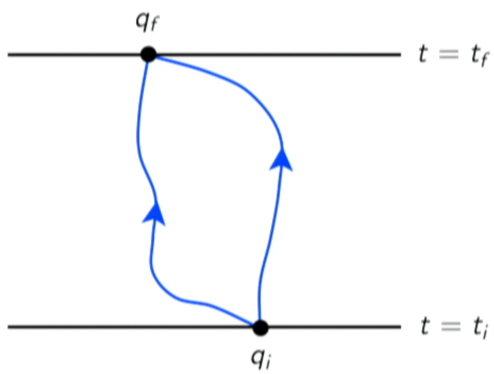
Hamilton's Principle of extremal action

- <u>The Problem</u>:

"Find the path $q(t)$ passing through the given values q_i at t_i and q_f at t_f that extremizes the value of the action."



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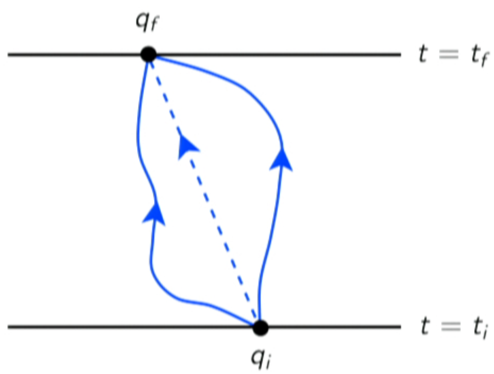

- <u>The Solution</u>:

  The extremal path satisfies the Euler-Lagrange equations iff that solution passes through the given values q_i at t_i and q_f at t_f.

Boundary conditions & Green's functions

- Recall the harmonic oscillator example:

$$m\ddot{q} + m\omega^2 q = \sum_{n=1}^{N} \lambda_n Q_n^{(h)}(t) + \sum_{n=1}^{N} \frac{\lambda_n^2}{M} \int_{t_i}^{t_f} dt' \left[\frac{G_{\text{ret}}^{(n)}(t-t') + G_{\text{adv}}^{(n)}(t-t')}{2} \right] q(t')$$

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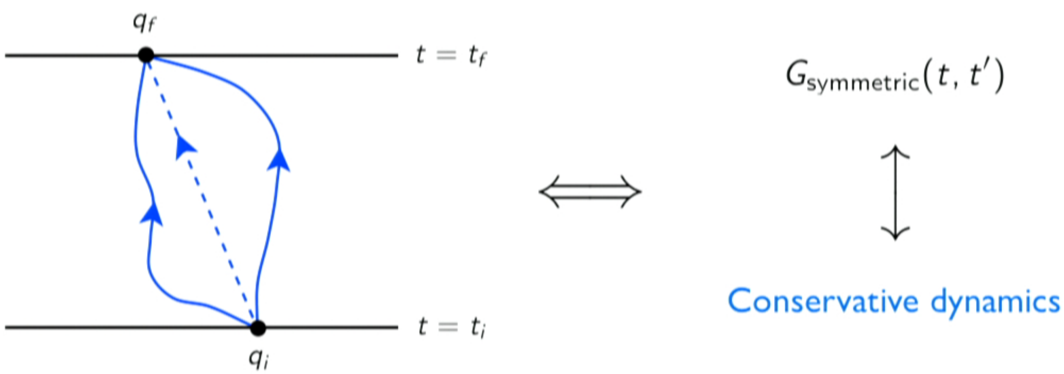
The above Green's function is symmetric in time, which is the one for systems having boundary conditions in time (recall from Sturm-Liouville theory)



A naive approach

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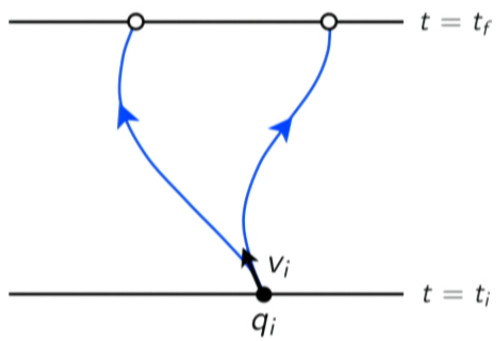

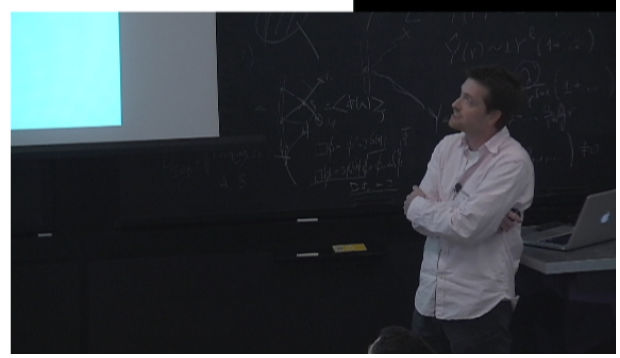

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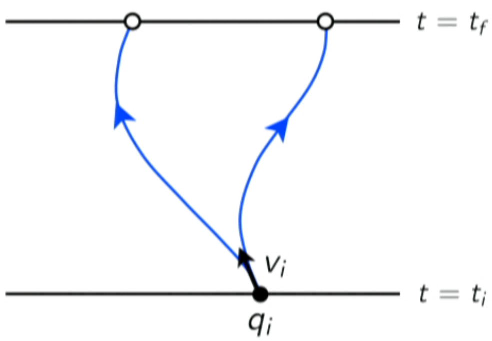

$$q(t, \epsilon) = q(t, 0) + \epsilon\eta(t)$$

$$0 = \left.\frac{dS[q]}{d\epsilon}\right|_{\epsilon=0} = \int_{t_i}^{t_f} dt\, \eta(t)\left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right)_0 + \left[\eta(t)\left(\frac{\partial L}{\partial \dot{q}}\right)_0\right]_{t=t_i}^{t_f}$$

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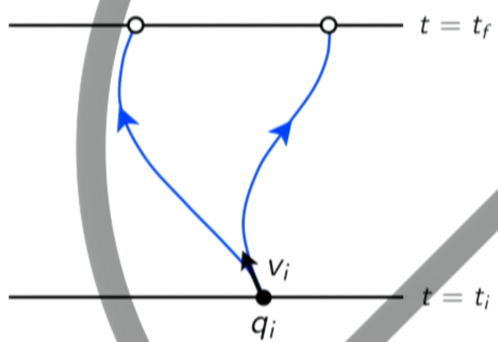

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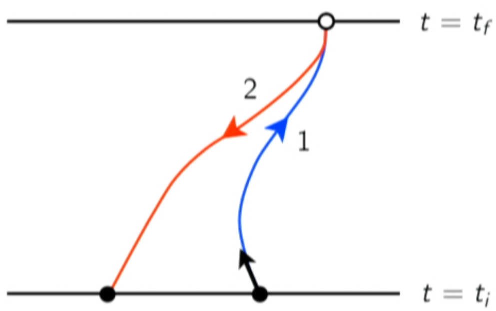
$$\Longrightarrow \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \delta(t - t_f) \left[\frac{\partial L}{\partial \dot{q}} \right]_{t=t_f}$$

Hamilton's Principle & initial conditions (I) *(in preparation)*

- Introduce <u>two</u> paths such that:

  1) Both paths have vanishing displacements at the initial time

  2) The coordinates and conjugate momenta of both paths are equal at the final time (continuity condition)



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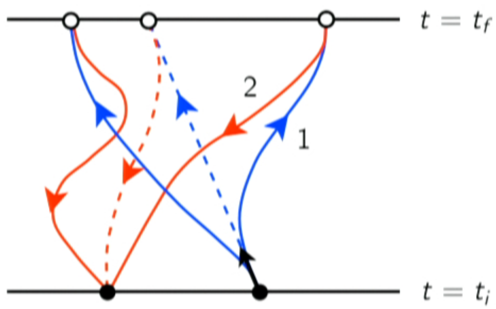
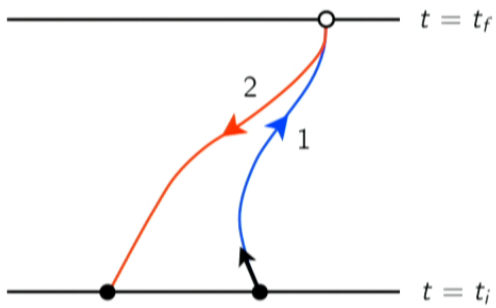


- After <u>all</u> variations are done, identify both paths with the physical one ("physical limit")

Hamilton's Principle & initial conditions (II) *(in preparation)*



- New action defined by the total line integral of the Lagrangian along both segments

$$S[q_1, q_2] \equiv \int_{t_i}^{t_f} dt\, L(q_1, \dot{q}_1) + \int_{t_f}^{t_i} dt\, L(q_2, \dot{q}_2)$$

$$S[q_1, q_2] = \int_{t_i}^{t_f} dt \left\{ L(q_1, \dot{q}_1) - L(q_2, \dot{q}_2) \right\}$$

Hamilton's Principle & initial conditions (III) *(in preparation)*

- Hamilton's Principle: Extremize the new action $S[q_1, q_2]$

  - Convenient to make a change of variables:

$$q_+ = \frac{q_1 + q_2}{2} \qquad q_- = q_1 - q_2$$

$$p_+ = \frac{\partial \Lambda}{\partial \dot{q}_-} \qquad p_- = \frac{\partial \Lambda}{\partial \dot{q}_+}$$

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- Virtual paths are given by:

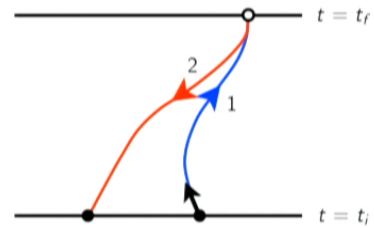
$$q_+(t,\epsilon) = q_+(t,0) + \epsilon\eta_+(t) \qquad q_-(t,\epsilon) = q_-(t,0) + \epsilon\eta_-(t)$$

- Variation of the new action:

$$\left.\frac{dS[q_+, q_-]}{d\epsilon}\right|_{\epsilon=0} = \int_{t_i}^{t_f} dt \left\{ \eta_+(t) \left(\frac{\partial \Lambda}{\partial q_+} - \frac{d}{dt}\frac{\partial \Lambda}{\partial \dot{q}_+} \right)_0 + \eta_-(t) \left(\frac{\partial \Lambda}{\partial q_-} - \frac{d}{dt}\frac{\partial \Lambda}{\partial \dot{q}_-} \right)_0 \right\}$$

$$+ \Big[\eta_+(t) p_-(t) + \eta_-(t) p_+(t) \Big]_{t=t_i}^{t_f}$$

Hamilton's Principle & initial conditions (IV) *(in preparation)*



- Conditions at the time boundaries

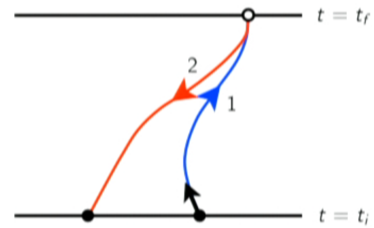
  - Vanishing displacements at initial time

$$\eta_1(t_i) = 0 = \eta_2(t_i) \quad \Longrightarrow \quad \eta_+(t_i) = 0 = \eta_-(t_i)$$

  - Continuity of coordinates at final time

$$\eta_2(t_f) = \eta_1(t_f) \quad \Longrightarrow \quad \eta_-(t_f) = 0$$

Hamilton's Principle & initial conditions (IV) *(in preparation)*



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$$\eta_2(t_f) = \eta_1(t_f) \implies \eta_-(t_f) = 0$$

  - Continuity of conjugate momenta at final time

$$p_2(t_f) = p_1(t_f) \implies p_-(t_f) = 0$$

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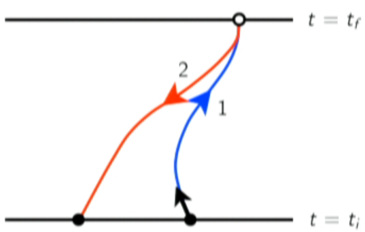

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  - Continuity of coordinates at final time

$$\eta_2(t_f) = \eta_1(t_f) \implies \eta_-(t_f) = 0$$

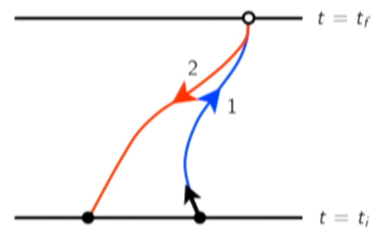
  - Continuity of conjugate momenta at final time

$$p_2(t_f) = p_1(t_f) \implies p_-(t_f) = 0$$

- Boundary contributions to action

$$\Big[\eta_+(t)p_-(t) + \eta_-(t)p_+(t)\Big]_{t=t_i}^{t_f} = 0$$

Hamilton's Principle & initial conditions (V) *(in preparation)*



Equations of motion

- With the boundary term eliminated:

$$\left.\frac{dS[q_+, q_-]}{d\epsilon}\right|_{\epsilon=0} = \int_{t_i}^{t_f} dt \left\{ \eta_+(t) \left(\frac{\partial \Lambda}{\partial q_+} - \frac{d}{dt}\frac{\partial \Lambda}{\partial \dot{q}_+} \right)_0 + \eta_-(t) \left(\frac{\partial \Lambda}{\partial q_-} - \frac{d}{dt}\frac{\partial \Lambda}{\partial \dot{q}_-} \right)_0 \right\}$$

- The action is stationary or extremal when

$$\left.\frac{dS[q_+, q_-]}{d\epsilon}\right|_{\epsilon=0} = 0 \quad \Longrightarrow \quad \begin{aligned} \frac{d}{dt}\left(\frac{\partial \Lambda}{\partial \dot{q}_+}\right) - \frac{\partial \Lambda}{\partial q_+} = 0 \\ \frac{d}{dt}\left(\frac{\partial \Lambda}{\partial \dot{q}_-}\right) - \frac{\partial \Lambda}{\partial q_-} = 0 \end{aligned}$$

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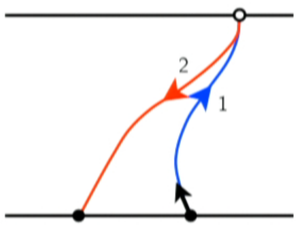

☐ Equations of motion

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- Lastly, identify both paths as the physical one, *q(t)* -- the "physical limit"

$$\Longrightarrow \quad q_-(t) \to 0, \;\; q_+(t) \to q(t)$$

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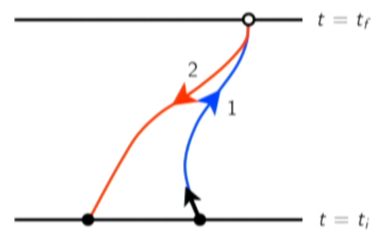

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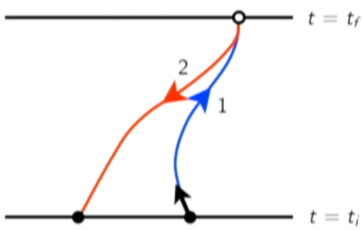
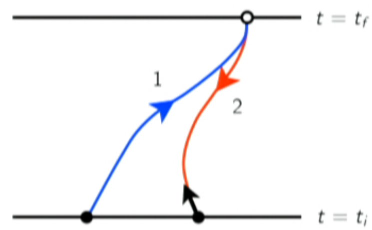
Identifying the paths: The "physical limit" (1)

- An ambiguity in associating physical initial data for a physical path with two unphysical paths



$q_1(t_i) = q_i$



$q_2(t_i) = q_i$

Identifying the paths: The "physical limit" (1)

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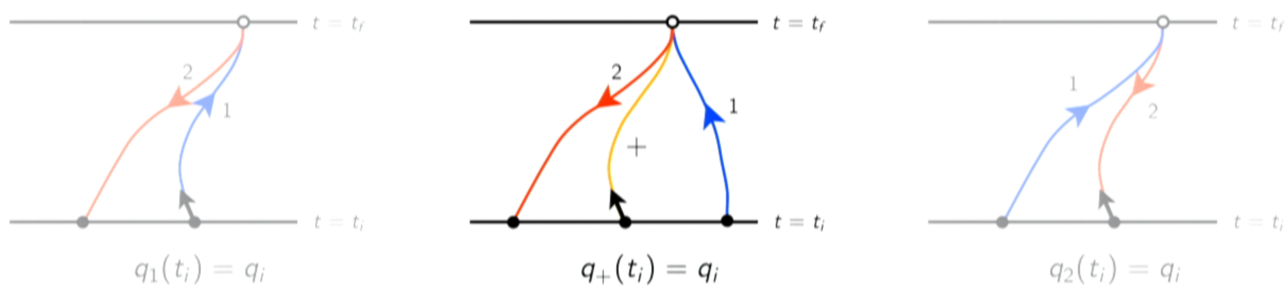

- Natural to identify physical initial data with $q_+(t_i)$

$$\text{Physical limit} \implies q_-(t) \to 0, \; q_+(t) \to q(t)$$

- Make this a convention

Identifying the paths: The "physical limit" (II)

- A short-cut for deriving equations of motion/forces in physical limit:

  - In the physical limit, only the Euler-Lagrange (EL) eqn for the + variable survives

Physical limit

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$$\Longrightarrow \quad \frac{d}{dt}\left(\frac{\partial \Lambda}{\partial \dot{q}_+}\right) = \frac{dp_-}{dt} = \frac{\partial \Lambda}{\partial q_+} \to 0$$

$$\frac{d}{dt}\left(\frac{\partial \Lambda}{\partial \dot{q}_-}\right) = \frac{dp_+}{dt} = \frac{\partial \Lambda}{\partial q_-} \longrightarrow \frac{dp}{dt} = \frac{\partial L}{\partial q}$$

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Short-cut: The EL equation for the + variable is equivalently given by

$$0 = \left.\frac{\delta S[q_+, q_-]}{\delta q_-(t)}\right|_{q_-=0, q_+=q}$$

Therefore, only terms of the new action linear in q_- contribute to forces in the physical limit

Example: Simple harmonic oscillator

- Lagrangian for a unit-mass oscillator

$$L(q, \dot{q}) = \frac{1}{2}\dot{q}^2 - \frac{1}{2}\omega^2 q^2 \implies \Lambda(q_1, q_2, \dot{q}_1, \dot{q}_2) = L(q_1, \dot{q}_1) - L(q_2, \dot{q}_2)$$

- Equations of motion for q_+ and q_-

$$\ddot{q}_+ + \omega^2 q_+ = 0$$

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- "Final" conditions for q_-

$$\text{Continuity} \implies q_-(t_f) = 0$$

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$$\begin{aligned} \ddot q_+ + \omega^2 q_+ &= 0 \\ \ddot q_- + \omega^2 q_- &= 0 \end{aligned} \implies \begin{aligned} q_+(t) &= A_+\cos\omega(t-t_i) + B_+\sin\omega(t-t_i) \\ q_-(t) &= A_-\cos\omega(t_f-t) + B_-\sin\omega(t_f-t) \end{aligned}$$

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$$\begin{aligned} q_+(t_i) &= q_i \\ \dot q_+(t_i) &= v_i \end{aligned} \implies q_+(t) = q_i\cos\omega(t-t_i) + \frac{v_i}{\omega}\sin\omega(t-t_i)$$

Example: Forced harmonic oscillator

- Add an external driving force *F(t)* -- one for each path $F_1(t)$, $F_2(t)$ & identify with *F(t)* at end

$$L(q, \dot{q}) = \frac{1}{2}\dot{q}^2 - \frac{1}{2}\omega^2 q^2 + qF(t)$$

$$\Longrightarrow \quad \Lambda = L(q_1, \dot{q}_1) - L(q_2, \dot{q}_2) = \dot{q}_+ \dot{q}_- - \omega^2 q_+ q_- + q_- F_+(t) + q_+ F_-(t)$$

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$$\implies q_-(t) = \int_{t_i}^{t_f} dt\, G_{\text{adv}}(t-t')F_-(t')$$

- Initial conditions for q_+ $\qquad q_+(t_i) = q_i \qquad \dot{q}_+(t_i) = v_i$

$$\implies q_+(t) = q_i \cos\omega(t-t_i) + \frac{v_i}{\omega}\sin\omega(t-t_i) + \int_{t_i}^{t_f} dt\, G_{\text{ret}}(t-t')F_+(t')$$

Plus-minus variables vs. (1,2) variables

- The plus-minus variables have a clear, dynamical interpretation

Plus variable:

- Evolves forward in time due to initial conditions
- Equals the physical solution for source-free dynamics
- Corresponds to physical variable in physical limit

Minus variable:

- Evolves backward in time due to continuity conditions at final time
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Minus variable:

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- The (1,2) variables have no clear interpretation

- e.g., Forced harmonic oscillator:

$$q_1(t) = q_i \cos\omega(t - t_i) + \frac{v_i}{\omega}\sin\omega(t - t_i)$$

$$+ \int_{t_i}^{t_f} dt' \left\{ \left[\frac{G_{\rm ret}(t-t') + G_{\rm adv}(t-t')}{2} \right] F_1(t') + \left[\frac{G_{\rm ret}(t-t') - G_{\rm adv}(t-t')}{2} \right] F_2(t') \right\}$$

$$q_2(t) = q_1(t) \text{ with } (1 \leftrightarrow 2)$$

New Hamiltonian (1)

- Recall the new Lagrangian

$$\Lambda(q_1, q_2, \dot{q}_1, \dot{q}_2) \equiv L(q_1, \dot{q}_1) - L(q_2, \dot{q}_2)$$

- Conjugate momenta

$$p_1 = \frac{\partial L(q_1, \dot{q}_1)}{\partial \dot{q}_1} = \frac{\partial \Lambda}{\partial \dot{q}_1} \qquad p_2 = \frac{\partial L(q_2, \dot{q}_2)}{\partial \dot{q}_2} = -\frac{\partial \Lambda}{\partial \dot{q}_2}$$

- Legendre transform both Lagrangians to get the new Hamiltonian

$$H(q_1, p_1) = p_1 \dot{q}_1 - L(q_1, \dot{q}_1) \qquad H(q_2, p_2) = p_2 \dot{q}_2 - L(q_2, \dot{q}_2)$$

$$A(q_1, p_1, q_2, p_2) = p_1 \dot{q}_1 - p_2 \dot{q}_2 - \Lambda(q_1, q_2, \dot{q}_1, \dot{q}_2)$$

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- A hint of internal structure for the path variables

- Can define a (2d Minkowski) "metric"

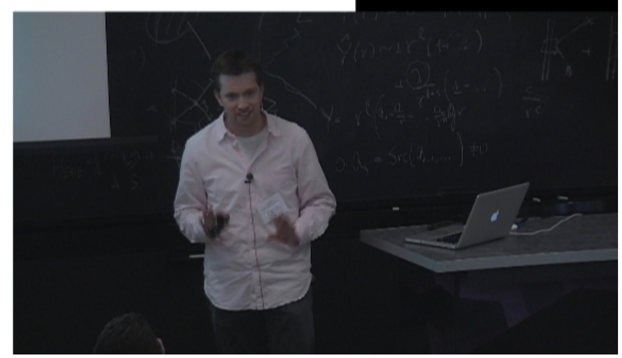
$$c^{ab} = c_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad A(q_a, p_a) = c^{ab} p_a \dot{q}_b - \Lambda(q_a, \dot{q}_a)$$

New Hamiltonian (II)

- This "metric" behaves like a real metric
  - A diffeomorphism of the path variables implies

$$q_a = q_a(q'_b) \implies \dot{q}_a = \frac{\partial q_a}{\partial q'_i}\dot{q}'_i \qquad p_a = \frac{\partial q_a}{\partial q'_i} p'_i \implies c^{ab} p_a \dot{q}_b = c'^{ij} p'_i \dot{q}'_j$$

$$c^{ab} = \frac{\partial q'_i}{\partial q_a}\frac{\partial q'_j}{\partial q_b} c'^{ij}$$



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  - In +/- variables, the new Hamiltonian is

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- Hamiltonian in general path coordinates

$$A(q_a, p_a) \equiv p_a \dot{q}^a - \Lambda(q_a, \dot{q}_a)$$

- Hamilton's equations & Poisson brackets

$$\dot{q}_a = \frac{\partial A}{\partial p^a} = \{A, q_a\} \qquad \dot{p}_a = -\frac{\partial A}{\partial q^a} = \{A, p_a\} \qquad \{f, g\} \equiv \frac{\partial f}{\partial p_a}\frac{\partial g}{\partial q^a} - \frac{\partial f}{\partial q^a}\frac{\partial g}{\partial p_a}$$

Open systems

- Many EFTs are mean to describe open systems resulting from integrating out some degrees of freedom in one way or another

  - Extended charge distribution coupled to electromagnetic field

  - Gravitational waves emitted by the inspiral of a compact binary

  - Viscous fluid flow

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- The resulting effective action describes an open (i.e., dissipative) system

  - Usual formulation of Hamilton's Principle will fail dramatically

  - Must use the new Hamilton's Principle to correctly account for causality, dissipation, initial data, etc.

Reprise: A simple open system example (1)

- N harmonic oscillators $\{Q_n(t)\}$ coupled bi-linearly to $q(t)$

$$L[q, \{Q_n\}] = L_q[q] + \sum_{n=1}^{N} \left(\frac{M\dot{Q}_n^2}{2} - \frac{M\Omega_n^2 Q_n^2}{2} + \lambda_n q Q_N \right)$$

$$\implies \Lambda[q_1, q_2, \{Q_{1,n}\}, \{Q_{2,n}\}] \equiv L[q_1, \{Q_{1,n}\}] - L[q_2, \{Q_{2,n}\}]$$

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- "Integrate out" $Q_n(t)$ subject to initial conditions

$$\ddot{Q}_{\pm,n} + \Omega_n^2 Q_{\pm,n} = \frac{\lambda_n}{M} q_\pm \qquad \Longrightarrow \qquad \begin{aligned} Q_{+,n}^{\text{soln}}(t) &= Q_{+,n}^{(h)}(t) + \frac{\lambda_n}{M} \int_{t_i}^{t_f} dt'\, G_{\text{ret}}^{(n)}(t-t') q_+(t') \\ Q_{-,n}^{\text{soln}}(t) &= \frac{\lambda_n}{M} \int_{t_i}^{t_f} dt'\, G_{\text{adv}}^{(n)}(t-t') q_-(t') \end{aligned}$$

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Reprise: A simple open system example (II)

- Effective action

$$S_{\text{eff}}[q_1, q_2] = S_q[q_1] - S_q[q_2] + \sum_{n=1}^{N} \frac{\lambda_n}{2} \int_{t_i}^{t_f} dt' \left(q_- Q_{+,n}^{\text{soln}} + q_+ Q_{-,n}^{\text{soln}} \right)$$

- Effective action in a compact notation

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Reprise: A simple open system example (II)

- Effective action

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- In the +/- basis

$$c^{ab} = c_{ab} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad G^{ab}_{(n)}(t-t') = \begin{pmatrix} 0 & G^{\text{adv}}_{(n)}(t-t') \\ G^{\text{ret}}_{(n)}(t-t') & 0 \end{pmatrix}$$

$$a, b = \{+, -\}$$

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Reprise: A simple open system example (III)

- Equation of motion for $q(t)$ is (in the physical limit)

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- A little massaging gives

$$m\ddot{q} + m\left(\omega^2 + \sum_{n=1}^{N} \frac{\lambda_n^2}{M\Omega_n^2}\right) q + \int_{t_i}^{t} dt'\, \gamma(t-t')\dot{q}(t') = F(t)$$

$$\gamma(t-t') = \sum_{n=1}^{N} \frac{\lambda_n^2}{M\Omega_n^2} \cos\Omega_n(t-t') \qquad F(t) = \sum_{n=1}^{N}\left(\frac{\lambda_N}{2} Q_n^{(h)}(t) - \frac{\lambda_n^2}{M\Omega_n^2}\cos\Omega_n(t-t_i) q(t_i)\right)$$

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- Resulting solution evolves causally and specified entirely by initial data

- History-dependent dissipation term ("non-Markovian" evolution)

Connection to quantum theory

- The new action is the one that arises in the classical limit of the corresponding "in-in" or "closed-time-path (CTP)" quantum theory

- "In-in" or "CTP" formulation of quantum theory

  - Is an initial value formulation of path integral quantization

  - Developed by Schwinger, Keldysh, Feynman, Vernon, Hu, Calzetta, Cooper,...

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$$Z_{\text{in}-\text{in}}[J_1, J_2] \equiv \int \mathcal{D}q_1(t)\mathcal{D}q_2(t) \, \exp\left\{ \frac{i}{\hbar}(S[q_1] - S[q_2]) + \frac{i}{\hbar}\int dt(J_1 q_1 - J_2 q_2) \right\}$$

  - Approach the classical limit

$$\begin{aligned} S_{\text{classical}} &= S[q_1] - S[q_2] + \int dt(J_1 q_1 - J_2 q_2) + O(\hbar) \\ &= S[q_1, q_2] + O(\hbar) \end{aligned}$$

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  - On the other hand, path integral quantization based on the new action

$$Z[J_1, J_2] \equiv \int \mathcal{D}q_1(t)\mathcal{D}q_2(t) \, \exp\left\{ \frac{i}{\hbar} S[q_1, q_2] + \frac{i}{\hbar}\int dt(J_1 q_1 - J_2 q_2) \right\} = Z_{\text{in}-\text{in}}[J_1, J_2]$$

Summary of Hamilton's Principle for IVPs

- ☐ Usual formulation does not properly describe initial value problems, dissipation, or open systems

- ☐ New formulation presented here fixes these shortcomings:

  - Variational calculus for initial value problems

  - New action is classical limit of "in-in"/"closed-time-path" quantum theory

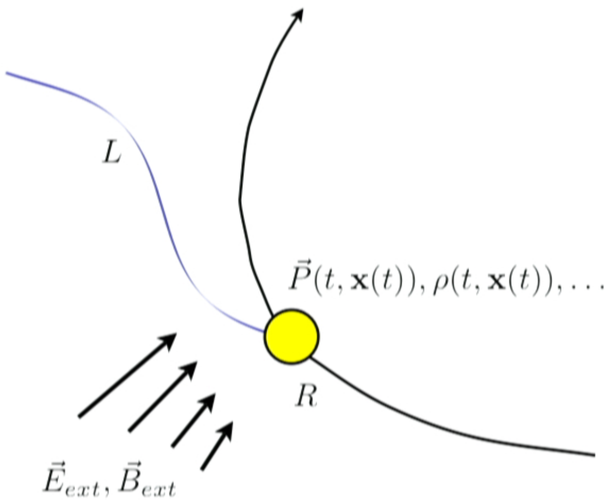
- ☐ New action is appropriate for describing open systems in EFT

<u>New "mantra" of classical mechanics:</u>

New Lagrangian, Hamiltonian, and Routhian can be constructed and applied to general dissipative systems

Motion of an extended charge

- Consider the motion of an extended (spherical) charge distribution



Metal

Dielectric

Superconductor

Metamaterial

Spinning dust

etc.

- A complete description of the motion and radiation is hopelessly complicated...

CG, Leibovich & Rothstein, PRL (2010)

Motion of an extended charge in EFT (1)

- The EFT for this system:

Identify the relevant degrees of freedom and their symmetries

$\{z^\mu(\lambda), A_\mu(x^\alpha)\}$

Lorentz (Poincare) invariance
[Rotational, time-reversal & parity invariance]

Gauge invariance

Reparameterization invariance of worldline

Write most general action consistent with symmetries (i.e., derivative expansion)

$$S[z^\mu, A_\mu] = -\frac{1}{4}\int_x F^{\alpha\beta}F_{\alpha\beta} - m\int d\tau + e\int d\tau\, u^\alpha A_\alpha(z) + C_d \int d\tau\, u^{[\alpha}a^{\beta]}F_{\alpha\beta}(z)$$

$$+ C_e \int d\tau\, F^{\alpha\beta}(z)F_{\alpha\beta}(z) + C_m \int d\tau\, u^\alpha F_{\alpha\beta}(z) F^\beta{}_\gamma(z) u^\gamma + \cdots$$

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- Power counting

$C_d \sim eR^2 \qquad C_e, C_m \sim R^3 \qquad \Longrightarrow$ No $O(R)$ terms in action!

CG, Leibovich, Rothstein, PRL (2010)

Motion of an extended charge in EFT (II)

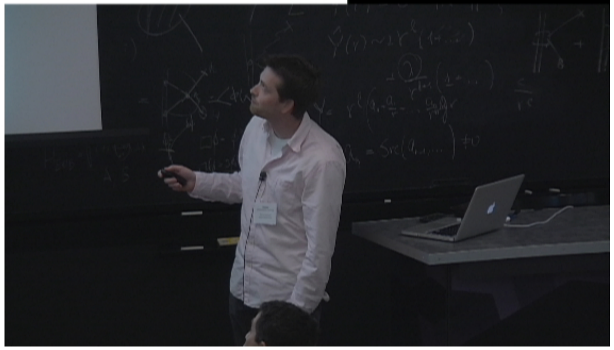
Matching to a specific theory, model, or data

Perfectly conducting sphere of charge

$$C_d = \frac{1}{10} e R^2$$

Spherical shell of charge

$$C_d = \frac{1}{6} e R^2$$



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Compute stuff: Radiation reaction force due to finite size of charge distribution

- New action

$$S[z^\mu_{1,2}, A^\mu_{1,2}] \equiv S[z^\mu_1, A^\mu_1] - S[z^\mu_2, A^\mu_2]$$

$$= -\frac{1}{4}\int_x (F_1^2 - F_2^2) + e\int_x (j_1^\alpha A_{1\alpha} - j_2^\alpha A_{2\alpha}) + C_d \int_x (\sigma_1^{\alpha\beta} F_{1\alpha\beta} - \sigma_2^{\alpha\beta} F_{2\alpha\beta}) + \cdots$$

$$j^\alpha_{1,2}(x;z] \equiv \int d\lambda\, \delta^4(x - z_{1,2}) u^\alpha_{1,2} \qquad \sigma^{\alpha\beta}_{1,2}(x;z] \equiv \int d\lambda\, \delta^4(x - z_{1,2}) u^{[\alpha}_{1,2} a^{\beta]}_{1,2}$$

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- New action in a compact notation

$$S[z^\mu_{1,2}, A^\mu_{1,2}] = -\frac{1}{4}\int_x F_a^{\alpha\beta} F^a_{\alpha\beta} + e\int_x j_a^\alpha A^a_\alpha + C_d \int_x \sigma_a^{\alpha\beta} F^a_{\alpha\beta} + \cdots$$

Motion of an extended charge in EFT (III)

- Integrate out the vector field using Feynman diagrams and rules

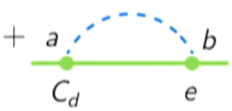
$$= D^{ab}_{\mu\nu}(x - x')$$

$$= \left. \frac{\delta^n S}{\delta A^{a_1}_{\alpha_1}(x) \cdots \delta A^{a_n}_{\alpha_n}(x)} \right|_{A_\mu = 0}$$

- In terms of Feynman diagrams, the effective action is

$$S_{\text{eff}}[z^{\mu}_{1,2}] = -m \int (d\tau_1 - d\tau_2) +$$

$$+$$



$$+ O(R^3)$$

- Diagram for leading order radiation reaction

$$= \frac{e^2}{6\pi} \int d\tau \, z_{-\alpha} (\dot{a}^{\alpha}_{+} + u^{\alpha}_{+} u^{\beta}_{+} \dot{a}_{+\beta}) + O(z^2_{-})$$

$$0 = \left. \frac{\delta S}{\delta z^{\mu}_{-}(\tau)} \right|_{z_- = 0, z_+ = z} \quad \Longrightarrow \quad F^{\alpha}_{\text{ALD}}(\tau) = \frac{e^2}{6\pi} (\dot{a}^{\alpha} + u^{\alpha} u^{\beta} \dot{a}_{\beta})$$

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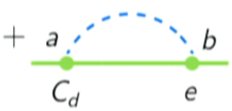

$$+ O(R^3)$$

- Diagram for leading order radiation reaction

$$= \frac{e^2}{6\pi} \int d\tau\, z_{-\alpha} (\dot{a}^\alpha_+ + u^\alpha_+ u^\beta_+ \dot{a}_{+\beta}) + O(z_-^2)$$

$$0 = \frac{\delta S}{\delta z^\mu_-(\tau)}\Bigg|_{z_- = 0, z_+ = z} \qquad \Longrightarrow \qquad F^\alpha_{\rm ALD}(\tau) = \frac{e^2}{6\pi}(\dot{a}^\alpha + u^\alpha$$

- Compare to effective action from usual Hamilton's Principle

$$S_{\rm eff}[z^\mu] = -m \int d\tau - \frac{e^2}{8\pi} \int d\tau\, z^\alpha \dot{a}_\alpha$$

Take-home points

- Most EFTs rely on building an action and thus on Hamilton's Principle

- Hamilton's Principle of Extremal Action, as usually formulated, has shortcomings:

  1) Is applicable only to boundary values in time

  2) Cannot generally incorporate dissipative effects

  3) Fails to describe open systems

- Hamilton's Principle can be reformulated to address these shortcomings

  - A variational calculus for initial value problems

  - Provides Lagrangian, Hamiltonian, Routhian formulations for open/dissipative systems

- Reformulation of Hamilton's Principle + EFT provides a powerful framework

Conclusion

- Reformulation of Hamilton's Principle is successfully applied in EFT:

1) Finite size corrections to radiation reaction force on an extended charge
   CG, Leibovich & Rothstein, PRL (2010)
2) 2.5 & 3.5 PN radiation reaction forces
   CG, & Tiglio, PRD (2010); CG & Leibovich (in preparation)
3) 4 PN tail contribution to potential See talk by Foffa & Sturani
   Foffa & Sturani (arXiv:1111.5488)
4) Gravitational self-force and waveform at 1st order in mass ratio
   CG & Hu, PRD (2009)
5) Scalar self-force and waveforms through 3rd order in mass ratio
   CG, CQG (in publication) (2011)