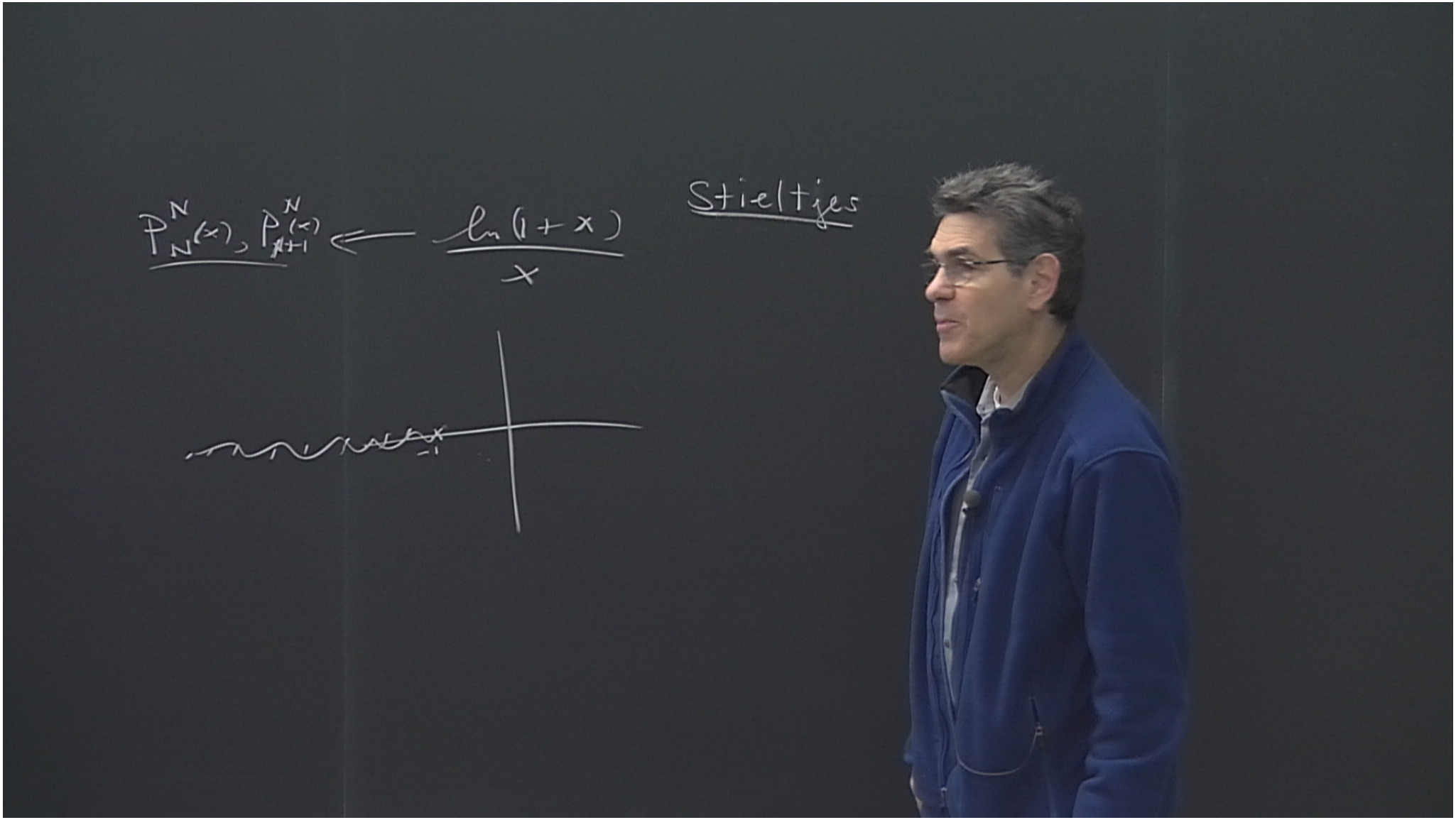
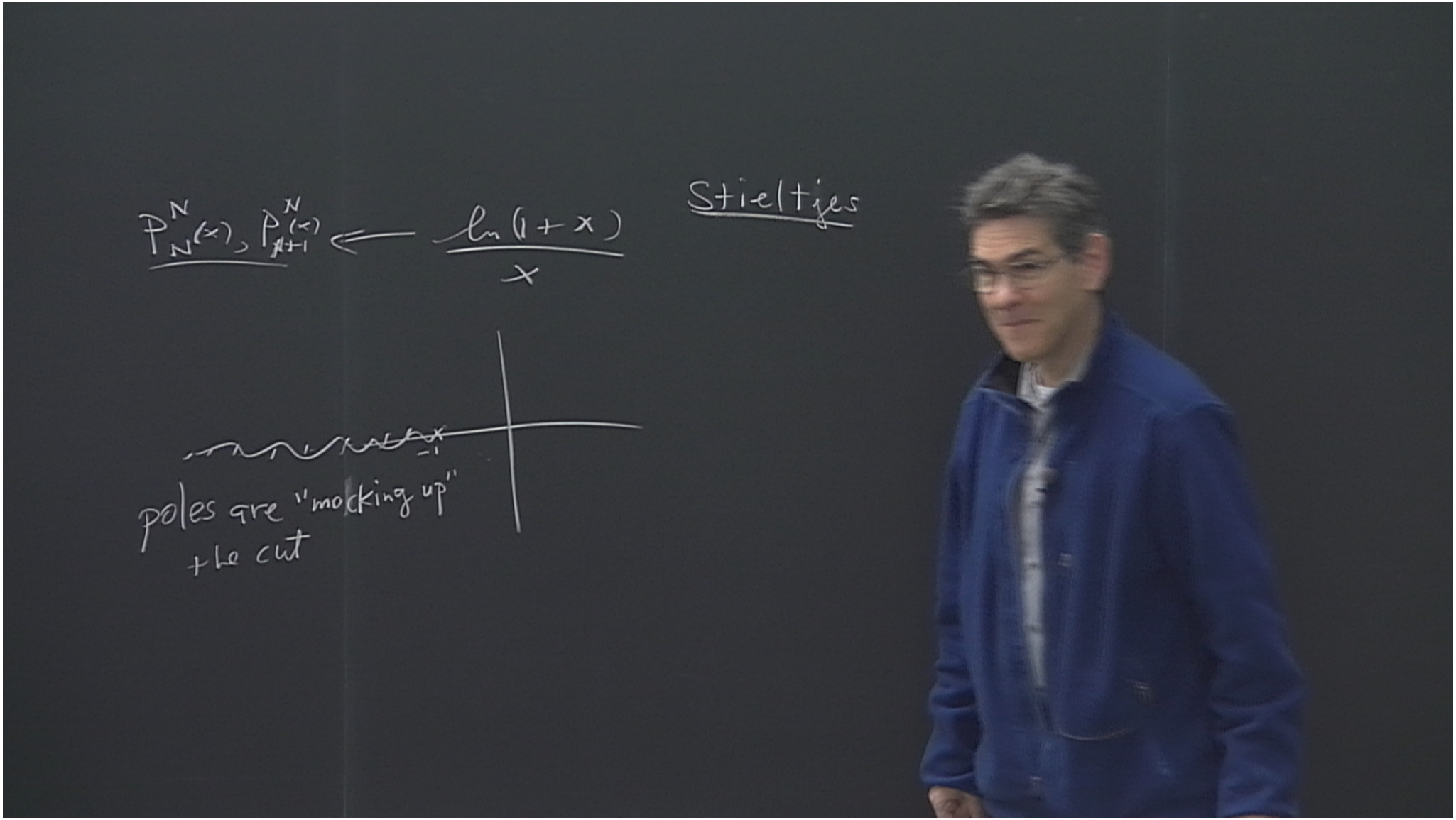
Title: Mathematical Physics - Lecture 7

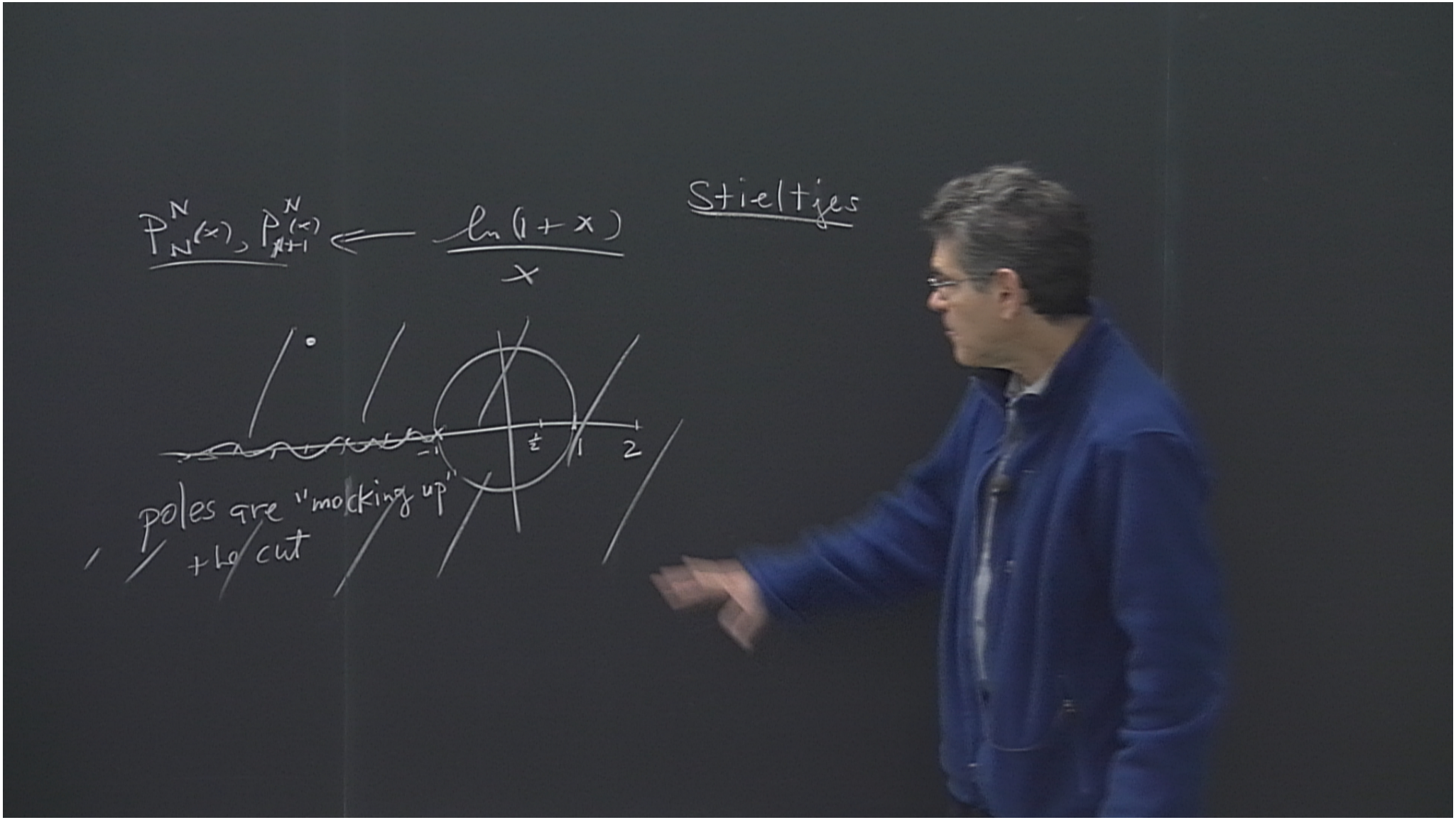
Date: Nov 29, 2011 09:00 AM

URL: http://pirsa.org/11110048

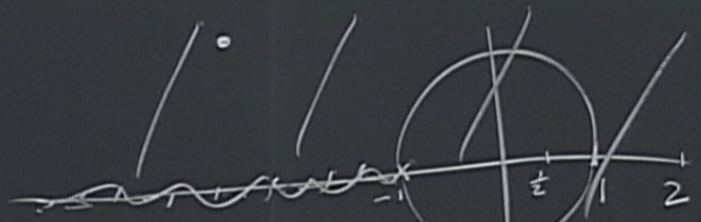
Abstract:


$P_N^N(x), P_{N+1}^N(x) \leftarrow \frac{\ln(1+x)}{x}$
Stieltjes
-1

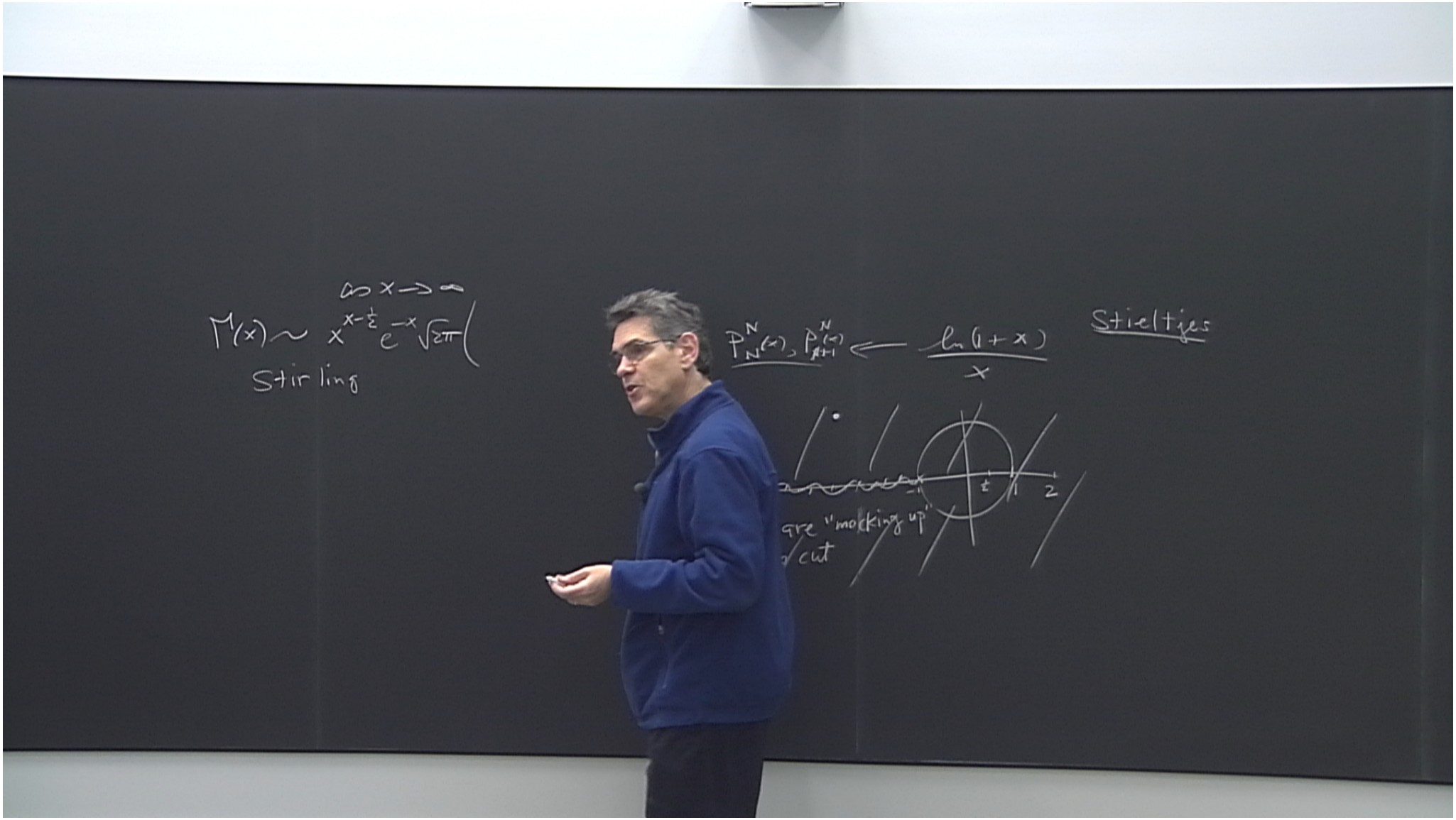

$P_N^N(x), P_{N+1}^N(x) \Leftarrow \frac{\ln(1+x)}{x}$
Stieltjes
-1
poles are "mocking up" the cut

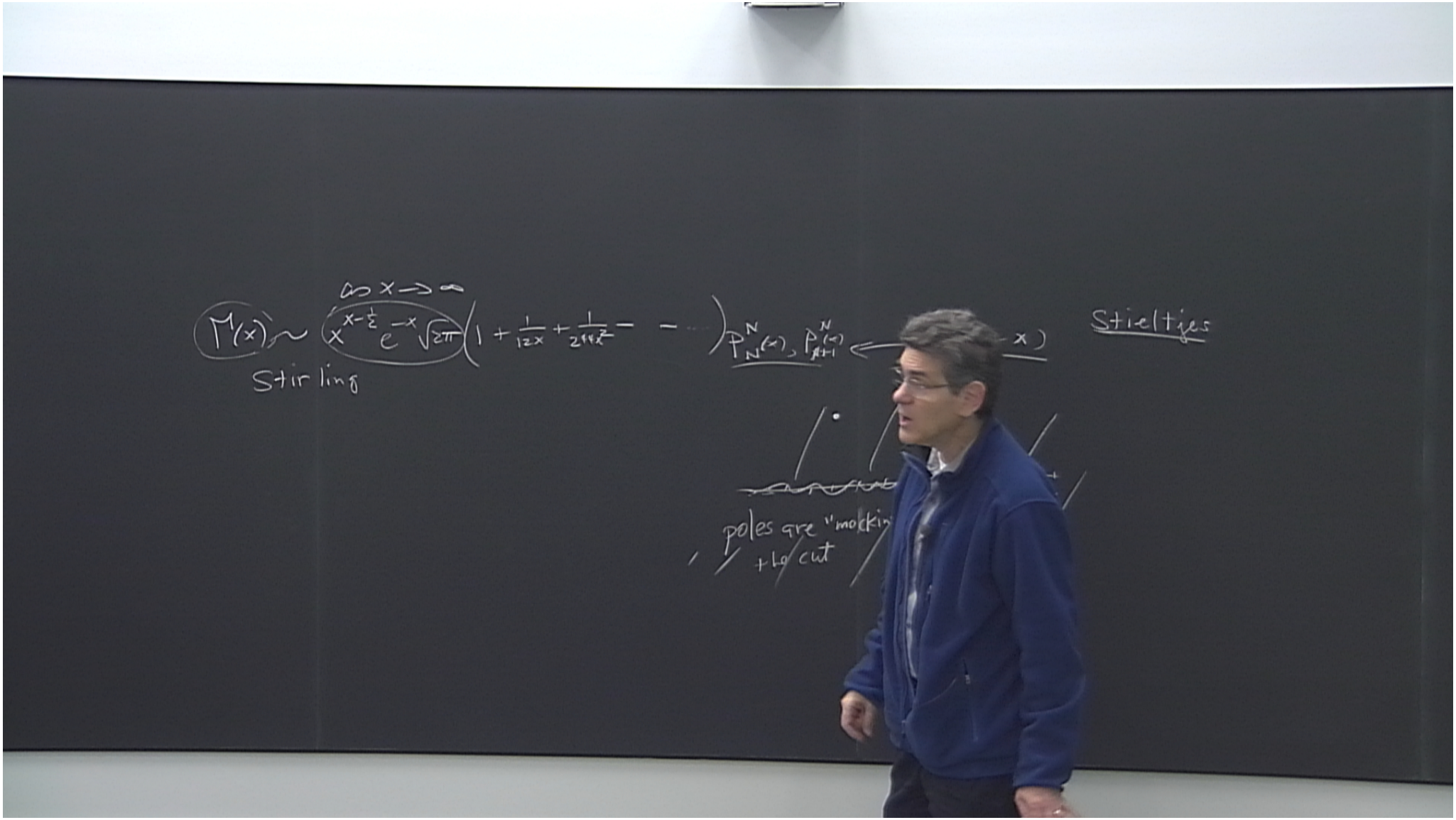

$P_N^N(x), P_{N+1}^N(x) \leftarrow \frac{\ln(1+x)}{x}$
Stieltjes
-1
1/2
1
2
poles are "mocking up"
+ log cut

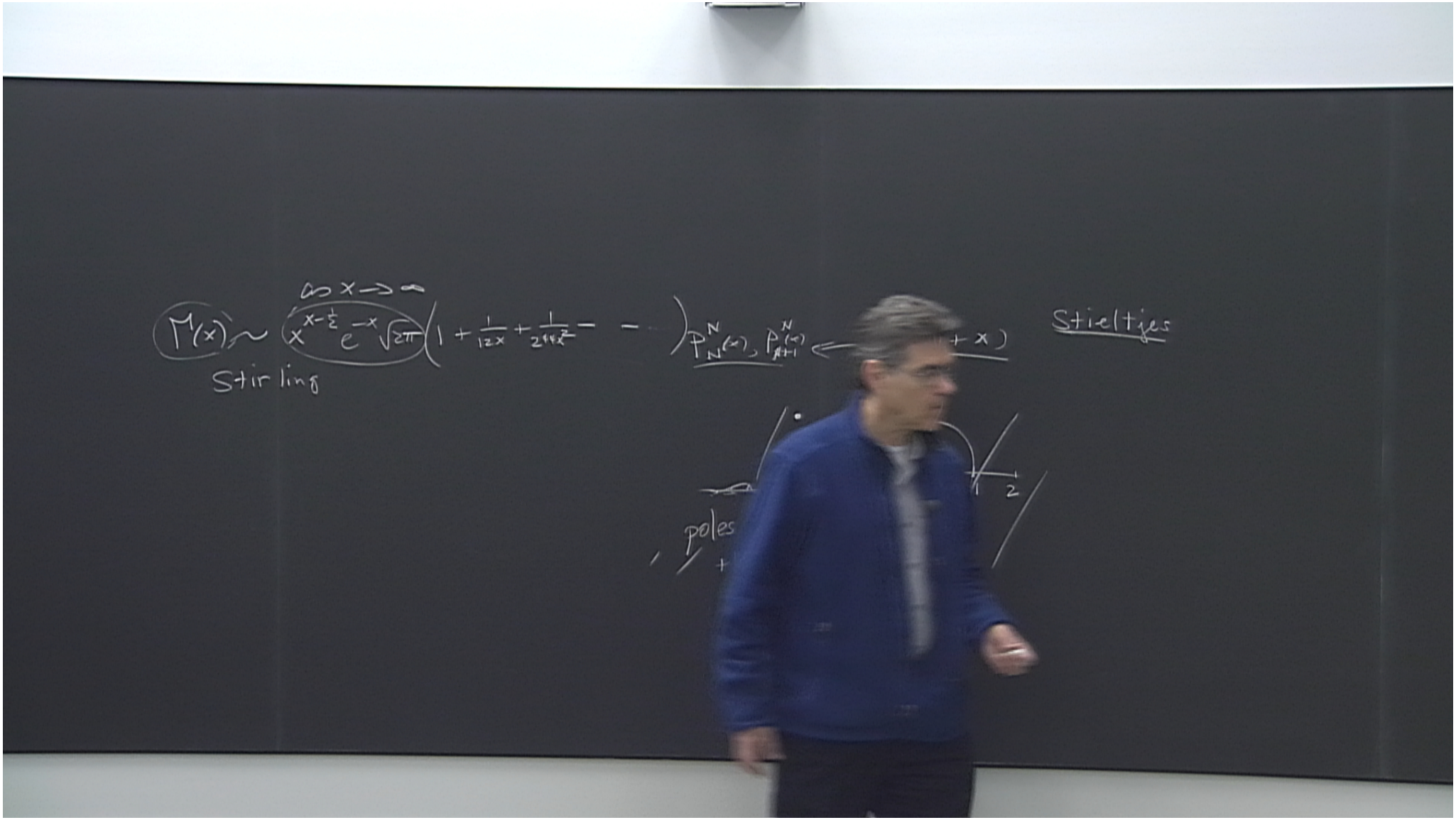
$P_N^N(x)$, $P_{N+1}^N(x)$ ⟵ $\frac{\ln(1+x)}{x}$ Stieltjes

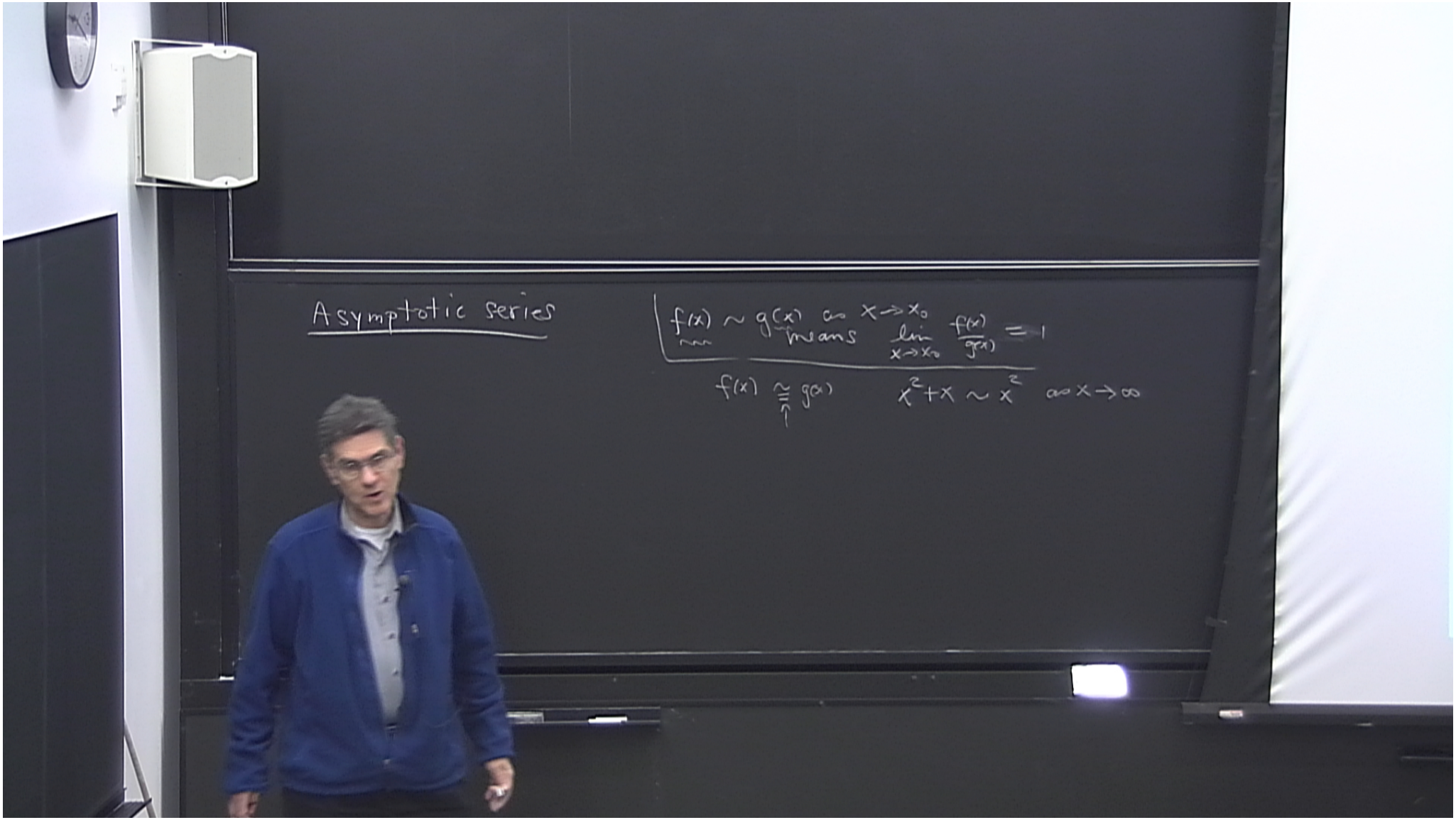


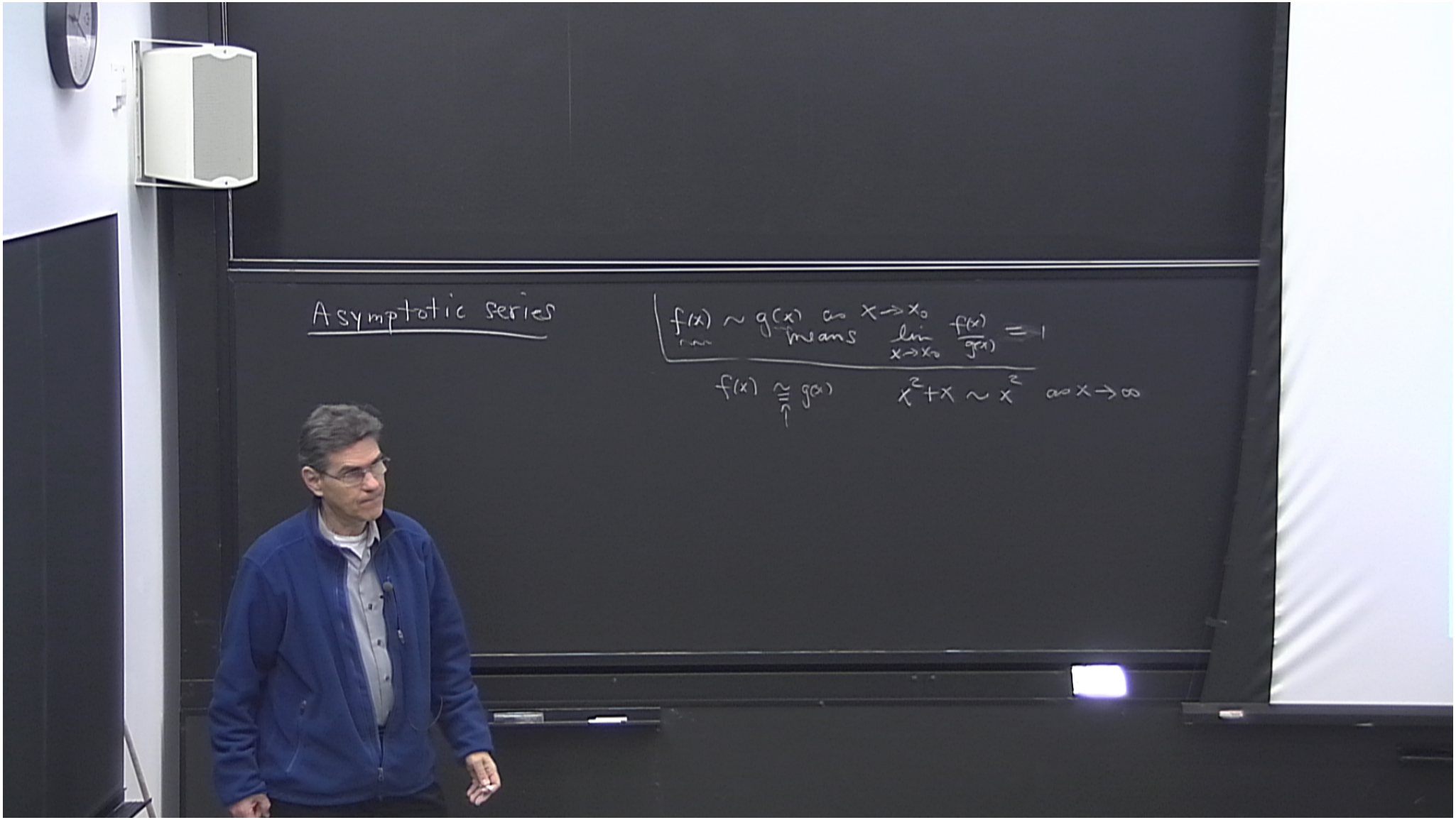
poles are "mocking up" + Log cut

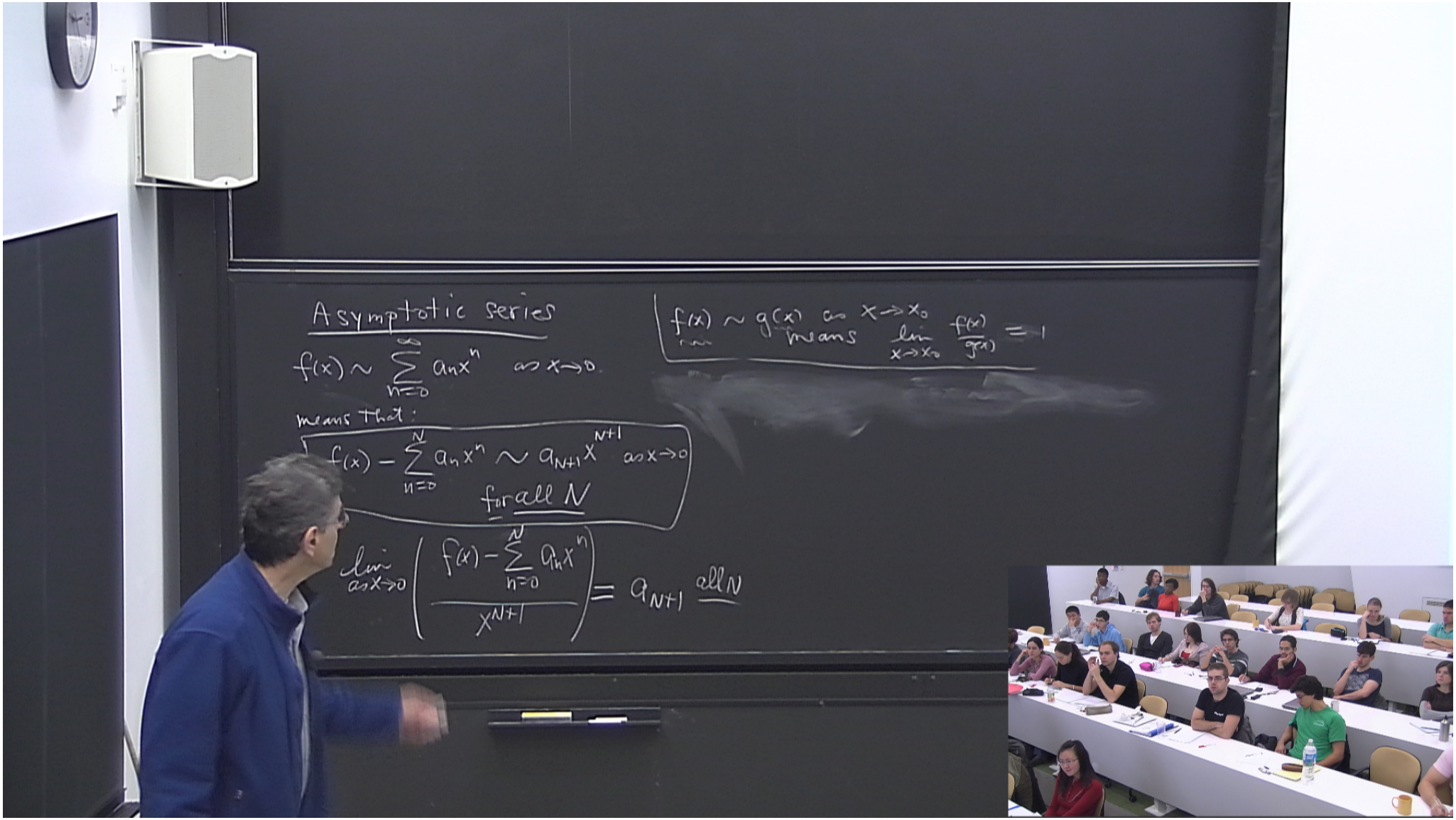

Stirling
Stieltjes
are "mocking up"
cut

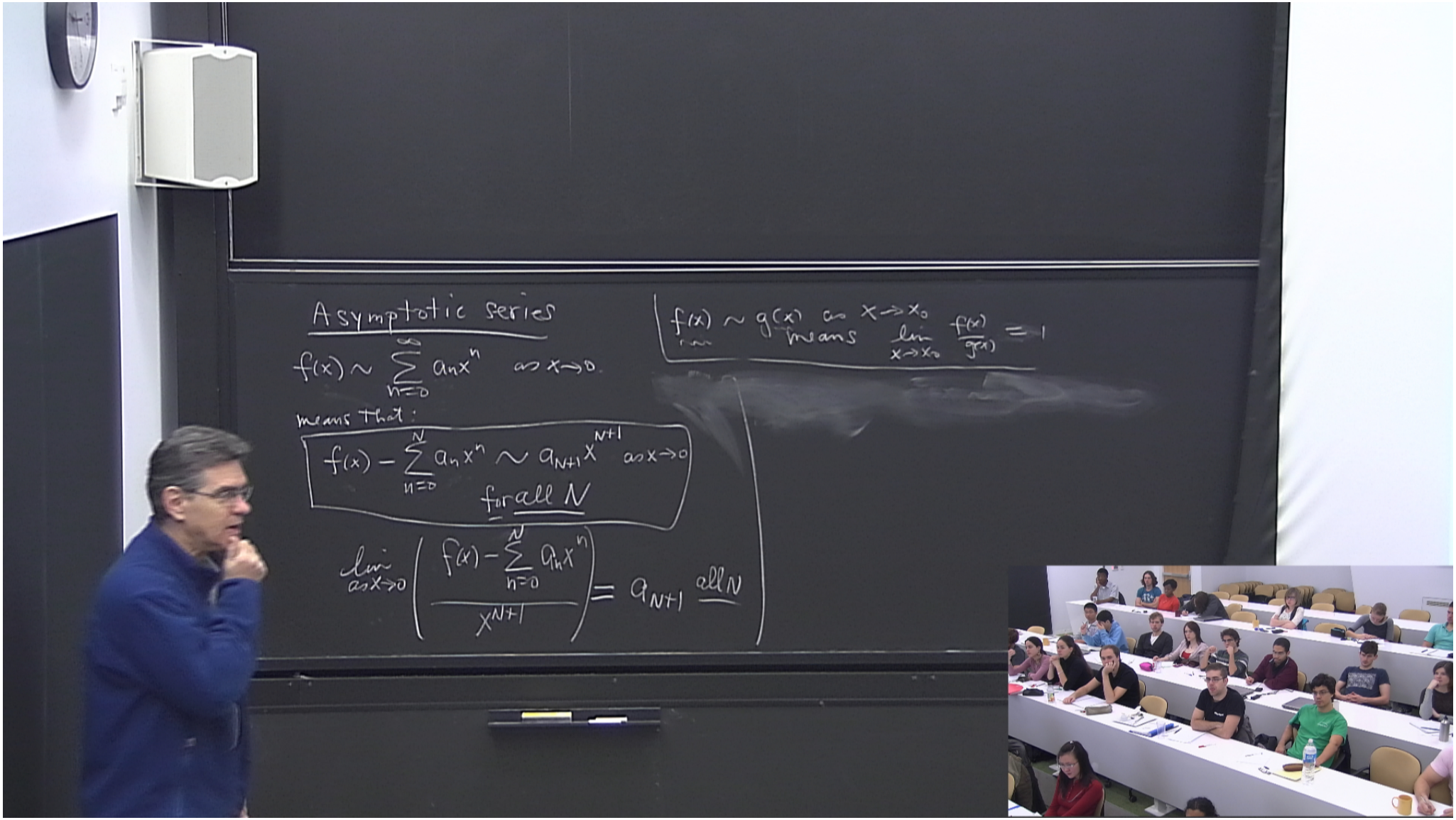

as x → ∞
Γ(x) ~ x^{x-1/2} e^{-x} √(2π) (1 + 1/12x + 1/288x^2 − ⋯)
Stirling
P_N^N(x), P_{N+1}^N(x)
x)
Stieltjes
poles are "mokkin
+ branch cut

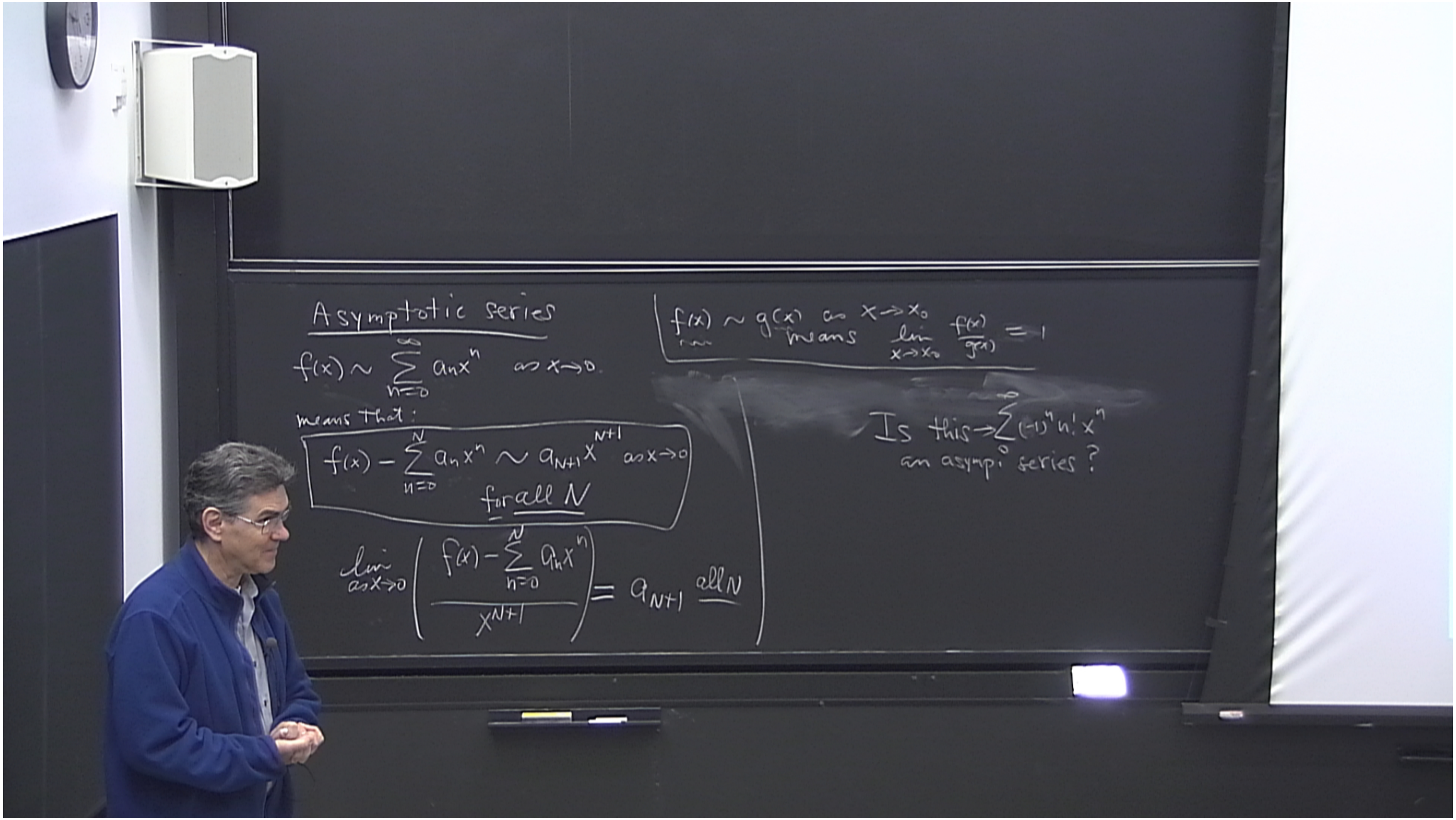

as x → ∞
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Stirling
P_N^N(x), P_{N+1}^N(x)
+ x)
Stieltjes
poles
1
2

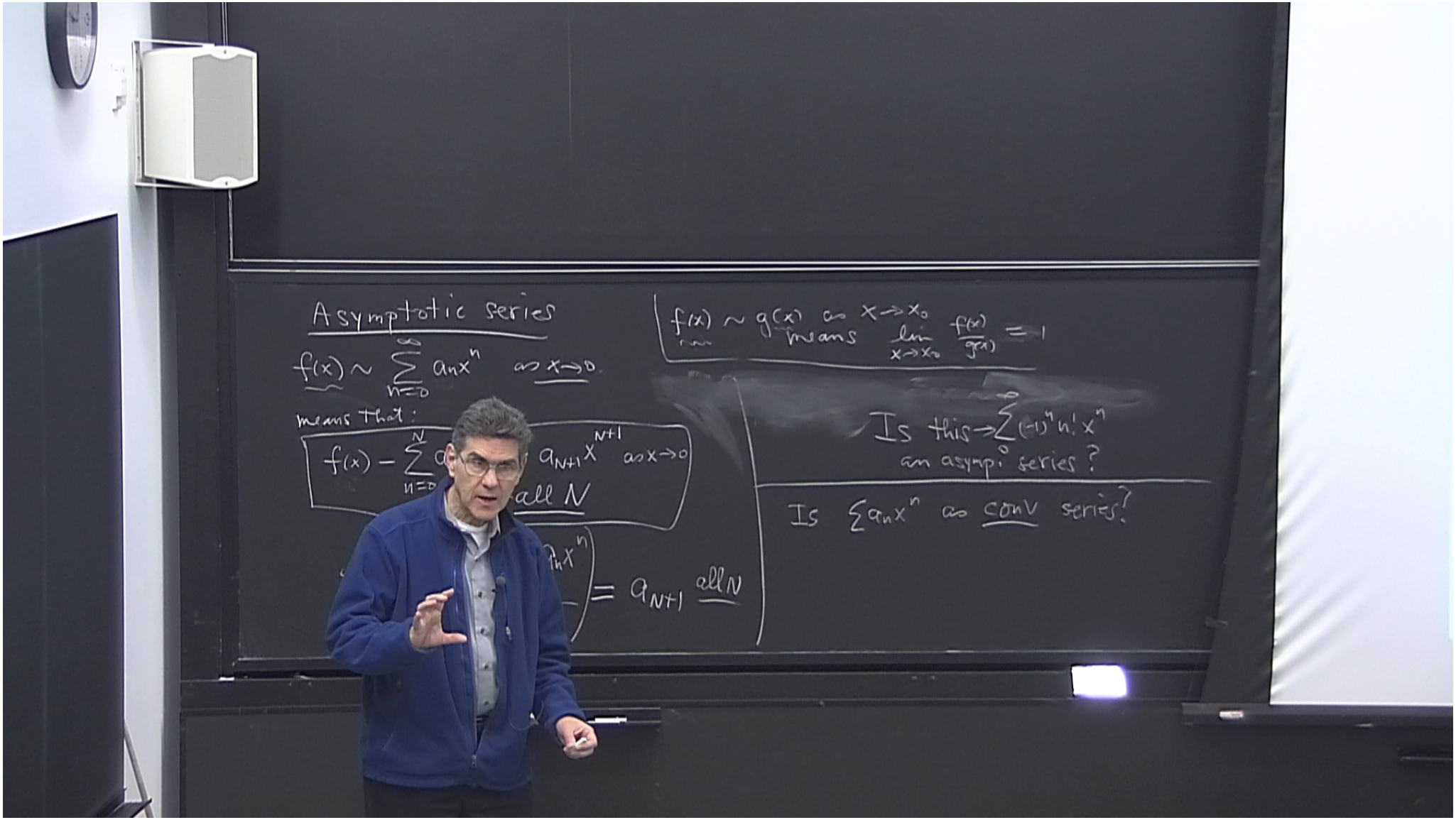

Asymptotic series
$f(x) \sim g(x)$ as $x \to x_0$ means $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1$
$f(x) \cong g(x)$
$x^2 + x \sim x^2$ as $x \to \infty$

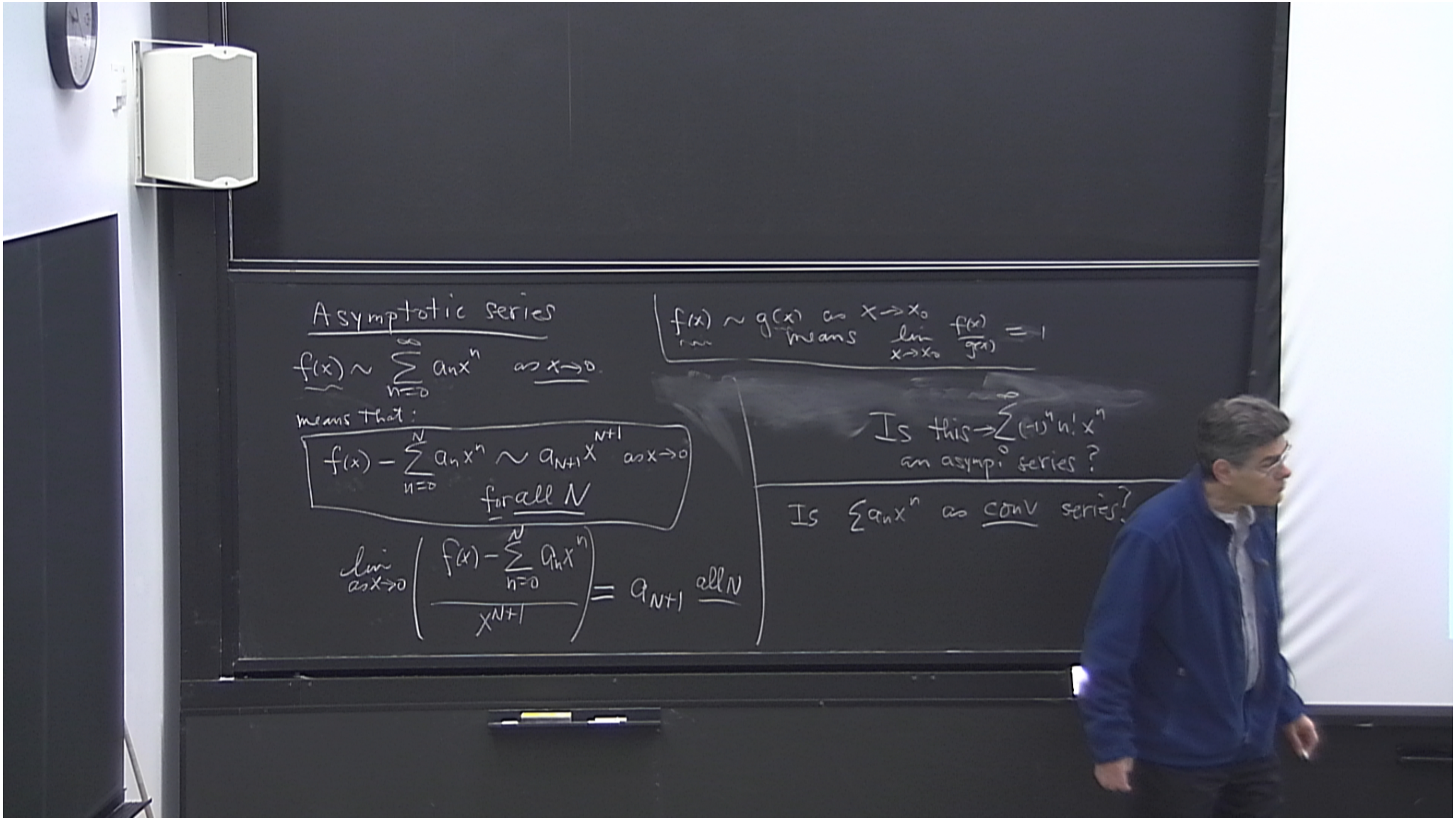

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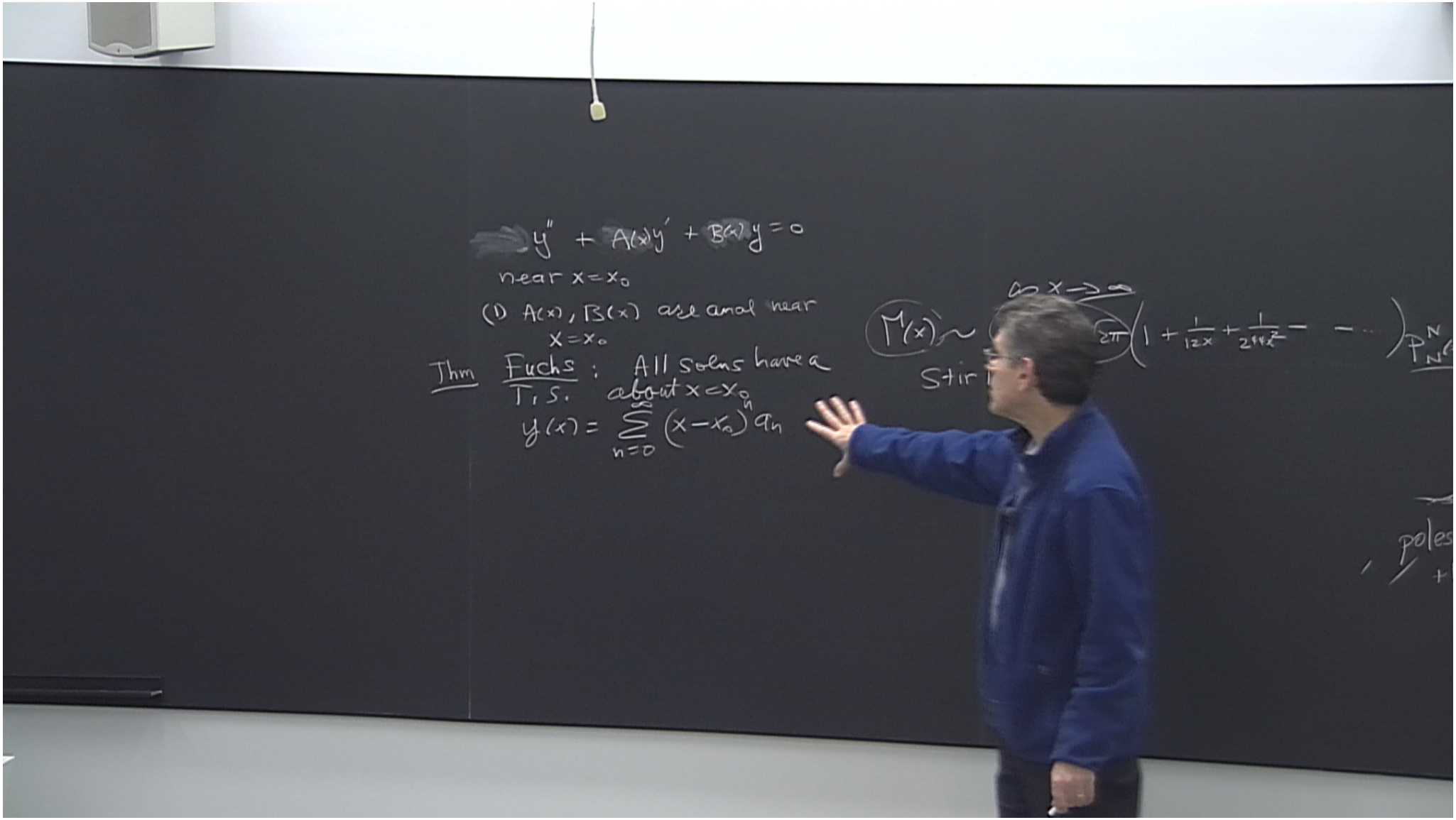

Asymptotic series
$f(x) \sim \sum_{n=0}^{\infty} a_n x^n$ as $x \to 0$
means that:
$f(x) - \sum_{n=0}^{N} a_n x^n \sim a_{N+1} x^{N+1}$ as $x \to 0$
for all N
$\lim_{x \to 0} \left(\frac{f(x) - \sum_{n=0}^{N} a_n x^n}{x^{N+1}} \right) = a_{N+1}$ all N
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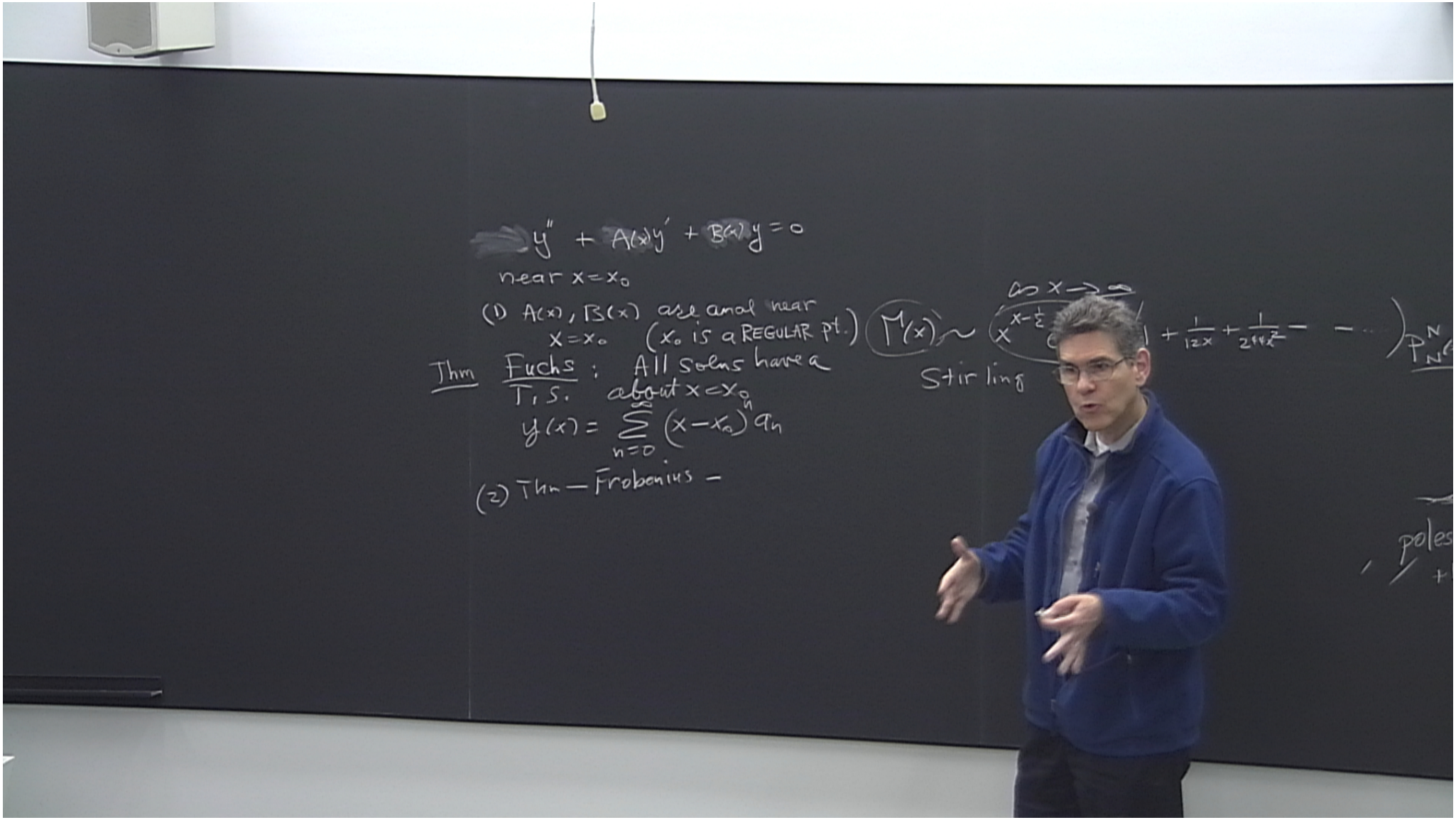

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Is this $\to \sum_{0}^{\infty} (-1)^n n! x^n$
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$y'' + A(x)y' + B(x)y = 0$
near $x = x_0$
(1) $A(x)$, $B(x)$ are anal near $x = x_0$
Thm Fuchs: All solns have a T.S. about $x = x_0$
$y(x) = \sum_{n=0}^{\infty} (x - x_0)^n a_n$
$\Gamma(x) \sim$
$\sqrt{2\pi}\left(1 + \frac{1}{12x} + \frac{1}{2!(12x)^2} - \cdots\right)$
Stir
poles


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(2) Thm — Frobenius —
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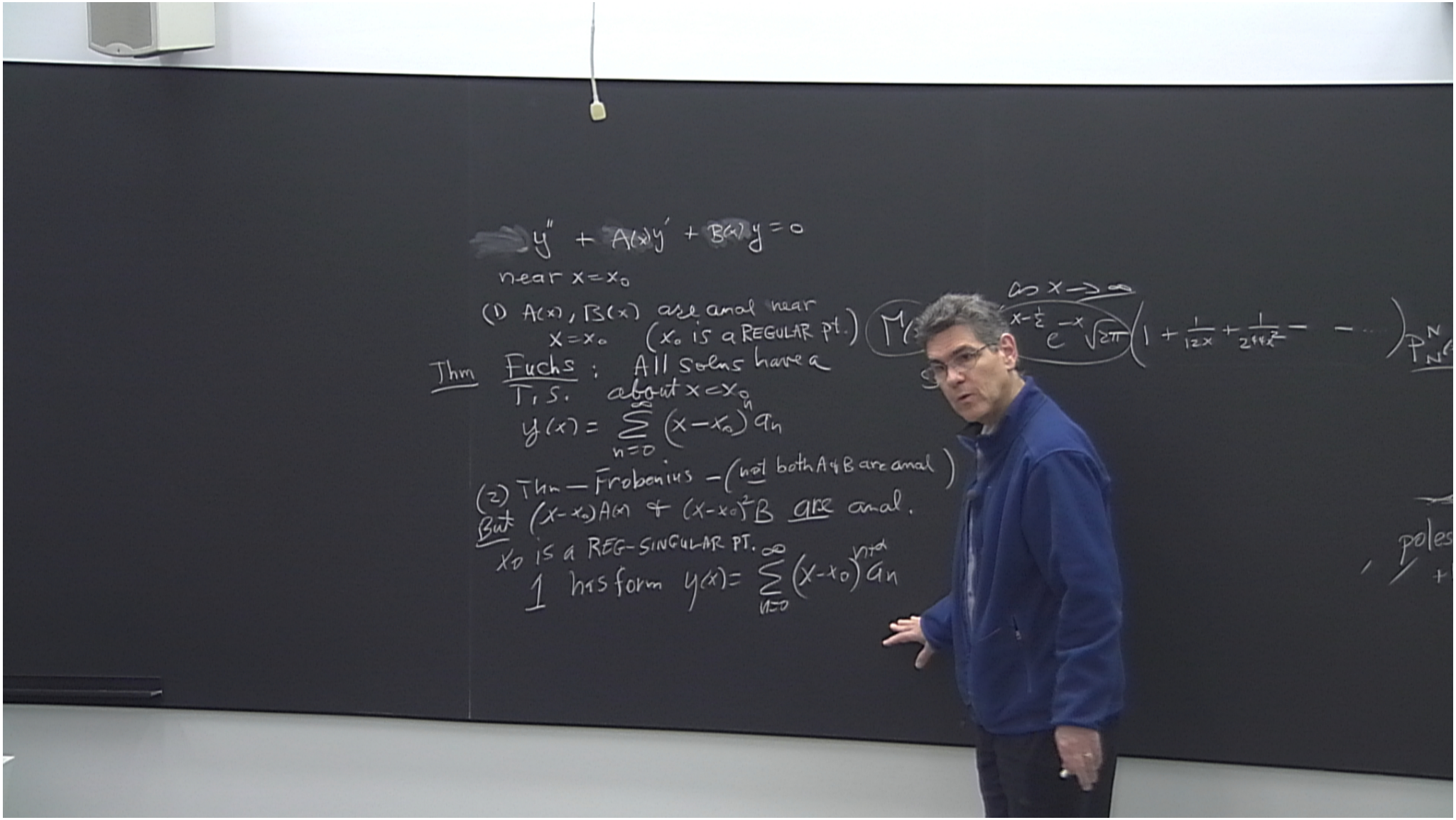
(2) Thm — Frobenius — (not both A & B are anal

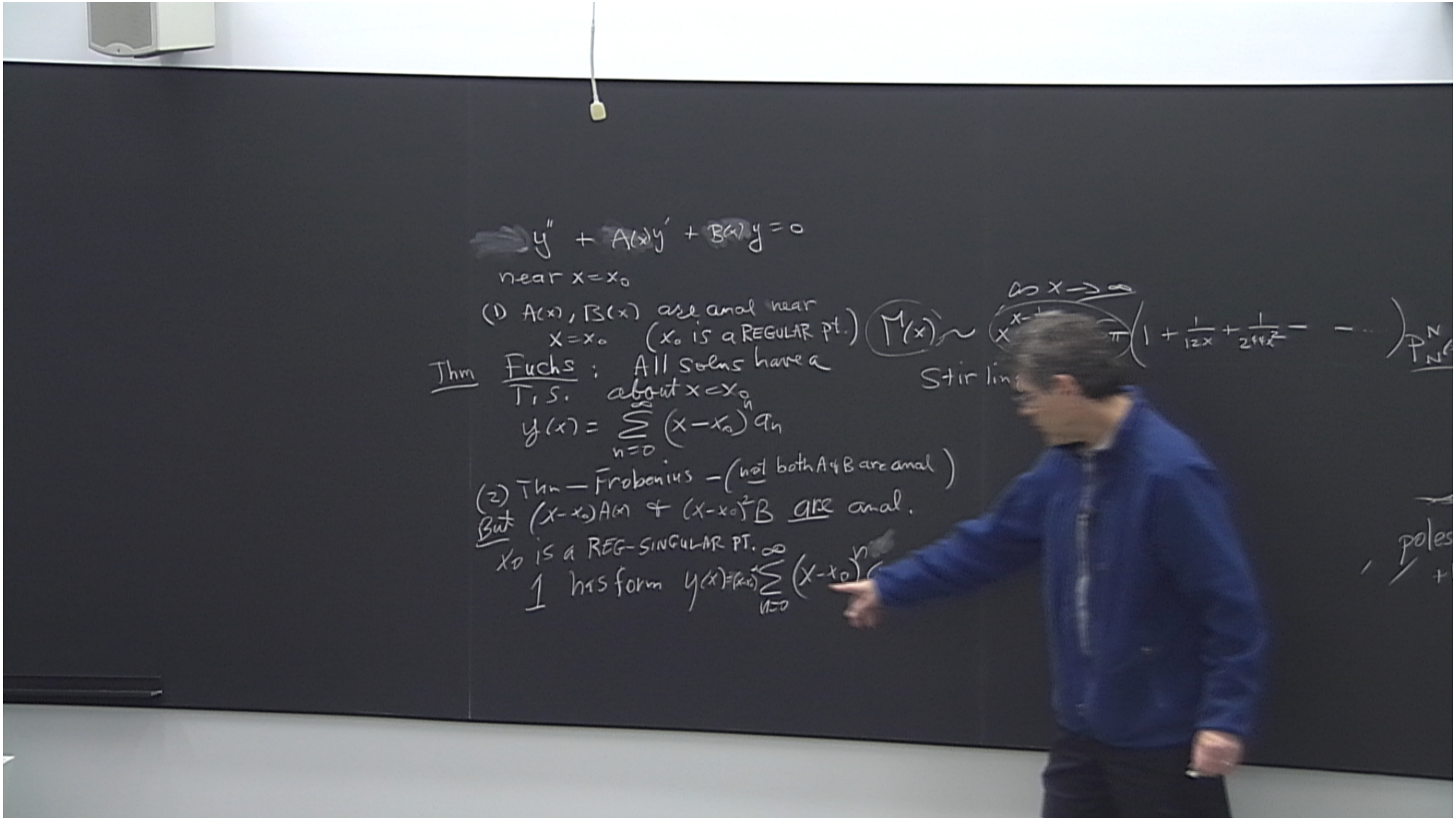
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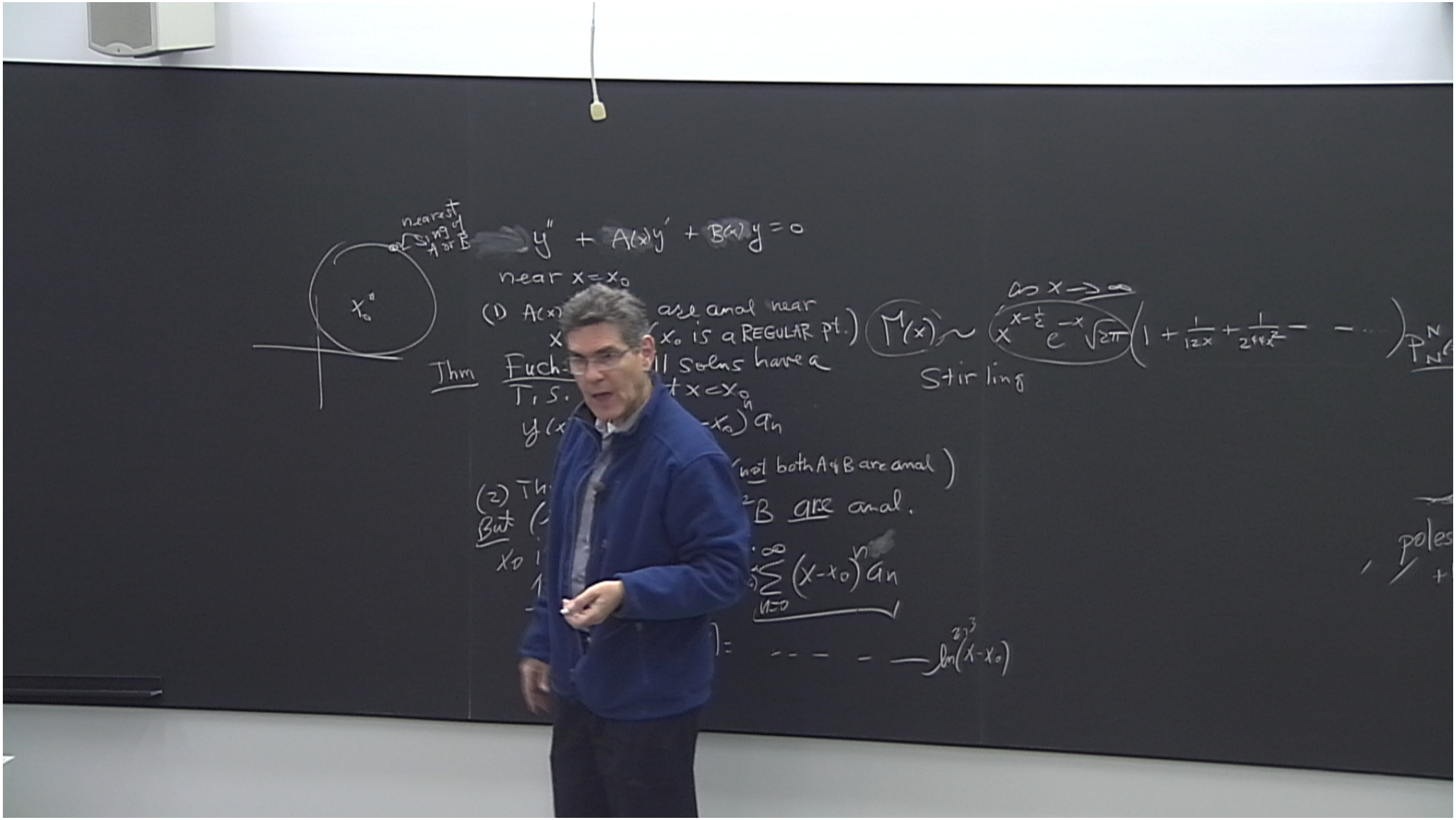
x_0 is a REG-SINGULAR PT.

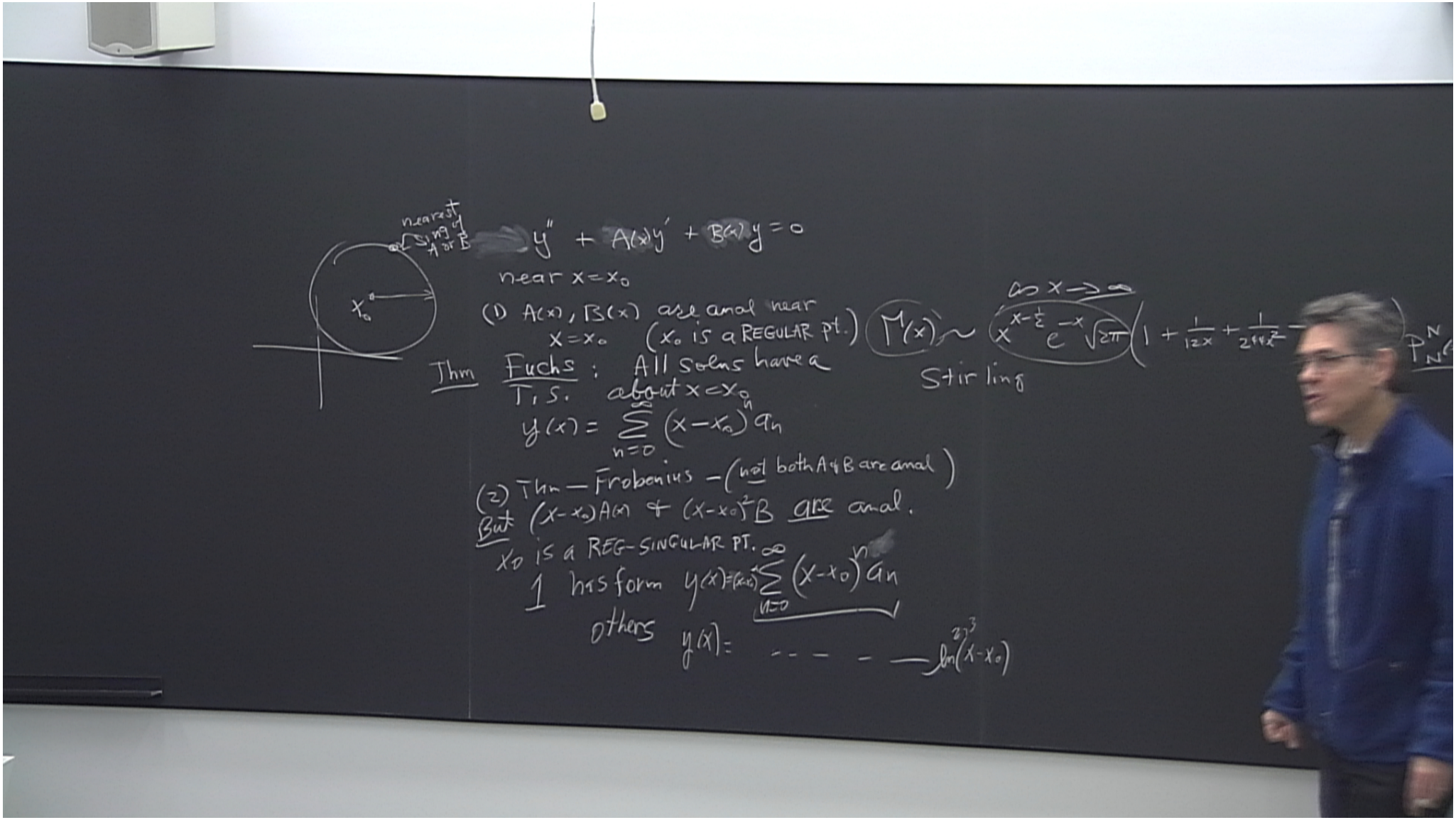
$\Gamma(x) \sim$ Stirling

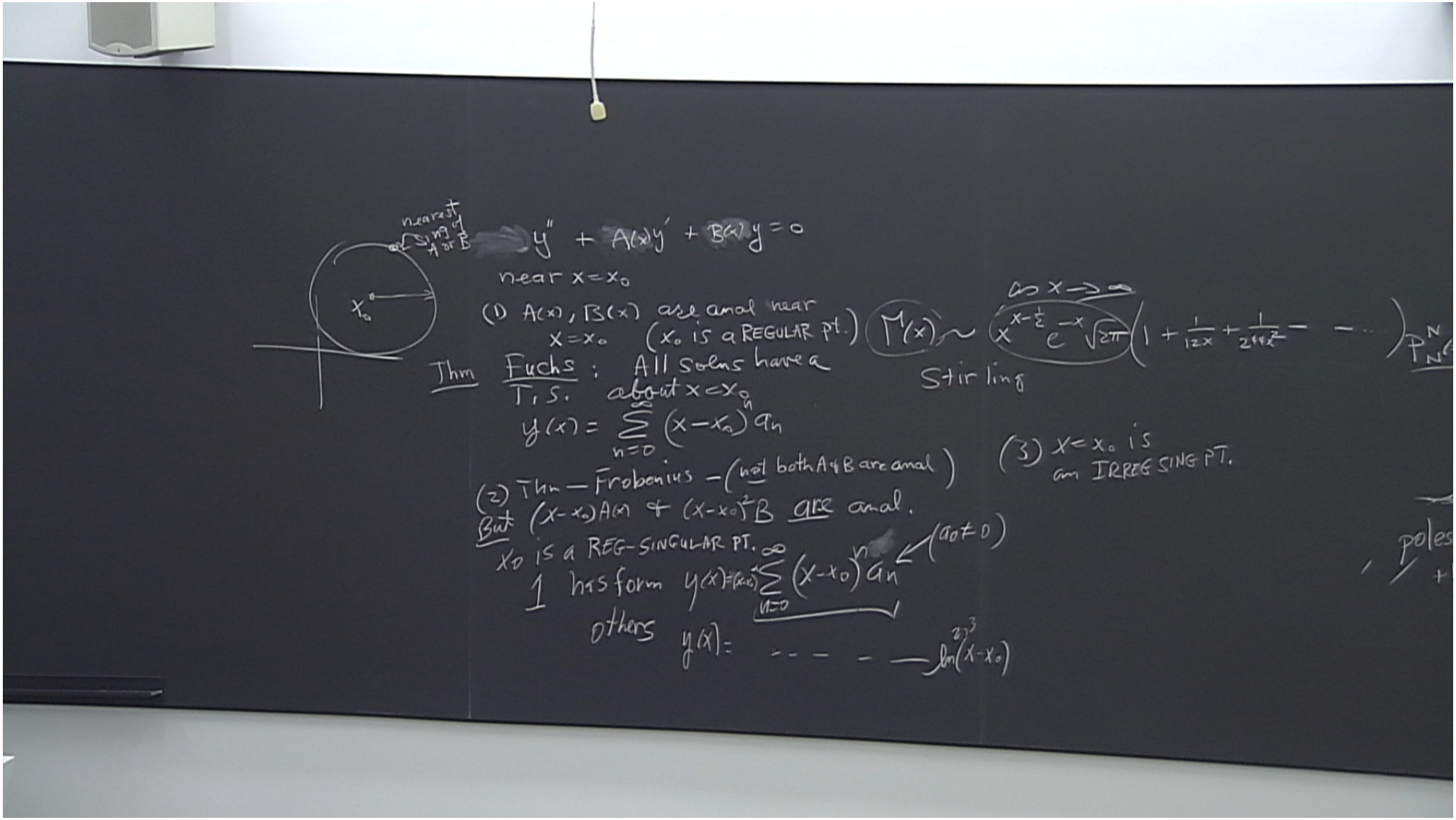


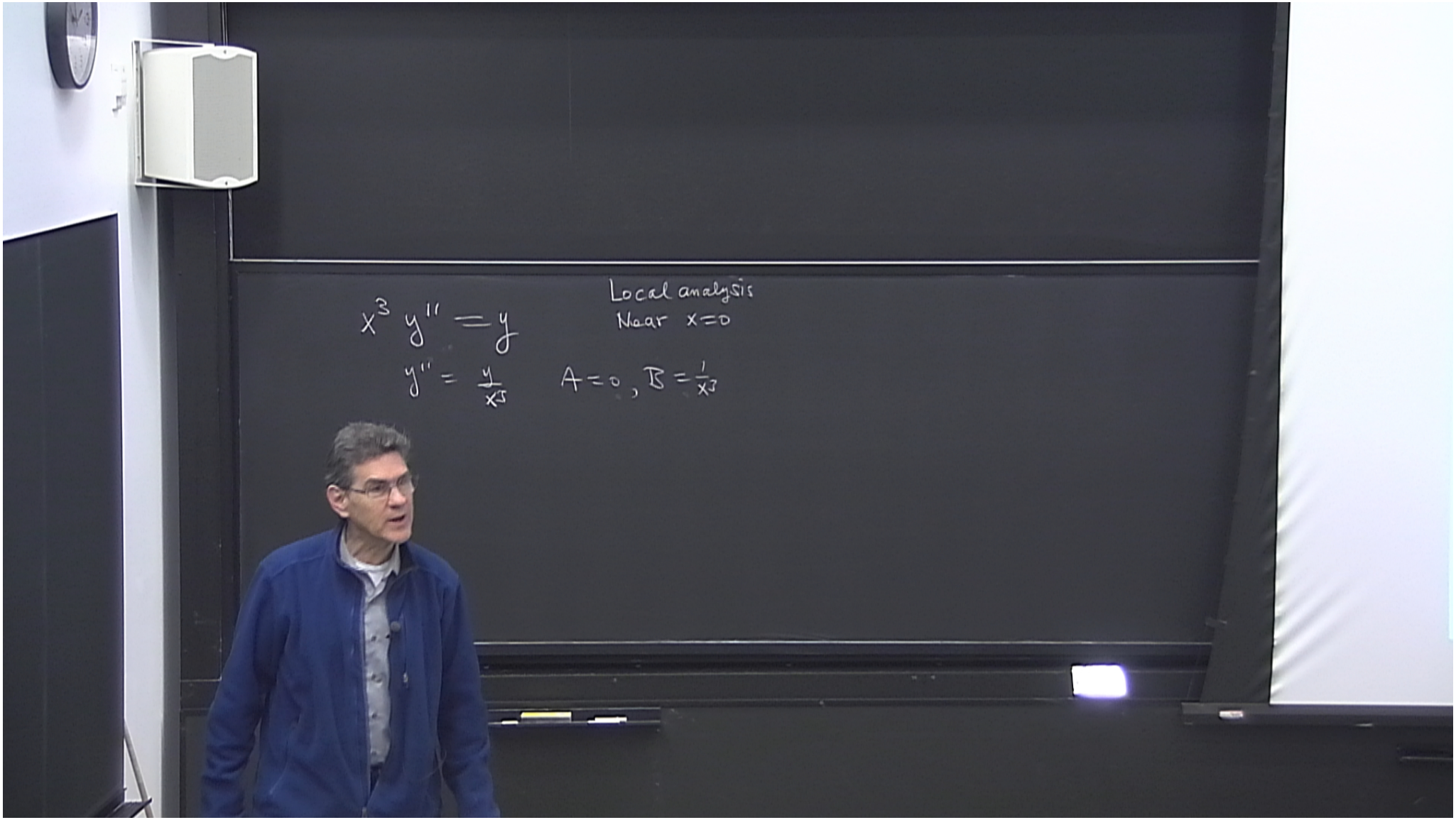

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But $(x - x_0)A(x)$ & $(x - x_0)^2 B$ are anal.
x_0 is a REG-SINGULAR PT.
1 has form $y(x) = \sum_{n=0}^{\infty} (x - x_0)^{n+d} a_n$
as $x \to \infty$
$e^{-x} \sqrt{2\pi} \left(1 + \frac{1}{12x} + \frac{1}{2!(12x)^2} - \cdots\right)$
poles

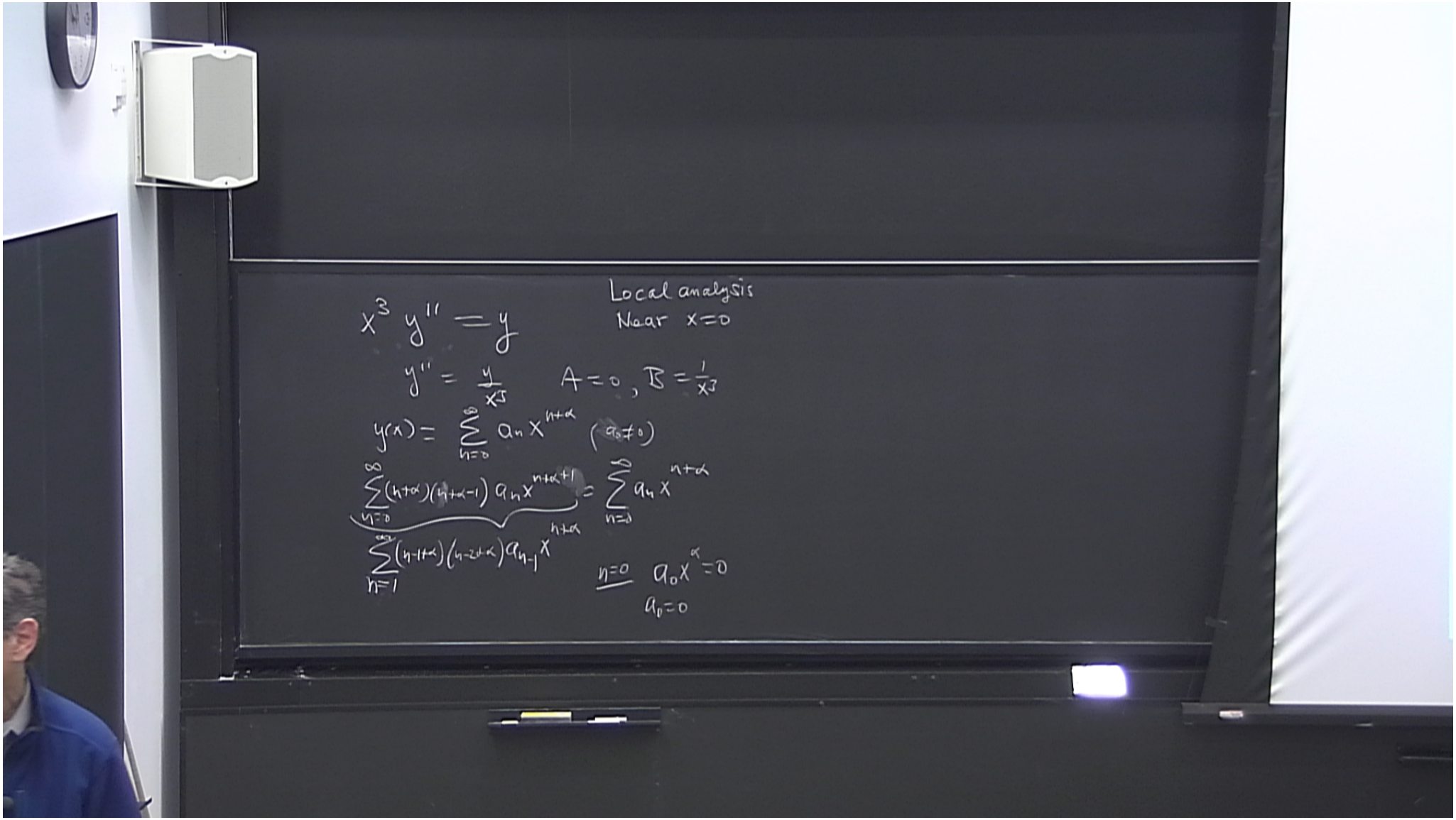

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$\Gamma(x) \sim$
Stir lin
poles

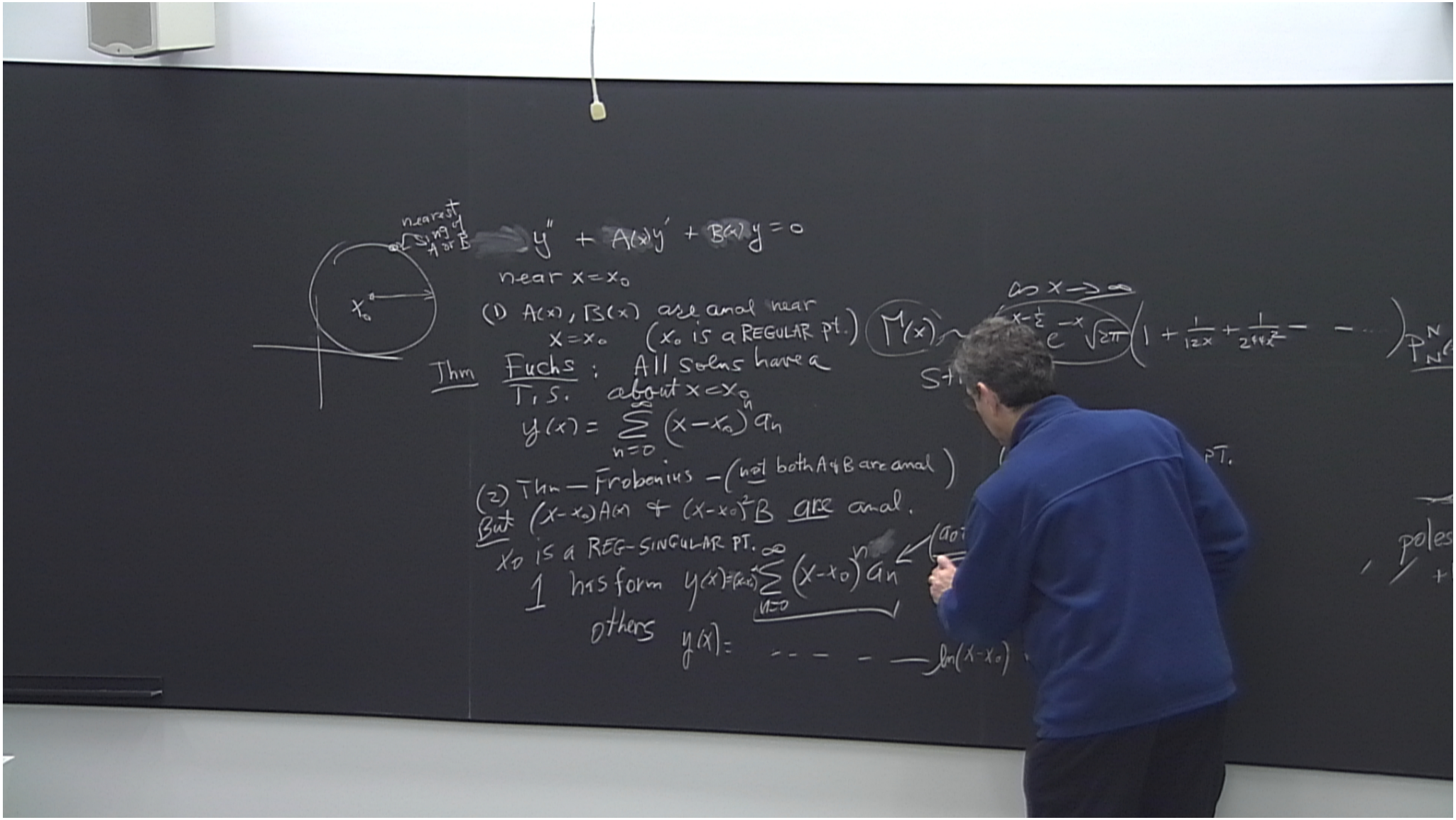

nearest singularity of A or B
$y'' + A(x)y' + B(x)y = 0$
near $x = x_0$
x_0
(1) A(x) ... are anal near
x_0 is a REGULAR pt.)
Thm Fuch... ll Solns have a
T.S. ... $x = x_0$
$y(x ... -x_0)^n a_n$
(2) Th...
But (... not both A & B are anal)
x_0 ... B are anal.
$\sum_{n=0}^{\infty} (x-x_0)^n a_n$
$\Gamma(x) \sim x^{x-\frac{1}{2}} e^{-x} \sqrt{2\pi} \left(1 + \frac{1}{12x} + \frac{1}{2!(12x)^2} - \cdots\right)$
as $x \to \infty$
Stirling
poles
$\ln(x-x_0)$

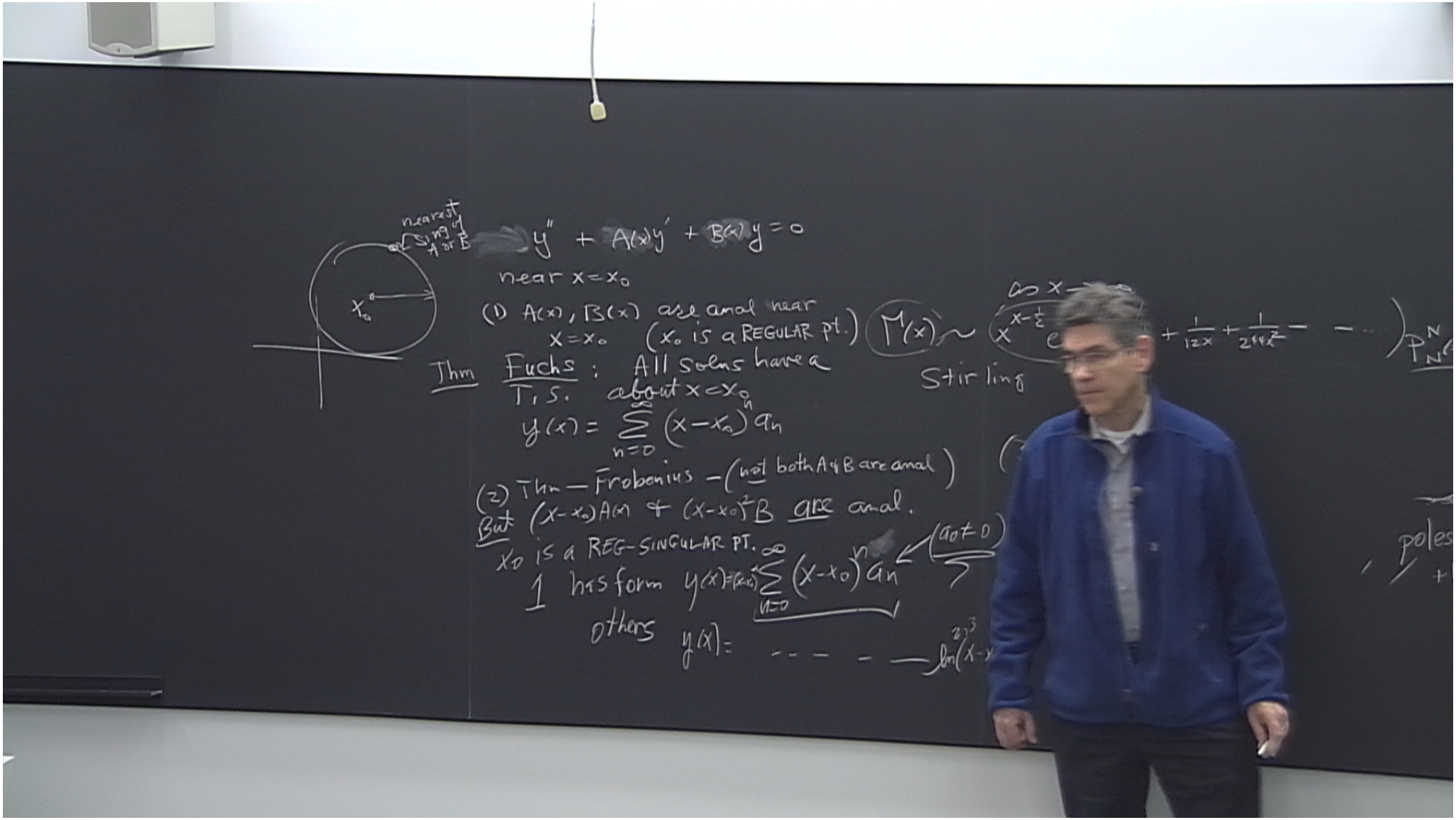

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others $y(x) =$ ---
$\ln(x - x_0)$
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$\Gamma(x) \sim x^{x-\frac{1}{2}} e^{-x} \sqrt{2\pi} \left(1 + \frac{1}{12x} + \frac{1}{2!(12x)^2} - \right.$
Stirling

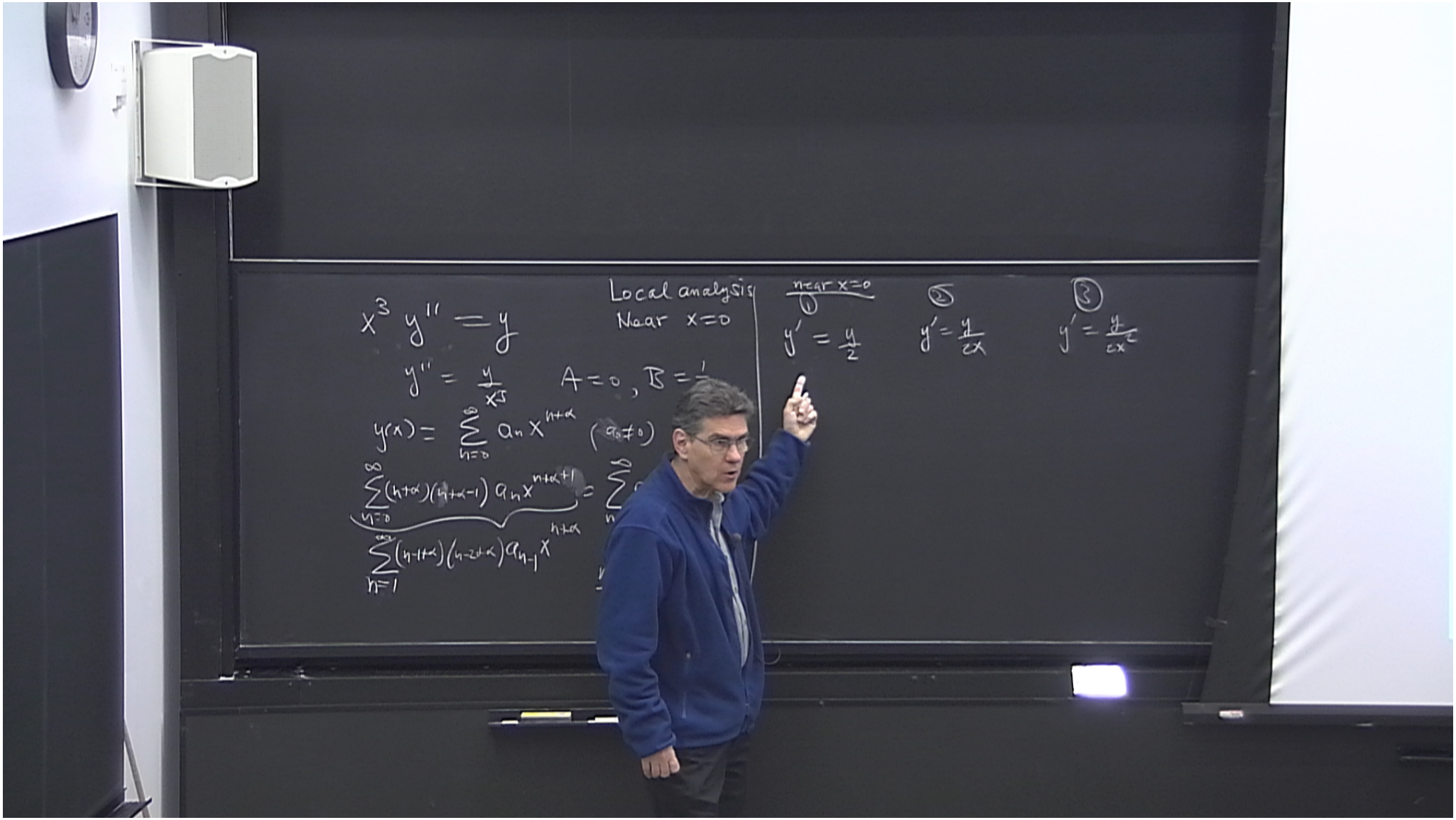

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But $(x-x_0)A(x)$ & $(x-x_0)^2 B$ are anal.
x_0 is a REG-SINGULAR PT.
1 has form $y(x) = \sum_{n=0}^{\infty} (x-x_0)^n a_n$ $(a_0 \neq 0)$
others $y(x) =$ ---
(3) $x = x_0$ is an IRREG SING PT.
$\Gamma(x) \sim x^{x-\frac{1}{2}} e^{-x} \sqrt{2\pi} \left(1 + \frac{1}{12x} + \frac{1}{2!(12x)^2} - \cdots\right)$ as $x \to \infty$
Stirling
poles

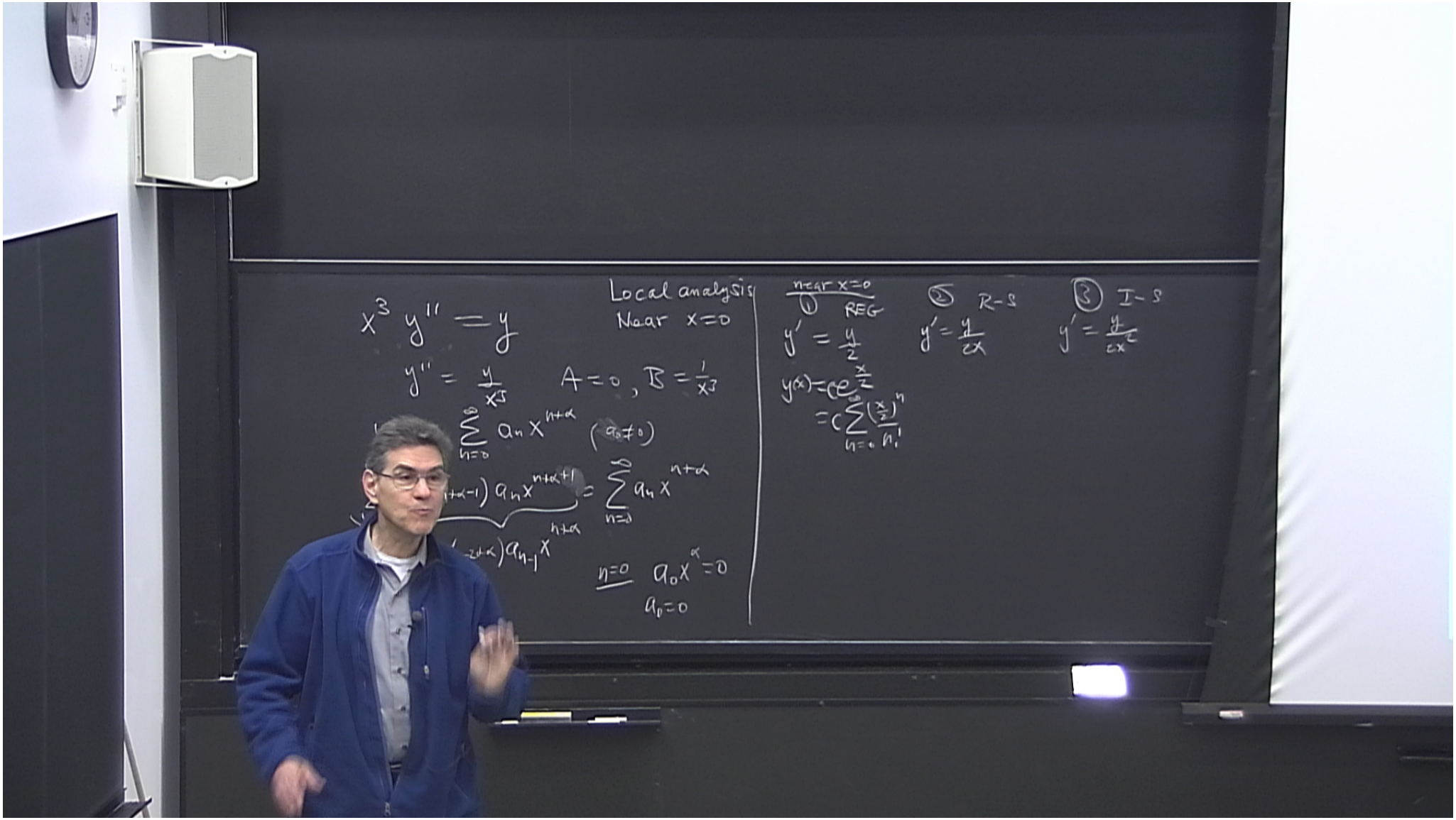

$x^3 y'' = y$
Local analysis
Near $x=0$
$y'' = \frac{y}{x^3}$
$A=0, B=\frac{1}{x^3}$

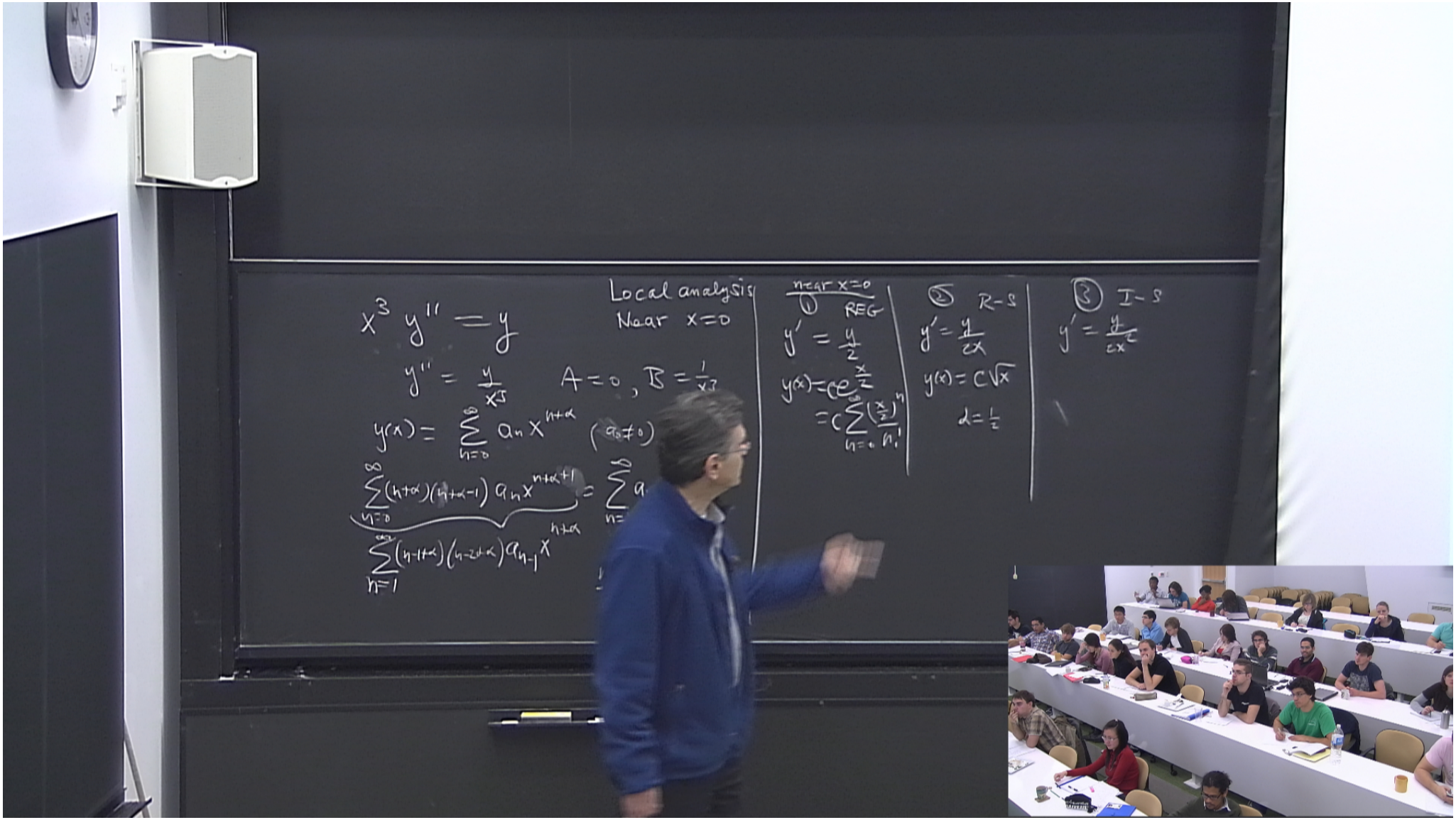

Local analysis
Near x=0
$x^3 y'' = y$
$y'' = \frac{y}{x^3}$ $A=0$, $B=\frac{1}{x^3}$
$y(x) = \sum_{n=0}^{\infty} a_n x^{n+\alpha}$ $(a_0 \neq 0)$
$\sum_{n=0}^{\infty}(n+\alpha)(n+\alpha-1)\, a_n x^{n+\alpha+1} = \sum_{n=0}^{\infty} a_n x^{n+\alpha}$
$\sum_{n=1}^{\infty}(n-1+\alpha)(n-2+\alpha) a_{n-1} x^{n+\alpha}$
$n=0$ $a_0 x^{\alpha} = 0$
$a_0 = 0$

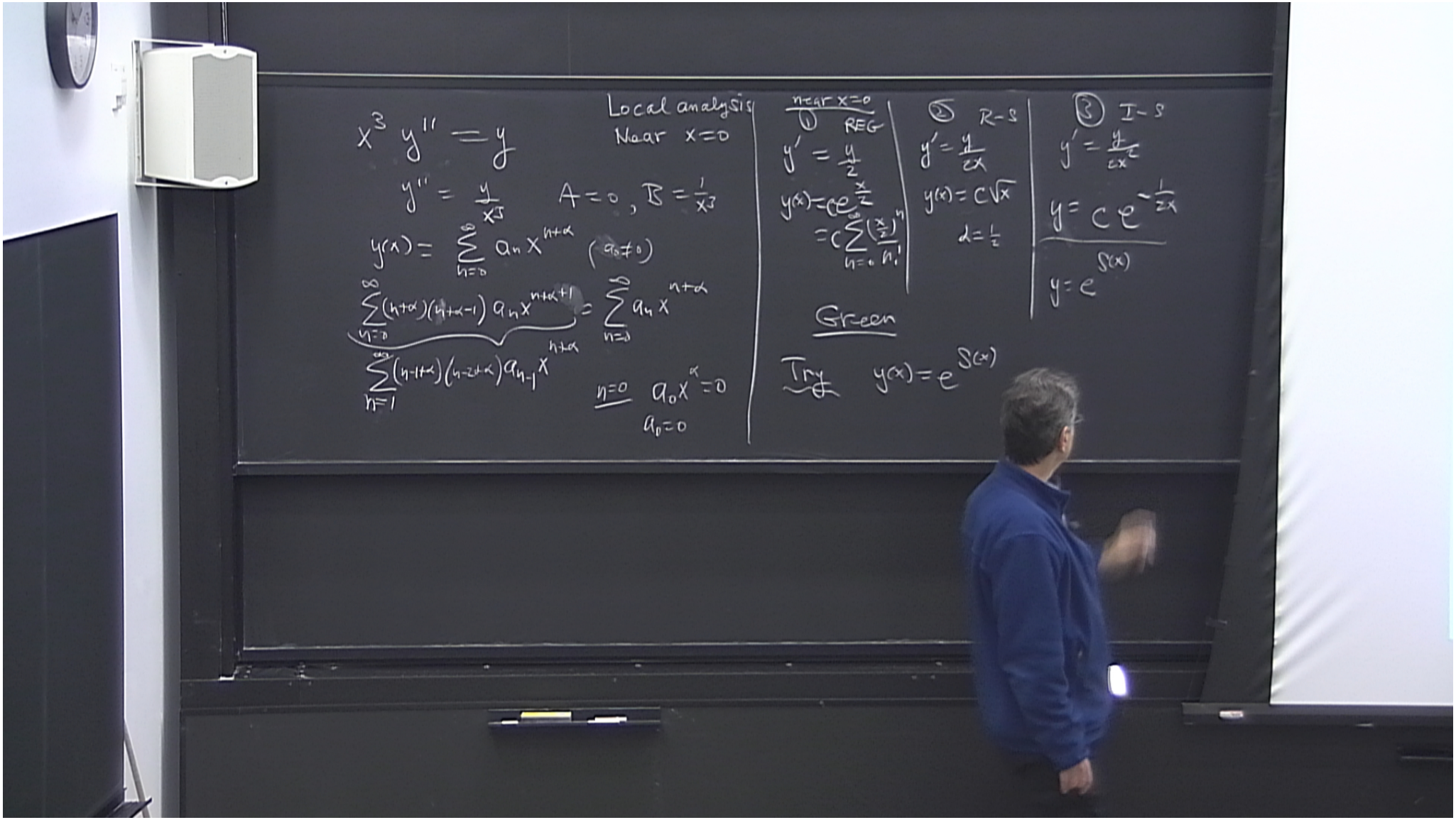

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$x \to \infty$
P_N
poles

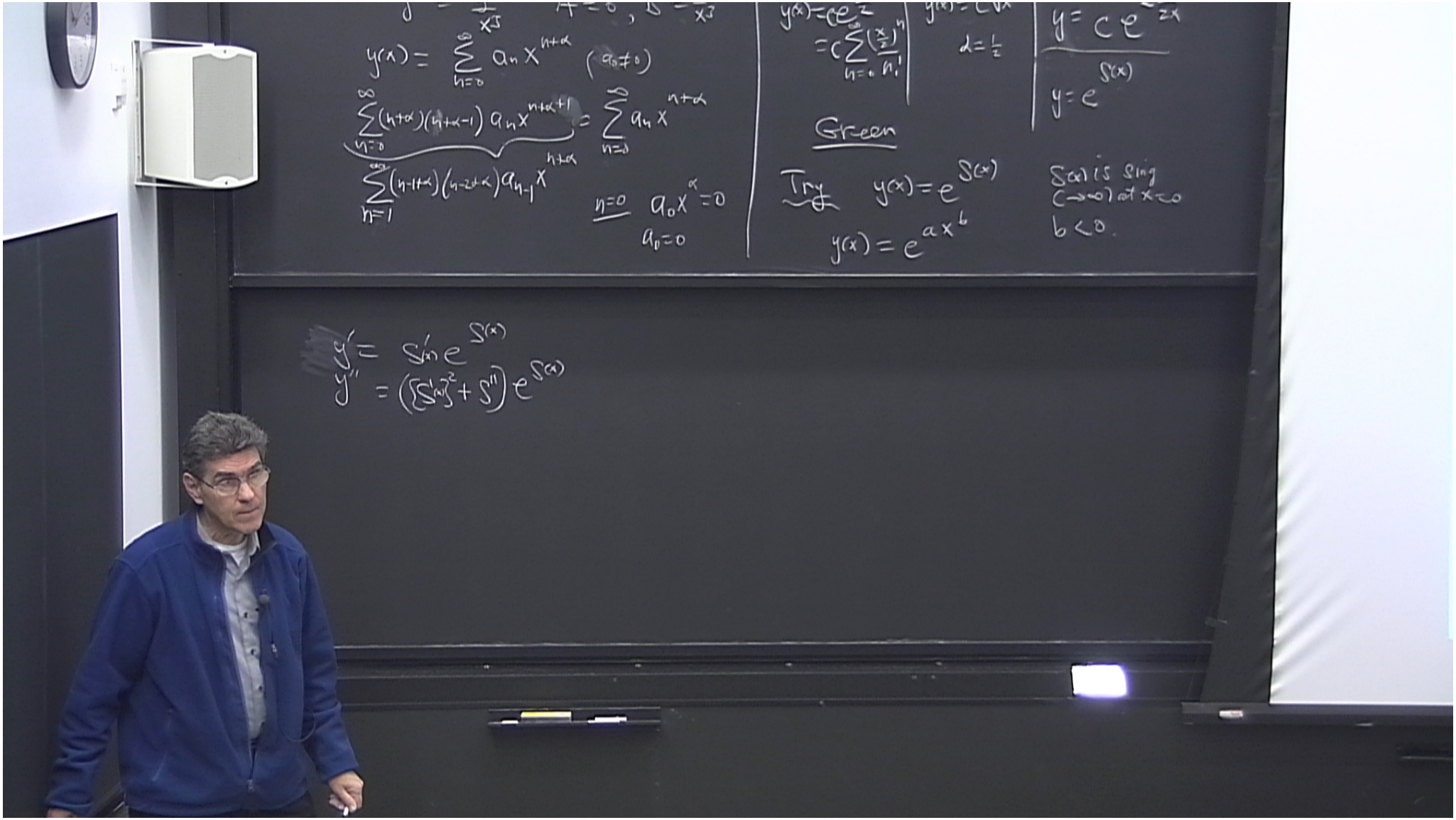

nearest singularity of A or B
x_0
$y'' + A(x)y' + B(x)y = 0$
near $x = x_0$
(1) $A(x)$, $B(x)$ are anal near $x = x_0$ (x_0 is a REGULAR pt.)
Thm Fuchs: All solns have a T.S. about $x = x_0$
$y(x) = \sum_{n=0}^{\infty} (x - x_0)^n a_n$
(2) Thm — Frobenius — (not both A & B are anal)
But $(x - x_0)A(x)$ & $(x - x_0)^2 B$ are anal.
x_0 is a REG-SINGULAR PT.
1 has form $y(x) = \sum_{n=0}^{\infty} (x - x_0)^n a_n$ ($a_0 \neq 0$)
others $y(x) =$ ---
$\Gamma(x) \sim$
Stirling
poles

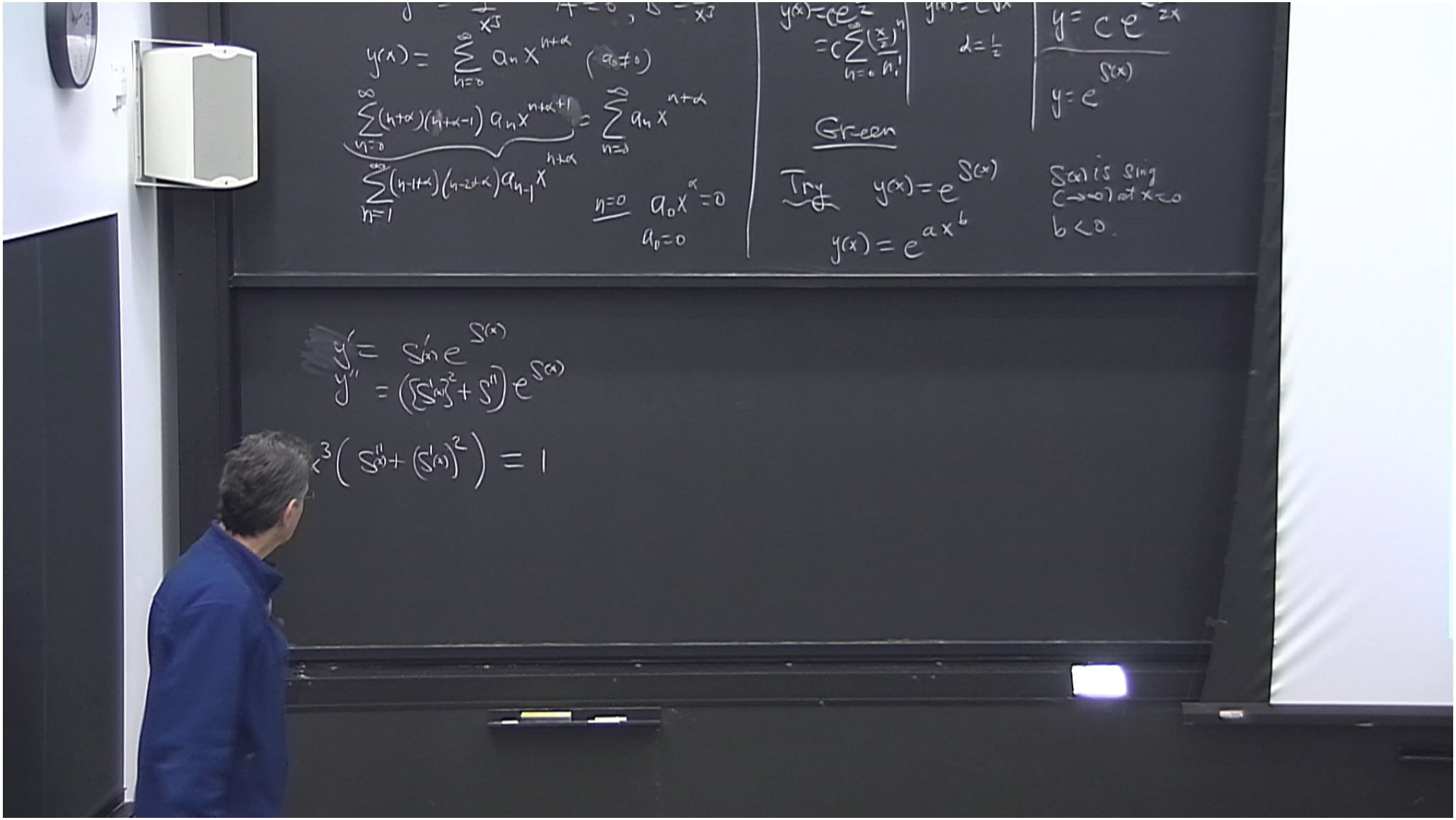

Local analysis
Near x=0
near x=0
y'' = y/x³
A=0, B=1
y(x) = Σ a_n x^{n+α} (a_0 ≠ 0)
y' = y/2
y' = y/cx
y' = y/cx²

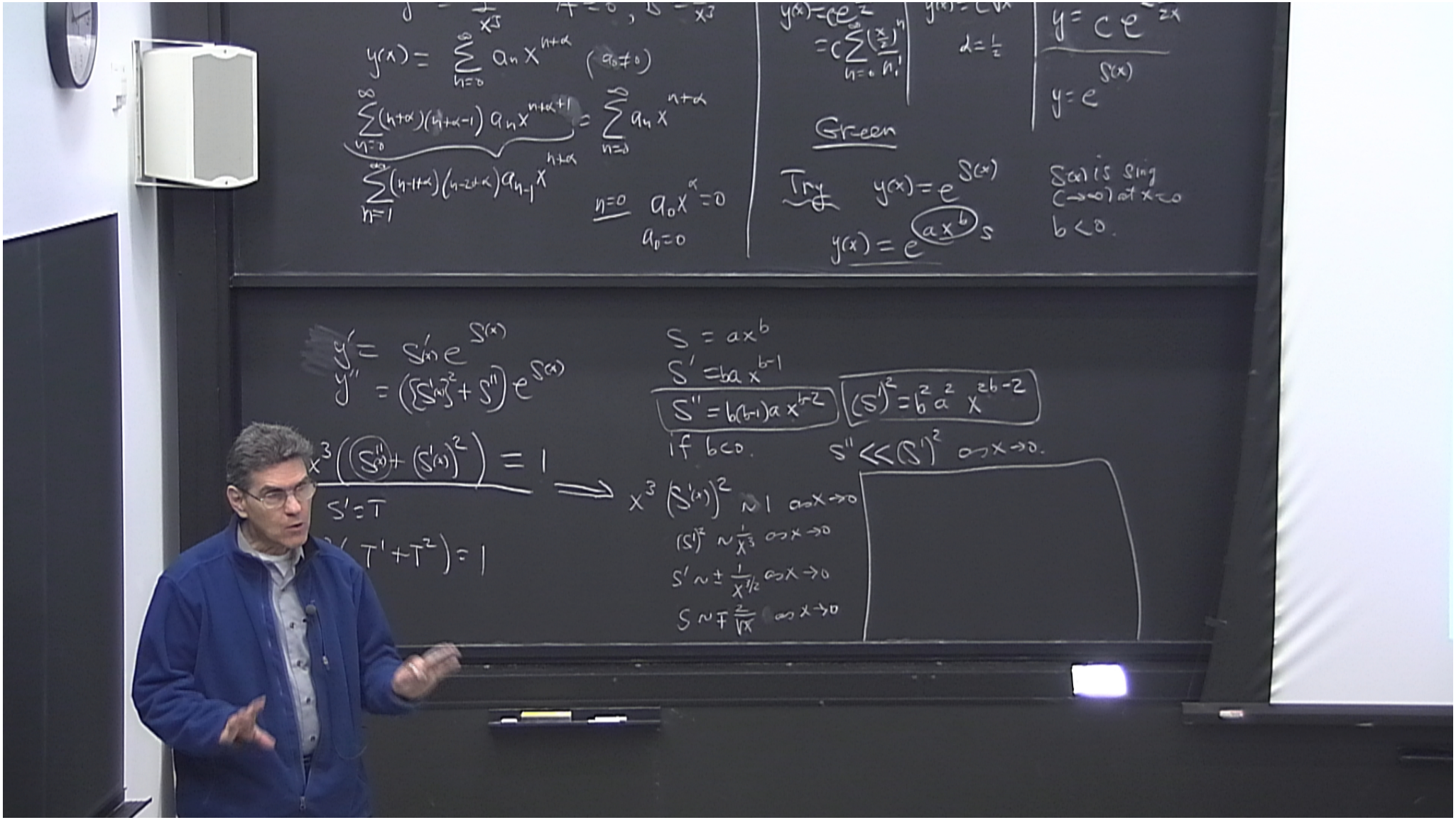

Local analysis
Near x=0
near x=0
① REG
② R-S
③ I-S
A=0, B=1/x³
(a₀≠0)
a₀=0

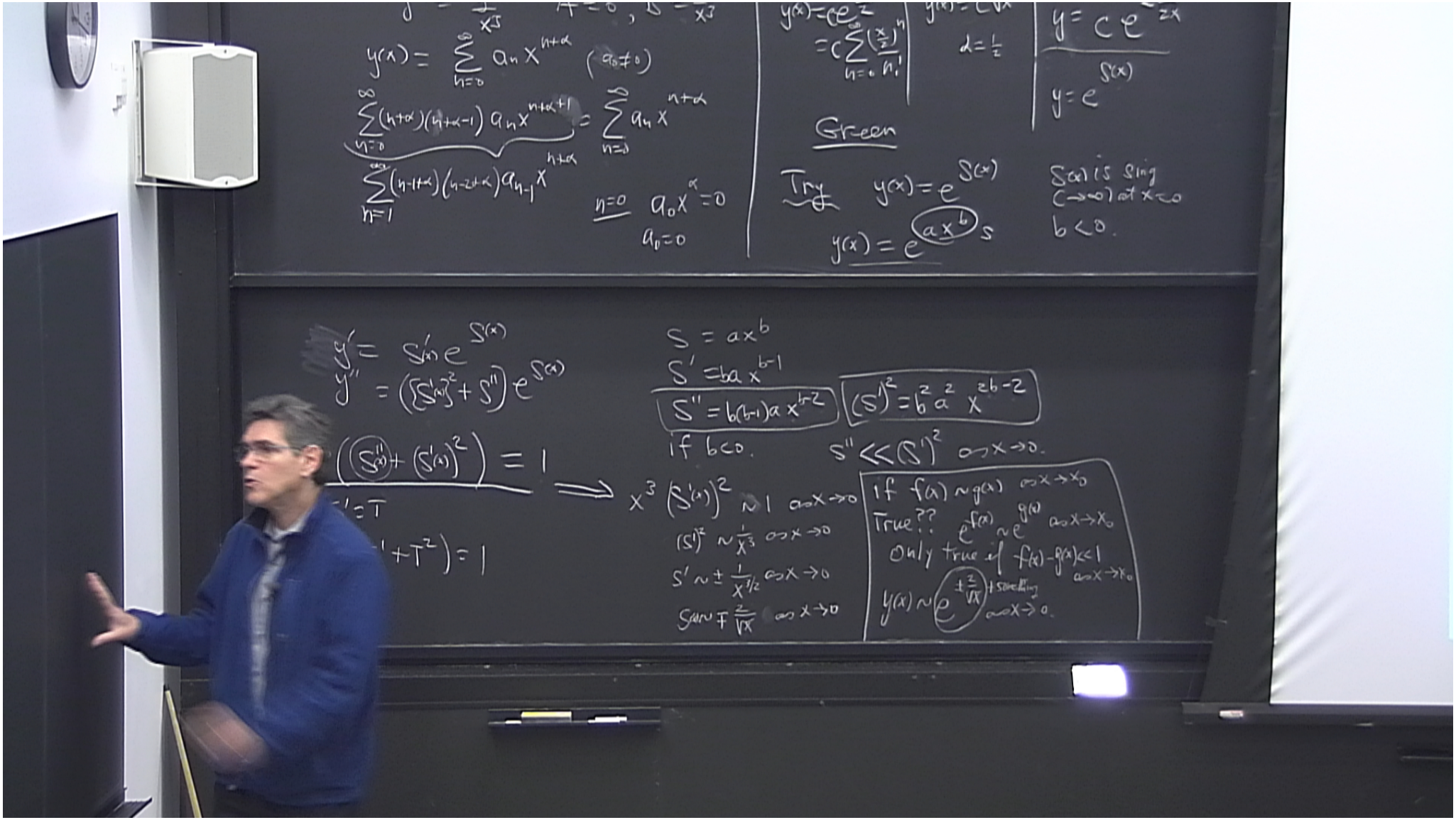

Local analysis
Near x=0
near x=0
① REG
② R-S
③ I-S

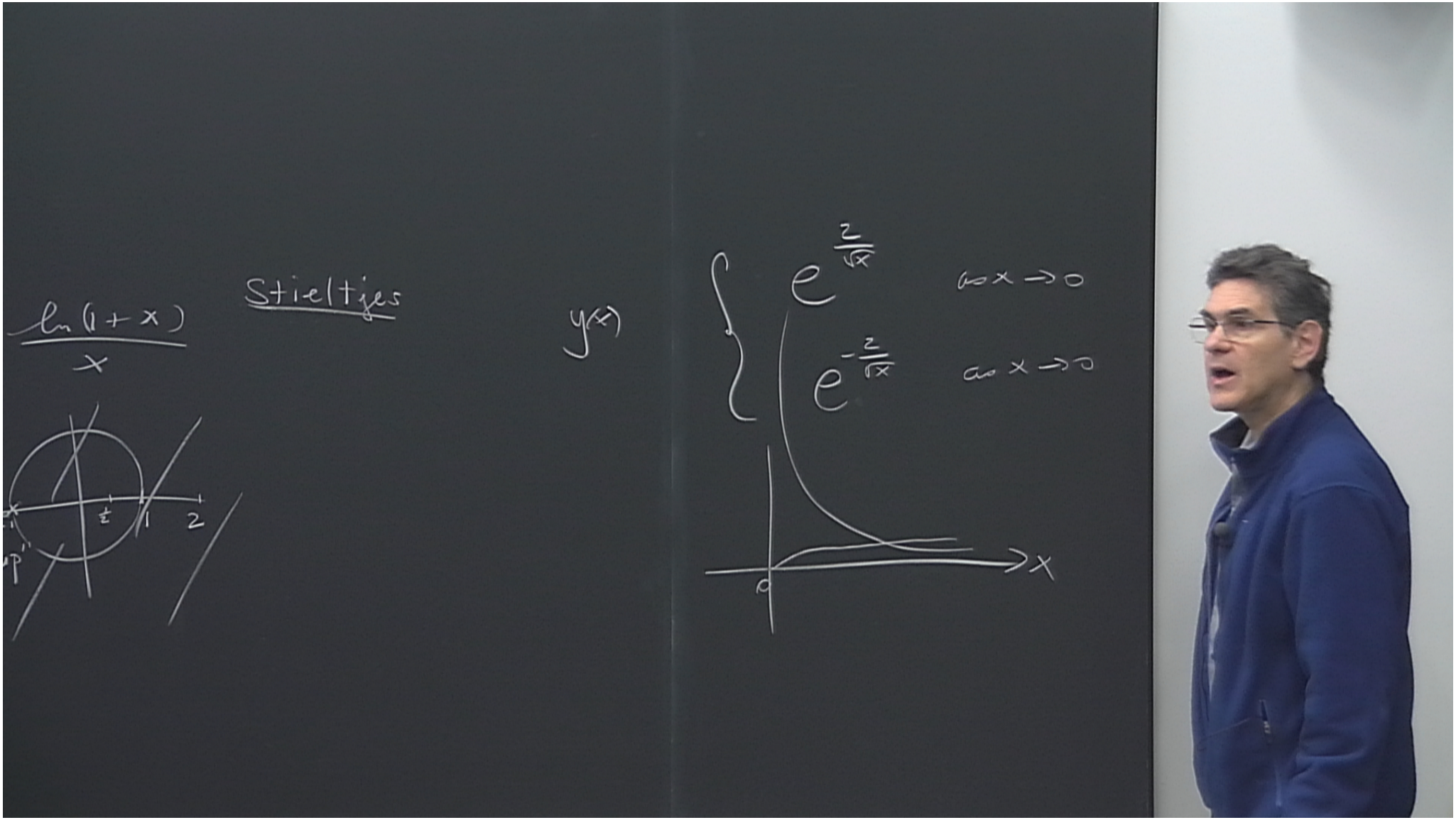

Local analysis
Near x=0
near x=0
① REG
② R-S
③ I-S
Green
Try

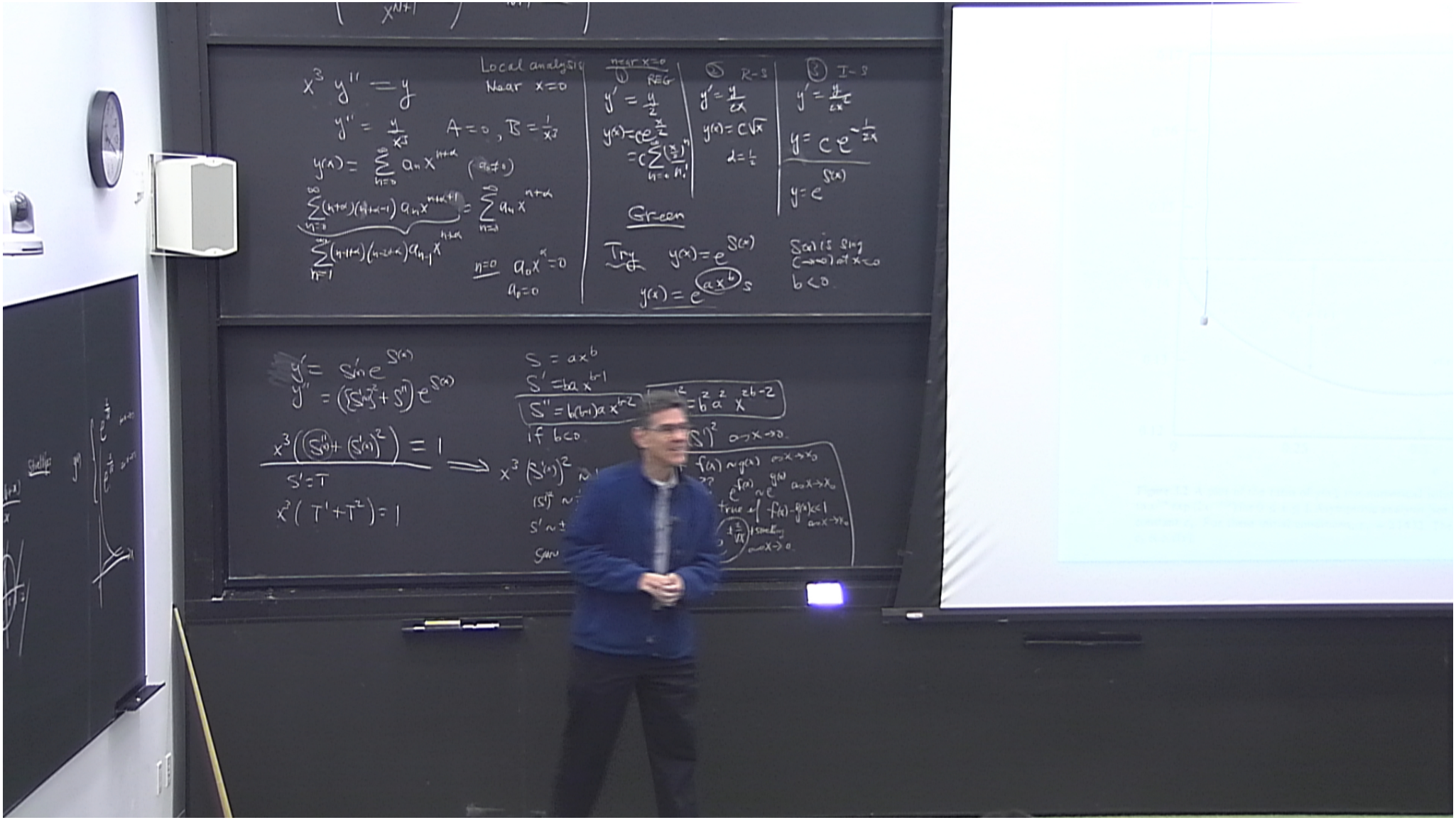

$y(x) = \sum_{n=0}^{\infty} a_n x^{n+\alpha} \quad (a_0 \neq 0)$
$\sum_{n=0}^{\infty} (n+\alpha)(n+\alpha-1)\, a_n x^{n+\alpha-1} = \sum_{n=0}^{\infty} a_n x^{n+\alpha}$
$\sum_{n=1}^{\infty} (n-1+\alpha)(n-2+\alpha)\, a_{n-1} x^{n+\alpha-1}$
$n=0 \quad a_0 x^{\alpha} = 0$
$a_0 = 0$
Green
Try $y(x) = e^{S(x)}$
$y(x) = e^{ax^b}$
$S(x)$ is sing ($\to \infty$) at $x=0$
$b < 0$.
$y = ce^{2x}$
$y = e^{S(x)}$
$y' = S' e^{S(x)}$
$y'' = (S'^2 + S'') e^{S(x)}$

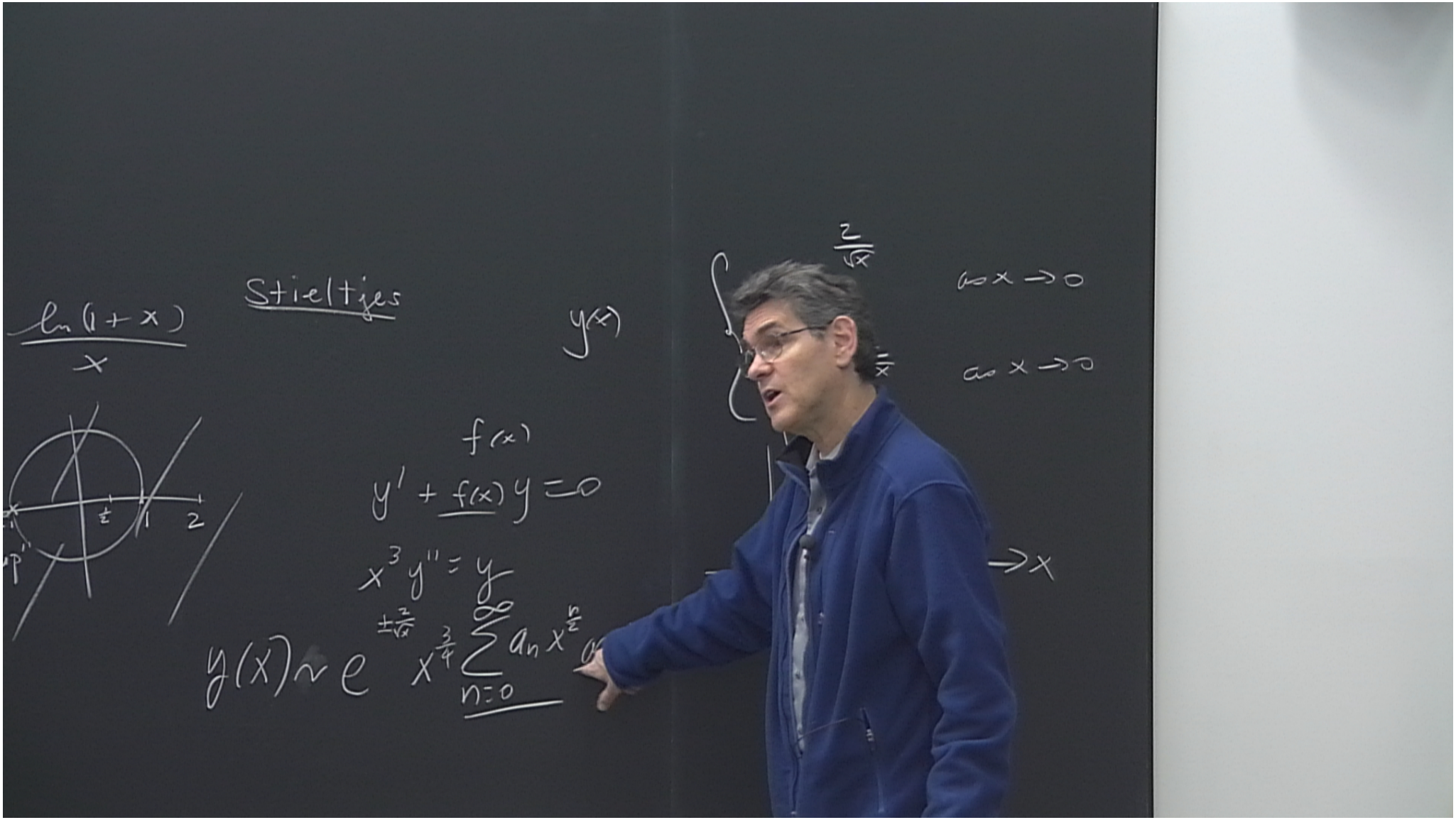

$y(x) = \sum_{n=0}^{\infty} a_n x^{n+\alpha} \quad (a_0 \neq 0)$
$\sum_{n=0}^{\infty} (n+\alpha)(n+\alpha-1) a_n x^{n+\alpha-1} = \sum_{n=0}^{\infty} a_n x^{n+\alpha}$
$\sum_{n=1}^{\infty} (n-1+\alpha)(n-2+\alpha) a_{n-1} x^{n+\alpha-1}$
$n=0 \quad a_0 x^{\alpha-1} = 0$
$a_0 = 0$
Green
Try $y(x) = e^{S(x)}$
$y(x) = e^{ax^b}$
$S(x)$ is sing ($\to \infty$) at $x=0$
$b < 0$.
$y = ce^{2x}$
$y = e^{S(x)}$
$y' = S' e^{S(x)}$
$y'' = (S'^2 + S'') e^{S(x)}$
$x^3 (S'' + (S')^2) = 1$

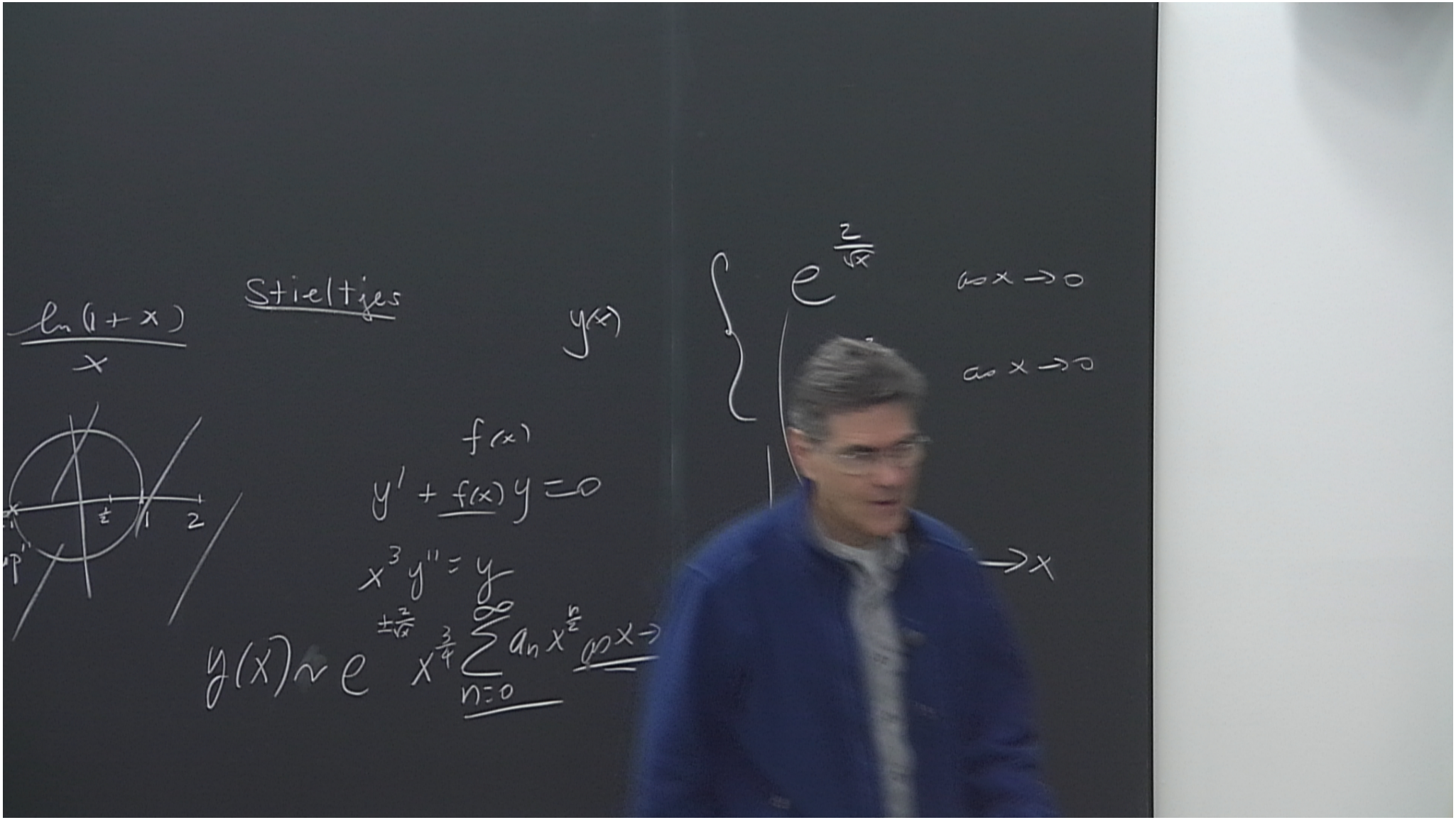

$y(x) = \sum_{n=0}^{\infty} a_n x^{n+\alpha}$ $(a_0 \neq 0)$
$\sum_{n=0}^{\infty}(n+\alpha)(n+\alpha-1)\,a_n x^{n+\alpha-1} = \sum_{n=0}^{\infty} a_n x^{n+\alpha}$
$\sum_{n=1}^{\infty}(n-1+\alpha)(n-2+\alpha)\,a_{n-1} x^{n+\alpha-1}$
$n=0$ $a_0 x^{\alpha-1} = 0$
$a_0 = 0$
$y(x) = ce^{\frac{x}{2}} = c\sum_{n=0}^{\infty}\frac{(\frac{x}{2})^n}{n!}$
$\alpha = \frac{1}{2}$
$y = ce^{2x}$
$y = e^{S(x)}$
Green
Try $y(x) = e^{S(x)}$
$y(x) = e^{ax^b}$ S
S(x) is sing $(\to\infty)$ at $x=0$
$b<0$.
$y' = S'(x)e^{S(x)}$
$y'' = (S'^2 + S'')e^{S(x)}$
$x^3\left(S'' + (S')^2\right) = 1$
$S' = T$
$T' + T^2 = 1$
$S = ax^b$
$S' = bax^{b-1}$
$S'' = b(b-1)ax^{b-2}$
if $b<0$.
$(S')^2 = b^2a^2x^{2b-2}$
$S'' \ll (S')^2$ as $x\to 0$.
$x^3(S')^2 \sim 1$ as $x\to 0$
$(S')^2 \sim \frac{1}{x^3}$ as $x\to 0$
$S' \sim \pm\frac{1}{x^{1/2}}$ as $x\to 0$
$S \sim \mp\frac{2}{\sqrt{x}}$ as $x\to 0$

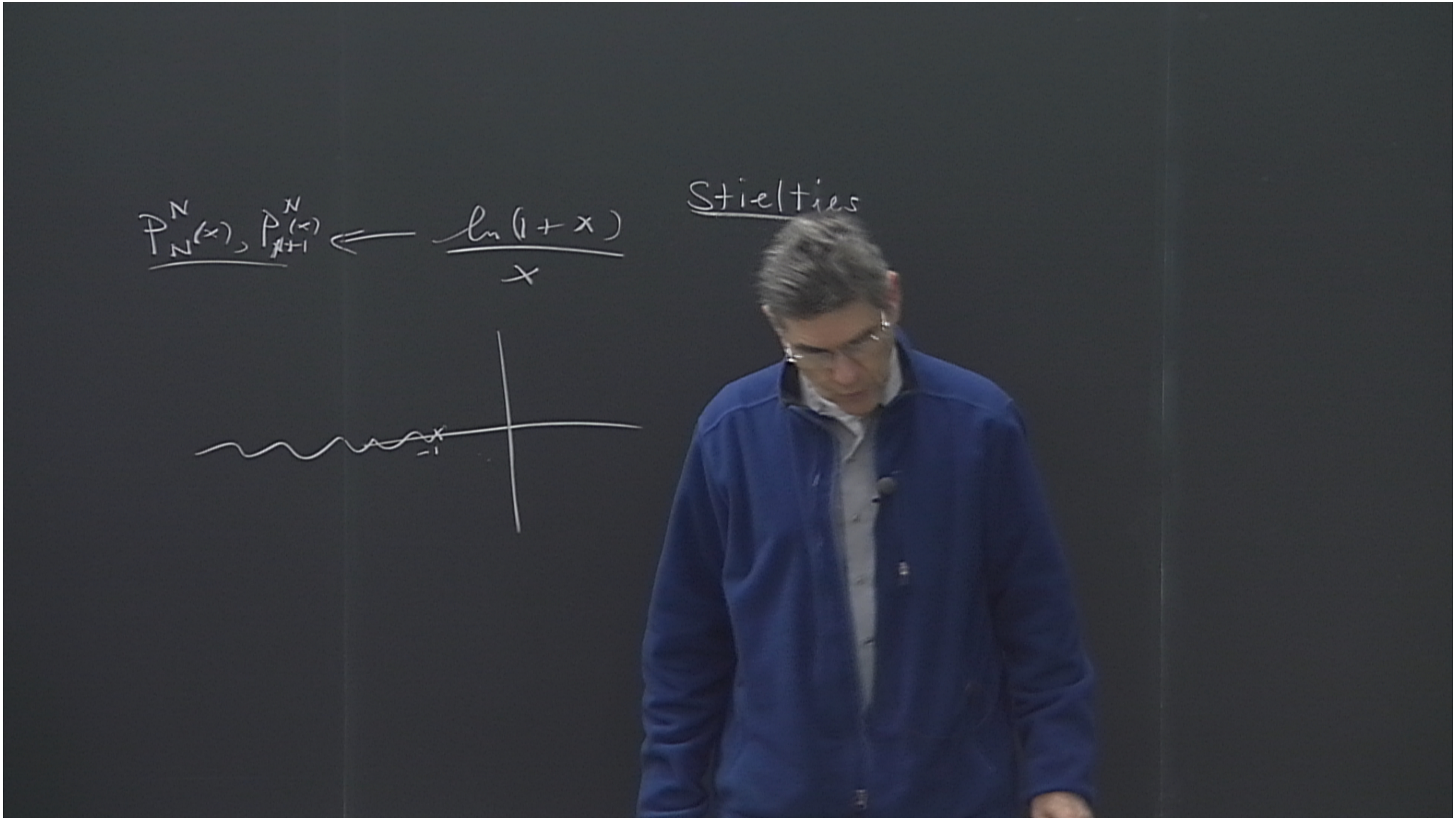

Green
Try
S(x) is sing (→∞) at x=0
b<0.
Sʹ = ba x^{b-1}
Sʹʹ = b(b-1) a x^{b-2}
(Sʹ)² = b² a² x^{2b-2}
if b<0.
Sʹʹ ≪ (Sʹ)² as x→0.
if f(x) ~ g(x) as x→x₀
True??
only true if f(x)-g(x)<<1 as x→x₀
+ something as x→0.


$\frac{\ln(1+x)}{x}$
Stieltjes
y(x)
$e^{\frac{2}{\sqrt{x}}}$
$x \to 0$
$e^{-\frac{2}{\sqrt{x}}}$
$x \to 0$
1
2
0
x


Local analysis
Near x=0
Green
Try
Shelling


Stieltjes
$\frac{\ln(1+x)}{x}$
$y(x)$
$f(x)$
$y' + f(x)\,y = 0$
$x^3 y'' = y$
$y(x) \sim e^{\pm \frac{2}{\sqrt{x}}} x^{\frac{3}{4}} \sum_{n=0}^{\infty} a_n x^{\frac{n}{2}}$
$\frac{2}{\sqrt{x}}$
$\frac{1}{x}$
$a_0 x \to 0$
$a_0 x \to 0$
$\to x$


$\frac{\ln(1+x)}{x}$
Stieltjes
$y(x)$
$e^{\frac{2}{\sqrt{x}}}$
as $x \to 0$
as $x \to 0$
$f(x)$
$y' + f(x)\,y = 0$
$x^3 y'' = y$
$\pm\frac{2}{\sqrt{x}}$
$y(x) \sim e \; x^{\frac{3}{4}} \sum_{n=0}^{\infty} a_n x^{\frac{n}{2}}$
as $x \to$
$\to x$


$P_N^N(x), P_{N+1}^N(x) \Leftarrow \frac{\ln(1+x)}{x}$
Stieltjes
-1