Title: Post-Newtonian equations of motion and radiation (standard approach)

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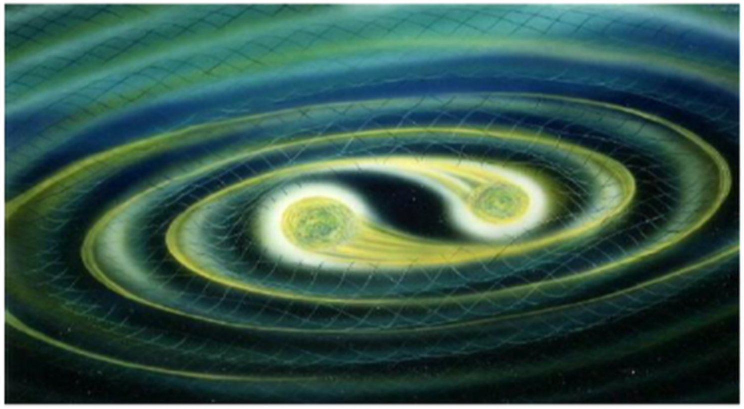
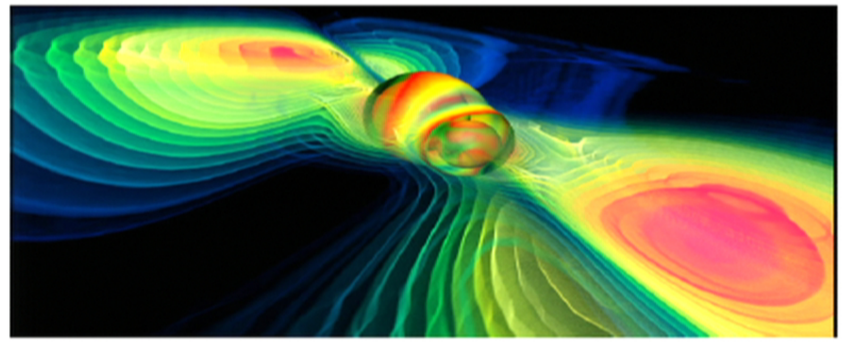
Abstract: High-accuracy templates predicted by general relativity for the gravitational waves generated by inspiralling compact binaries (binary star systems composed of neutron stars and/or black holes) have been developed using a mixed multipolar and post-Newtonian (MPN) formalism. In this talk we shall review the foundations of this formalism and its main results, including the equations of motion and radiation from compact binaries up to 3.5PN order. We shall also present some recent work on the comparison between post-Newtonian approximations and black hole perturbations applied to compact binaries in the small mass ratio limit.

The inspiral and merger of compact binaries



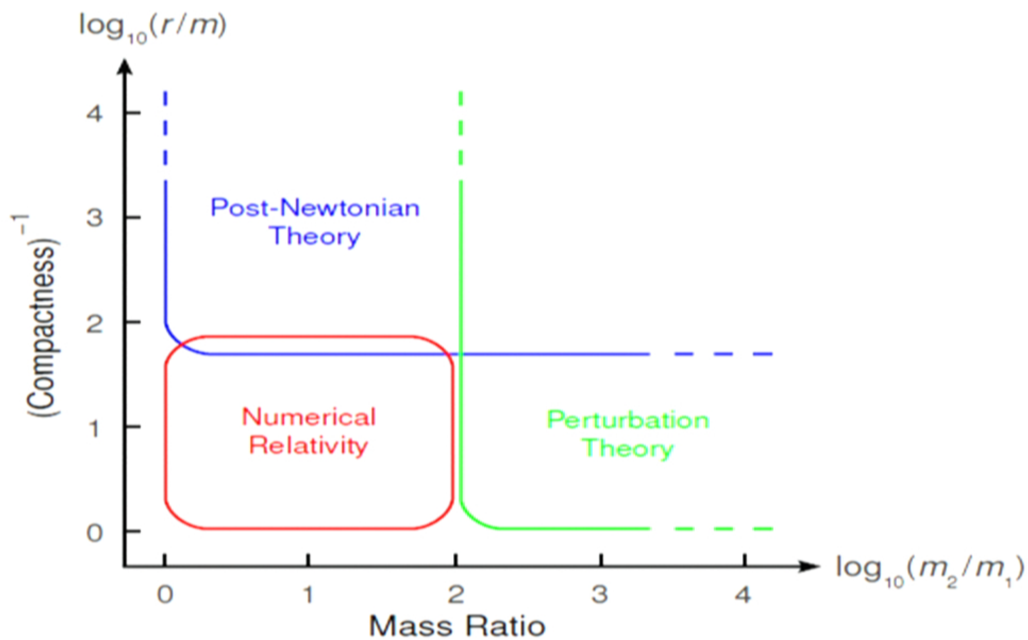
Neutron stars spiral and coalesce



Black holes spiral and coalesce

1. Neutron star ($M = 1.4\, M_\odot$) events will be detected by ground-based detectors LIGO/VIRGO/GEO
2. Stellar size black hole ($5\, M_\odot \lesssim M \lesssim 20\, M_\odot$) events will also be detected by ground-based detectors
3. Supermassive black hole ($10^5\, M_\odot \lesssim M \lesssim 10^8\, M_\odot$) events will be detected by the space-based detectors LISA/eLISA

Methods to compute gravitational-wave templates



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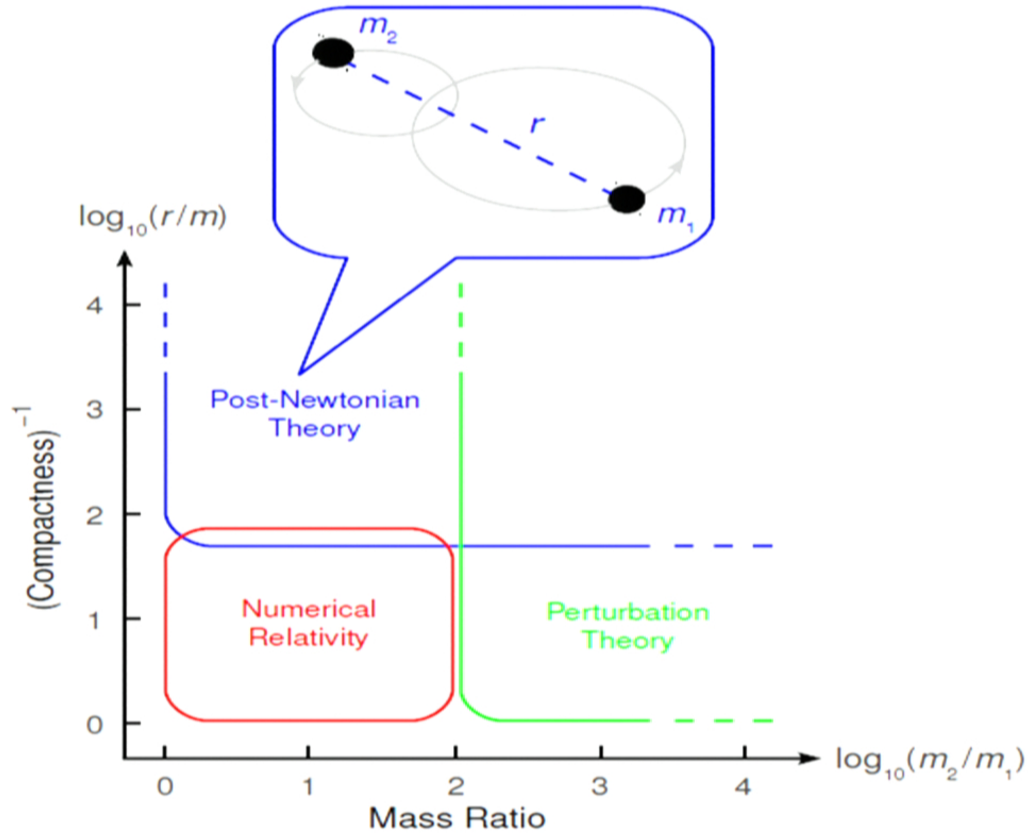

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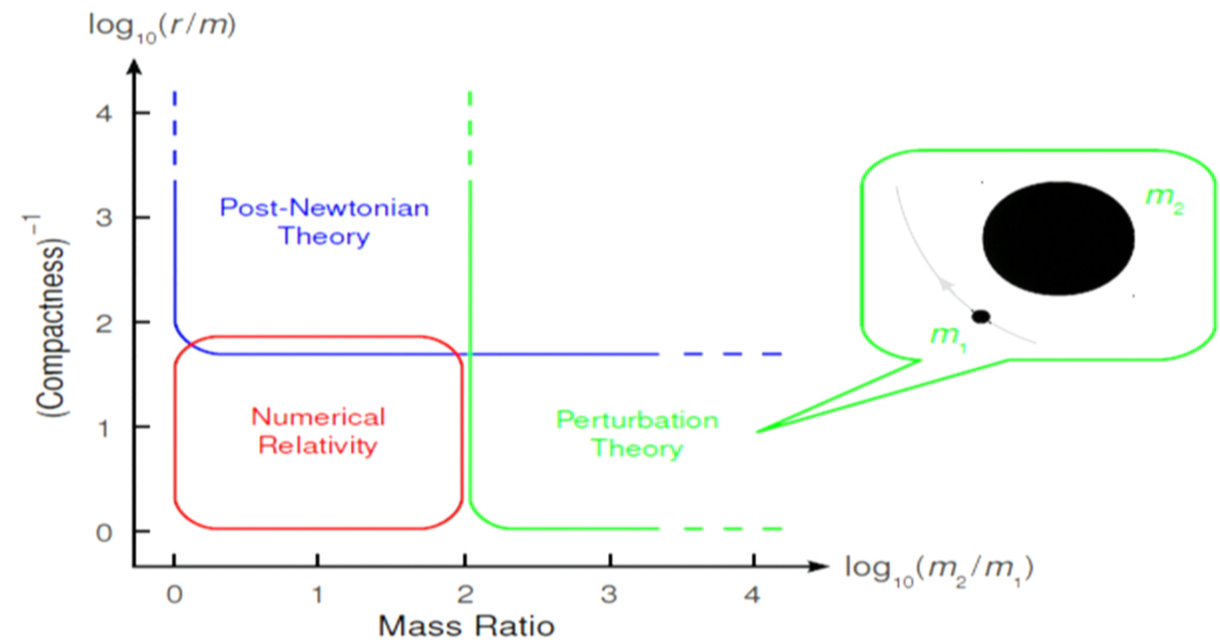

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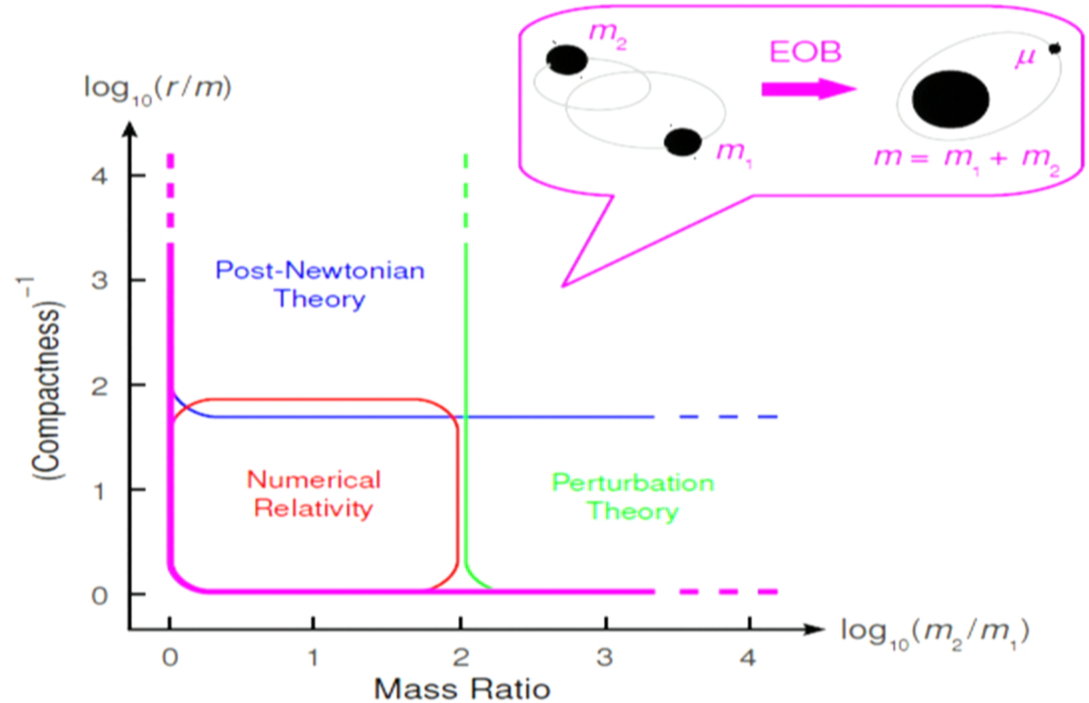

Short history of the traditional approach

Equations of motion

- 1PN equations of motion [Lorentz & Droste 1917, Einstein, Infeld & Hoffmann 1930]
- Radiation-reaction controvercy [Ehlers *et al* 1979, Walker & Will 1982]
- 2.5PN equations of motion and GR prediction for the binary pulsar [Damour & Deruelle 1982, Damour 1983]
- The 3mn Caltech paper [Cutler, Flanagan, Poisson, Thorne 1993]
- 3PN equations of motion [Jaranowski & Schäfer 1999, LB & Faye 2001, Itoh & Futamase 2003]
- Ambiguity parameters resolved [Damour, Jaranowski & Schäfer 2001, LB, Damour & Esposito-Farèse 2003]

Radiation field

- 1916 Einstein quadrupole formula
- 1940 Landau-Lifchitz formula
- 1960 Peters-Mathews formula
- Epstein-Wagoner-Thorne moments [Thorne 1980]
- 1PN wave generation [Wagoner & Will 1976, LB & Schäfer 1989]
- Blanchet-Damour moments [LB & Damour 1989, LB 1995, 1998]
- 2PN wave generation [LB, Damour, Iyer, Will & Wiseman 1995]
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Wave-generation and radiation-reaction

Einstein's field equations

$$\partial_\nu h^{\mu\nu} = 0 \qquad \text{(harmonic gauge condition)}$$

$$\Box h^{\mu\nu} = \frac{16\pi G}{c^4}\tau^{\mu\nu} \qquad \text{(flat space-time wave equation)}$$

Pseudo-tensor of matter and gravitational fields

$$\tau^{\mu\nu} = \underbrace{|g| T^{\mu\nu}}_{\text{matter part}} + \underbrace{\Lambda^{\mu\nu}(h, \partial h, \partial^2 h)}_{\text{gravitational part}}$$

1. Radiation-reaction problem: solve the EFE inside the compact-support source to get the reaction forces acting the an isolated source
2. Wave generation problem: solve the EFE in vacuum outside the source (including the regions at infinity) to get the waveform as a functional of the source parameters

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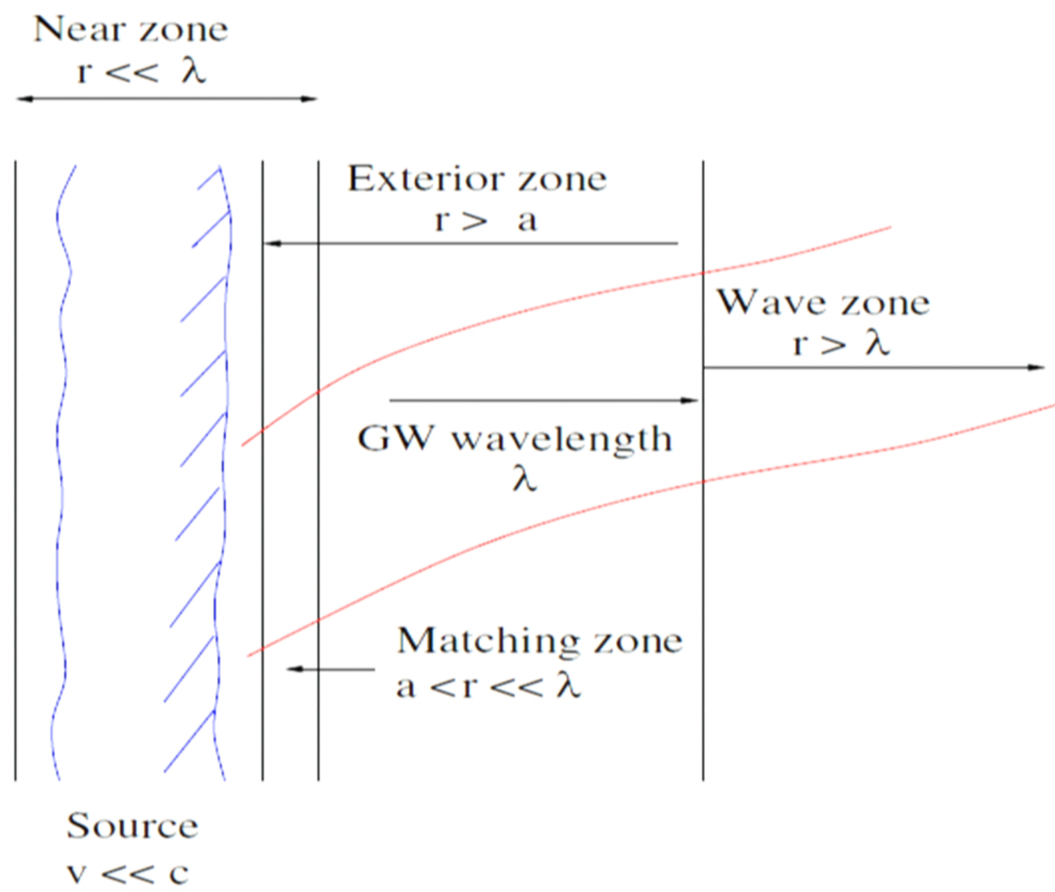
Multipolar post-Newtonian and matching approach



Multipolar post-Newtonian and matching approach

1. Solve the EFE in the vacuum region outside the source

$$\begin{aligned} \partial_\nu h^{\mu\nu} &= 0 \\ \Box h^{\mu\nu} &= \Lambda^{\mu\nu}(h) \end{aligned}$$

in the form of a general multipolar-post-Minkowskian expansion

2. Look for the general solution of the EFE in the source's near zone in the form of a post-Newtonian expansion
3. Match the MPM expansion and the PN expansion in the overlapping exterior part of the near zone

The matching will uniquely determine

- the multipole moments in terms of the source
- the radiation reaction force acting in the source

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Multipolar post-Minkowskian expansion [LB & Damour 1986, 1988]

1. Look for the general multipolar expansion $\mathcal{M}(h)$ generated outside the source in the form [Bonnor 1959, Bonnor & Rotenberg 1961]

$$\mathcal{M}(h) = \underbrace{G\, h_{(1)} + G^2\, h_{(1)} + \cdots + G^n\, h_{(n)} + \cdots}_{\text{formal post-Minkowskian expansion}}$$

2. Start from the general multipolar solution of the vacuum field equation at linear order [Thorne 1980]

$$h_{(1)}[M_L, S_L] = \sum_{\ell=0}^{\infty} \partial_L \left(\frac{1}{r} \underbrace{M_L(t - r/c)}_{\text{mass-type moment}} \right) + \varepsilon\, \partial_L \left(\frac{1}{r} \underbrace{S_L(t - r/c)}_{\text{current-type moment}} \right)$$

3. Iterate that solution using a regularization scheme to cope with the singularity of the multipole expansion when $r \to 0$

$$\mathrm{FP}\Box^{-1}_{\mathrm{ret}} f \equiv \text{Finite Part}_{B\to 0}\Box^{-1}_{\mathrm{ret}}\left[(r/r_0)^B f\right]$$

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A plug-and-grind algorithm

At n-th post-Minkowskian order we need to solve

$$\partial_\nu h^{\mu\nu}_{(n)} = 0$$

$$\Box h^{\mu\nu}_{(n)} = \Lambda^{\mu\nu}(\underbrace{h_{(1)}, \cdots h_{(n-1)}}_{\text{previously known}})$$

The solution is found to be

$$h^{\mu\nu}_{(n)} = \mathrm{FP}\Box^{-1}_{\mathrm{Ret}}\Lambda^{\mu\nu}_{(n)} + q^{\mu\nu}_{(n)}$$

The term $q^{\mu\nu}_{(n)}$ is required to guarantee the harmonic gauge condition $\partial_\nu h^{\mu\nu}_{(n)} = 0$

The above solution is the most general solution of the vacuum EFE outside the source. It is parametrized by two and only two sets of multipole moments

$$\left\{M_L(t), S_L(t)\right\}$$

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Matching to a post-Newtonian source

The general PN expansion inside the source's near zone is

$$\bar{h}^{\mu\nu}(\mathbf{x}, t, c) = \sum_{p \geq 2} \frac{1}{c^p} h_p^{\mu\nu}(\mathbf{x}, t, \ln c)$$

Matching asymptotic expansions formally requires

$$\overline{\mathcal{M}(h^{\mu\nu})} \equiv \mathcal{M}(\bar{h}^{\mu\nu})$$

- The left side is the NZ expansion ($r \to 0$) of the exterior MPM field
- The right side is the multipole expansion ($r \to \infty$) of the inner PN field

The matching equation has been implemented at any post-Minkowskian order in the exterior field and any PN order in the inner field. It gives the formal multipolar-post-Newtonian solution valid everywhere inside and outside the source

General solution for the exterior field [LB 1995, 1998]

The general multipole expansion outside the source is obtained as

$$\mathcal{M}(h^{\mu\nu}) = \mathrm{FP}\Box^{-1}_{\mathrm{ret}}\mathcal{M}(\Lambda^{\mu\nu}) + \underbrace{\sum_{\ell=0}^{\infty}\partial_L\left\{\frac{F_L^{\mu\nu}(t-r/c)}{r}\right\}}_{\text{homogeneous retarded solution}}$$

where
$$F_L^{\mu\nu}(t) = \mathrm{FP}\int \mathrm{d}^3\mathbf{x}\,\hat{x}_L\int_{-1}^{1}\delta_\ell(z)\underbrace{\overline{\tau}^{\mu\nu}(\mathbf{x},t-zr/c)}_{\text{PN expansion of the pseudo-tensor}}$$

The FP procedure plays the role of an UV regularization in the non-linearity term but an IR regularization in the multipole moments

From this one obtains the source multipole moments $\{M_L(t), S_L(t)\}$ at any PN order solving the wave generation problem

Two equivalent PN wave generation formalisms

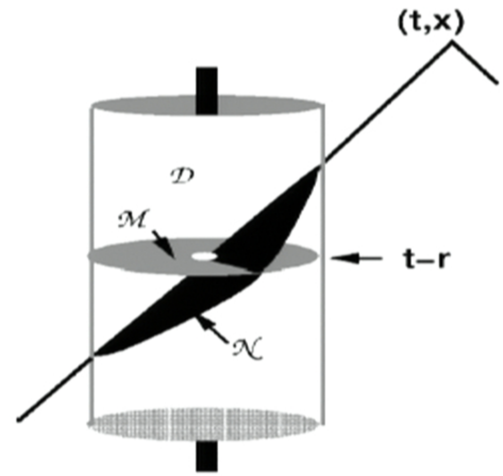
The field equations are integrated in the exterior of an extended PN source by means of a multipolar expansion

BD multipole moments

$$M_L^{\mu\nu}(t) = \underset{B=0}{\text{Finite Part}} \int \mathrm{d}^3x \, x_L \, \overline{\tau}^{\mu\nu}(\mathbf{x}, t)$$

WW multipole moments [Will & Wiseman 1996]

$$W_L^{\mu\nu}(t) = \int_{\mathcal{M}} \mathrm{d}^3x \, x_L \, \overline{\tau}^{\mu\nu}(\mathbf{x}, t)$$



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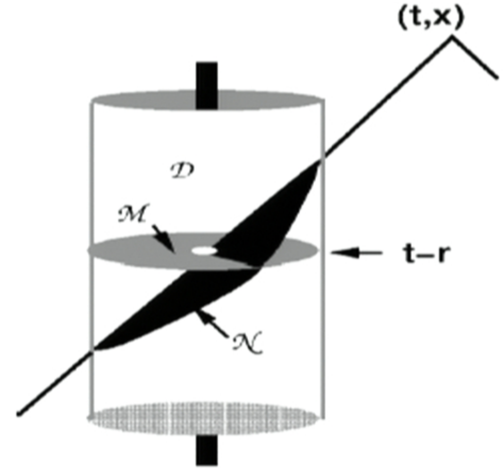

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General solution for the inner PN field [Poujade & LB 2002]

The inner PN field in the source's near zone is fully determined as

$$\bar{h}^{\mu\nu} = \mathrm{FP}\,\Box^{-1}_{\mathrm{Ret}}\bar{\tau}^{\mu\nu} + \underbrace{\sum_{\ell=0}^{\infty} \partial_L \left\{ \frac{R_L^{\mu\nu}(t-r/c) - R_L^{\mu\nu}(t+r/c)}{r} \right\}}_{\text{homogeneous antisymmetric solution}}$$

$$\text{where} \quad R_L^{\mu\nu}(t) = \mathrm{FP} \int \mathrm{d}^3\mathbf{x}\, \hat{x}_L \int_1^{\infty} \gamma_\ell(z) \mathcal{M}(\tau^{\mu\nu})(\mathbf{x}, t - zr/c)$$

This solves the radiation-reaction problem for the isolated matter source

1. Standard radiation reaction effects starting at 2.5PN order are contained in the first term
2. Nonlinear radiation reaction effects (tails) are contained in the second term and start at 4PN order

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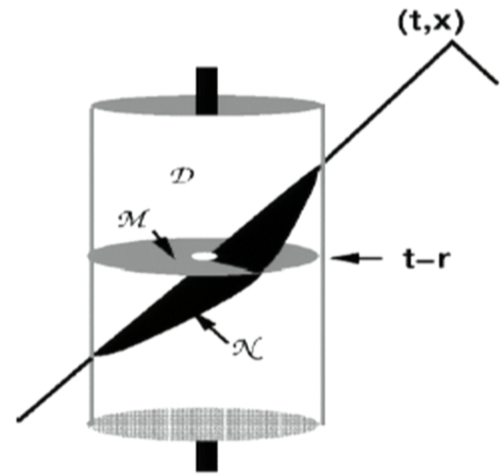

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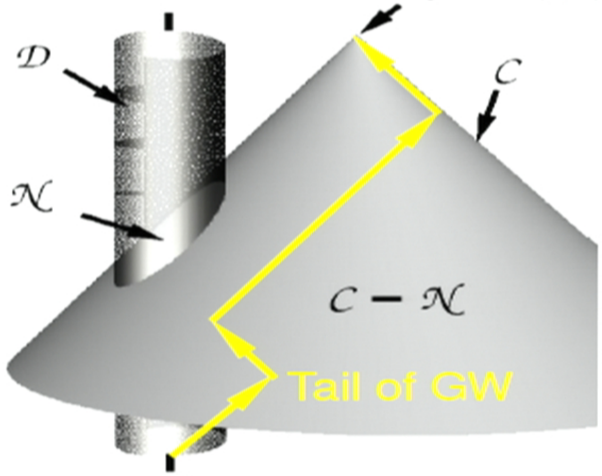
Tail modification of the radiation-reaction [LB & Damour 1988]



$$F^i_{\substack{\text{radiation}\\ \text{reaction}}} = -\frac{2}{5c^5}\rho x^j \left[M^{(5)}_{ij}(t) + \frac{4GM}{c^3}\int_{-\infty}^{t} \mathrm{d}t' \underbrace{\ln\left(\frac{t-t'}{2r}\right)}_{\text{logarithms appear at 4PN order}} M^{(7)}_{ij}(t') \right]$$

Radiative moments at future null infinity

Asymptotic waveform is parametrized by radiative moments [Thorne 1980]

$$h_{ij}^{TT} = \frac{1}{R}\sum_{\ell=2}^{\infty} N_{L-2}\underbrace{U_{ijL-2}(T-R/c)}_{\text{mass-type}} + \varepsilon_{ab(i}N_{aL-1}\underbrace{V_{j)bL-2}(T-R/c)}_{\text{current-type}}$$

Applying the plug-and-grind algorithm

$$\begin{aligned} U_{ij}(t) &= I_{ij}^{(2)}(t) + \underbrace{\frac{2GM}{c^3}\int_0^{+\infty} d\tau\, M_{ij}^{(4)}(t-\tau)\left[\ln\left(\frac{c\tau}{2r_0}\right)+\frac{11}{12}\right]}_{\text{1.5PN tail integral}} \\ &+ \frac{G}{c^5}\left\{\underbrace{-\frac{2}{7}\int_0^{+\infty} d\tau\, M_{a<i}^{(3)} M_{j>a}^{(3)}(t-\tau)}_{\text{2.5PN memory integral}} + \text{instantaneous terms}\right\} \\ &+ \underbrace{\frac{2G^2M^2}{c^6}\int_0^{+\infty} d\tau\, M_{ij}^{(5)}(t-\tau)\left[\ln^2\left(\frac{c\tau}{2r_0}\right)+\frac{57}{70}\ln\left(\frac{c\tau}{2r_0}\right)+\frac{124627}{44100}\right]}_{\text{3PN tail-of-tail integral}} \end{aligned}$$

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Application to compact binary inspiral

1. Specialize the previous PN solution to systems of point particles

$$T^{\mu\nu}(x) = \sum_A \int_{-\infty}^{+\infty} \mathrm{d}\tau_A \, p_A^{(\mu} u_A^{\nu)} \frac{\delta^{(4)}(x - y_A)}{\sqrt{-g_A}} + \text{(spin contributions)}$$

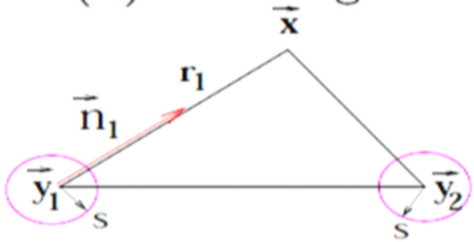
2. Suplement the calculation by a self-field regularization
   - Hadamard's regularization
   - Dimensional regularization
3. The self-field regularization should be applied conjointly with the FP regularization, say in the multipole moments

$$\mathrm{FP} \int \frac{\mathrm{d}^d \mathbf{x}}{{\ell_0}^{d-3}} \left(\frac{|\mathbf{x}|}{r_0} \right)^B F(\mathbf{x})$$

4. The IR scale r_0 and UV scale ℓ_0 should disappear at the end of the calculation

Hadamard self-field regularization [Hadamard 1932; Schwartz 1978]

Let $F(\mathbf{x})$ be a singular function at source points $\mathbf{y}_1$ and $\mathbf{y}_2$ such that when $r_1 \to 0$



$$F(\mathbf{x}) = \sum_{a_0 \leq a \leq N} r_1^a \underset{1}{f}_a(\mathbf{n}_1) + o(r_1^N)$$

Hadamard partie finie of the singular function F at point 1

$$(F)_1 \equiv \int \frac{d\Omega_1}{4\pi} \underset{1}{f}_0(\mathbf{n}_1)$$

Hadamard partie finie of the divergent integral $\int d^3\mathbf{x}\, F$

$$\mathrm{Pf} \int d^3\mathbf{x}\, F(\mathbf{x}) \equiv \lim_{s \to 0} \left\{ \int_{\mathbb{R}^3 \setminus B_1(s) \cup B_2(s)} d^3\mathbf{x} F(\mathbf{x}) - \mathrm{div}(s) \right\}$$

A detailed account on Hadamard regularization and associated distributional forms is given in [LB & Faye 2000]

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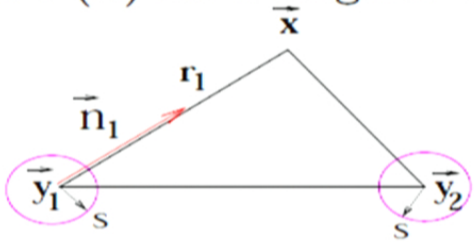

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Self-field regularization ambiguities at 3PN order

Hadamard's regularization is non-distributive, in the sense that the regularization of a product differs from the product of regularizations

$$(F\,G)_1 \neq (F)_1\,(G)_1$$

As a result the computation cannot be complete. A few regularization ambiguities appear at the 3PN order, but Hadamard's regularization works well up to 2PN order and can be implemented to compute most of the terms at 3PN order

All terms but for a few regularization ambiguities have been computed with Hadamard's regularization

- γ, λ in the equations of motion [Jaranowski & Schäfer 1999; LB & Faye 2000, 2001]
- ξ, κ and ζ in the radiation field [LB, Iyer & Joguet 2002; LB & Iyer 2004]

The regularization ambiguities do not have a direct physical meaning. They are due to a mathematical deficiency of Hadamard's regularization

Computation of the regularization ambiguity parameters

- Following dimensional regularization one solves Einstein's field equations in

$$d = 3 + \varepsilon \quad \text{spatial dimensions}$$

with source terms involving Dirac's d-dimensional function $\delta^{(d)}(\mathbf{x} - \mathbf{y}_A)$

- One does not compute the result in d dimensions from scratch, but only the *difference* between dimensional regularization and Hadamard's one, say

$$\mathcal{D}F(\mathbf{y}_1) \equiv \underbrace{F^{(d)}(\mathbf{y}_1)}_{\text{value in } d \text{ dimensions}} - \underbrace{(F)_1}_{\text{Hadamard's partie finie}}$$

- That difference will typically be of the form (when $\varepsilon \to 0$)

$$\mathcal{D}F(\mathbf{y}_1) = \frac{a_{-1}}{\varepsilon} + a_0 + \mathcal{O}(\varepsilon)$$

- The ambiguities in Hadamard's regularization are precisely associated with the occurence of poles $\propto 1/\varepsilon$ at 3PN order in dimensional regularization

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Dimensional regularization of the equations of motion

- The 3PN acceleration in Hadamard's regularization is

$$\mathbf{a}_1[\lambda] = \mathbf{a}_1^{\text{pure Hadamard Schwartz}} + \underbrace{\Delta\mathbf{a}_1[\gamma,\lambda]}_{\text{ambiguity part}}$$

- while in dimensional regularization ($d = 3 + \varepsilon$) it reads

$$\mathbf{a}_1^{(d)} = \mathbf{a}_1^{\text{pure Hadamard Schwartz}} + \mathcal{D}\mathbf{a}_1$$

- We require physical equivalence between dimensional and Hadamard regularizations. We look for a shift $\mathbf{y}_A \rightarrow \mathbf{y}_A + \boldsymbol{\eta}_A$ such that

$$\mathbf{a}_1[\gamma,\lambda] = \lim_{\varepsilon\rightarrow 0}\left[\mathbf{a}_1^{(d)} + \delta_{\boldsymbol{\eta}}\,\mathbf{a}_1\right]$$

- This requirement determines the particle's worldlines shift $\boldsymbol{\eta}_A = \mathcal{O}(1/\varepsilon)$ and fixes uniquely

$$\boxed{\gamma = \frac{41}{24}, \qquad \lambda = -\frac{1987}{3080}}$$

Dimensional regularization of the radiation field

- The 3PN quadrupole moment in Hadamard's regularization reads

$$I_{ij}[\xi, \kappa, \zeta] = I_{ij}^{\text{pure Hadamard Schwartz}} + \underbrace{\Delta I_{ij}[\xi, \kappa, \zeta]}_{\text{ambiguity part}}$$

- while in dimensional regularization it reads

$$I_{ij}^{(d)} = I_{ij}^{\text{pure Hadamard Schwartz}} + \mathcal{D} I_{ij}$$

- Physical equivalence between the dimensional and Hadamard results means

$$I_{ij}[\xi, \kappa, \zeta] = \lim_{\varepsilon \to 0} \left[I_{ij}^{(d)} + \delta_{\boldsymbol{\eta}} I_{ij} \right]$$

with *the same shift* η_A as for the equations of motion

- This uniquely determines

$$\boxed{\xi = -\frac{9871}{9240}, \quad \kappa = 0, \quad \zeta = -\frac{7}{33}}$$

Poincaré invariance of the 3PN EOM and radiation field

1. The ambiguity γ can be computed by requiring that the EOM in harmonic coordinates be explicitly invariant under a global Poincaré transformation developed to 3PN order

2. The ambiguity ζ in the radiation field can be obtained by comparing the result of two-body calculations in the limit $m_2 \rightarrow 0$ with the case of a boosted Schwarzschild black hole (BBH). Computing the 3PN quadrupole of the BBH we recover exactly the result of dimensional regularization which is an in direct test of the global Poincaré invariance of the 3PN wave generation

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3.5PN equations of motion of compact binary systems

The acceleration of one of the particle is expressed in fully reduced form as a functional of the positions and velocities of the bodies

$$\frac{\mathrm{d}v_1^i}{\mathrm{d}t} = -\frac{Gm_2}{r_{12}^2}n_{12}^i$$
$$+ \frac{1}{c^2}\left\{\left[\frac{5G^2m_1m_2}{r_{12}^3} + \frac{4G^2m_2^2}{r_{12}^3} + \frac{Gm_2}{r_{12}^2}\left(\frac{3}{2}(n_{12}v_2)^2 - v_1^2 + 4(v_1v_2) - 2v_2^2\right)\right]n_{12}^i\right\}$$
$$+ \underbrace{\frac{1}{c^4}[\cdots]}_{\text{2PN term}} + \underbrace{\frac{1}{c^5}[\cdots]}_{\text{2.5PN radiation reaction}} + \underbrace{\frac{1}{c^6}[\cdots]}_{\text{3PN term}} + \underbrace{\frac{1}{c^7}[\cdots]}_{\text{3.5PN radiation reaction}} + \mathcal{O}\left(\frac{1}{c^8}\right)$$

At 3PN order the EOM in harmonic coordinates contain logarithms with associated UV regularization scales but these are pure gauge

The 3PN equations have also been recovered by a surface-integral approach [Itoh & Futamase 2003] and by the effective field theory [Foffa & Sturani 2011]

3.5PN orbital phase of inspiralling compact binaries

The total gravitational-wave energy flux for circular orbits is obtained as

$$F(x) = -\frac{32c^5}{5G}\nu^2 x^5 \Bigg\{ 1 + \left(-\frac{1247}{336} - \frac{35}{12}\nu\right) x + \overbrace{4\pi x^{3/2}}^{\text{1.5PN tail}} + \left(-\frac{44711}{9072} + \frac{9271}{504}\nu + \frac{65}{18}\nu^2\right) x^2 + \underbrace{[\cdots]\, x^{5/2}}_{\text{2.5PN tail}} + \underbrace{[\cdots]\, x^3}_{\substack{\text{3PN} \\ \text{includes a tail-of-tail}}} + \underbrace{[\cdots]\, x^{7/2}}_{\text{3.5PN tail}} + \mathcal{O}\left(x^4\right) \Bigg\}$$

where the gauge-invariant PN parameter $x = (Gm\omega/c^3)^{3/2}$ and the symmetric mass ratio $\nu = m_1 m_2/m^2$

The 3.5PN expansion of the orbital phase for quasi-circular orbits is deduced from an energy balance argument

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\mathcal{F}$$

The energy, angular momentum and linear momentum fluxes for eccentric orbits and associated balance equations are known at 3PN order [Arun et al 2008, 2009]

3.5PN dominant gravitational wave mode

The modes of the GW amplitude are defined by the spherical harmonics expansion

$$h_+ - \mathrm{i} h_\times = \sum_{\ell=2}^{+\infty} \sum_{m=-\ell}^{\ell} h^{\ell m}\, Y^{\ell m}_{-2}(\Theta, \Phi)$$

The dominant quadrupole 22 mode is known at 3.5PN order

$$\begin{aligned} h^{22} &= \frac{2G\, m\, \nu\, x}{R\, c^2} \sqrt{\frac{16\pi}{5}}\, e^{-2\mathrm{i}\psi} \Bigg\{ 1 + x\left(-\frac{107}{42} + \frac{55\nu}{42}\right) + 2\pi x^{3/2} \\ &\quad + x^2\left(-\frac{2173}{1512} - \frac{1069\nu}{216} + \frac{2047\nu^2}{1512}\right) \\ &\quad + [\cdots]\, x^{5/2} + [\cdots]\, x^3 + [\cdots]\, x^{7/2} + \mathcal{O}\left(x^4\right) \Bigg\} \end{aligned}$$

Tail contributions are factorized out in the phase variable

$$\psi = \phi - \frac{2GM\omega}{c^3} \ln\left(\frac{\omega}{\omega_0}\right)$$

Spin-orbit effects in the orbital frequency evolution

$$\frac{\dot{\omega}}{\omega^2} = \frac{96}{5}\nu\, x^{5/2}\Big\{1 + x\,[\cdots] + x^{3/2}\,[\cdots] + x^2\,[\cdots] + x^{5/2}\,[\cdots] + x^3\,[\cdots]$$

spin-orbit (SO) and spin-spin (SS) terms [Kidder, Will & Wiseman 1993, Kidder 1995]

$$+ \overbrace{x^{3/2}\left[-\frac{47}{3}s_z - \frac{25}{4}\Delta\sigma_z\right] + x^2[\mathrm{SS}]}$$

1PN correction to the SO term [LB, Buonanno & Faye 2006]

$$+ \overbrace{x^{5/2}\left[\left(-\frac{5861}{144} + \frac{1001}{12}\nu\right)s_z + \left(-\frac{809}{84} + \frac{281}{8}\nu\right)\Delta\sigma_z\right]}$$

tail correction to the SO term [LB, Buonanno & Faye 2011]

1PN correction to the SS

$$+ \overbrace{x^3\left[-\frac{188\pi}{3}s_z - \frac{151\pi}{6}\Delta\sigma_z\right]} + \overbrace{x^3[\mathrm{SS}]} + \mathcal{O}(x^{7/2})$$

Next-to-leading SS terms in both the EOM and energy flux have been computed in [Steinhoff, Hergt & Schäfer 2008] and within the effective field theory [Porto & Rothstein 2006, 2008]

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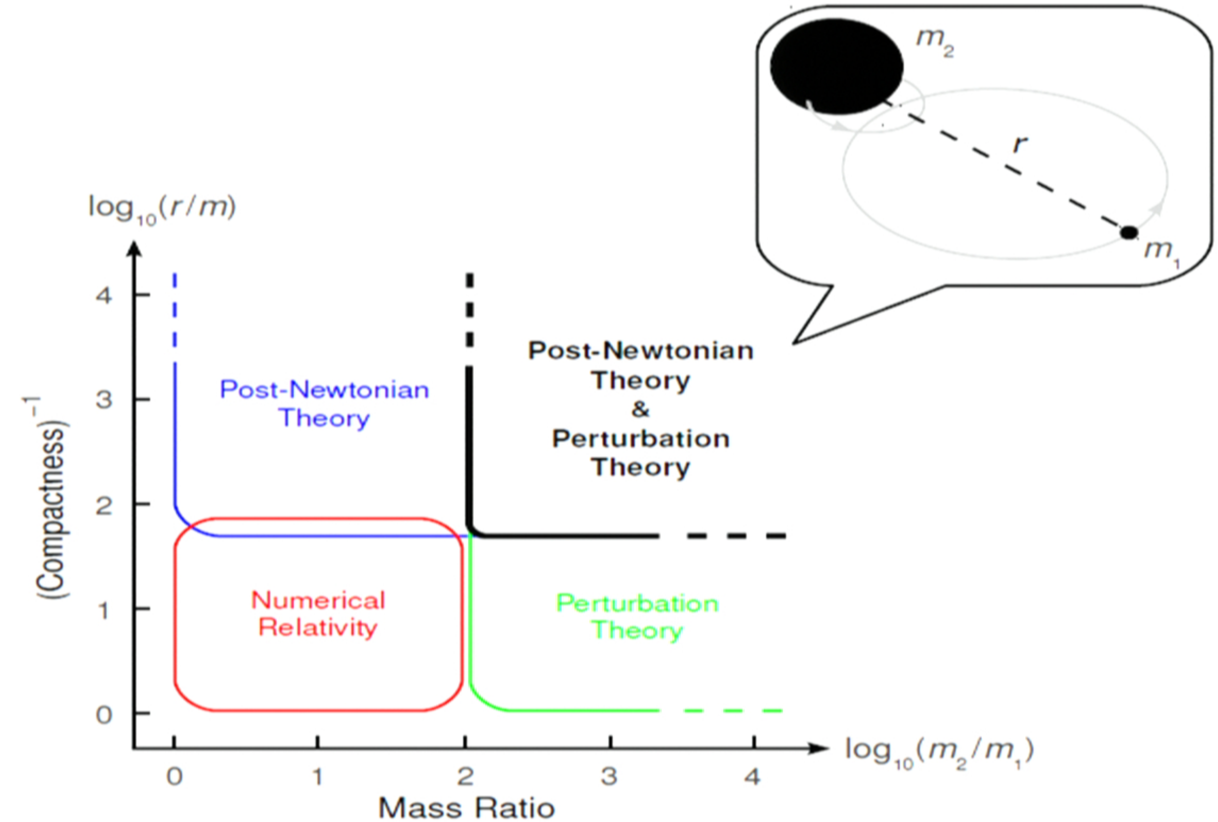
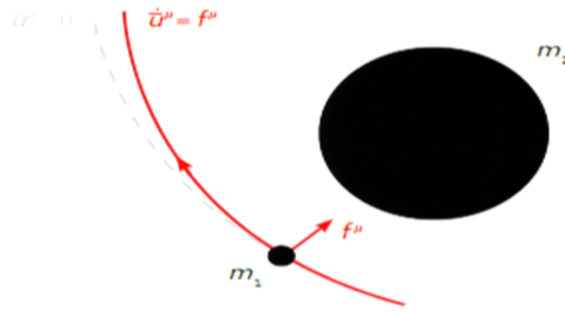
POST-NEWTONIAN VERSUS SELF-FORCE PREDICTIONS

Common regime of validity of SF and PN



General problem of the self-force

- A particle is moving on a background space-time
- Its own stress-energy tensor modifies the background gravitational field
- Because of the "back-reaction" the motion of the particle deviates from a background geodesic hence the appearance of a self force



The self acceleration of the particle is proportional to its mass

$$\frac{D\bar{u}^\mu}{d\tau} = f^\mu = \mathcal{O}\left(\frac{m_1}{m_2}\right)$$

The gravitational self force includes both dissipative (radiation reaction) and conservative effects.

Why and how comparing PN and SF predictions?

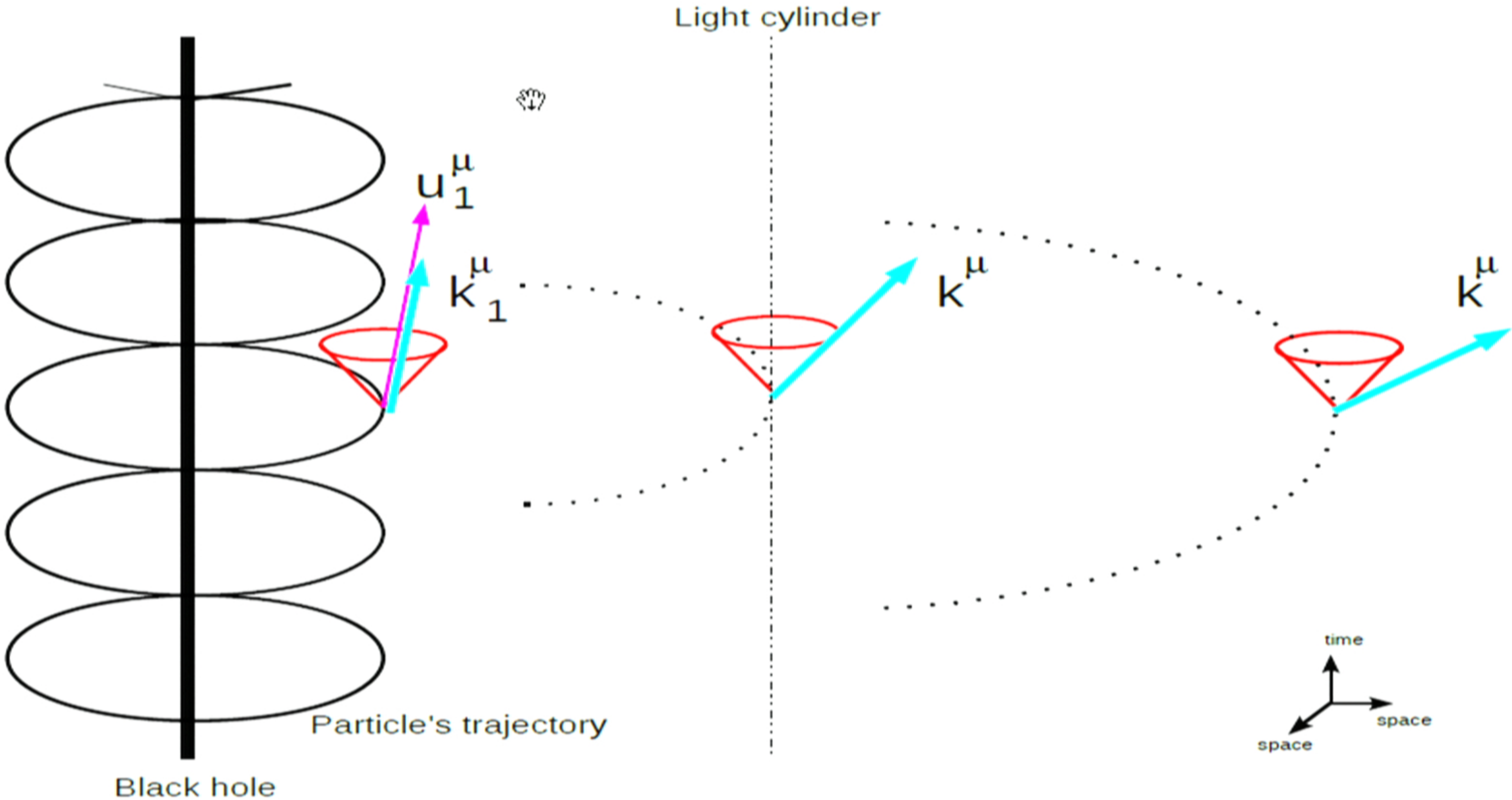
Both the PN and SF approaches use a self-field regularization for point particles followed by a renormalization. However, the prescription are very different

1. SF theory is based on a prescription for the Green function G_R that is at once regular and causal
2. PN theory uses dimensional regularization and it was shown that subtle issues appear at the 3PN order due to the appearance of poles $\propto (d-3)^{-1}$

How can we make a meaningful comparison?

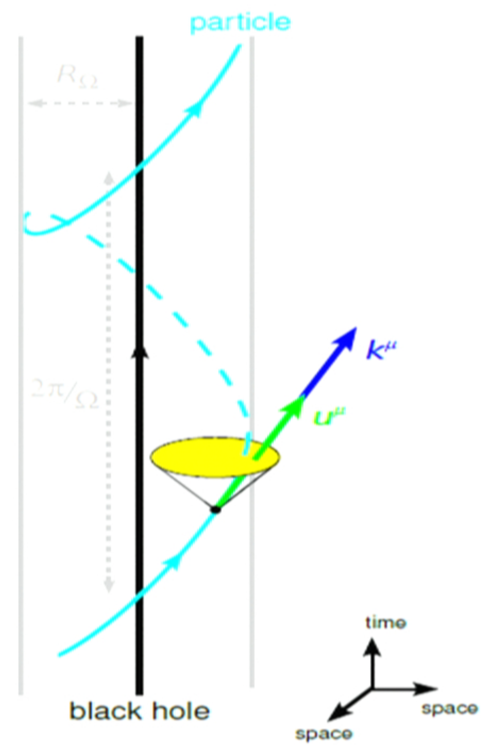
1. To restrict attention to the conservative part of the dynamics
2. To find a gauge-invariant observable computable in both formalisms

Circular orbits admit a helical Killing vector



Choice of a gauge-invariant observable [Detweiler 2008]

1. For exactly circular orbits the geometry admits a helical Killing vector with
$$k^\mu \partial_\mu = \partial_t + \Omega\, \partial_\varphi \quad \text{(asymptotically)}$$
2. The four-velocity of the particle is necessarily tangent to the Killing vector hence
$$u_1^\mu = u_1^T\, k_1^\mu$$
3. The relation $u_1^T(\Omega)$ is well-defined in both PN and SF approaches and is gauge-invariant



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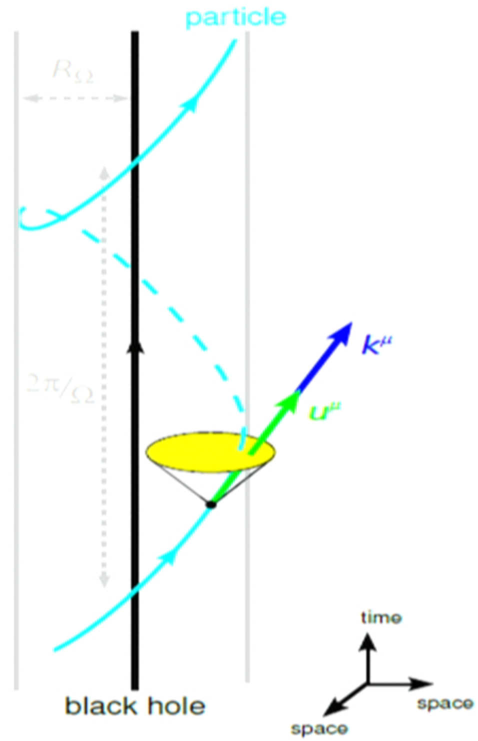
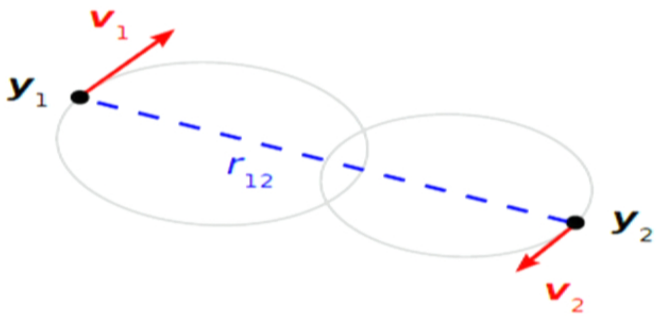


Post-Newtonian calculation

In a coordinate system such that $k^\mu \partial_\mu = \partial_t + \Omega\, \partial_\varphi$ everywhere this invariant quantity reduces to the zero component of the particle's four-velocity,

$$u_1^t = \Big(- \underbrace{\mathrm{Reg}_1 \left[g_{\mu\nu} \right]}_{\text{regularized metric}} \frac{v_1^\mu v_1^\nu}{c^2} \Big)^{-1/2}$$



One crucially needs a self-field regularization

- Hadamard regularization will yield an ambiguity at 3PN order
- Dimensional regularization will be free of any ambiguity at 3PN order

Result at 3PN order [LB, Detweiler, Le Tiec & Whiting 2010a]

- The 3PN result is expressed in terms of $x = \left(\frac{GM\Omega}{c^3}\right)^{3/2}$ as

$$u^T = 1 + A_0\,x + A_1\,x^2 + A_2\,x^3 + \underbrace{A_3\,x^4}_{\text{3PN}} + o(x^4)$$

- The coefficients depend on mass ratios $\nu = m_1 m_2/M^2$, $\Delta = (m_1 - m_2)/M$

$$\begin{aligned} A_3 &= \frac{2835}{256} + \frac{2835}{256}\Delta - \left[\frac{2183}{48} - \frac{41}{64}\pi^2\right]\nu - \left[\frac{12199}{384} - \frac{41}{64}\pi^2\right]\Delta\,\nu \\ &+ \quad \text{other terms} \end{aligned}$$

- We find that the poles $\propto \varepsilon^{-1}$ cancel out

ADM mass and angular momentum [Le Tiec, LB & Whiting 2011]

1. The ADM mass M and angular momentum J of the circular-orbit binary are computed through 3PN order augmented by 4PN and 5PN logarithmic contributions
2. We explicitly check through 3PN + 4PN/5PN$_{\log}$ that they obey the relation
$$\frac{\partial M}{\partial \Omega} = \Omega \frac{\partial J}{\partial \Omega}$$
used in computations of the binary evolution based on a sequence of quasi-equilibrium configurations, e.g. [Gourgoulhon *et al* 2002]
3. However we find that they obey also related to Detweiler's redshift observables $z_1 = 1/u_1^T$ and $z_2 = 1/u_2^T$ by
$$\begin{aligned} \frac{\partial M}{\partial m_1} - \Omega \frac{\partial J}{\partial m_1} &= z_1 \\ \frac{\partial M}{\partial m_2} - \Omega \frac{\partial J}{\partial m_2} &= z_2 \end{aligned}$$
These relations tell how the ADM quantities change when the individual masses m_1 and m_2 of the particles vary (keeping the frequency Ω fixed)

First law of binary black hole mechanics

These relations can be summarized in the first law of binary black hole or binary point-particle mechanics

$$\boxed{\delta M - \Omega\,\delta J = z_1\,\delta m_1 + z_2\,\delta m_2}$$

Generalized law of black hole mechanics [Friedman, Uryū & Shibata 2002]

$$\delta Q = \int_\Sigma \left[\bar{\mu}\,\Delta(\mathrm{d}M_\mathrm{b}) + \bar{T}\,\Delta(\mathrm{d}S) + v^\alpha \Delta(\mathrm{d}C_\alpha)\right] + \sum_n \frac{\kappa_n}{8\pi}\,\delta A_n$$

When one assumes the existence of the HKV,

$$\delta Q = \delta M - \Omega\,\delta J$$

An interesting consequence of the first law is the remarkably simple relation

$$\boxed{M - 2\Omega J = m_1 z_1 + m_2 z_2}$$

Higher PN terms in the binary's energy [Le Tiec, LB & Whiting 2011]

The first law can be used to compute new PN coefficients in the binary's binding energy $E = M - m_1 - m_2$

$$\begin{aligned} E &= -\frac{1}{2} m \nu x \left\{ 1 + \left(-\frac{3}{4} - \frac{\nu}{12} \right) x + \left(-\frac{27}{8} + \frac{19}{8}\nu - \frac{\nu^2}{24} \right) x^2 \right. \\ &+ \left(-\frac{675}{64} + \left[\frac{34445}{576} - \frac{205}{96}\pi^2 \right] \nu - \frac{155}{96}\nu^2 - \frac{35}{5184}\nu^3 \right) x^3 \\ &+ \left(-\frac{3969}{128} + \nu\, e_4(\nu) + \frac{448}{15}\nu \ln x \right) x^4 \\ &+ \left(-\frac{45927}{512} + \nu\, e_5(\nu) + \left[-\frac{4988}{35} - 6565\nu \right] \nu \ln x \right) x^5 \\ &\left. + \left(-\frac{264627}{1024} + \nu\, e_6(\nu) + \nu\, e_6^{\ln}(\nu) \ln x \right) x^6 \right\} \end{aligned}$$

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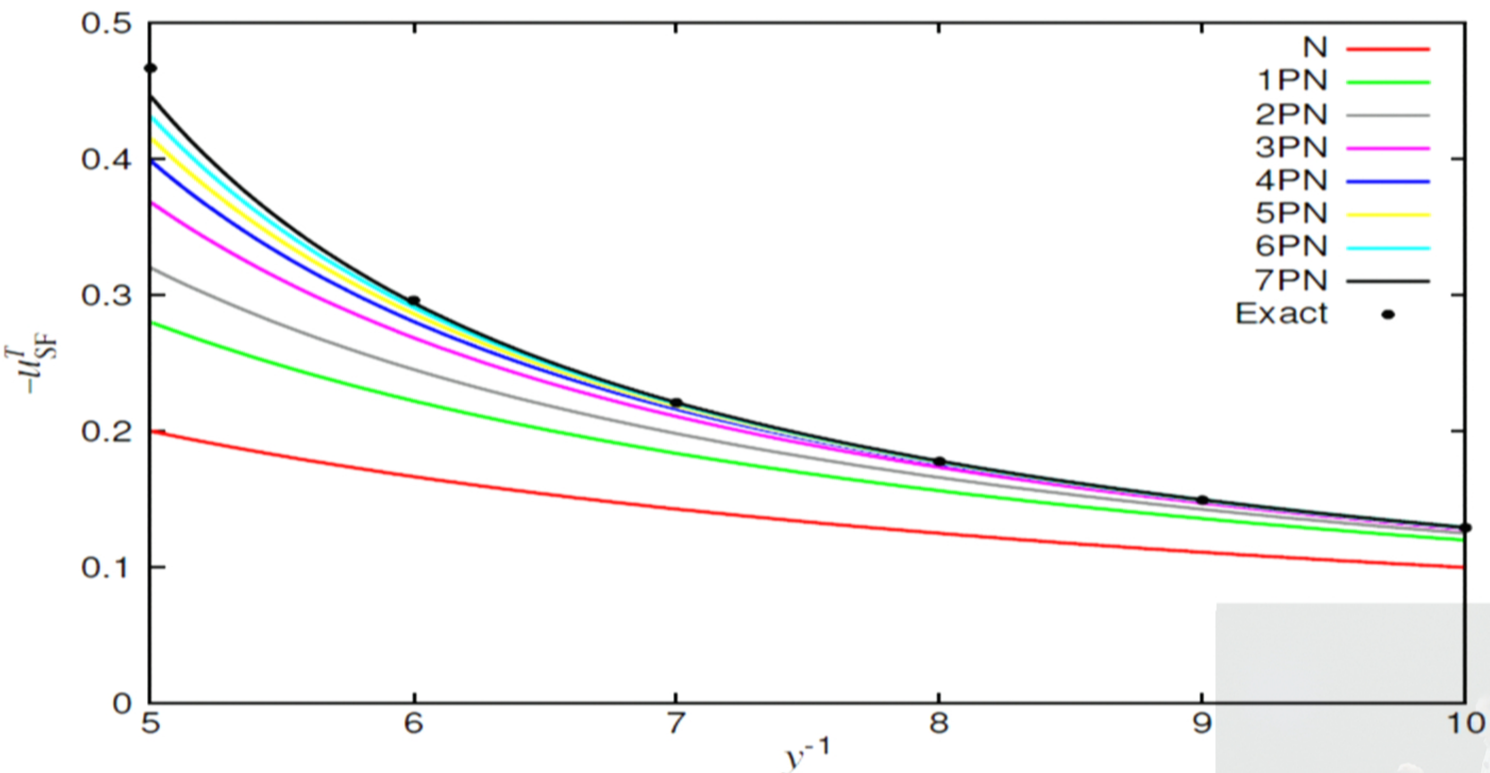
& + \left(-\frac{45927}{512} - 55.13\,\nu + \left[-\frac{4988}{35} - 6565\nu\right]\nu \ln x\right) x^5 \\
& + \left(-\frac{264627}{1024} + 588.\,\nu - 1144.\,\nu \ln x\right) x^6 + \mathcal{O}(\nu^2) \Bigg\}
\end{aligned}
$$

Comparison between PN and SF predictions



High-order PN fit to the numerical self-force

- Post-Newtonian coefficients are fitted up to 7PN order

PN coefficient	SF value
a_4	$-114.34747(5)$
a_5	$-245.53(1)$
a_6	$-695(2)$
b_6	$+339.3(5)$
a_7	$-5837(16)$

- The 3PN prediction agrees with the SF value with 7 significant digits

3PN value	SF fit
$a_3 = -\frac{121}{3} + \frac{41}{32}\pi^2 = -27.6879026\cdots$	-27.68790