Title: Scale Without Conformal Invariance

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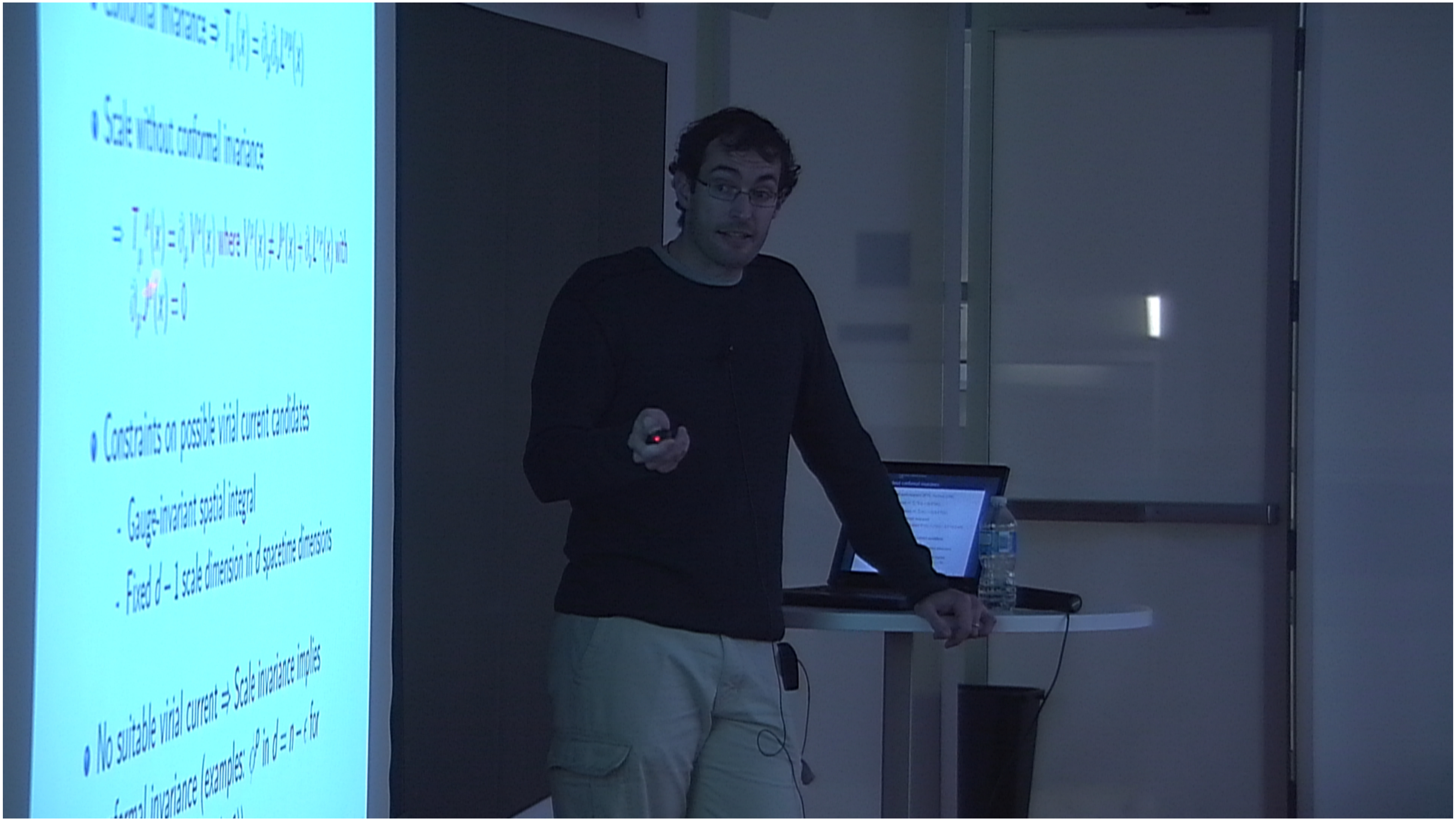
Abstract: We investigate the theoretical implications of scale without conformal invariance in quantum field theory. We argue that the RG flows of such theories correspond to recurrent behaviors, i.e. limit cycles or ergodicity. We discuss the implications for the a-theorem and show how dilatation generators do generate dilatations. Finally, we discuss possible well-behaved non-conformal scale-invariant examples.


Scale without conformal invariance
⇒ T_μ^μ(x) = ∂_μ V^μ(x) where V^μ(x) = J^μ(x) − ∂_ν L^{νμ}(x) with
∂_μ J^μ(x) = 0
Constraints on possible virial current candidates
- Gauge-invariant spatial integral
- Fixed d − 1 scale dimension in d spacetime dimensions
No suitable virial current ⇒ Scale invariance implies

Scale without Conformal Invariance

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November 29, 2011
The Perimeter Institute

based on
arXiv:1106.2540 [hep-th], arXiv:1107.3840 [hep-th] and arXiv:1110.1634 [hep-th]
with Benjamín Grinstein and Andreas Stergiou

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Scale and conformal invariance in two dimensions

Does scale invariance imply conformal invariance ?

- Polchinski following Zamolodchikov Polchinski (1988) & Zamolodchikov (1986)
  - Unitarity
  - Finiteness of EM tensor correlation functions

  ⇒ Scale invariance implies conformal invariance from conservation of EM tensor

"Counter" examples

- Non-linear σ model Hull, Townsend (1986)
  - Non-existence of EM tensor two-point correlation functions
- Theory of elasticity Cardy, Riva (2005)
  - Non-reflection-positive

Scale invariance implies conformal invariance

Scale and conformal invariance in $d > 2$ dimensions

Does scale invariance imply conformal invariance ?

- No proof *à la* Polchinski
  - Conservation of EM tensor $\Rightarrow$ Not enough information

  $\Rightarrow$ Scale invariance does not necessarily imply conformal invariance

No interesting counterexamples

- AdS/CFT Kerr-AdS black holes in $d = 5, 7$ dimensions Awad, Johnson (1999)
  - Conformal invariance broken to scale invariance by black hole rotation
- Maxwell theory in $d \neq 4$ dimensions Jackiw, Pi (2011) & El-Showk, Nakayama, Rychkov (2011)
  - Free field theory
  - Scale invariance broken by interactions

Scale and conformal invariance in $d > 2$ dimensions

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- No proof *à la* Polchinski
  - Conservation of EM tensor $\Rightarrow$ Not enough information
  - $\Rightarrow$ Scale invariance does not necessarily imply conformal invariance

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  - Free field theory
  - Scale invariance broken by interactions

Study of non-conformal scale-invariant QFTs

Scale invariance does not necessarily imply conformal invariance but no proper counterexamples $\Rightarrow$ Possible proof !

- Without proper counterexamples
  - Physical implications of non-conformal scale-invariant QFTs (correlation functions in non-conformal scale-invariant QFTs versus CFTs ?)
- With proper counterexamples
  - Scale invariance conditions weaker than conformal invariance conditions (plentiful examples ?)

Uncharted territory !

Outline

1. Historical review
2. Scale versus conformal invariance
   - Preliminaries
   - Scale invariance and new improved energy-momentum tensor
   - RG flows along scale-invariant trajectories
   - Scale invariance and recurrent behaviors
   - Scale invariance, gradient flows and a-theorem
   - Why dilatation generators generate dilatations
3. Scale-invariant trajectories
   - Polchinski–Dorigoni–Rychkov argument
   - Systematic approach
   - Scheme-dependence of β-functions
   - Examples
4. Discussion and conclusion
   - Features and future work

Preliminaries ($d > 2$)

- Dilatation current Polchinski (1988)
  - $\mathcal{D}^\mu(x) = x^\nu T_\nu{}^\mu(x) - V^\mu(x)$
  - $T_\nu{}^\mu(x)$ any symmetric EM tensor following from spacetime nature of scale transformations
  - $V^\mu(x)$ local operator (virial current) contributing to scale dimensions of fields
  - Freedom in choice of $T_\nu{}^\mu(x)$ compensated by freedom in choice of $V^\mu(x)$
- Scale invariance $\Rightarrow T_\mu{}^\mu(x) = \partial_\mu V^\mu(x)$

- Conformal current Polchinski (1988)
  - $\mathcal{C}^{\mu}_{\nu}(x) = v^{\nu}(x) T_{\nu}{}^{\mu}(x) - \partial \cdot v(x) V'^{\mu}(x) + \partial_{\nu}\partial \cdot v(x) L^{\nu\mu}(x)$
  - $T_{\nu}{}^{\mu}(x)$ any symmetric EM tensor following from spacetime nature of conformal transformations
  - $V'^{\mu}(x)$ local operator corresponding to ambiguity in choice of dilatation current
  - $L^{\nu\mu}(x)$ local symmetric operator correcting position dependence of scale factor
  - $\partial \cdot v(x)$ scale factor (general linear function of x^{μ})
  - Freedom in choice of $T_{\nu}{}^{\mu}(x)$ compensated by freedom in choice of $V'^{\mu}(x)$ and $L^{\nu\mu}(x)$
- Conformal invariance $\Rightarrow$ $T_{\mu}{}^{\mu}(x) = \partial_{\mu} V'^{\mu}(x) = \partial_{\mu}\partial_{\nu} L^{\nu\mu}(x)$
- Conformal invariance $\Rightarrow$ Existence of symmetric traceless energy-momentum tensor

Scale without conformal invariance

Non-conformal scale-invariant QFTs Polchinski (1988)

- Scale invariance $\Rightarrow T_\mu{}^\mu(x) = \partial_\mu V^\mu(x)$
- Conformal invariance $\Rightarrow T_\mu{}^\mu(x) = \partial_\mu \partial_\nu L^{\nu\mu}(x)$
- Scale without conformal invariance

  $\Rightarrow T_\mu{}^\mu(x) = \partial_\mu V^\mu(x)$ where $V^\mu(x) \neq J^\mu(x) + \partial_\nu L^{\nu\mu}(x)$ with $\partial_\mu J^\mu(x) = 0$
- Constraints on possible virial current candidates
  - Gauge-invariant spatial integral
  - Fixed $d-1$ scale dimension in d spacetime dimensions
- No suitable virial current $\Rightarrow$ Scale invariance implies conformal invariance (examples: ϕ^p in $d = n - \epsilon$ for $(p, n) = (6, 3), (4, 4)$ and $(3, 6)$)

Virial current candidates

Most general classically scale-invariant renormalizable theory in $d = 4$ spacetime dimensions Jack, Osborn (1985)

$$\begin{aligned}\mathcal{L} = &- \mu^{-\epsilon} Z_A \frac{1}{4g_A^2} F^A_{\mu\nu} F^{A\mu\nu} + \frac{1}{2} Z^{\frac{1}{2}}_{ab} Z^{\frac{1}{2}}_{ac} D_\mu \phi_b D^\mu \phi_c \\ &+ \frac{1}{2} Z^{\frac{1}{2}*}_{ij} Z^{\frac{1}{2}}_{ik} \bar{\psi}_j i \bar{\sigma}^\mu D_\mu \psi_k - \frac{1}{2} Z^{\frac{1}{2}*}_{ij} Z^{\frac{1}{2}}_{ik} D_\mu \bar{\psi}_j i \bar{\sigma}^\mu \psi_k \\ &- \frac{1}{4!} \mu^\epsilon (\lambda Z^\lambda)_{abcd} \phi_a \phi_b \phi_c \phi_d \\ &- \frac{1}{2} \mu^{\frac{\epsilon}{2}} (y Z^y)_{a|ij} \phi_a \psi_i \psi_j - \frac{1}{2} \mu^{\frac{\epsilon}{2}} (y Z^y)^*_{a|ij} \phi_a \bar{\psi}_i \bar{\psi}_j\end{aligned}$$

- $A^A_\mu(x)$ gauge fields
- $\phi_a(x)$ real scalar fields
- $\psi^\alpha_i(x)$ Weyl fermions
- Dimensional regularization ($d = 4 - \epsilon$)

Virial current candidates

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\begin{aligned}
\mathcal{L} = & -\mu^{-\epsilon} Z_A \frac{1}{4g_A^2} F_{\mu\nu}^A F^{A\mu\nu} + \frac{1}{2} Z_{ab}^{\frac{1}{2}} Z_{ac}^{\frac{1}{2}} D_\mu \phi_b D^\mu \phi_c \\
& + \frac{1}{2} Z_{ij}^{\frac{1}{2}*} Z_{ik}^{\frac{1}{2}} \bar{\psi}_j i \bar{\sigma}^\mu D_\mu \psi_k - \frac{1}{2} Z_{ij}^{\frac{1}{2}*} Z_{ik}^{\frac{1}{2}} D_\mu \bar{\psi}_j i \bar{\sigma}^\mu \psi_k \\
& - \frac{1}{4!} \mu^\epsilon (\lambda Z^\lambda)_{abcd} \phi_a \phi_b \phi_c \phi_d \\
& - \frac{1}{2} \mu^{\frac{\epsilon}{2}} (y Z^y)_{a|ij} \phi_a \psi_i \psi_j - \frac{1}{2} \mu^{\frac{\epsilon}{2}} (y Z^y)^*_{a|ij} \phi_a \bar{\psi}_i \bar{\psi}_j
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$$

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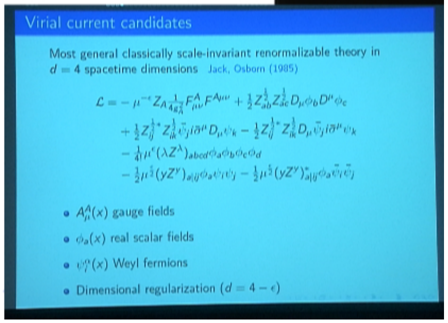
- Dimensional regularization ($d = 4 - \epsilon$)



Virial current candidates and new improved EM tensor

- Virial current $V^\mu(x) = Q_{ab}\phi_a D^\mu \phi_b - P_{ij}\bar{\psi}_i i\bar{\sigma}^\mu \psi_j$
  - $Q_{ba} = -Q_{ab}$
  - $P^*_{ji} = -P_{ij}$

- New improved energy-momentum tensor $\Theta_\nu{}^\mu(x)$ Callan, Coleman, Jackiw (1970)
  - Finite
  - Not renormalized
  - Anomalous trace Robertson (1991)

$$\begin{aligned}\Theta_\mu{}^\mu(x) =& \frac{\beta_A}{2g_A^3} F^A_{\mu\nu} F^{A\mu\nu} + \gamma_{aa'} D^2 \phi_a \phi_{a'} \\ &- \gamma^*_{i'i} \bar{\psi}_i i\bar{\sigma}^\mu D_\mu \psi_{i'} + \gamma_{ii'} D_\mu \bar{\psi}_i i\bar{\sigma}^\mu \psi_{i'} \\ &- \frac{1}{4!}(\beta_{abcd} - \gamma_{a'a}\lambda_{a'bcd} - \gamma_{b'b}\lambda_{ab'cd} \\ &\qquad - \gamma_{c'c}\lambda_{abc'd} - \gamma_{d'd}\lambda_{abcd'})\phi_a\phi_b\phi_c\phi_d \\ &- \frac{1}{2}(\beta_{a|ij} - \gamma_{a'a} y_{a'|ij} - \gamma_{i'i} y_{a|i'j} - \gamma_{j'j} y_{a|ij'})\phi_a\psi_i\psi_j + \text{h.c.}\end{aligned}$$

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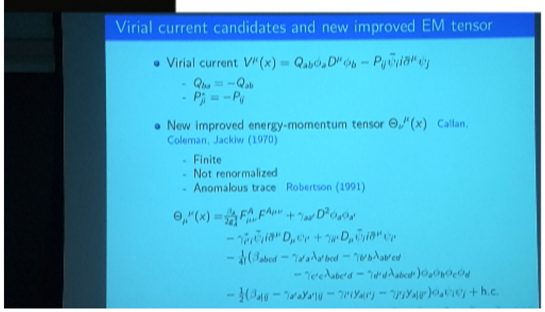

- β-functions from vertex corrections and wavefunction renormalizations ($d = 4$ spacetime dimensions)
  - RG time $t = \ln(\mu_0/\mu)$

$$\beta_A = -\frac{dg_A}{dt} = \gamma_A g_A \quad \text{(no sum)}$$

$$\begin{aligned}\beta_{abcd} &= -\frac{d\lambda_{abcd}}{dt} \\ &= -(\lambda\gamma^{\lambda})_{abcd} + \gamma_{a'a}\lambda_{a'bcd} + \gamma_{b'b}\lambda_{ab'cd} + \gamma_{c'c}\lambda_{abc'd} + \gamma_{d'd}\lambda_{abcd'}\end{aligned}$$

$$\beta_{a|ij} = -\frac{dy_{a|ij}}{dt} = -(y\gamma^{y})_{a|ij} + \gamma_{a'a}y_{a'|ij} + \gamma_{i'i}y_{a|i'j} + \gamma_{j'j}y_{a|ij'}$$

- Divergence of dilatation current

$$\begin{aligned}\partial_\mu \mathcal{D}^\mu(x) =& \frac{\beta_A}{2g_A^3} F^A_{\mu\nu} F^{A\mu\nu} + (\gamma_{aa'} + Q_{aa'}) D^2 \phi_a \phi_{a'} \\ & - (\gamma^*_{i'i} + P^*_{i'i}) \bar{\psi}_i i \bar{\sigma}^\mu D_\mu \psi_{i'} + (\gamma_{ii'} + P_{ii'}) D_\mu \bar{\psi}_i i \bar{\sigma}^\mu \psi_{i'} \\ & - \frac{1}{4!}(\beta_{abcd} - \gamma_{a'a}\lambda_{a'bcd} - \gamma_{b'b}\lambda_{ab'cd} \\ & \qquad - \gamma_{c'c}\lambda_{abc'd} - \gamma_{d'd}\lambda_{abcd'})\phi_a\phi_b\phi_c\phi_d \\ & - \frac{1}{2}(\beta_{a|ij} - \gamma_{a'a} y_{a'|ij} - \gamma_{i'i} y_{a|i'j} - \gamma_{j'j} y_{a|ij'})\phi_a \psi_i \psi_j + \text{h.c.}\end{aligned}$$

- Conserved dilatation current $\partial_\mu \mathcal{D}^\mu(x) = 0$ (up to EOMs)

$$\begin{aligned}\beta_A &= 0 \\ \beta_{abcd} &= -Q_{a'a}\lambda_{a'bcd} - Q_{b'b}\lambda_{ab'cd} - Q_{c'c}\lambda_{abc'd} - Q_{d'd}\lambda_{abcd'} \\ \beta_{a|ij} &= -Q_{a'a} y_{a'|ij} - P_{i'i} y_{a|i'j} - P_{j'j} y_{a|ij'}\end{aligned}$$

- Conserved conformal current $\partial_\mu \mathcal{C}^\mu_\nu(x) = 0$ (up to EOMs)

$$\beta_A = \beta_{abcd} = \beta_{a|ij} = 0$$

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- Conserved conformal current $\partial_\mu \mathcal{C}^\mu_\nu(x) = 0$ (up to EOMs)

$$\beta_A = \beta_{abcd} = \beta_{a|ij} = 0$$

Interlude: Current conservation

- Divergence of current $J^\mu(x)$ without use of EOMs Collins (1984)

$$\partial_\mu J^\mu(x) = \Delta_{\text{EOM}} + \Delta_{\text{Classical}} + \Delta_{\text{Anomaly}}$$

- Green's function of elementary fields with current $J^\mu(x)$ and Ward identity
  - ✓ $\Delta_{\text{EOM}} \Rightarrow$ Expected contact terms from Ward identity
  - ✗ $\Delta_{\text{Classical}} \Rightarrow$ Usual non-anomalous classical violation
  - ✗ $\Delta_{\text{Anomaly}} \Rightarrow$ Possible anomalous violation in divergent Green's function
- Example: $\mathcal{L} = -\frac{1}{4}F^A_{\mu\nu}F^{A\mu\nu} + \bar{\psi}_i(i\gamma^\mu D_\mu \delta_{ij} - M_{ij})\psi_j$
  - Vector current $J_V^{\mu a}(x) = \bar{\psi}\gamma^\mu t^a \psi$ with $\Delta_{\text{EOM}} \neq 0$, $\Delta_{\text{Classical}} = i\bar{\psi}[M, t^a]\psi$ and $\Delta_{\text{Anomaly}} = 0$
  - Axial current $J_A^{\mu a}(x) = \bar{\psi}\frac{1}{2}[\gamma^\mu, \gamma^5]t^a\psi$ with $\Delta_{\text{EOM}} \neq 0$, $\Delta_{\text{Classical}} = i\bar{\psi}\gamma^5\{M, t^a\}\psi$ and $\Delta_{\text{Anomaly}} = \frac{1}{2}\bar{\psi}\{\gamma^\mu, \gamma^5\}t^a D_\mu\psi - \frac{1}{2}D_\mu\bar{\psi}\{\gamma^\mu, \gamma^5\}t^a\psi$

Virial current and unitarity bounds

- New improved energy-momentum tensor ⇒ Finite and not renormalized Callan, Coleman, Jackiw (1970)
- Operators related to EOMs ⇒ Finite and not renormalized Politzer (1980) & Robertson (1991)
- Virial current ⇒ Finite and not renormalized
  - Unconserved current with scale dimension exactly 3
- Unitarity bounds for conformal versus scale-invariant QFTs Grinstein, Intriligator, Rothstein (2008)
- Non-trivial virial current ⇒ Non-conformal scale-invariant QFTs

- Divergence of dilatation current

$$\partial_\mu \mathcal{D}^\mu(x) = \frac{\beta_A}{2g_A^3} F^A_{\mu\nu} F^{A\mu\nu} + (\gamma_{aa'} + Q_{aa'}) D^2 \phi_a \phi_{a'}$$
$$- (\gamma^*_{i'i} + P^*_{i'i}) \bar{\psi}_i i \bar{\sigma}^\mu D_\mu \psi_{i'} + (\gamma_{ii'} + P_{ii'}) D_\mu \bar{\psi}_i i \bar{\sigma}^\mu \psi_{i'}$$
$$- \frac{1}{4!}(\beta_{abcd} - \gamma_{a'a}\lambda_{a'bcd} - \gamma_{b'b}\lambda_{ab'cd} - \gamma_{c'c}\lambda_{abc'd} - \gamma_{d'd}\lambda_{abcd'})\phi_a\phi_b\phi_c\phi_d$$
$$- \frac{1}{2}(\beta_{a|ij} - \gamma_{a'a} y_{a'|ij} - \gamma_{i'i} y_{a|i'j} - \gamma_{j'j} y_{a|ij'})\phi_a \psi_i \psi_j + \text{h.c.}$$

- Conserved dilatation current $\partial_\mu \mathcal{D}^\mu(x) = 0$ (up to EOMs)

$$\beta_A = 0$$
$$\beta_{abcd} = -Q_{a'a}\lambda_{a'bcd} - Q_{b'b}\lambda_{ab'cd} - Q_{c'c}\lambda_{abc'd} - Q_{d'd}\lambda_{abcd'}$$
$$\beta_{a|ij} = -Q_{a'a} y_{a'|ij} - P_{i'i} y_{a|i'j} - P_{j'j} y_{a|ij'}$$

- Conserved conformal current $\partial_\mu \mathcal{C}^\mu_\nu(x) = 0$ (up to EOMs)

$$\beta_A = \beta_{abcd} = \beta_{a|ij} = 0$$

Virial current and unitarity bounds

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RG flows along scale-invariant trajectories

Scale-invariant solution $(g_A, \lambda_{abcd}, y_{a|ij}) \Rightarrow$ RG trajectory

$$\begin{aligned}\bar{g}_A(t) &= g_A\\ \bar{\lambda}_{abcd}(t) &= \widehat{Z}_{a'a}(t)\widehat{Z}_{b'b}(t)\widehat{Z}_{c'c}(t)\widehat{Z}_{d'd}(t)\lambda_{a'b'c'd'}\\ \bar{y}_{a|ij}(t) &= \widehat{Z}_{a'a}(t)\widehat{Z}_{i'i}(t)\widehat{Z}_{j'j}(t)y_{a'|i'j'}\end{aligned}$$

$$\begin{aligned}\widehat{Z}_{aa'}(t) &= \left(e^{Qt}\right)_{aa'}\\ \widehat{Z}_{ii'}(t) &= \left(e^{Pt}\right)_{ii'}\end{aligned}$$

- $(\bar{g}_A(t,g,\lambda,y), \bar{\lambda}_{abcd}(t,g,\lambda,y), \bar{y}_{a|ij}(t,g,\lambda,y))$ also scale-invariant solution
- Q_{ab} and P_{ij} constant along RG trajectory
- $\widehat{Z}_{ab}(t)$ orthogonal and $\widehat{Z}_{ij}(t)$ unitary $\Rightarrow$ Always non-vanishing β-functions along scale-invariant trajectory

- Divergence of dilatation current

$$
\begin{aligned}
\partial_\mu \mathcal{D}^\mu(x) = & \frac{\beta_A}{2g_A^3} F^A_{\mu\nu} F^{A\mu\nu} + (\gamma_{aa'} + Q_{aa'}) D^2 \phi_a \phi_{a'} \\
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\end{aligned}
$$

- Conserved dilatation current $\partial_\mu \mathcal{D}^\mu(x) = 0$ (up to EOMs)

$$
\begin{aligned}
\beta_A &= 0 \\
\beta_{abcd} &= -Q_{a'a} \lambda_{a'bcd} - Q_{b'b} \lambda_{ab'cd} - Q_{c'c} \lambda_{abc'd} - Q_{d'd} \lambda_{abcd'} \\
\beta_{a|ij} &= -Q_{a'a} y_{a'|ij} - P_{i'i} y_{a|i'j} - P_{j'j} y_{a|ij'}
\end{aligned}
$$

- Conserved conformal current $\partial_\mu \mathcal{C}^\mu_\nu(x) = 0$ (up to EOMs)

$$
\beta_A = \beta_{abcd} = \beta_{a|ij} = 0
$$

RG flows along scale-invariant trajectories

Scale-invariant solution $(g_A, \lambda_{abcd}, y_{a|ij}) \Rightarrow$ RG trajectory

$$\bar{g}_A(t) = g_A$$
$$\bar{\lambda}_{abcd}(t) = \widehat{Z}_{a'a}(t)\widehat{Z}_{b'b}(t)\widehat{Z}_{c'c}(t)\widehat{Z}_{d'd}(t)\lambda_{a'b'c'd'}$$
$$\bar{y}_{a|ij}(t) = \widehat{Z}_{a'a}(t)\widehat{Z}_{i'i}(t)\widehat{Z}_{j'j}(t)y_{a'|i'j'}$$

$$\widehat{Z}_{aa'}(t) = \left(e^{Qt}\right)_{aa'}$$
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- $(\bar{g}_A(t,g,\lambda,y), \bar{\lambda}_{abcd}(t,g,\lambda,y), \bar{y}_{a|ij}(t,g,\lambda,y))$ also scale-invariant solution
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& - \frac{1}{4!} (\beta_{abcd} - \gamma_{a'a} \lambda_{a'bcd} - \gamma_{b'b} \lambda_{ab'cd} \\
& \qquad - \gamma_{c'c} \lambda_{abc'd} - \gamma_{d'd} \lambda_{abcd'}) \phi_a \phi_b \phi_c \phi_d \\
& - \frac{1}{2} (\beta_{a|ij} - \gamma_{a'a} y_{a'|ij} - \gamma_{i'i} y_{a|i'j} - \gamma_{j'j} y_{a|ij'}) \phi_a \psi_i \psi_j + \text{h.c.}
\end{aligned}
$$

- Conserved dilatation current $\partial_\mu \mathcal{D}^\mu(x) = 0$ (up to EOMs)

$$
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\beta_A &= 0 \\
\beta_{abcd} &= -Q_{a'a} \lambda_{a'bcd} - Q_{b'b} \lambda_{ab'cd} - Q_{c'c} \lambda_{abc'd} - Q_{d'd} \lambda_{abcd'} \\
\beta_{a|ij} &= -Q_{a'a} y_{a'|ij} - P_{i'i} y_{a|i'j} - P_{j'j} y_{a|ij'}
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- Conserved conformal current $\partial_\mu \mathcal{C}^\mu_\nu(x) = 0$ (up to EOMs)

$$
\beta_A = \beta_{abcd} = \beta_{a|ij} = 0
$$

Scale invariance and recurrent behaviors

RG flows along scale-invariant trajectories $\Rightarrow$ Recurrent behaviors !

Lorenz (1963,1964), Wilson (1971) & Kogut, Wilson (1974)

- Virial current $\Rightarrow$ Transformation in symmetry group of kinetic terms ($SO(N_S) \times U(N_F)$)
  - Q_{ab} antisymmetric and P_{ij} antihermitian $\Rightarrow$ Purely imaginary eigenvalues
  - $\widehat{Z}_{ab}(t)$ and $\widehat{Z}_{ij}(t)$ in $SO(N_S) \times U(N_F)$

$\Rightarrow$ Periodic (limit cycle) or quasi-periodic (ergodicity) scale-invariant trajectories

RG flows along scale-invariant trajectories

Scale-invariant solution $(g_A, \lambda_{abcd}, y_{a|ij}) \Rightarrow$ RG trajectory

$$\begin{aligned}
\bar{g}_A(t) &= g_A \\
\bar{\lambda}_{abcd}(t) &= \widehat{Z}_{a'a}(t)\widehat{Z}_{b'b}(t)\widehat{Z}_{c'c}(t)\widehat{Z}_{d'd}(t)\lambda_{a'b'c'd'} \\
\bar{y}_{a|ij}(t) &= \widehat{Z}_{a'a}(t)\widehat{Z}_{i'i}(t)\widehat{Z}_{j'j}(t)y_{a'|i'j'}
\end{aligned}$$

$$\begin{aligned}
\widehat{Z}_{aa'}(t) &= \left(e^{Qt}\right)_{aa'} \\
\widehat{Z}_{ii'}(t) &= \left(e^{Pt}\right)_{ii'}
\end{aligned}$$

- $(\bar{g}_A(t, g, \lambda, y), \bar{\lambda}_{abcd}(t, g, \lambda, y), \bar{y}_{a|ij}(t, g, \lambda, y))$ also scale-invariant solution
- Q_{ab} and P_{ij} constant along RG trajectory
- $\widehat{Z}_{ab}(t)$ orthogonal and $\widehat{Z}_{ij}(t)$ unitary $\Rightarrow$ Always non-vanishing β-functions along scale-invariant trajectory

Recurrent behaviors

Intuition from $\mathcal{D}^\mu(x) = x^\nu \Theta_\nu{}^\mu(x) - V^\mu(x)$

- RG flow $\Rightarrow$ Generated by scale transformation $(x^\nu \Theta_\nu{}^\mu(x))$
- RG flow $\Rightarrow$ Related to virial current through conservation of dilatation current
- Virial current $\Rightarrow$ Generates internal transformation of the fields
  - Internal transformation in compact group $SO(N_S) \times U(N_F)$

  $\Rightarrow$ Rotate back to or close to identity
- RG flow return back to or close to identity $\Rightarrow$ Recurrent behavior

Scale-invariant trajectories ?

RG flows $\sim$ Field redefinitions $\Rightarrow$ Scale-invariant trajectories or fixed points ?

- RG-time-dependent field redefinitions $\Rightarrow$ Generates RG flows
  Wegner (1974) & Latorre, Morris (2001)
  - RG-time-dependent field redefinitions $\Rightarrow$ All exact RG flows (Wilson, Wegner, Polchinski, etc.)

β-function operators $\sim$ Redundant operators $\Rightarrow$ Scale-invariant trajectories or fixed points ?

- Wavefunction renormalization operators $\Rightarrow$ Redundant operators
  - Redundant β-function operators necessary for scale invariance

Non-conformal scale-invariant QFTs $\Rightarrow$ Non-trivial RG flows (recurrent behaviors)

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Scale invariance, gradient flows and *a*-theorem

- Gradient flow

$$\beta_i(g) = -\frac{dg_i}{dt} = G_{ij}(g)\frac{\partial c(g)}{\partial g_j}$$

  - G_{ij} positive-definite metric
  - Potential $c(g)$ function of couplings

- Potential $c(g)$ monotonically decreasing along RG trajectory

$$\frac{dc(g(t))}{dt} = -G^{ij}(g)\beta_i\beta_j \leq 0$$

  - Recurrent behaviors (scale-invariant trajectories) $\not\Leftrightarrow$ Gradient flows (scale implies conformal invariance) Wallace, Zia (1975)

- *a*-theorem Barnes, Intriligator, Wecht, Wright (2004)
  - RG flow $\Rightarrow$ Irreversible process (integrating out DOFs)
  - $c(g) \sim$ measure of number of massless DOFs
  - *a*-theorem $\Rightarrow$ weak ($c_{IR} < c_{UV}$), stronger ($\frac{dc}{dt} \leq 0$), ~~strongest~~ (RG flows as gradient flows)

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Why dilatation generators generate dilatations

Dilatation generators do not generate dilatations in non-scale-invariant QFTs Coleman, Jackiw (1971)

- Quantum anomalies at low orders
  - Anomalous dimensions

  ⇒ Possible to absorb into redefinition of scale dimensions of fields

  ✓ Preserve scale invariance

- Quantum anomalies at high orders
  - β-functions

  ⇒ Not possible to absorb

  ✗ Break scale invariance

- Divergence of dilatation current

$$\begin{aligned}
\partial_\mu \mathcal{D}^\mu(x) = & \frac{\beta_A}{2g_A^3} F^A_{\mu\nu} F^{A\mu\nu} + (\gamma_{aa'} + Q_{aa'}) D^2 \phi_a \phi_{a'} \\
& - (\gamma^*_{i'i} + P^*_{i'i}) \bar{\psi}_i i \bar{\sigma}^\mu D_\mu \psi_{i'} + (\gamma_{ii'} + P_{ii'}) D_\mu \bar{\psi}_i i \bar{\sigma}^\mu \psi_{i'} \\
& - \frac{1}{4!} (\beta_{abcd} - \gamma_{a'a} \lambda_{a'bcd} - \gamma_{b'b} \lambda_{ab'cd} \\
& \qquad - \gamma_{c'c} \lambda_{abc'd} - \gamma_{d'd} \lambda_{abcd'}) \phi_a \phi_b \phi_c \phi_d \\
& - \frac{1}{2} (\beta_{a|ij} - \gamma_{a'a} y_{a'|ij} - \gamma_{i'i} y_{a|i'j} - \gamma_{j'j} y_{a|ij'}) \phi_a \psi_i \psi_j + \text{h.c.}
\end{aligned}$$

- Conserved dilatation current $\partial_\mu \mathcal{D}^\mu(x) = 0$ (up to EOMs)

$$\begin{aligned}
\beta_A &= 0 \\
\beta_{abcd} &= -Q_{a'a} \lambda_{a'bcd} - Q_{b'b} \lambda_{ab'cd} - Q_{c'c} \lambda_{abc'd} - Q_{d'd} \lambda_{abcd'} \\
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\end{aligned}$$

- Conserved conformal current $\partial_\mu \mathcal{C}^\mu_\nu(x) = 0$ (up to EOMs)

$$\beta_A = \beta_{abcd} = \beta_{a|ij} = 0$$

Why dilatation generators generate dilatations in non-conformal scale-invariant QFTs ?

- β-functions on scale-invariant trajectories
  - Both vertex correction and wavefunction renormalization contributions
  - Very specific form for vertex correction contribution
  - Equivalent in form to wavefunction renormalization contribution (redundant operators)

  $\Rightarrow$ Also possible to absorb into redefinition of scale dimensions of fields

  ✔ Preserve scale invariance !

Ward identity for scale invariance

Callan-Symanzik equation for effective action Callan (1970) & Symanzik (1970)

$$\left[M\frac{\partial}{\partial M}+\beta_i\frac{\partial}{\partial g_i}+\gamma_j{}^i\int d^4x\,\varphi_i(x)\frac{\delta}{\delta\varphi_j(x)}\right]\Gamma[\varphi(x),g,M]=0$$

- In non-scale-invariant QFTs
  - ✔ Anomalous dimensions
  - ✗ β-functions
- In CFTs
  - ✔ Anomalous dimensions
  - ✔ Vanishing β-functions

$$\left[M\frac{\partial}{\partial M}+(\gamma_j{}^i+Q_j{}^i)\int d^4x\,\varphi_i(x)\frac{\delta}{\delta\varphi_j(x)}\right]\Gamma[\varphi(x),g,M]=0$$

- In non-conformal scale-invariant QFTs
  - ✔ Anomalous dimensions
  - ✔ β-functions (redundant operators)

Poincaré algebra augmented with dilatation charge

- β-functions on scale-invariant trajectories
  - Quantum-mechanical generation of scale dimensions
  - Appropriate scale dimensions required by virial current

  $\Rightarrow$ Conserved dilatation current $\mathcal{D}^{\mu}(x)$

- Poincaré algebra with dilatation charge $D = \int d^3x\, \mathcal{D}^0(x)$

$$
\begin{aligned}
[M_{\mu\nu}, M_{\rho\sigma}] &= -i(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho} - \eta_{\mu\sigma}M_{\nu\rho}) \\
[M_{\mu\nu}, P_{\rho}] &= -i(\eta_{\mu\rho}P_{\nu} - \eta_{\nu\rho}P_{\mu}) \\
[D, P_{\mu}] &= -iP_{\mu}
\end{aligned}
$$

- Algebra action on fields $\mathcal{O}_I(x)$

$$
\begin{aligned}
[M_{\mu\nu}, \mathcal{O}_I(x)] &= -i(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu} + \Sigma_{\mu\nu})\mathcal{O}_I(x) \\
[P_{\mu}, \mathcal{O}_I(x)] &= -i\partial_{\mu}\mathcal{O}_I(x) \\
[D, \mathcal{O}_I(x)] &= -i(x \cdot \partial + \Delta)\mathcal{O}_I(x)
\end{aligned}
$$

- New classical scale dimensions of fields due to virial current

$$[D, \phi_a(x)] = -i(x \cdot \partial + 1)\phi_a(x) - iQ_{ab}\phi_b(x)$$
$$[D, \psi_i(x)] = -i(x \cdot \partial + \tfrac{3}{2})\psi_i(x) - iP_{ij}\psi_j(x)$$

  - How do non-conformal scale-invariant QFTs know about new scale dimensions ?

  $\Rightarrow$ Generated by β-functions !

- Quantum-mechanical scale dimensions of fields

$$\Delta_{ab} = \delta_{ab} + \gamma_{ab} + Q_{ab}$$
$$\Delta_{ij} = \tfrac{3}{2}\delta_{ij} + \gamma_{ij} + P_{ij}$$

Scale-invariant trajectories ??

β-functions $\sim$ Anomalous dimensions $\Rightarrow$ Scale-invariant trajectories or fixed points ?

- Shift β-functions away $\Rightarrow$ Scheme change
  - ✗ Non-conformal scale-invariant QFTs with traceless EM tensor
- Shift β-functions away $\Rightarrow$ Global shift
  - ✗ Conformal fixed points become conformal trajectories

Non-conformal scale-invariant QFTs $\Rightarrow$ Non-trivial RG flows

Non-conformal scale-invariant correlation functions

- Scalar fields $\mathcal{O}_I(x)$ with scale dimensions Δ_I

$$\langle \mathcal{O}_I(x_1)\mathcal{O}_J(x_2)\rangle = \frac{g_{IJ}}{(x_1-x_2)^{\Delta_I+\Delta_J}}$$

$$\langle \mathcal{O}_I(x_1)\mathcal{O}_J(x_2)\mathcal{O}_K(x_3)\rangle = \sum_{\substack{\delta_1+\delta_2+\delta_3=\\ \Delta_I+\Delta_J+\Delta_K}} \frac{c_{IJK}^{\delta_1\delta_2\delta_3}}{(x_1-x_2)^{\delta_1}(x_2-x_3)^{\delta_2}(x_3-x_1)^{\delta_3}}$$

  - Non-vanishing two-point functions with $\Delta_I \neq \Delta_J$ contrary to CFTs

- Two-point correlation functions of fundamental real scalar fields

$$\langle \phi_a(x)\phi_b(0)\rangle = \left[(x^2)^{-\frac{\Delta}{2}}\, G^{\phi}\, (x^2)^{-\frac{\Delta^T}{2}}\right]_{ab}$$

  - G^{ϕ} constant real symmetric matrix

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Polchinski–Dorigoni–Rychkov argument at one loop

Non-conformal scale-invariant β-functions

$$\beta_{abcd} = \mathcal{Q}_{abcd}$$
$$\beta_{a|ij} = \mathcal{P}_{a|ij}$$

$$\mathcal{Q}_{abcd} = -Q_{a'a}\lambda_{a'bcd} - Q_{b'b}\lambda_{ab'cd} - Q_{c'c}\lambda_{abc'd} - Q_{d'd}\lambda_{abcd'}$$
$$\mathcal{P}_{a|ij} = -Q_{a'a}y_{a'|ij} - P_{i'i}y_{a|i'j} - P_{j'j}y_{a|ij'}$$

- Real scalar fields only Polchinski (1988)
  - $\mathcal{Q}_{abcd}\beta_{abcd}^{(\mathrm{one-loop})} = 0 \Rightarrow \mathcal{Q}_{abcd} = 0$

  $\Rightarrow$ Scale invariance implies conformal invariance
- Real scalar fields and Weyl fermions Dorigoni, Rychkov (2009)
  - $\mathcal{P}^*_{a|ij}\beta_{a|ij}^{(\mathrm{one-loop})} = 0 \Rightarrow \mathcal{P}_{a|ij} = 0$
  - $\mathcal{Q}_{abcd}\beta_{abcd}^{(\mathrm{one-loop})} = 0$ using $\mathcal{P}_{a|ij} = 0 \Rightarrow \mathcal{Q}_{abcd} = 0$

  $\Rightarrow$ Scale invariance implies conformal invariance

Polchinski–Dorigoni–Rychkov argument at two loops

- Real scalar fields only JFF, Grinstein, Stergiou (2011)
  - $\mathcal{Q}_{abcd}\beta_{abcd}^{(\text{two-loop})} = 0 \Rightarrow \mathcal{Q}_{abcd} = 0$

  $\Rightarrow$ Scale invariance implies conformal invariance

- One real scalar field only and Weyl fermions *ibid*
  - $\mathcal{P}_{a|ij}^{*}\beta_{a|ij}^{(\text{two-loop})} = 0 \Rightarrow \mathcal{P}_{a|ij} = 0$
  - $\mathcal{Q}_{abcd} \equiv 0$

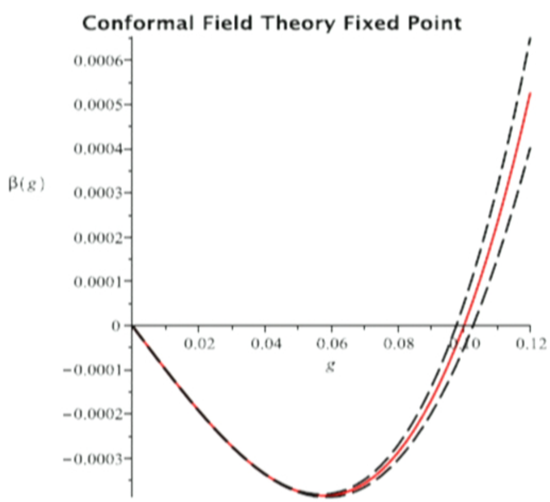
  $\Rightarrow$ Scale invariance implies conformal invariance (also at all loops)

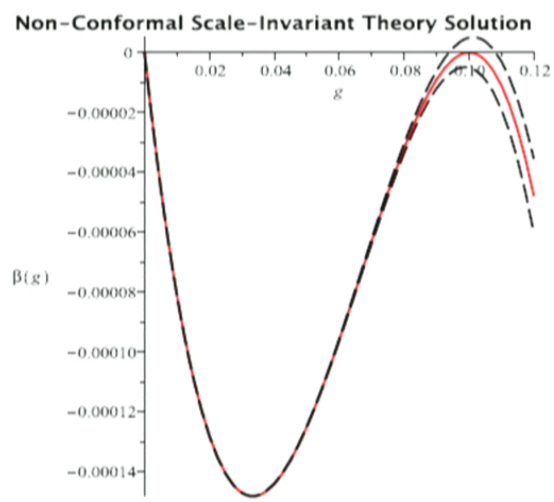
- Real scalar fields and Weyl fermions *ibid*
  - $\mathcal{P}_{a|ij}^{*}\beta_{a|ij}^{(\text{two-loop})} \neq 0$

  $\Rightarrow$ Scale invariance does NOT imply conformal invariance
  - Obstruction due to $y^3\lambda$ and $y\lambda^2$ terms (also obstruction to gradient flow interpretation Wallace, Zia (1975))

Schematically,

- Conformal field theory fixed point
  - Stable with respect to higher-order corrections
- Non-conformal scale-invariant theory solution
  - Would-be conformal fixed point at lowest order unstable with respect to higher-order corrections





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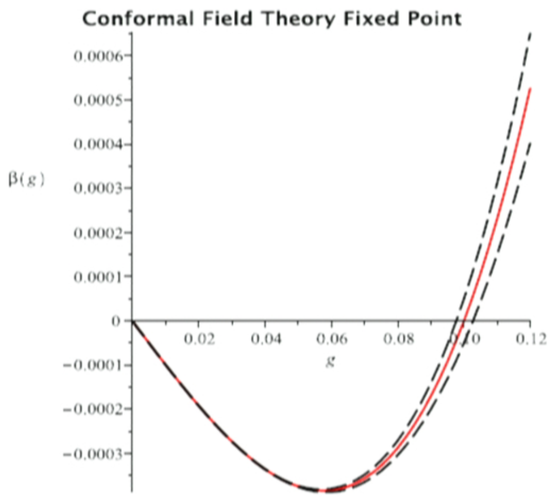

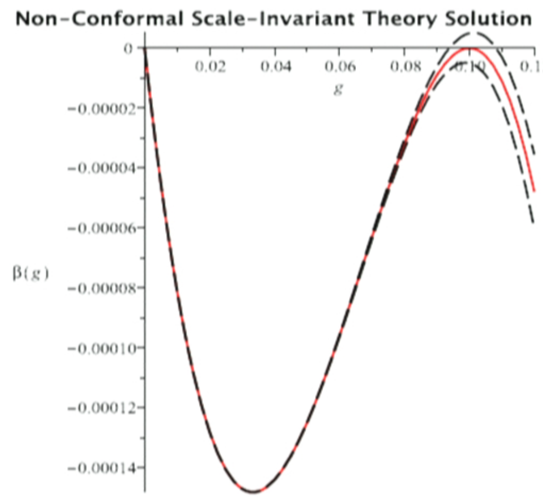

Systematic approach

Scale-invariant trajectories at weak coupling

$$g_A = \sum_{n\geq 1} g_A^{(n)} \epsilon^{n-\frac{1}{2}} \qquad \lambda_{abcd} = \sum_{n\geq 1} \lambda_{abcd}^{(n)} \epsilon^{n} \qquad y_{a|ij} = \sum_{n\geq 1} y_{a|ij}^{(n)} \epsilon^{n-\frac{1}{2}}$$

$$Q_{ab} = \sum_{n\geq 2} Q_{ab}^{(n)} \epsilon^{n} \qquad P_{ij} = \sum_{n\geq 2} P_{ij}^{(n)} \epsilon^{n}$$

- ϵ small parameter
  - Obvious choice in $d = 4 - \epsilon$
  - One-loop gauge coupling β-function coefficient in $d = 4$ Banks, Zaks (1982)
- Form of expansions determined by β-functions
  - For coupling constants $\Rightarrow$ Lowest-order terms in β-functions (would-be conformal fixed points)
  - For virial current $\Rightarrow$ Higher-order terms in β-functions due to Polchinski–Dorigoni–Rychkov argument

Interlude: Scheme-(in)dependence of β-functions

Change of scheme $\bar{\lambda} = \lambda + a^{(1)}\lambda^2 + a^{(2)}\lambda^3 + \cdots$

$$\beta = b^{(1)}\lambda^2 + b^{(2)}\lambda^3 + b^{(3)}\lambda^4 + \cdots$$
$$\bar{\beta} = \bar{b}^{(1)}\bar{\lambda}^2 + \bar{b}^{(2)}\bar{\lambda}^3 + \bar{b}^{(3)}\bar{\lambda}^4 + \cdots$$

$$\bar{b}^{(1)} = b^{(1)}$$
$$\bar{b}^{(2)} = b^{(2)}$$
$$\bar{b}^{(3)} = b^{(3)} + f(b^{(1)}, b^{(2)}, a^{(1)}, a^{(2)})$$

- One coupling constant case
  - Scheme-independence $\Rightarrow$ Only two lowest-order terms
  - Scheme-dependence $\Rightarrow$ All higher-order terms
  - Scheme-dependent terms can all be set to vanish
  - High-precision numerical analysis possible (but useless)

Change of scheme

$$(\bar{g}_A, \bar{\lambda}_{abcd}, \bar{y}_{a|ij}) = (g_A, \lambda_{abcd}, y_{a|ij}) + (a_A^{(1)}, a_{abcd}^{(1)}, a_{a|ij}^{(1)}) + \cdots$$

$$(\beta_A, \beta_{abcd}, \beta_{a|ij}) = (b_A^{(1)}, b_{abcd}^{(1)}, b_{a|ij}^{(1)}) + (b_A^{(2)}, b_{abcd}^{(2)}, b_{a|ij}^{(2)}) + \cdots$$
$$(\bar{\beta}_A, \bar{\beta}_{abcd}, \bar{\beta}_{a|ij}) = (\bar{b}_A^{(1)}, \bar{b}_{abcd}^{(1)}, \bar{b}_{a|ij}^{(1)}) + (\bar{b}_A^{(2)}, \bar{b}_{abcd}^{(2)}, \bar{b}_{a|ij}^{(2)}) + \cdots$$

$$(\bar{b}_A^{(1)}, \bar{b}_{abcd}^{(1)}, \bar{b}_{a|ij}^{(1)}) = (b_A^{(1)}, b_{abcd}^{(1)}, b_{a|ij}^{(1)})$$
$$(\bar{b}_A^{(2)}, \bar{b}_{abcd}^{(2)}, \bar{b}_{a|ij}^{(2)}) \neq (b_A^{(2)}, b_{abcd}^{(2)}, b_{a|ij}^{(2)})$$

- General case
  - Scheme-independence ⇒ Only lowest-order terms
  - Scheme-dependence ⇒ All higher-order terms
  - Scheme-dependent terms cannot all be set to vanish
  - High-precision numerical analysis not possible (but useful)

Examples

Physical $d = 4$ case

- No proper example yet $\Rightarrow$ Maybe none ?
  - Technically difficult to generate β-functions
- $SU(2)$ gauge theory with two real scalars (singlet) and two active flavors of Weyl fermions (fundamental)
  - Unbounded-from-below scalar potential
  - Only found numerically $\Rightarrow$ Trustworthy ?

Unphysical $d = 4 - \epsilon$ case

- Two real scalars and one Dirac fermion
  - Limit cycle (bounded-from-below scalar potential, CP conservation, vacuum at origin of field space)
  - Unsatisfactory unless condensed matter example in $\epsilon \to 1$ limit (universality class ?)

Features and future work

Physics of non-conformal scale-invariant QFTs

- Less constrained than CFTs
- β-functions $\sim$ Anomalous dimensions
- Rare RG flows (recurrent behaviors)
  - RG flows $\neq$ Gradient flows
  - Strongest version of a-theorem violated

Future work

- Phenomenological applications
  - Cyclic unparticle physics JFF, Grinstein, Stergiou (2011)
- Generic examples (with β-functions at higher order) ?

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