Title: Recent Developments in Generalized ADM Formalism for the Dynamics of Compact Binaries with Spin

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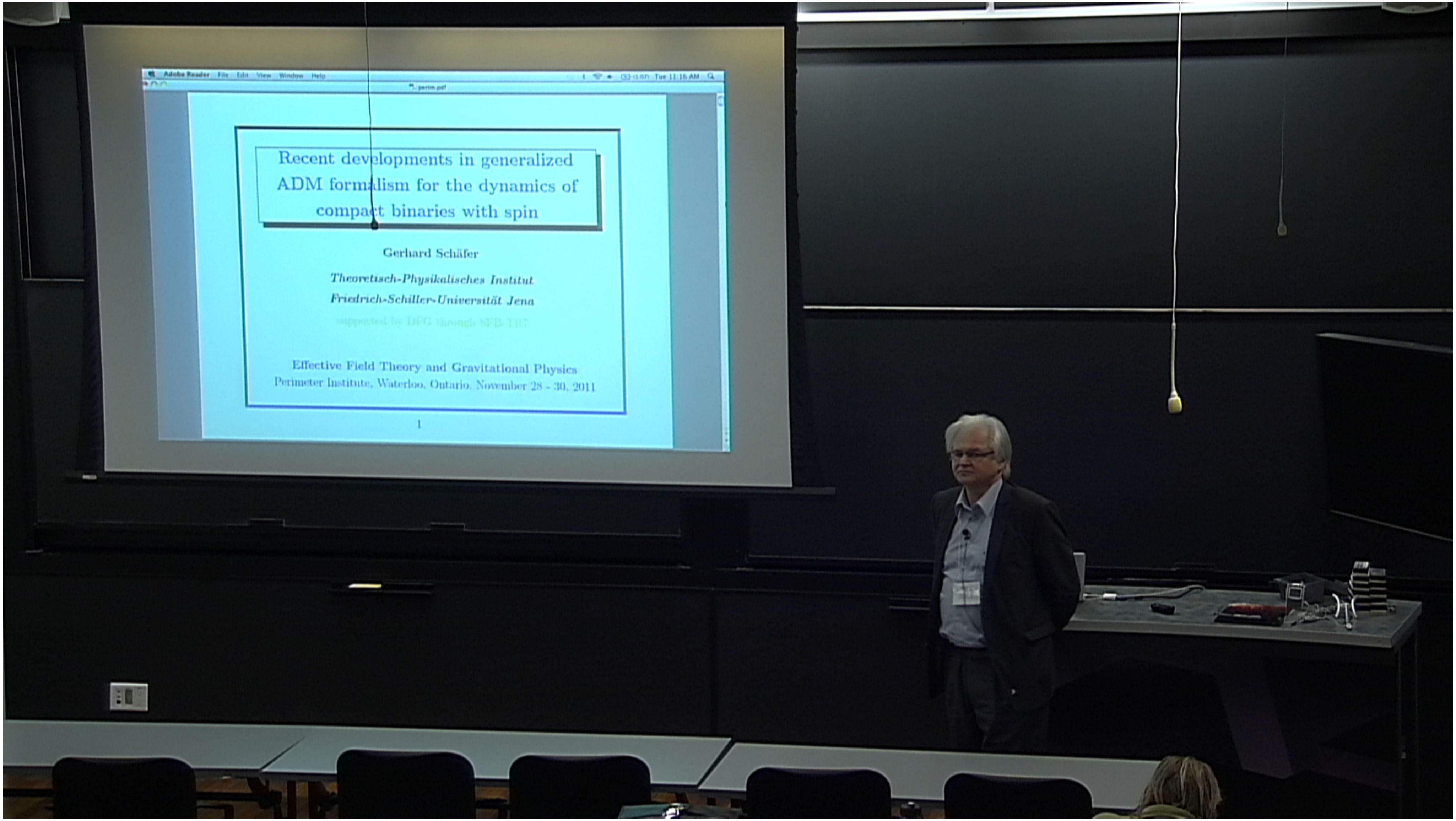
Abstract: Based on tetrad-generalized canonical formalism by Arnowitt, Deser, and Misner most recent achievements in analytic calculations of higher order post-Newtonian Hamiltonians for spinning binary black holes and neutron stars are presented. The results of the generalized ADM formalism are put into mathematical relationship with those obtained within the Effective Field Theory approach.


Recent developments in generalized ADM formalism for the dynamics of compact binaries with spin
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Effective Field Theory and Gravitational Physics
Perimeter Institute, Waterloo, Ontario, November 28 - 30, 2011
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Outline

- Spin in Minkowski Space (introductory)
- Higher Order PN Hamiltonians (summary)
- Spin and Gravity (general)
- Hamiltonian Formulation for Spinning Objects
- Comparison with Results from EFT (recent preprint)
- Dirac Delta Functions and Black Holes

sometimes: $c = 1\,,\ G = 1$

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Results based on the Hamiltonian approach

point-mass dynamics:

3PN: Damour/Jaranowski/GS (2001)

next-to-leading order rad. reaction (3.5PN): Jaranowski/GS (1997)

1PM: Ledvinka/GS/Bicak (2008)

linear-in-spin coupling: Steinhoff/GS (2009)

spin-orbit coupling (spin counted 1/c):

next-to-leading order (2.5PN): Damor/Jaranowski/GS (2008)

next-to-next-to-leading order (3.5PN): Hartung/Steinhoff (2011)

leading order radiation reaction (4PN): Steinhoff/Wang (2010)

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Results based on the Hamiltonian approach

point-mass dynamics:

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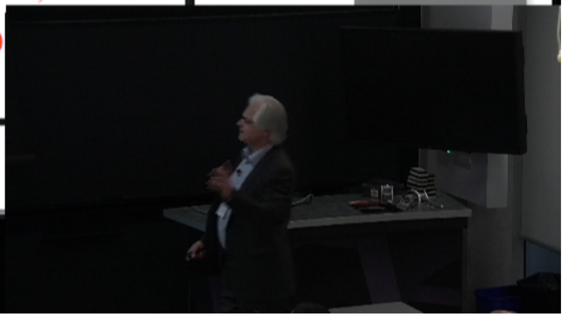

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spin(1)-spin(2) coupling (spin counted 1/c):

next-to-leading order (3PN): Steinhoff/Hergt/GS (2008)

next-to-next-to-leading order (4PN): Hartung/Steinhoff (2011)

lead. ord. rad. reaction (4.5PN): Wang/Steinhoff/Zeng/GS (2011)

spin(1)-spin(1) coupling (spin counted 1/c):

next-to-leading order for black holes and neutron stars (3PN):
Steinhoff/Hergt/GS (2008, 2010)

higher-order-in-spin coupling:

up to 3rd order in spin: $S(a)^n S(b)^{3-n} p(c)$: Hergt/GS (2008)

up to 4th order in spin: $S(a)^n S(b)^{4-n}$: Hergt/GS (2008)

spinning test-particle in Kerr: Barausse/Racine/Buonanno (2009)

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el. charged particle with spin $(m,\, e,\, \mathbf{S})$

$$\frac{dp_\mu}{d\tau} = eF_{\mu\nu}u^\nu$$

$$\eta_{\mu\nu}\frac{dw^\nu}{d\tau} = g\frac{e}{2m}F_{\mu\nu}w^\nu + (g-2)\frac{e}{2m^2}F_{\lambda\nu}u^\lambda w^\nu p_\mu$$

$$u^\mu = \tfrac{dx^\mu}{d\tau}, \quad -u^\mu u^\mu \eta_{\mu\nu} = 1, \quad p_\mu = mu_\mu, \quad -p_\mu p_\nu \eta^{\mu\nu} = m^2$$

$$w^\mu w^\nu \eta_{\mu\nu} = m^2\mathbf{S}^2, \quad p_\mu w^\mu = 0$$

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$$\frac{dp_\mu}{d\tau} = eF_{\mu\nu}u^\nu$$

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$$w^\mu w^\nu \eta_{\mu\nu} = m^2\mathbf{S}^2, \quad p_\mu w^\mu = 0$$

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canonical variables

total angular momentum: $\mathbf{J} = \hat{\mathbf{X}} \times \mathbf{P} + \hat{\mathbf{S}}$

Lorentz boost: $\mathbf{K} = -\mathrm{t}\mathbf{P} + \mathrm{H}\hat{\mathbf{X}} - \frac{1}{H+m}\hat{\mathbf{S}} \times \mathbf{P}$

Hamiltonian: $H = \sqrt{m^2 + \mathbf{P}^2}$

center-of-energy: $\bar{\mathbf{X}} = \hat{\mathbf{X}} - \frac{1}{(H+m)H}\,\hat{\mathbf{S}} \times \mathbf{P}$

$$\mathbf{K} = -\mathrm{t}\mathbf{P} + H\bar{\mathbf{X}}$$

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$$w^0 = \mathbf{P} \cdot \mathbf{J} = \mathbf{P} \cdot \hat{\mathbf{S}}$$

$$\mathbf{w} = H\mathbf{J} + \mathbf{P} \times \mathbf{K} = H\hat{\mathbf{S}} - \tfrac{1}{H+m}\mathbf{P} \times (\hat{\mathbf{S}} \times \mathbf{P})$$

$$w^\mu = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} P_\nu J_{\alpha\beta} \qquad J_{kl} = \epsilon_{klm} J_m \qquad J_{0i} = K_i$$

$$w^\mu w_\mu \equiv -(w^0)^2 + \mathbf{w}^2 = m^2 \hat{\mathbf{S}}^2$$

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Poincaré algebra

$$\{P_i, \mathrm{H}\} = \{J_i, H\} = 0$$

$$\{J_i, P_j\} = \varepsilon_{ijk}\, P_k\,, \;\; \{J_i, J_j\} = \varepsilon_{ijk}\, J_k$$

$$\{J_i, G_j\} = \varepsilon_{ijk}\, G_k$$

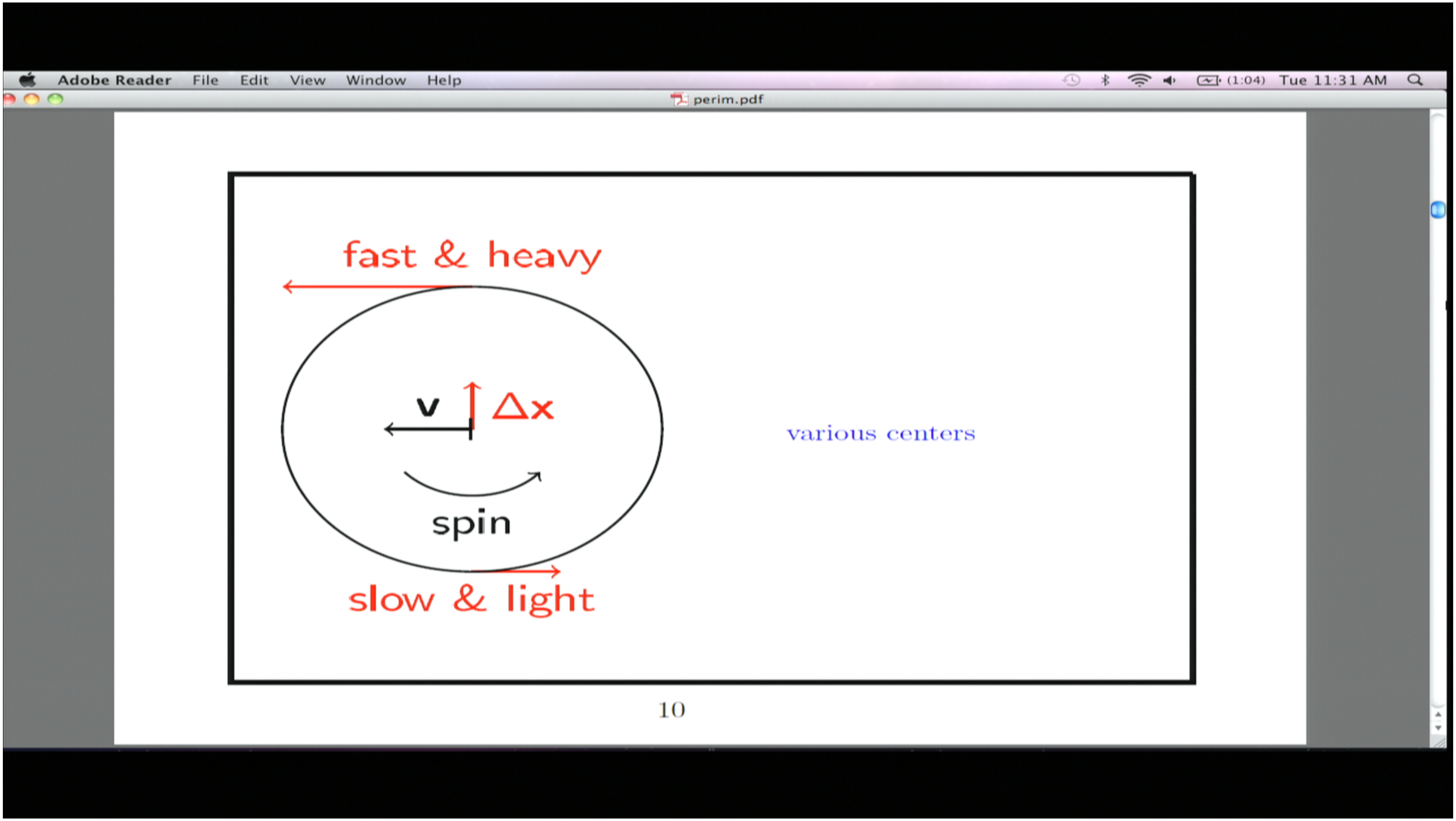
$$\{\mathbf{G}, H\} = \mathbf{P}$$

$$\{G_i, P_j\} = \tfrac{1}{c^2}\, H\, \delta_{ij}$$

$$\{G_i, G_j\} = -\tfrac{1}{c^2}\, \varepsilon_{ijk}\, J_k$$

$$\mathbf{K} = -t\,\mathbf{P} + \mathbf{G}$$

$$d\mathbf{K}/dt = \partial\mathbf{K}/\partial t + \{\mathbf{K}, H\} = -\mathbf{P} + \{\mathbf{G}, H\} = 0$$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fast & heavy
v
Δx
spin
slow & light
various centers
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center-of-spin: $\hat{\mathbf{X}}$; $\{\hat{X}^i, \hat{X}^j\} = 0$ (Newton-Wigner coordinates)

center-of-energy: $\bar{\mathbf{X}} = \hat{\mathbf{X}} - \frac{1}{(H+m)H}\, \hat{\mathbf{S}} \times \mathbf{P}$

center-of-inertia: $\mathbf{X} = \hat{\mathbf{X}} + \frac{1}{(H+m)m}\, \hat{\mathbf{S}} \times \mathbf{P}$

center-of-inertia: $S^{\mu\nu} P_\nu = 0$

center-of-energy: $\bar{S}^{\mu\nu} n_\nu = 0, \qquad n_\mu = (-1, 0, 0, 0)$

center-of-spin: $m\hat{S}^{\mu\nu} n_\nu + \hat{S}^{\mu\nu} P_\nu = 0$

minimal radius of particle with spin: $\frac{\hat{S}}{mc}$

Compton wavelength: $2\frac{\hbar/2}{mc}$

radius of Kerr ring singularity (Boyer-Lindquist coord's): $\frac{\hat{S}}{mc}$

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many-particle systems with interaction

$$\mathbf{P} = \sum_a \mathbf{p}_a$$

$$\mathbf{J} = \sum_a (\mathbf{r}_a \times \mathbf{p}_a + \mathbf{s}_a)$$

$$M = \sqrt{H^2 - \mathbf{P}^2}, \qquad H = \sqrt{M^2 + \mathbf{P}^2}$$

$$\hat{\mathbf{X}} = \frac{\mathbf{G}}{H} + \frac{1}{M(H+M)}(\mathbf{J} - \frac{\mathbf{G}}{H} \times \mathbf{P}) \times \mathbf{P}$$

$$\{\hat{X}^i, \hat{X}^j\} = \{P^i, P^j\} = 0, \quad \{\hat{X}^i, P^j\} = \delta^{ij}$$

$$\{M, \hat{X}^j\} = \{M, P^j\} = \{M, H\} = 0$$

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free particles with spin:

$$H = \sum_a h_a, \qquad h_a = \sqrt{m_a^2 + \mathbf{p}_a^2}$$

$$\mathbf{G} = \sum_a \left(h_a \mathbf{r}_a - \frac{1}{h_a + m_a} \mathbf{s}_a \times \mathbf{p}_a\right)$$

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Higher Order PN Hamiltonians (summary)

spin not counted 1/c

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binary black holes to 3.5 PN order

$$
\begin{aligned}
H(t) &= m_1c^2 + m_2c^2 + H_N + H_{1PN} \\
&+ H_{2PN} + H_{3PN} + \ldots \\
&+ \frac{1}{c^5} H_{2.5PN}(t) + \frac{1}{c^7} H_{3.5PN}(t) + \ldots
\end{aligned}
$$

$\hat{H} = (H - Mc^2)/\mu, \qquad \mu = m_1 m_2/M, \qquad M = m_1 + m_2$

$\nu = \mu/M, \qquad 0 \leq \nu \leq 1/4$

test particles: $\nu = 0$, equal masses: $\nu = 1/4$

CMF: $\mathbf{p}_1 + \mathbf{p}_2 = 0, \qquad \mathbf{p} = \mathbf{p}_1/\mu, \qquad r = r_{12} = |\mathbf{x}_1 - \mathbf{x}_2|,$

$p_r = (\mathbf{n} \cdot \mathbf{p}), \qquad \mathbf{q} = (\mathbf{x}_1 - \mathbf{x}_2)/GM, \qquad \mathbf{n} = \mathbf{n}_{12} = \mathbf{q}/|\mathbf{q}|$

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$$\hat{H}_N = \frac{p^2}{2} - \frac{1}{q}$$

$$c^2 \hat{H}_{1PN} = \frac{1}{8}(3\nu - 1)p^4 - \frac{1}{2}[(3+\nu)p^2 + \nu p_r^2]\frac{1}{q} + \frac{1}{2q^2}$$

$$\begin{aligned}
c^4 \hat{H}_{2PN} &= \frac{1}{16}(1 - 5\nu + 5\nu^2)p^6 \\
&+ \frac{1}{8}[(5 - 20\nu - 3\nu^2)p^4 - 2\nu^2 p_r^2 p^2 - 3\nu^2 p_r^4]\frac{1}{q} \\
&+ \frac{1}{2}[(5 + 8\nu)p^2 + 3\nu p_r^2]\frac{1}{q^2} - \frac{1}{4}(1 + 3\nu)\frac{1}{q^3}
\end{aligned}$$

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$$
\begin{aligned}
c^6 \hat{H}_{3PN} &= \frac{1}{128}(-5 + 35\nu - 70\nu^2 + 35\nu^3)p^8 \\
&+ \frac{1}{16}[(-7 + 42\nu - 53\nu^2 - 5\nu^3)p^6 + (2 - 3\nu)\nu^2 p_r^2 p^4 \\
&+ 3(1-\nu)\nu^2 p_r^4 p^2 - 5\nu^3 p_r^6]\frac{1}{q} \\
&+ [\frac{1}{16}(-27 + 136\nu + 109\nu^2)p^4 + \frac{1}{16}(17 + 30\nu)\nu p_r^2 p^2 \\
&+ \frac{1}{12}(5 + 43\nu)\nu p_r^4]\frac{1}{q^2} \\
&+ [\left(-\frac{25}{8} + \left(\frac{1}{64}\pi^2 - \frac{335}{48}\right)\nu - \frac{23}{8}\nu^2\right)p^2 \\
&+ \left(-\frac{85}{16} - \frac{3}{64}\pi^2 - \frac{7}{4}\nu\right)\nu p_r^2]\frac{1}{q^3} + [\frac{1}{8} + \left(\frac{109}{12} - \frac{21}{32}\pi^2\right)\nu]\frac{1}{q^4}
\end{aligned}
$$

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spin-gravity interaction ($\mathbf{S} \equiv \hat{\mathbf{S}}$)

leading order spin orbit

$$H_{\rm SO}^{1PN} = \frac{G}{c^2} \sum_a \sum_{b \neq a} \frac{1}{r_{ab}^2} (\mathbf{S}_a \times \mathbf{n}_{ab}) \cdot \left[\frac{3m_b}{2m_a} \mathbf{p}_a - 2\mathbf{p}_b \right]$$

leading order spin(1)-spin(2)

$$H_{\rm S_1S_2}^{1PN} = \frac{G}{c^2} \sum_a \sum_{b \neq a} \frac{1}{2r_{ab}^3} \left[3(\mathbf{S}_a \cdot \mathbf{n}_{ab})(\mathbf{S}_b \cdot \mathbf{n}_{ab}) - (\mathbf{S}_a \cdot \mathbf{S}_b) \right]$$

leading order spin(1)-spin(1)

$$H_{\rm S_1S_1}^{1PN} = \frac{G}{c^2} \frac{1}{2r_{12}^3} \left[3(\mathbf{S}_1 \cdot \mathbf{n}_{12})(\mathbf{S}_1 \cdot \mathbf{n}_{12}) - (\mathbf{S}_1 \cdot \mathbf{S}_1) \right]$$

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$$
\begin{aligned}
H_{SO}^{2PN} &= \frac{G}{c^4 r^2}\Bigg[-((\mathbf{p}_1 \times \mathbf{S}_1)\cdot \mathbf{n}_{12})\Bigg[\frac{5 m_2 \mathbf{p}_1^2}{8 m_1^3} + \frac{3(\mathbf{p}_1 \cdot \mathbf{p}_2)}{4 m_1^2} \\
&- \frac{3\mathbf{p}_2^2}{4 m_1 m_2} + \frac{3(\mathbf{p}_1 \cdot \mathbf{n}_{12})(\mathbf{p}_2 \cdot \mathbf{n}_{12})}{4 m_1^2} + \frac{3(\mathbf{p}_2 \cdot \mathbf{n}_{12})^2}{2 m_1 m_2}\Bigg] \\
&+ ((\mathbf{p}_2 \times \mathbf{S}_1)\cdot \mathbf{n}_{12})\left[\frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1 m_2} + \frac{3(\mathbf{p}_1 \cdot \mathbf{n}_{12})(\mathbf{p}_2 \cdot \mathbf{n}_{12})}{m_1 m_2}\right] \\
&+ ((\mathbf{p}_1 \times \mathbf{S}_1)\cdot \mathbf{p}_2)\left[\frac{2(\mathbf{p}_2 \cdot \mathbf{n}_{12})}{m_1 m_2} - \frac{3(\mathbf{p}_1 \cdot \mathbf{n}_{12})}{4 m_1^2}\right]\Bigg] \\
&+ \frac{G^2}{c^4 r^3}\Bigg[-((\mathbf{p}_1 \times \mathbf{S}_1)\cdot \mathbf{n}_{12})\left[\frac{11 m_2}{2} + \frac{5 m_2^2}{m_1}\right] \\
&+ ((\mathbf{p}_2 \times \mathbf{S}_1)\cdot \mathbf{n}_{12})\left[6 m_1 + \frac{15 m_2}{2}\right]\Bigg] + (1 \leftrightarrow 2)
\end{aligned}
$$

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$$
\begin{aligned}
H^{2PN}_{S_1S_2} &= (G/2m_1m_2c^4r^3)[3((\mathbf{p}_1 \times \mathbf{S}_1)\cdot\mathbf{n}_{12})((\mathbf{p}_2 \times \mathbf{S}_2)\cdot\mathbf{n}_{12})/2 \\
&+ 6((\mathbf{p}_2 \times \mathbf{S}_1)\cdot\mathbf{n}_{12})((\mathbf{p}_1 \times \mathbf{S}_2)\cdot\mathbf{n}_{12}) \\
&- 15(\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{S}_2\cdot\mathbf{n}_{12})(\mathbf{p}_1\cdot\mathbf{n}_{12})(\mathbf{p}_2\cdot\mathbf{n}_{12}) \\
&- 3(\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{S}_2\cdot\mathbf{n}_{12})(\mathbf{p}_1\cdot\mathbf{p}_2) + 3(\mathbf{S}_1\cdot\mathbf{p}_2)(\mathbf{S}_2\cdot\mathbf{n}_{12})(\mathbf{p}_1\cdot\mathbf{n}_{12}) \\
&+ 3(\mathbf{S}_2\cdot\mathbf{p}_1)(\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{p}_2\cdot\mathbf{n}_{12}) + 3(\mathbf{S}_1\cdot\mathbf{p}_1)(\mathbf{S}_2\cdot\mathbf{n}_{12})(\mathbf{p}_2\cdot\mathbf{n}_{12}) \\
&+ 3(\mathbf{S}_2\cdot\mathbf{p}_2)(\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{p}_1\cdot\mathbf{n}_{12}) - 3(\mathbf{S}_1\cdot\mathbf{S}_2)(\mathbf{p}_1\cdot\mathbf{n}_{12})(\mathbf{p}_2\cdot\mathbf{n}_{12}) \\
&+ (\mathbf{S}_1\cdot\mathbf{p}_1)(\mathbf{S}_2\cdot\mathbf{p}_2) - (\mathbf{S}_1\cdot\mathbf{p}_2)(\mathbf{S}_2\cdot\mathbf{p}_1)/2 + (\mathbf{S}_1\cdot\mathbf{S}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)/2] \\
&+ (3/2m_1^2r^3)[-((\mathbf{p}_1 \times \mathbf{S}_1)\cdot\mathbf{n}_{12})((\mathbf{p}_1 \times \mathbf{S}_2)\cdot\mathbf{n}_{12}) \\
&+ (\mathbf{S}_1\cdot\mathbf{S}_2)(\mathbf{p}_1\cdot\mathbf{n}_{12})^2 - (\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{S}_2\cdot\mathbf{p}_1)(\mathbf{p}_1\cdot\mathbf{n}_{12})] \\
&+ (3/2m_2^2r^3)[-((\mathbf{p}_2 \times \mathbf{S}_2)\cdot\mathbf{n}_{12})((\mathbf{p}_2 \times \mathbf{S}_1)\cdot\mathbf{n}_{12}) \\
&+ (\mathbf{S}_1\cdot\mathbf{S}_2)(\mathbf{p}_2\cdot\mathbf{n}_{12})^2 - (\mathbf{S}_2\cdot\mathbf{n}_{12})(\mathbf{S}_1\cdot\mathbf{p}_2)(\mathbf{p}_2\cdot\mathbf{n}_{12})] \\
&+ (6G^2(m_1+m_2)/c^4r^4)[(\mathbf{S}_1\cdot\mathbf{S}_2) - 2(\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{S}_2\cdot\mathbf{n}_{12})]
\end{aligned}
$$

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$$
\begin{aligned}
H^{2PN}_{S_1S_1} = \frac{G}{c^4 r^3}\Bigg[& -\frac{5m_2}{4m_1^3}\left(\mathbf{p}_1\cdot\mathbf{S}_1\right)^2 + \frac{m_2}{m_1^3}\mathbf{p}_1^2\mathbf{S}_1^2 - \frac{21m_2}{8m_1^3}\left(\mathbf{p}_1\cdot\mathbf{n}\right)^2\mathbf{S}_1^2 \\
& - \frac{3m_2}{8m_1^3}\mathbf{p}_1^2\left(\mathbf{S}_1\cdot\mathbf{n}\right)^2 + \frac{15m_2}{4m_1^3}\left(\mathbf{p}_1\cdot\mathbf{n}\right)\left(\mathbf{S}_1\cdot\mathbf{n}\right)\left(\mathbf{p}_1\cdot\mathbf{S}_1\right) - \frac{3}{4m_1m_2}\mathbf{p}_2^2\mathbf{S}_1^2 \\
& + \frac{9}{4m_1m_2}\mathbf{p}_2^2\left(\mathbf{S}_1\cdot\mathbf{n}\right)^2 - \frac{1}{4m_1^2}\left(\mathbf{p}_1\cdot\mathbf{p}_2\right)\mathbf{S}_1^2 - \frac{9}{4m_1^2}\left(\mathbf{p}_1\cdot\mathbf{p}_2\right)\left(\mathbf{S}_1\cdot\mathbf{n}\right)^2 \\
& + \frac{3}{2m_1^2}\left(\mathbf{p}_1\cdot\mathbf{S}_1\right)\left(\mathbf{p}_2\cdot\mathbf{S}_1\right) - \frac{3}{2m_1^2}\left(\mathbf{p}_1\cdot\mathbf{n}\right)\left(\mathbf{p}_2\cdot\mathbf{S}_1\right)\left(\mathbf{S}_1\cdot\mathbf{n}\right) \\
& - \frac{3}{2m_1^2}\left(\mathbf{p}_2\cdot\mathbf{n}\right)\left(\mathbf{p}_1\cdot\mathbf{S}_1\right)\left(\mathbf{S}_1\cdot\mathbf{n}\right) + \frac{15}{4m_1^2}\left(\mathbf{p}_1\cdot\mathbf{n}\right)\left(\mathbf{p}_2\cdot\mathbf{n}\right)\mathbf{S}_1^2 \\
& - \frac{15}{4m_1^2}\left(\mathbf{p}_1\cdot\mathbf{n}\right)\left(\mathbf{p}_2\cdot\mathbf{n}\right)\left(\mathbf{S}_1\cdot\mathbf{n}\right)^2 \Bigg] \\
& - \frac{G^2 m_2}{2c^4 r^4}\left[5\left(1+\frac{m_2}{m_1}\right)\left(\left(\mathbf{S}_1\cdot\mathbf{n}\right)^2 - \mathbf{S}_1^2\right) + 4\left(1+\frac{2m_2}{m_1}\right)\left(\mathbf{S}_1\cdot\mathbf{n}\right)^2\right]
\end{aligned}
$$

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NLO center-of-mass:

$$\mathbf{G}_{\rm SO}^{2PN} = -\sum_a \frac{\mathbf{P}_a^2}{8m_a^3}(\mathbf{P}_a \times \mathbf{S}_a)$$

$$+ \sum_a \sum_{b\neq a} \frac{m_b}{4m_a r_{ab}}\left[((\mathbf{P}_a \times \mathbf{S}_a)\cdot \mathbf{n}_{ab})\frac{5\mathbf{x}_a + \mathbf{x}_b}{r_{ab}} - 5(\mathbf{P}_a \times \mathbf{S}_a)\right]$$

$$+ \sum_a \sum_{b\neq a} \frac{1}{r_{ab}}\left[\frac{3}{2}(\mathbf{P}_b \times \mathbf{S}_a) - \frac{1}{2}(\mathbf{n}_{ab} \times \mathbf{S}_a)(\mathbf{P}_b \cdot \mathbf{n}_{ab})\right.$$

$$\left. - ((\mathbf{P}_a \times \mathbf{S}_a)\cdot \mathbf{n}_{ab})\frac{\mathbf{x}_a + \mathbf{x}_b}{r_{ab}}\right]$$

$$\mathbf{G}_{\rm SS}^{2PN} = \frac{1}{2}\sum_a \sum_{b\neq a}\left\{ [3(\mathbf{S}_a \cdot \mathbf{n}_{ab})(\mathbf{S}_b \cdot \mathbf{n}_{ab}) - (\mathbf{S}_a \cdot \mathbf{S}_b)]\frac{\mathbf{x}_a}{r_{ab}^3} + (\mathbf{S}_b \cdot \mathbf{n}_{ab})\frac{\mathbf{S}_a}{r_{ab}^2}\right\}$$

$$
\begin{aligned}
H \;=\;& H_N + H_{1PN} + H_{2PN} + H_{2.5PN} + H_{3PN} + H_{3.5PN} \\
+\;& H^{1PN}_{SO} + H^{2PN}_{SO} + H^{3PN}_{SO} + H^{3.5PN}_{SO} \\
+\;& H^{1PN}_{S_1S_2} + H^{2PN}_{S_1S_2} + H^{3PN}_{S_1S_2} + H^{3.5PN}_{S_1S_2} \\
+\;& H^{1PN}_{S_1S_1} + H^{1PN}_{S_2S_2} + H^{2PN}_{S_1S_1} + H^{2PN}_{S_2S_2} \\
+\;& H_{p_1S_2^3} + H_{p_2S_1^3} + H_{p_1S_1S_2^2} + H_{p_2S_2S_1^2} \\
+\;& H_{S_1S_2^3} + H_{S_1^3S_2} + H_{S_1^2S_2^2}
\end{aligned}
$$

$$H = H_{1PM}$$

$$H = H(t) = H[p, q, h^{\mathrm{TT}}(x; p', q'), \pi^{\mathrm{TT}}(x; p', q')] = H(p, q, p', q')$$

tetrad field e_a^μ: $e_a^\mu e_{b\mu} = \eta_{ab} \qquad e_{a\mu} e_{b\nu} \eta^{ab} = g_{\mu\nu} = g_{\nu\mu}$

local rotations: $e_a'^\mu = L^b{}_a e_b^\mu \qquad L^a{}_c \eta_{ab} L^b{}_d = \eta_{cd}$

linear connection ω_μ^{ab}: $D_\mu \phi \equiv \partial_\mu \phi + \frac{1}{2} \omega_\mu^{ab} G_{[ab]} \phi$

local rotations: $\omega_\mu'^{ab} = L^a{}_c L^b{}_d \omega_\mu^{cd} + L^a{}_d \partial_\mu L^{bd} \qquad \partial_\mu \equiv \frac{\partial}{\partial x^\mu}$

curvature tensor $R^{ab}{}_{\mu\nu}$: $D_\mu D_\nu \phi - D_\nu D_\mu \phi = R^{ab}{}_{\mu\nu} G_{[ab]} \phi$

$$R^{ab}{}_{\mu\nu} = \partial_\mu \omega_\nu^{ab} - \partial_\nu \omega_\mu^{ab} + \omega_\nu^{ac} \omega_\mu^{bd} \eta_{cd} - \omega_\mu^{ac} \omega_\nu^{bd} \eta_{cd}$$

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Lagrangian for gravity

$$\mathcal{L}_G = \frac{1}{16\pi} \det(e^c_\gamma) e^\mu_a e^\nu_b R^{ab}{}_{\mu\nu}(\omega) + \partial_\mu \mathcal{C}^\mu$$

vacuum Einstein equations:

$$0 = \frac{\delta \mathcal{L}_G}{\delta e^\mu_a} e_{a\nu} \equiv 2\det(e^c_\gamma)(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R)$$

$$0 = \frac{\delta \mathcal{L}_G}{\delta \omega^{ab}_\mu} \quad \Rightarrow \quad \omega^{ab}_\mu = \omega^{ab}_\mu(e, \partial_\nu e) \qquad \text{no torsion !}$$

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Lagrangian for spinning objects

$$\mathcal{L}_M = \int d\tau \left[\left(p_\mu - \frac{1}{2} S_{ab}\, \omega_\mu{}^{ab} \right) \frac{dz^\mu}{d\tau} + \frac{1}{2} S_{ab} \frac{d\theta^{ab}}{d\tau} \right] \delta_{(4)}$$

$$\mathcal{L}_C = \int d\tau \left[\lambda_1^a p^b S_{ab} + \lambda_{2[i]} \Lambda^{[i]a} p_a - \frac{\lambda_3}{2} (p^2 + m^2) \right] \delta_{(4)}$$

$$d\theta^{ab} = \Lambda_C^{\ a} d\Lambda^{Cb} = -d\theta^{ba}$$

matter action: $W_M = \int d^4x \, [\mathcal{L}_M + \mathcal{L}_C]$

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equations of motion

$$\frac{DS_{ab}}{D\tau} = 0$$

$$\frac{Dp_\mu}{D\tau} = -\frac{1}{2}\mathrm{R}^{(4)}_{\mu\rho ab}u^\rho S^{ab}$$

$$u^\mu \equiv \frac{dz^\mu}{d\tau} = \lambda_3 p^\mu$$

$$\sqrt{-g}T^{\mu\nu} = \int d\tau \left[\lambda_3 p^\mu p^\nu \delta_{(4)} + \left(u^{(\mu}S^{\nu)\alpha}\delta_{(4)}\right)_{||\alpha}\right]$$

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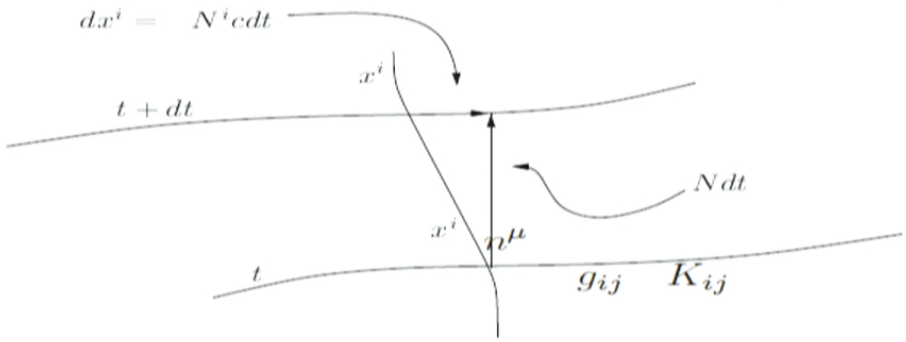
3+1 splitting of spacetime

$$n^\mu = (1, -N^i)/N$$

$$n_\mu = (-N, 0, 0, 0)$$



$$K_{ij} = -N\Gamma^0_{ij} = -N g^{0\mu}(g_{i\mu,j} + g_{j\mu,i} - g_{ij,\mu})/2$$

$$ds^2 = -(Ncdt)^2 + g_{ij}(dx^i + N^i cdt)(dx^j + N^j cdt)$$

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solution of the matter constraints

$$n^\mu = (1, -N^i)/N, \qquad n_\mu = (-N, 0, 0, 0)$$

$$\lambda_3 : \quad np \equiv n^\mu p_\mu = -\sqrt{m^2 + \gamma^{ij} p_i p_j}\,, \qquad \gamma^{ik} g_{kj} = \delta^i_j$$

$$\lambda_1 : \quad nS_i \equiv n^\mu S_{\mu i} = \frac{p_k \gamma^{kj} S_{ji}}{np} = g_{ij} n S^j$$

$$\lambda_2 : \quad \Lambda^{[j](0)} = \Lambda^{[j](i)} \frac{p_{(i)}}{p^{(0)}}, \qquad \Lambda^{[0]a} = -\frac{p^a}{m}$$

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time gauge for the tetrads

$$e^{\mu}_{(0)} = n^{\mu}, \qquad e^{0}_{(0)} = \frac{1}{N}, \qquad e^{i}_{(0)} = -\frac{N^i}{N}$$

$$g_{ij} = e_i^{(m)} e_{(m)j}$$

$$\mathcal{L}_{MC} = -N\mathcal{H}^{\text{matter}} + N^i \mathcal{H}_i^{\text{matter}}$$

$$\mathcal{H}^{\text{matter}} = -np\delta - K^{ij}\frac{p_i n S_j}{np}\delta - (nS^k\delta)_{;k}$$

$$\mathcal{H}_i^{\text{matter}} = (p_i + K_{ij} n S^j)\delta + \left(\frac{1}{2}\gamma^{mk} S_{ik}\delta + \delta_i^{(k}\gamma^{l)m}\frac{p_k n S_l}{np}\delta\right)_{;m}$$

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transformation to canonical matter variables

$$z^i = \hat{z}^i - \frac{nS^i}{m - np}, \quad nS_i = -\frac{p_k \gamma^{kj} \hat{S}_{ji}}{m}$$

$$S_{ij} = \hat{S}_{ij} - \frac{p_i n S_j}{m - np} + \frac{p_j n S_i}{m - np}$$

$$\Lambda^{[i](j)} = \hat{\Lambda}^{[i](k)} \left(\delta_{kj} + \frac{p_{(k)} p^{(j)}}{m(m - np)} \right)$$

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$$P_i = p_i + K_{ij} n S^j + \hat{A}^{kl} e_{(j)k} e^{(j)}_{l,i} - \left(\frac{1}{2} S_{kj} + \frac{p_{(k} n S_{j)}}{np} \right) \Gamma^{kj}{}_i$$

$$g_{ik} g_{jl} \hat{A}^{kl} = \frac{1}{2} \hat{S}_{ij} + \frac{m p_{(i} n S_{j)}}{np(m - np)}$$

$$S^{ab} S_{ab} = \hat{S}_{(i)(j)} \hat{S}_{(i)(j)} = 2 \hat{S}_{(i)} \hat{S}_{(i)} = 2 s^2 = \text{const}$$

$$\hat{\Lambda}^{(i)}_{[k]} \hat{\Lambda}^{[k](j)} = \delta_{ij}$$

$$d\hat{\theta}^{(i)(j)} \equiv \hat{\Lambda}^{(i)}_{[k]} d\hat{\Lambda}^{[k](j)} = -d\hat{\theta}^{(j)(i)}$$

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adding Lagrangian of gravity

$$\hat{\mathcal{L}}_{MK} = P_i \dot{\hat{z}}^i \delta + \frac{1}{2} \hat{S}_{(i)(j)} \dot{\hat{\theta}}^{(i)(j)} \delta$$

$$\hat{\mathcal{L}}_{GK} = \hat{A}^{ij} e_{(k)i} e^{(k)}_{j,0} \delta$$

$$\hat{\mathcal{L}}_{GK} + \mathcal{L}_G = \frac{1}{8\pi} [\pi^{ij} + 8\pi \hat{A}^{ij} \delta] e_{(k)i} e^{(k)}_{j,0} \; + \; \mathcal{L}_{GC} - \frac{1}{16\pi} \mathcal{E}_{i,i}$$

$$\mathcal{E}_i = g_{ij,j} - g_{jj,i}$$

total energy: $E = \frac{1}{16\pi} \oint d^2 s_i \; \mathcal{E}_i = \frac{1}{16\pi} \int d^3 x \; \mathcal{E}_{i,i}$

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$$\mathcal{L}_{GC} = -N\mathcal{H}^{\rm field} + N^i\mathcal{H}_i^{\rm field}$$

$$\mathcal{H}^{\rm field} = -\frac{1}{16\pi\sqrt{\gamma}}\left[\gamma \mathrm{R} + \frac{1}{2}\left(g_{ij}\pi^{ij}\right)^2 - g_{ij}g_{kl}\pi^{ik}\pi^{jl}\right]$$

$$\mathcal{H}_i^{\rm field} = \frac{1}{8\pi}g_{ij}\pi^{jk}{}_{;k}$$

$$\pi^{ij} = \sqrt{\gamma}(\gamma^{ij}\gamma^{kl} - \gamma^{ik}\gamma^{jl})K_{kl} \qquad \gamma \equiv \det(g_{ij})$$

$$W_G = \int d^4x\left[\frac{1}{8\pi}\pi^{ij}e_{(k)i}e^{(k)}_{j,0} - N\mathcal{H}^{\rm field} + N^i\mathcal{H}_i^{\rm field} - \frac{1}{16\pi}\mathcal{E}_{i,i}\right]$$

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$$\mathcal{L}_{GC} = -N\mathcal{H}^{\text{field}} + N^i\mathcal{H}_i^{\text{field}}$$

$$\mathcal{H}^{\text{field}} = -\frac{1}{16\pi\sqrt{\gamma}}\left[\gamma \mathrm{R} + \frac{1}{2}\left(g_{ij}\pi^{ij}\right)^2 - g_{ij}g_{kl}\pi^{ik}\pi^{jl}\right]$$

$$\mathcal{H}_i^{\text{field}} = \frac{1}{8\pi}g_{ij}\pi^{jk}{}_{;k}$$

$$\pi^{ij} = \sqrt{\gamma}(\gamma^{ij}\gamma^{kl} - \gamma^{ik}\gamma^{jl})K_{kl} \qquad \gamma \equiv \det(g_{ij})$$

$$W_G = \int d^4x\left[\frac{1}{8\pi}\pi^{ij}e_{(k)i}e^{(k)}_{j,0} - N\mathcal{H}^{\text{field}} + N^i\mathcal{H}_i^{\text{field}} - \frac{1}{16\pi}\mathcal{E}_{i,i}\right]$$

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spatially symmetric time gauge for the tetrads

$$e_{(k)i}e^{(k)}_{j,\mu} = B^{kl}_{ij} g_{kl,\mu} + \frac{1}{2} g_{ij,\mu} \qquad B^{kl}_{ij} = B^{(kl)}_{[ij]}$$

$$e_{(i)j} = e_{ij} = e_{ji}$$

$$e_{ij}e_{jk} = g_{ik} \qquad e_{ij} = \sqrt{(g_{kl})}$$

$$\pi^{ij}_{\mathrm{can}} = \pi^{ij} + 8\pi \hat{A}^{(ij)}\delta + 16\pi B^{ij}_{kl}\hat{A}^{[kl]}\delta$$

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spacetime-coordinates conditions

$$3g_{ij,j} - g_{jj,i} = 0, \qquad \pi^{ii}_{\rm can} = 0$$

$$g_{ij} = \Psi^4 \delta_{ij} + h^{\rm TT}_{ij}, \qquad \pi^{ij}_{\rm can} = \tilde{\pi}^{ij}_{\rm can} + \pi^{ij\,\rm TT}_{\rm can}$$

transverse traceless: $h^{\rm TT}_{ii} = \pi^{ii\,\rm TT}_{\rm can} = h^{\rm TT}_{ij,j} = \pi^{ij\,\rm TT}_{{\rm can},j} = 0$

$$\tilde{\pi}^{ij}_{\rm can} = V^i_{{\rm can},j} + V^j_{{\rm can},i} - \frac{2}{3}\delta_{ij} V^k_{{\rm can},k}$$

constraints: $\mathcal{H}^{\rm field} + \mathcal{H}^{\rm matter} = 0, \qquad \mathcal{H}^{\rm field}_i + \mathcal{H}^{\rm matter}_i = 0$

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total action in canonical form

$$W = \frac{1}{16\pi}\int d^4x\,\pi^{ij\mathrm{TT}}_{\mathrm{can}} h^{\mathrm{TT}}_{ij,0} + \int dt\left[P_i \dot{\hat{z}}^i + \frac{1}{2}\hat{S}_{(i)(j)}\dot{\hat{\theta}}^{(i)(j)} - E\right]$$

Hamiltonian: $E \equiv H_{\mathrm{ADM}} = -\frac{1}{2\pi}\int d^3x\,\Delta\Psi\left[\hat{z}^i, P_i, \hat{S}_{(i)(j)}, h^{\mathrm{TT}}_{ij}, \pi^{ij\mathrm{TT}}_{\mathrm{can}}\right]$

$$\{\hat{z}^i, P_j\} = \delta_{ij}, \qquad \{\hat{S}_{(i)}, \hat{S}_{(j)}\} = \epsilon_{ijk}\hat{S}_{(k)}$$

$$\{h^{\mathrm{TT}}_{ij}(\mathbf{x},t), \pi^{kl\mathrm{TT}}_{\mathrm{can}}(\mathbf{x}',t)\} = 16\pi\delta^{\mathrm{TT}kl}_{ij}\delta(\mathbf{x}-\mathbf{x}')$$

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total action in canonical form

$$W = \frac{1}{16\pi}\int d^4x \pi_{\rm can}^{ij{\rm TT}} h_{ij,0}^{\rm TT} + \int dt \left[P_i \dot{\hat{z}}^i + \frac{1}{2}\hat{S}_{(i)(j)}\dot{\hat{\theta}}^{(i)(j)} - E \right]$$

Hamiltonian: $E \equiv H_{\rm ADM} = -\frac{1}{2\pi}\int d^3x\ \Delta\Psi \left[\hat{z}^i, P_i, \hat{S}_{(i)(j)}, h_{ij}^{\rm TT}, \pi_{\rm can}^{ij{\rm TT}}\right]$

$$\{\hat{z}^i, P_j\} = \delta_{ij}, \qquad \{\hat{S}_{(i)}, \hat{S}_{(j)}\} = \epsilon_{ijk}\hat{S}_{(k)}$$

$$\{h_{ij}^{\rm TT}(\mathbf{x},t), \pi_{\rm can}^{kl{\rm TT}}(\mathbf{x}',t)\} = 16\pi\delta_{ij}^{{\rm TT}kl}\delta(\mathbf{x}-\mathbf{x}')$$

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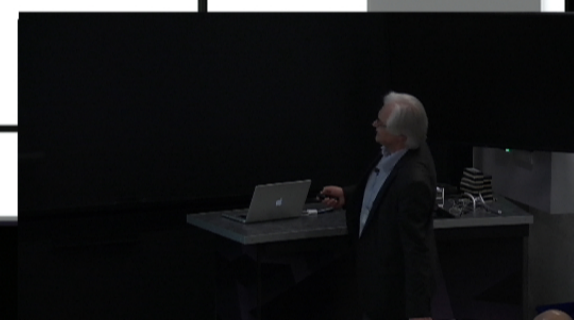

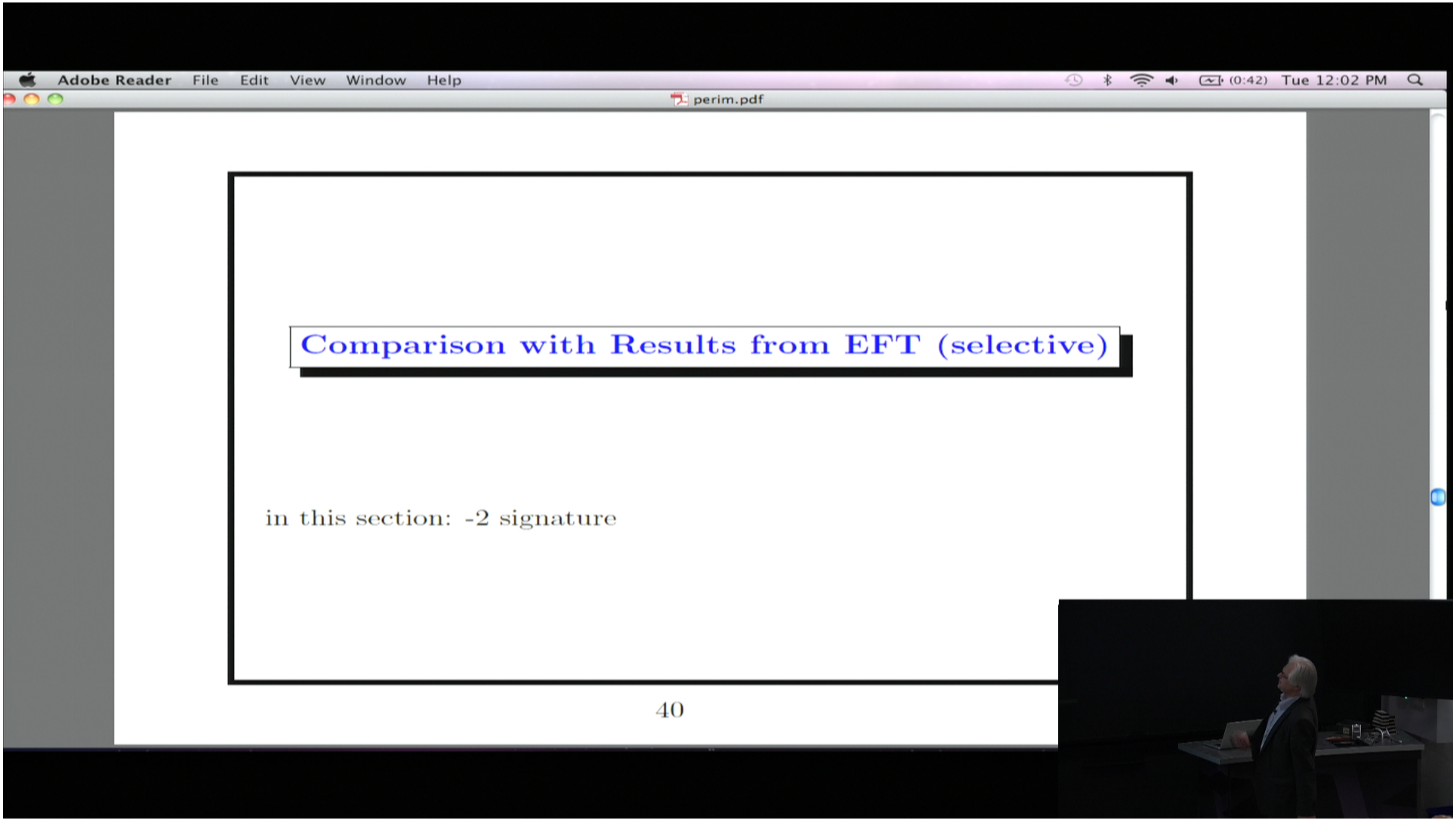
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Comparison with Results from EFT (selective)
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$$
\begin{aligned}
L^{\mathrm{NLO}}_{\mathrm{SO(L)}} = {} & \frac{Gm_2}{r_{12}^2}\boldsymbol{S}_1 \cdot \Bigg[\boldsymbol{v}_1 \times \boldsymbol{n}_{12}\left(\frac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v}_2 - \frac{1}{2}v_2^2 - \frac{3}{2}(\boldsymbol{v}_1\cdot\boldsymbol{n}_{12})(\boldsymbol{v}_2\cdot\boldsymbol{n}_{12})\right) \\
& + \boldsymbol{v}_2 \times \boldsymbol{n}_{12}\left(\boldsymbol{v}_1\cdot\boldsymbol{v}_2 - v_2^2 + 3(\boldsymbol{v}_1\cdot\boldsymbol{n}_{12})(\boldsymbol{v}_2\cdot\boldsymbol{n}_{12})\right) \\
& + \boldsymbol{v}_1 \times \boldsymbol{v}_2\left(\frac{1}{2}\boldsymbol{v}_1\cdot\boldsymbol{n}_{12} + \boldsymbol{v}_2\cdot\boldsymbol{n}_{12}\right)\Bigg] \\
& + \frac{G^2m_2}{r_{12}^3}\boldsymbol{S}_1 \cdot \left[\boldsymbol{v}_1 \times \boldsymbol{n}_{12}\left(2m_1 - \frac{1}{2}m_2\right) + \boldsymbol{v}_2 \times \boldsymbol{n}_{12}(2m_2)\right] \\
& + \frac{Gm_2}{r_{12}^2}\Bigg[S_1^{(0)(i)} n_{12}^i\left(1 - \frac{3}{2}\boldsymbol{v}_1\cdot\boldsymbol{v}_2 + \frac{3}{2}v_2^2 - \frac{3}{2}(\boldsymbol{v}_1\cdot\boldsymbol{n}_{12})(\boldsymbol{v}_2\cdot\boldsymbol{n}_{12})\right) \\
& + S_1^{(0)(i)} v_2^i\left(-\frac{3}{2}\boldsymbol{v}_1\cdot\boldsymbol{n}_{12}\right)\Bigg] \\
& + \frac{Gm_2}{r_{12}}\left[\frac{3}{2}\dot{S}_1^{(0)(i)} v_2^i\right] - \frac{G^2m_2}{r_{12}^3} S_1^{(0)(i)} n_{12}^i\left[m_1 + 2m_2\right]
\end{aligned}
$$

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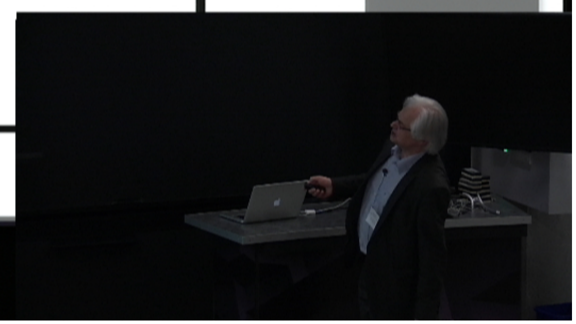

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$$L^{\rm NLO}_{\rm SO(PR)} = -\frac{Gm_2}{r_{12}^2}\Bigg[\Bigg\{S_1^{(i)(0)}\Bigg(1 + 2\mathbf{v}_2^2 - 2\mathbf{v}_1\cdot\mathbf{v}_2 - \frac{3}{2}(\mathbf{v}_2\cdot\boldsymbol{n}_{12})^2$$

$$-\frac{G}{r}(3m_1 + 2m_2)\Bigg) + \Bigg(1 - \frac{3}{2}(\mathbf{v}_2\cdot\boldsymbol{n}_{12})^2 + \frac{G}{2r}(4m_1 - m_2)\Bigg) S_1^{(i)(j)}\mathbf{v}_1^j$$

$$-\Bigg(2 - 2\mathbf{v}_1\cdot\mathbf{v}_2 - 3(\mathbf{v}_2\cdot\boldsymbol{n}_{12})^2 + 2\mathbf{v}_2^2 - \frac{G}{2r}(2m_1 + 5m_2)\Bigg) S_1^{(i)(j)}\mathbf{v}_2^j\Bigg\}\boldsymbol{n}_{12}^i$$

$$+ S_1^{(i)(0)}(\mathbf{v}_1 - \mathbf{v}_2)^i\mathbf{v}_2\cdot\boldsymbol{n}_{12} + S_1^{(i)(j)}\mathbf{v}_1^j\mathbf{v}_2^i\mathbf{v}_2\cdot\boldsymbol{n}_{12}\Bigg] \quad + \quad (1\leftrightarrow 2)$$

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$$
\begin{aligned}
V^{\mathrm{NLO}}_{S_1S_2(\mathrm{PR})} = & -\frac{G}{r_{12}^3}\Big[(\delta^{ij} - 3n_{12}^i n_{12}^j)\Big(S_1^{(i)(0)}S_2^{(j)(0)} + \frac{1}{2}\boldsymbol{v}_1\cdot\boldsymbol{v}_2 S_1^{(i)(k)}S_2^{(j)(k)} \\
& + v_1^m v_2^k S_1^{(i)(k)} S_2^{(j)(m)} - v_1^k v_2^m S_1^{(i)(k)} S_2^{(j)(m)} + S_1^{(i)(0)} S_2^{(j)(k)}(v_2^k - v_1^k) \\
& + S_1^{(i)(k)} S_2^{(j)(0)}(v_1^k - v_2^k)\Big) \\
& + \frac{1}{2} S_1^{(k)(i)} S_2^{(k)(j)}\Big(3\boldsymbol{v}_1\cdot\boldsymbol{n}_{12}\boldsymbol{v}_2\cdot\boldsymbol{n}_{12}(\delta^{ij} - 5n_{12}^i n_{12}^j) \\
& + 3\boldsymbol{v}_1\cdot\boldsymbol{n}_{12}(v_2^j n_{12}^i + v_2^i n_{12}^j) + 3\boldsymbol{v}_2\cdot\boldsymbol{n}_{12}(v_1^j n_{12}^i + v_1^i n_{12}^j) - v_1^i v_2^j - v_2^i v_1^j\Big) \\
& + (3n_{12}^l \boldsymbol{v}_2\cdot\boldsymbol{n}_{12} - v_2^l) S_1^{(0)(k)} S_2^{(k)(l)} + (3n_{12}^l \boldsymbol{v}_1\cdot\boldsymbol{n}_{12} - v_1^l) S_2^{(0)(k)} S_1^{(k)(l)}\Big] \\
& + \left(\frac{G}{r_{12}^3} - \frac{3(m_1+m_2)G^2}{r_{12}^4}\right) S_1^{(j)(k)} S_2^{(j)(i)}(\delta^{ki} - 3n_{12}^k n_{12}^i) \\
& + \frac{Gm_2}{r_{12}^2} n_{12}^j S_1^{(j)(0)} - \frac{Gm_1}{r_{12}^2} n_{12}^j S_2^{(j)(0)}
\end{aligned}
$$

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$$p_\mu = m g_{\mu\nu} \frac{u^\nu}{\sqrt{u_\lambda u^\lambda}}, \qquad g^{\mu\nu} p_\mu p_\nu = m^2, \qquad u^0 = 1$$

$$\begin{aligned}
\{z^i_I, p_{Jj}\} &= \delta_{ij}\delta_{IJ} \\
\{S_{Iab}, S_{Icd}\} &= S_{Ica}\eta_{bd} - S_{Ida}\eta_{bc} + S_{Idb}\eta_{ac} - S_{Icb}\eta_{ad} \\
\{S_{Iab}, \Lambda^A_{I\,c}\} &= \eta_{bc}\Lambda^A_{I\,a} - \eta_{ac}\Lambda^A_{I\,b}
\end{aligned}$$

remaining supplementary conditions:

$$\begin{aligned}
S_{ab}u^b &= 0 \quad \Leftrightarrow \quad S_{(i)b}u^b = 0 \quad \Leftrightarrow \quad S_{(i)(0)} + S_{(i)(j)}\frac{u^{(j)}}{u^{(0)}} = 0 \\
\Lambda^{[i]a}u_a &= 0 \quad \Leftrightarrow \quad \Lambda^{[i](0)} + \Lambda^{[i](j)}\frac{u_{(j)}}{u_{(0)}} = 0 \\
\Lambda_{[0]a} &= \frac{u_a}{\sqrt{u_b u^b}}
\end{aligned}$$

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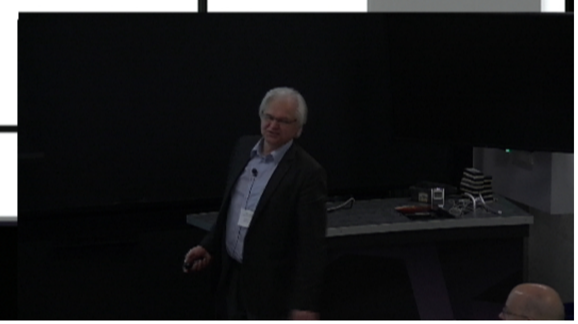

$$\{\Phi_A, \Phi_B\} \equiv C_{AB}$$

$$\{Q_1, Q_2\}_{\mathrm{D}} = \{Q_1, Q_2\} - \{Q_1, \Phi_A\} C^{AB} \{\Phi_B, Q_2\}$$

$$\dot{z}^i_I = \{z^i_I, H_{\mathrm{eff}}\}_{\mathrm{D}}\,, \qquad \dot{p}_{Ii} = \{p_{Ii}, H_{\mathrm{eff}}\}_{\mathrm{D}}\,, \qquad \dot{S}_{I(i)(j)} = \{S_{I(i)(j)}, H_{\mathrm{eff}}\}_{\mathrm{D}}$$

aimed at variables:

$$\{\hat{z}^i_I, \hat{p}_{Jj}\}_{\mathrm{D}} = \delta^i_j \delta_{IJ}$$

$$\{\hat{\Lambda}^{[i](j)}_I, \hat{S}^{(m)(n)}_I\}_{\mathrm{D}} = -\delta_{jm} \hat{\Lambda}^{[i](n)}_I + \delta_{jn} \hat{\Lambda}^{[i](m)}_I$$

$$\{\hat{S}^{(i)(j)}_I, \hat{S}^{(k)(l)}_I\}_{\mathrm{D}} = \delta_{jl} \hat{S}^{(i)(k)}_I - \delta_{jk} \hat{S}^{(i)(l)}_I + \delta_{ik} \hat{S}^{(j)(l)}_I - \delta_{il} \hat{S}^{(j)(k)}_I$$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

$$u_I^a = e^a{}_\mu(z_I) u_I^\mu = \eta^{a\mu} \left(\eta_{\mu\nu} + \frac{1}{2} h_{\mu\nu} - \frac{1}{8} h_{\mu\rho} \eta^{\rho\sigma} h_{\sigma\nu} + \dots \right) u_I^\nu$$

$$e_{(0)0} = 1 + \frac{1}{2} h_{00} = 1 - G\frac{m_1}{r_1} - G\frac{m_2}{r_2} + \dots$$

$$e_{(0)i} = e_{(i)0} = \frac{1}{2} h_{0i} = 2Gv_1^i \frac{m_1}{r_1} - GS_1^{(i)(j)} \frac{n_1^j}{r_1^2} + (1 \leftrightarrow 2) + \dots$$

$$e_{(i)j} = -\delta_{ij} + \frac{1}{2} h_{ij} = -\delta_{ij} \left(1 + G\frac{m_1}{r_1} + G\frac{m_2}{r_2} \right) + \dots$$

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$0 = \Phi_A$

$$
\begin{aligned}
0 &= \Lambda_1^{[0](i)} - \frac{u_1^{(i)}}{\sqrt{\left(u_1^{(0)}\right)^2 - \left(u_1^{(i)}\right)^2}} \\
&= \Lambda_1^{[0](i)} - (1 - \boldsymbol{v}_1^2)^{-\frac{1}{2}} v_1^i - 2G(v_1^i - v_2^i)\frac{m_2}{r_{12}} - G S_2^{ij} \frac{n_{12}^j}{r_{12}^2} + \dots \\
&= \Lambda_1^{[0](i)} - \frac{p_{1i}}{m_1} - \frac{G}{2r_{12}} \left(3p_{2i} - 2\frac{m_2}{m_1} p_{1i} + (\boldsymbol{n}_{12} \cdot \boldsymbol{p}_2) n_{12}^i \right) \\
&+ \frac{G n_{12}^j}{r_{12}^2} \left(S_2^{(i)(j)} + \frac{m_2}{m_1} S_1^{(i)(j)} \right) + \dots
\end{aligned}
$$

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$0 = \Phi_A$

$$
\begin{aligned}
0 &= \Lambda_1^{[0](i)} - \frac{u_1^{(i)}}{\sqrt{\left(u_1^{(0)}\right)^2 - \left(u_1^{(i)}\right)^2}} \\
&= \Lambda_1^{[0](i)} - (1 - \boldsymbol{v}_1^2)^{-\frac{1}{2}} v_1^i - 2G(v_1^i - v_2^i)\frac{m_2}{r_{12}} - G S_2^{ij} \frac{n_{12}^j}{r_{12}^2} + \dots \\
&= \Lambda_1^{[0](i)} - \frac{p_{1i}}{m_1} - \frac{G}{2r_{12}}\left(3p_{2i} - 2\frac{m_2}{m_1}p_{1i} + (\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)n_{12}^i\right) \\
&+ \frac{G n_{12}^j}{r_{12}^2}\left(S_2^{(i)(j)} + \frac{m_2}{m_1} S_1^{(i)(j)}\right) + \dots
\end{aligned}
$$

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$$\mathbf{p}_I = \hat{\mathbf{p}}_I$$

$$z_1^i = \hat{z}_1^i - \left[\frac{1}{2m_1^2} p_{1k} \hat{S}_{1(i)(k)} \left(1 - \frac{\boldsymbol{p}_1^2}{4m_1^2} \right) - G \frac{m_2}{m_1^2} \frac{p_{1k} \hat{S}_{1(i)(k)}}{\hat{r}_{12}} \right.$$

$$+ \frac{3}{2} G \frac{p_{2k} \hat{S}_{1(i)(k)}}{m_1 \hat{r}_{12}}$$

$$+ \frac{G}{2} \frac{\hat{n}_{12}^k (\hat{\boldsymbol{n}}_{12} \cdot \boldsymbol{p}_2) \hat{S}_{1(i)(k)}}{m_1 \hat{r}_{12}} + G \frac{m_2}{m_1^2} \frac{\hat{S}_{1(k)(l)} \hat{S}_{1(i)(l)} \hat{n}_{12}^k}{\hat{r}_{12}^2}$$

$$\left. + G \frac{\hat{n}_{12}^k \hat{S}_{1(i)(l)} \hat{S}_{2(k)(l)}}{m_1 \hat{r}_{12}^2} \right] + \ldots$$

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$$
\begin{aligned}
S_{1(i)(j)} \;=\;& \hat{S}_{1(i)(j)} - \Bigg[\frac{p_{1[i}\hat{S}_{1(j)](k)}p_{1k}}{m_1^2}\left(1 - \frac{\boldsymbol{p}_1^2}{4m_1^2}\right) \\
&- \frac{2Gm_2}{m_1^2 \hat{r}_{12}} p_{1[i}\hat{S}_{1(j)](k)}p_{1k} \\
&+ \frac{3G}{m_1 \hat{r}_{12}} p_{1[i}\hat{S}_{1(j)](k)}p_{2k} + \frac{G}{m_1 \hat{r}_{12}} p_{1[i}\hat{S}_{1(j)](k)}\hat{n}_{12}^k(\hat{\boldsymbol{n}}_{12}\cdot\boldsymbol{p}_2) \\
&+ \frac{2Gm_2}{m_1^2 \hat{r}_{12}^2} p_{1[i}\hat{S}_{1(j)](l)}\hat{S}_{1(k)(l)}\hat{n}_{12}^k \\
&+ \frac{2G}{m_1 \hat{r}_{12}^2} p_{1[i}\hat{S}_{1(j)](l)}\hat{S}_{2(k)(l)}\hat{n}_{12}^k \Bigg] + \ldots
\end{aligned}
$$

Hergt/Steinhoff/GS: arXiv:1110.2094

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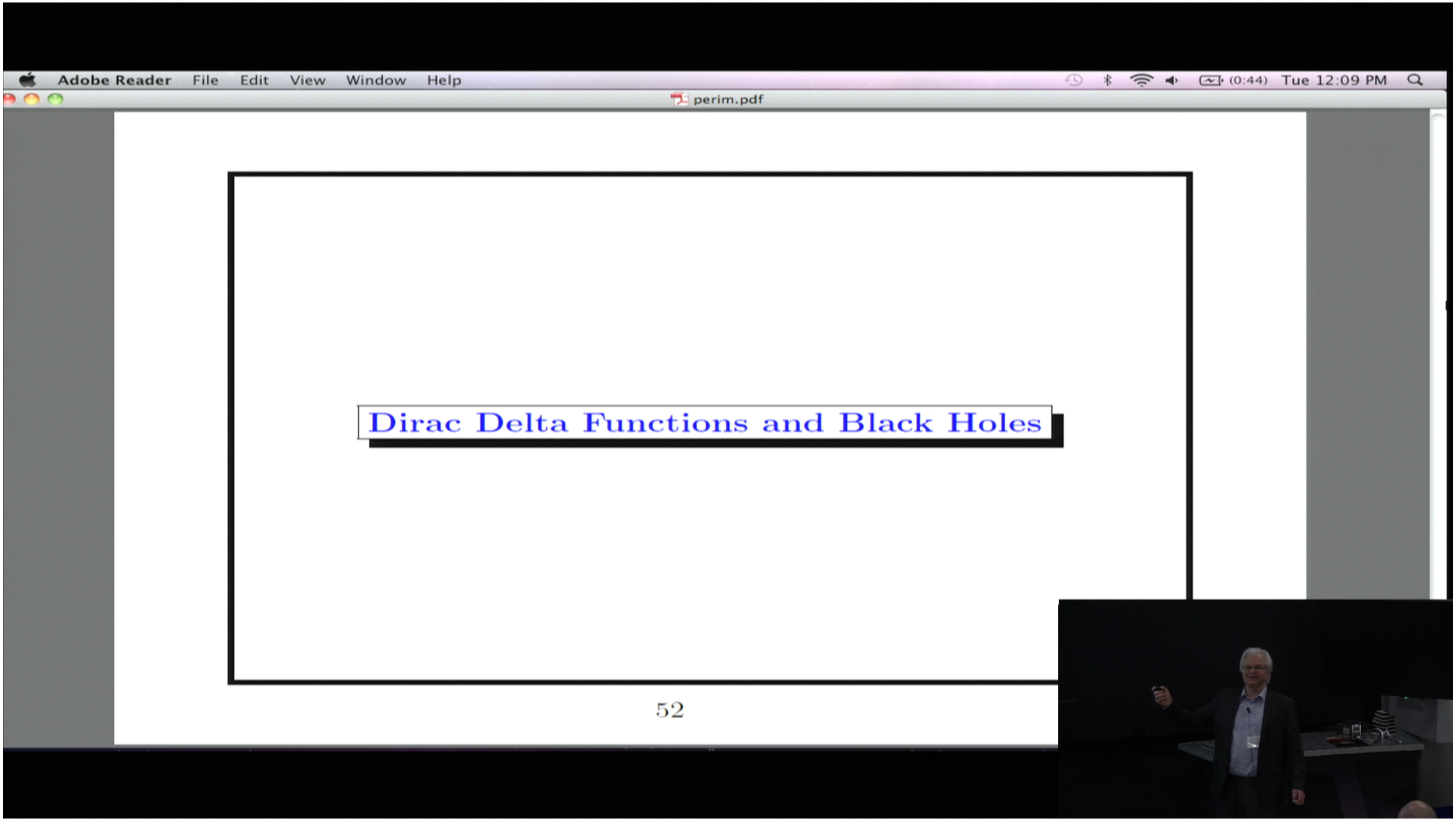
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Dirac Delta Functions and Black Holes
52

independent field variables

$$g_{ij} = \left(1 + \frac{1}{8}\phi\right)^4 \delta_{ij} + h_{ij}^{\mathrm{TT}}$$

$$\pi^{ii} = 0, \qquad \pi^{ij} = -\gamma^{1/2}(K^{ij} - \gamma^{ij}K), \qquad \pi^i_i = \pi^{ij}h_{ij}^{\mathrm{TT}}$$

unique decomposition: $\pi^{ij} = \tilde{\pi}^{ij} + \pi^{ij\mathrm{TT}}$

$$\tilde{\pi}^{ij} = \partial_i \pi^j + \partial_j \pi^i - \frac{2}{3}\delta_{ij}\partial_k \pi^k$$

$\pi^{ij\mathrm{TT}} c^3/16\pi G$: canonical conjugate to h_{ij}^{TT}

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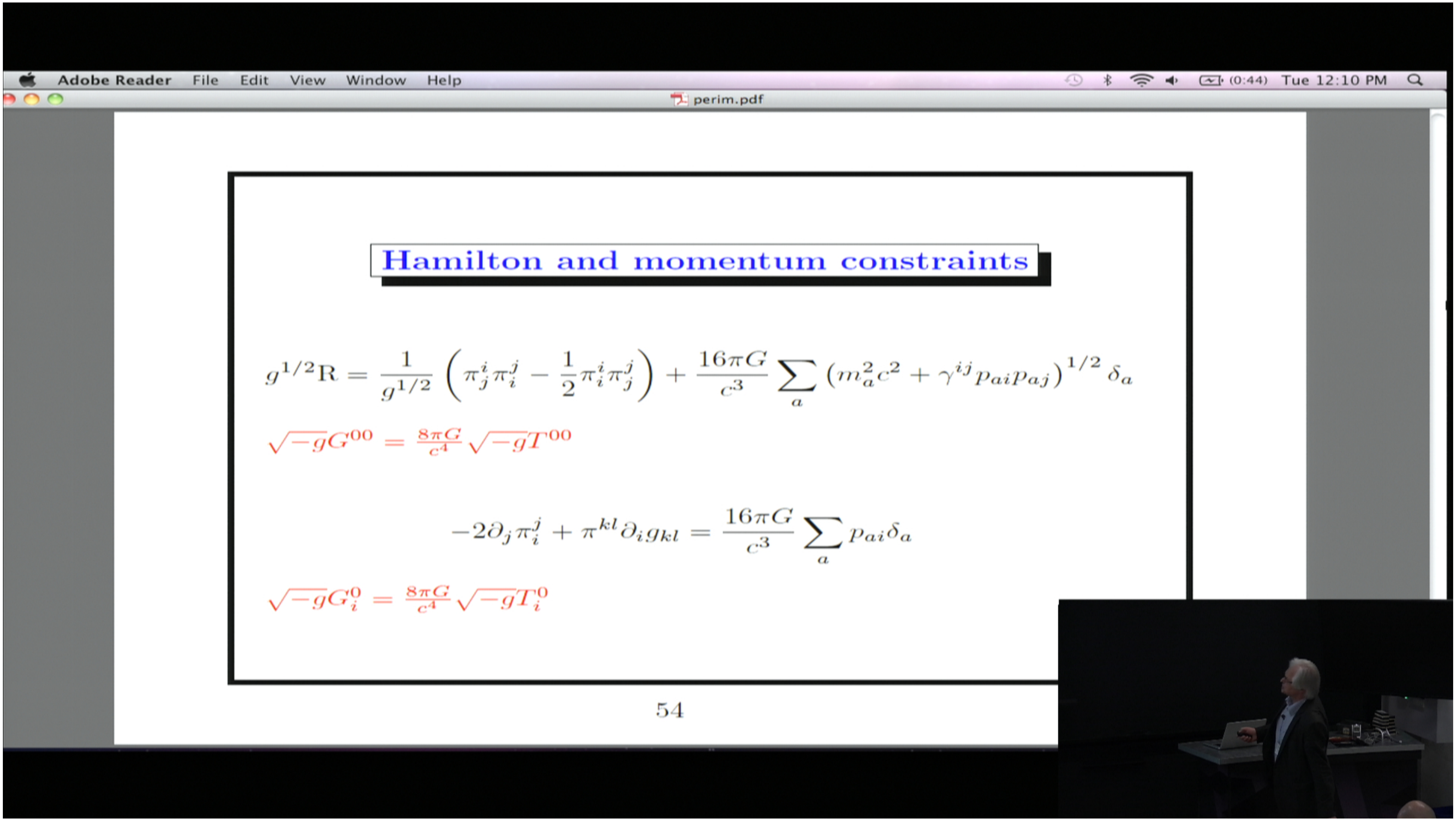
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perim.pdf
Hamilton and momentum constraints
$g^{1/2}\mathrm{R} = \frac{1}{g^{1/2}}\left(\pi^i_j\pi^j_i - \frac{1}{2}\pi^i_i\pi^j_j\right) + \frac{16\pi G}{c^3}\sum_a \left(m_a^2c^2 + \gamma^{ij}p_{ai}p_{aj}\right)^{1/2}\delta_a$
$\sqrt{-g}G^{00} = \frac{8\pi G}{c^4}\sqrt{-g}T^{00}$
$-2\partial_j\pi^j_i + \pi^{kl}\partial_i g_{kl} = \frac{16\pi G}{c^3}\sum_a p_{ai}\delta_a$
$\sqrt{-g}G^0_i = \frac{8\pi G}{c^4}\sqrt{-g}T^0_i$
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isolated BH

$$ds^2 = -\left(\frac{1-\frac{Gm}{2rc^2}}{1+\frac{Gm}{2rc^2}}\right)^2 c^2dt^2 + \left(1+\frac{Gm}{2rc^2}\right)^4 \delta_{ij}dx^i dx^j$$

$$= -\left(\frac{1-\frac{Gm}{2Rc^2}}{1+\frac{Gm}{2Rc^2}}\right)^2 c^2dt^2 + \left(1+\frac{Gm}{2Rc^2}\right)^4 \delta_{ij}dX^i dX^j$$

symmetry transformation (inversion): $Rr = \left(\frac{Gm}{2c^2}\right)^2$

$R^2 = X^iX^i, \quad r^2 = x^ix^i$

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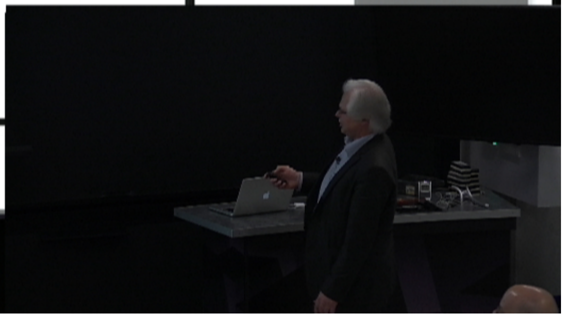


Brill-Lindquist BHs

maximally sliced

$$ds^2 = -\left(\frac{1 - \frac{\beta_1 G}{2r_1c^2} - \frac{\beta_2 G}{2r_2c^2}}{1 + \frac{\alpha_1 G}{2r_1c^2} + \frac{\alpha_2 G}{2r_2c^2}}\right)^2 c^2dt^2 + \left(1 + \frac{\alpha_1 G}{2r_1c^2} + \frac{\alpha_2 G}{2r_2c^2}\right)^4 d\mathbf{x}^2$$

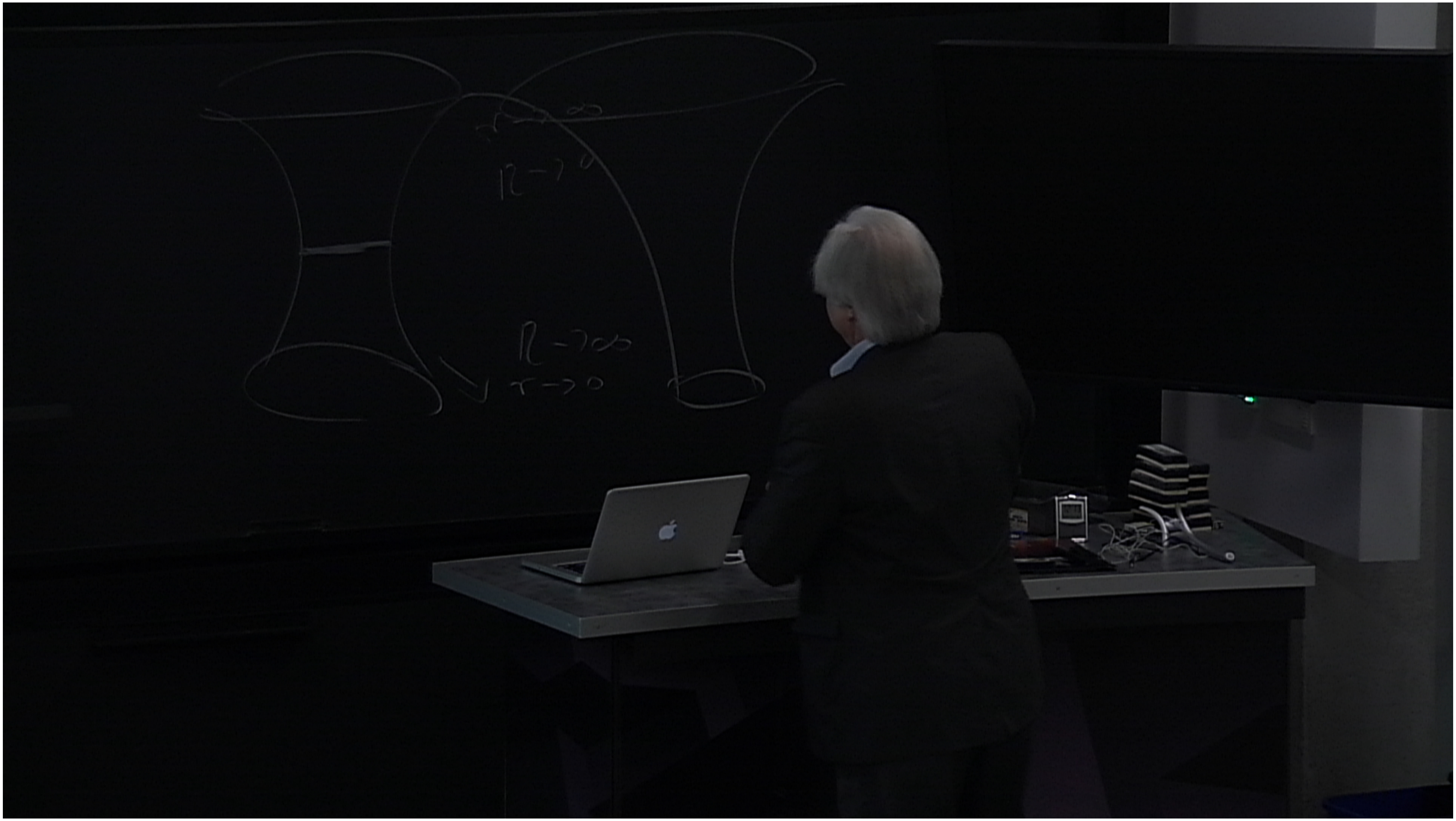
total energy: $$E_{ADM} = -\frac{c^4}{2\pi G}\oint_{i_0} ds_i\partial_i\Psi = (\alpha_1 + \alpha_2)c^2$$

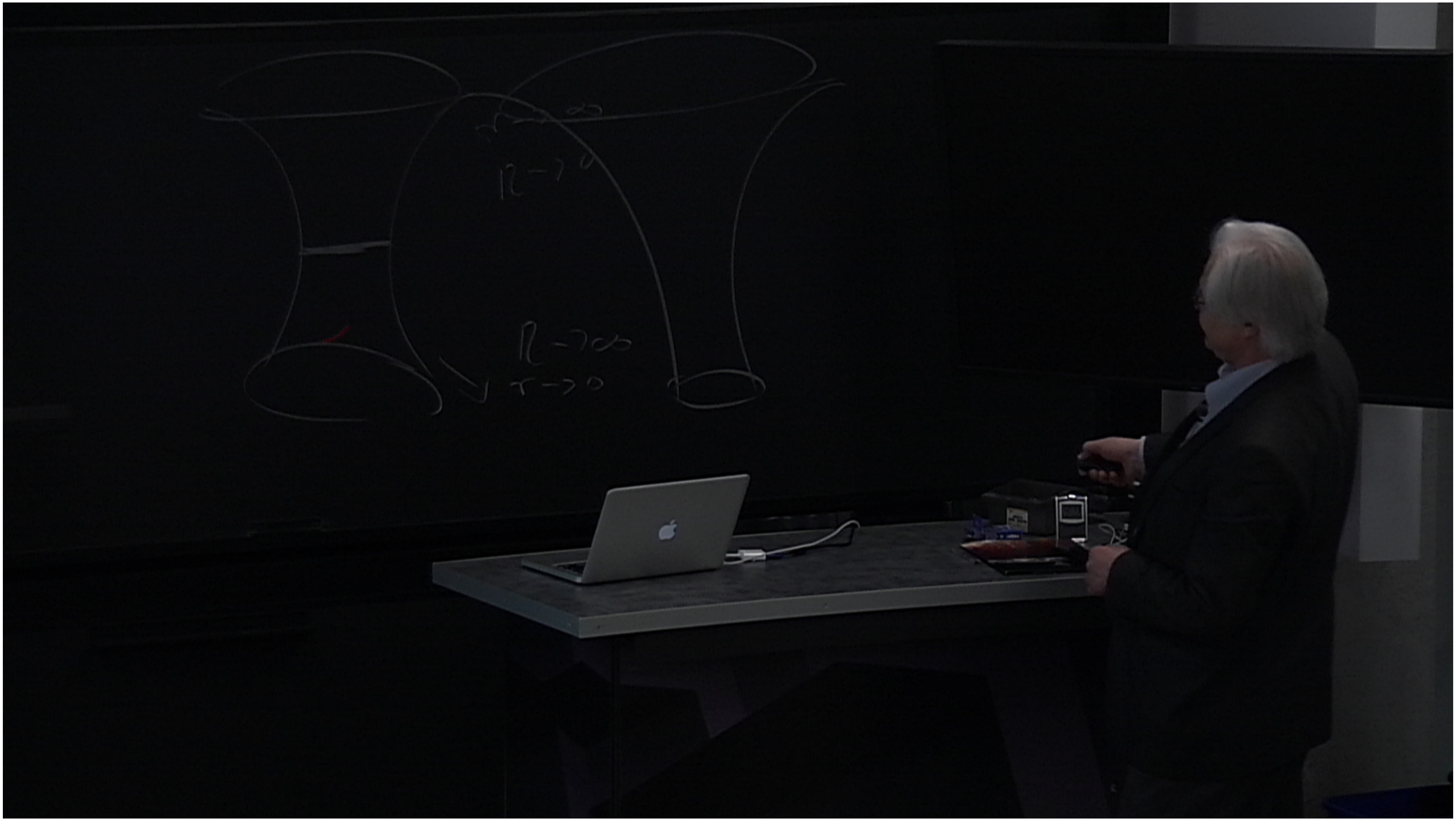
$$\Psi = 1 + \frac{\alpha_1 G}{2r_1c^2} + \frac{\alpha_2 G}{2r_2c^2}$$

inversion map of the three-metric at the throat of black hole 1

$$\mathbf{r}_1' = \mathbf{r}_1\alpha_1^2G^2/4c^4r_1^2$$

$$\mathbf{r}_1' = \mathbf{x}' - \mathbf{x}_1, \quad \mathbf{r}_1 = \mathbf{x} - \mathbf{x}_1, \quad r_1 = |\mathbf{x} - \mathbf{x}_1|$$





$$dl^2 = \Psi^4 d\mathbf{x}^2 = \left(1 + \frac{\alpha_1 G}{2r_1 c^2} + \frac{\alpha_2 G}{2r_2 c^2}\right)^4 d\mathbf{x}^2$$

$$dl^2 = \Psi'^4 d\mathbf{x}'^2 = \left(1 + \frac{\alpha_1 G}{2r'_1 c^2} + \frac{\alpha_1 \alpha_2 G^2}{4r_2 r'_1 c^4}\right)^4 d\mathbf{x}'^2$$

$$\mathbf{r}_2 = \frac{\alpha_1^2 G^2}{4c^4}\frac{\mathbf{r}'_1}{r'^2_1} + \mathbf{r}_{12}, \qquad \mathbf{r}_{12} = \mathbf{r}_1 - \mathbf{r}_2$$

$$m_1 = -\frac{c^2}{2\pi G}\oint_{i_0} ds'_i \partial'_i \Psi' = \alpha_1 + \frac{\alpha_1 \alpha_2 G}{2r_{12} c^2}$$

$$\Psi' = 1 + \frac{\alpha_1 G}{2r'_1 c^2} + \frac{\alpha_1 \alpha_2 G^2}{4r_2 r'_1 c^4}$$

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$$-\left(1+\frac{1}{8}\,\phi\right)\Delta\phi = \frac{16\pi G}{c^2}\sum_a m_a\delta_a \qquad (h_{ij}^{\mathrm{TT}} = \pi^{ij\mathrm{TT}} = p_{ai} = 0)$$

$$\phi = \frac{4G}{c^2}\left(\frac{\alpha_1}{r_1}+\frac{\alpha_2}{r_2}\right)$$

$$\alpha_a = m_a - \frac{m_a+m_b}{2} + \frac{c^2 r_{ab}}{G}\left(\sqrt{1+\frac{m_a+m_b}{c^2 r_{ab}/G}+\left(\frac{m_a-m_b}{2c^2 r_{ab}/G}\right)^2}-1\right)$$

$$H_{\mathrm{BL}} = (\alpha_1+\alpha_2)\;c^2 = (m_1+m_2)\;c^2 - G\;\frac{\alpha_1\alpha_2}{r_{12}}$$

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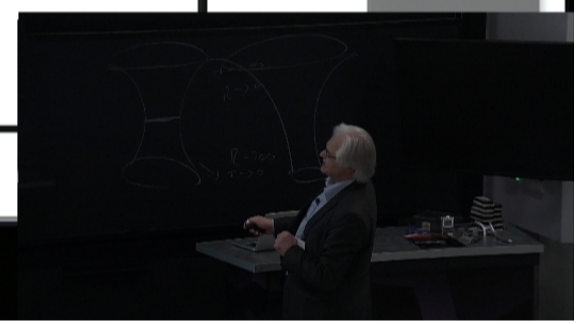

Metric in d-dimensional conformally flat space:

$$g_{ij} = \left(1 + \frac{1}{4}\frac{d-2}{d-1}\phi\right)^{\frac{4}{d-2}} \delta_{ij}$$

$$\phi = \frac{4G}{c^2}\frac{\Gamma(\frac{d-2}{2})}{\pi^{\frac{d-2}{2}}}\left(\frac{\alpha_1}{r_1^{d-2}} + \frac{\alpha_2}{r_2^{d-2}}\right)$$

$$\Psi = 1 + \frac{1}{4}\frac{d-2}{d-1}\phi$$

$$-\Delta^{-1}\delta = \frac{\Gamma((d-2)/2)}{4\pi^{d/2}} r^{2-d}$$

$$\Psi = 1 + \frac{G(d-2)\Gamma((d-2)/2)}{c^2(d-1)\pi^{(d-2)/2}} \left(\frac{\alpha_1}{r_1^{d-2}} + \frac{\alpha_2}{r_2^{d-2}} \right)$$

$$\left(1 + \frac{G(d-2)\Gamma((d-2)/2)}{c^2(d-1)\pi^{(d-2)/2}} \left(\frac{\alpha_1}{r_1^{d-2}} + \frac{\alpha_2}{r_2^{d-2}} \right)\right) \alpha_1 \delta_1 = m_1 \delta_1$$

$1 < d < 2$

$$\left(1 + \frac{G(d-2)\Gamma((d-2)/2)}{c^2(d-1)\pi^{(d-2)/2}} \frac{\alpha_2}{r_{12}^{d-2}} \right) \alpha_1 \delta_1 = m_1 \delta_1$$

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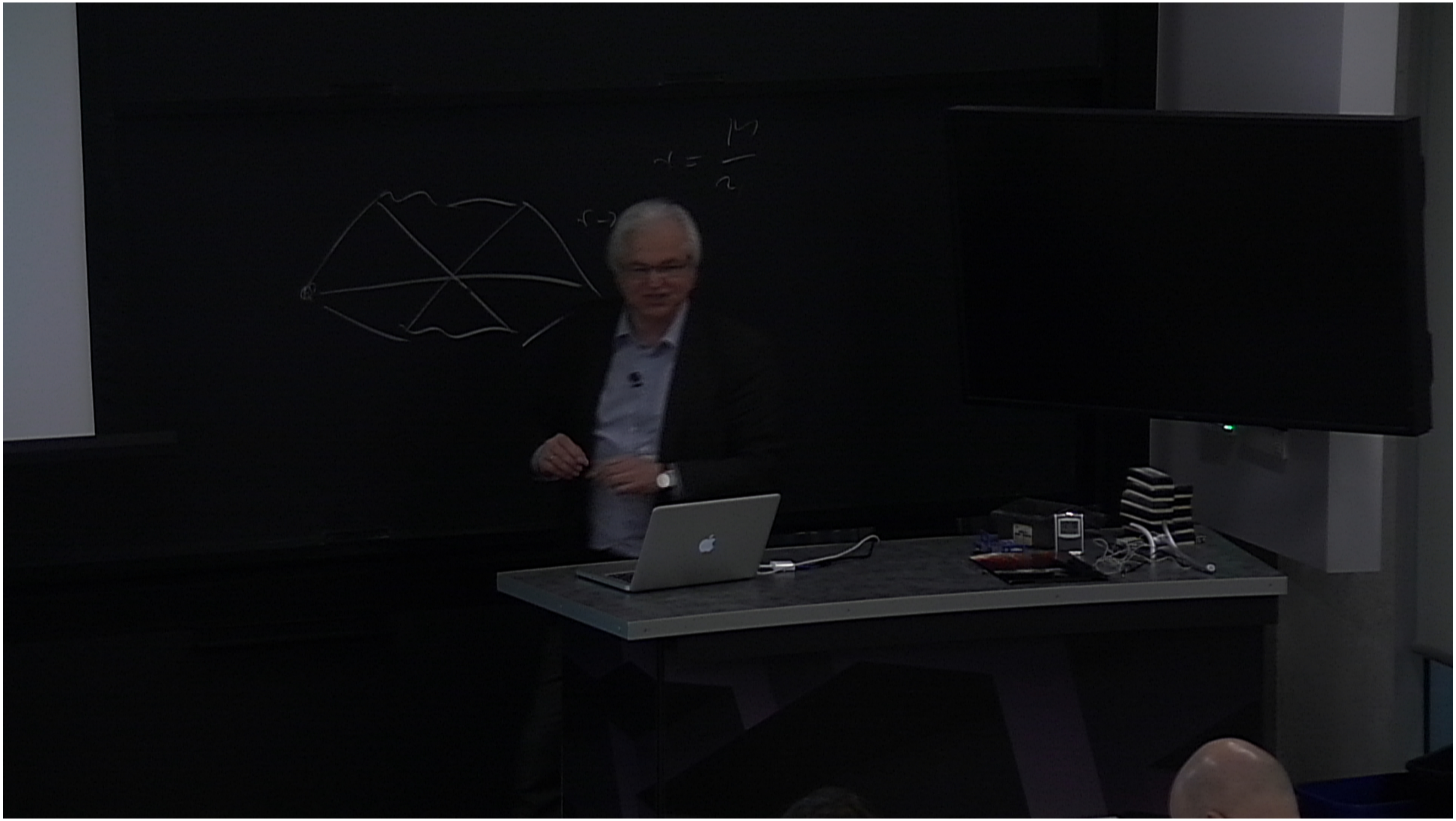

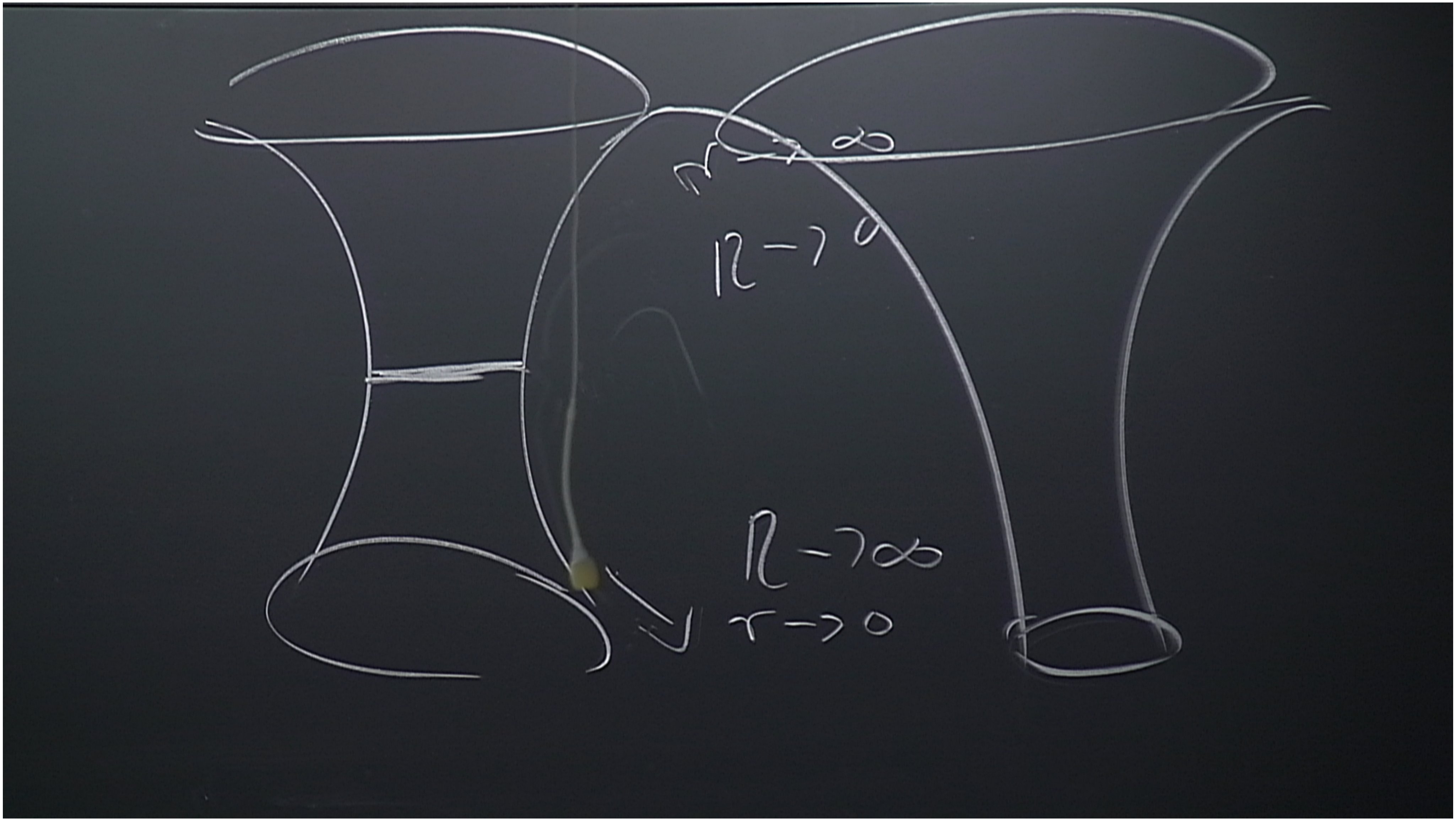
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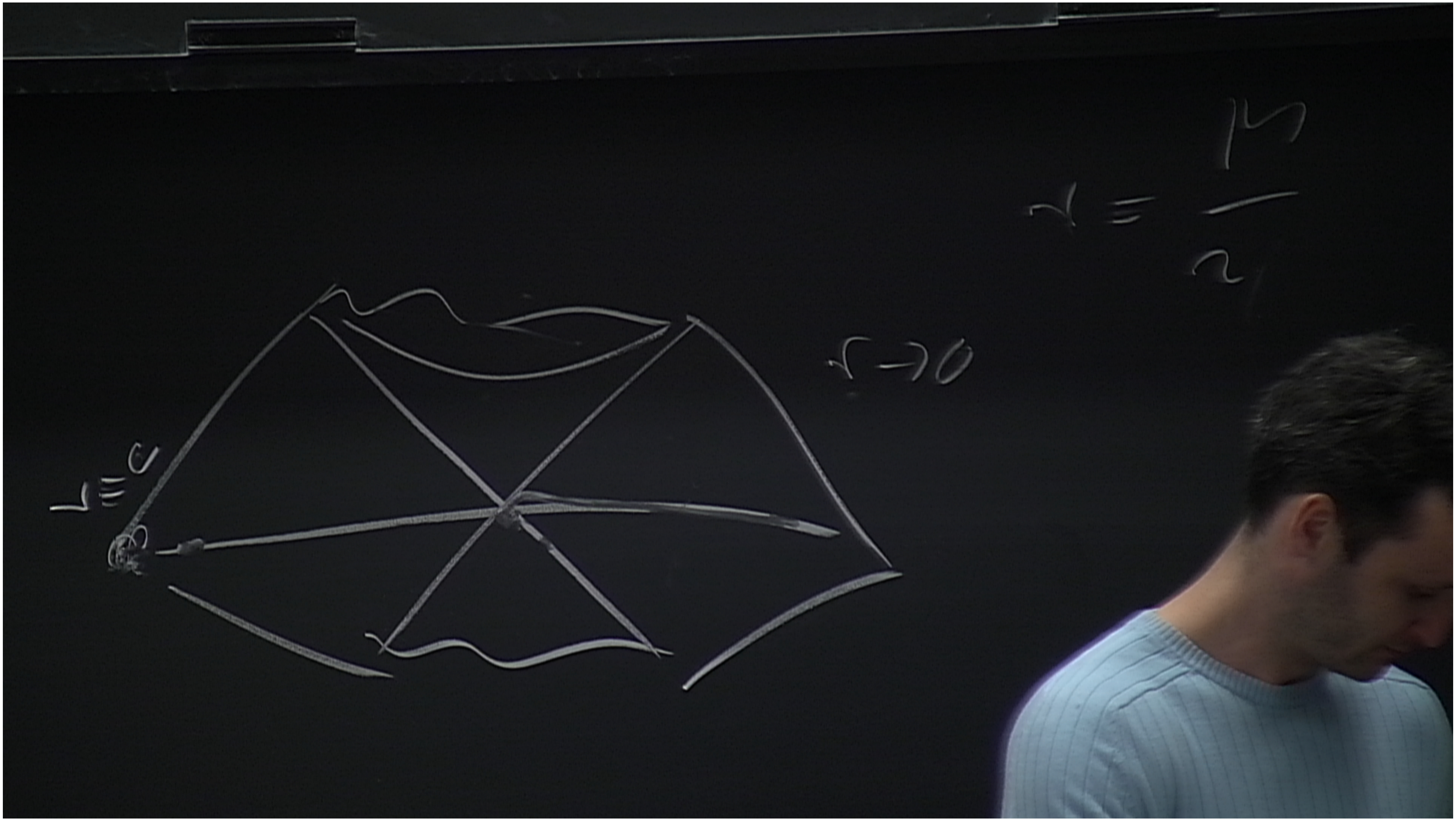
R → 0
R → ∞
r → 0


r ≡ M/2
r → 0