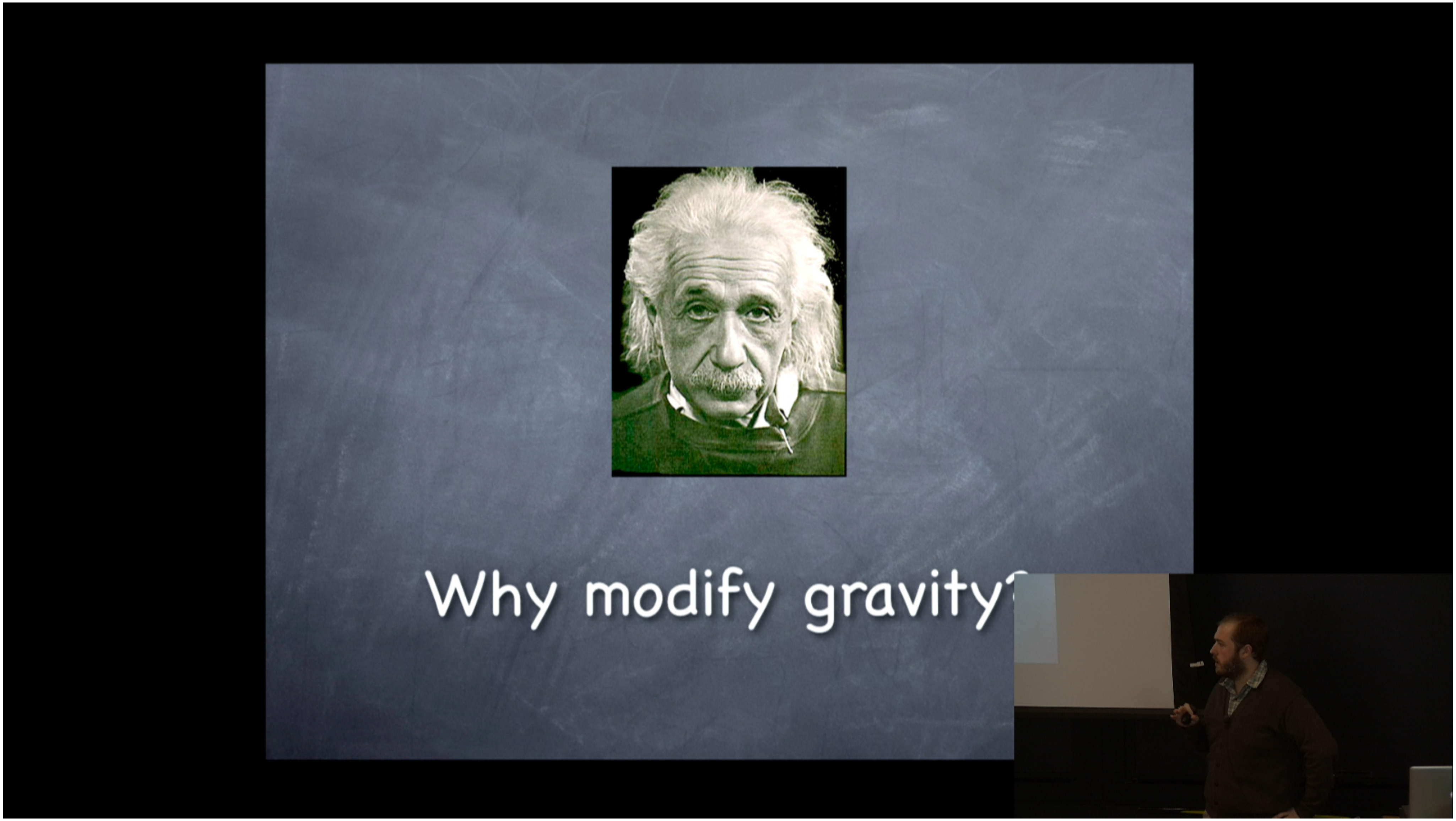
Title: Spatially Covariant Theories of a Transverse, Traceless Graviton

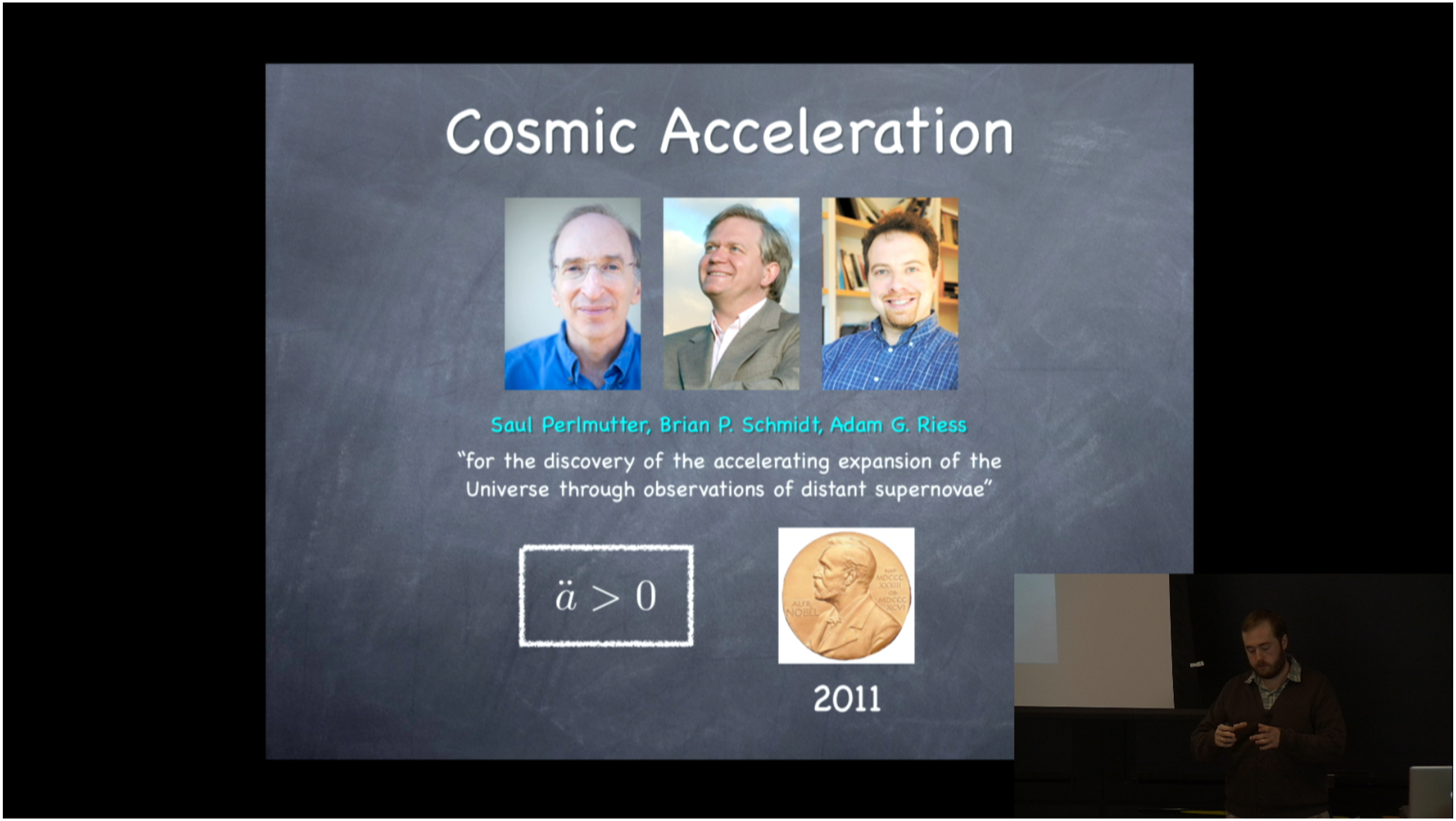
Date: Nov 15, 2011 11:00 AM

URL: http://pirsa.org/11110077

Abstract: General relativity is a covariant theory of two transverse, traceless graviton degrees of freedom. According to a theorem of Hojman, Kuchar, and Teitelboim, modifications of general relativity must either introduce new degrees of freedom or violate the principle of general covariance. In my talk, I will discuss modifications of general relativity that retain the same number of gravitational degrees of freedom, and therefore explicitly break general covariance. Motivated by cosmology, the modifications of interest maintain spatial covariance. Demanding consistency of the theory forces the physical Hamiltonian density to obey an analogue of the renormalization group equation, which encodes the invariance of the theory under flow through the space of conformally equivalent spatial metrics.



Why modify gravity?



Cosmic Acceleration

Saul Perlmutter, Brian P. Schmidt, Adam G. Riess

"for the discovery of the accelerating expansion of the Universe through observations of distant supernovae"

$$\boxed{\ddot{a} > 0}$$

2011

Cosmological Constant Problems

Expectation	Apparent Reality
$\rho_\Lambda \approx M_{Pl}^4$	$\rho_\Lambda \approx (meV)^4 \approx 10^{-120} M_{Pl}^4$

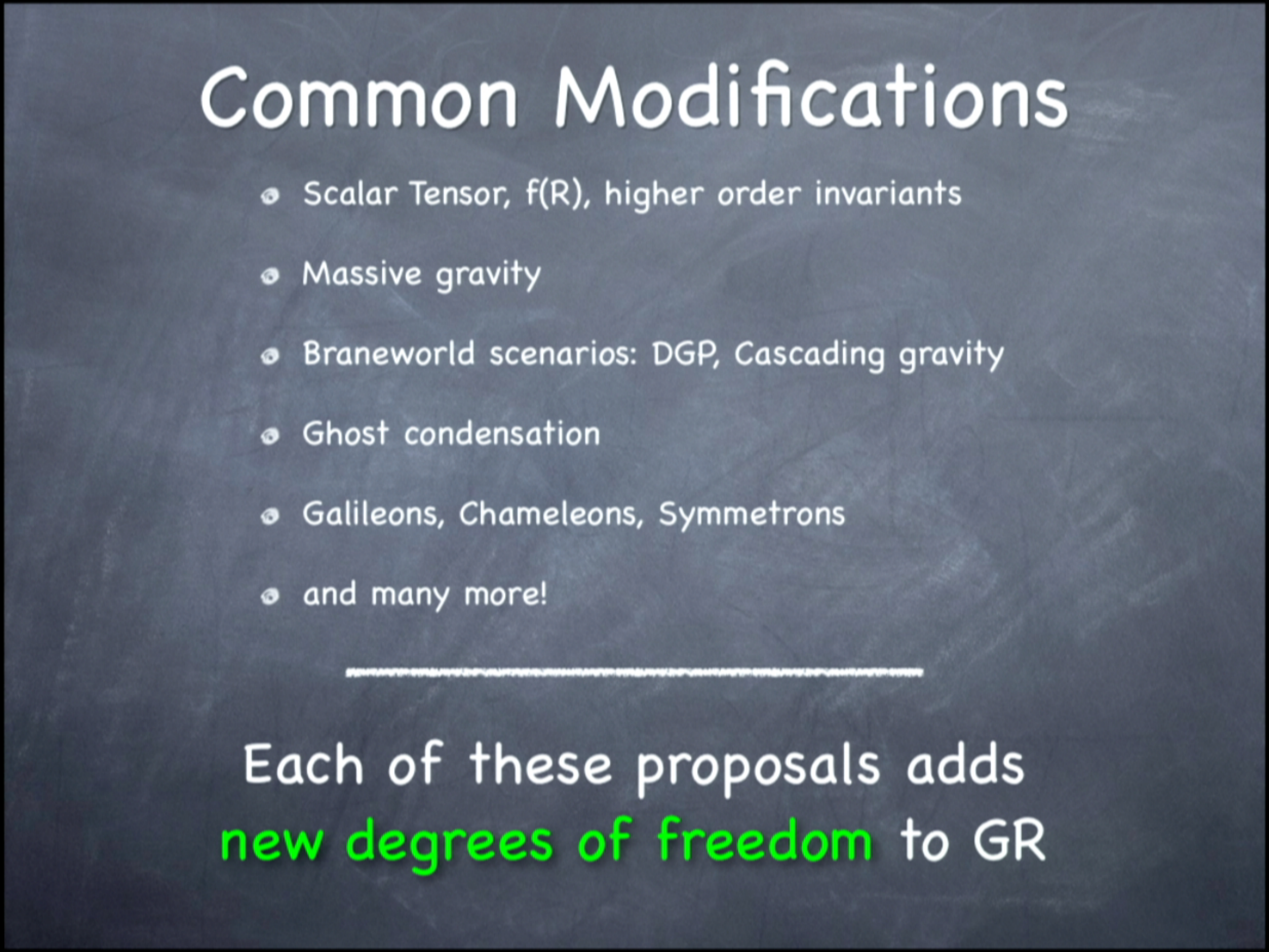
- Cosmological Constant Problem - Why is ρ_Λ so small?
  - Cancellation of zero-point energies requires extreme fine-tuning
  - Weinberg's "No-Go" Theorem: Cannot relax dynamically to $\rho_\Lambda \approx 0$

    S. Weinberg, Rev.Mod.Phys. 61 (1989) 1-23

- Coincidence Problem - Why is the dark energy density comparable to the present matter density?
  - Weinberg's second "No-Go" Theorem: Cannot relax to $\rho_\Lambda \approx \rho_m$ today without extreme fine-tuning S. Weinberg, astro-ph 0005265

Common Modifications

- Scalar Tensor, f(R), higher order invariants
- Massive gravity
- Braneworld scenarios: DGP, Cascading gravity
- Ghost condensation
- Galileons, Chameleons, Symmetrons
- and many more!



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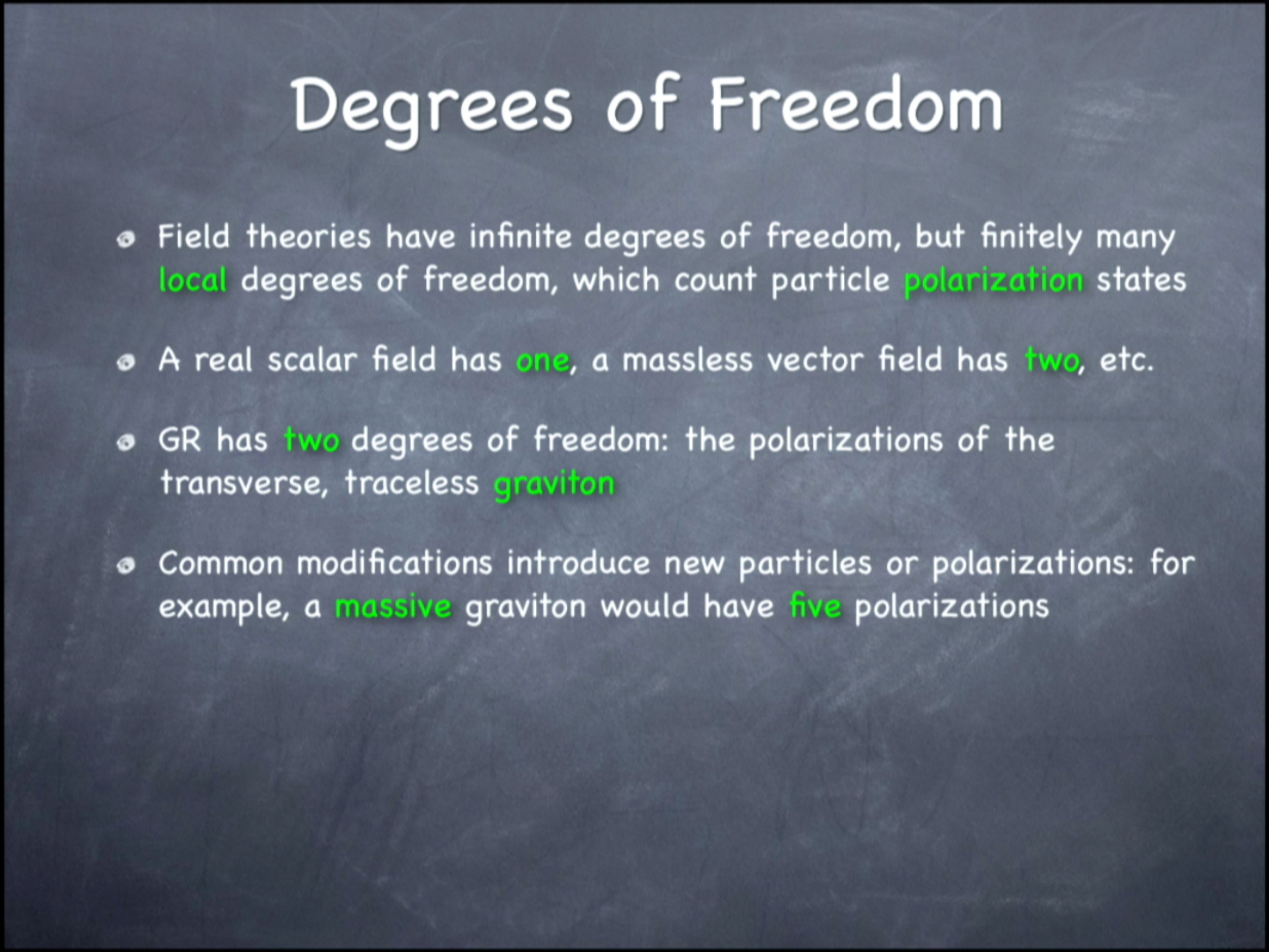
Each of these proposals adds
new degrees of freedom to GR



Degrees of Freedom

- Field theories have infinite degrees of freedom, but finitely many local degrees of freedom, which count particle polarization states
- A real scalar field has one, a massless vector field has two, etc.
- GR has two degrees of freedom: the polarizations of the transverse, traceless graviton
- Common modifications introduce new particles or polarizations: for example, a massive graviton would have five polarizations

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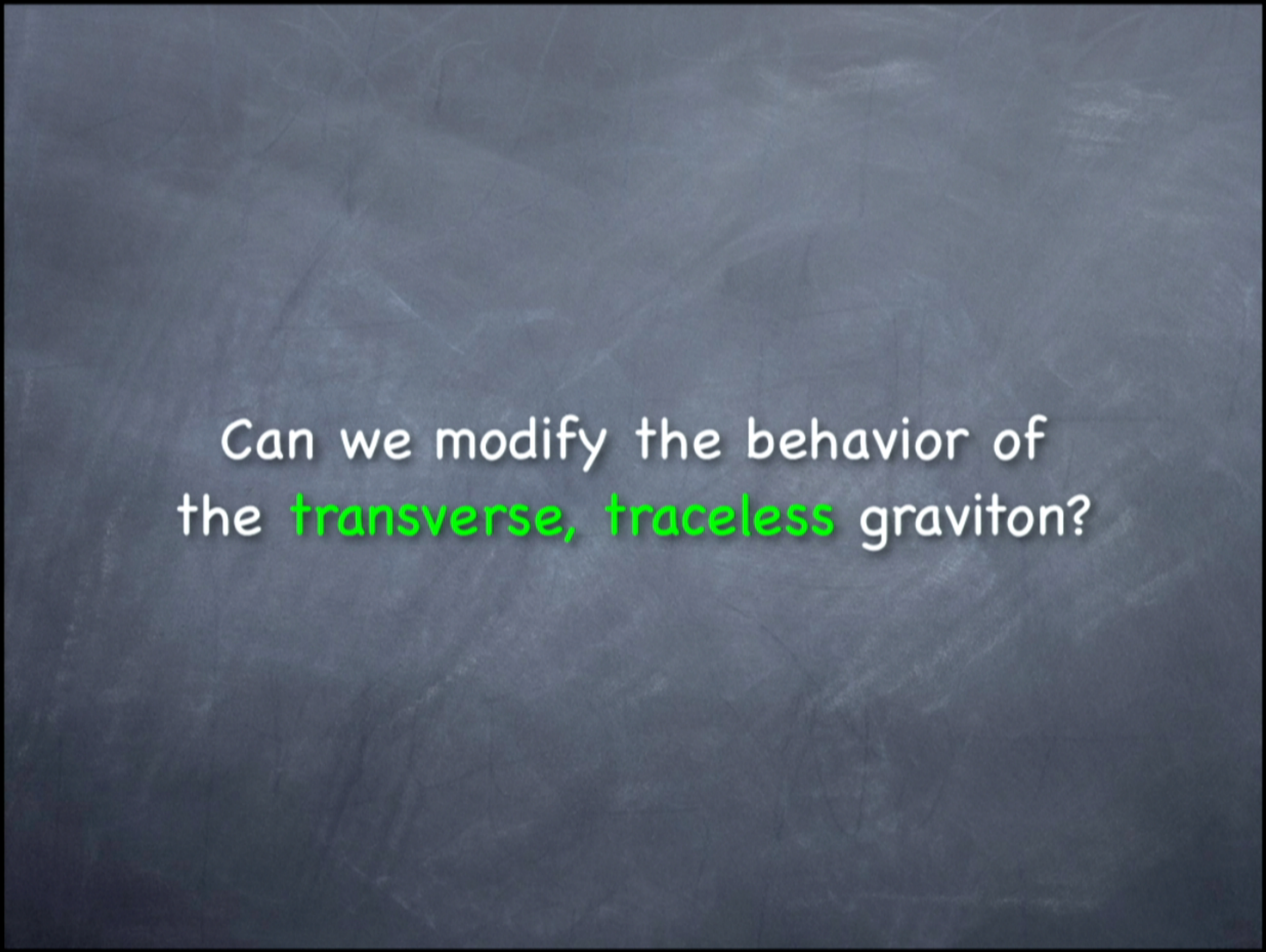
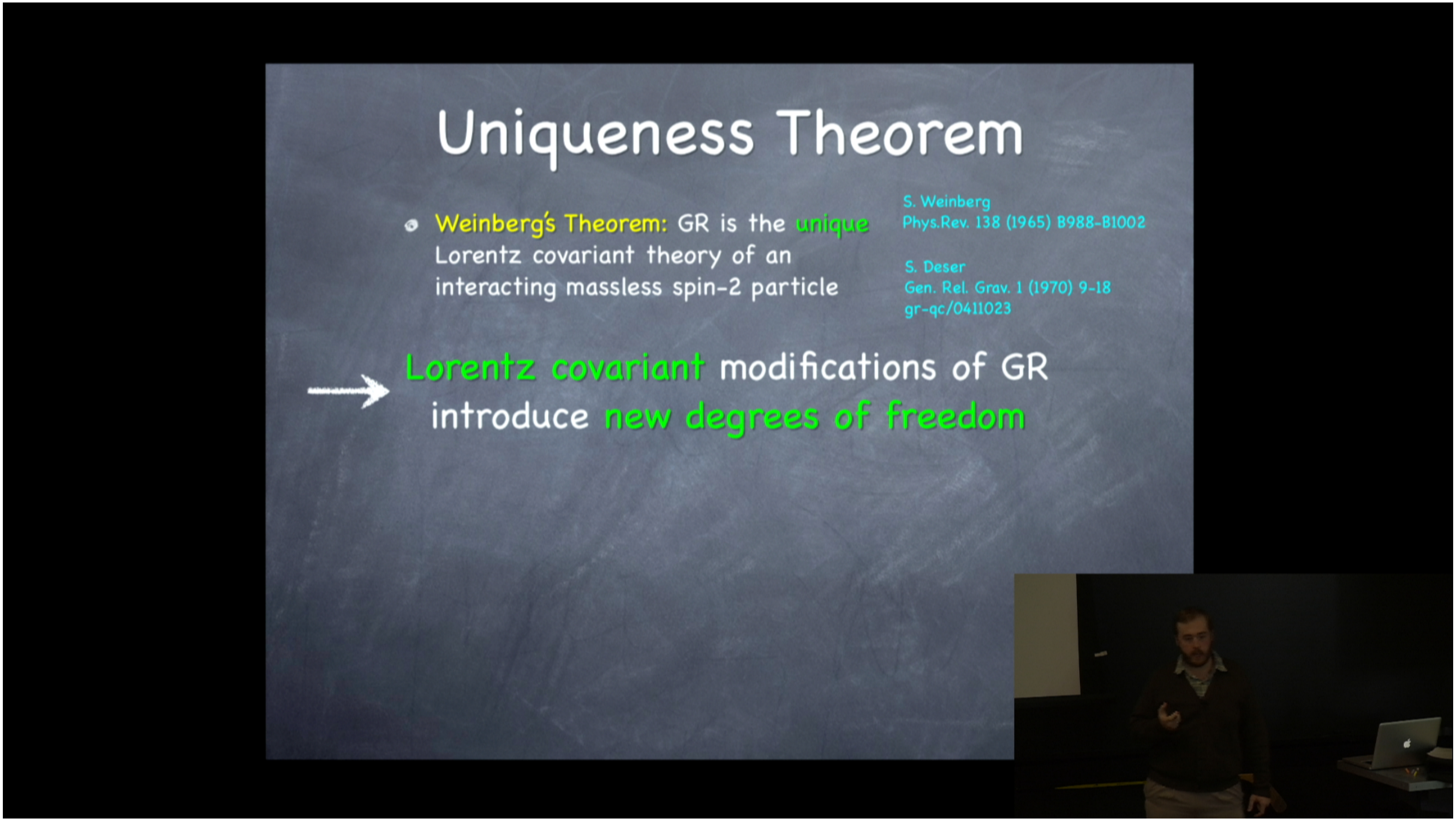
Could General Relativity be modified without new degrees of freedom?



Can we modify the behavior of the transverse, traceless graviton?



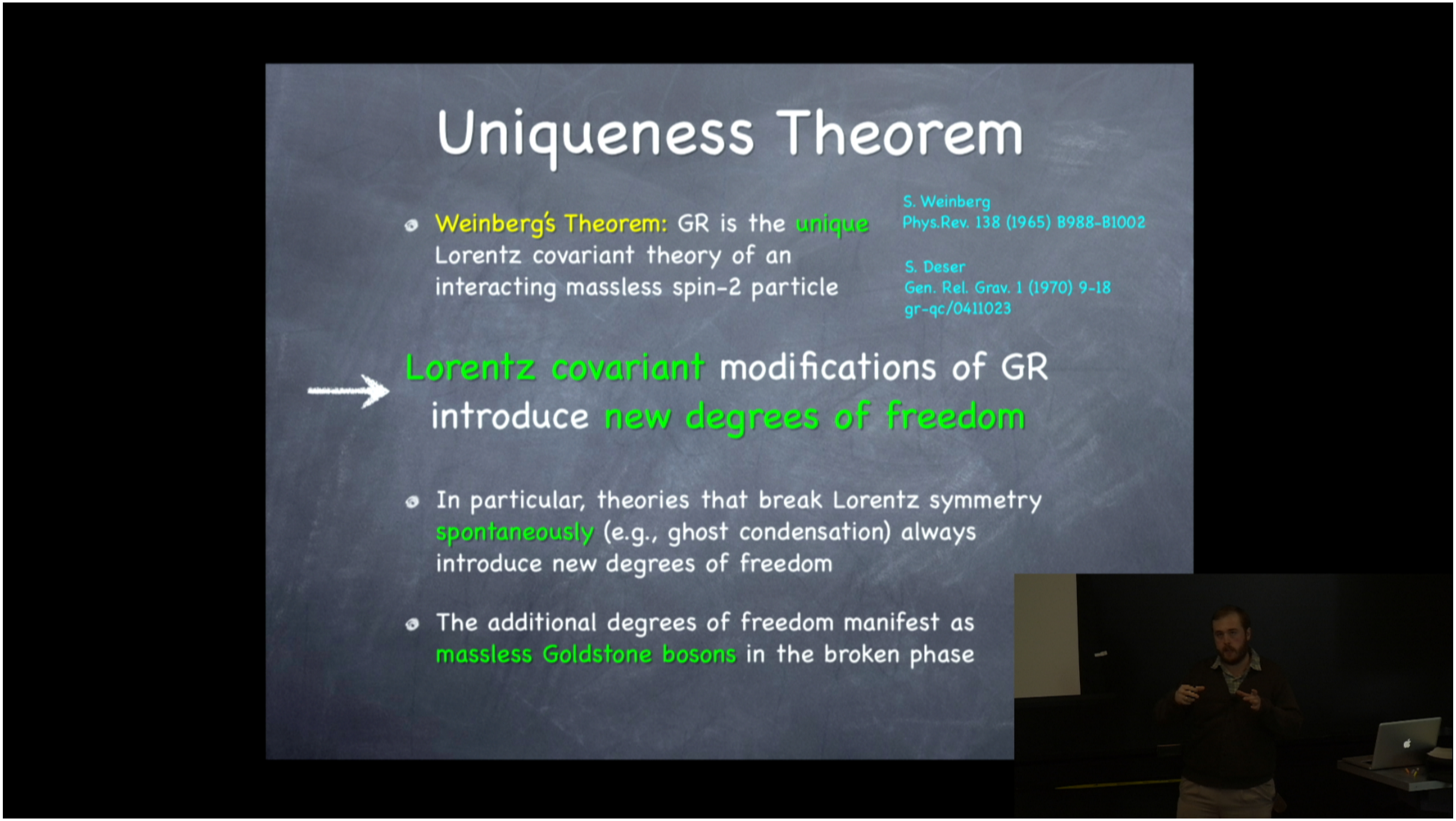
Uniqueness Theorem

- **Weinberg's Theorem:** GR is the unique Lorentz covariant theory of an interacting massless spin-2 particle

S. Weinberg
Phys.Rev. 138 (1965) B988-B1002

S. Deser
Gen. Rel. Grav. 1 (1970) 9-18
gr-qc/0411023

→ Lorentz covariant modifications of GR introduce new degrees of freedom



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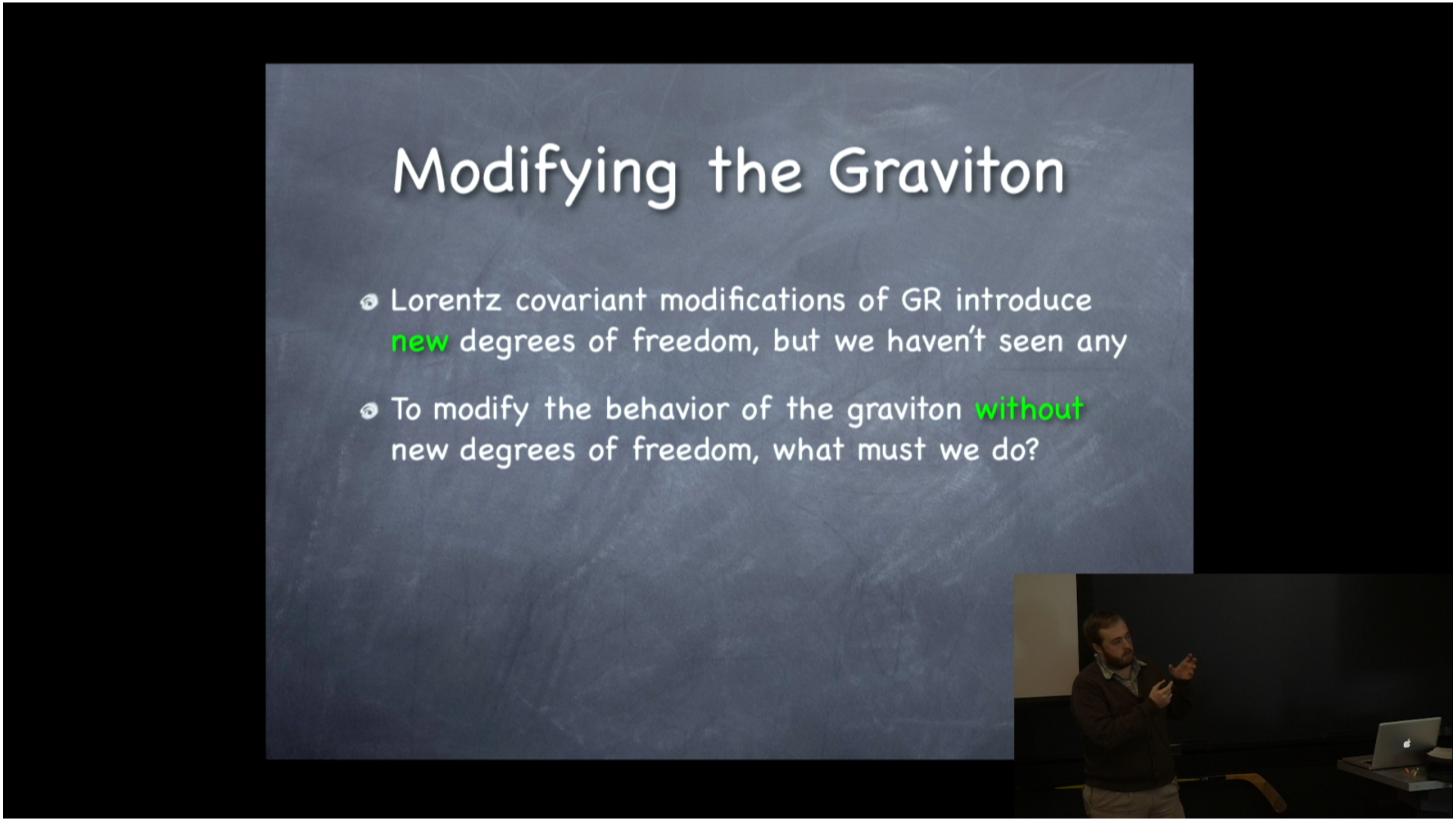
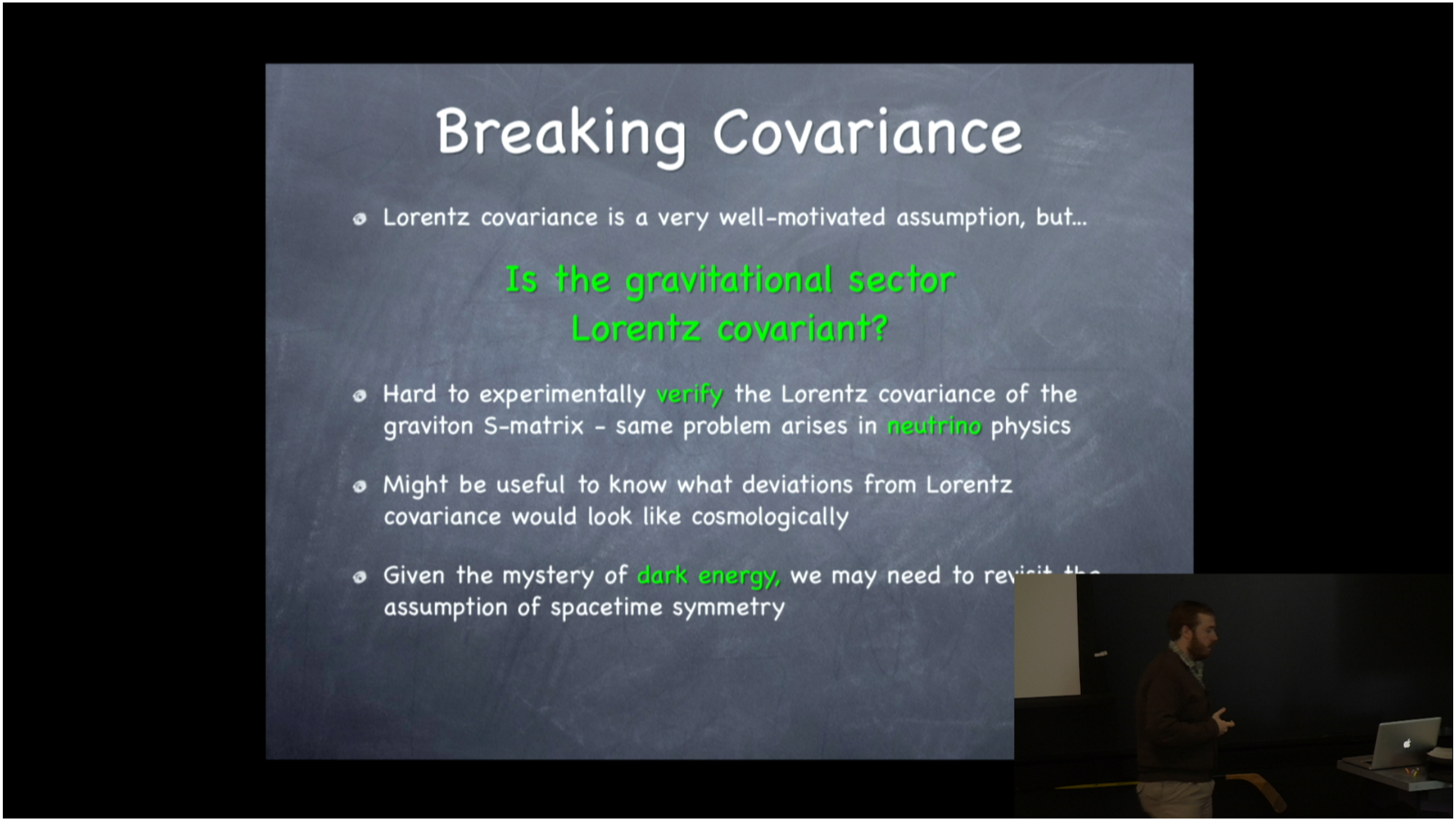
→ Lorentz covariant modifications of GR introduce new degrees of freedom

- In particular, theories that break Lorentz symmetry spontaneously (e.g., ghost condensation) always introduce new degrees of freedom
- The additional degrees of freedom manifest as massless Goldstone bosons in the broken phase



Modifying the Graviton

- Lorentz covariant modifications of GR introduce new degrees of freedom, but we haven't seen any
- To modify the behavior of the graviton without new degrees of freedom, what must we do?



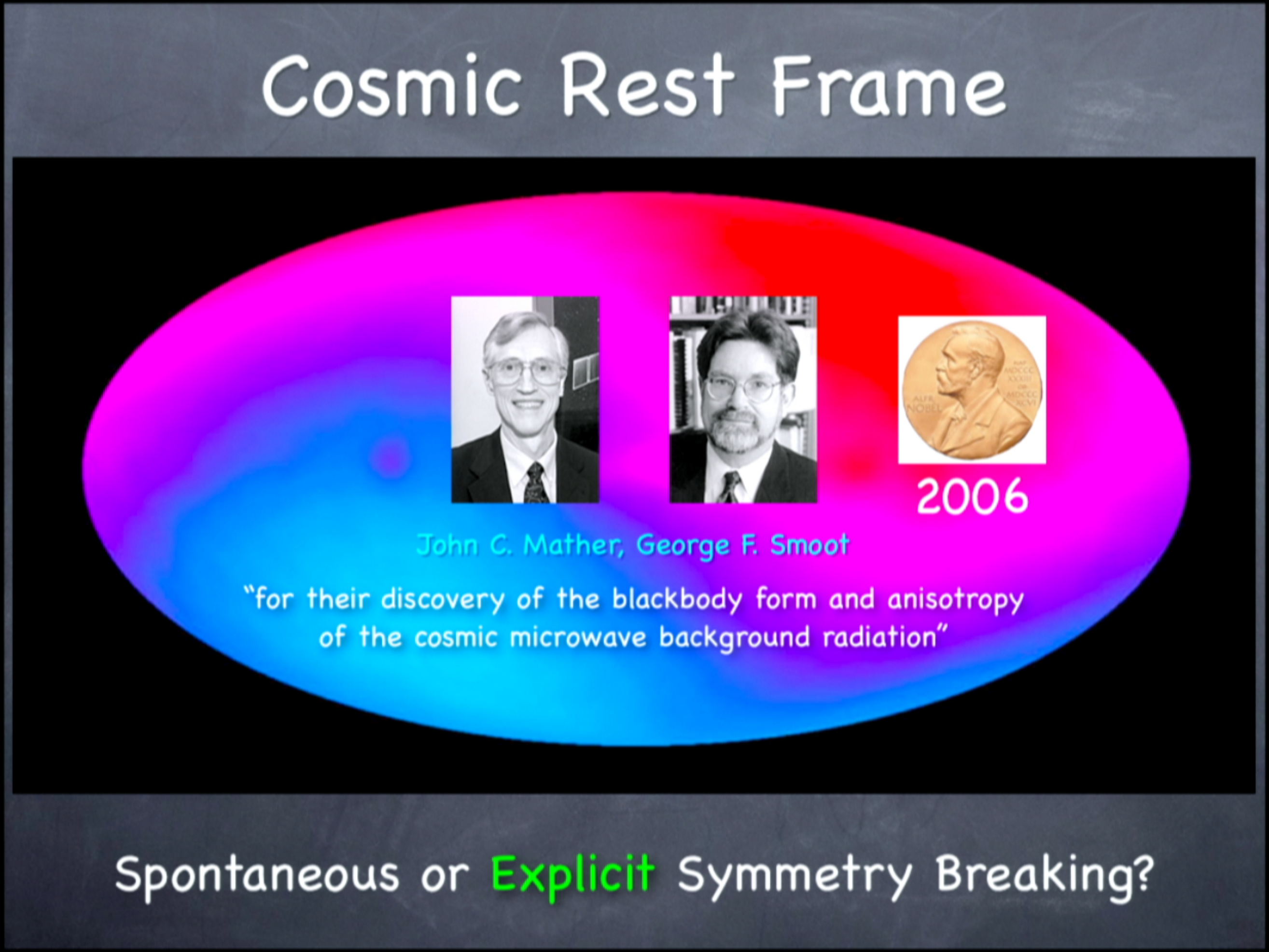
Breaking Covariance

- Lorentz covariance is a very well-motivated assumption, but...

Is the gravitational sector Lorentz covariant?

- Hard to experimentally **verify** the Lorentz covariance of the graviton S-matrix - same problem arises in **neutrino** physics
- Might be useful to know what deviations from Lorentz covariance would look like cosmologically
- Given the mystery of **dark energy,** we may need to rev assumption of spacetime symmetry

Cosmic Rest Frame

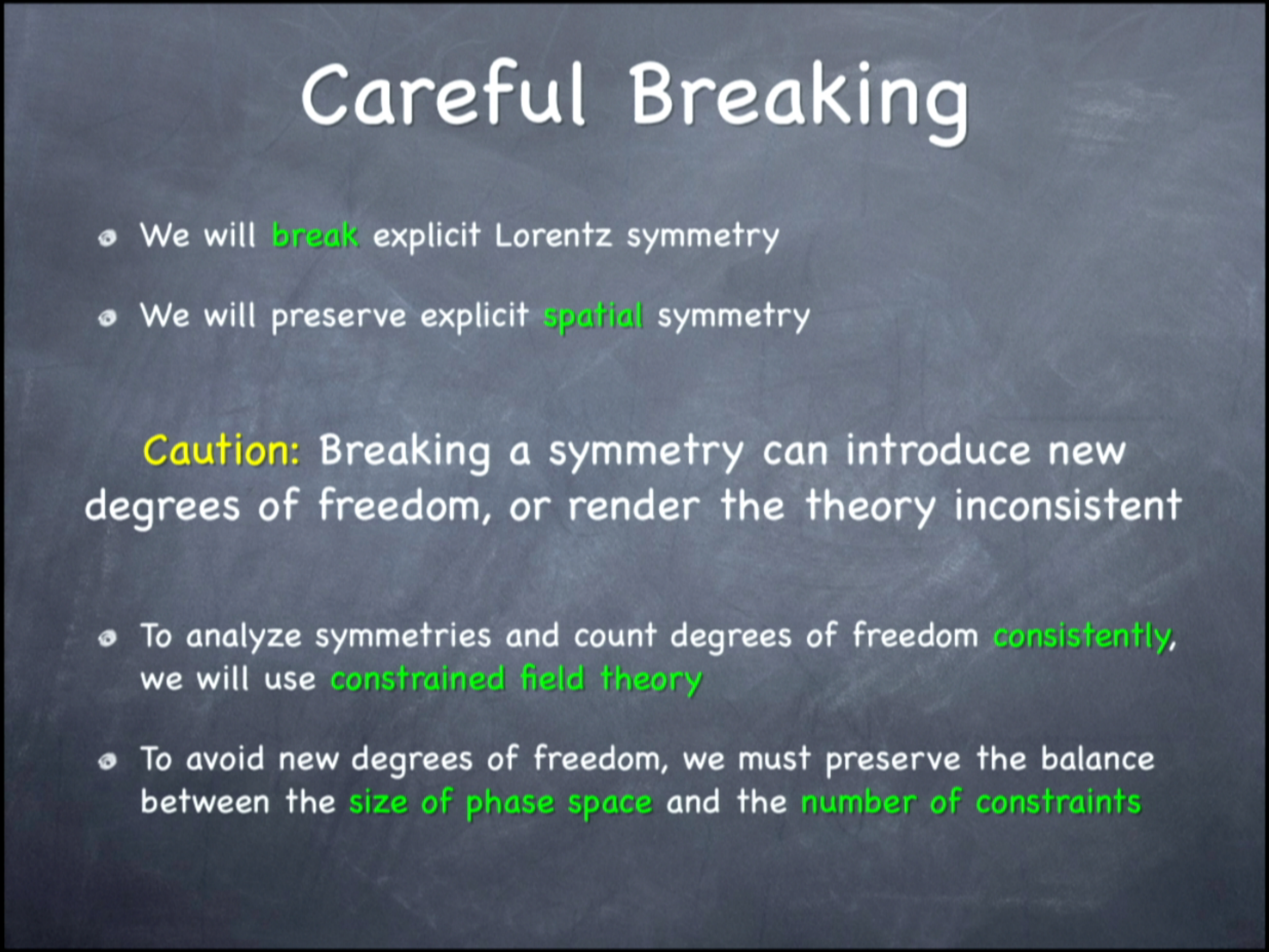


2006

John C. Mather, George F. Smoot

"for their discovery of the blackbody form and anisotropy
of the cosmic microwave background radiation"

Spontaneous or **Explicit** Symmetry Breaking?

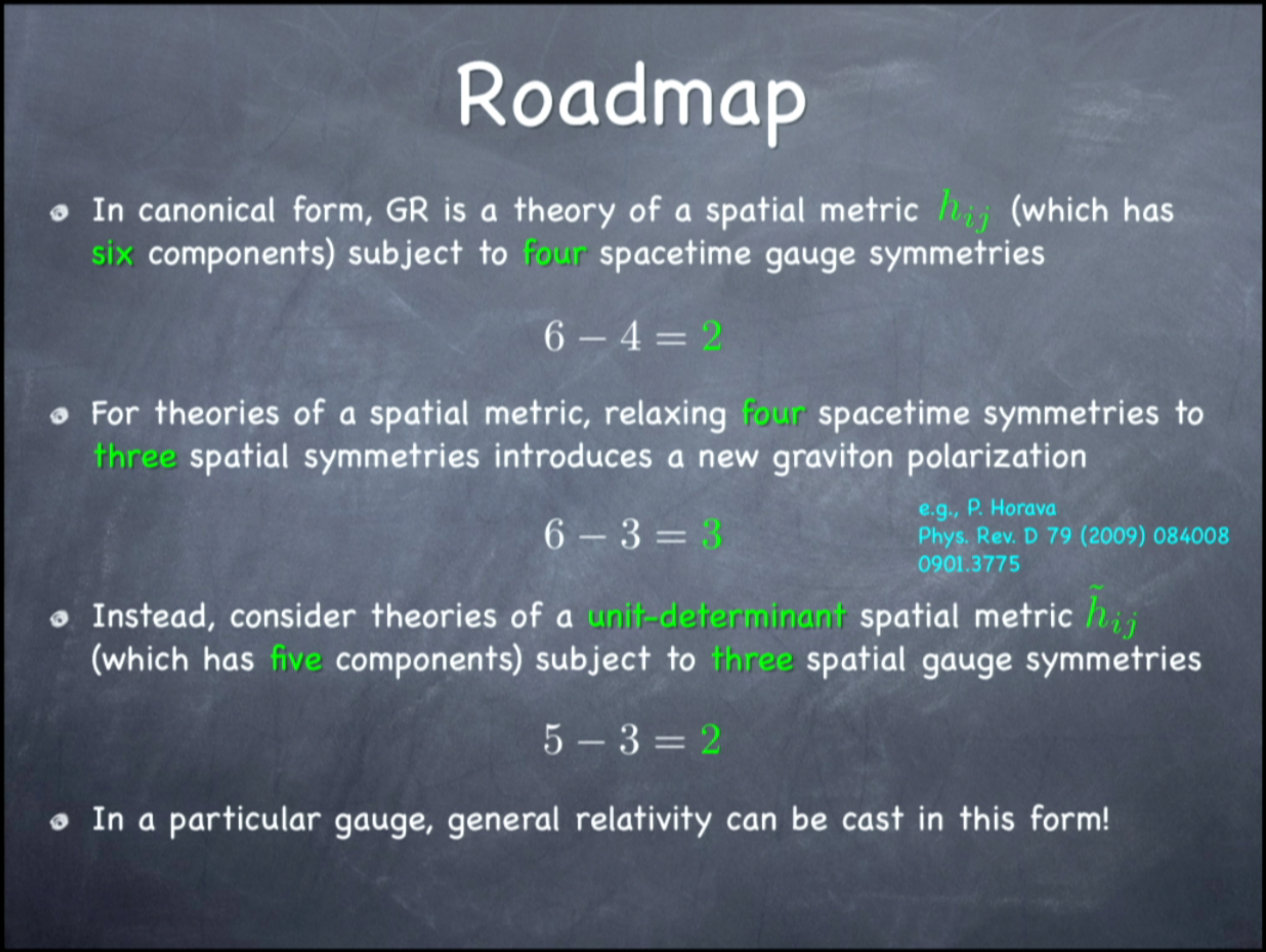


Careful Breaking

- We will break explicit Lorentz symmetry
- We will preserve explicit spatial symmetry

Caution: Breaking a symmetry can introduce new degrees of freedom, or render the theory inconsistent

- To analyze symmetries and count degrees of freedom consistently, we will use constrained field theory
- To avoid new degrees of freedom, we must preserve the balance between the size of phase space and the number of constraints



Roadmap

- In canonical form, GR is a theory of a spatial metric h_{ij} (which has six components) subject to four spacetime gauge symmetries

$$6 - 4 = 2$$

- For theories of a spatial metric, relaxing four spacetime symmetries to three spatial symmetries introduces a new graviton polarization

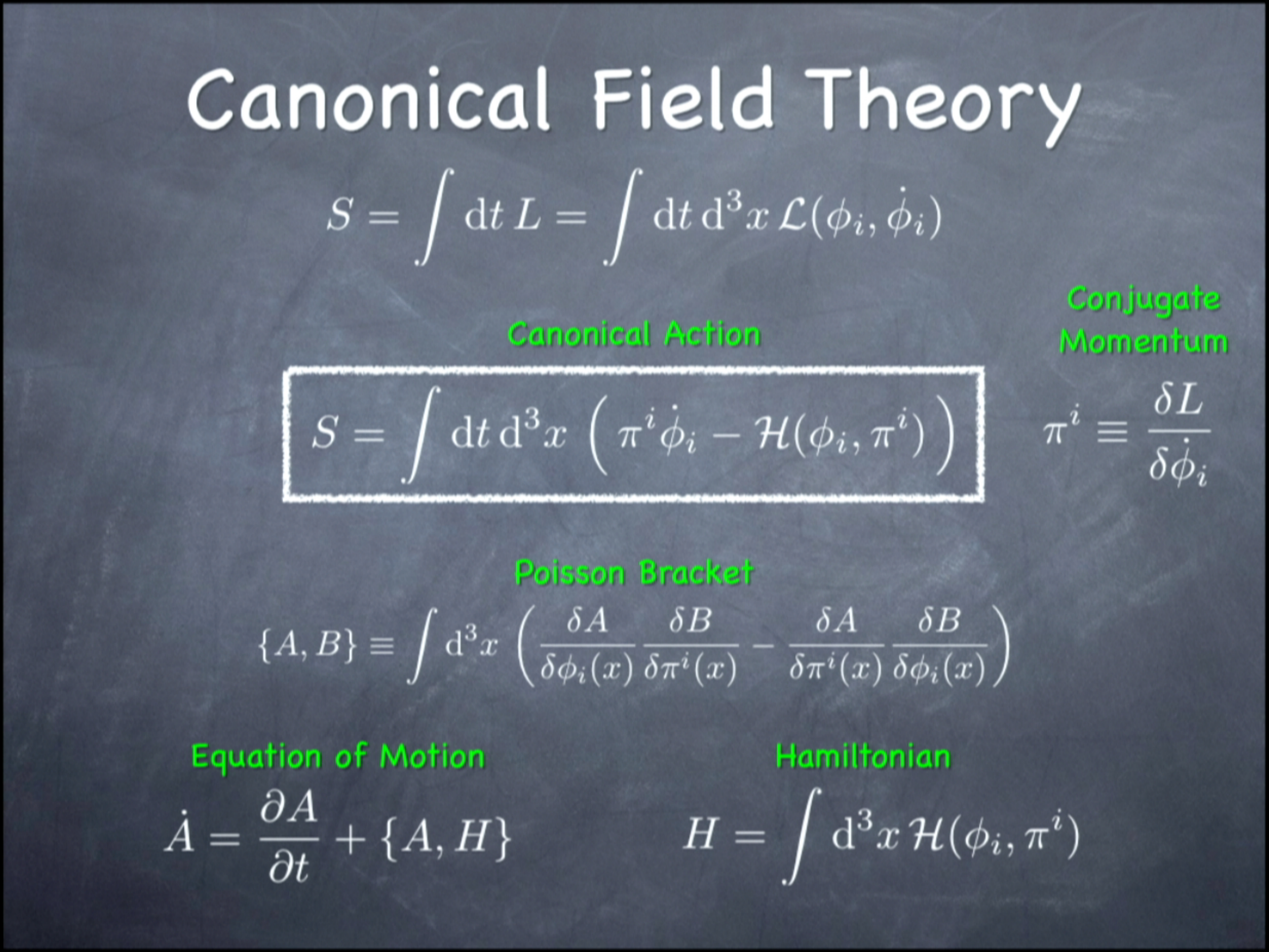
$$6 - 3 = 3$$

e.g., P. Horava
Phys. Rev. D 79 (2009) 084008
0901.3775

- Instead, consider theories of a unit-determinant spatial metric $\tilde{h}_{ij}$ (which has five components) subject to three spatial gauge symmetries

$$5 - 3 = 2$$

- In a particular gauge, general relativity can be cast in this form!



Canonical Field Theory

$$S = \int \mathrm{d}t\, L = \int \mathrm{d}t\, \mathrm{d}^3x\, \mathcal{L}(\phi_i, \dot{\phi}_i)$$

Canonical Action

$$\boxed{S = \int \mathrm{d}t\, \mathrm{d}^3x\, \left(\pi^i \dot{\phi}_i - \mathcal{H}(\phi_i, \pi^i) \right)}$$

Conjugate Momentum

$$\pi^i \equiv \frac{\delta L}{\delta \dot{\phi}_i}$$

Poisson Bracket

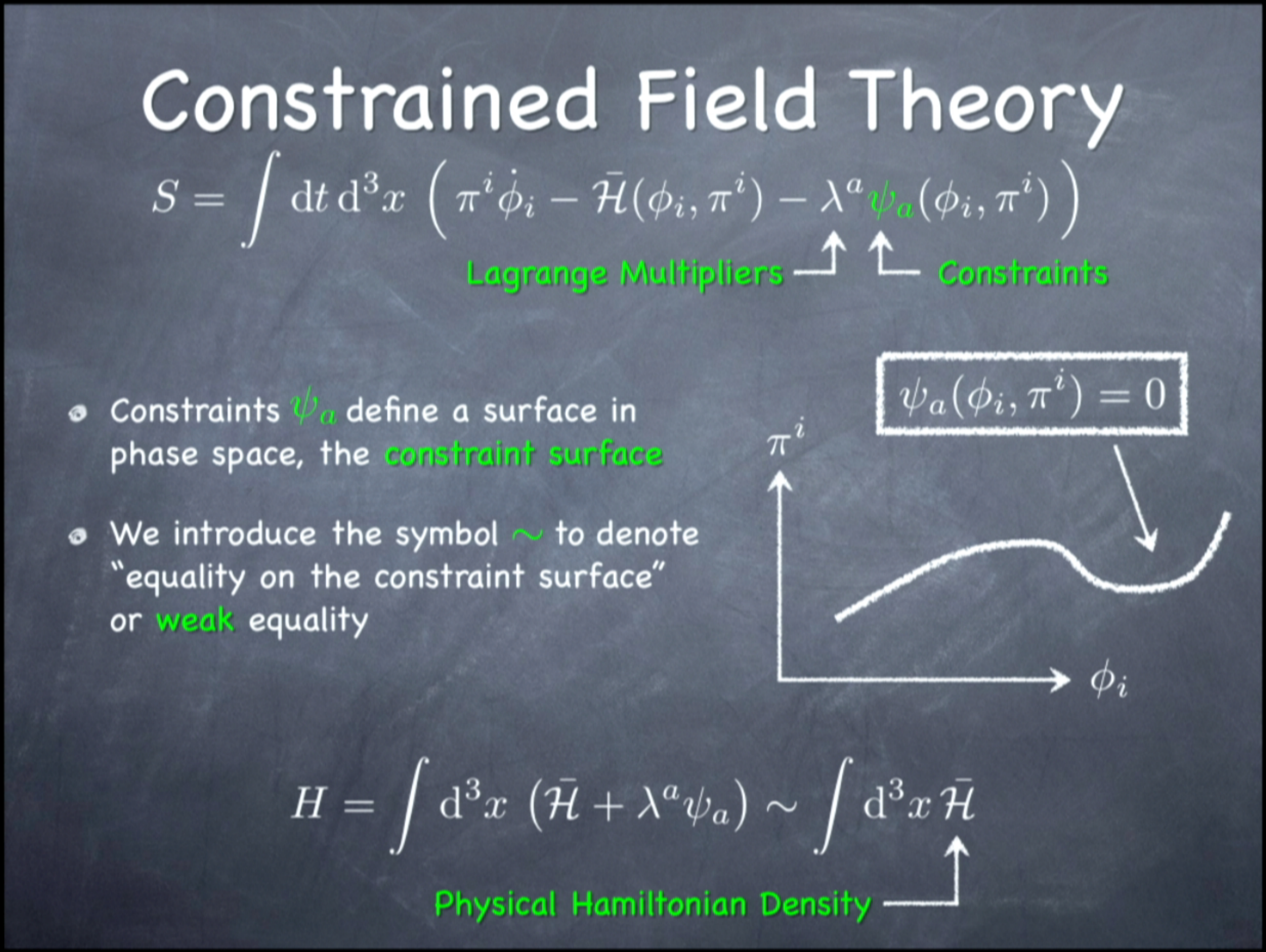
$$\{A, B\} \equiv \int \mathrm{d}^3x \left(\frac{\delta A}{\delta \phi_i(x)} \frac{\delta B}{\delta \pi^i(x)} - \frac{\delta A}{\delta \pi^i(x)} \frac{\delta B}{\delta \phi_i(x)} \right)$$

Equation of Motion

$$\dot{A} = \frac{\partial A}{\partial t} + \{A, H\}$$

Hamiltonian

$$H = \int \mathrm{d}^3x\, \mathcal{H}(\phi_i, \pi^i)$$



Constrained Field Theory

$$S = \int \mathrm{d}t\, \mathrm{d}^3x \, \left(\pi^i \dot{\phi}_i - \bar{\mathcal{H}}(\phi_i, \pi^i) - \lambda^a \psi_a(\phi_i, \pi^i) \right)$$

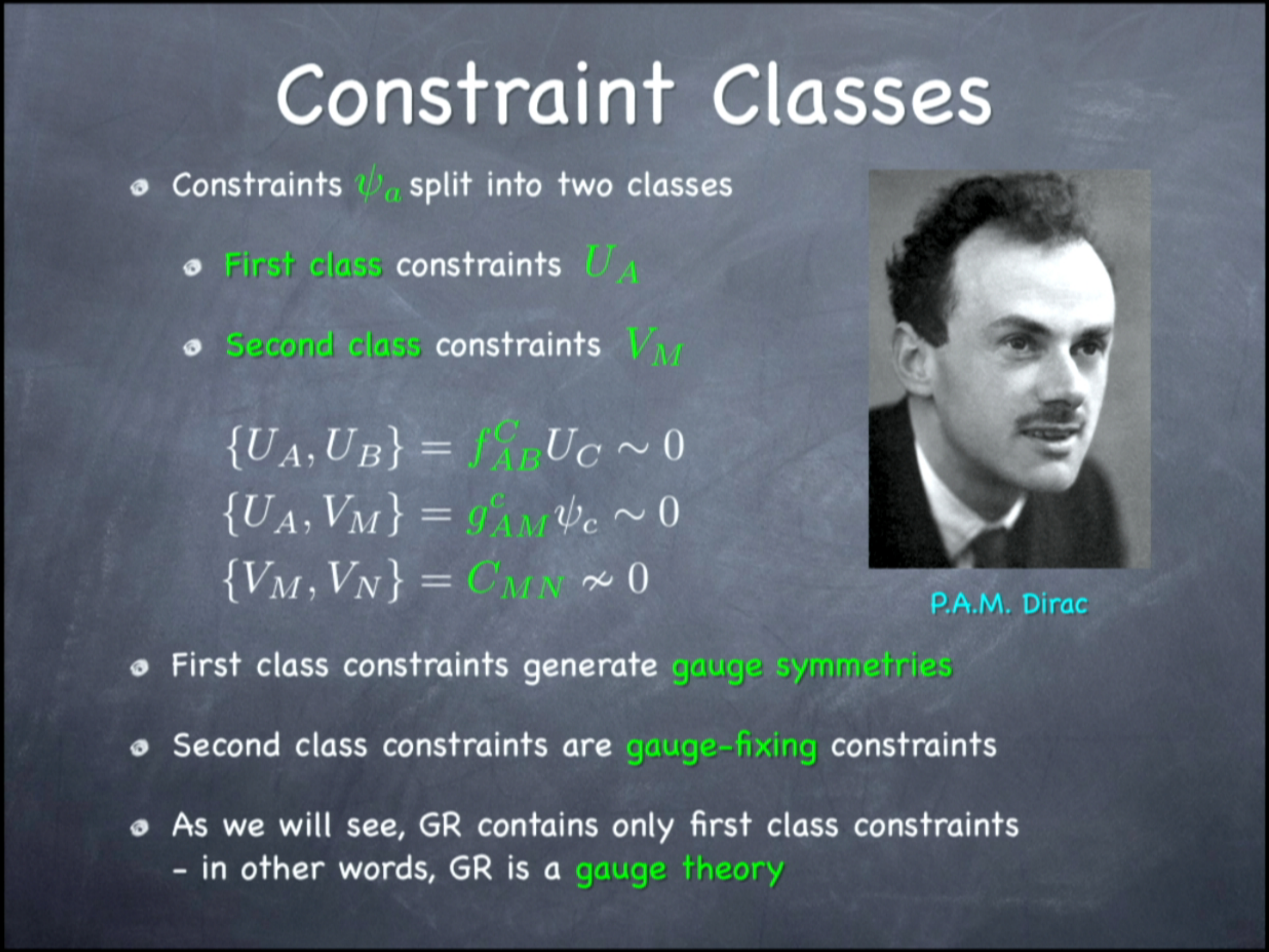
Lagrange Multipliers — Constraints

- Constraints ψ_a define a surface in phase space, the constraint surface
- We introduce the symbol $\sim$ to denote "equality on the constraint surface" or weak equality

$$\boxed{\psi_a(\phi_i, \pi^i) = 0}$$

$$H = \int \mathrm{d}^3x \, (\bar{\mathcal{H}} + \lambda^a \psi_a) \sim \int \mathrm{d}^3x \, \bar{\mathcal{H}}$$

Physical Hamiltonian Density



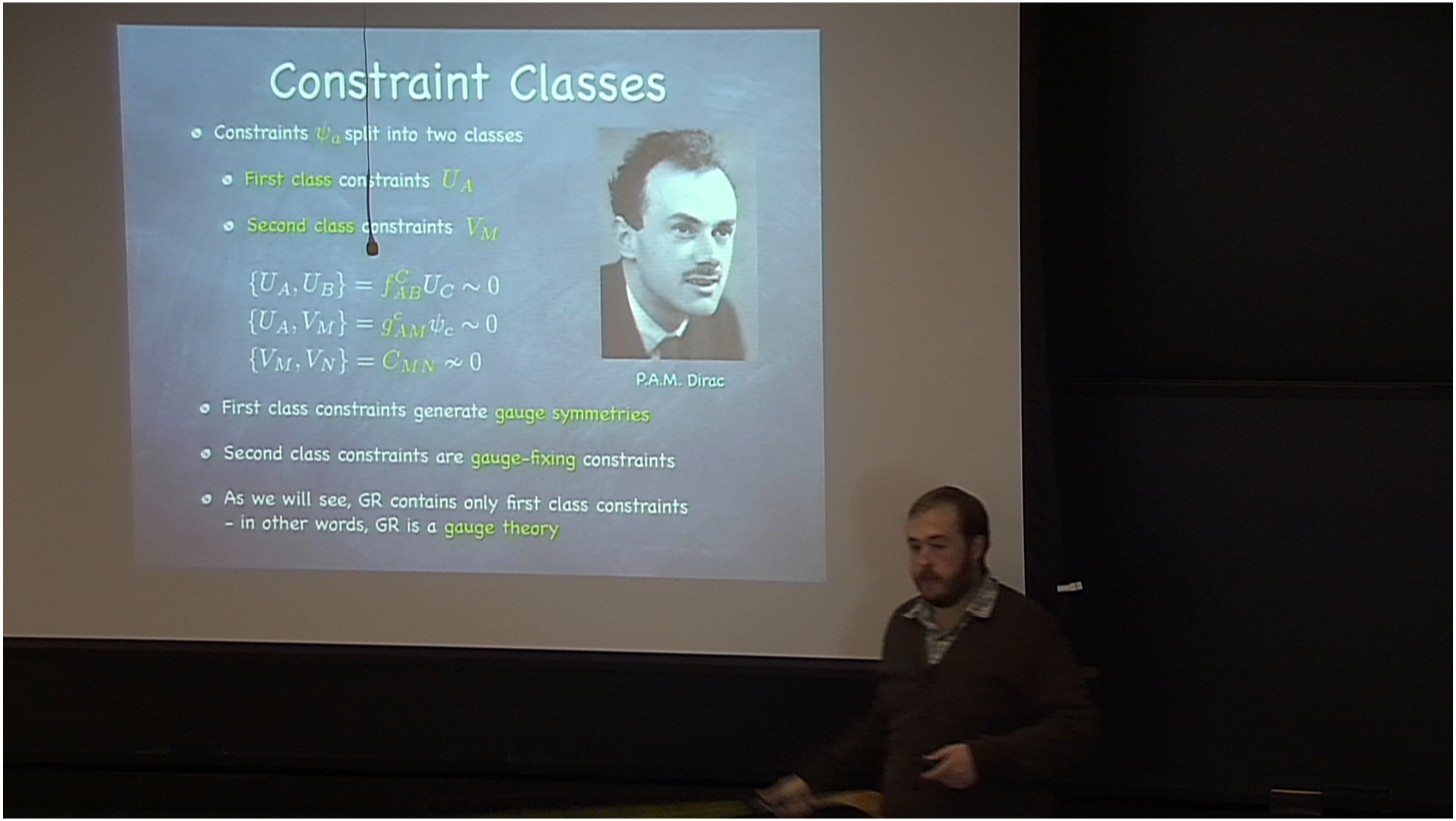
Constraint Classes

- Constraints ψ_a split into two classes
  - First class constraints U_A
  - Second class constraints V_M

$$
\begin{aligned}
\{U_A, U_B\} &= f^C_{AB} U_C \sim 0 \\
\{U_A, V_M\} &= g^c_{AM} \psi_c \sim 0 \\
\{V_M, V_N\} &= C_{MN} \nsim 0
\end{aligned}
$$

P.A.M. Dirac

- First class constraints generate gauge symmetries
- Second class constraints are gauge-fixing constraints
- As we will see, GR contains only first class constraints - in other words, GR is a gauge theory



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Maxwell Theory

$$S = -\frac{1}{4}\int \mathrm{d}^4x\, F_{\mu\nu}F^{\mu\nu} \qquad F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$$

Space-Time Split

$$\boxed{A_\mu \to A_0, A_i}$$

Only A_i is dynamical

$$S = \int \mathrm{d}t\, \mathrm{d}^3x \left(\frac{1}{2}\left(\dot{A}_i - \partial_i A_0\right)^2 - \frac{1}{4}F_{ij}F^{ij}\right)$$

Canonical Momentum

$$\pi^i \equiv \frac{\delta L}{\delta \dot{A}_i} = \dot{A}_i - \partial_i A_0$$

$-E_i$

Legendre Transform

$$\mathcal{H} \equiv \pi^i \dot{A}_i -$$

$$\boxed{\mathcal{H} = \frac{1}{2}\pi^i\pi_i - \frac{1}{4}F_{ij}F^{ij} + \pi^i\partial_i A_0}$$

Canonical E&M

$$\mathcal{L} = \pi^i \dot{A}_i - \mathcal{H} \qquad \mathcal{H} = \frac{1}{2}\pi^i\pi_i - \frac{1}{4}F_{ij}F^{ij} + \pi^i\partial_i A_0$$

$$S = \int \mathrm{d}t\,\mathrm{d}^3x \left(\pi^i \dot{A}_i - \frac{1}{2}\pi^i\pi_i + \frac{1}{4}F_{ij}F^{ij} - \pi^i\partial_i A_0\right)$$

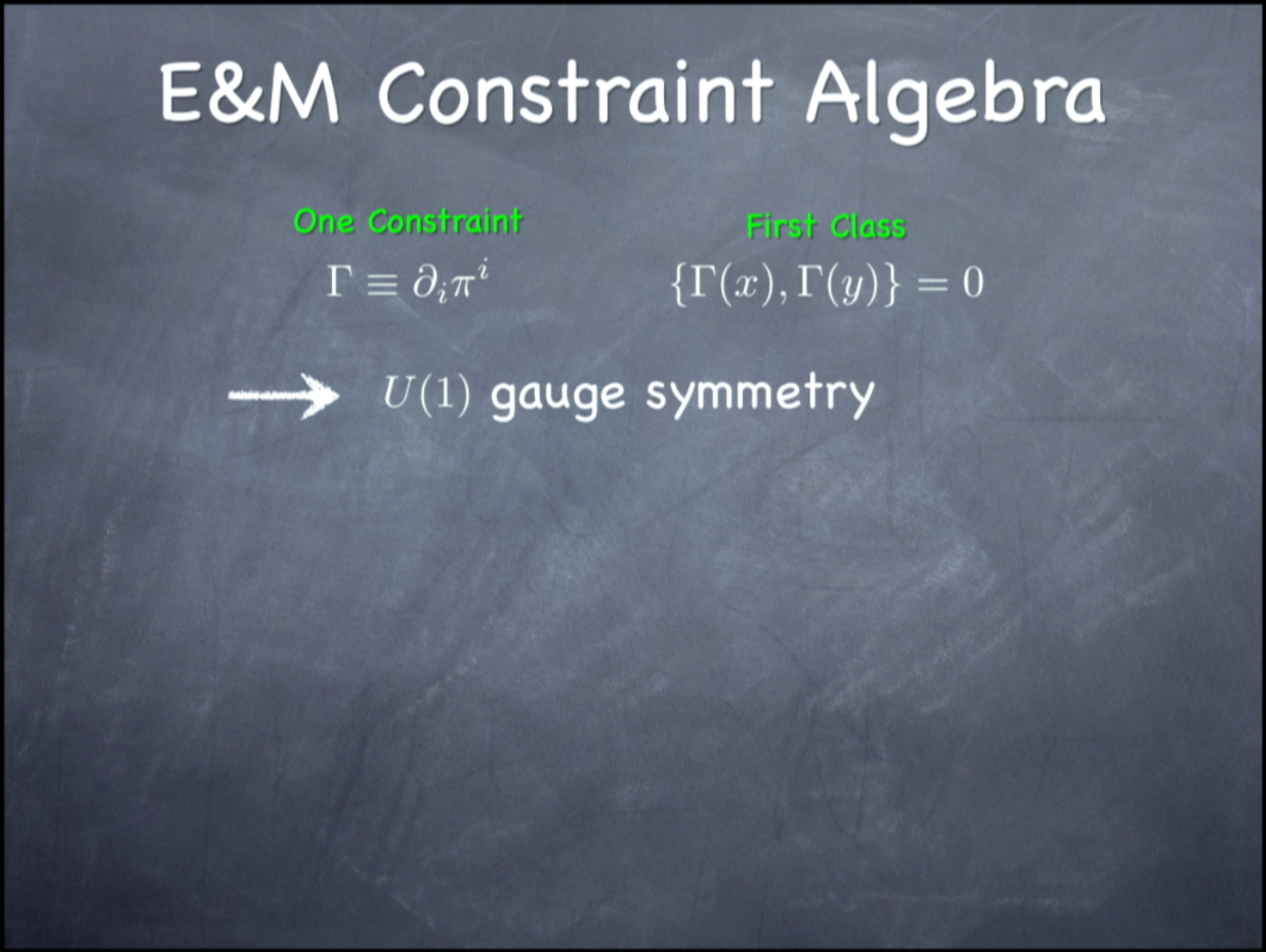
Integrate by parts, drop boundary term

$$\boxed{S = \int \mathrm{d}t\,\mathrm{d}^3x \left(\pi^i \dot{A}_i - \frac{1}{2}\pi^i\pi_i + \frac{1}{4}F_{ij}F^{ij} + A_0\partial_i\pi^i\right)}$$

A_0 is a Lagrange multiplier enforcing Gauss's Law: $\boxed{\partial_i\pi^i \sim 0}$

Poisson Bracket

$$\{C, D\} \equiv \int \mathrm{d}^3x \left(\frac{\delta C}{\delta A_i(x)}\frac{\delta D}{\delta \pi^i(x)} - \frac{\delta C}{\delta \pi^i(x)}\frac{\delta D}{\delta A_i(x)}\right)$$



E&M Constraint Algebra

One Constraint

$$\Gamma \equiv \partial_i \pi^i$$

First Class

$$\{\Gamma(x), \Gamma(y)\} = 0$$

$\longrightarrow$ $U(1)$ gauge symmetry

E&M Constraint Algebra

One Constraint

$$\Gamma \equiv \partial_i \pi^i$$

First Class

$$\{\Gamma(x), \Gamma(y)\} = 0$$

→ $U(1)$ gauge symmetry

Radiation/Coulomb Gauge

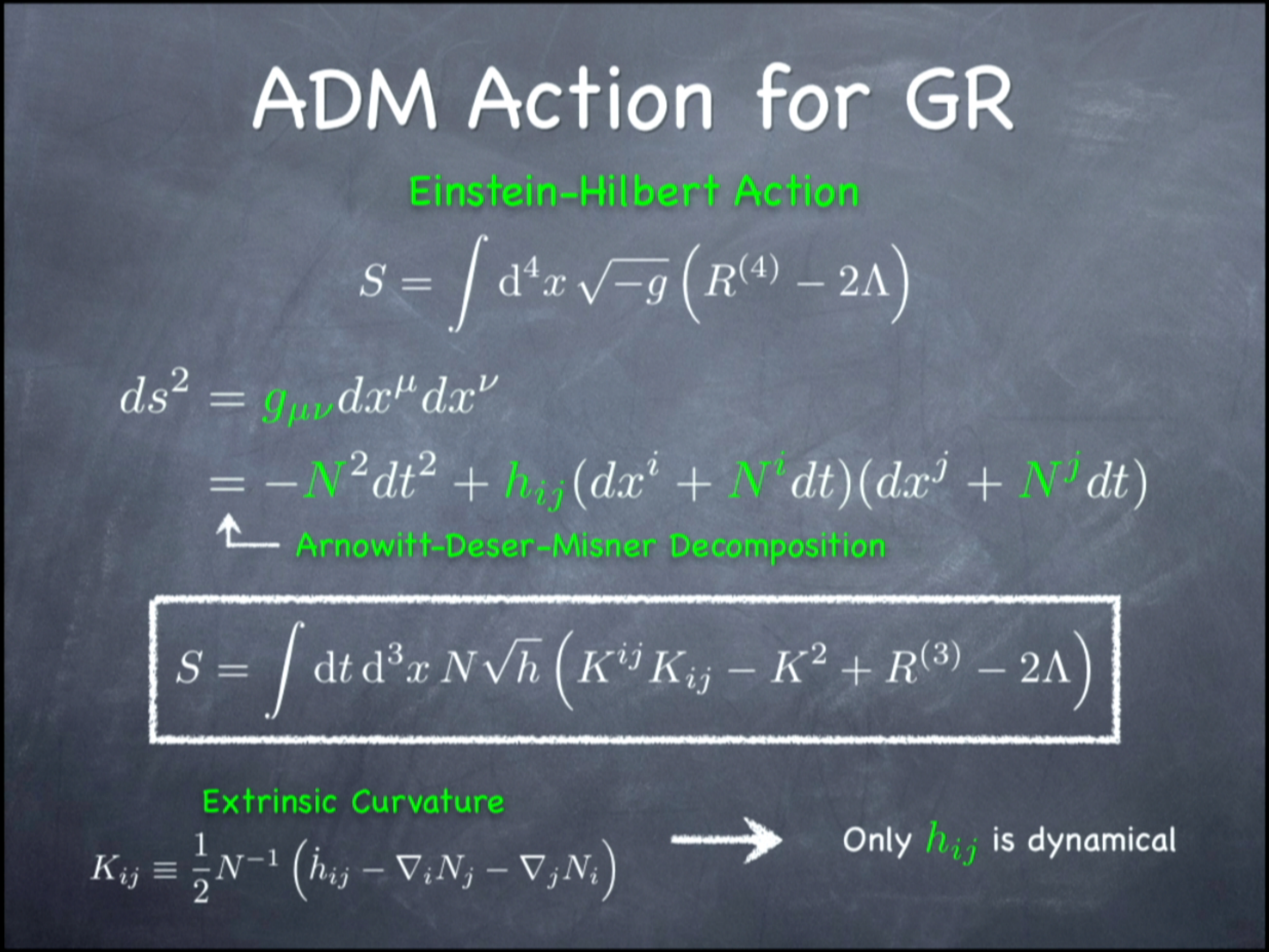
Gauge-fixing Constraint

$$\chi \equiv \partial_i A^i$$

Both Constraints Second Class

$$\{\chi(x), \Gamma(y)\} = \partial_{x^i} \partial_{y^i} \delta^3(x-y) \not\sim 0$$

→ No residual gauge symmetry



ADM Action for GR

Einstein-Hilbert Action

$$S = \int \mathrm{d}^4x \, \sqrt{-g} \left(R^{(4)} - 2\Lambda \right)$$

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\ &= -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \end{aligned}$$

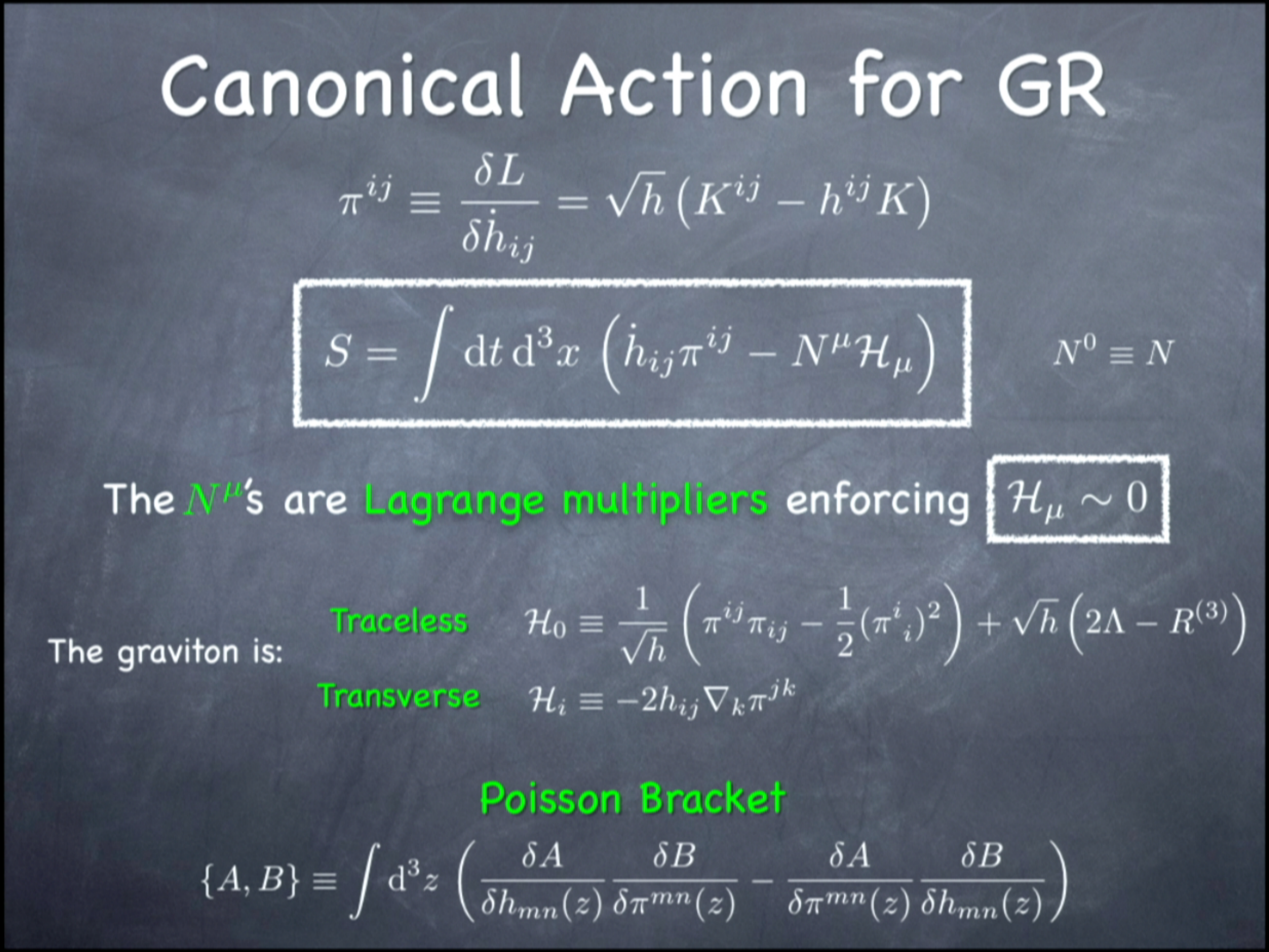
Arnowitt-Deser-Misner Decomposition

$$\boxed{S = \int \mathrm{d}t \, \mathrm{d}^3x \, N\sqrt{h} \left(K^{ij}K_{ij} - K^2 + R^{(3)} - 2\Lambda \right)}$$

Extrinsic Curvature

$$K_{ij} \equiv \frac{1}{2} N^{-1} \left(\dot{h}_{ij} - \nabla_i N_j - \nabla_j N_i \right)$$

→ Only h_{ij} is dynamical



Canonical Action for GR

$$\pi^{ij} \equiv \frac{\delta L}{\delta \dot{h}_{ij}} = \sqrt{h}\left(K^{ij} - h^{ij}K\right)$$

$$\boxed{S = \int \mathrm{d}t\, \mathrm{d}^3x \left(\dot{h}_{ij}\pi^{ij} - N^\mu \mathcal{H}_\mu\right)} \qquad N^0 \equiv N$$

The N^μ's are **Lagrange multipliers** enforcing $\boxed{\mathcal{H}_\mu \sim 0}$

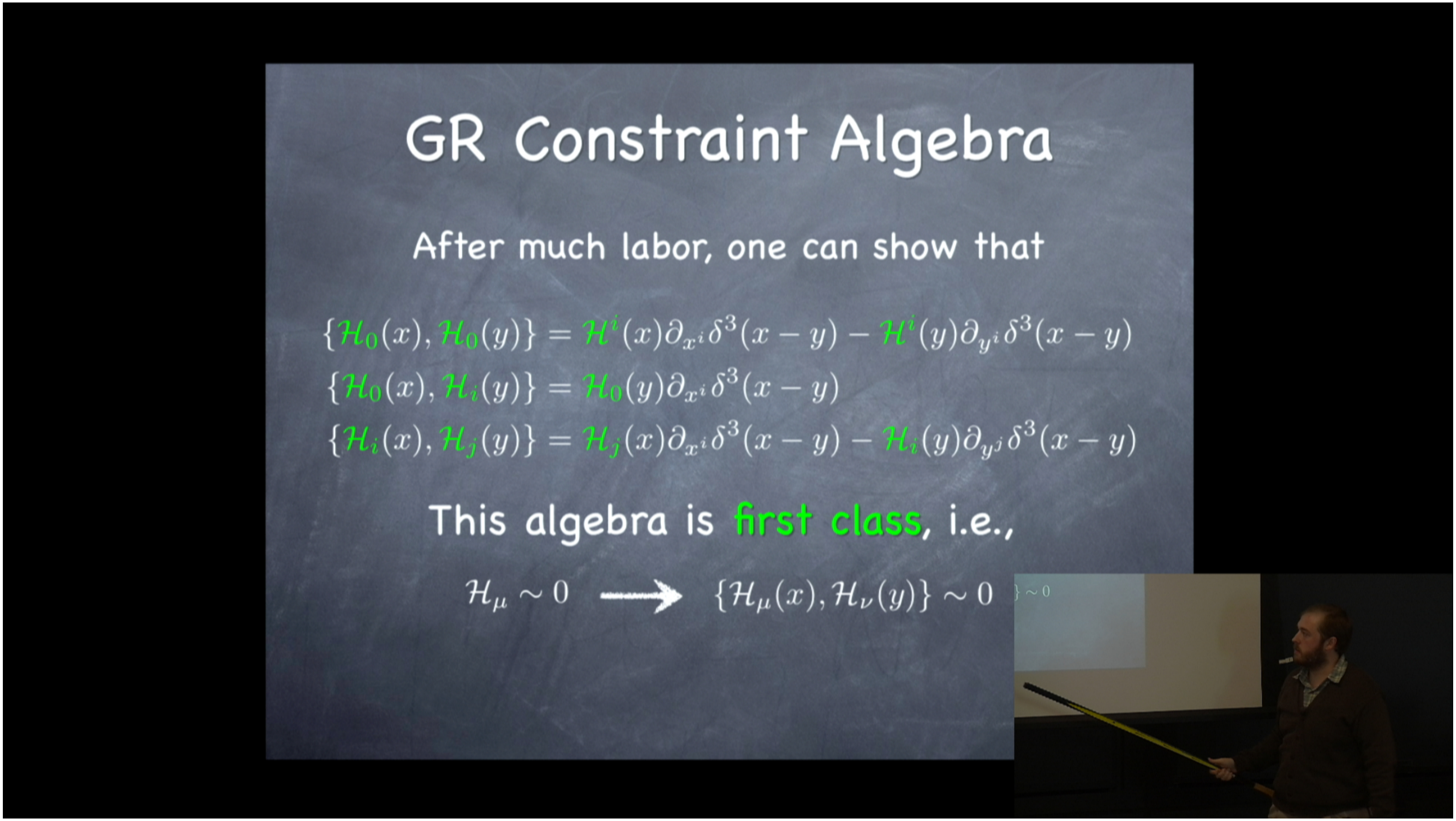
The graviton is:

Traceless $\mathcal{H}_0 \equiv \frac{1}{\sqrt{h}}\left(\pi^{ij}\pi_{ij} - \frac{1}{2}(\pi^i{}_i)^2\right) + \sqrt{h}\left(2\Lambda - R^{(3)}\right)$

Transverse $\mathcal{H}_i \equiv -2h_{ij}\nabla_k\pi^{jk}$

Poisson Bracket

$$\{A, B\} \equiv \int \mathrm{d}^3z \left(\frac{\delta A}{\delta h_{mn}(z)}\frac{\delta B}{\delta \pi^{mn}(z)} - \frac{\delta A}{\delta \pi^{mn}(z)}\frac{\delta B}{\delta h_{mn}(z)}\right)$$



GR Constraint Algebra

After much labor, one can show that

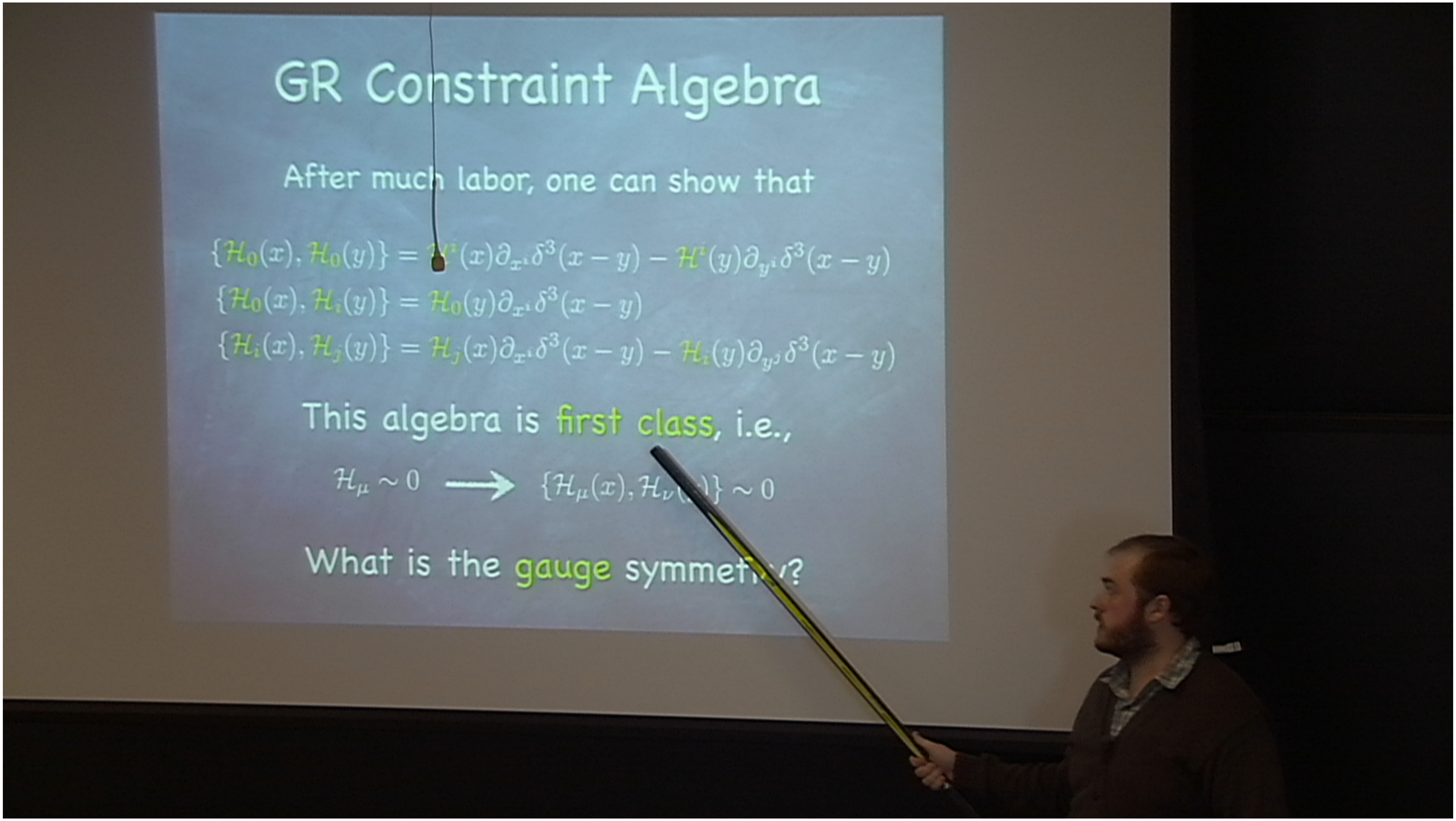
$$\{\mathcal{H}_0(x), \mathcal{H}_0(y)\} = \mathcal{H}^i(x)\partial_{x^i}\delta^3(x-y) - \mathcal{H}^i(y)\partial_{y^i}\delta^3(x-y)$$

$$\{\mathcal{H}_0(x), \mathcal{H}_i(y)\} = \mathcal{H}_0(y)\partial_{x^i}\delta^3(x-y)$$

$$\{\mathcal{H}_i(x), \mathcal{H}_j(y)\} = \mathcal{H}_j(x)\partial_{x^i}\delta^3(x-y) - \mathcal{H}_i(y)\partial_{y^j}\delta^3(x-y)$$

This algebra is first class, i.e.,

$$\mathcal{H}_\mu \sim 0 \longrightarrow \{\mathcal{H}_\mu(x), \mathcal{H}_\nu(y)\} \sim 0$$



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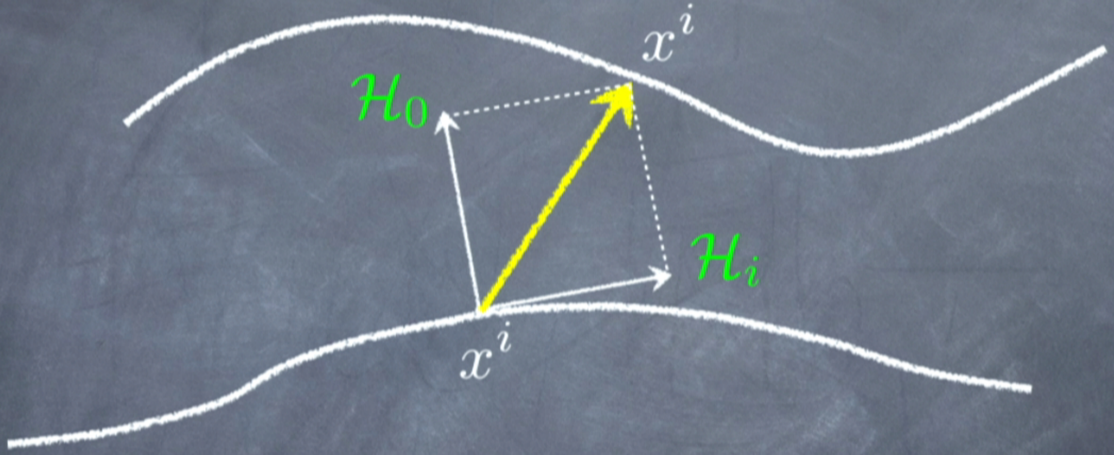
What is the gauge symmetry?

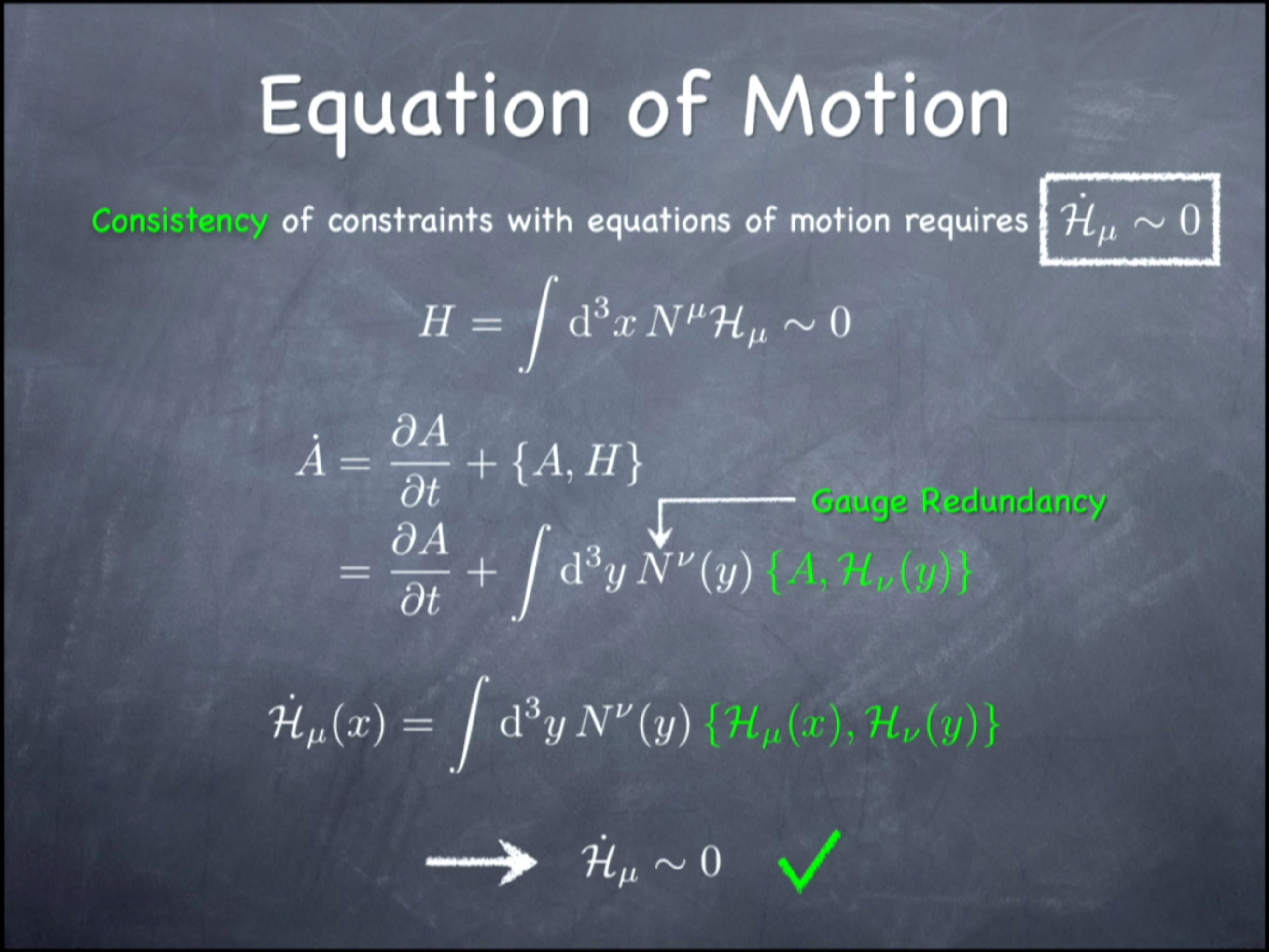
General Covariance



- The $\mathcal{H}_\mu$'s generate the deformations of a spacelike hypersurface in a Riemannian spacetime
- This "general covariance" algebra encodes the local Lorentz covariance of a canonical action
- GR is the unique minimal representation; this result complements Weinberg's Theorem

C. Teitelboim
Annals Phys. 79 (1973) 542-557

S. Hojman, K. Kuchar, C. Teitelboim
Annals Phys. 96 (1976) 88-135



Equation of Motion

Consistency of constraints with equations of motion requires $\boxed{\dot{\mathcal{H}}_\mu \sim 0}$

$$H = \int \mathrm{d}^3x\, N^\mu \mathcal{H}_\mu \sim 0$$

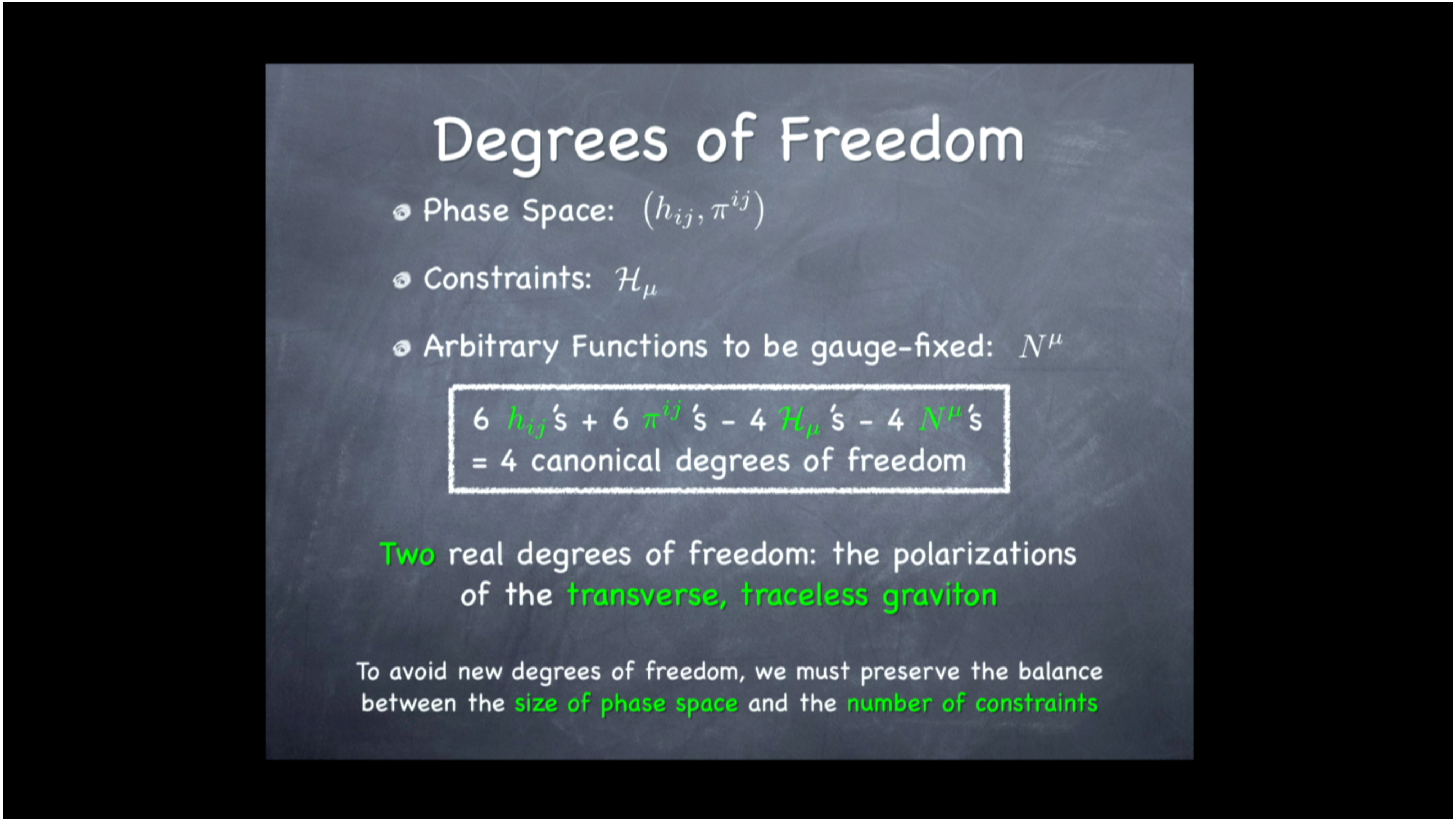
$$\dot{A} = \frac{\partial A}{\partial t} + \{A, H\}$$

Gauge Redundancy

$$= \frac{\partial A}{\partial t} + \int \mathrm{d}^3y\, N^\nu(y)\, \{A, \mathcal{H}_\nu(y)\}$$

$$\dot{\mathcal{H}}_\mu(x) = \int \mathrm{d}^3y\, N^\nu(y)\, \{\mathcal{H}_\mu(x), \mathcal{H}_\nu(y)\}$$

$$\longrightarrow \quad \dot{\mathcal{H}}_\mu \sim 0 \quad ✓$$



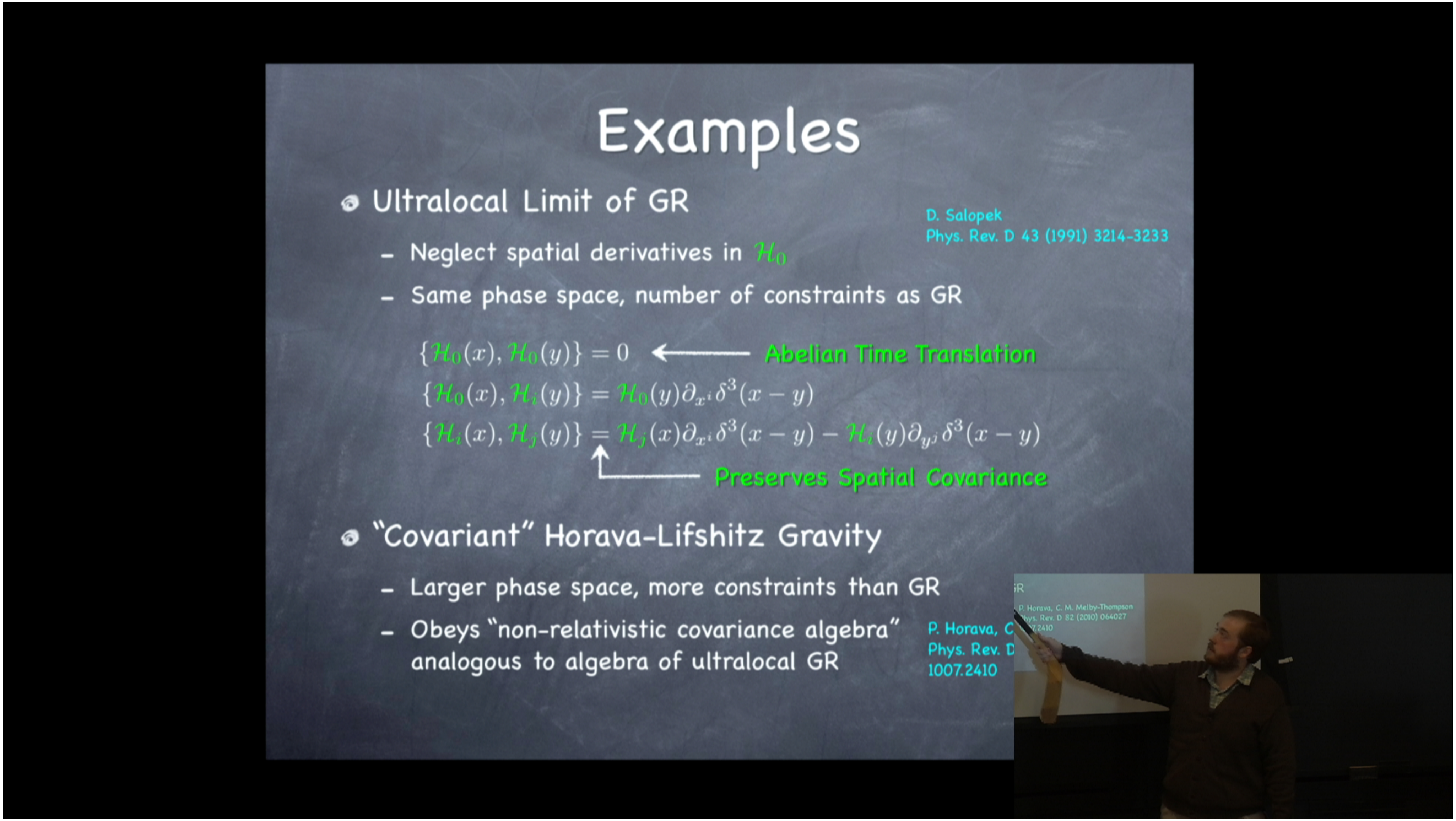
Degrees of Freedom

- Phase Space: (h_{ij}, π^{ij})
- Constraints: $\mathcal{H}_\mu$
- Arbitrary Functions to be gauge-fixed: N^μ

$$\boxed{\begin{array}{l} 6\ h_{ij}\text{'s} + 6\ \pi^{ij}\text{'s} - 4\ \mathcal{H}_\mu\text{'s} - 4\ N^\mu\text{'s} \\ = 4 \text{ canonical degrees of freedom} \end{array}}$$

Two real degrees of freedom: the polarizations of the transverse, traceless graviton

To avoid new degrees of freedom, we must preserve the balance between the size of phase space and the number of constraints



Examples

- Ultralocal Limit of GR
  - Neglect spatial derivatives in $\mathcal{H}_0$
  - Same phase space, number of constraints as GR

D. Salopek
Phys. Rev. D 43 (1991) 3214-3233

$$\{\mathcal{H}_0(x), \mathcal{H}_0(y)\} = 0 \longleftarrow \text{Abelian Time Translation}$$

$$\{\mathcal{H}_0(x), \mathcal{H}_i(y)\} = \mathcal{H}_0(y)\partial_{x^i}\delta^3(x-y)$$

$$\{\mathcal{H}_i(x), \mathcal{H}_j(y)\} = \mathcal{H}_j(x)\partial_{x^i}\delta^3(x-y) - \mathcal{H}_i(y)\partial_{y^j}\delta^3(x-y)$$

Preserves Spatial Covariance

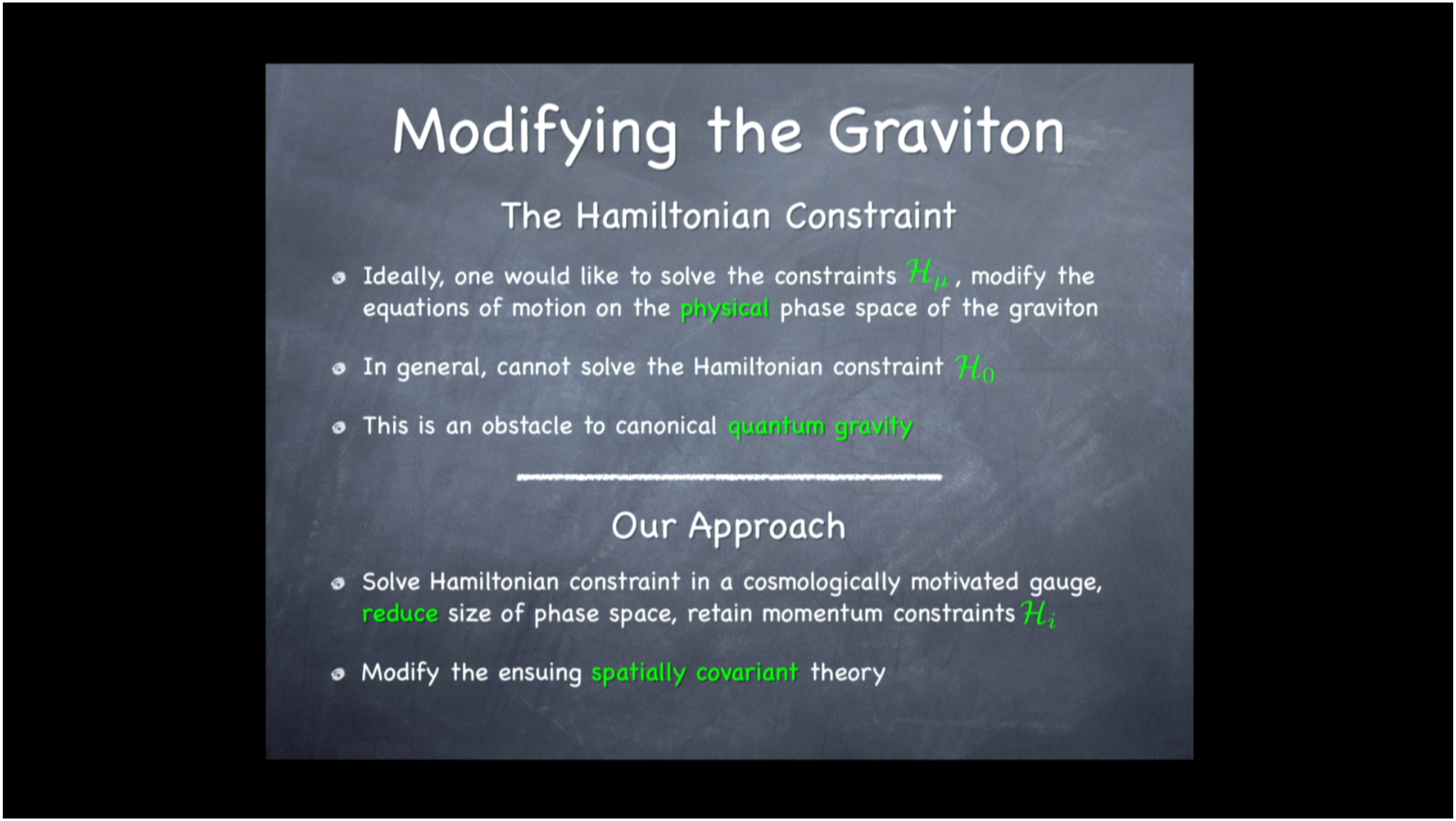
- "Covariant" Horava-Lifshitz Gravity
  - Larger phase space, more constraints than GR
  - Obeys "non-relativistic covariance algebra" analogous to algebra of ultralocal GR

P. Horava, C
Phys. Rev. D
1007.2410

Modifying the Graviton

The Hamiltonian Constraint

- Ideally, one would like to solve the constraints $\mathcal{H}_\mu$, modify the equations of motion on the physical phase space of the graviton
- In general, cannot solve the Hamiltonian constraint $\mathcal{H}_0$
- This is an obstacle to canonical quantum gravity



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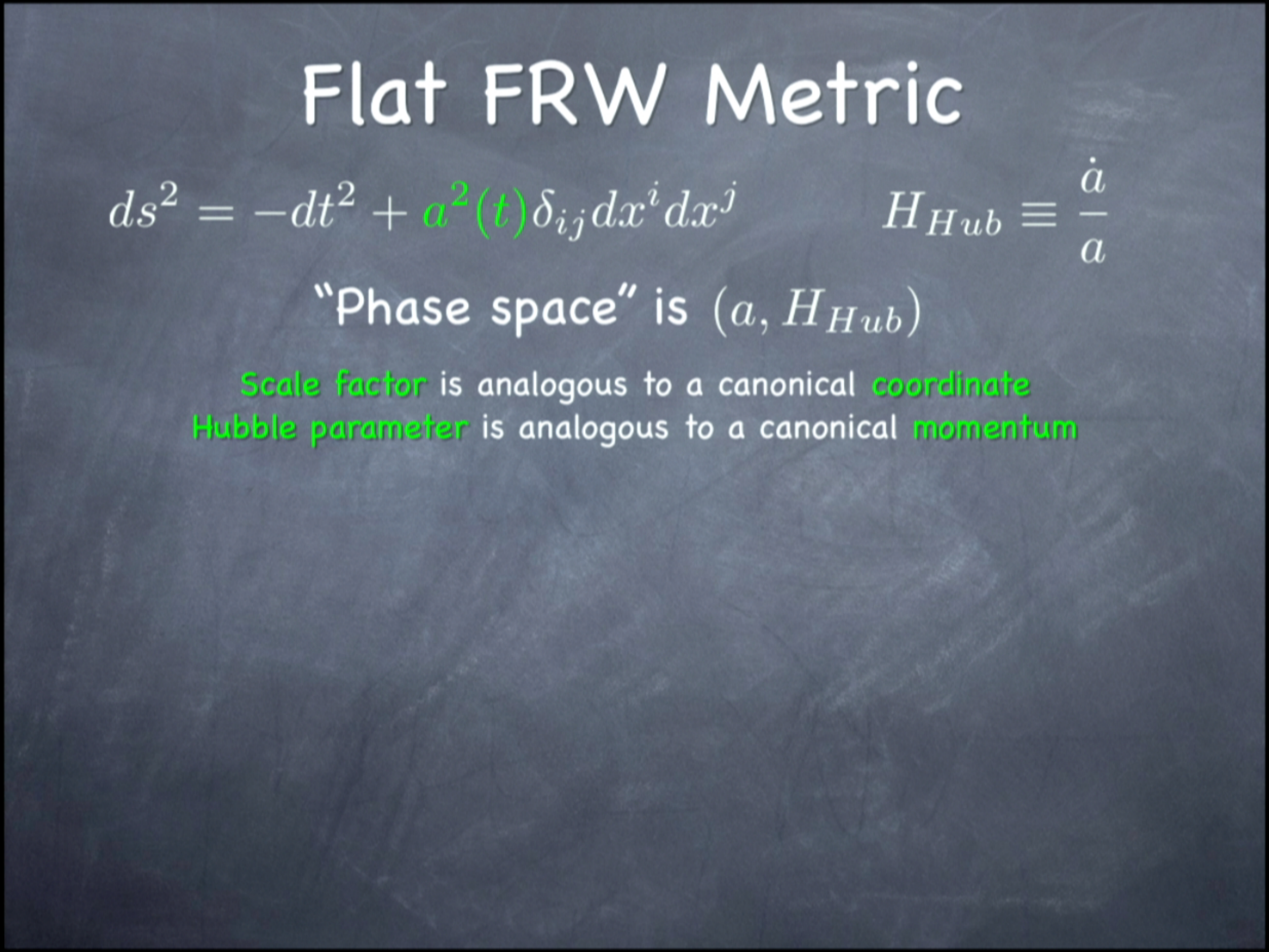
Our Approach

- Solve Hamiltonian constraint in a cosmologically motivated gauge, reduce size of phase space, retain momentum constraints $\mathcal{H}_i$
- Modify the ensuing spatially covariant theory

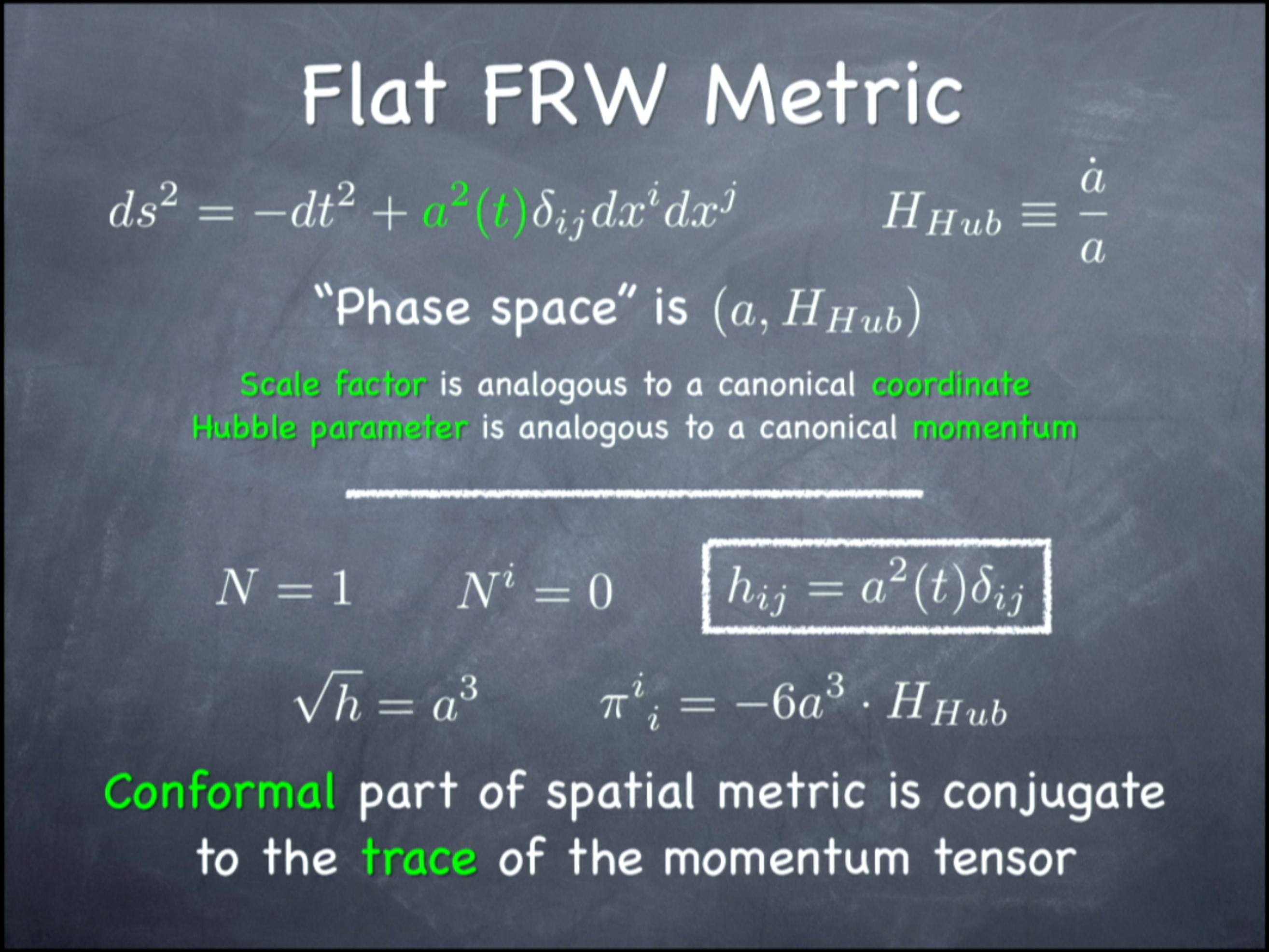


Flat FRW Metric

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j \qquad H_{Hub} \equiv \frac{\dot{a}}{a}$$

"Phase space" is (a, H_{Hub})

Scale factor is analogous to a canonical coordinate
Hubble parameter is analogous to a canonical momentum



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Scale factor is analogous to a canonical **coordinate**
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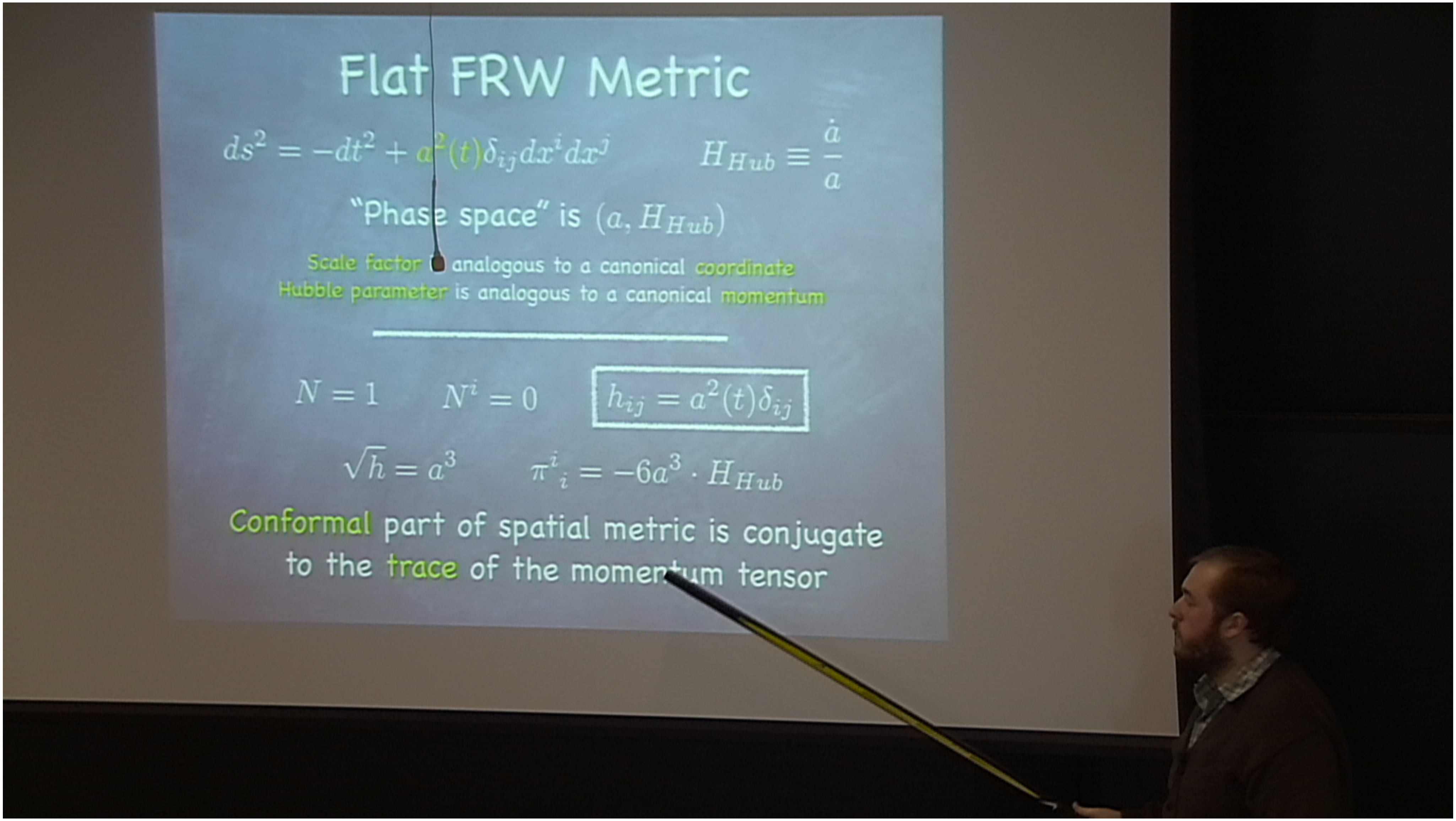
$$N = 1 \qquad N^i = 0 \qquad \boxed{h_{ij} = a^2(t)\delta_{ij}}$$

$$\sqrt{h} = a^3 \qquad \pi^i{}_i = -6a^3 \cdot H_{Hub}$$

Conformal part of spatial metric is conjugate to the **trace** of the momentum tensor



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Conformal Decomposition

Split spatial metric h_{ij} into a volume
factor ω and a unit-determinant metric $\tilde{h}_{ij}$

Conformal
Phase Space

$$\omega \equiv \sqrt{h} \qquad \pi_\omega = \frac{2\pi^i{}_i}{3\omega}$$

Unit-Determinant
Phase Space

$$\det \tilde{h}_{ij} = 1 \qquad h_{ij}\tilde{\pi}^{ij} = 0$$

$$\boxed{h_{ij} \equiv \tilde{h}_{ij} \cdot \omega^{2/3} \qquad \pi^{ij} = \frac{\tilde{\pi}^{ij}}{\omega^{2/3}} + \frac{1}{2}\tilde{h}^{ij} \cdot \pi_\omega \cdot \omega^{1/3}}$$

$$\left(h_{ij}, \pi^{ij}\right) \longrightarrow \left(\omega, \pi_\omega\right), \left(\tilde{h}_{ij}, \tilde{\pi}^{ij}\right)$$

This decomposition is completely general

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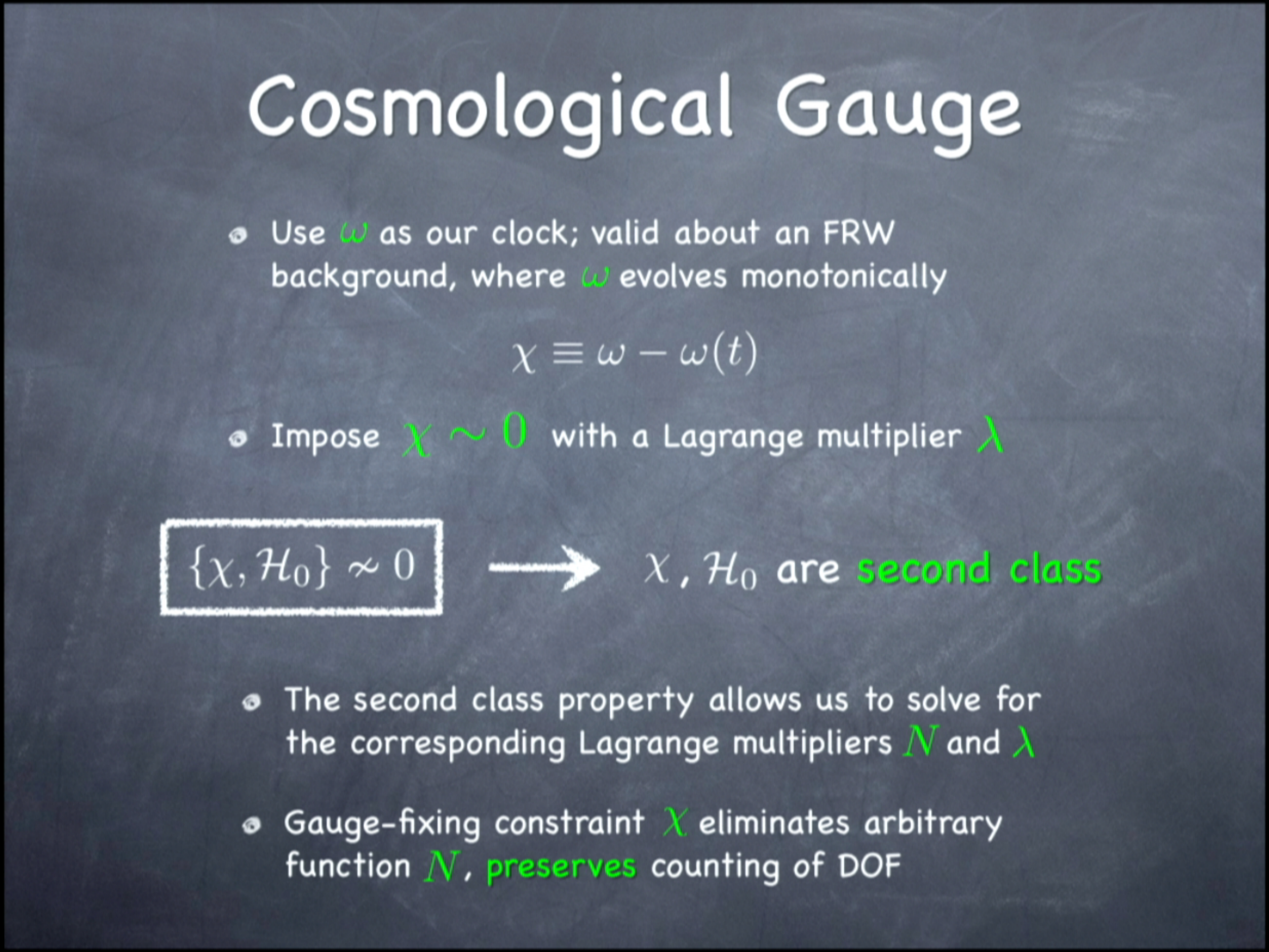


Cosmological Gauge

- Use ω as our clock; valid about an FRW background, where ω evolves monotonically

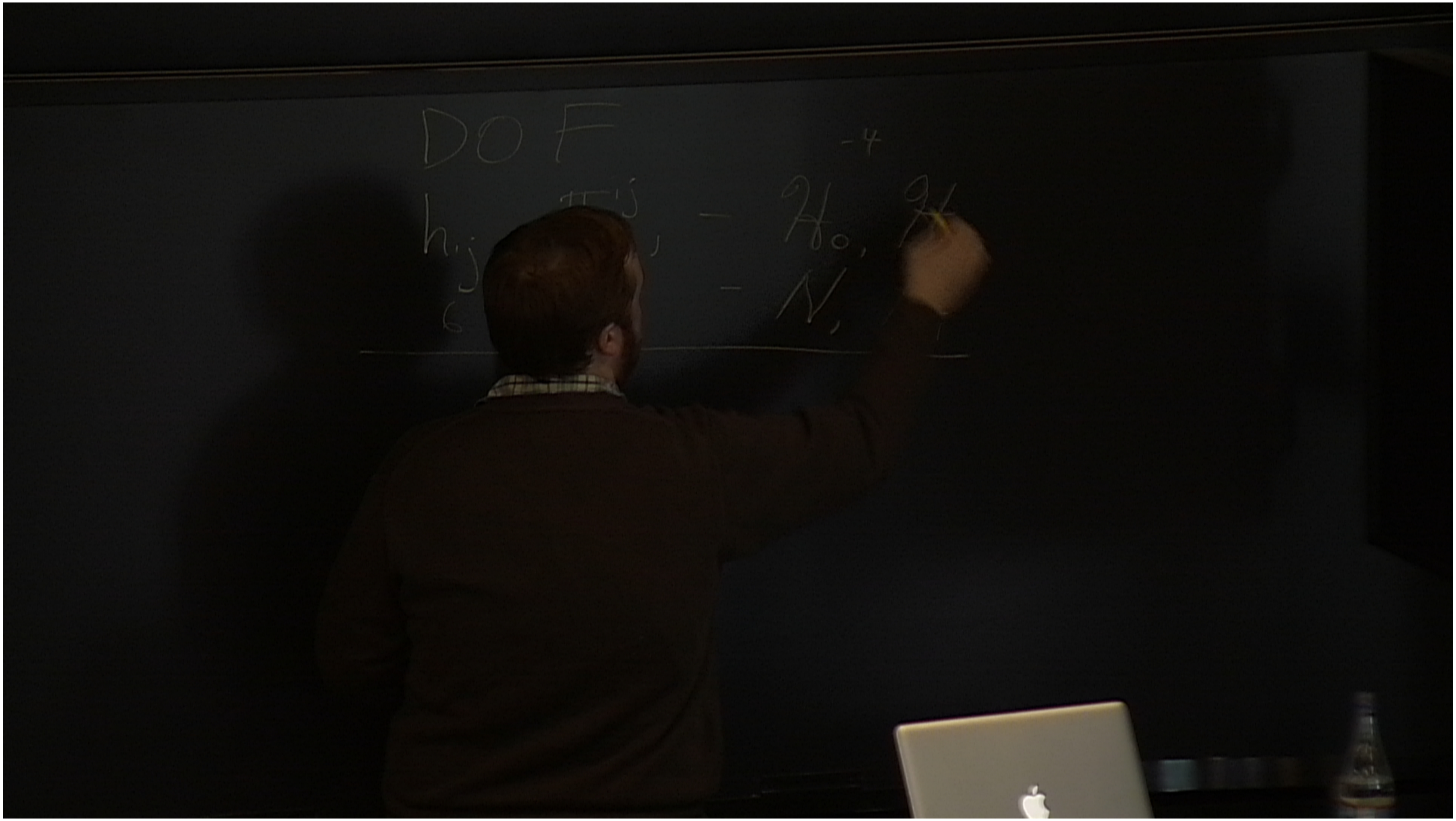
$$\chi \equiv \omega - \omega(t)$$

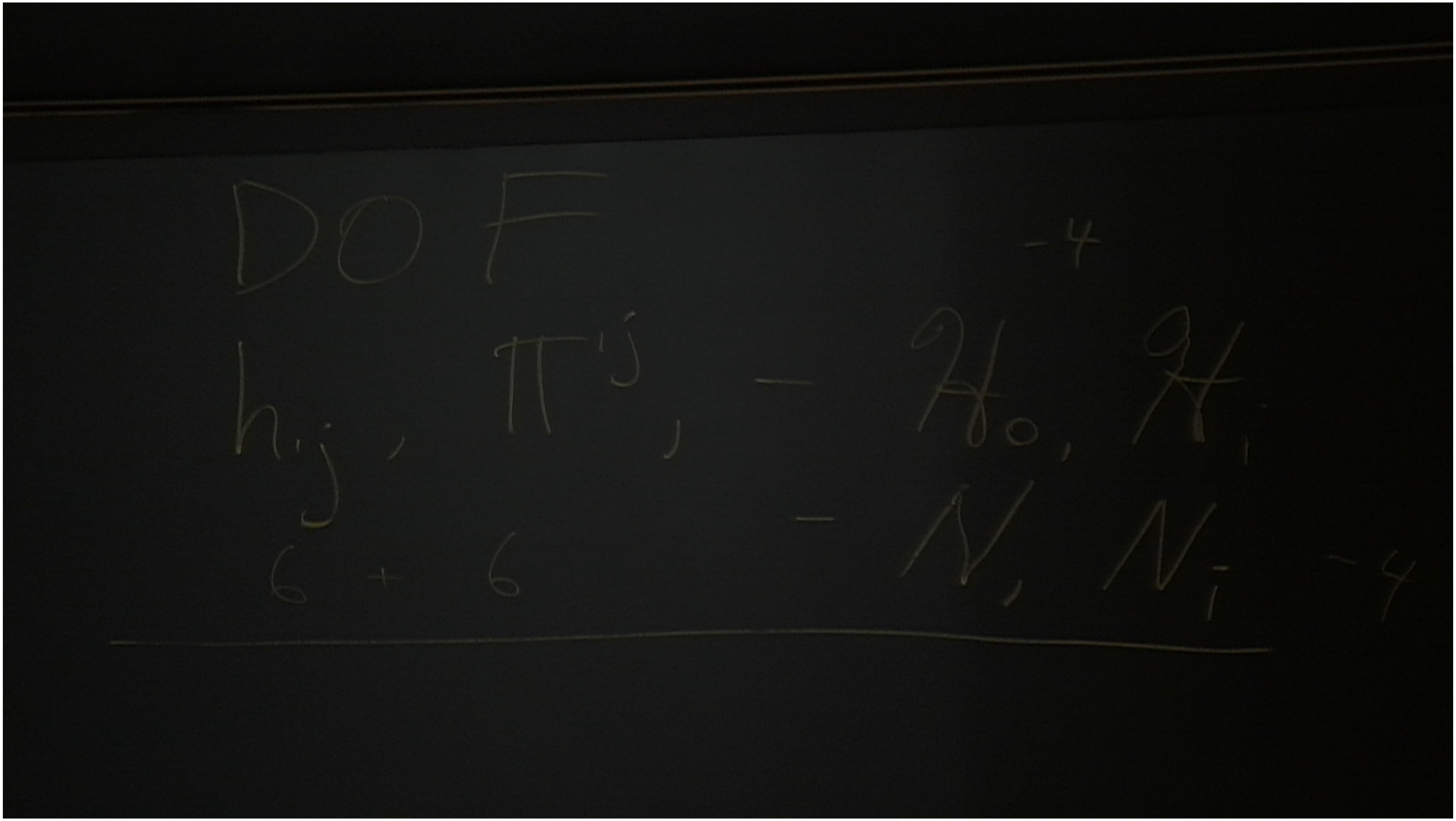
- Impose $\chi \sim 0$ with a Lagrange multiplier λ

$$\boxed{\{\chi, \mathcal{H}_0\} \not\approx 0} \longrightarrow$$

χ, $\mathcal{H}_0$ are **second class**

- The second class property allows us to solve for the corresponding Lagrange multipliers N and λ
- Gauge-fixing constraint χ eliminates arbitrary function N, preserves counting of DOF

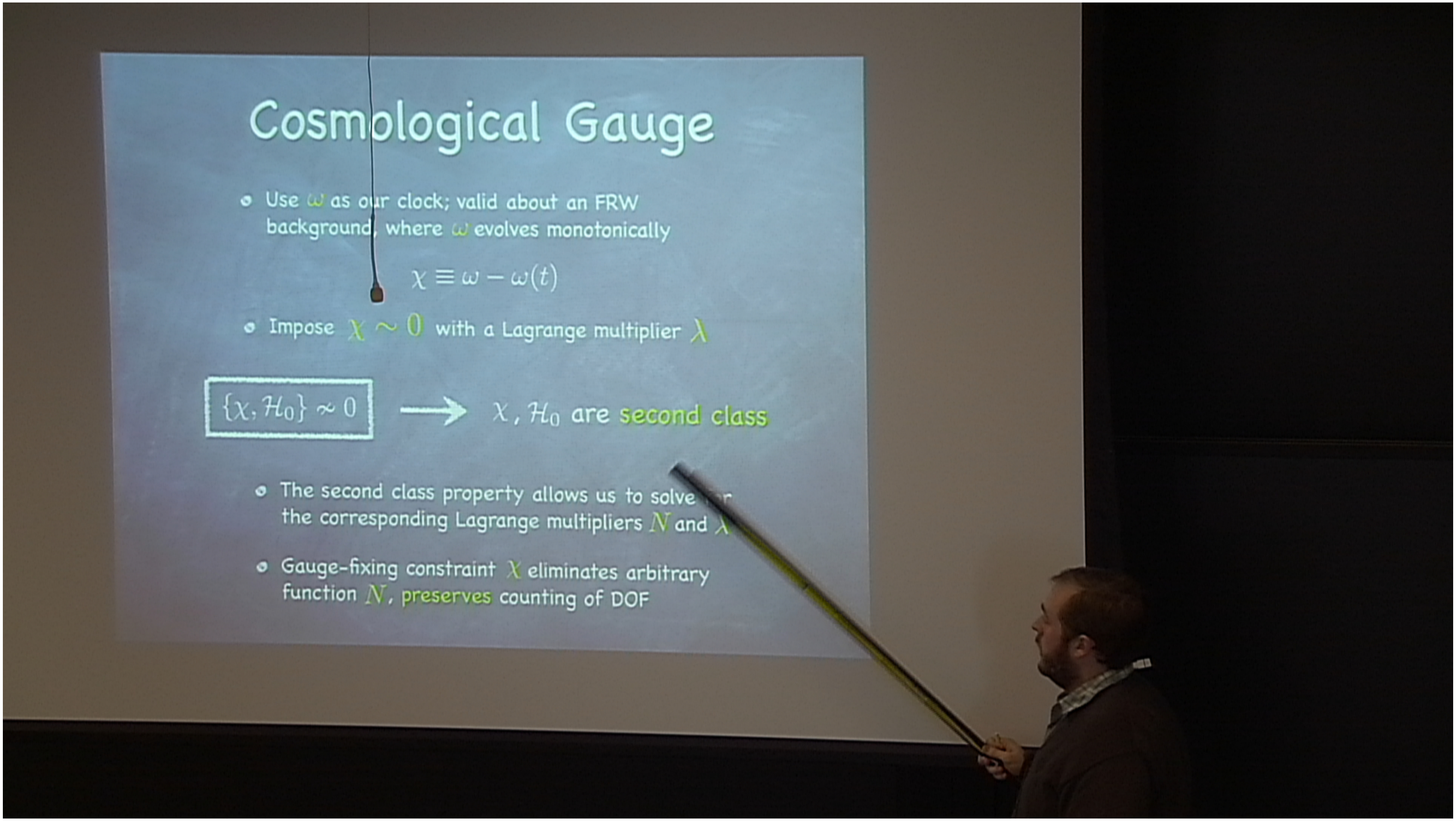

DOF
-4
h_ij   π^ij ,   —   ℋ_0 , ℋ
6   —   N ,



DOF

$$h_{ij}, \quad \pi^{ij}, \quad - \overset{-4}{\mathcal{H}_0, \mathcal{H}_i}$$

$$6 + 6 \qquad - N, N_i \quad -4$$



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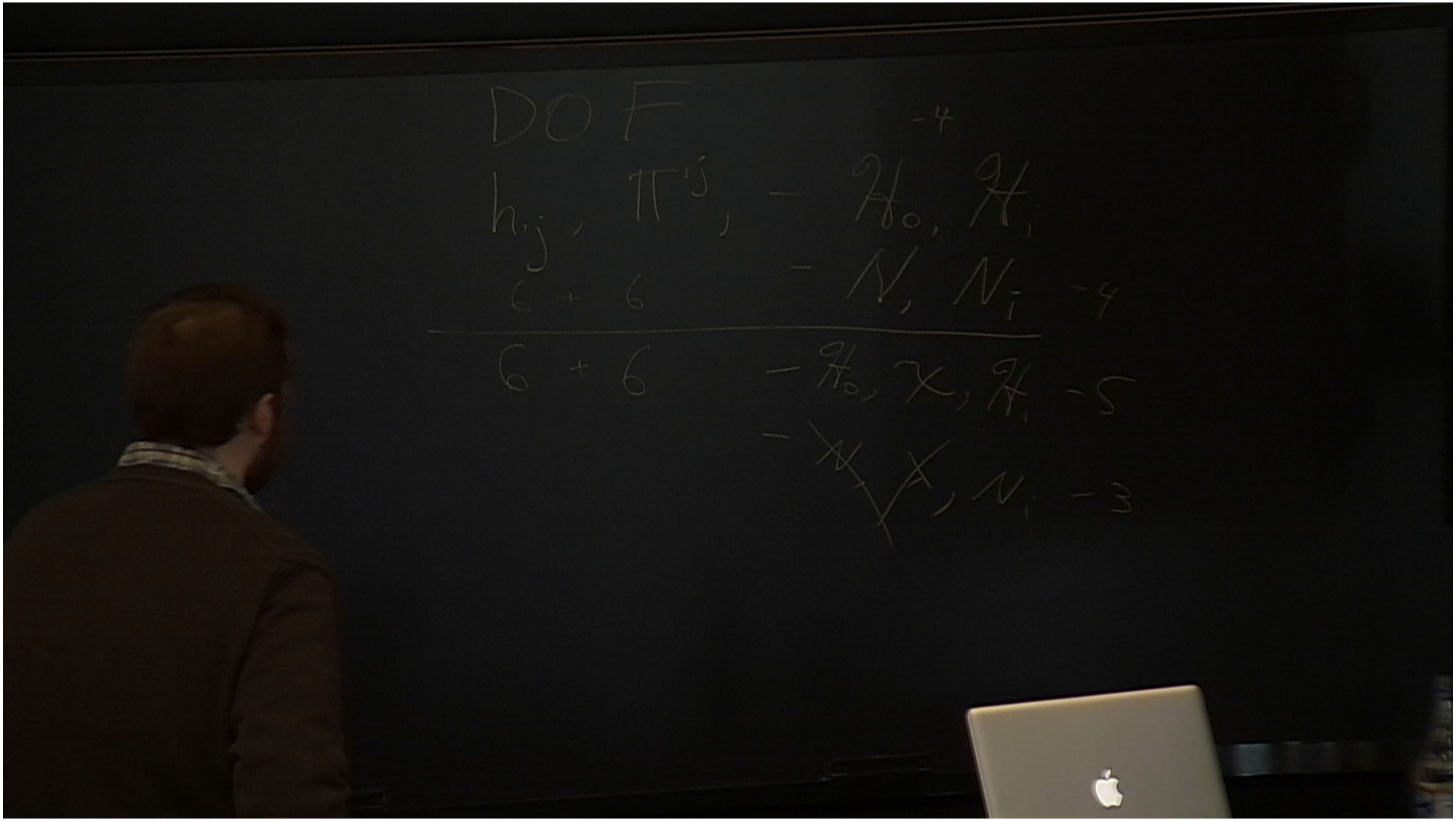


DOF

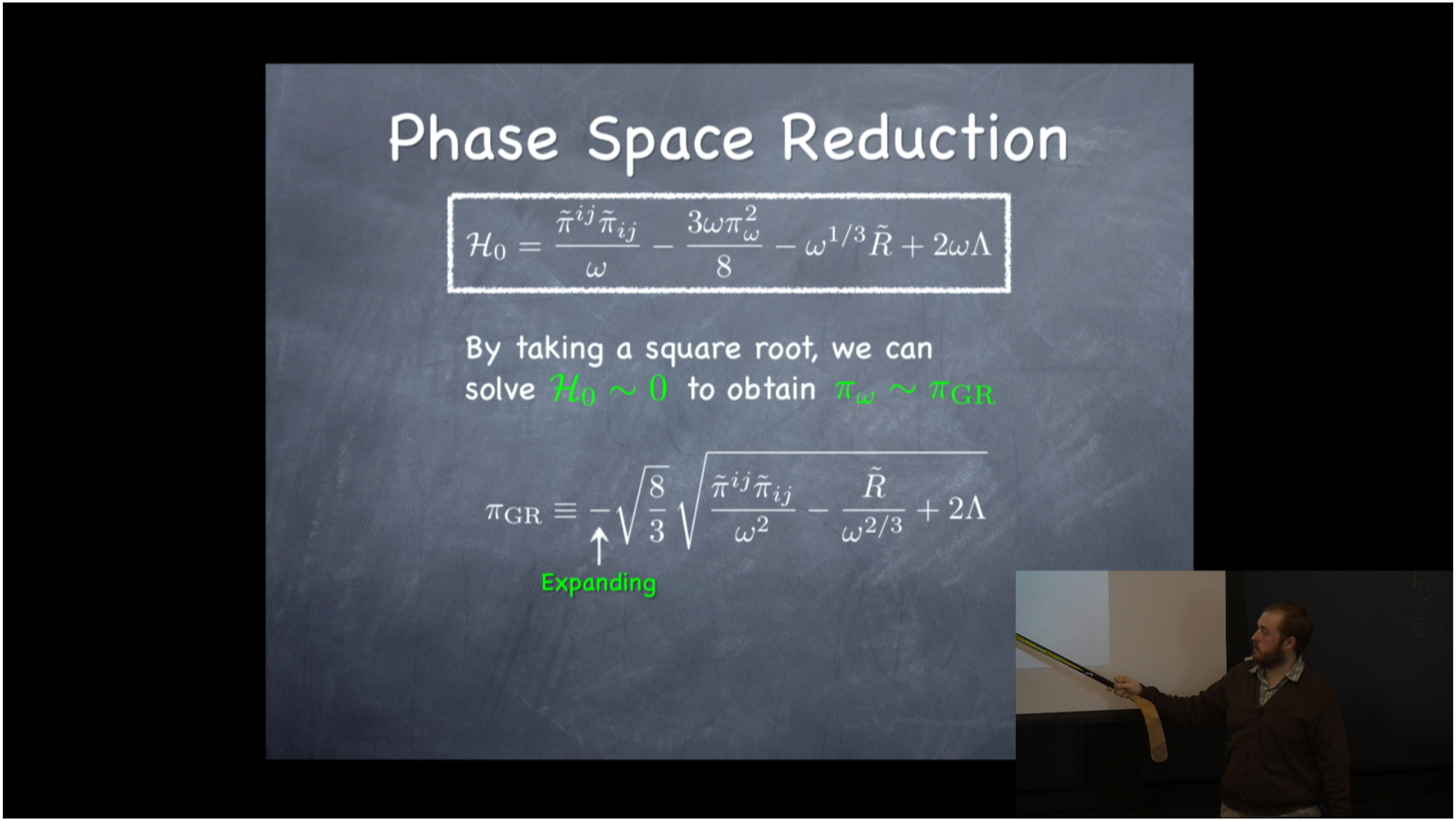
h_{ij}, π^{ij}, — $\mathcal{H}_0, \mathcal{H}_i$ -4

— N, N_i -4

$6 + 6$

$6 + 6$ — $\mathcal{H}_0, \chi, \mathcal{H}_i$ -5

— $\cancel{N}, \cancel{\chi}, N_i$ -3

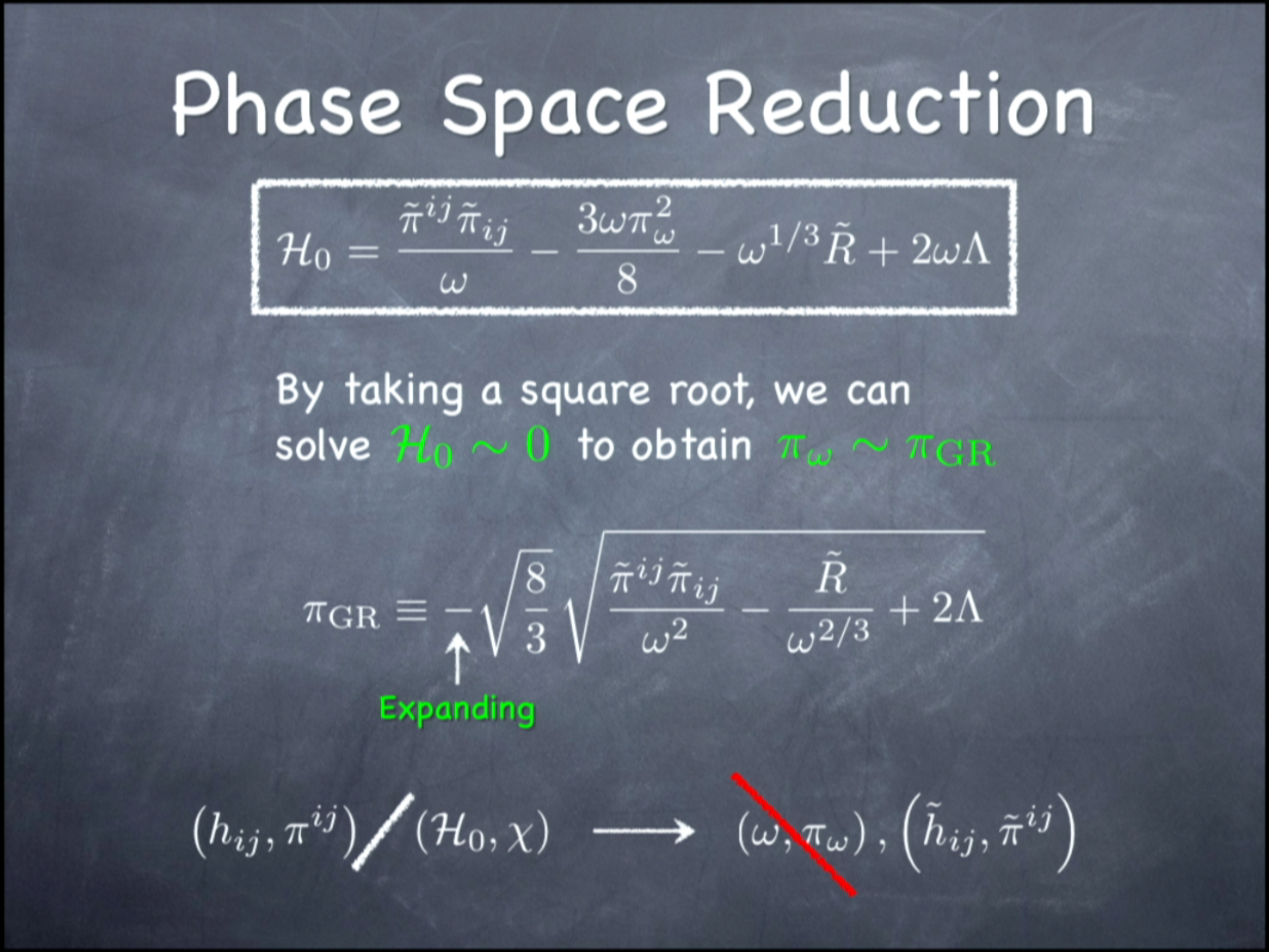


Phase Space Reduction

$$\boxed{\mathcal{H}_0 = \frac{\tilde{\pi}^{ij}\tilde{\pi}_{ij}}{\omega} - \frac{3\omega\pi_\omega^2}{8} - \omega^{1/3}\tilde{R} + 2\omega\Lambda}$$

By taking a square root, we can solve $\mathcal{H}_0 \sim 0$ to obtain $\pi_\omega \sim \pi_{\mathrm{GR}}$

$$\pi_{\mathrm{GR}} \equiv \underset{\substack{\uparrow \\ \text{Expanding}}}{-}\sqrt{\frac{8}{3}}\sqrt{\frac{\tilde{\pi}^{ij}\tilde{\pi}_{ij}}{\omega^2} - \frac{\tilde{R}}{\omega^{2/3}} + 2\Lambda}$$



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$$(h_{ij},\pi^{ij})\,/\,(\mathcal{H}_0,\chi) \longrightarrow \cancel{(\omega,\pi_\omega)}\,,\left(\tilde{h}_{ij},\tilde{\pi}^{ij}\right)$$

Spatially Covariant GR

$$\boxed{S = \int \mathrm{d}t\,\mathrm{d}^3x\,\left(\dot{\tilde{h}}_{ij}\tilde{\pi}^{ij} + \dot{\omega}\pi_\omega - N^i\tilde{\mathcal{H}}_i\right)}$$

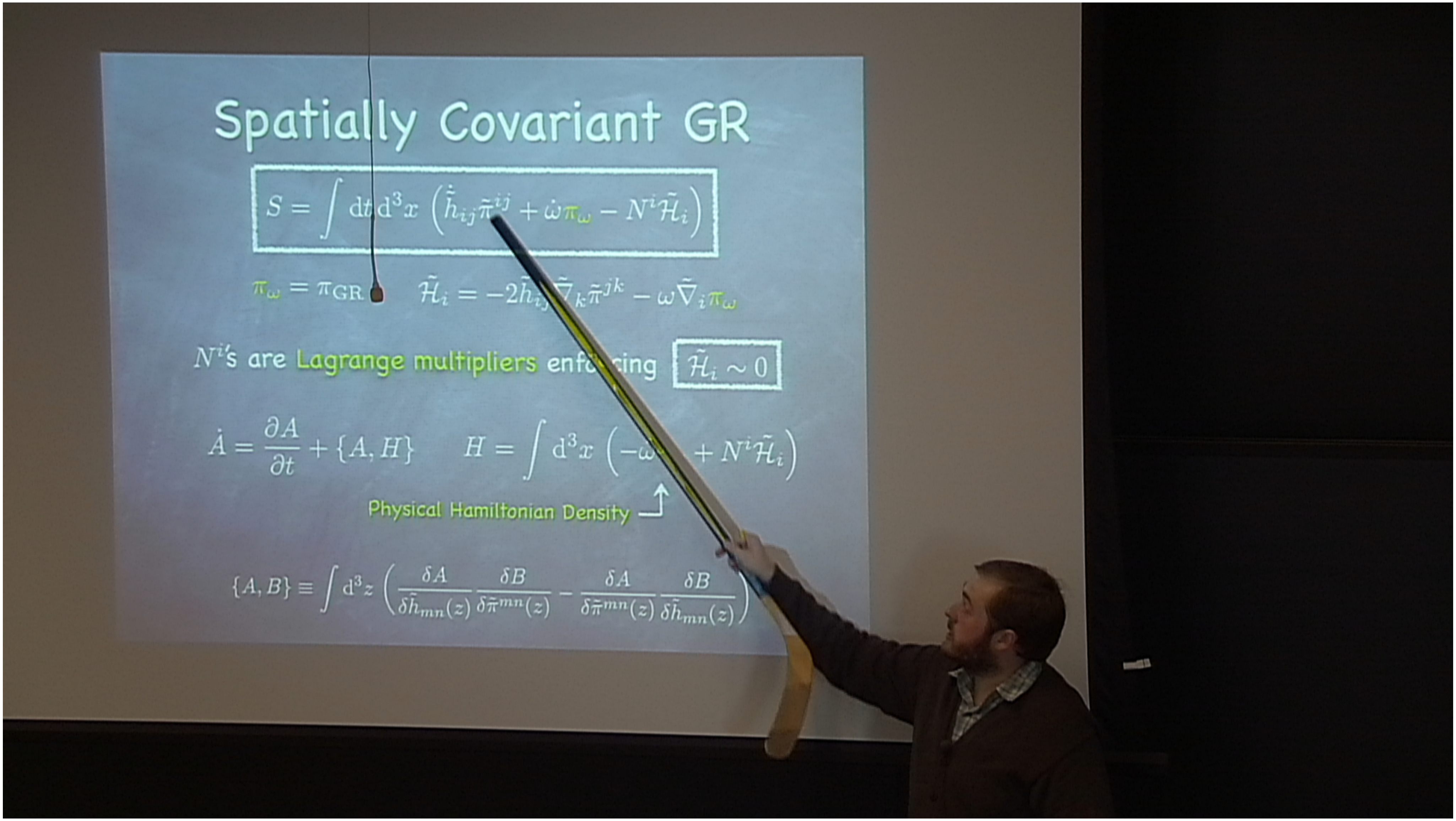
$$\pi_\omega = \pi_{\mathrm{GR}} \qquad \tilde{\mathcal{H}}_i = -2\tilde{h}_{ij}\tilde{\nabla}_k\tilde{\pi}^{jk} - \omega\tilde{\nabla}_i\pi_\omega$$

N^i's are **Lagrange multipliers** enforcing $\boxed{\tilde{\mathcal{H}}_i \sim 0}$

$$\dot{A} = \frac{\partial A}{\partial t} + \{A, H\} \qquad H = \int \mathrm{d}^3x\,\left(\underset{\uparrow}{-\dot{\omega}\pi_\omega} + N^i\tilde{\mathcal{H}}_i\right)$$

Physical Hamiltonian Density

$$\{A, B\} \equiv \int \mathrm{d}^3z\,\left(\frac{\delta A}{\delta\tilde{h}_{mn}(z)}\frac{\delta B}{\delta\tilde{\pi}^{mn}(z)} - \frac{\delta A}{\delta\tilde{\pi}^{mn}(z)}\frac{\delta B}{\delta\tilde{h}_{mn}(z)}\right)$$



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Degrees of Freedom

- Phase Space: $\left(\tilde{h}_{ij}, \tilde{\pi}^{ij}\right)$
- First Class Constraints: $\tilde{\mathcal{H}}_i$

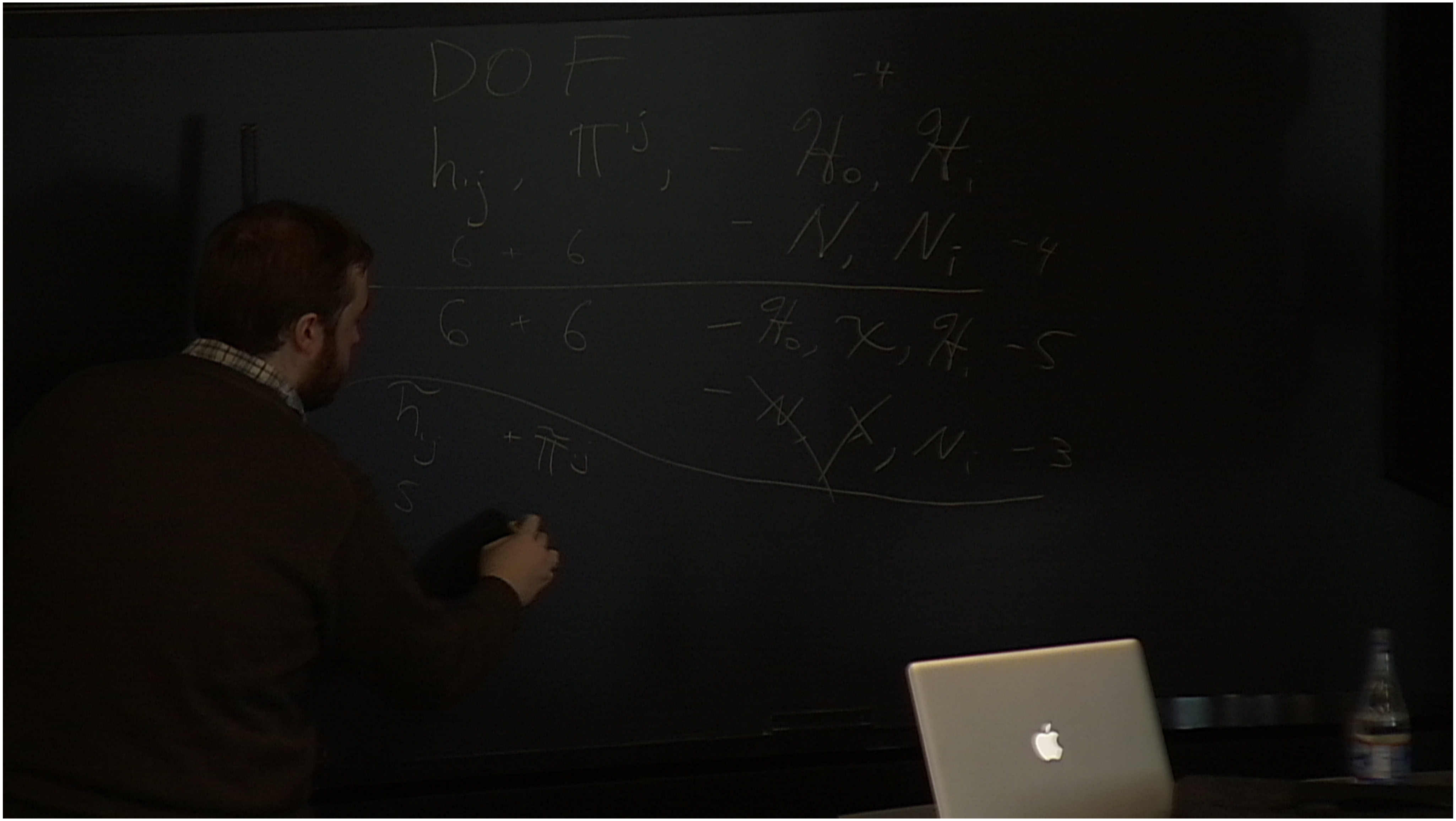
$$\boxed{\{\tilde{\mathcal{H}}_i(x), \tilde{\mathcal{H}}_j(y)\} \sim 0}$$

- Arbitrary Functions: N^i

5 $\tilde{h}_{ij}$'s + 5 $\tilde{\pi}^{ij}$'s - 3 $\tilde{\mathcal{H}}_i$'s - 3 N^i's
= 4 canonical degrees of freedom

Counting depends **solely** on

- Size of phase space
- First class constraint structure



DOF

-4

$h_{ij}, \; \pi^{ij}, \quad - \; \mathcal{H}_0, \; \mathcal{H}_i$

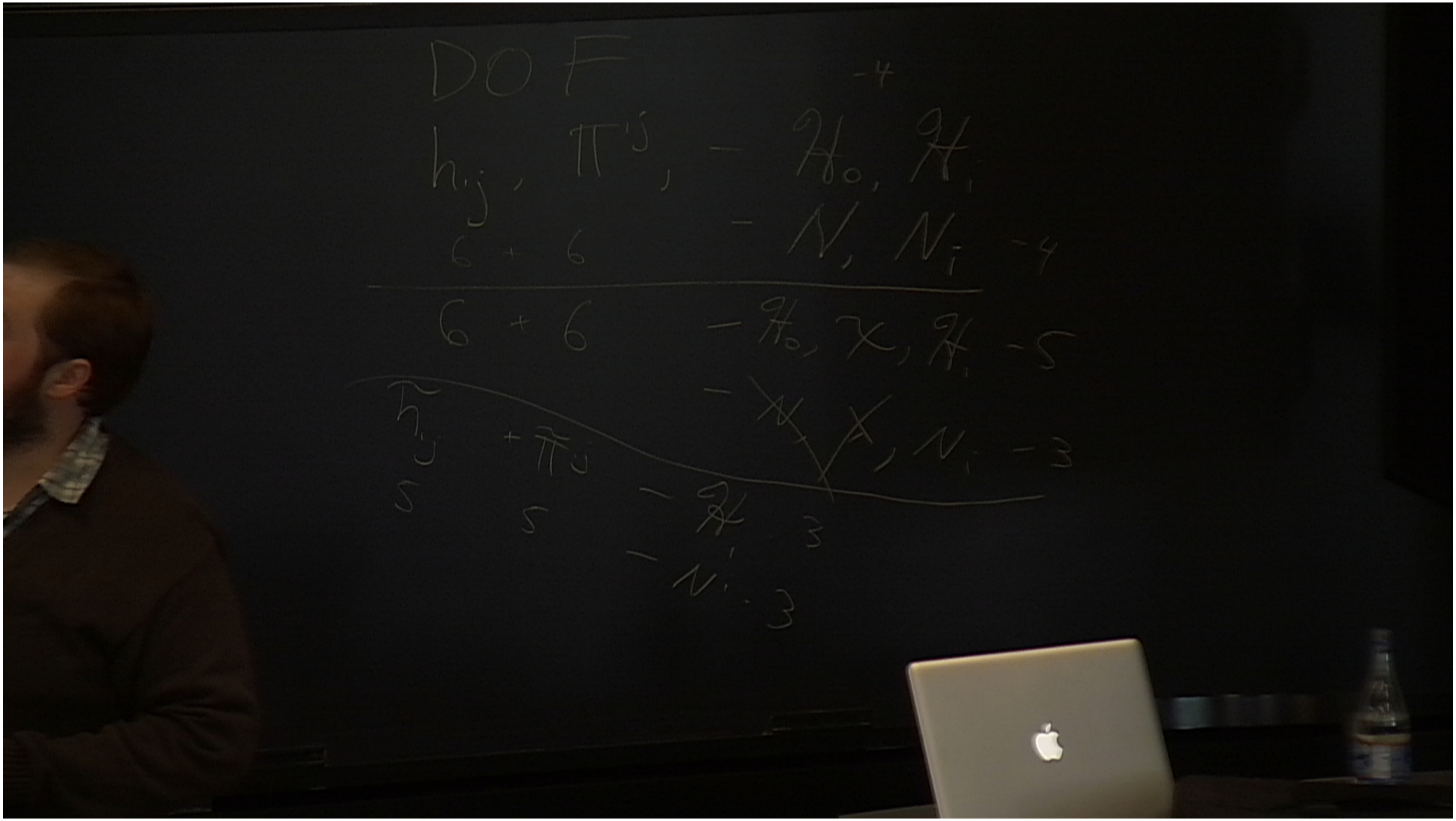
$6 + 6 \quad\quad - \; N, \; N_i \quad -4$

$6 + 6 \quad\quad - \; \mathcal{H}_0, \; \chi, \; \mathcal{H}_i \quad -5$

$- \; \cancel{N}, \; \cancel{\chi}, \; N_i \quad -3$

$\tilde{h}_{ij} + \tilde{\pi}^{ij}$

5



DOF

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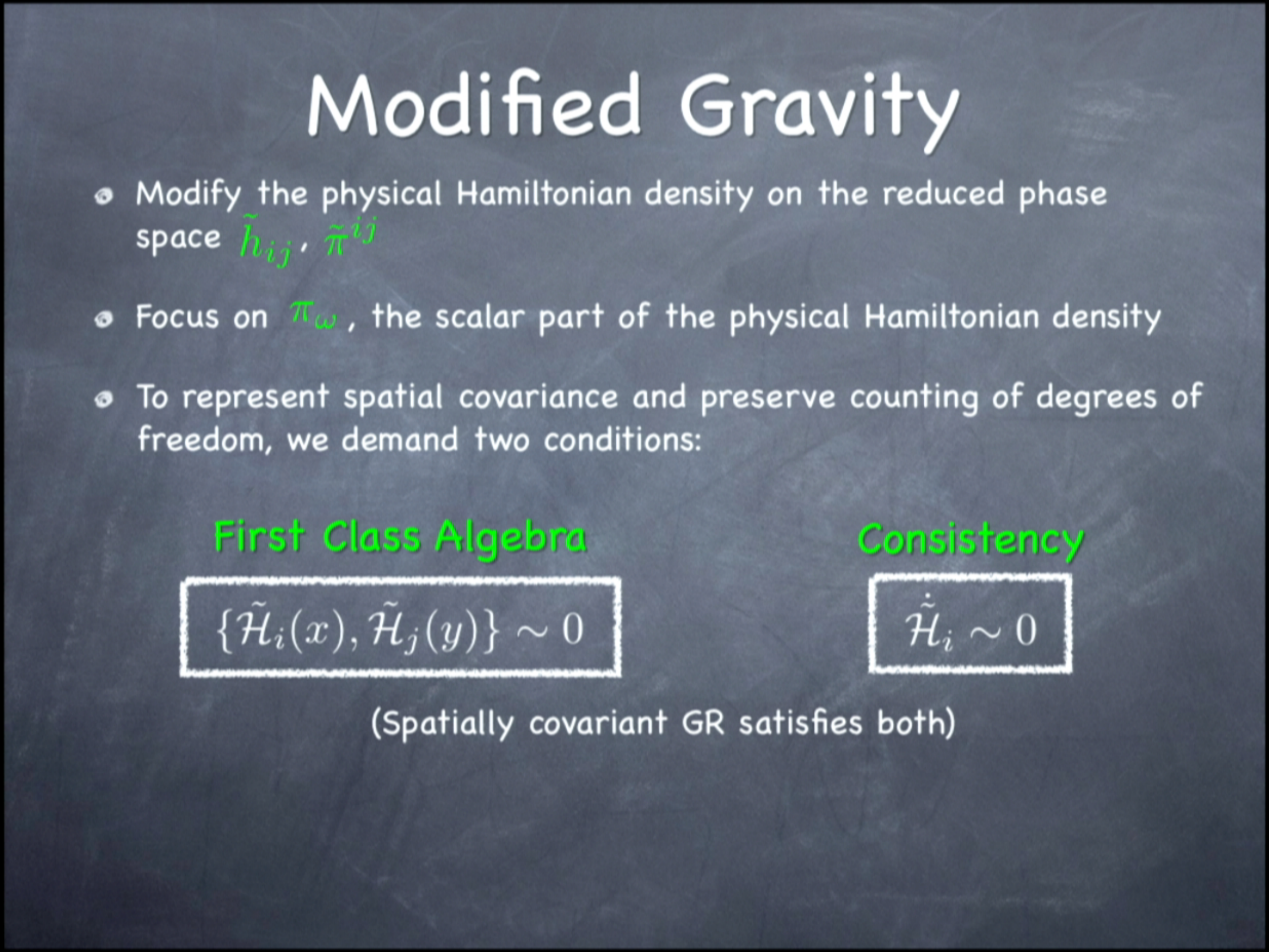
$\tilde{h}_{ij} + \tilde{\pi}^{ij} \quad - \mathcal{H}_i \quad 3$

$5 \quad 5$

$- N_i \quad 3$



Modified Gravity

- Modify the physical Hamiltonian density on the reduced phase space $\tilde{h}_{ij}$, $\tilde{\pi}^{ij}$
- Focus on π_ω, the scalar part of the physical Hamiltonian density
- To represent spatial covariance and preserve counting of degrees of freedom, we demand two conditions:

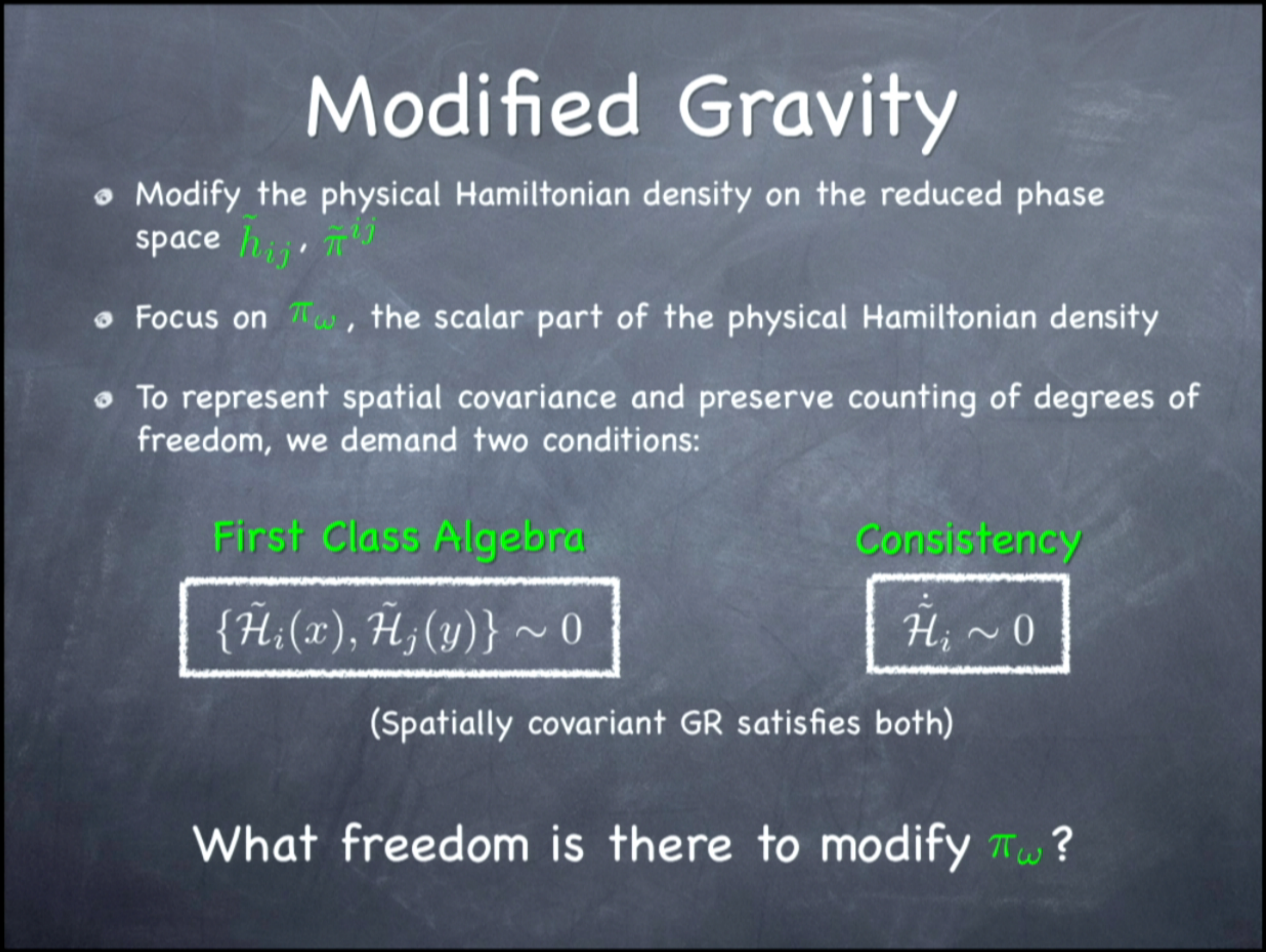
First Class Algebra

$$\boxed{\{\tilde{\mathcal{H}}_i(x), \tilde{\mathcal{H}}_j(y)\} \sim 0}$$

Consistency

$$\boxed{\dot{\tilde{\mathcal{H}}}_i \sim 0}$$

(Spatially covariant GR satisfies both)



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(Spatially covariant GR satisfies both)

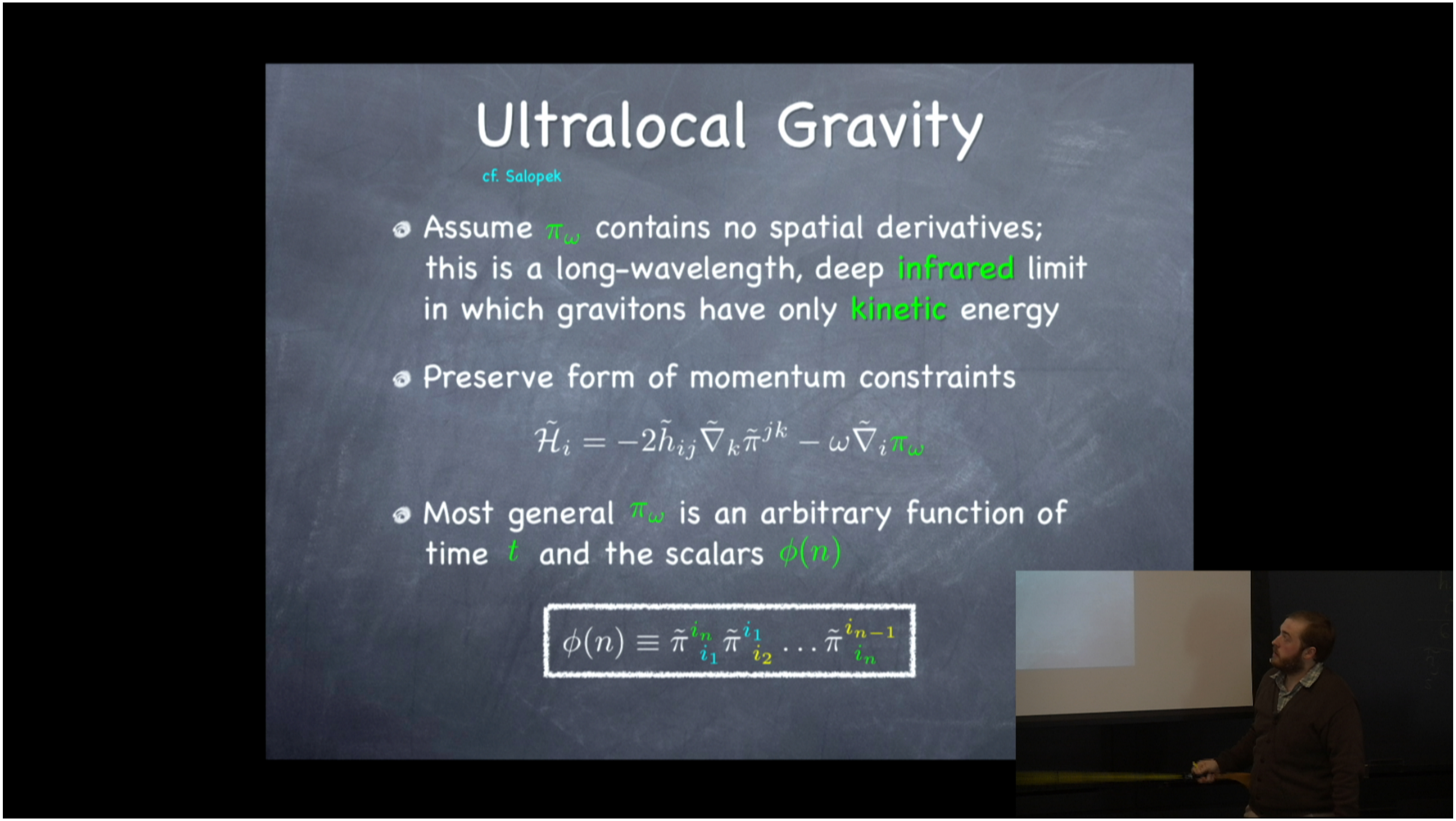
What freedom is there to modify π_ω?

Gradient Expansion

- Expand in powers of spatial derivatives $\tilde{\nabla}_i$

$$\pi_\omega = \pi_0 + \pi_1(\tilde{\nabla}_i) + \pi_2(\tilde{\nabla}_i \tilde{\nabla}_j) + \ldots$$

- Aside from the momentum constraints $\tilde{\mathcal{H}}_i$, there are no vector quantities, so π_1 vanishes
- Taking only the lowest order term yields an **ultralocal** theory of gravity



Ultralocal Gravity

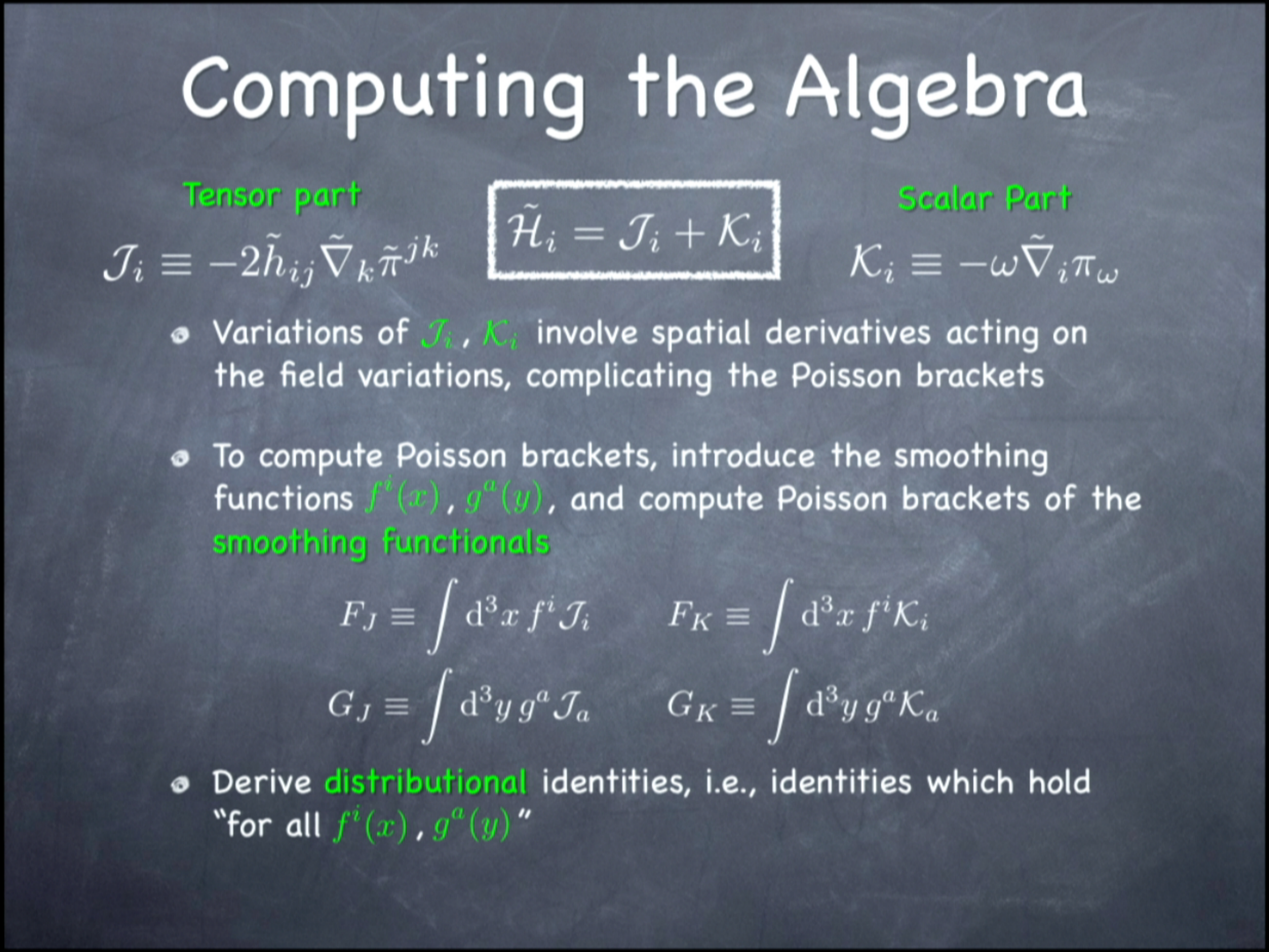
cf. Salopek

- Assume π_ω contains no spatial derivatives; this is a long-wavelength, deep infrared limit in which gravitons have only kinetic energy
- Preserve form of momentum constraints

$$\tilde{\mathcal{H}}_i = -2\tilde{h}_{ij}\tilde{\nabla}_k\tilde{\pi}^{jk} - \omega\tilde{\nabla}_i\pi_\omega$$

- Most general π_ω is an arbitrary function of time t and the scalars $\phi(n)$

$$\boxed{\phi(n) \equiv \tilde{\pi}^{i_n}{}_{i_1}\tilde{\pi}^{i_1}{}_{i_2}\dots\tilde{\pi}^{i_{n-1}}{}_{i_n}}$$



Computing the Algebra

Tensor part

$$\mathcal{J}_i \equiv -2\tilde{h}_{ij}\tilde{\nabla}_k\tilde{\pi}^{jk}$$

$$\boxed{\tilde{\mathcal{H}}_i = \mathcal{J}_i + \mathcal{K}_i}$$

Scalar Part

$$\mathcal{K}_i \equiv -\omega\tilde{\nabla}_i\pi_\omega$$

- Variations of $\mathcal{J}_i$, $\mathcal{K}_i$ involve spatial derivatives acting on the field variations, complicating the Poisson brackets
- To compute Poisson brackets, introduce the smoothing functions $f^i(x)$, $g^a(y)$, and compute Poisson brackets of the **smoothing functionals**

$$F_J \equiv \int \mathrm{d}^3x\, f^i \mathcal{J}_i \qquad F_K \equiv \int \mathrm{d}^3x\, f^i \mathcal{K}_i$$

$$G_J \equiv \int \mathrm{d}^3y\, g^a \mathcal{J}_a \qquad G_K \equiv \int \mathrm{d}^3y\, g^a \mathcal{K}_a$$

- Derive **distributional** identities, i.e., identities which hold “for all $f^i(x)$, $g^a(y)$ ”

Sample Bracket

$$F_J \equiv \int \mathrm{d}^3x\, f^i \mathcal{J}_i \qquad\qquad G_J \equiv \int \mathrm{d}^3y\, g^a \mathcal{J}_a$$

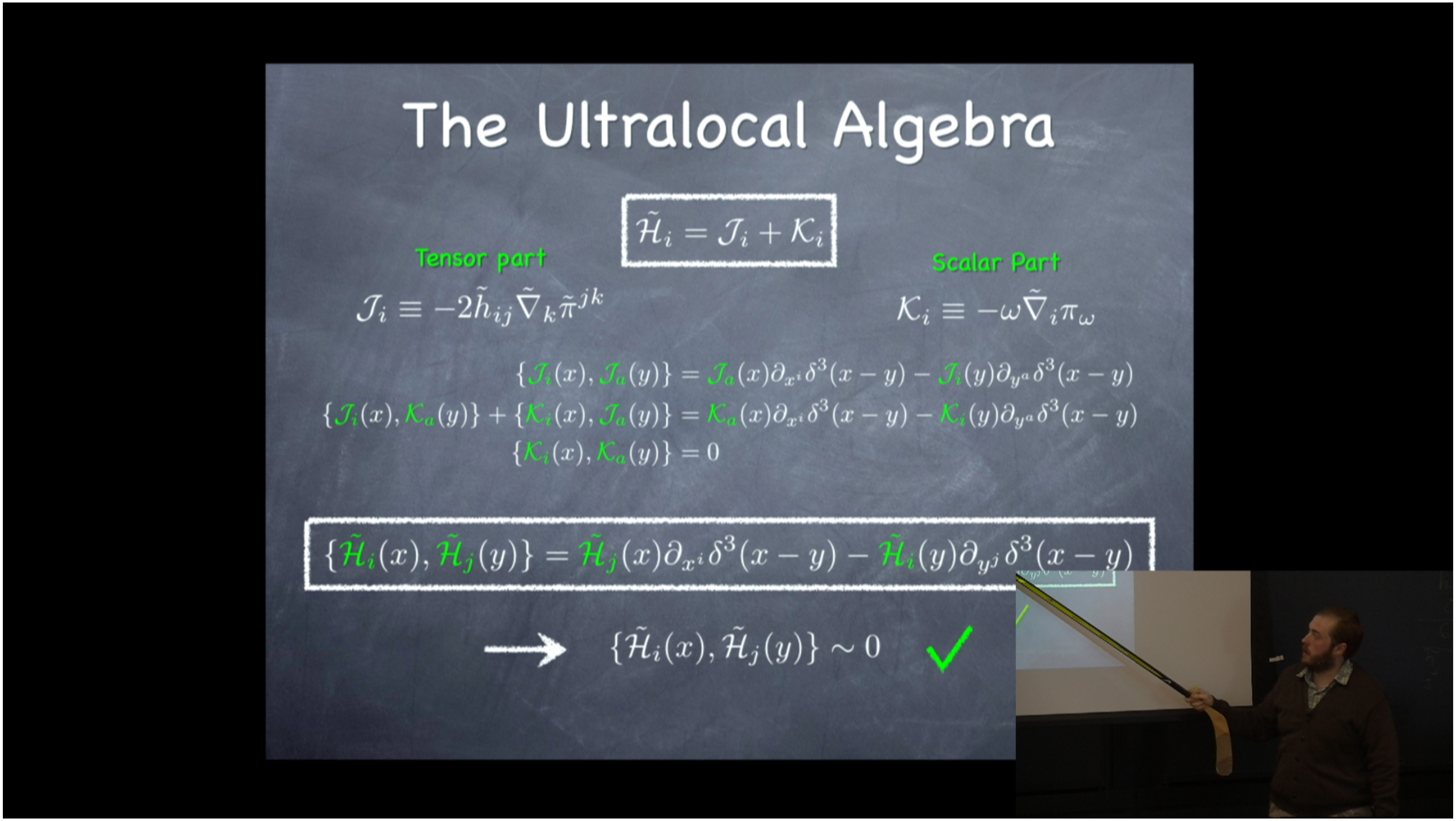
$$\{F_J, G_J\} = \int \mathrm{d}^3x\, \mathrm{d}^3y\, f^i(x) g^a(y) \{\mathcal{J}_i(x), \mathcal{J}_a(y)\}$$

$$\frac{\delta F_J}{\delta \tilde{h}_{mn}} = 2\, \tilde{\delta}_{ij}^{mn}\, \tilde{\pi}^{jk} \tilde{\nabla}_k f^i - \tilde{\nabla}_i \left(f^i \tilde{\pi}^{mn}\right) - \frac{2}{3} \tilde{\pi}^{mn} \tilde{\nabla}_i f^i \qquad \frac{\delta F_J}{\delta \tilde{\pi}^{mn}} = 2\, \tilde{\delta}_{mn}^{jk}\, \tilde{h}_{ij} \tilde{\nabla}_k f^i$$

$$\begin{aligned}\{F_J, G_J\} = 2 \int \mathrm{d}^3z\, \Big\{ &\left(\tilde{\nabla}_c f^i\right) \left(\tilde{\nabla}_i g^a\right) \tilde{h}_{ab} \tilde{\pi}^{bc} - \left(\tilde{\nabla}_k g^a\right) \left(\tilde{\nabla}_a f^i\right) \tilde{h}_{ij} \tilde{\pi}^{jk} \\ &+ \left(\tilde{\nabla}_k f^i\right) \tilde{\nabla}_a \left(g^a \tilde{h}_{ij} \tilde{\pi}^{jk}\right) - \left(\tilde{\nabla}_c g^a\right) \tilde{\nabla}_i \left(f^i \tilde{h}_{ab} \tilde{\pi}^{bc}\right) \Big\}\end{aligned}$$

$$\{F_J, G_J\} = \int \mathrm{d}^3x\, \mathrm{d}^3y\, f^i(x) g^a(y) \left(\mathcal{J}_a(x) \partial_{x^i} \delta^3(x-y) - \mathcal{J}_i(y) \partial_{y^a} \delta^3(x-y)\right)$$

$$\boxed{\{\mathcal{J}_i(x), \mathcal{J}_a(y)\} = \mathcal{J}_a(x) \partial_{x^i} \delta^3(x-y) - \mathcal{J}_i(y) \partial_{y^a} \delta^3(x-y)}$$



The Ultralocal Algebra

$$\boxed{\tilde{\mathcal{H}}_i = \mathcal{J}_i + \mathcal{K}_i}$$

Tensor part

$$\mathcal{J}_i \equiv -2\tilde{h}_{ij}\tilde{\nabla}_k\tilde{\pi}^{jk}$$

Scalar Part

$$\mathcal{K}_i \equiv -\omega\tilde{\nabla}_i\pi_\omega$$

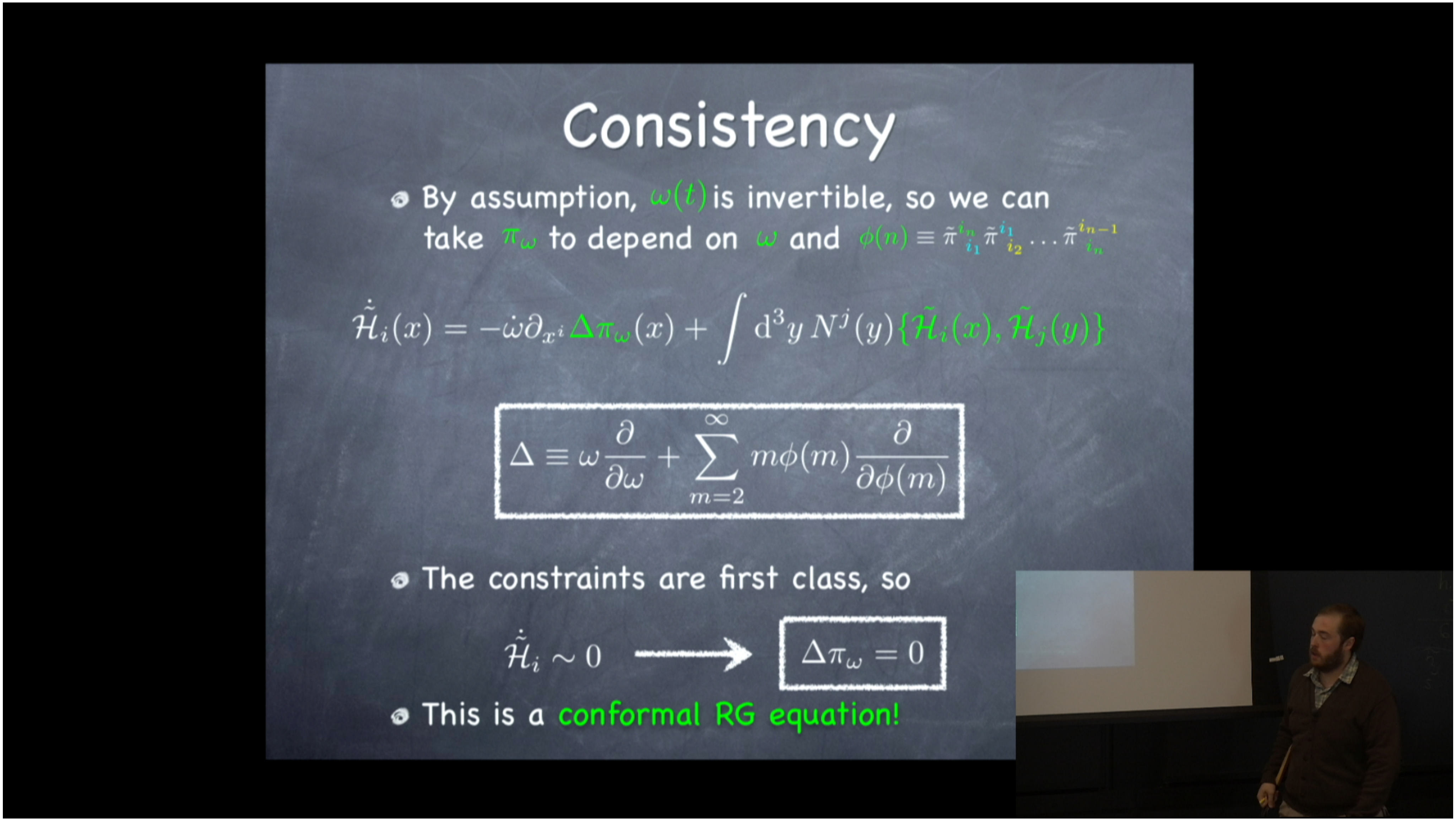
$$\{\mathcal{J}_i(x), \mathcal{J}_a(y)\} = \mathcal{J}_a(x)\partial_{x^i}\delta^3(x-y) - \mathcal{J}_i(y)\partial_{y^a}\delta^3(x-y)$$

$$\{\mathcal{J}_i(x), \mathcal{K}_a(y)\} + \{\mathcal{K}_i(x), \mathcal{J}_a(y)\} = \mathcal{K}_a(x)\partial_{x^i}\delta^3(x-y) - \mathcal{K}_i(y)\partial_{y^a}\delta^3(x-y)$$

$$\{\mathcal{K}_i(x), \mathcal{K}_a(y)\} = 0$$

$$\boxed{\{\tilde{\mathcal{H}}_i(x), \tilde{\mathcal{H}}_j(y)\} = \tilde{\mathcal{H}}_j(x)\partial_{x^i}\delta^3(x-y) - \tilde{\mathcal{H}}_i(y)\partial_{y^j}\delta^3(x-y)}$$

$$\longrightarrow \quad \{\tilde{\mathcal{H}}_i(x), \tilde{\mathcal{H}}_j(y)\} \sim 0 \quad ✓$$



Consistency

- By assumption, $\omega(t)$ is invertible, so we can take π_ω to depend on ω and $\phi(n) \equiv \tilde{\pi}^{i_n}_{i_1} \tilde{\pi}^{i_1}_{i_2} \dots \tilde{\pi}^{i_{n-1}}_{i_n}$

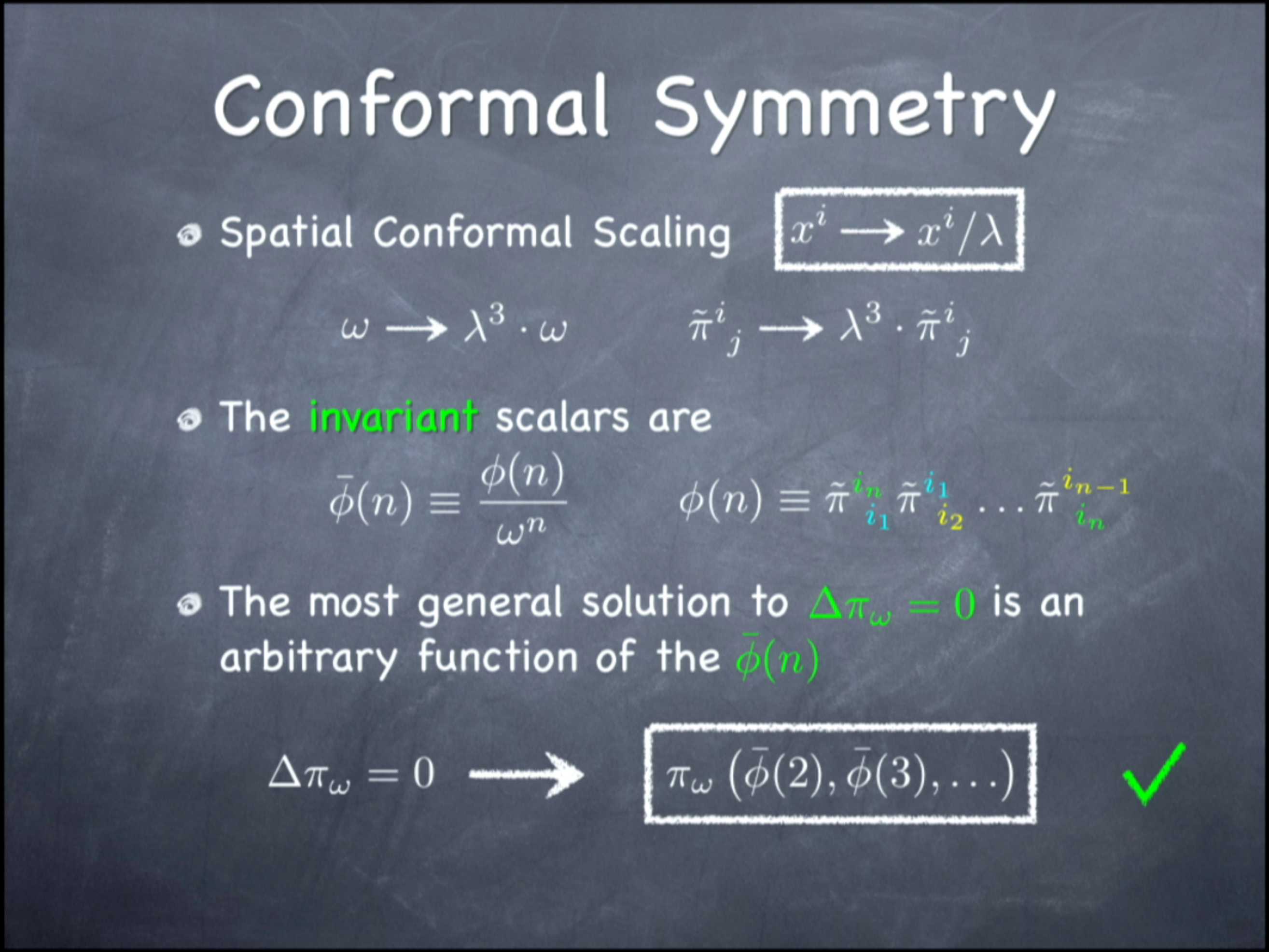
$$\dot{\tilde{\mathcal{H}}}_i(x) = -\dot{\omega}\partial_{x^i} \Delta\pi_\omega(x) + \int \mathrm{d}^3y\, N^j(y) \{\tilde{\mathcal{H}}_i(x), \tilde{\mathcal{H}}_j(y)\}$$

$$\boxed{\Delta \equiv \omega \frac{\partial}{\partial \omega} + \sum_{m=2}^{\infty} m\phi(m) \frac{\partial}{\partial \phi(m)}}$$

- The constraints are first class, so

$$\dot{\tilde{\mathcal{H}}}_i \sim 0 \longrightarrow \boxed{\Delta\pi_\omega = 0}$$

- This is a conformal RG equation!



Conformal Symmetry

- Spatial Conformal Scaling $\boxed{x^i \longrightarrow x^i/\lambda}$

$$\omega \longrightarrow \lambda^3 \cdot \omega \qquad \tilde{\pi}^i{}_j \longrightarrow \lambda^3 \cdot \tilde{\pi}^i{}_j$$

- The invariant scalars are

$$\bar{\phi}(n) \equiv \frac{\phi(n)}{\omega^n} \qquad \phi(n) \equiv \tilde{\pi}^{i_n}{}_{i_1} \tilde{\pi}^{i_1}{}_{i_2} \ldots \tilde{\pi}^{i_{n-1}}{}_{i_n}$$

- The most general solution to $\Delta\pi_\omega = 0$ is an arbitrary function of the $\bar{\phi}(n)$

$$\Delta\pi_\omega = 0 \longrightarrow \boxed{\pi_\omega\left(\bar{\phi}(2), \bar{\phi}(3), \ldots\right)} \quad ✓$$

Spatially Covariant GR

$$\boxed{S = \int \mathrm{d}t\, \mathrm{d}^3x \left(\dot{\tilde{h}}_{ij}\tilde{\pi}^{ij} + \dot{\omega}\pi_\omega - N^i\tilde{\mathcal{H}}_i \right)}$$

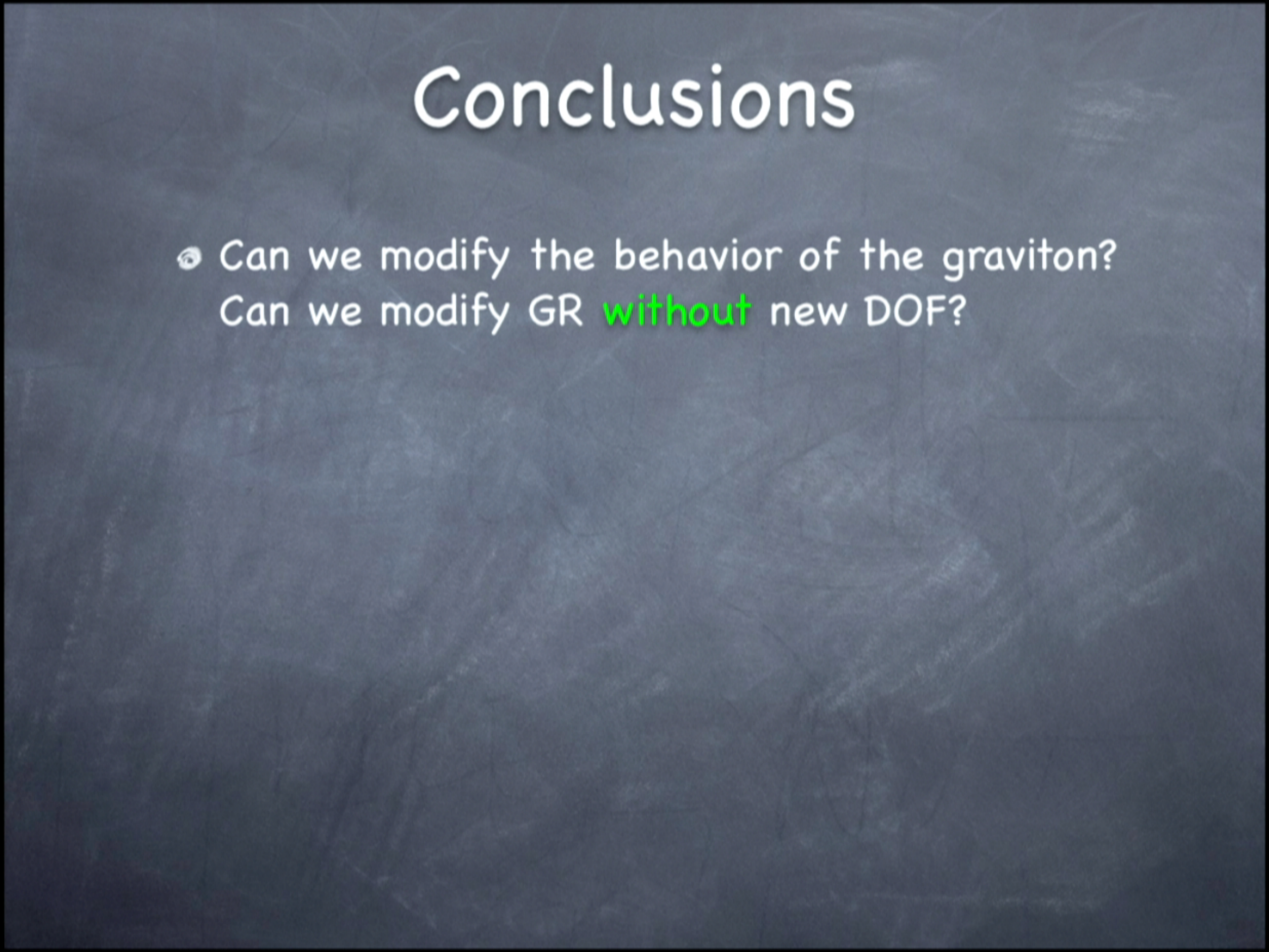
$$\pi_\omega = \pi_{\mathrm{GR}} \qquad \tilde{\mathcal{H}}_i = -2\tilde{h}_{ij}\tilde{\nabla}_k\tilde{\pi}^{jk} - \omega\tilde{\nabla}_i\pi_\omega$$

N^i's are Lagrange multipliers enforcing $\boxed{\tilde{\mathcal{H}}_i \sim 0}$

$$\dot{A} = \frac{\partial A}{\partial t} + \{A, H\} \qquad H = \int \mathrm{d}^3x \left(\underbrace{-\dot{\omega}\pi_\omega}_{} + N^i\tilde{\mathcal{H}}_i \right)$$

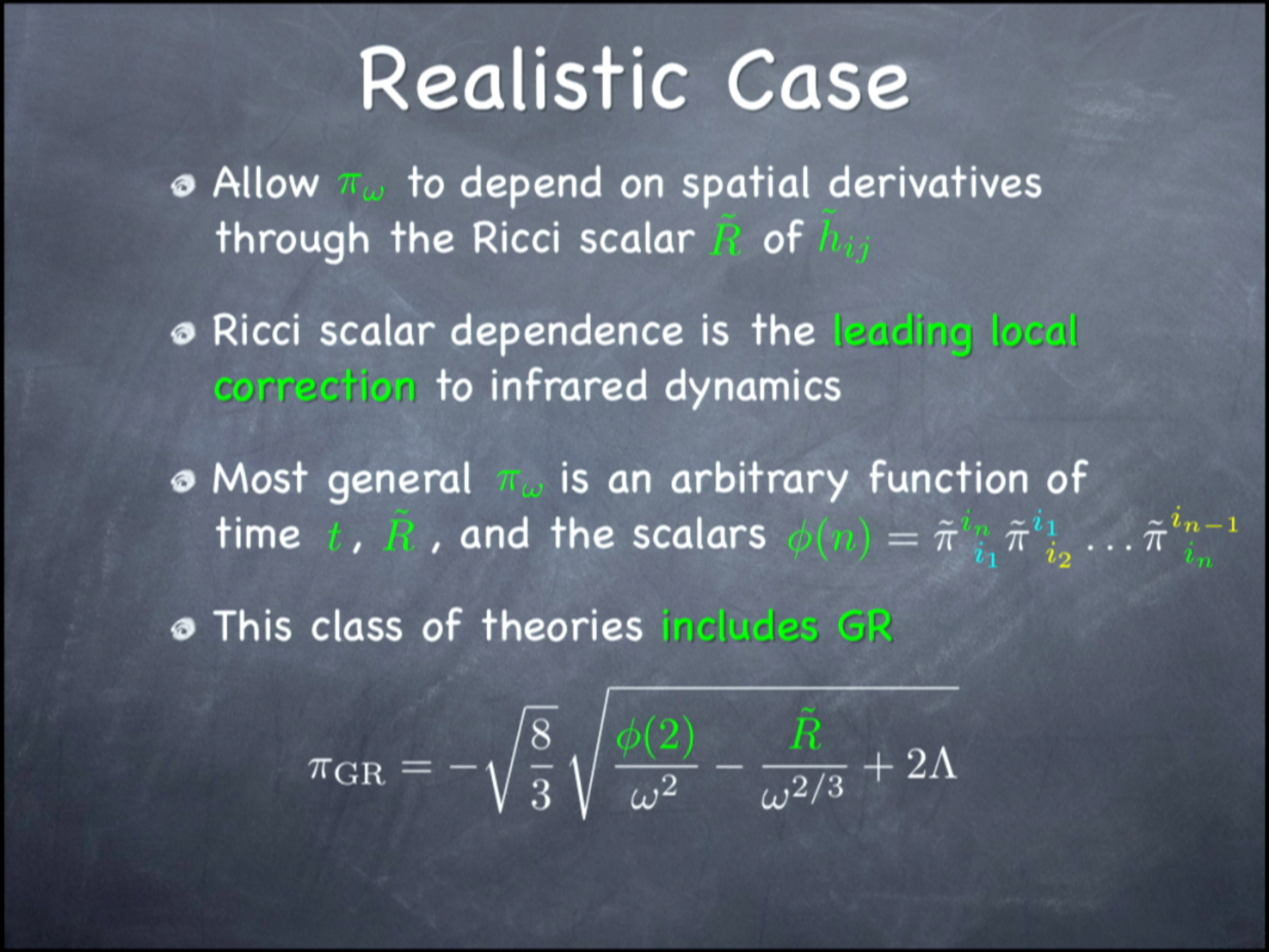
Physical Hamiltonian Density ↑

$$\{A, B\} \equiv \int \mathrm{d}^3z \left(\frac{\delta A}{\delta\tilde{h}_{mn}(z)}\frac{\delta B}{\delta\tilde{\pi}^{mn}(z)} - \frac{\delta A}{\delta\tilde{\pi}^{mn}(z)}\frac{\delta B}{\delta\tilde{h}_{mn}(z)} \right)$$



Conclusions

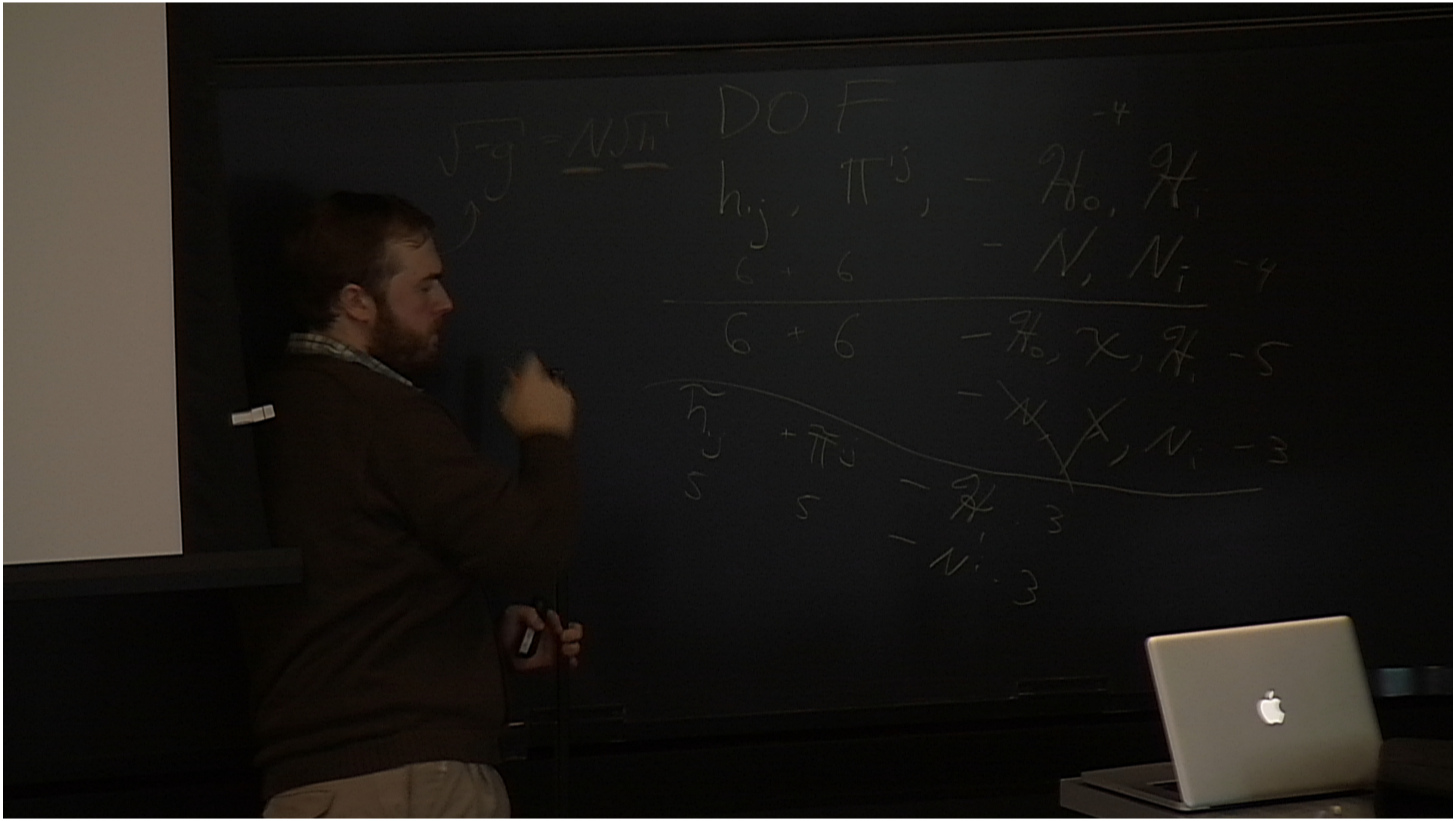
- Can we modify the behavior of the graviton? Can we modify GR without new DOF?

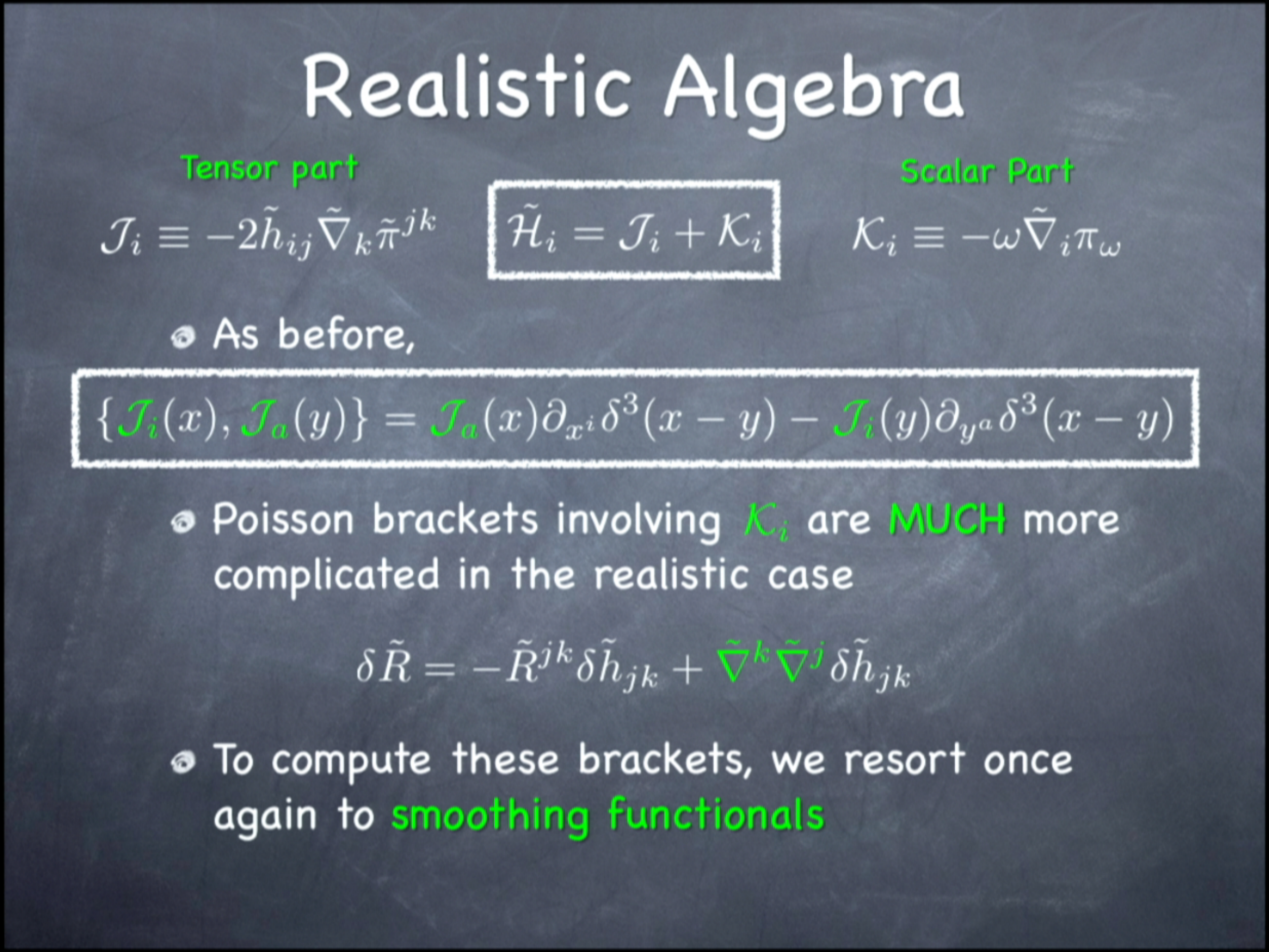


Realistic Case

- Allow π_ω to depend on spatial derivatives through the Ricci scalar $\tilde{R}$ of $\tilde{h}_{ij}$
- Ricci scalar dependence is the **leading local correction** to infrared dynamics
- Most general π_ω is an arbitrary function of time t, $\tilde{R}$, and the scalars $\phi(n) = \tilde{\pi}^{i_n}{}_{i_1} \tilde{\pi}^{i_1}{}_{i_2} \dots \tilde{\pi}^{i_{n-1}}{}_{i_n}$
- This class of theories **includes GR**

$$\pi_{\mathrm{GR}} = -\sqrt{\frac{8}{3}} \sqrt{\frac{\phi(2)}{\omega^2} - \frac{\tilde{R}}{\omega^{2/3}} + 2\Lambda}$$


$\sqrt{-g} = \underline{N}\underline{\sqrt{h}}$
DOF
-4
$h_{ij}\,,\ \pi^{ij}\,,\ -\ \mathcal{H}_0,\ \mathcal{H}_i$
$6 + 6\ \ -\ N,\ N_i\ -4$
$6 + 6\ \ -\ \mathcal{H}_0,\ \chi,\ \mathcal{H}_i\ -5$
$-\ \cancel{N},\ \cancel{\chi},\ N_i\ -3$
$\tilde{h}_{ij} + \tilde{\pi}^{ij}$
$5\ \ 5$
$-\ \mathcal{H}_i : 3$
$-\ N_i : 3$



Realistic Algebra

Tensor part

Scalar Part

$$\mathcal{J}_i \equiv -2\tilde{h}_{ij}\tilde{\nabla}_k\tilde{\pi}^{jk} \qquad \boxed{\tilde{\mathcal{H}}_i = \mathcal{J}_i + \mathcal{K}_i} \qquad \mathcal{K}_i \equiv -\omega\tilde{\nabla}_i\pi_\omega$$

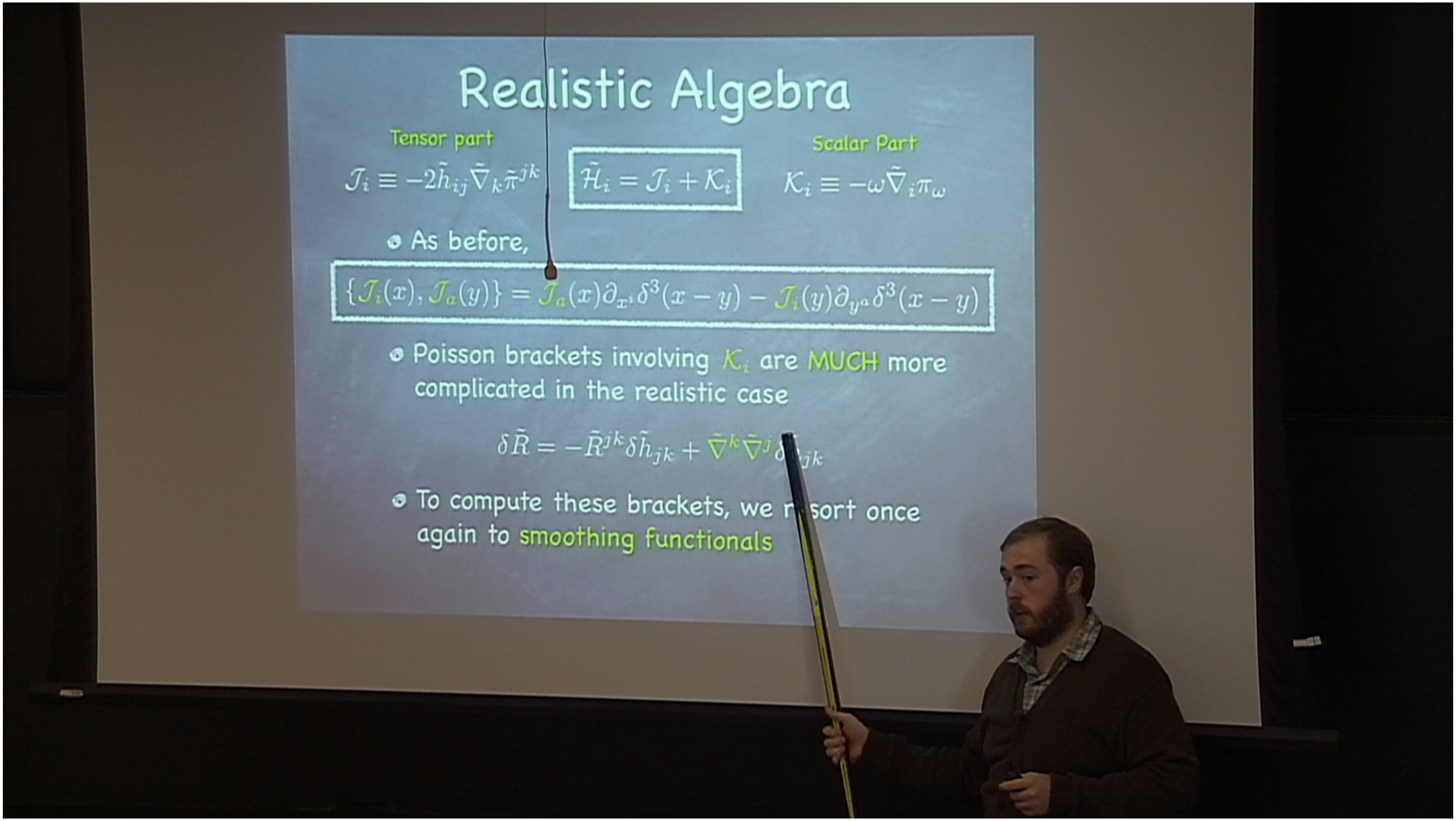
- As before,

$$\boxed{\{\mathcal{J}_i(x), \mathcal{J}_a(y)\} = \mathcal{J}_a(x)\partial_{x^i}\delta^3(x-y) - \mathcal{J}_i(y)\partial_{y^a}\delta^3(x-y)}$$

- Poisson brackets involving $\mathcal{K}_i$ are **MUCH** more complicated in the realistic case

$$\delta\tilde{R} = -\tilde{R}^{jk}\delta\tilde{h}_{jk} + \tilde{\nabla}^k\tilde{\nabla}^j\delta\tilde{h}_{jk}$$

- To compute these brackets, we resort once again to smoothing functionals



Realistic Algebra

Tensor part

$\mathcal{J}_i \equiv -2\tilde{h}_{ij}\tilde{\nabla}_k\tilde{\pi}^{jk}$

$\boxed{\tilde{\mathcal{H}}_i = \mathcal{J}_i + \mathcal{K}_i}$

Scalar Part

$\mathcal{K}_i \equiv -\omega\tilde{\nabla}_i\pi_\omega$

- As before,

$$\boxed{\{\mathcal{J}_i(x),\mathcal{J}_a(y)\} = \mathcal{J}_a(x)\partial_{x^i}\delta^3(x-y) - \mathcal{J}_i(y)\partial_{y^a}\delta^3(x-y)}$$

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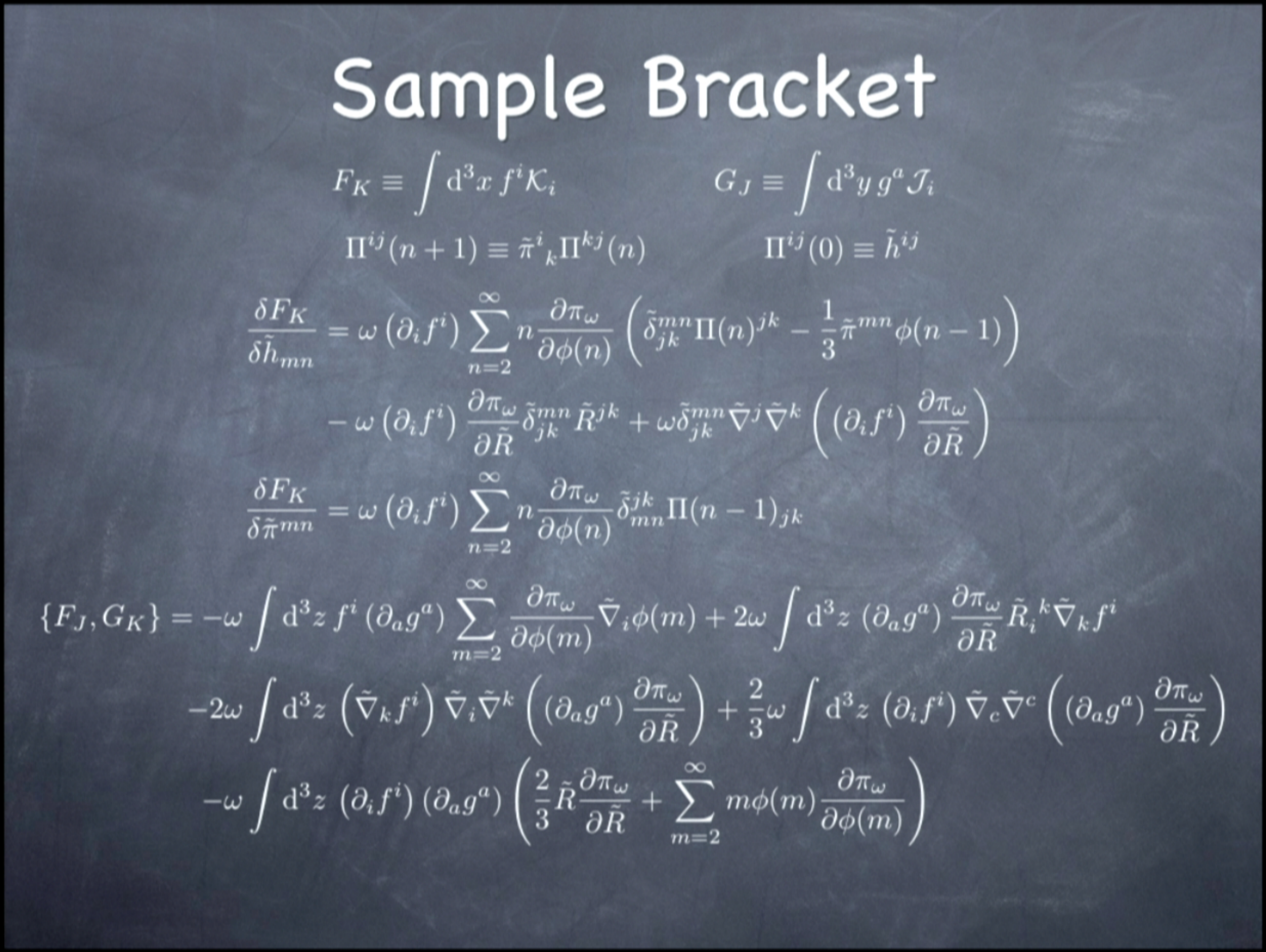


Sample Bracket

$$F_K \equiv \int \mathrm{d}^3x\, f^i \mathcal{K}_i \qquad G_J \equiv \int \mathrm{d}^3y\, g^a \mathcal{J}_i$$

$$\Pi^{ij}(n+1) \equiv \tilde{\pi}^i{}_k \Pi^{kj}(n) \qquad \Pi^{ij}(0) \equiv \tilde{h}^{ij}$$

$$\begin{aligned} \frac{\delta F_K}{\delta \tilde{h}_{mn}} &= \omega \left(\partial_i f^i\right) \sum_{n=2}^{\infty} n \frac{\partial \pi_\omega}{\partial \phi(n)} \left(\tilde{\delta}^{mn}_{jk} \Pi(n)^{jk} - \frac{1}{3} \tilde{\pi}^{mn} \phi(n-1) \right) \\ &\quad - \omega \left(\partial_i f^i\right) \frac{\partial \pi_\omega}{\partial \tilde{R}} \tilde{\delta}^{mn}_{jk} \tilde{R}^{jk} + \omega \tilde{\delta}^{mn}_{jk} \tilde{\nabla}^j \tilde{\nabla}^k \left(\left(\partial_i f^i\right) \frac{\partial \pi_\omega}{\partial \tilde{R}} \right) \end{aligned}$$

$$\frac{\delta F_K}{\delta \tilde{\pi}^{mn}} = \omega \left(\partial_i f^i\right) \sum_{n=2}^{\infty} n \frac{\partial \pi_\omega}{\partial \phi(n)} \tilde{\delta}^{jk}_{mn} \Pi(n-1)_{jk}$$

$$\begin{aligned} \{F_J, G_K\} = &-\omega \int \mathrm{d}^3z\, f^i \left(\partial_a g^a\right) \sum_{m=2}^{\infty} \frac{\partial \pi_\omega}{\partial \phi(m)} \tilde{\nabla}_i \phi(m) + 2\omega \int \mathrm{d}^3z\, \left(\partial_a g^a\right) \frac{\partial \pi_\omega}{\partial \tilde{R}} \tilde{R}_i{}^k \tilde{\nabla}_k f^i \\ &-2\omega \int \mathrm{d}^3z\, \left(\tilde{\nabla}_k f^i\right) \tilde{\nabla}_i \tilde{\nabla}^k \left(\left(\partial_a g^a\right) \frac{\partial \pi_\omega}{\partial \tilde{R}} \right) + \frac{2}{3}\omega \int \mathrm{d}^3z\, \left(\partial_i f^i\right) \tilde{\nabla}_c \tilde{\nabla}^c \left(\left(\partial_a g^a\right) \frac{\partial \pi_\omega}{\partial \tilde{R}} \right) \\ &-\omega \int \mathrm{d}^3z\, \left(\partial_i f^i\right) \left(\partial_a g^a\right) \left(\frac{2}{3} \tilde{R} \frac{\partial \pi_\omega}{\partial \tilde{R}} + \sum_{m=2}^{\infty} m \phi(m) \frac{\partial \pi_\omega}{\partial \phi(m)} \right) \end{aligned}$$

Sample Bracket

$$\{F_J, G_K\} = \int \mathrm{d}^3z\, f^i \mathcal{K}_i \partial_a g^a + \frac{4}{3}\omega \int \mathrm{d}^3z\, \tilde{\nabla}_k \left(\partial_i f^i\right) \tilde{\nabla}^k \left(\left(\partial_a g^a\right) \frac{\partial \pi_\omega}{\partial \tilde{R}} \right)$$

$$-\omega \int \mathrm{d}^3z\, \left(\partial_i f^i\right) \left(\partial_a g^a\right) \left(\frac{2}{3}\tilde{R} \frac{\partial \pi_\omega}{\partial \tilde{R}} + \sum_{m=2}^{\infty} m\phi(m) \frac{\partial \pi_\omega}{\partial \phi(m)} \right)$$

$$\{F_K, G_J\} = -\int \mathrm{d}^3z\, g^a \mathcal{K}_a \partial_i f^i - \frac{4}{3}\omega \int \mathrm{d}^3z\, \tilde{\nabla}_k \left(\partial_a g^a\right) \tilde{\nabla}^k \left(\left(\partial_i f^i\right) \frac{\partial \pi_\omega}{\partial \tilde{R}} \right)$$

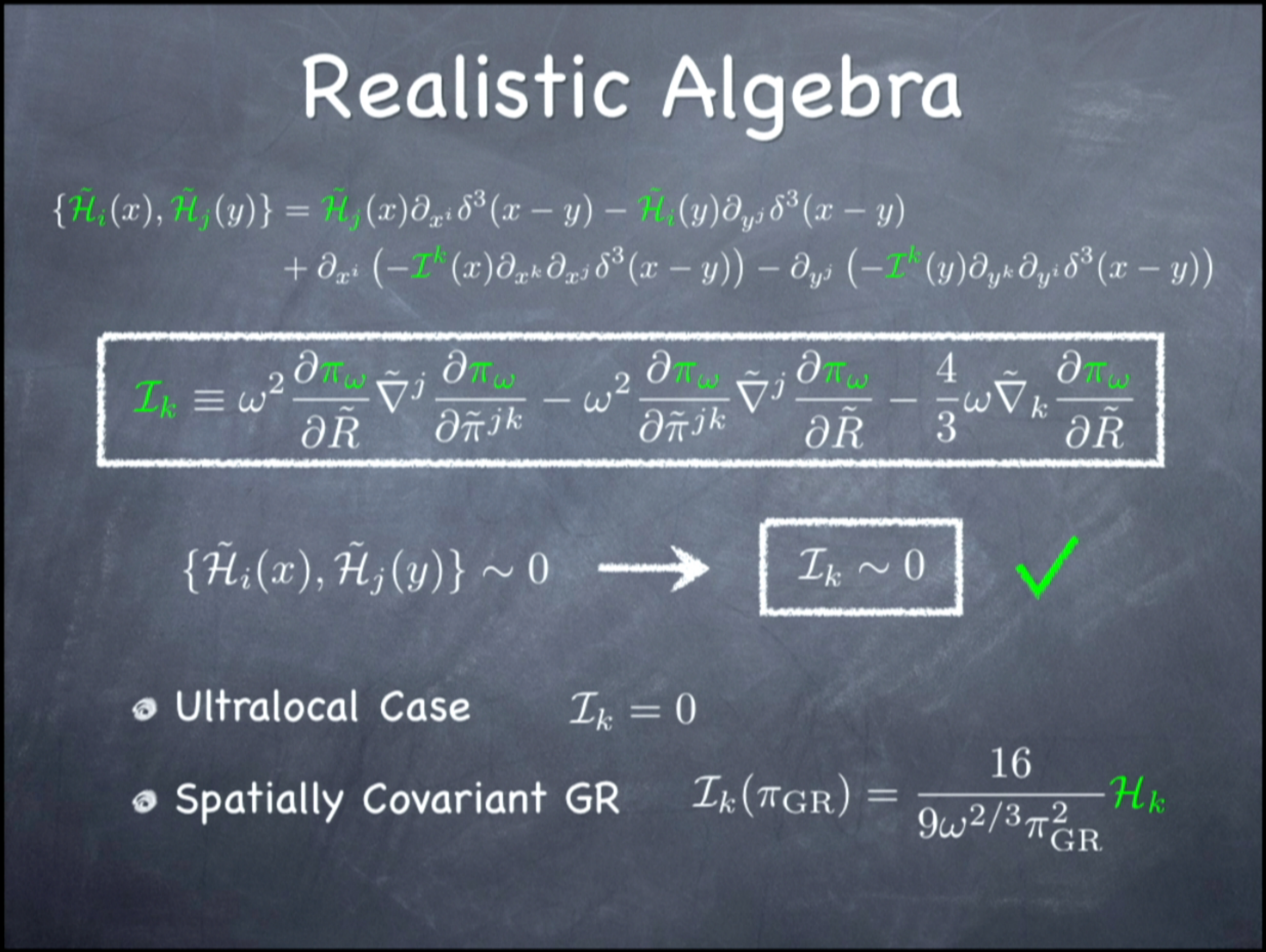
$$+\omega \int \mathrm{d}^3z\, \left(\partial_i f^i\right) \left(\partial_a g^a\right) \left(\frac{2}{3}\tilde{R} \frac{\partial \pi_\omega}{\partial \tilde{R}} + \sum_{m=2}^{\infty} m\phi(m) \frac{\partial \pi_\omega}{\partial \phi(m)} \right)$$

$$\{F_J, G_K\} + \{F_K, G_J\} = \int \mathrm{d}^3z\, f^i \mathcal{K}_i \partial_a g^a - \int \mathrm{d}^3z\, g^a \mathcal{K}_a \partial_i f^i$$

$$+\frac{4}{3}\omega \int \mathrm{d}^3z\, \tilde{\nabla}_k \left(\partial_i f^i\right) \tilde{\nabla}^k \left(\left(\partial_a g^a\right) \frac{\partial \pi_\omega}{\partial \tilde{R}} \right)$$

$$-\frac{4}{3}\omega \int \mathrm{d}^3z\, \tilde{\nabla}_k \left(\partial_a g^a\right) \tilde{\nabla}^k \left(\left(\partial_i f^i\right) \frac{\partial \pi_\omega}{\partial \tilde{R}} \right)$$

$$\boxed{\begin{aligned} &\{F_J, G_K\} + \{F_K, G_J\} = \int \mathrm{d}^3z\, f^i \mathcal{K}_a \partial_i g^a - \int \mathrm{d}^3z\, g^a \mathcal{K}_i \partial_a f^i \\ &+ \int \mathrm{d}^3z\, \left(\partial_i f^i\right) \left(\partial_k \partial_a g^a\right) \left(-\frac{4}{3}\omega \tilde{\nabla}_k \frac{\partial \pi_\omega}{\partial \tilde{R}} \right) - \int \mathrm{d}^3z\, \left(\partial_a g^a\right) \left(\partial_k \partial_i f^i\right) \left(-\frac{4}{3}\omega \tilde{\nabla}_k \frac{\partial \pi_\omega}{\partial \tilde{R}} \right) \end{aligned}}$$



Realistic Algebra

$$\{\tilde{\mathcal{H}}_i(x), \tilde{\mathcal{H}}_j(y)\} = \tilde{\mathcal{H}}_j(x)\partial_{x^i}\delta^3(x-y) - \tilde{\mathcal{H}}_i(y)\partial_{y^j}\delta^3(x-y)$$
$$+ \partial_{x^i}\left(-\mathcal{I}^k(x)\partial_{x^k}\partial_{x^j}\delta^3(x-y)\right) - \partial_{y^j}\left(-\mathcal{I}^k(y)\partial_{y^k}\partial_{y^i}\delta^3(x-y)\right)$$

$$\boxed{\mathcal{I}_k \equiv \omega^2 \frac{\partial \pi_\omega}{\partial \tilde{R}} \tilde{\nabla}^j \frac{\partial \pi_\omega}{\partial \tilde{\pi}^{jk}} - \omega^2 \frac{\partial \pi_\omega}{\partial \tilde{\pi}^{jk}} \tilde{\nabla}^j \frac{\partial \pi_\omega}{\partial \tilde{R}} - \frac{4}{3}\omega \tilde{\nabla}_k \frac{\partial \pi_\omega}{\partial \tilde{R}}}$$

$$\{\tilde{\mathcal{H}}_i(x), \tilde{\mathcal{H}}_j(y)\} \sim 0 \longrightarrow \boxed{\mathcal{I}_k \sim 0} \quad ✓$$

- Ultralocal Case $\mathcal{I}_k = 0$
- Spatially Covariant GR $\mathcal{I}_k(\pi_{\mathrm{GR}}) = \frac{16}{9\omega^{2/3}\pi_{\mathrm{GR}}^2}\mathcal{H}_k$

$$\mathcal{I}_k \equiv \omega^2 \frac{\partial \pi_\omega}{\partial \tilde{R}} \tilde{\nabla}^j \frac{\partial \pi_\omega}{\partial \tilde{\pi}^{jk}} - \omega^2 \frac{\partial \pi_\omega}{\partial \tilde{\pi}^{jk}} \tilde{\nabla}^j \frac{\partial \pi_\omega}{\partial \tilde{R}} - \frac{4}{3} \omega \tilde{\nabla}_k \frac{\partial \pi_\omega}{\partial \tilde{R}}$$

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$$\boxed{\pi_{\mathrm{GR}} = -\sqrt{\frac{8}{3}} \sqrt{\frac{\phi(2)}{\omega^2} - \frac{\tilde{R}}{\omega^{2/3}} + 2\Lambda}}$$

$$\mathcal{I}_k(\pi_{\mathrm{GR}}) = -\frac{4}{3} \omega \tilde{\nabla}_k \frac{\partial \pi_{\mathrm{GR}}}{\partial \tilde{R}} - \omega^2 \frac{\partial \pi_{\mathrm{GR}}}{\partial \tilde{R}} \frac{\partial \pi_{\mathrm{GR}}}{\partial \phi(2)} \mathcal{J}_k + 2\omega^2 \tilde{\pi}_{jk} \left(\frac{\partial \pi_{\mathrm{GR}}}{\partial \tilde{R}} \tilde{\nabla}^j \frac{\partial \pi_{\mathrm{GR}}}{\partial \phi(2)} - \frac{\partial \pi_{\mathrm{GR}}}{\partial \phi(2)} \tilde{\nabla}^j \frac{\partial \pi_{\mathrm{GR}}}{\partial \tilde{R}} \right)$$

$$\mathcal{I}_k \equiv \omega^2 \frac{\partial \pi_\omega}{\partial \tilde{R}} \tilde{\nabla}^j \frac{\partial \pi_\omega}{\partial \tilde{\pi}^{jk}} - \omega^2 \frac{\partial \pi_\omega}{\partial \tilde{\pi}^{jk}} \tilde{\nabla}^j \frac{\partial \pi_\omega}{\partial \tilde{R}} - \frac{4}{3}\omega \tilde{\nabla}_k \frac{\partial \pi_\omega}{\partial \tilde{R}}$$

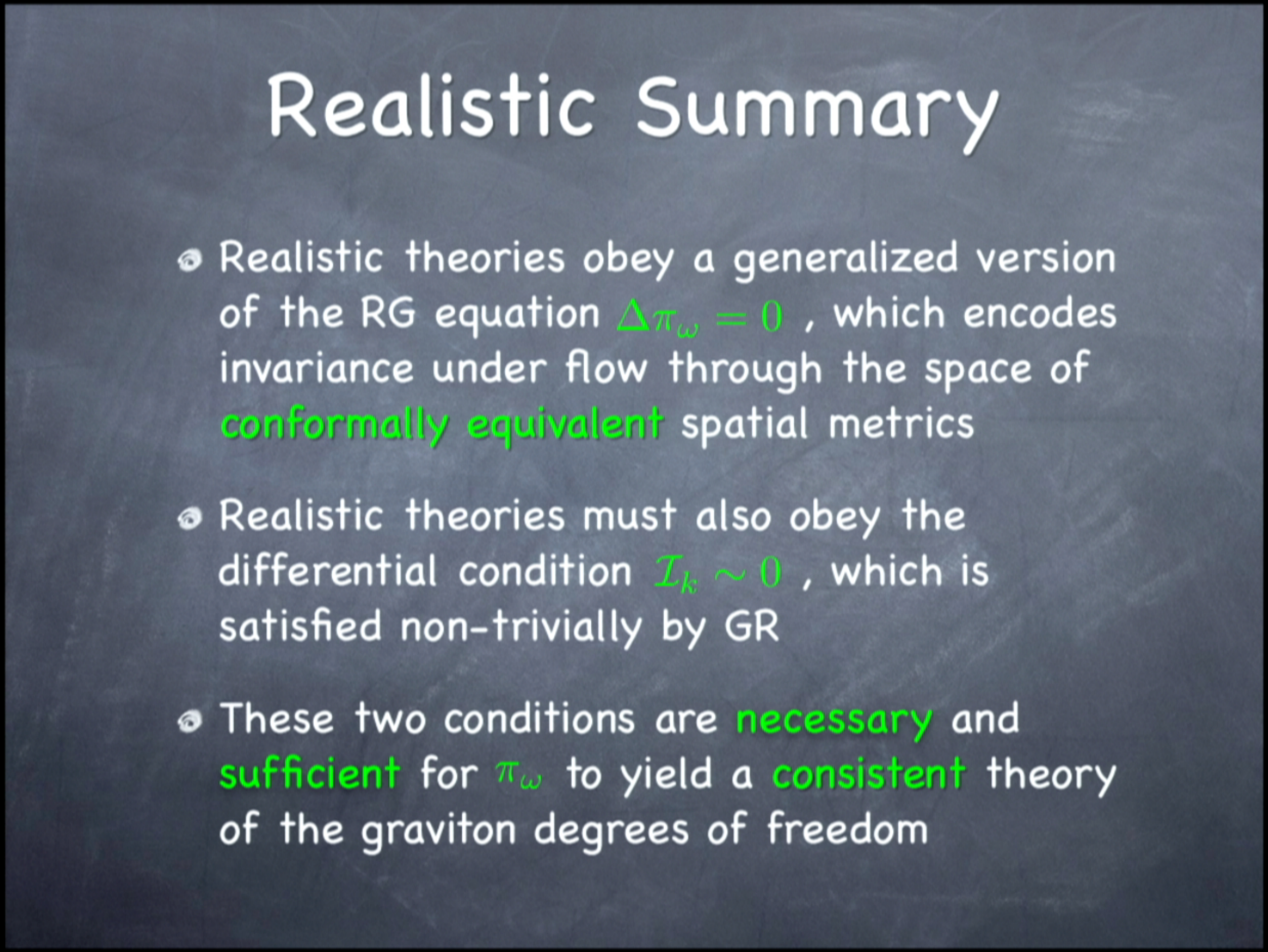
$$\boxed{\pi_{\mathrm{GR}} = -\sqrt{\frac{8}{3}} \sqrt{\frac{\phi(2)}{\omega^2} - \frac{\tilde{R}}{\omega^{2/3}} + 2\Lambda}}$$

$$\mathcal{I}_k(\pi_{\mathrm{GR}}) = -\frac{4}{3}\omega \tilde{\nabla}_k \frac{\partial \pi_{\mathrm{GR}}}{\partial \tilde{R}} - \omega^2 \frac{\partial \pi_{\mathrm{GR}}}{\partial \tilde{R}} \frac{\partial \pi_{\mathrm{GR}}}{\partial \phi(2)} \mathcal{J}_k + 2\omega^2 \tilde{\pi}_{jk} \left(\frac{\partial \pi_{\mathrm{GR}}}{\partial \tilde{R}} \tilde{\nabla}^j \frac{\partial \pi_{\mathrm{GR}}}{\partial \phi(2)} - \frac{\partial \pi_{\mathrm{GR}}}{\partial \phi(2)} \tilde{\nabla}^j \frac{\partial \pi_{\mathrm{GR}}}{\partial \tilde{R}} \right)$$

$$\mathcal{I}_k(\pi_{\mathrm{GR}}) = -\frac{4}{3}\omega \tilde{\nabla}_k \frac{\partial \pi_{\mathrm{GR}}}{\partial \tilde{R}} - \omega^2 \frac{\partial \pi_{\mathrm{GR}}}{\partial \tilde{R}} \frac{\partial \pi_{\mathrm{GR}}}{\partial \phi(2)} \mathcal{J}_k$$

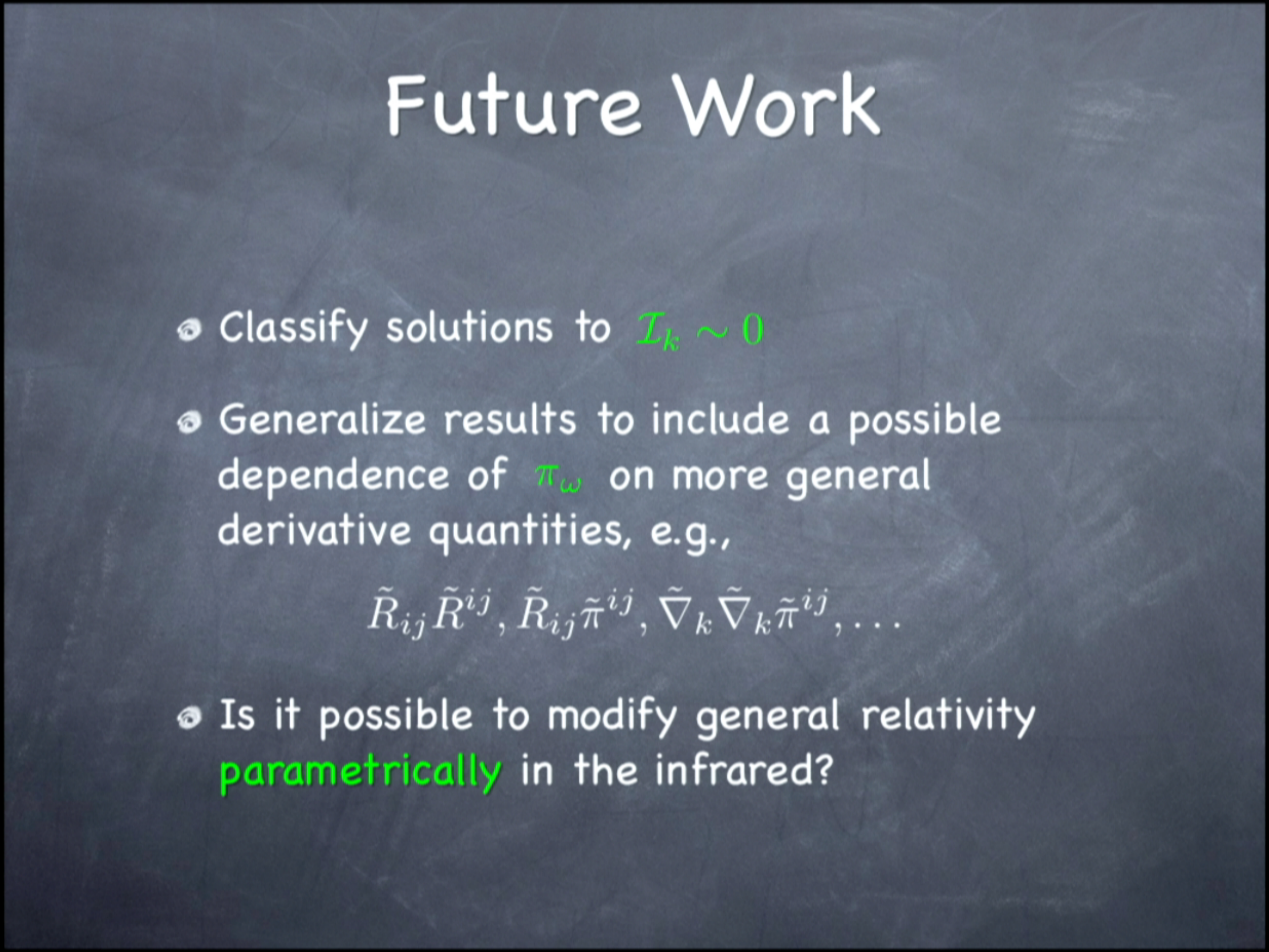
$$\mathcal{I}_k(\pi_{\mathrm{GR}}) = \frac{16}{9\omega^{2/3}\pi_{\mathrm{GR}}^2} \mathcal{K}_k + \frac{16}{9\omega^{2/3}\pi_{\mathrm{GR}}^2} \mathcal{J}_k$$

$$\boxed{\mathcal{I}_k(\pi_{\mathrm{GR}}) = \frac{16}{9\omega^{2/3}\pi_{\mathrm{GR}}^2} \mathcal{H}_k}$$



Realistic Summary

- Realistic theories obey a generalized version of the RG equation $\Delta\pi_\omega = 0$, which encodes invariance under flow through the space of conformally equivalent spatial metrics
- Realistic theories must also obey the differential condition $\mathcal{I}_k \sim 0$, which is satisfied non-trivially by GR
- These two conditions are necessary and sufficient for π_ω to yield a consistent theory of the graviton degrees of freedom



Future Work

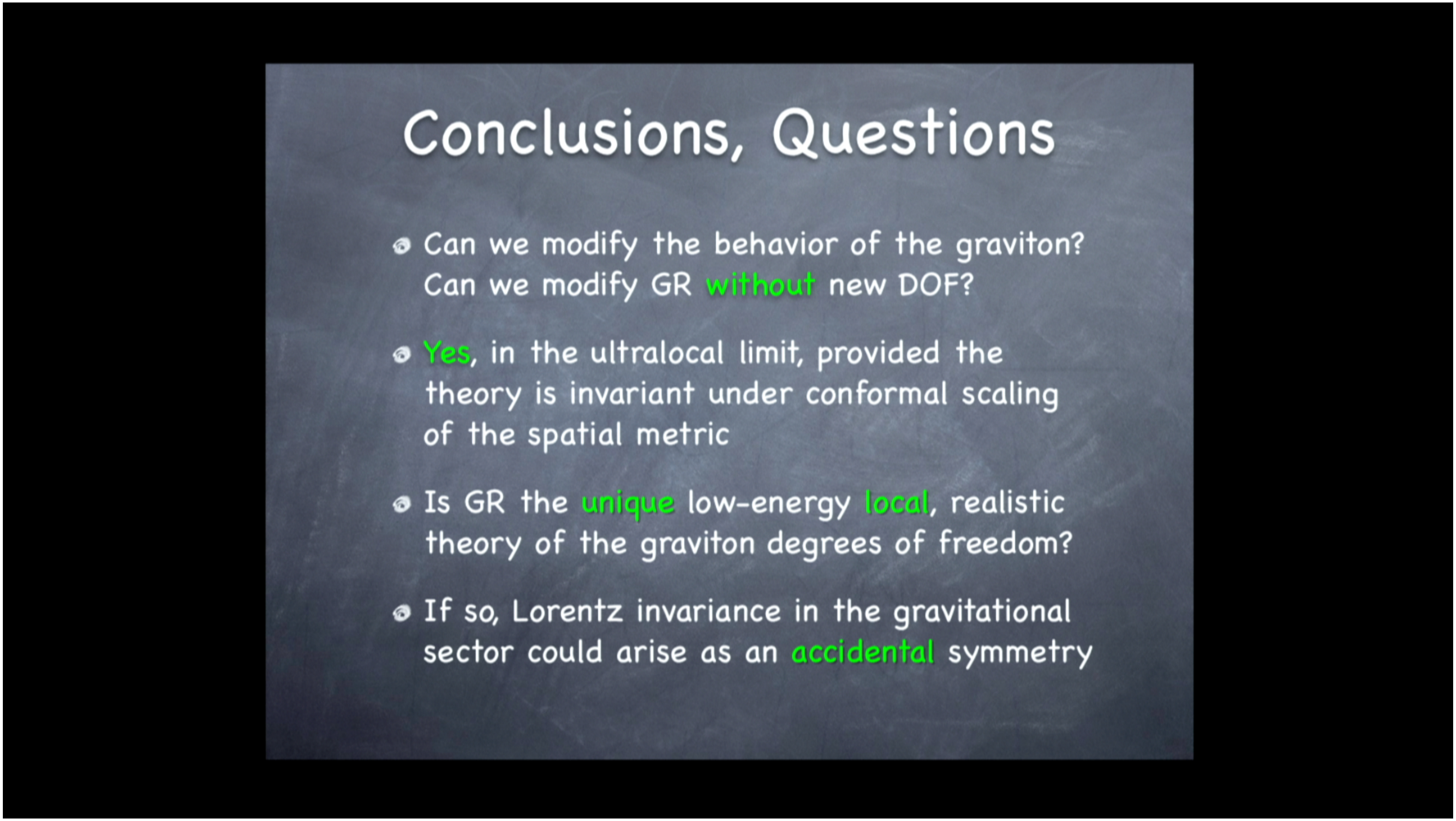
- Classify solutions to $\mathcal{I}_k \sim 0$
- Generalize results to include a possible dependence of π_ω on more general derivative quantities, e.g.,

$$\tilde{R}_{ij}\tilde{R}^{ij}, \tilde{R}_{ij}\tilde{\pi}^{ij}, \tilde{\nabla}_k\tilde{\nabla}_k\tilde{\pi}^{ij}, \ldots$$

- Is it possible to modify general relativity parametrically in the infrared?

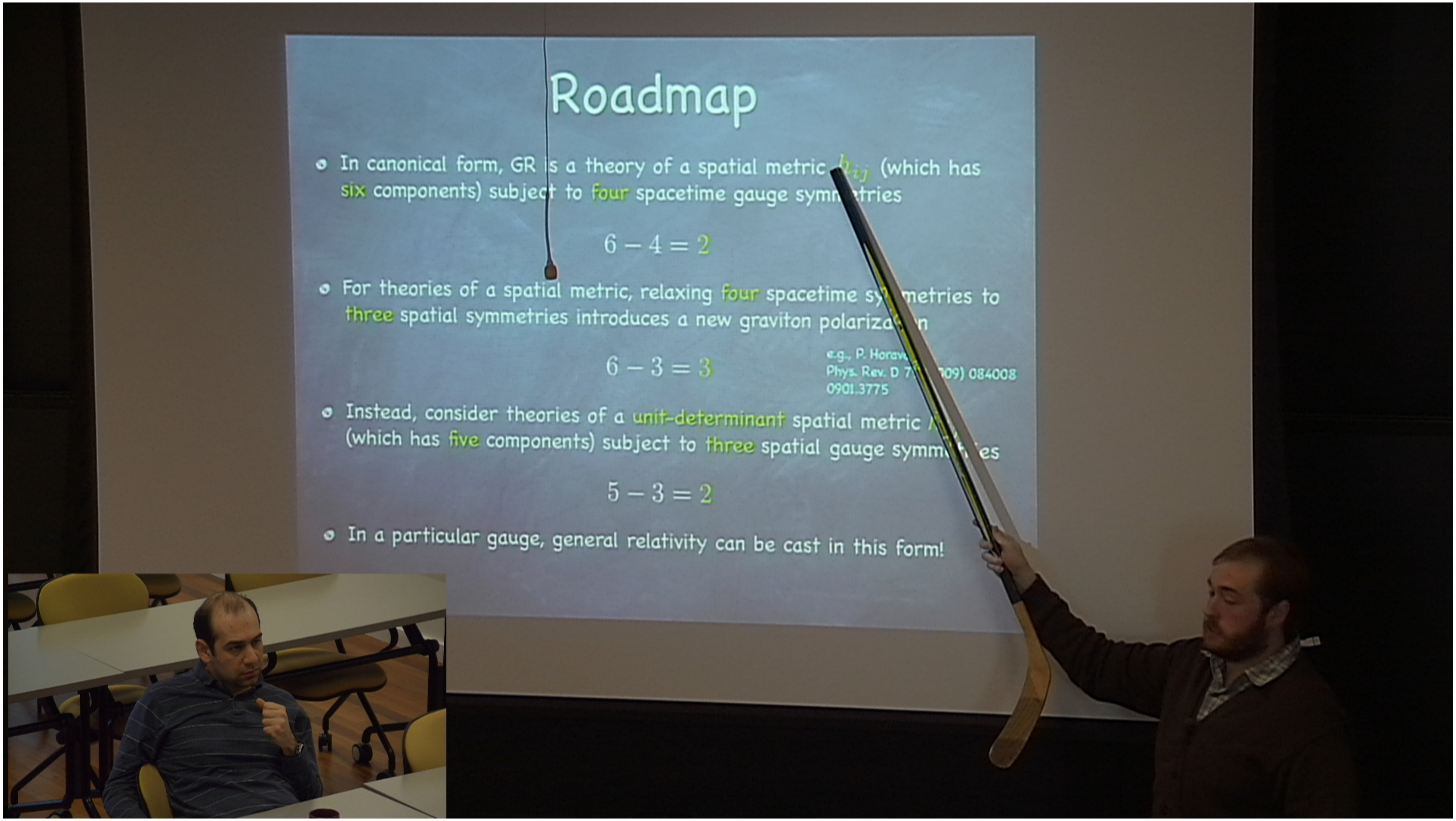
Conclusions, Questions

- Can we modify the behavior of the graviton? Can we modify GR without new DOF?
- Yes, in the ultralocal limit, provided the theory is invariant under conformal scaling of the spatial metric



Conclusions, Questions

- Can we modify the behavior of the graviton? Can we modify GR without new DOF?
- Yes, in the ultralocal limit, provided the theory is invariant under conformal scaling of the spatial metric
- Is GR the unique low-energy local, realistic theory of the graviton degrees of freedom?
- If so, Lorentz invariance in the gravitational sector could arise as an accidental symmetry



Roadmap

- In canonical form, GR is a theory of a spatial metric h_{ij} (which has six components) subject to four spacetime gauge symmetries

$$6-4=2$$

- For theories of a spatial metric, relaxing four spacetime symmetries to three spatial symmetries introduces a new graviton polarization

$$6-3=3$$

e.g., P. Horava,
Phys. Rev. D 7 09) 084008
0901.3775

- Instead, consider theories of a unit-determinant spatial metric (which has five components) subject to three spatial gauge symmetries

$$5-3=2$$

- In a particular gauge, general relativity can be cast in this form!