Title: Mathematical Physics - Lecture 6

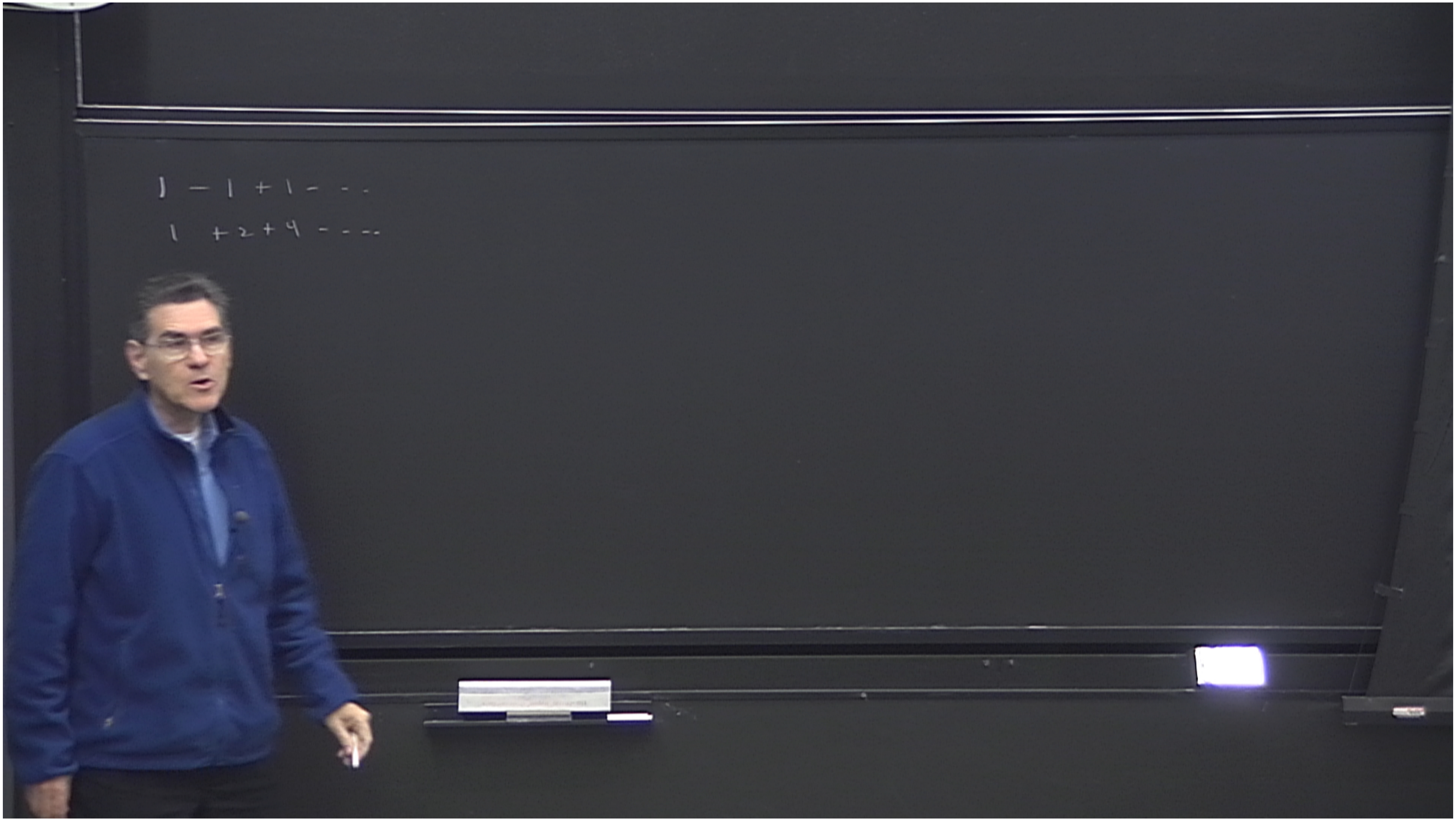
Date: Nov 28, 2011 09:00 AM

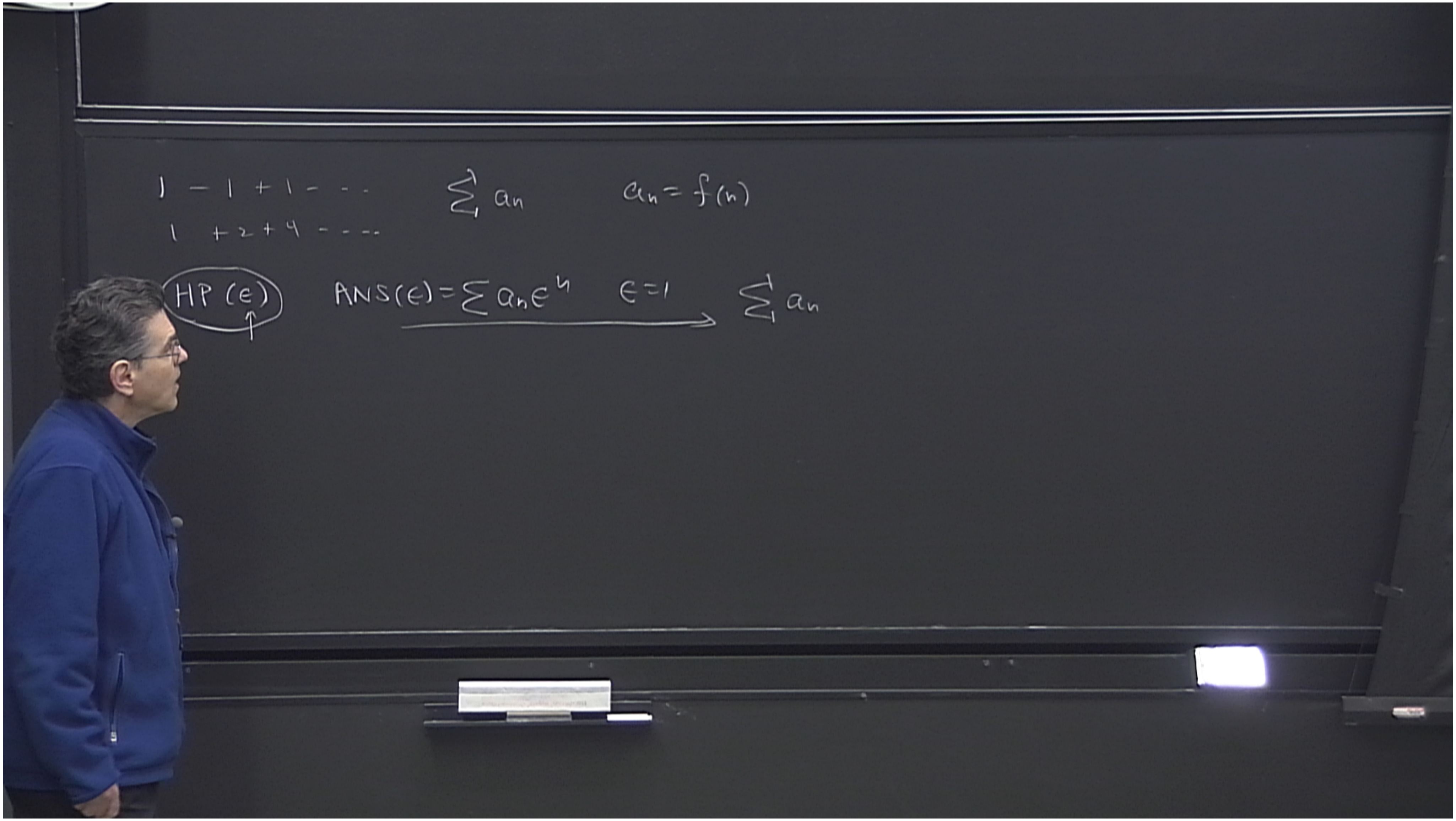
URL: http://pirsa.org/11110047

Abstract:

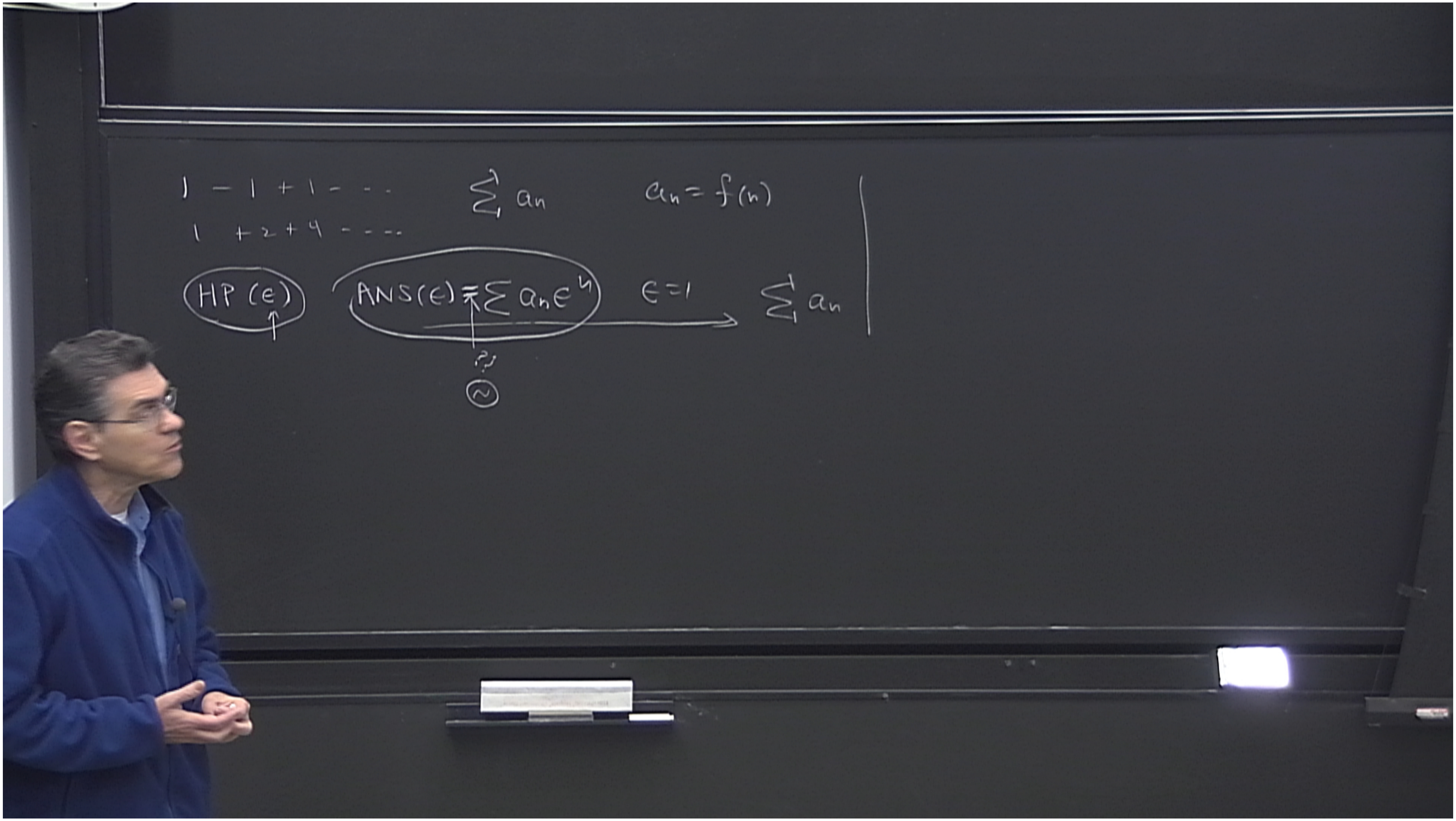
What if we don't know all the terms in the series??

WE USE CONTINUED FUNCTIONS!


1 − 1 + 1 − ...
1 + 2 + 4 − ...



$1 - 1 + 1 - \dots$
$1 + 2 + 4 - \dots$
$\sum_1^\infty a_n$
$a_n = f(n)$
$HP(\epsilon)$
$ANS(\epsilon) = \sum a_n \epsilon^n$
$\epsilon = 1$
$\sum_1^\infty a_n$


1 − 1 + 1 − - -
1 + 2 + 4 - - - -
Σ a_n
a_n = f(n)
HP (ε)
ANS(ε) = Σ a_n ε^n
ε = 1
Σ a_n

Continued exponentials

$$a_0 e^{a_1 z e^{a_2 z \cdots}}$$

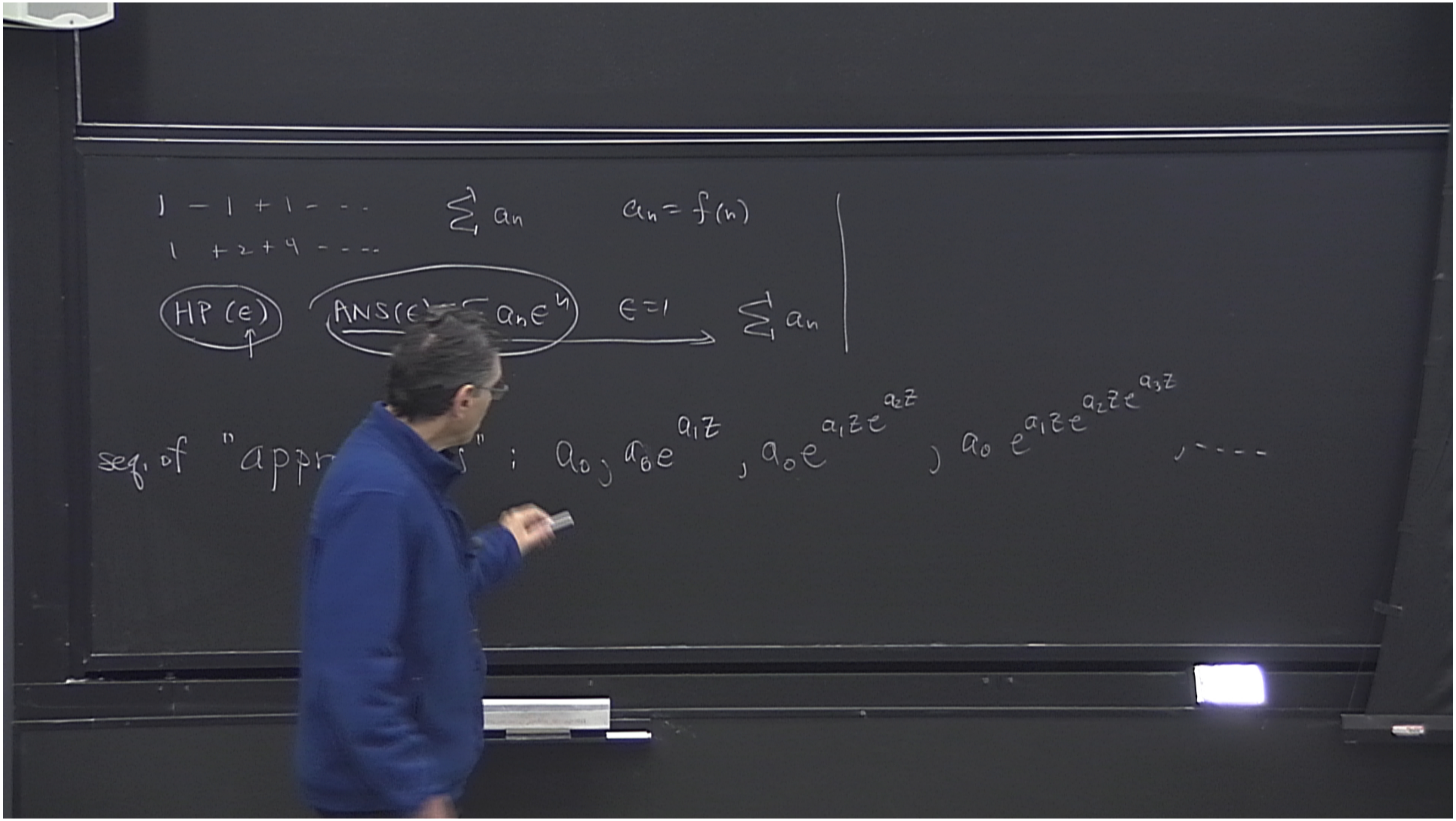
$$\sum_{n=0}^{\infty} c_n z^n$$

$$c_0 = a_0,$$

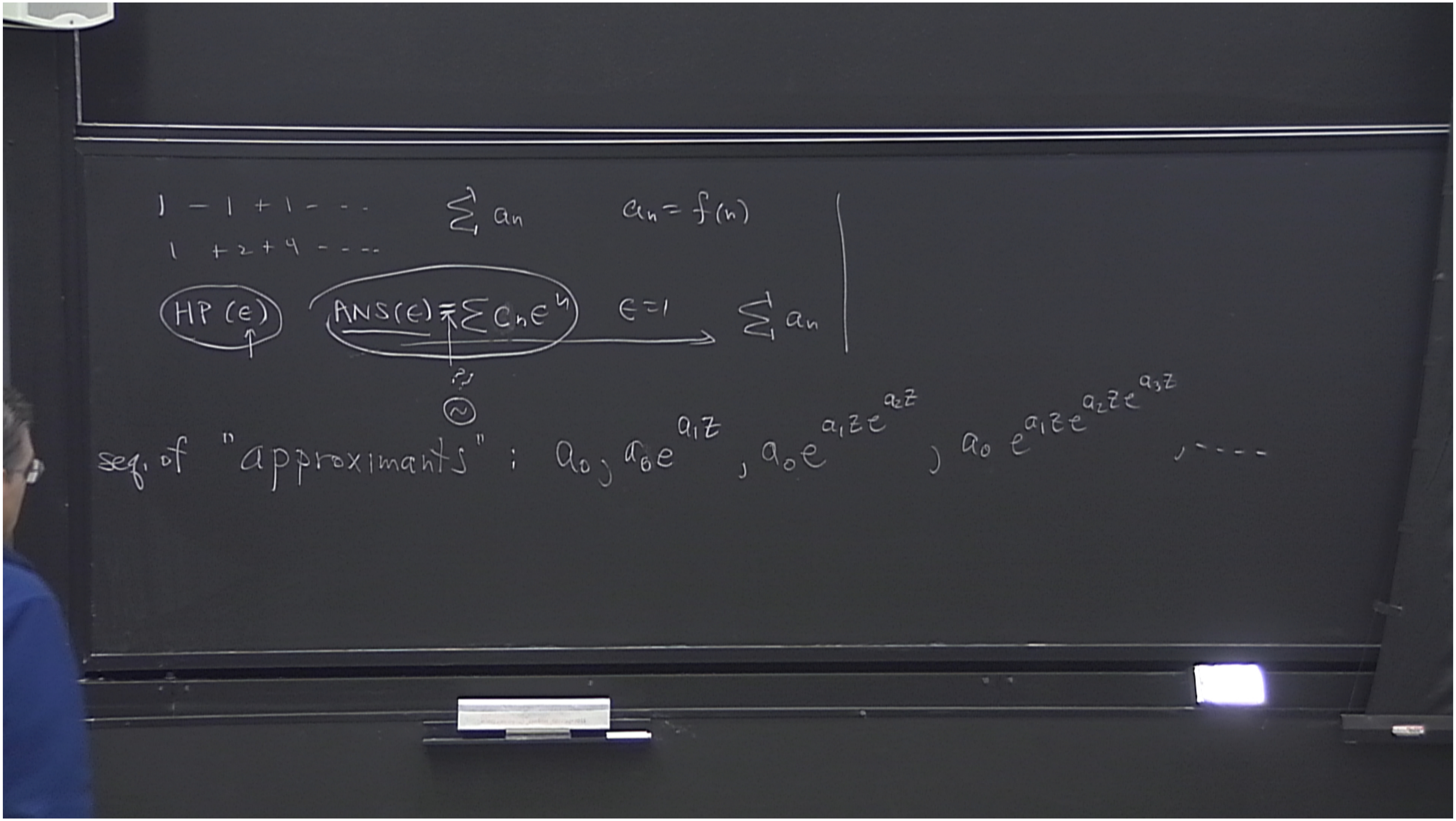
$$c_1 = a_1 a_0,$$

$$c_2 = a_0 a_1 a_2 + \frac{1}{2} a_0 a_1^2,$$

$$c_3 = a_0 a_1 a_2 a_3 + \frac{1}{2} a_0 a_1 a_2^2 + a_0 a_1^2 a_2 + \frac{1}{6} a_0 a_1^3$$



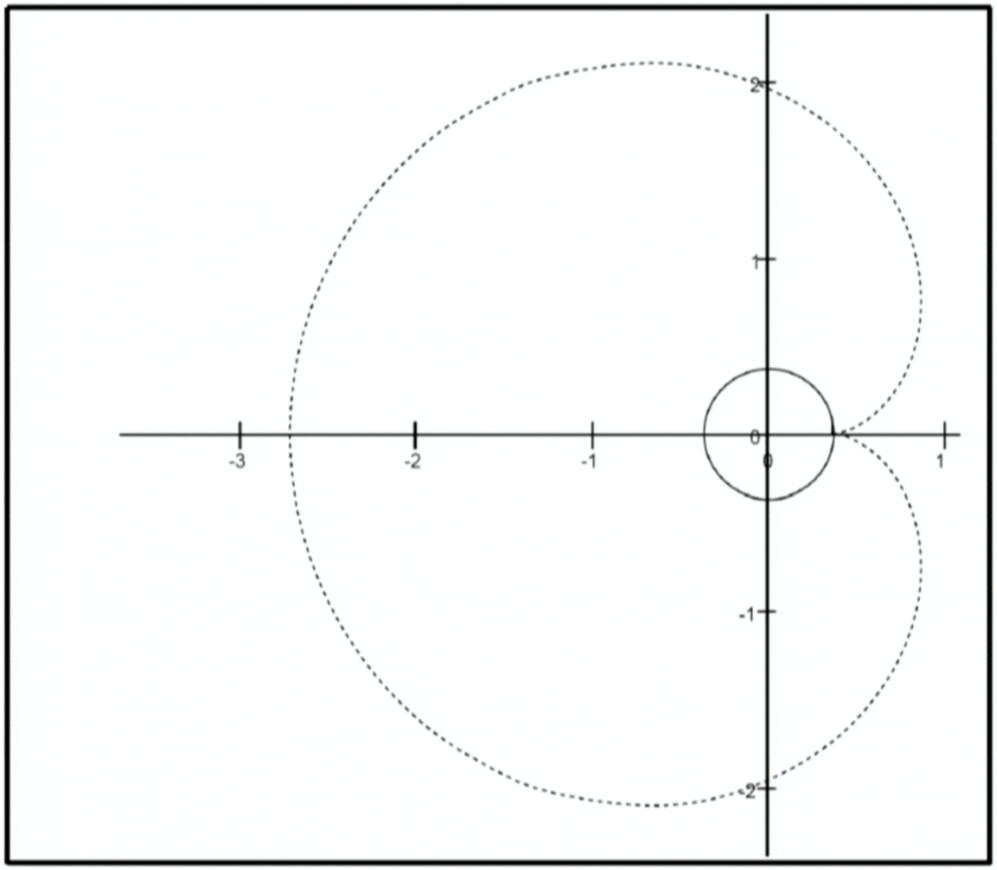
$1 - 1 + 1 - \cdots$
$\sum_1^\infty a_n$
$a_n = f(n)$
$1 + 2 + 4 - \cdots$
$HP(\epsilon)$
$ANS(\epsilon) = \sum a_n \epsilon^n$
$\epsilon = 1$
$\sum_1^\infty a_n$
seq. of "appr s" : $a_0, a_0 e^{a_1 z}, a_0 e^{a_1 z e^{a_2 z}}, a_0 e^{a_1 z e^{a_2 z e^{a_3 z}}}, \ldots$


$1 - 1 + 1 - \cdots$
$1 + 2 + 4 - \cdots$
$\sum_1^\infty a_n$
$a_n = f(n)$
$HP(\epsilon)$
$ANS(\epsilon) = \sum c_n \epsilon^n$
$\epsilon = 1$
$\sum_1^\infty a_n$
?!
$\sim$
seq. of "approximants" : a_0, $a_0 e^{a_1 z}$, $a_0 e^{a_1 z e^{a_2 z}}$, $a_0 e^{a_1 z e^{a_2 z e^{a_3 z}}}$, ...

Example

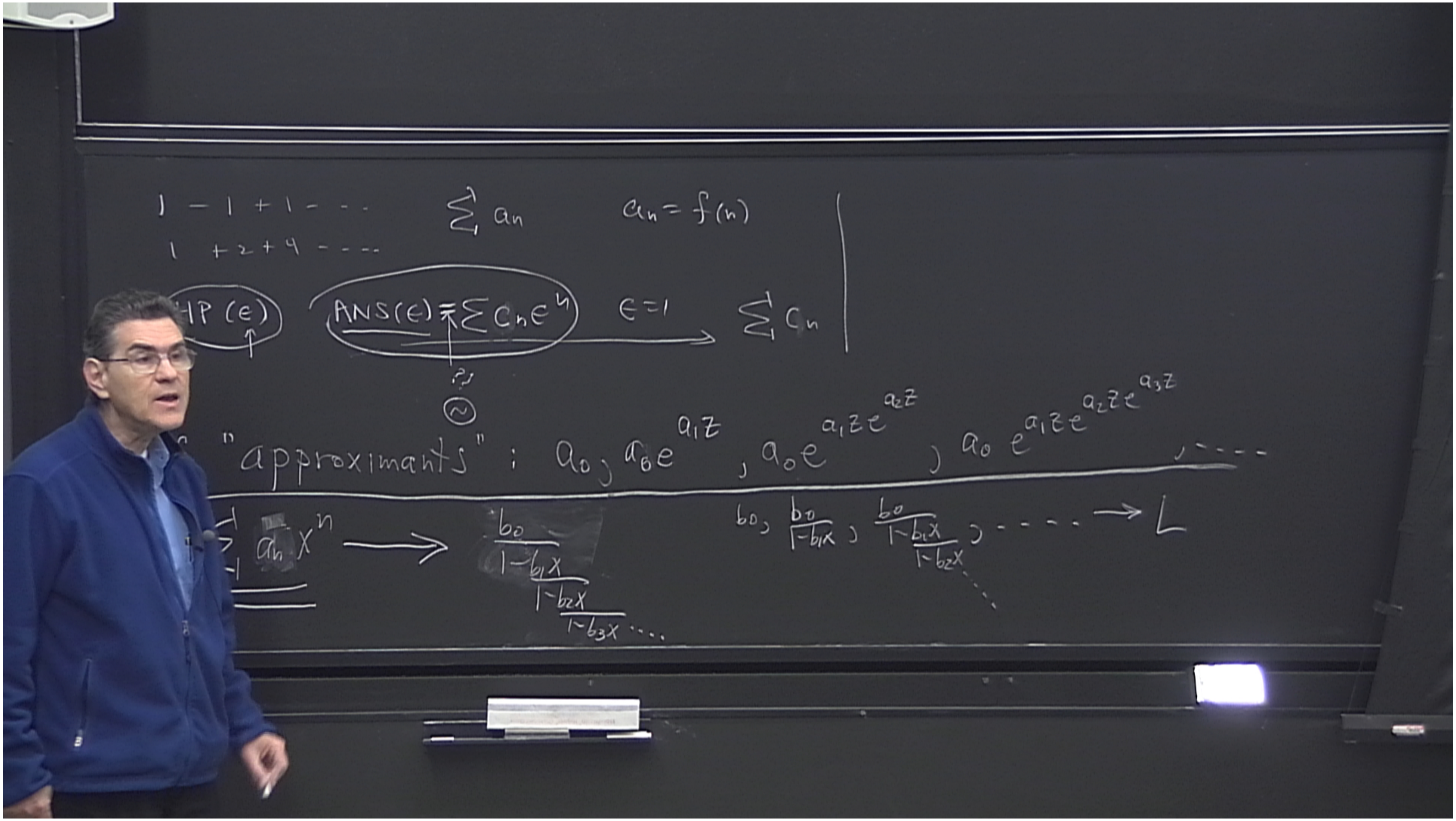
$$e^{ze^{ze^{ze^{z\cdots}}}} = \sum_{n=0}^{\infty} \frac{(n+1)^{n-1}}{n!} z^n$$

Region of convergence



Continued fractions ---
An IQ test

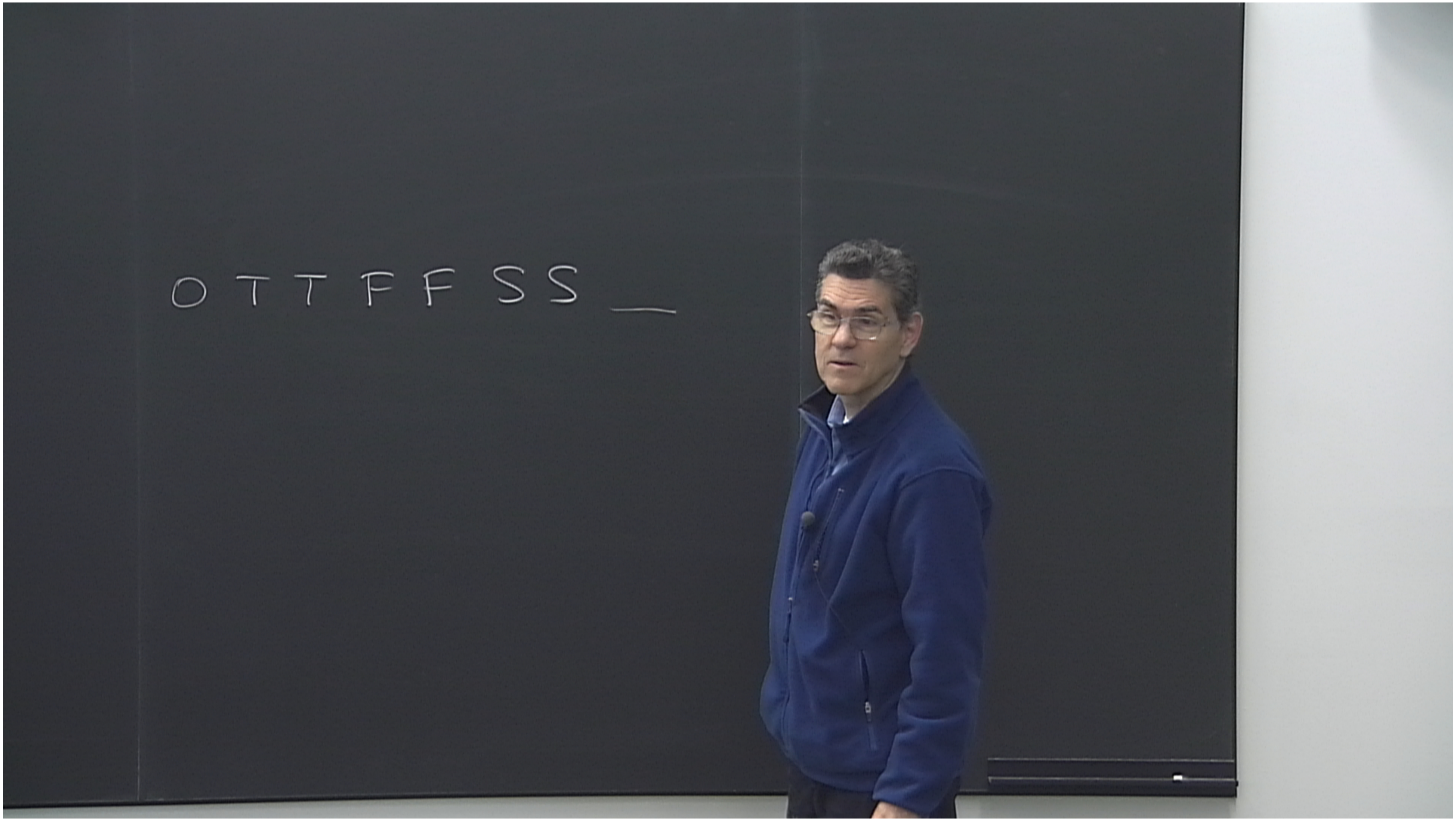
Continued fractions ---
An IQ test

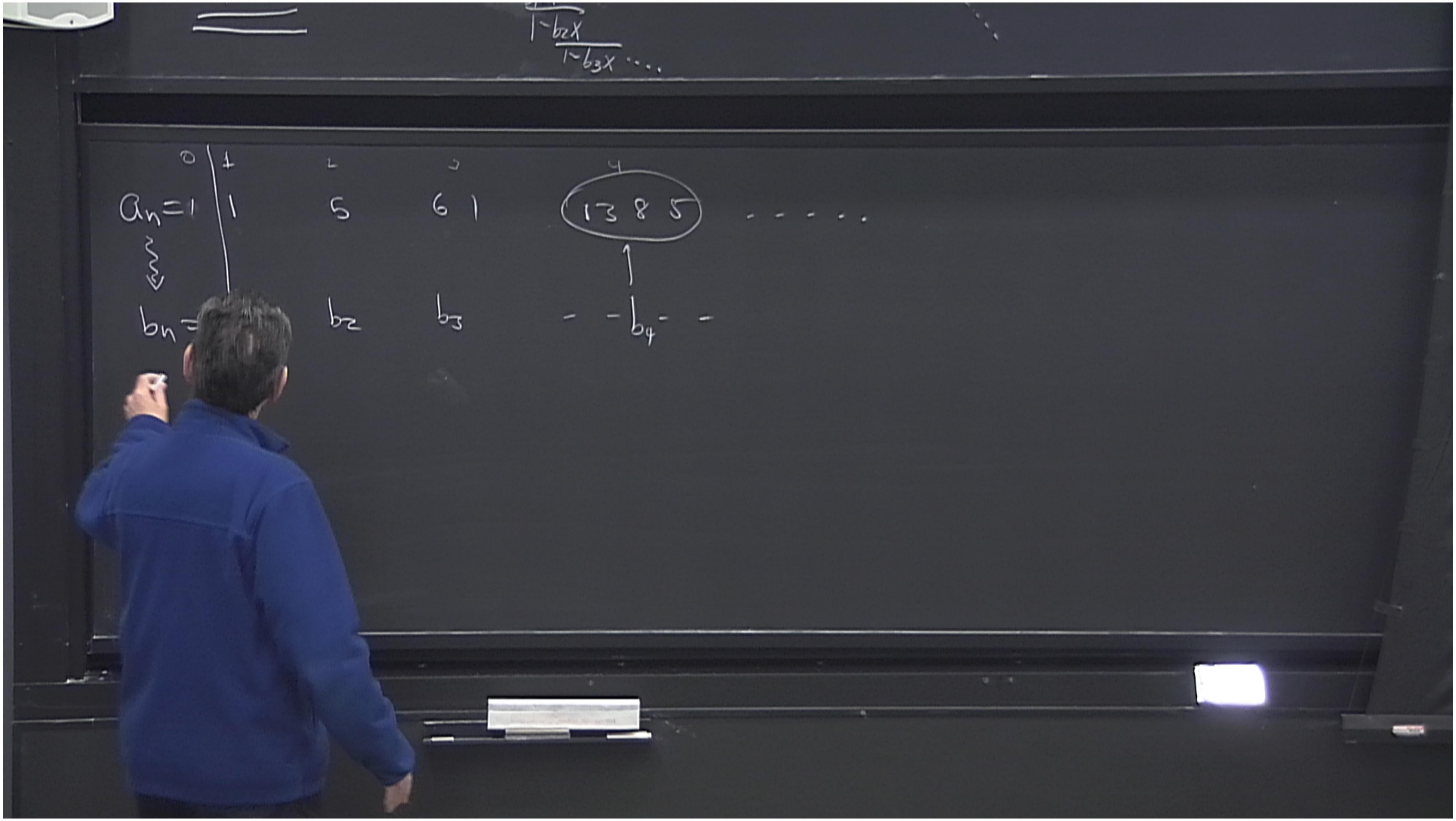

1 − 1 + 1 − − −
1 + 2 + 4 − − −
$\sum_1^\infty a_n$
$a_n = f(n)$
P (ε)
ANS(ε) ≈ $\sum c_n \epsilon^n$
ε = 1
$\sum_1^\infty c_n$
"approximants" : a_0, $a_0 e^{a_1 z}$, $a_0 e^{a_1 z e^{a_2 z}}$, $a_0 e^{a_1 z e^{a_2 z e^{a_3 z}}}$, ...
$\sum_1^\infty a_n x^n$
b_0, $\frac{b_0}{1-b_1 x}$, $\frac{b_0}{1-\frac{b_1 x}{1-b_2 x}}$, ... → L

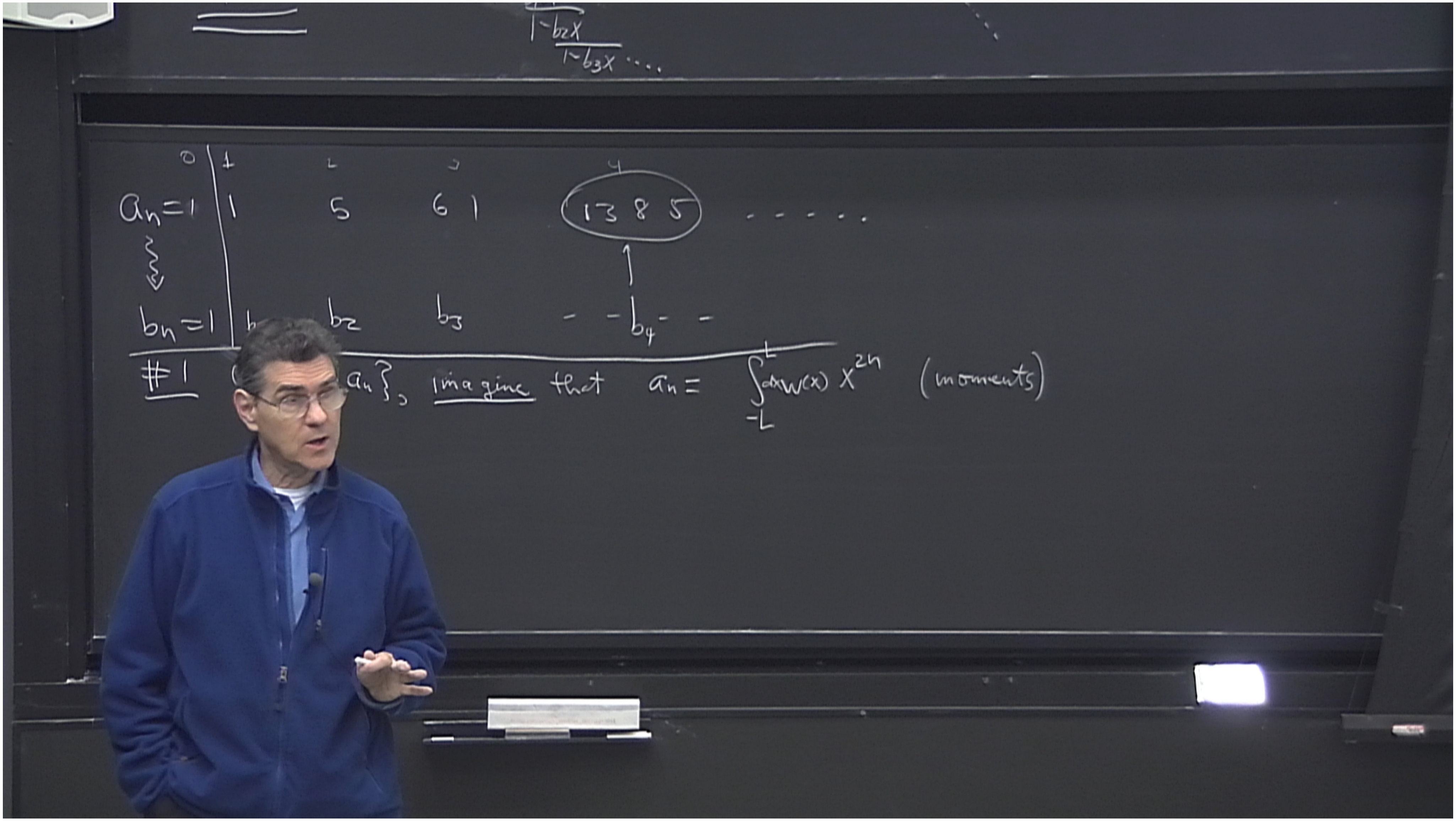
What's the next number in this sequence?

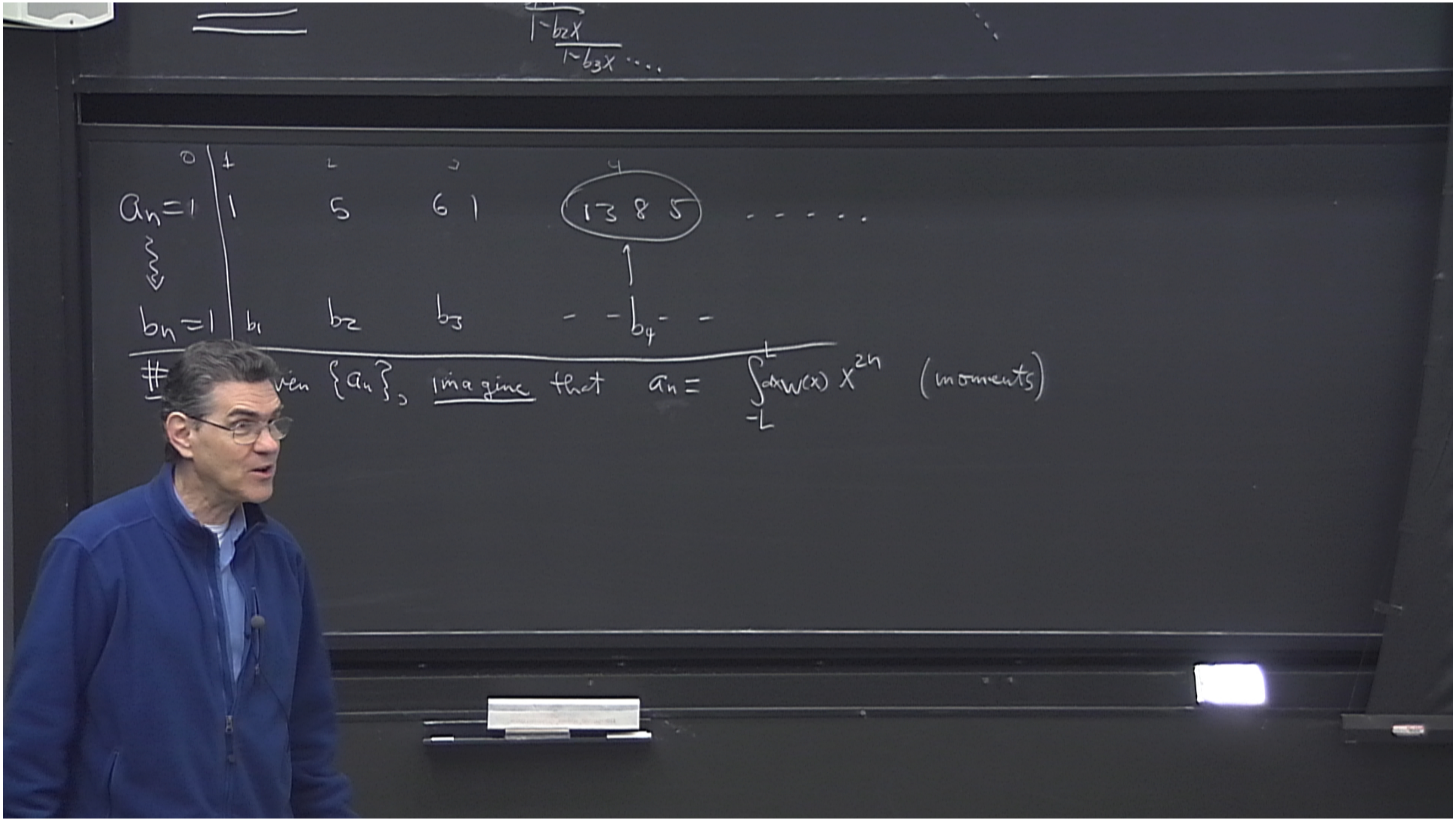
14, 34, 42, 72, 96, 110, 116, ??

1, 5, 61, ??

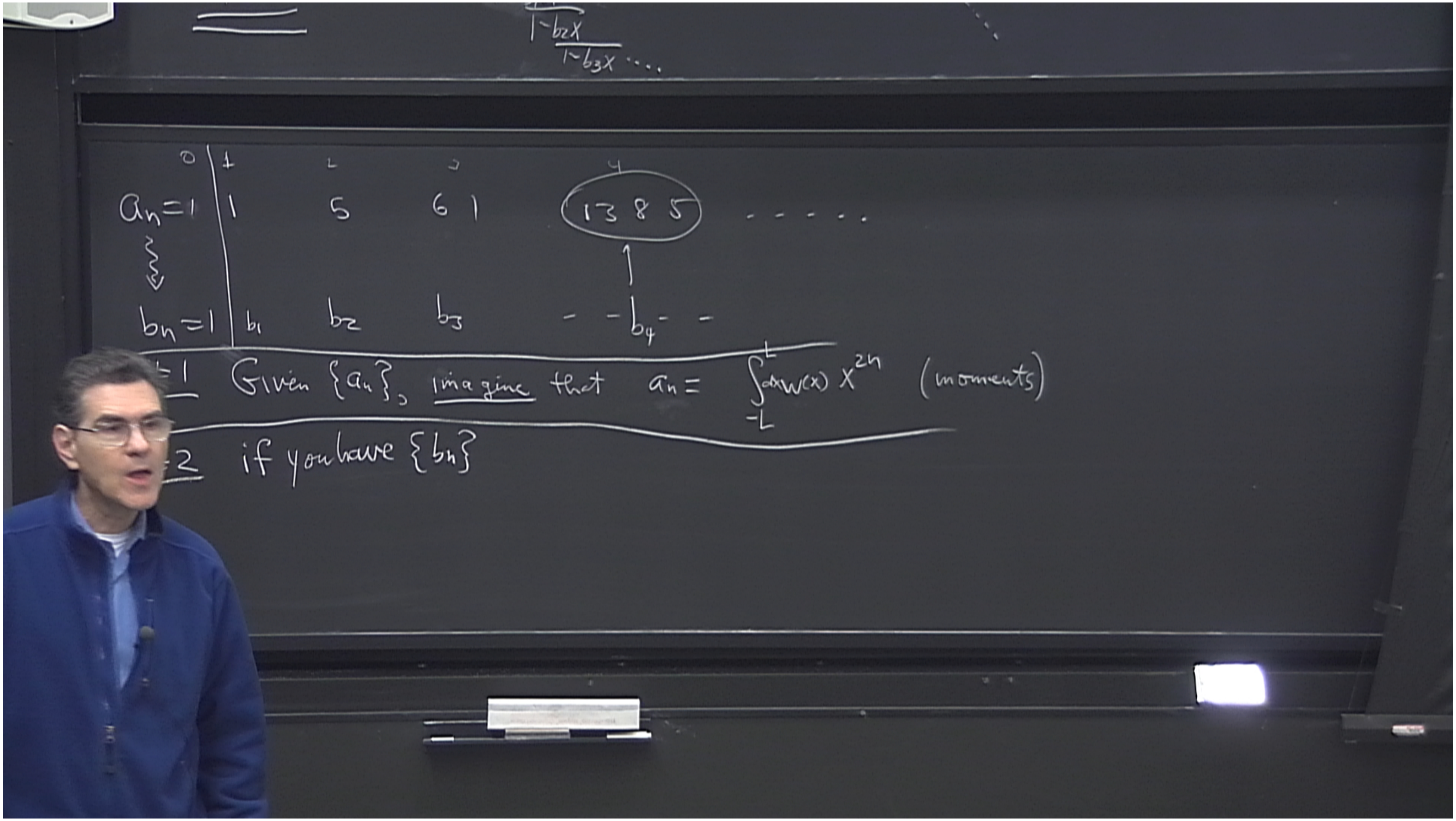

O T T F F S S _

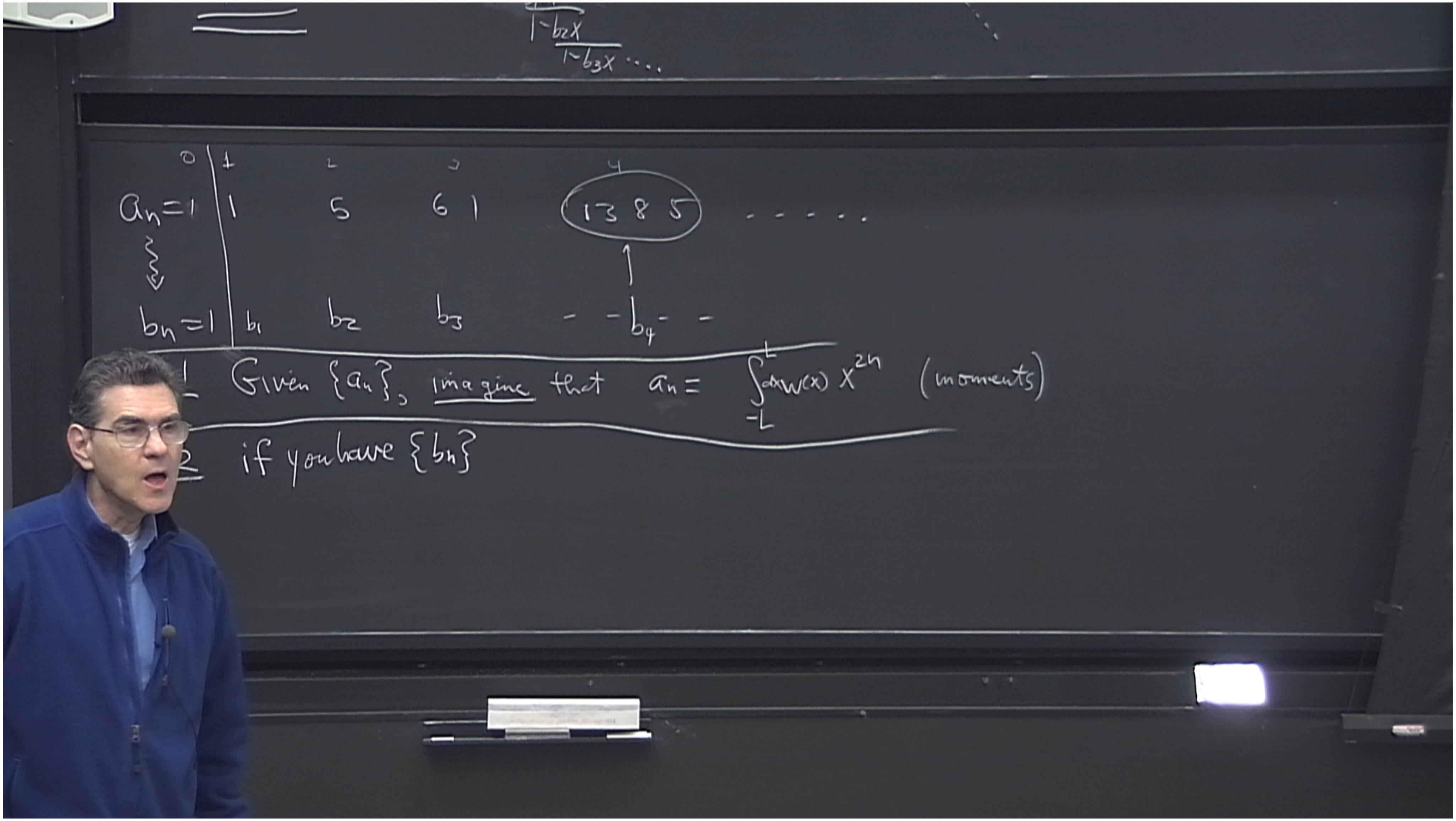

1-b2x
1-b3x ....
0 1 2 3 4
an=1 1 5 61 1385 .....
bn= b2 b3 -- b4 --


1-b2x
1-b3x ....
an=1 | 1 5 61 1385 .....
bn=1 | b2 b3 - - b4 - -
#1 an}, imagine that an = ∫dx w(x) x^2n (moments)

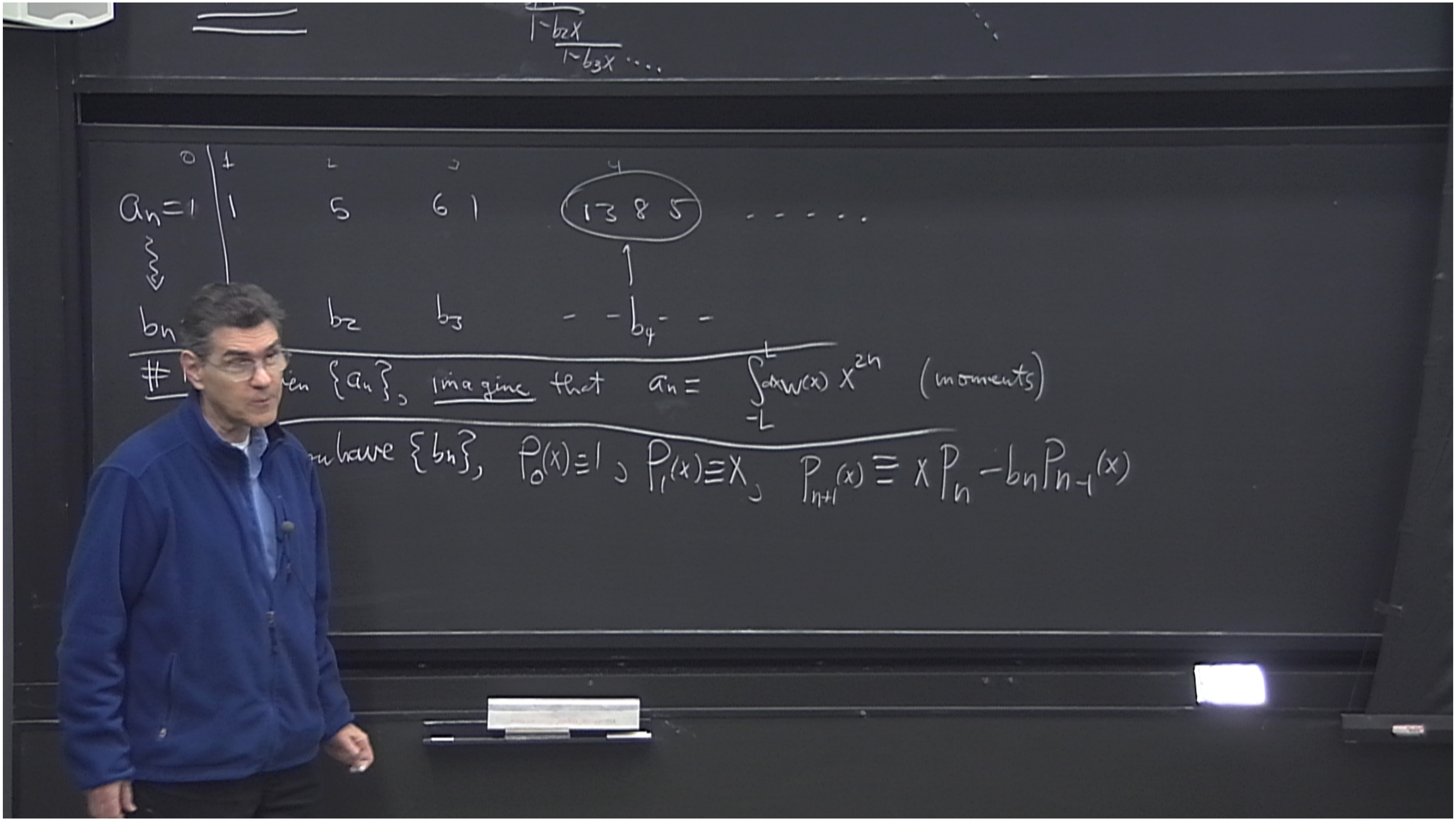


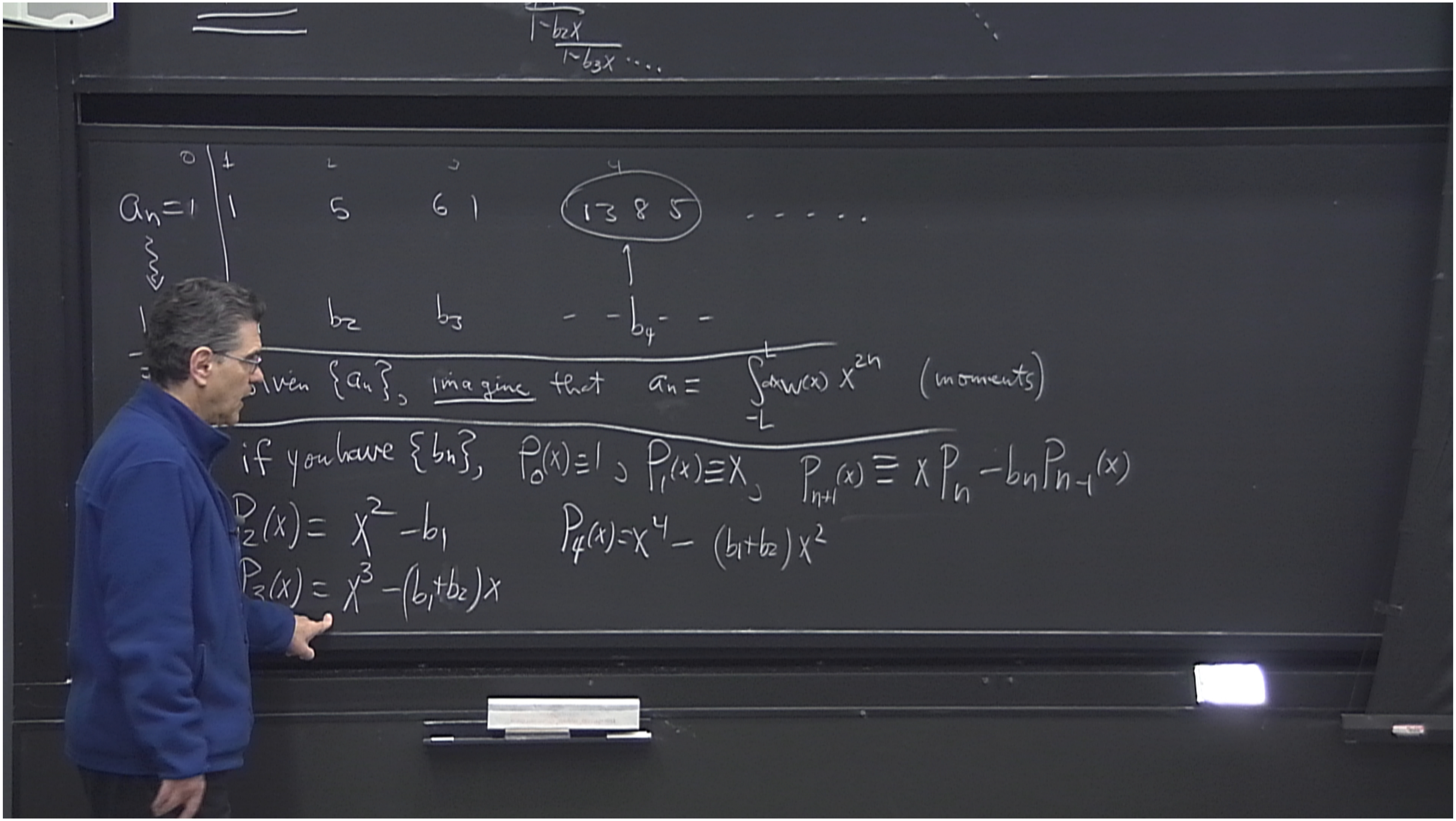
1-b2X
1-b3X ....
0 1 2 3 4
an=1 | 1 5 61 1385 .....
bn=1 | b1 b2 b3 - - b4 - -
ven {an}, imagine that an = ∫ dx w(x) X^2n (moments)
L
-L


1-b2X
1-b3X ....
an=1 | 1 5 61 1385 .....
bn=1 | b1 b2 b3 - - b4 - -
Given {an}, imagine that an = ∫ dx W(x) X^2n (moments)
if you have {bn}



1-b2X
1-b3X ....
0 1 2 3 4
an=1 1 5 61 1385 .....
bn=1 b1 b2 b3 - - b4 - -
1 Given {an}, imagine that an = ∫ dx w(x) x^2n (moments)
L
-L
2 if you have {bn}


1-b2x
1-b3x ....
an=1 | 1 5 61 1385 .....
bn b2 b3 - - b4 - -
imagine that an = ∫dxW(x) x^2n (moments)
have {bn}, P0(x)≡1, P1(x)≡X, Pn+1(x) ≡ x Pn - bnPn-1(x)


$1-b_2x$
$1-b_3x \cdots$
0 1 2 3 4
$a_n = 1$ | 1 5 61 1385 . . . . .
b_2 b_3 - - b_4 - -
given $\{a_n\}$, imagine that $a_n = \int_{-L}^{L} dx\, w(x)\, x^{2n}$ (moments)
if you have $\{b_n\}$, $P_0(x) \equiv 1$, $P_1(x) \equiv x$, $P_{n+1}(x) \equiv x P_n - b_n P_{n-1}(x)$
$P_2(x) = x^2 - b_1$
$P_4(x) = x^4 - (b_1 + b_2) x^2$
$P_3(x) = x^3 - (b_1 + b_2) x$

$1 - b_2 x$
$1 - b_3 x \cdots$

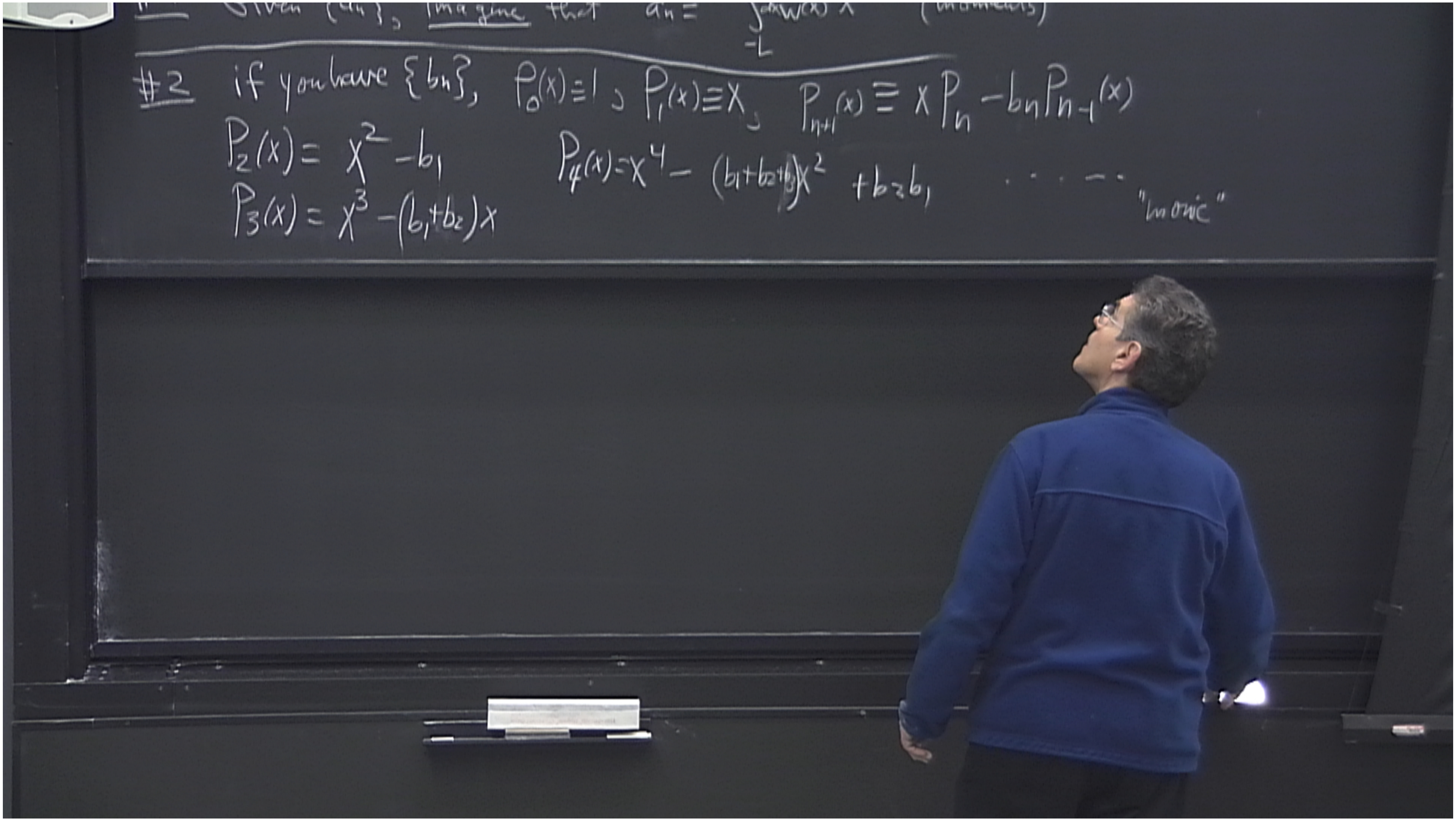
	0	1	2	3	4	
$a_n =$	1	1	5	61	1385	$\cdots\cdots$
$b_n =$	1	b_1	b_2	b_3	$-\,-\,b_4\,-\,-$	

$\#1$ Given $\{a_n\}$, imagine that $a_n = \int_{-L}^{L} dx\, w(x)\, x^{2n}$ (moments)

$\#2$ if you have $\{b_n\}$, $P_0(x) \equiv 1$, $P_1(x) \equiv x$, $P_{n+1}(x) \equiv x P_n - b_n P_{n-1}(x)$

$P_2(x) = x^2 - b_1$ $\qquad$ $P_4(x) = x^4 - (b_1 + b_2 + b_3) x^2 + b_2 b_1$

$P_3(x) = x^3 - (b_1 + b_2) x$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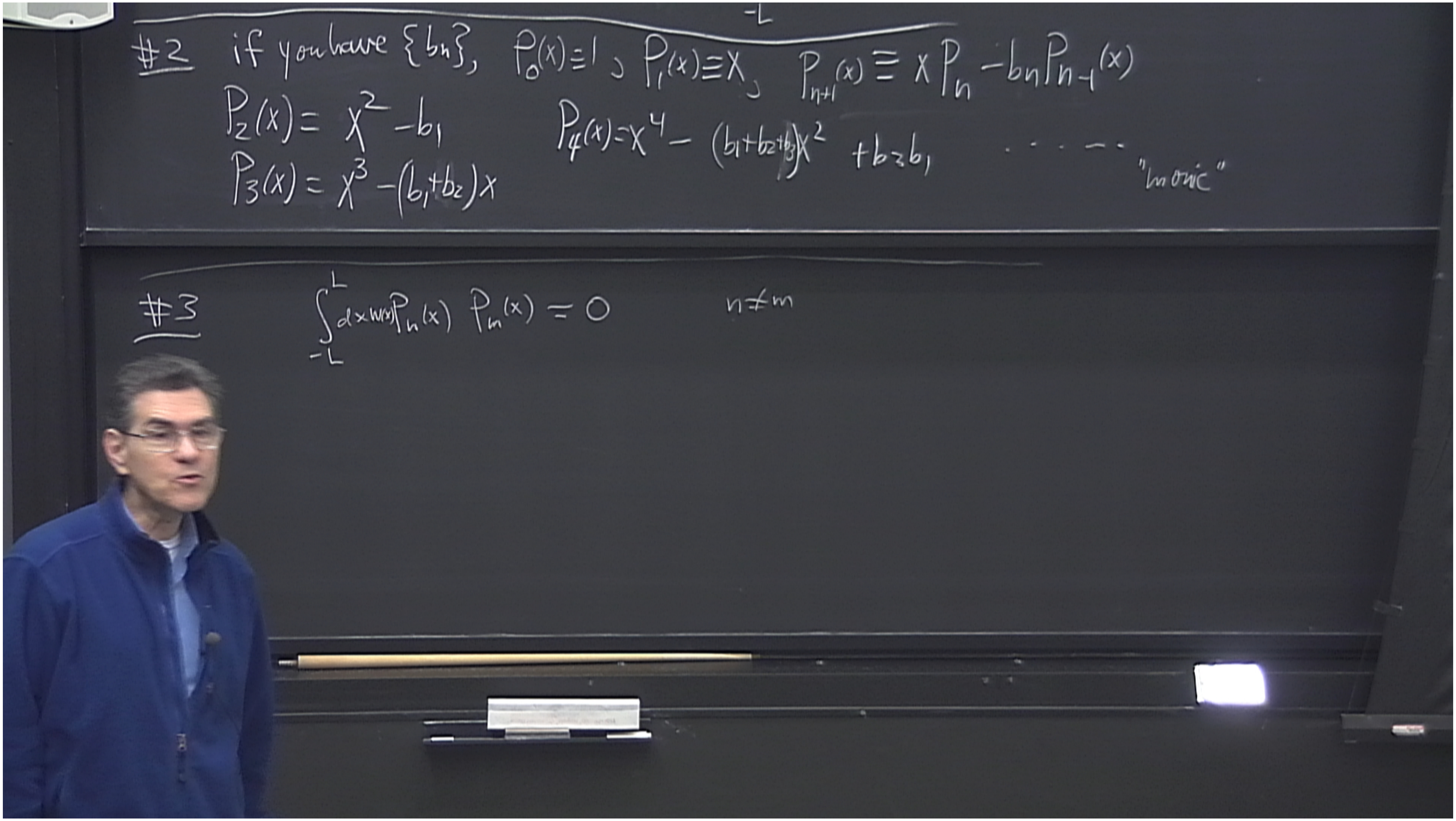
$P_2(x) = x^2 - b_1$
$P_4(x) = x^4 - (b_1+b_2+b_3)x^2 + b_3 b_1$ . . . . . .
$P_3(x) = x^3 - (b_1+b_2)x$
"monic"


$-L$
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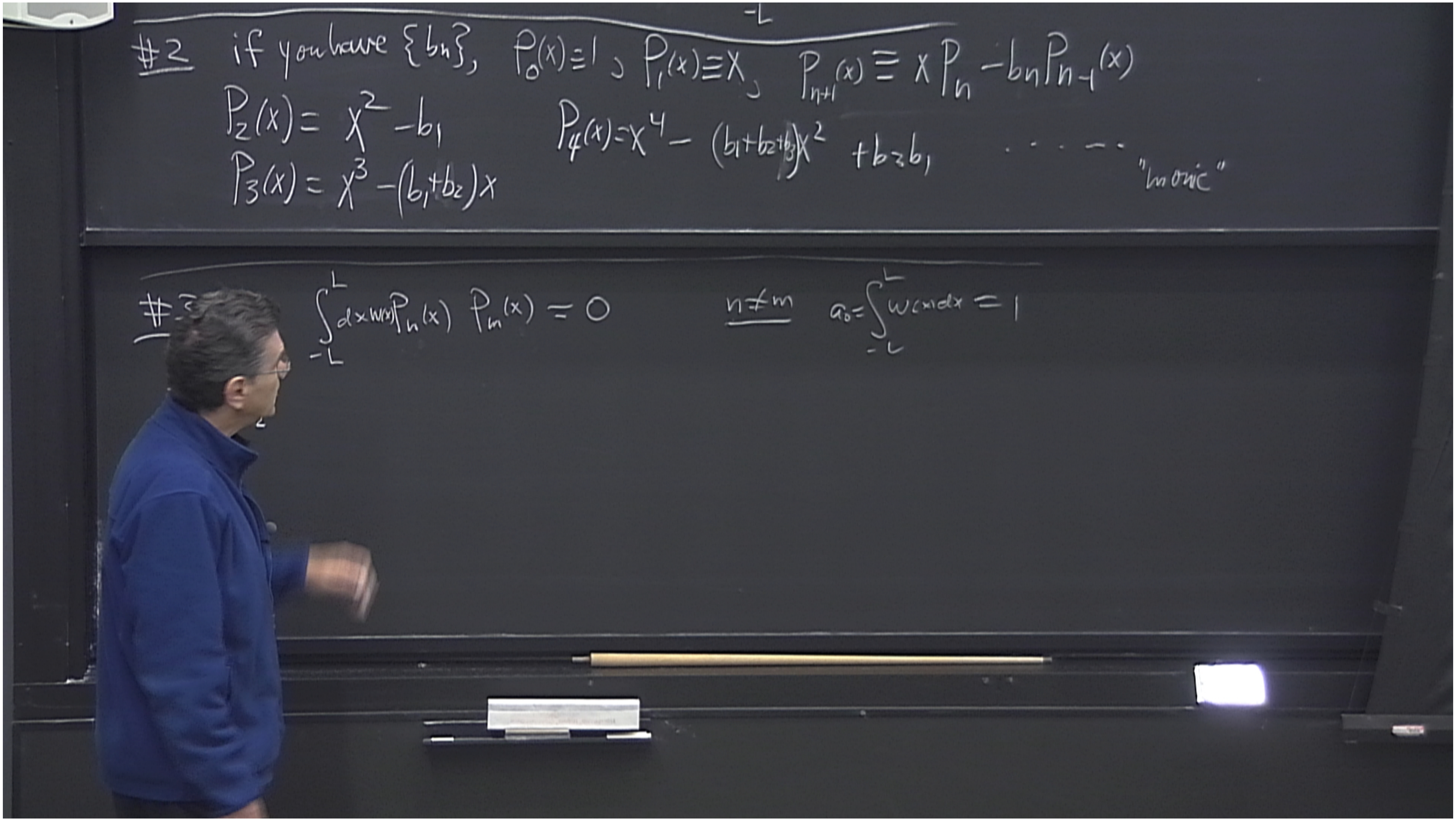
"monic"
#3 $\int_{-L}^{L} dx\, w(x) P_n(x)\, P_m(x) = 0$
$n \neq m$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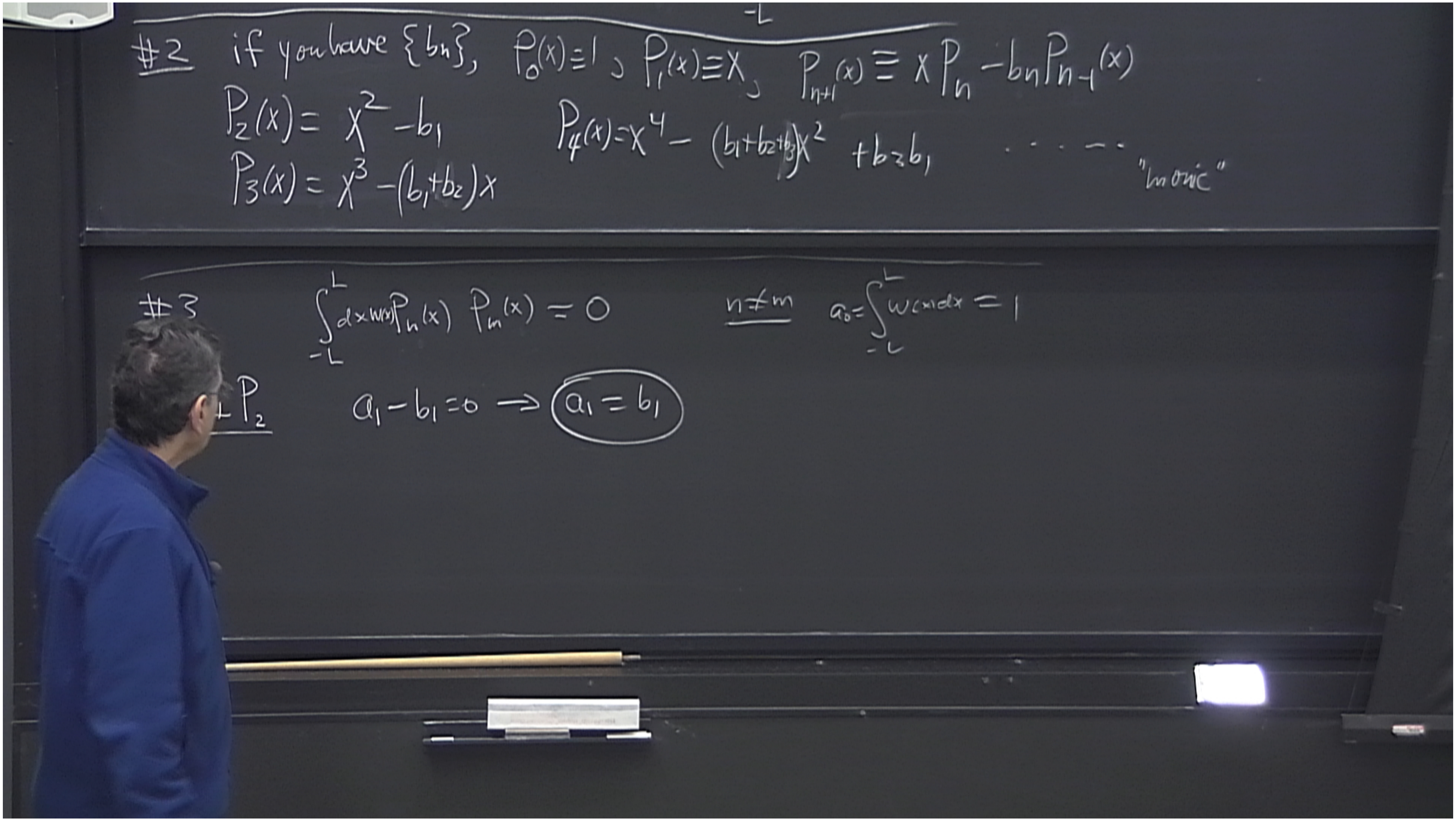
"monic"
#3 $\int_{-L}^{L} dx\, w(x) P_n(x)\, P_m(x) = 0$
$n \neq m$
$a_0 = \int_{-L}^{L} w(x) dx = 1$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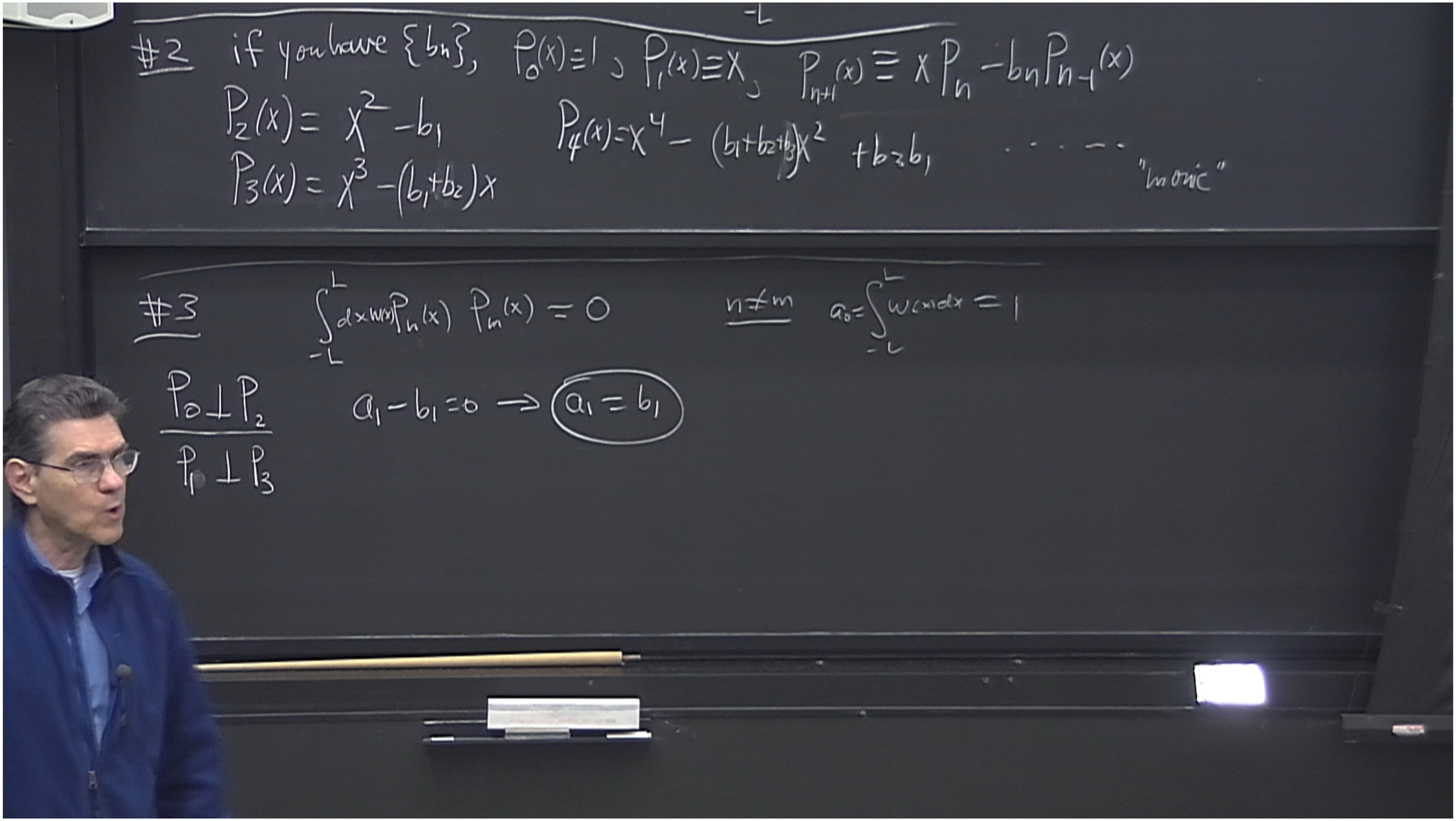
$P_3(x) = x^3 - (b_1+b_2)x$
#3 $\int_{-L}^{L} dx\, w(x) P_n(x) P_m(x) = 0$
$n \neq m$
$a_0 = \int_{-L}^{L} w(x) dx = 1$
$\perp P_2$
$a_1 - b_1 = 0 \rightarrow a_1 = b_1$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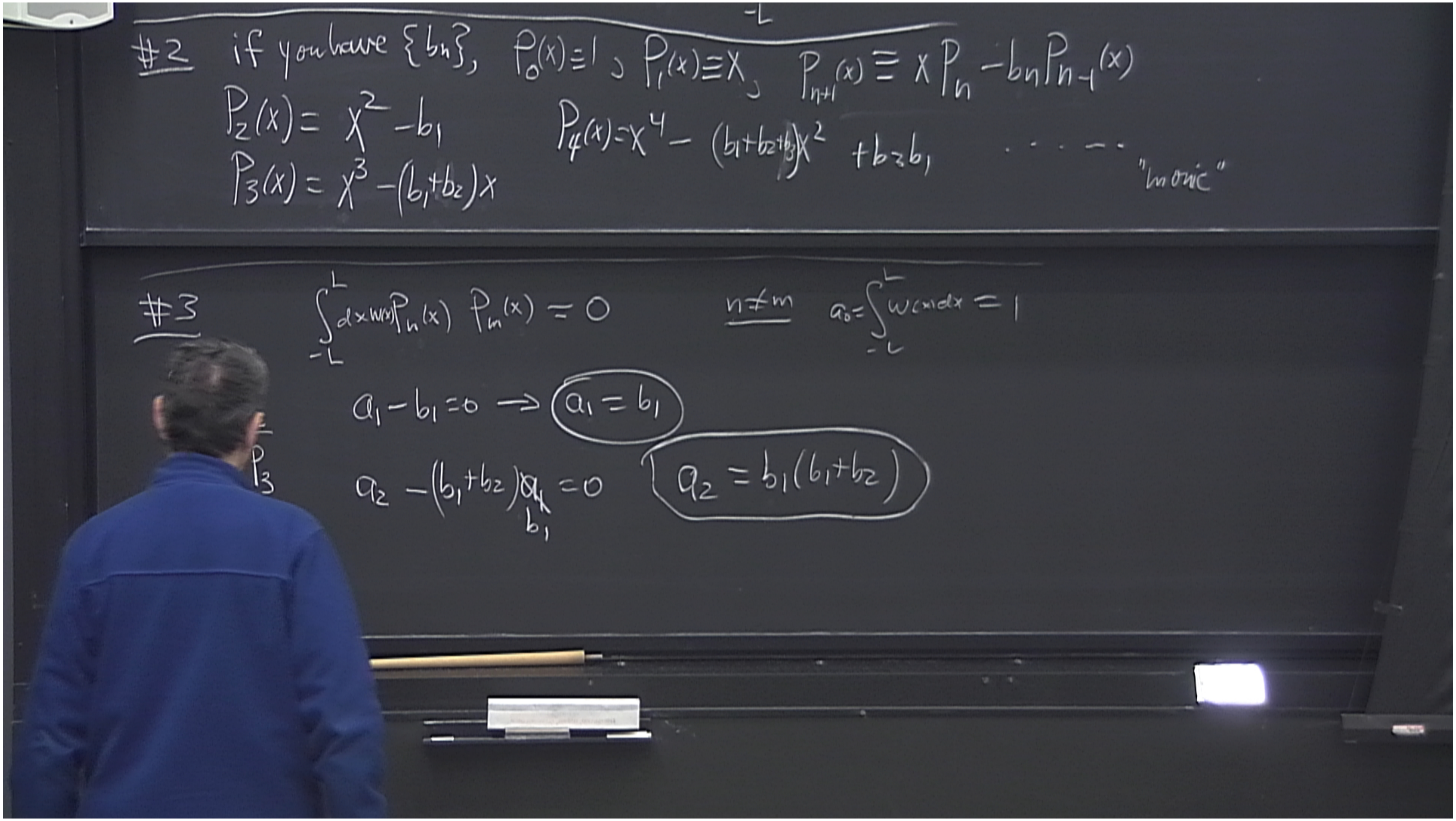
$P_4(x) = x^4 - (b_1 + b_2 + b_3) x^2 + b_3 b_1$ . . . . — .
"monic"
#3 $\int_{-L}^{L} dx\, w(x) P_n(x)\, P_m(x) = 0$
$n \neq m$
$a_0 = \int_{-L}^{L} w(x) dx = 1$
$P_0 \perp P_2$
$a_1 - b_1 = 0 \rightarrow a_1 = b_1$
$P_1 \perp P_3$



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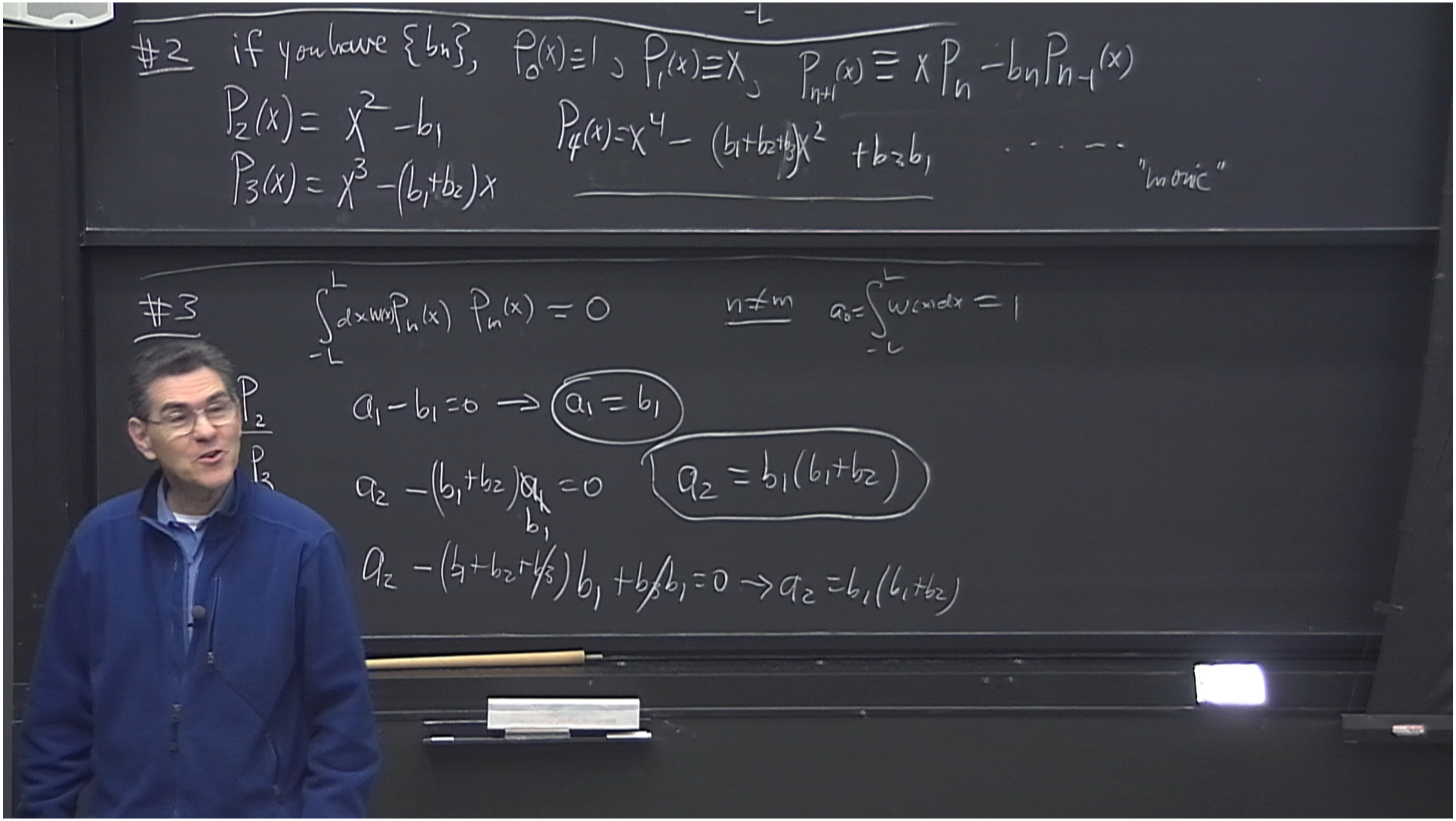
"monic"
#3
$\int_{-L}^{L} dx\, w(x) P_n(x) P_m(x) = 0$
$n \neq m$
$a_0 = \int_{-L}^{L} w(x) dx = 1$
$a_1 - b_1 = 0 \rightarrow a_1 = b_1$
P_3
$a_2 - (b_1 + b_2)\cancel{a_1} = 0$
b_1
$a_2 = b_1(b_1 + b_2)$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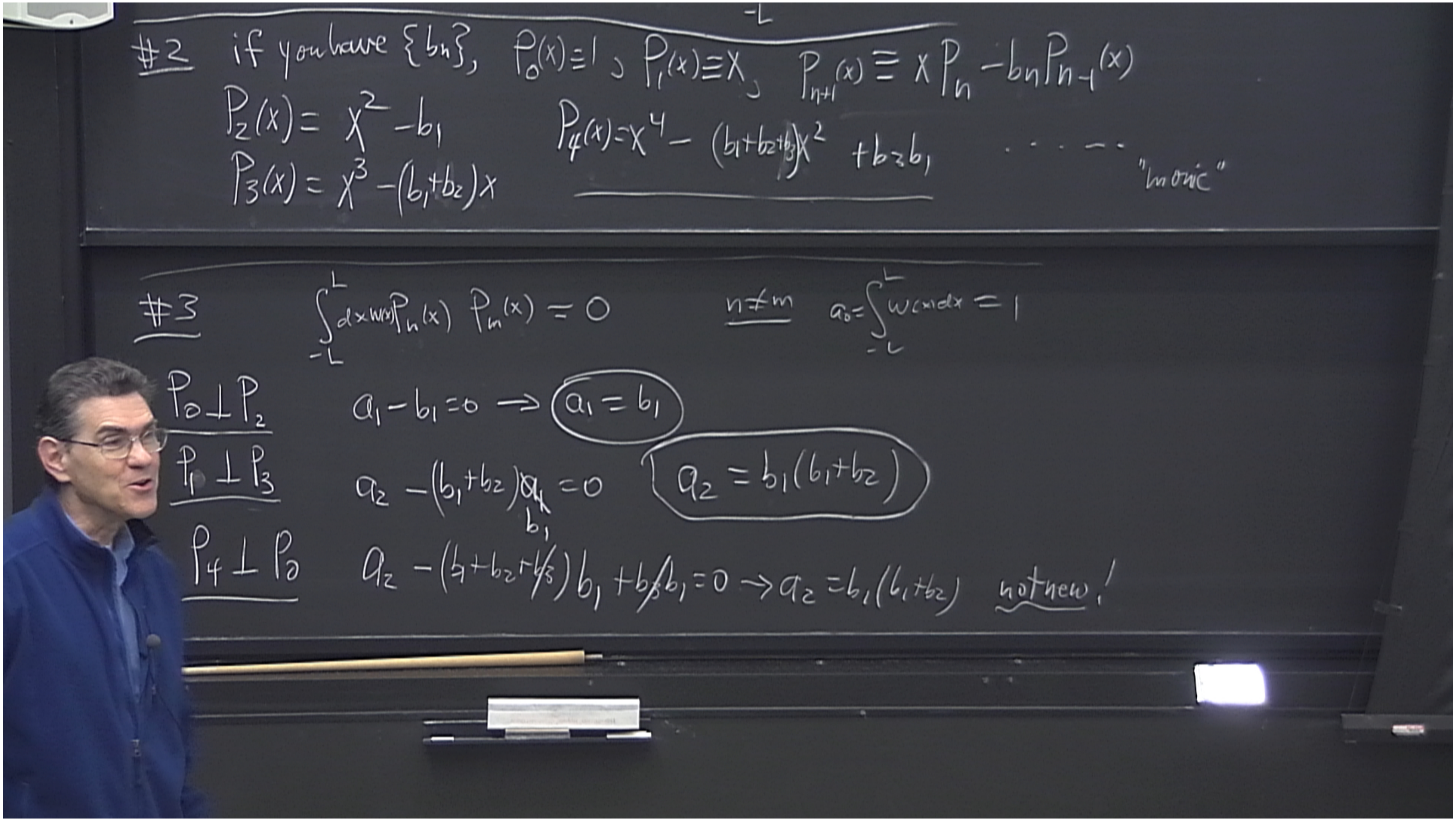
"monic"
#3
$\int_{-L}^{L} dx\, w(x) P_n(x)\, P_m(x) = 0$ $n \neq m$
$a_0 = \int_{-L}^{L} w(x)\, dx = 1$
P_2
P_3
$a_1 - b_1 = 0 \rightarrow a_1 = b_1$
$a_2 - (b_1 + b_2)\cancel{a_1} = 0$ $a_2 = b_1(b_1 + b_2)$
b_1
$a_2 - (b_1 + b_2 + \cancel{b_3}) b_1 + \cancel{b_3} b_1 = 0 \rightarrow a_2 = b_1(b_1 + b_2)$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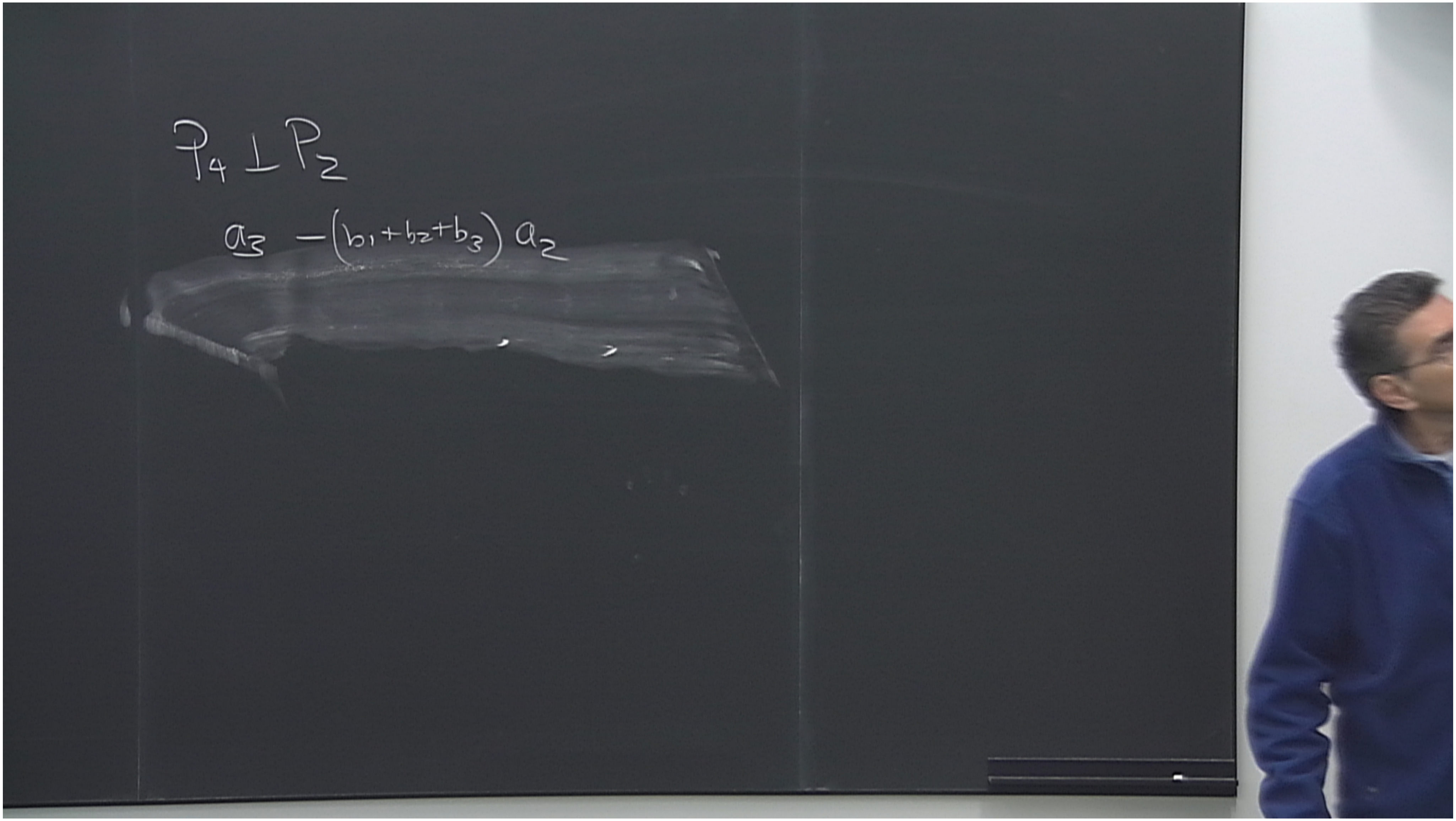
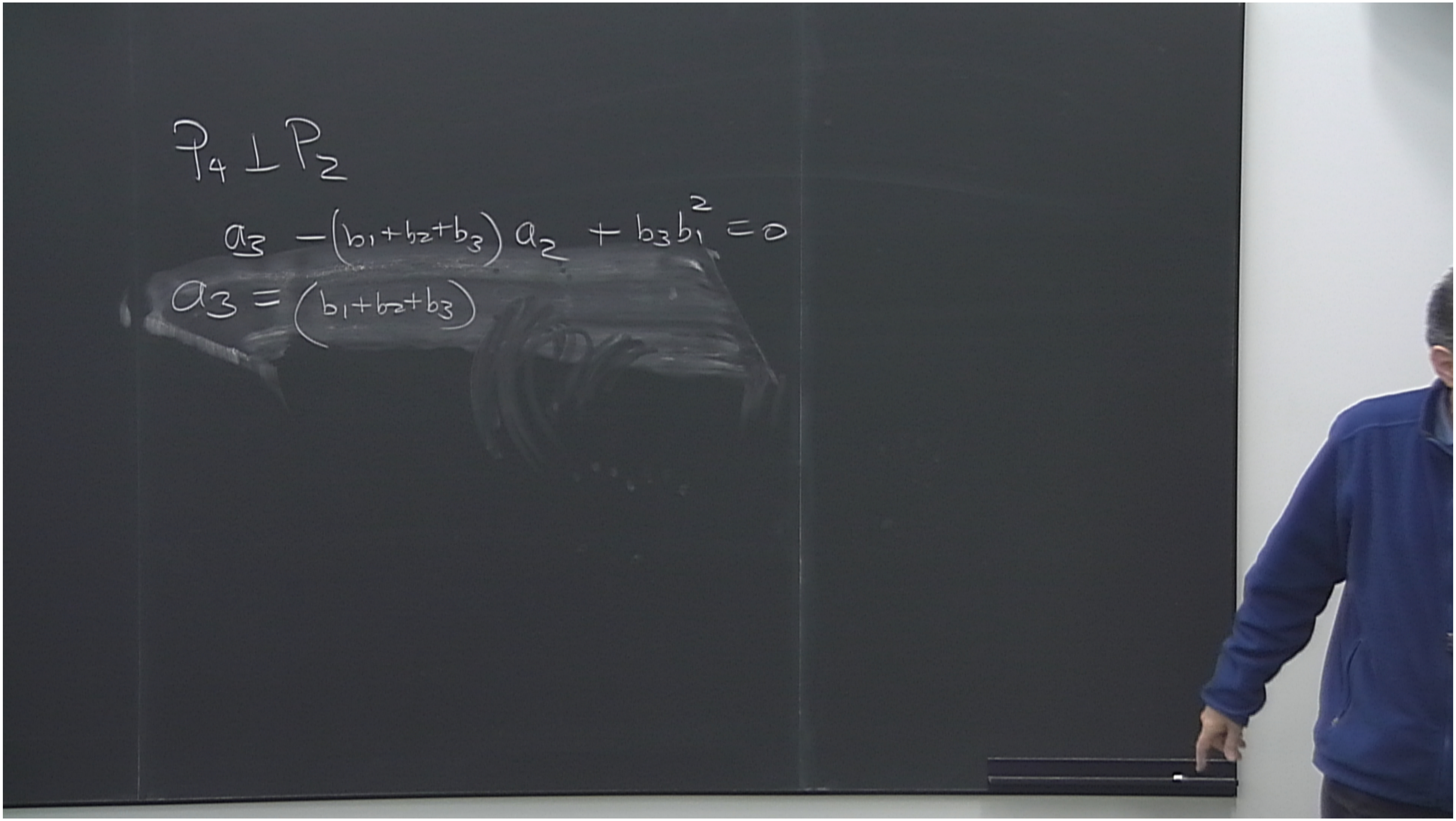
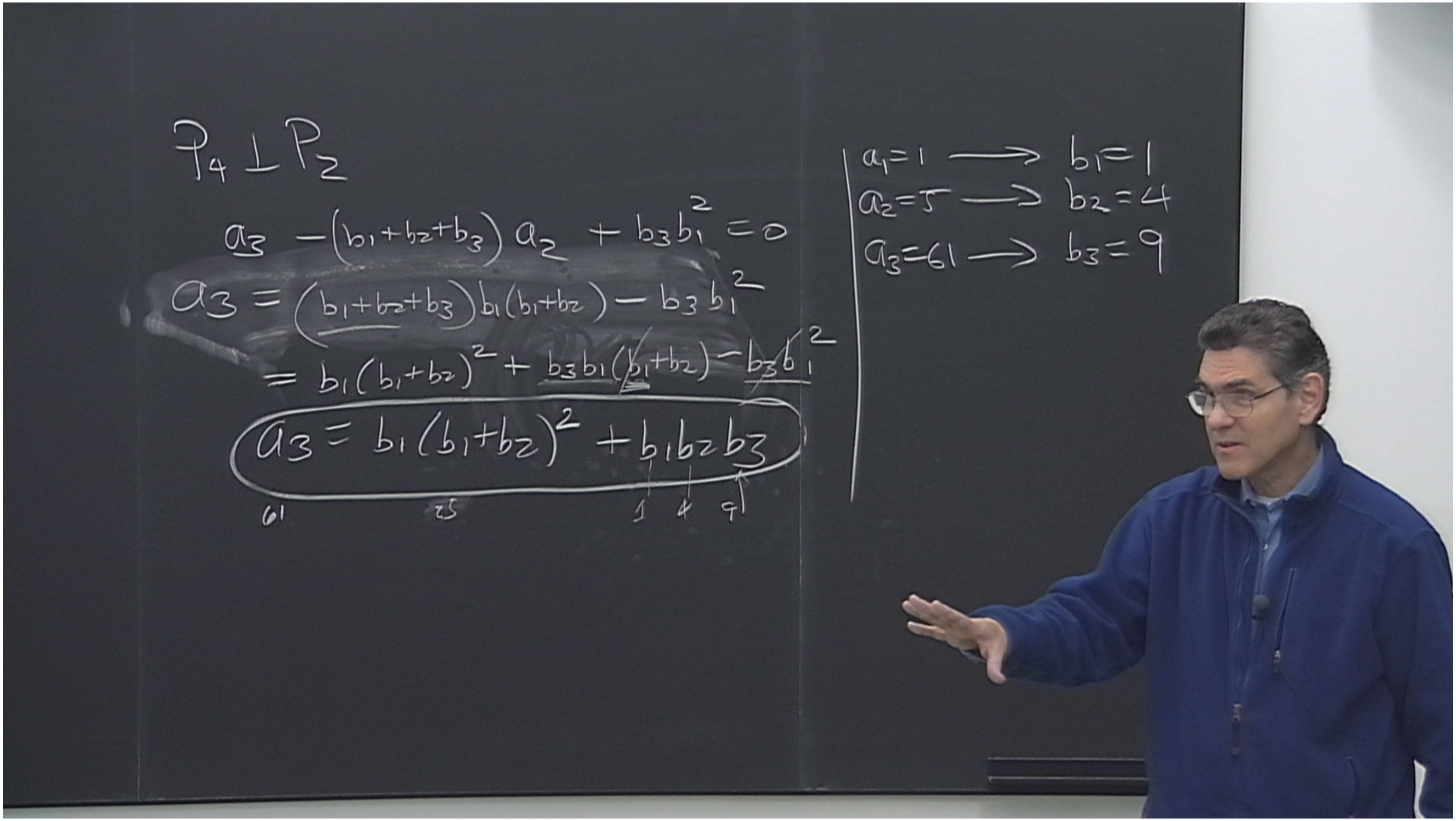
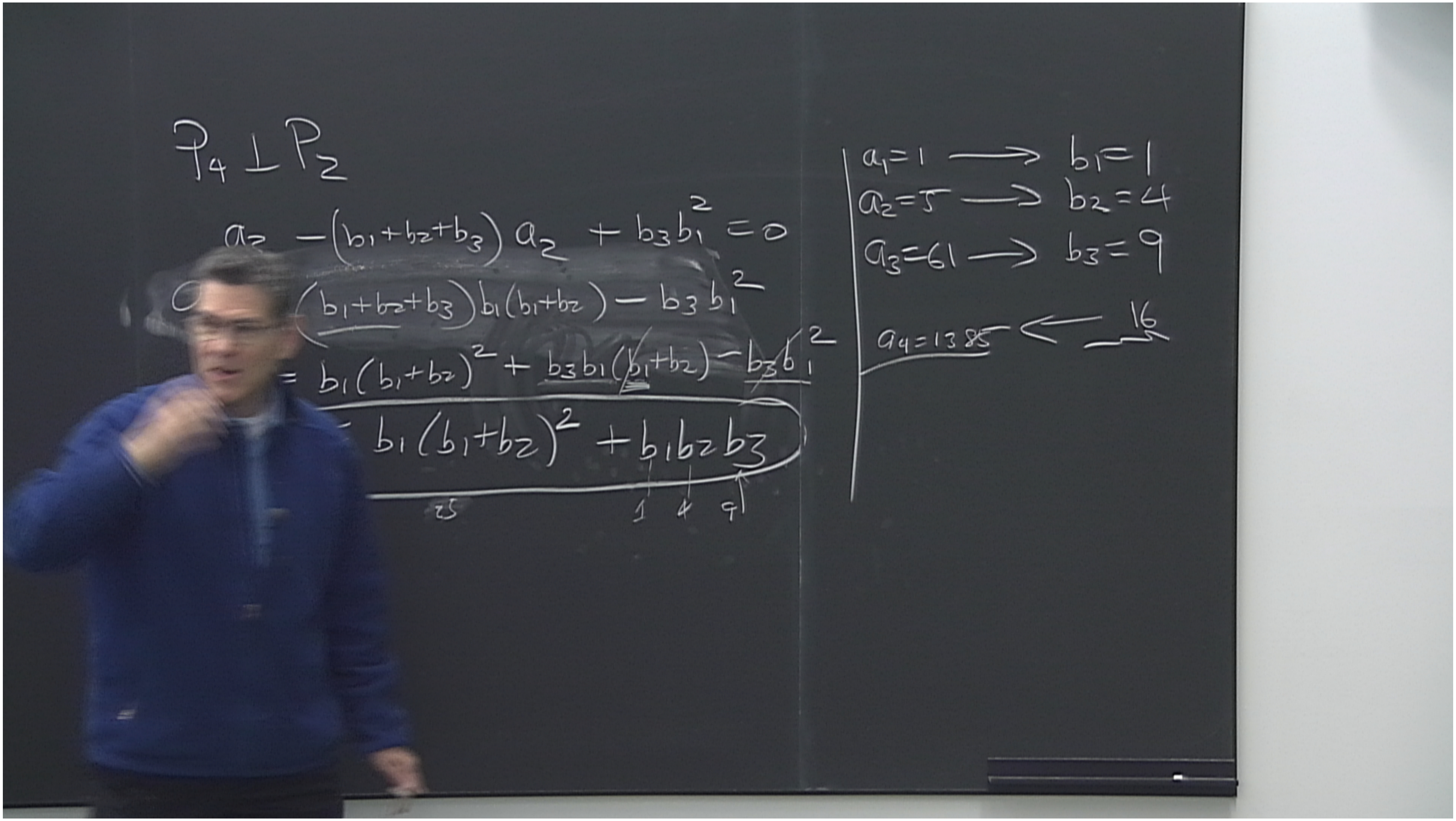
"monic"
#3 $\int_{-L}^{L} dx\, w(x) P_n(x)\, P_m(x) = 0$
$n \neq m$
$a_0 = \int_{-L}^{L} w(x)\, dx = 1$
$P_0 \perp P_2$
$a_1 - b_1 = 0 \rightarrow a_1 = b_1$
$P_1 \perp P_3$
$a_2 - (b_1 + b_2)\underset{b_1}{\cancel{a_1}} = 0$
$a_2 = b_1(b_1 + b_2)$
$P_4 \perp P_0$
$a_2 - (b_1 + b_2 + \cancel{b_3})\, b_1 + \cancel{b_3} b_1 = 0 \rightarrow a_2 = b_1(b_1 + b_2)$ not new!


$p_4 \perp P_2$
$a_3 - (b_1 + b_2 + b_3)\, a_2$

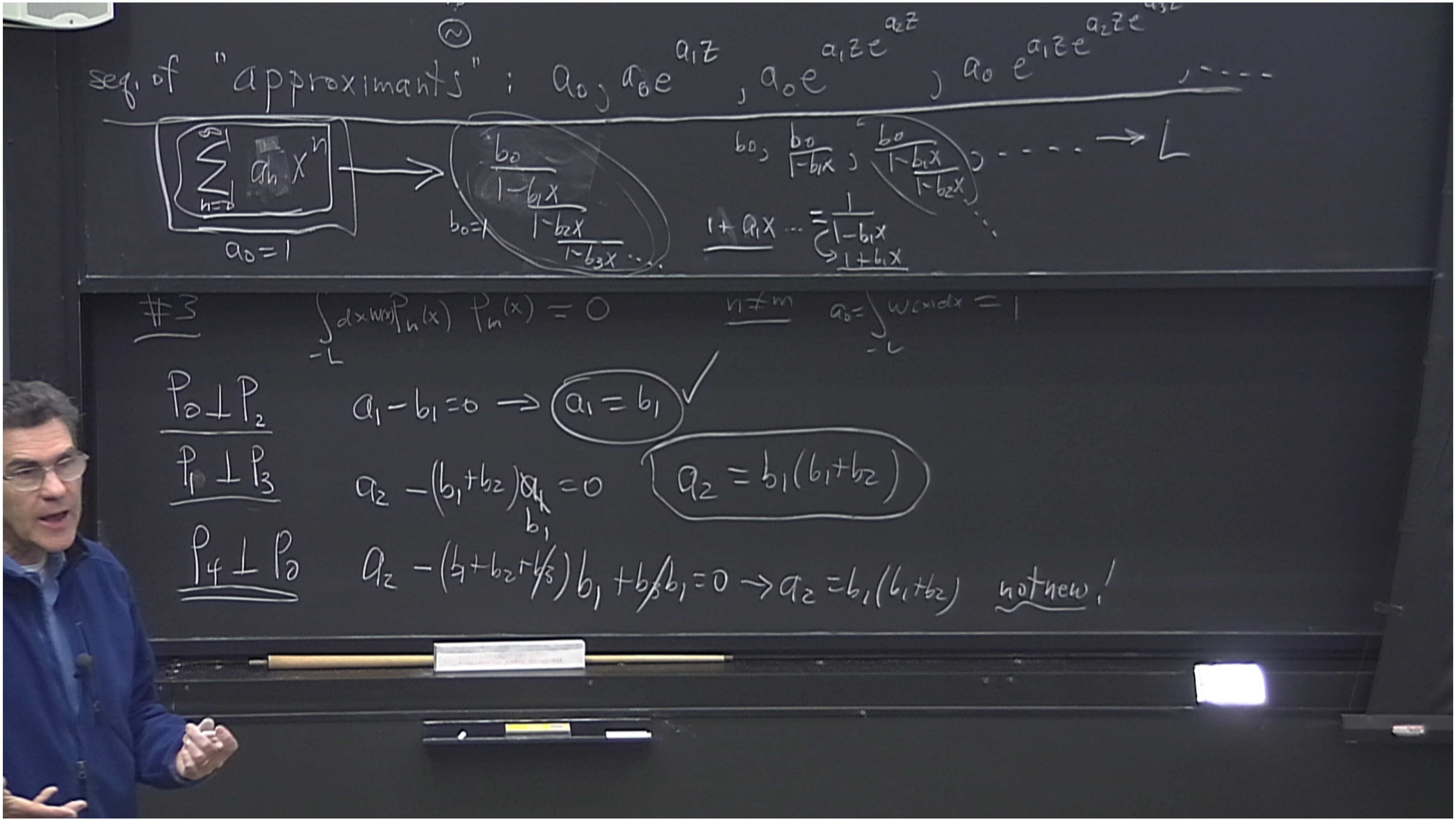

$p_4 \perp P_2$
$a_3 - (b_1+b_2+b_3)\, a_2 + b_3 b_1^2 = 0$
$a_3 = (b_1+b_2+b_3)$

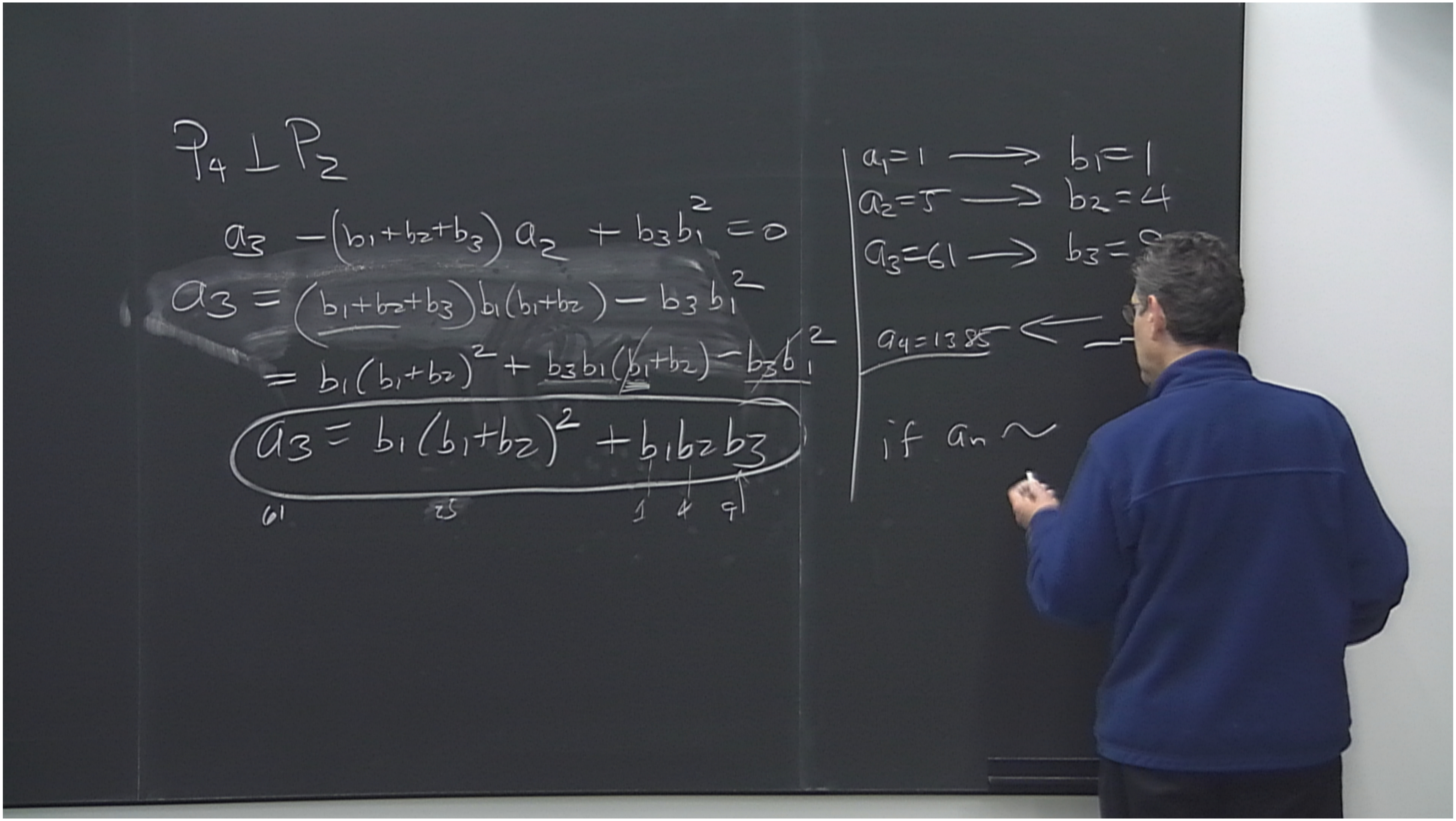


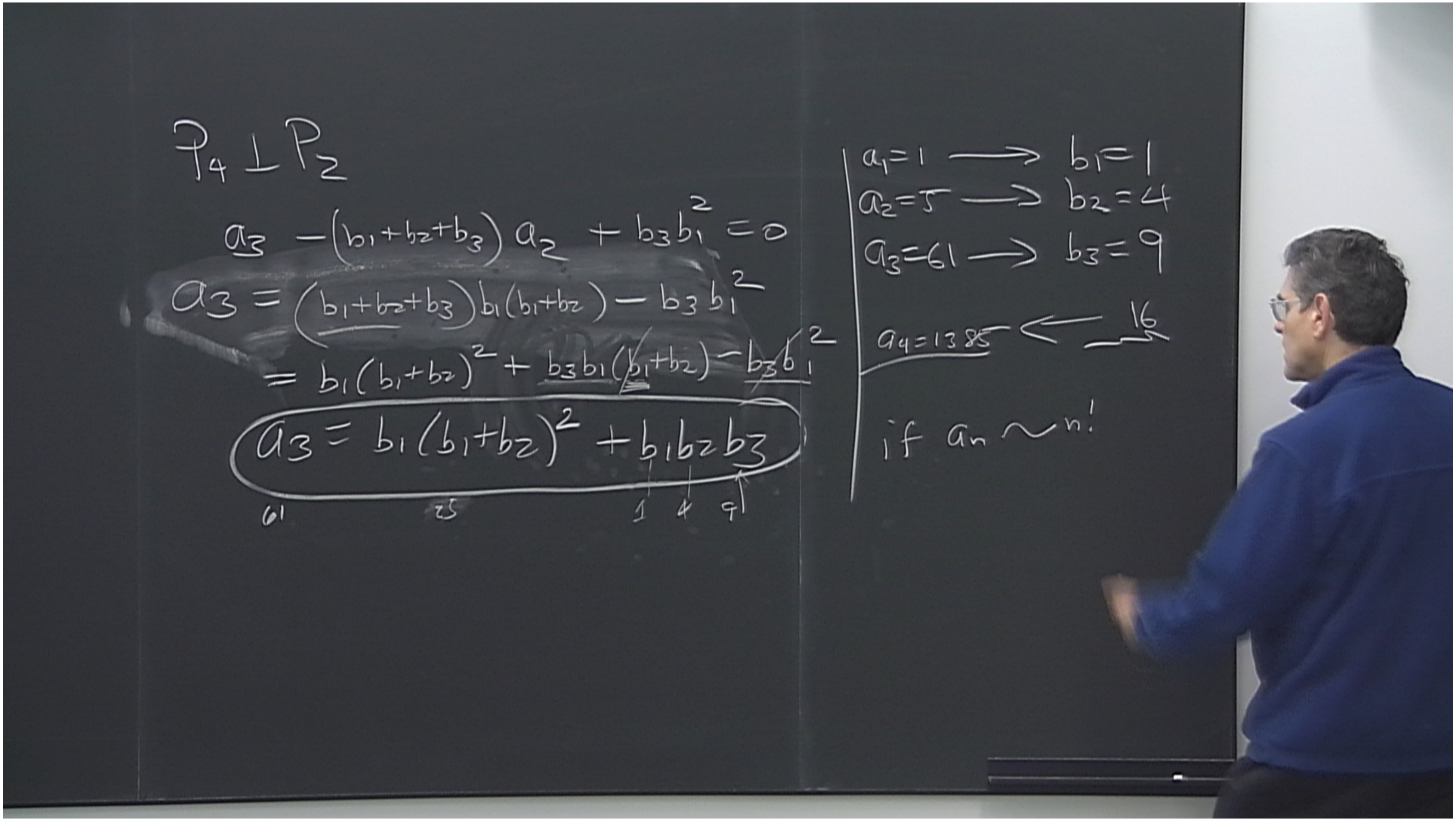
$P_4 \perp P_2$
$a_3 - (b_1+b_2+b_3)\,a_2 + b_3 b_1^2 = 0$
$a_3 = (b_1+b_2+b_3)\,b_1(b_1+b_2) - b_3 b_1^2$
$= b_1(b_1+b_2)^2 + b_3 b_1(b_1+b_2) - b_3 b_1^2$
$a_3 = b_1(b_1+b_2)^2 + b_1 b_2 b_3$
61 25 1 4 9
$a_1 = 1 \longrightarrow b_1 = 1$
$a_2 = 5 \longrightarrow b_2 = 4$
$a_3 = 61 \longrightarrow b_3 = 9$

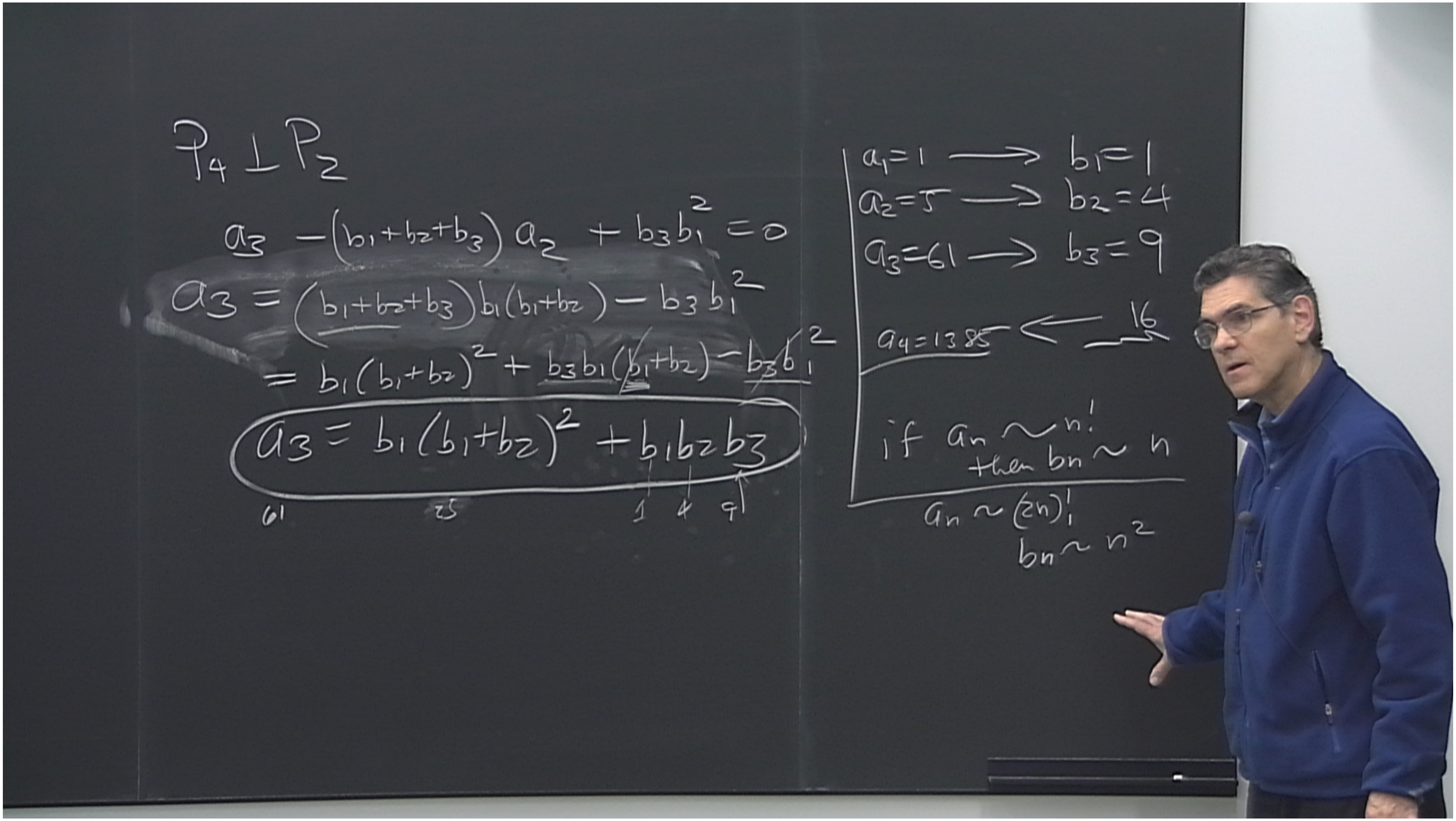


$P_4 \perp P_2$
$a_1 = 1 \longrightarrow b_1 = 1$
$a_2 = 5 \longrightarrow b_2 = 4$
$a_3 = 61 \longrightarrow b_3 = 9$
$a_4 = 1385 \longleftarrow 16$

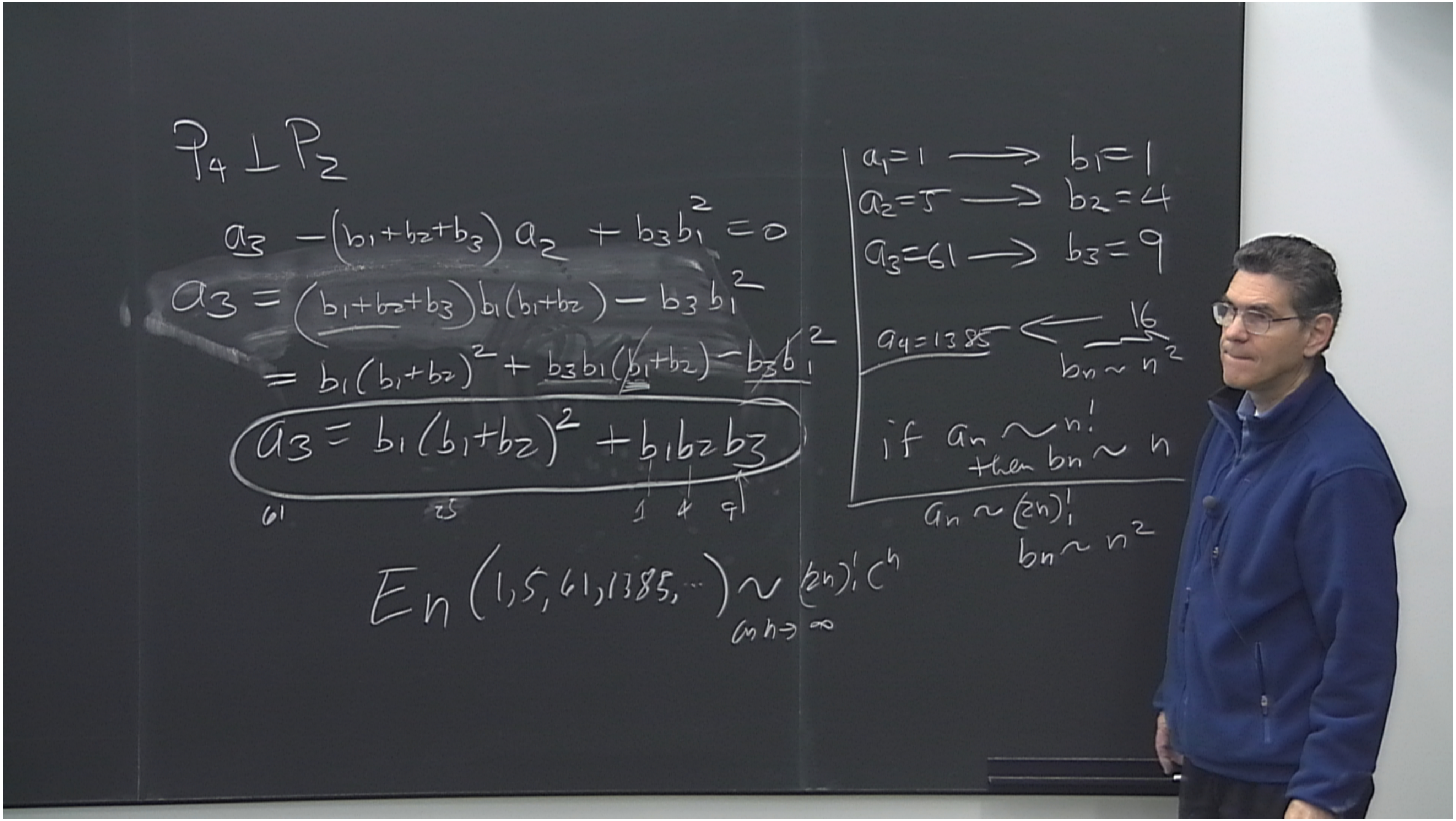

seq. of "approximants": a_0, $a_0 e^{a_1 z}$, $a_0 e^{a_1 z e^{a_2 z}}$, $a_0 e^{a_1 z e^{a_2 z e^{a_3 z}}}$, ...
$\sum_{n=0}^{\infty} a_n x^n \rightarrow \frac{b_0}{1-\frac{b_1 x}{1-\frac{b_2 x}{1-b_3 x \cdots}}}$
$a_0 = 1$
$b_0 = 1$
b_0, $\frac{b_0}{1-b_1 x}$, $\frac{b_0}{1-\frac{b_1 x}{1-b_2 x}}$, ... $\rightarrow L$
$1 + a_1 x \cdots = \frac{1}{1-b_1 x}$
$1 + b_1 x$
#3
$\int_{-L} dx\, w(x) P_n(x)\, P_m(x) = 0$
$n \neq m$
$a_0 = \int_{-L} w(x) dx = 1$
$P_0 \perp P_2$
$a_1 - b_1 = 0 \rightarrow a_1 = b_1$
$P_1 \perp P_3$
$a_2 - (b_1 + b_2) b_1 = 0$
$a_2 = b_1(b_1 + b_2)$
$P_4 \perp P_0$
$a_2 - (b_1 + b_2 + \cancel{b_3}) b_1 + \cancel{b_3} b_1 = 0 \rightarrow a_2 = b_1(b_1 + b_2)$
not new!

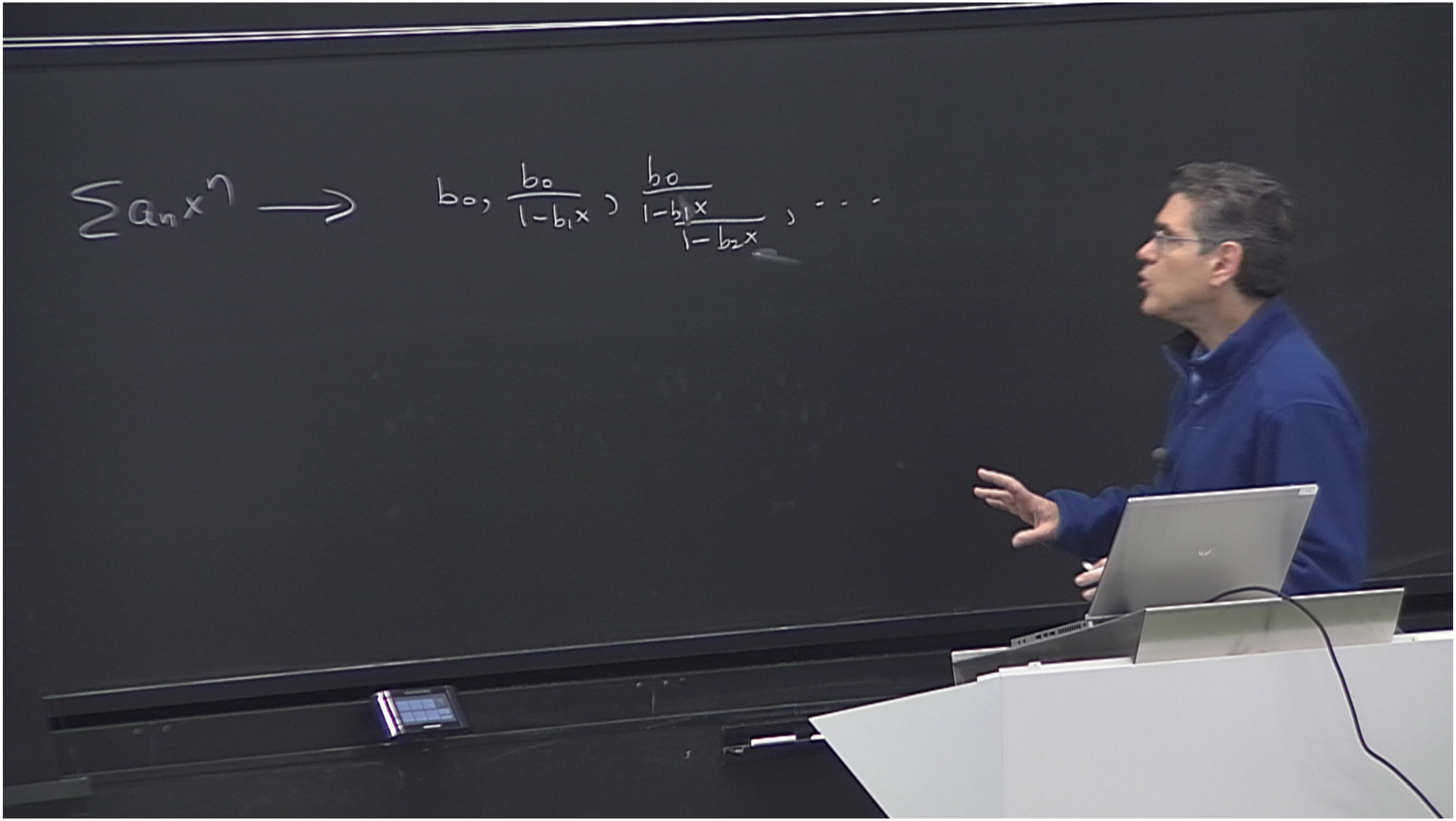

P4 ⊥ P2
a3 − (b1 + b2 + b3) a2 + b3 b1^2 = 0
a3 = (b1 + b2 + b3) b1 (b1 + b2) − b3 b1^2
= b1 (b1 + b2)^2 + b3 b1 (b1 + b2) − b3 b1^2
a3 = b1 (b1 + b2)^2 + b1 b2 b3
61 25 1 4 9
a1 = 1 ⟶ b1 = 1
a2 = 5 ⟶ b2 = 4
a3 = 61 ⟶ b3 =
a4 = 1385 ⟵
if an ~

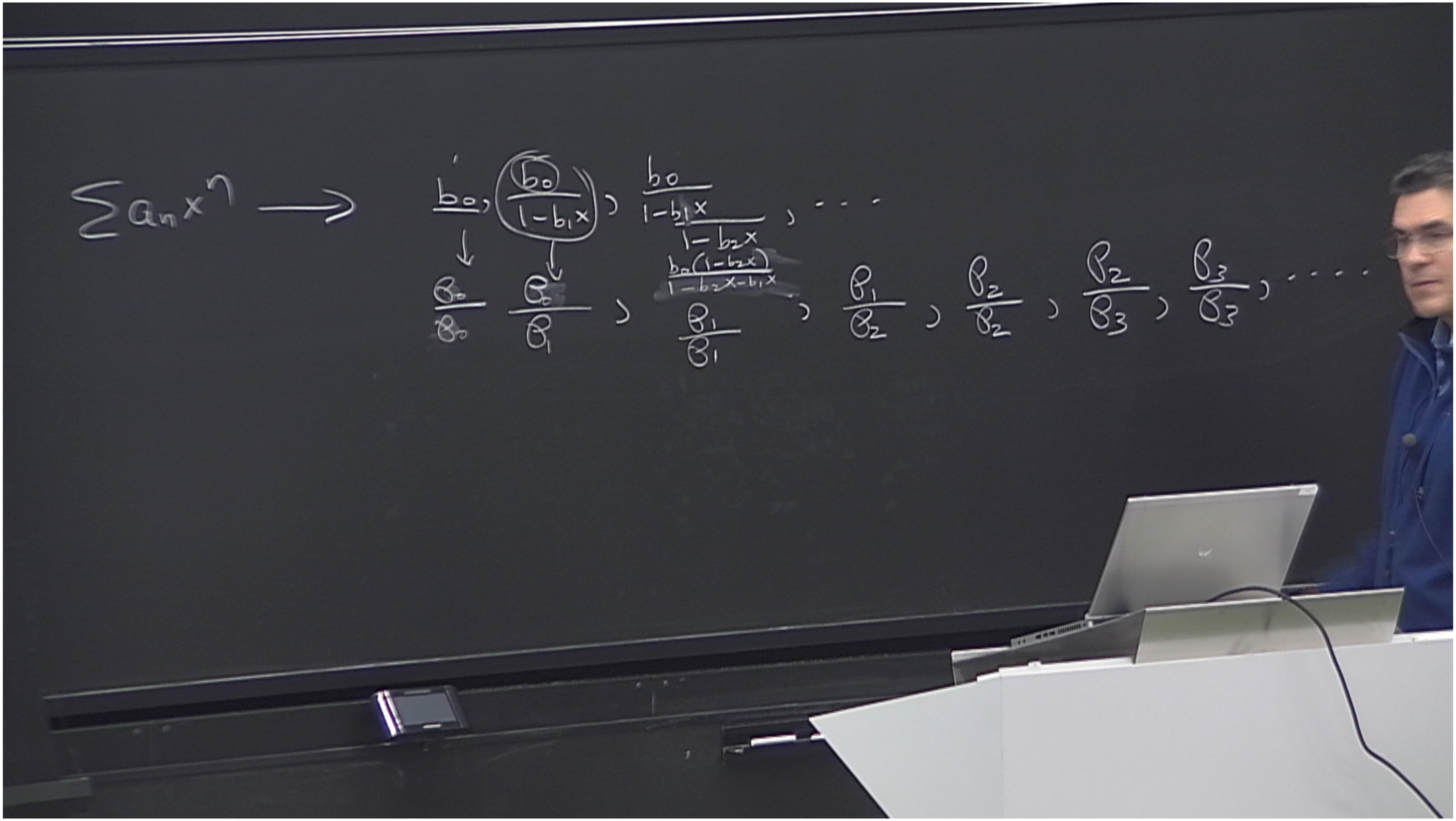

P4 ⊥ P2
a3 − (b1+b2+b3) a2 + b3 b1^2 = 0
a3 = (b1+b2+b3) b1(b1+b2) − b3 b1^2
= b1(b1+b2)^2 + b3 b1(b1+b2) − b3 b1^2
a3 = b1(b1+b2)^2 + b1 b2 b3
61
25
1 4 9
a1=1 → b1=1
a2=5 → b2=4
a3=61 → b3=9
a4=1385 ← 16
if an ~ n!

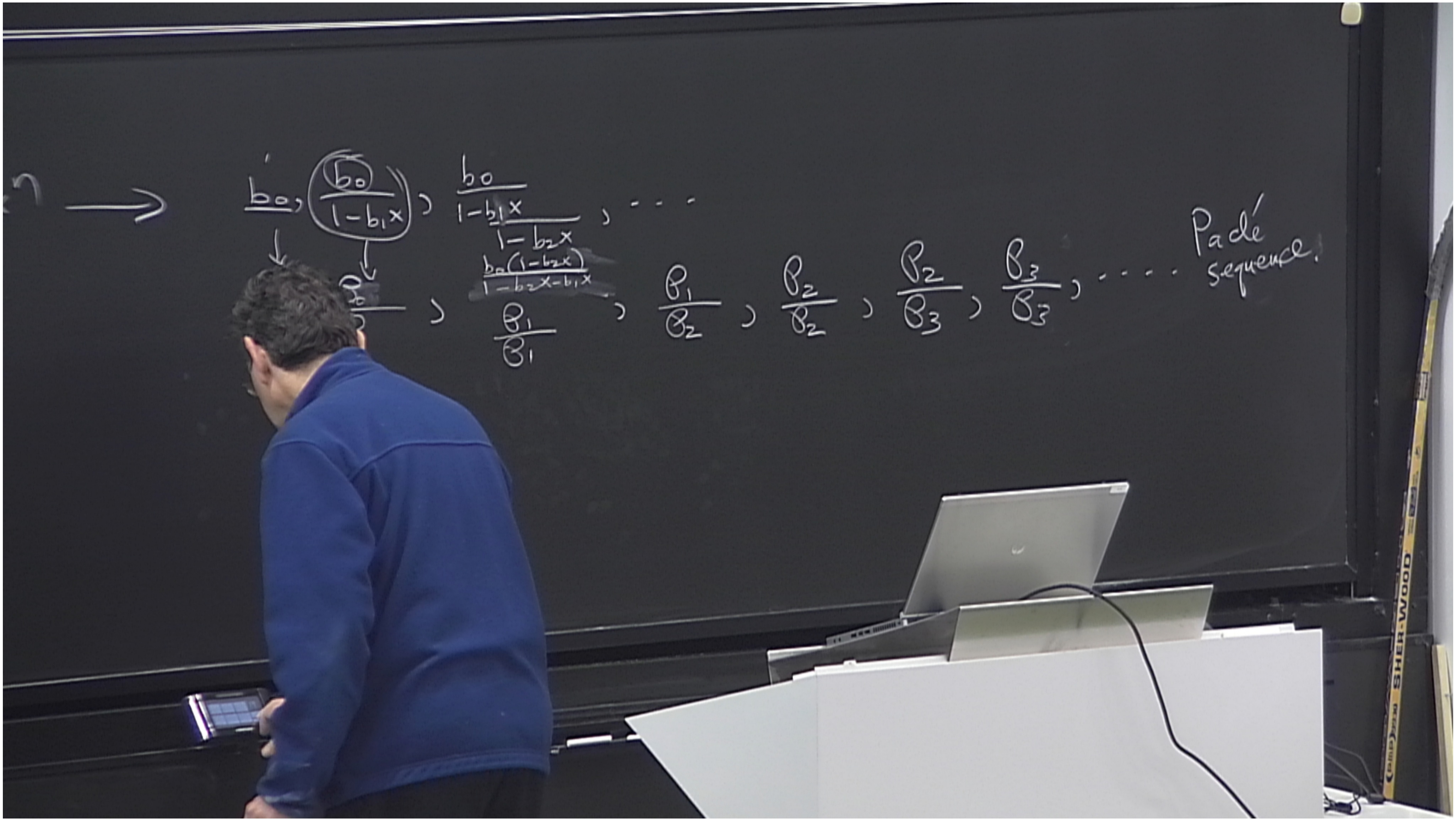


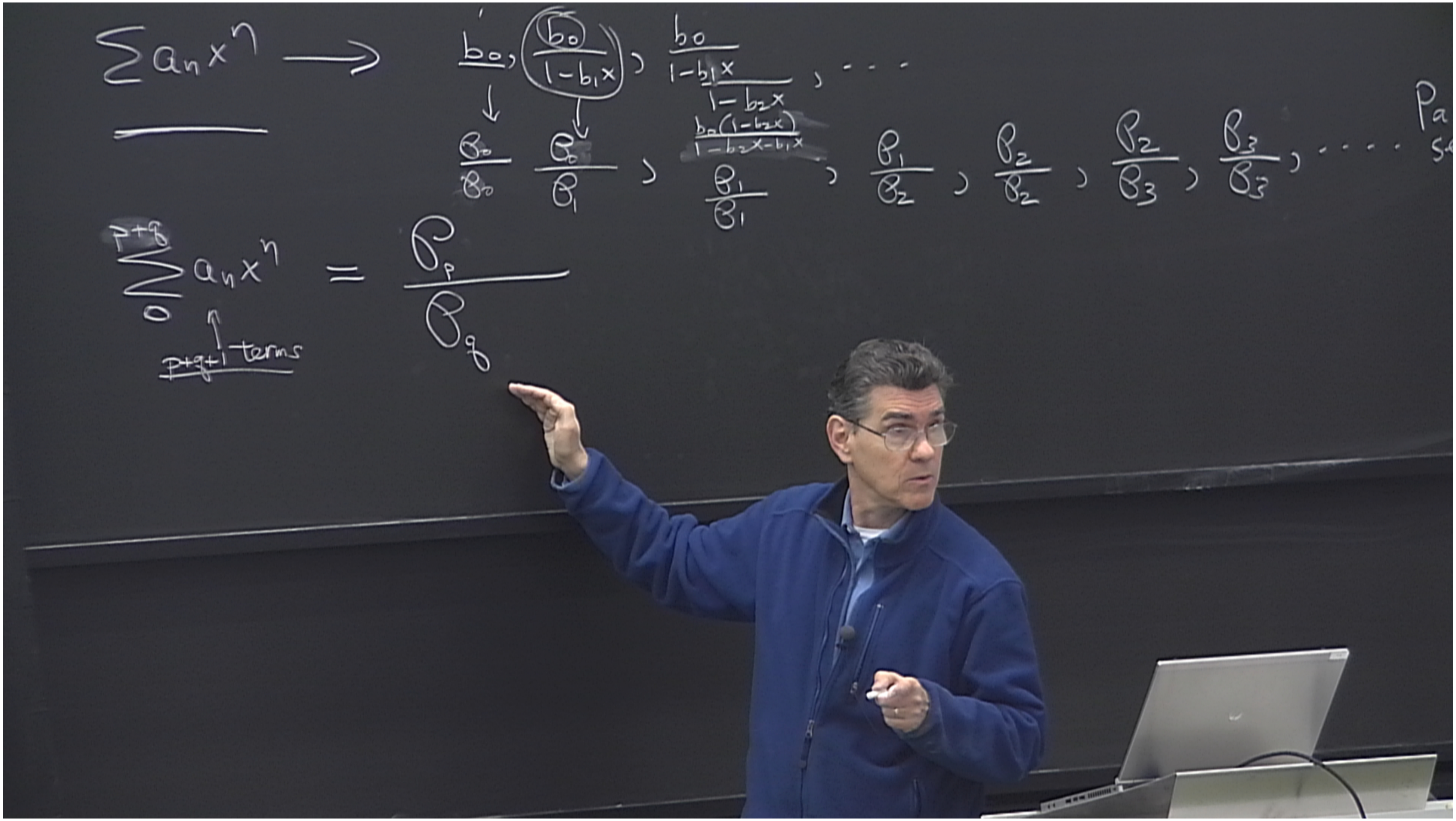
$P_4 \perp P_2$
$a_3 - (b_1+b_2+b_3)\,a_2 + b_3 b_1^2 = 0$
$a_3 = (b_1+b_2+b_3)\,b_1(b_1+b_2) - b_3 b_1^2$
$= b_1(b_1+b_2)^2 + b_3 b_1(\cancel{b_1}+b_2) - \cancel{b_3 b_1^2}$
$a_3 = b_1(b_1+b_2)^2 + b_1 b_2 b_3$
61 25 1 4 9
$a_1 = 1 \longrightarrow b_1 = 1$
$a_2 = 5 \longrightarrow b_2 = 4$
$a_3 = 61 \longrightarrow b_3 = 9$
$a_4 = 1385 \longleftarrow 16$
if $a_n \sim n!$
then $b_n \sim n$
$a_n \sim (2n)!$
$b_n \sim n^2$

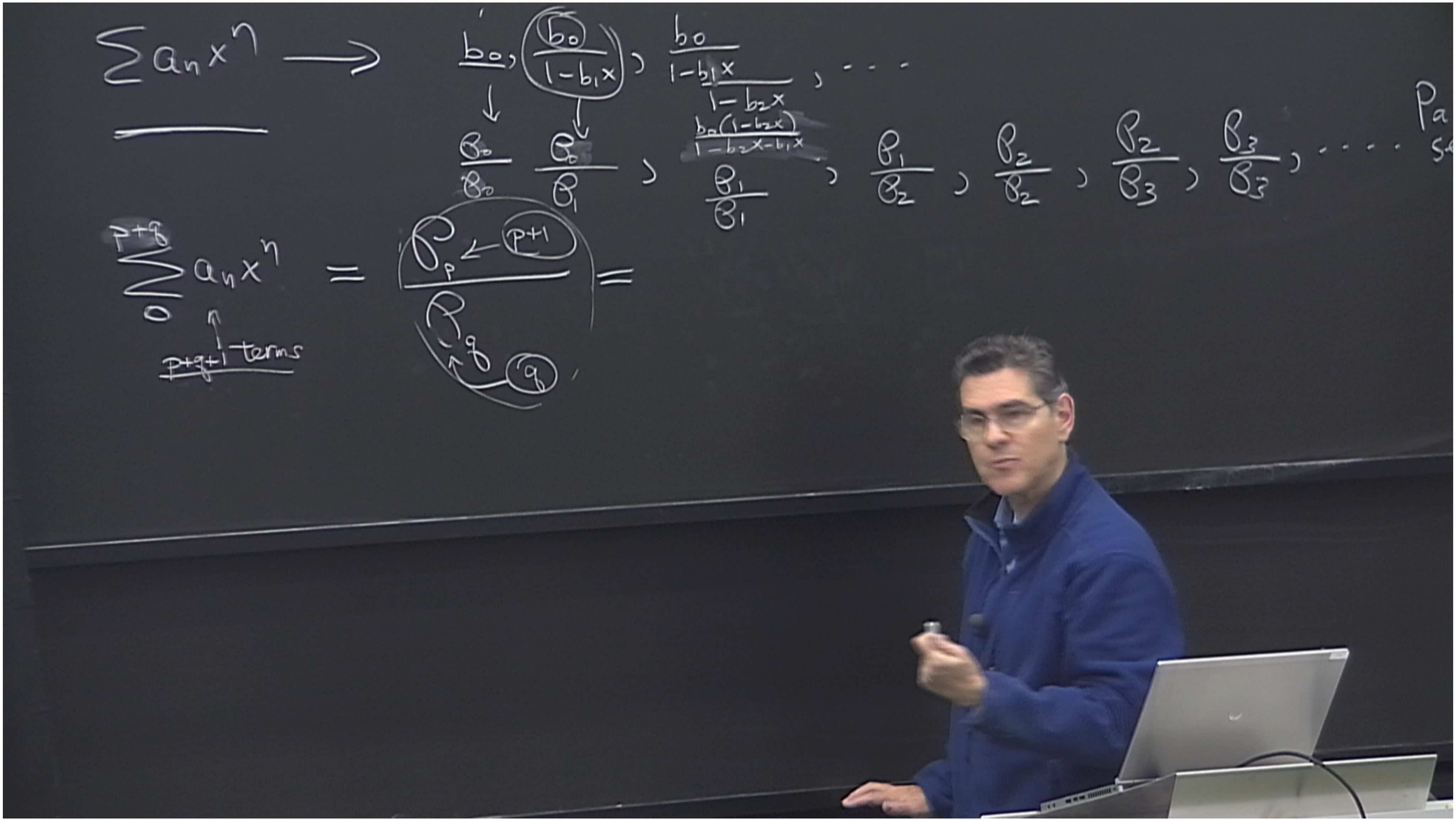

P4 ⊥ P2
a3 − (b1+b2+b3) a2 + b3 b1^2 = 0
a3 = (b1+b2+b3) b1 (b1+b2) − b3 b1^2
= b1 (b1+b2)^2 + b3 b1 (b1+b2) − b3 b1^2
a3 = b1 (b1+b2)^2 + b1 b2 b3
61
25
1 4 9
En (1, 5, 61, 1385, ...) ~ (2n)! c^n
as n → ∞
a1 = 1 → b1 = 1
a2 = 5 → b2 = 4
a3 = 61 → b3 = 9
a4 = 1385 ← 16
bn ~ n^2
if an ~ n!
then bn ~ n
an ~ (2n)!
bn ~ n^2

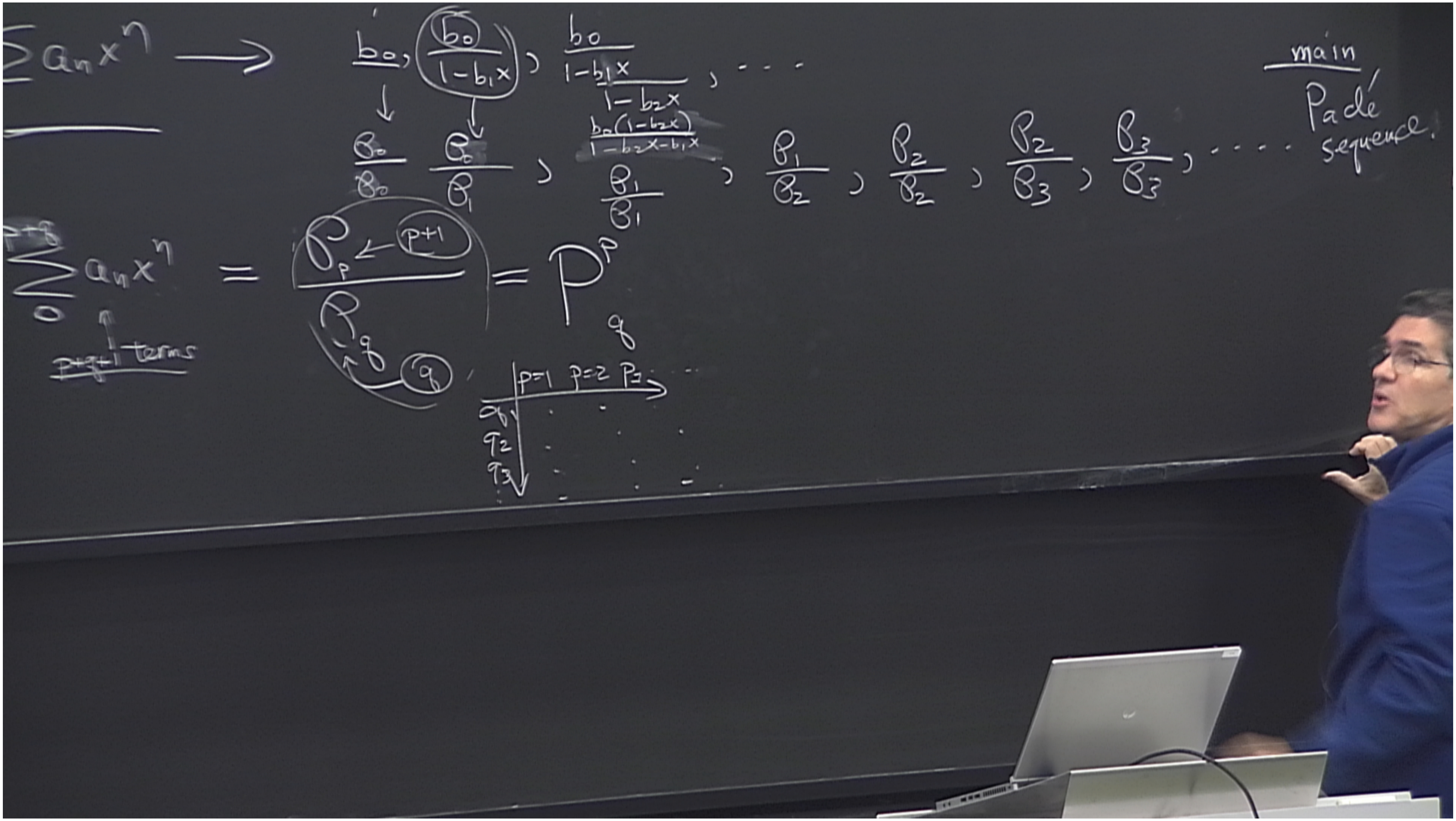

$\sum a_n x^n \longrightarrow b_0, \frac{b_0}{1-b_1 x}, \frac{b_0}{1-\frac{b_1 x}{1-b_2 x}}, \dots$

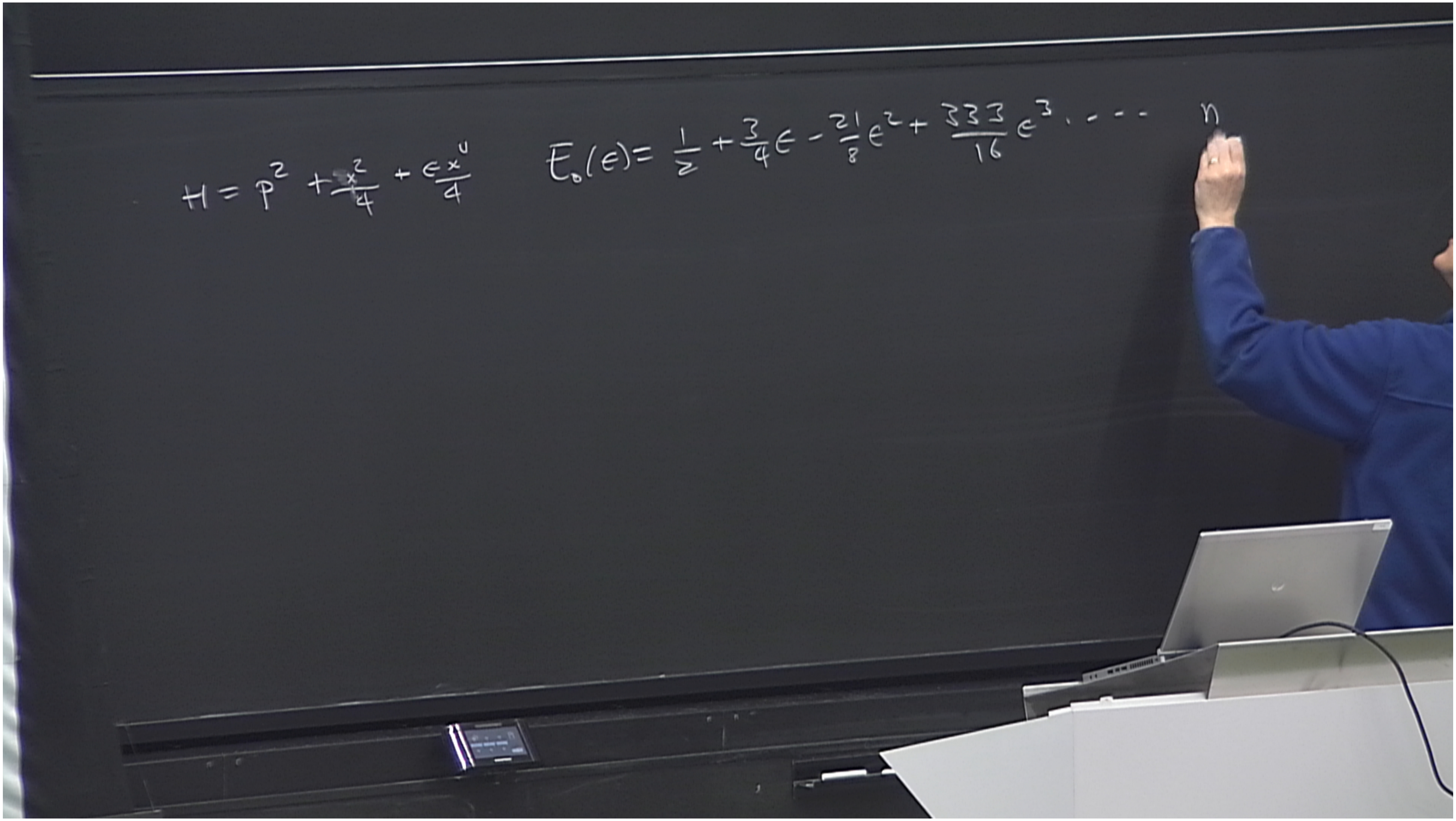

$\sum a_n x^n \longrightarrow$
$b_0, \frac{b_0}{1-b_1x}, \frac{b_0}{1-\frac{b_1x}{1-b_2x}}, \ldots$
$\frac{B_0}{B_0}, \frac{B_0}{B_1}, \frac{B_1}{B_1}, \frac{B_1}{B_2}, \frac{B_2}{B_2}, \frac{B_2}{B_3}, \frac{B_3}{B_3}, \ldots$

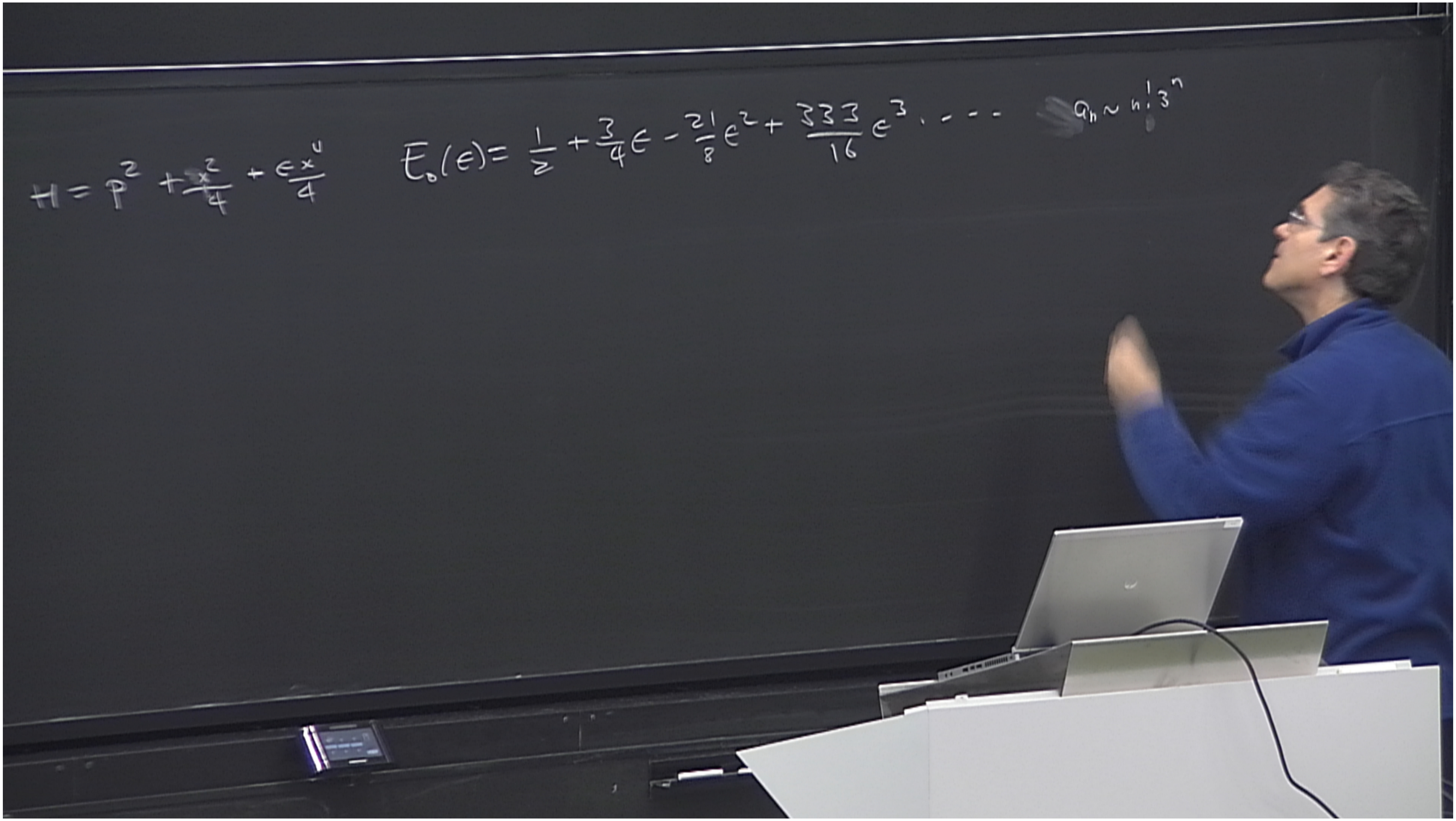

Padé
sequence.


$\sum a_n x^n \longrightarrow b_0, \frac{b_0}{1-b_1x}, \frac{b_0}{1-\frac{b_1x}{1-b_2x}}, \ldots$
$\frac{b_0(1-b_2x)}{1-b_2x-b_1x}$
$\sum_0^{p+q} a_n x^n = \frac{P_p}{Q_q}$
p+q+1 terms

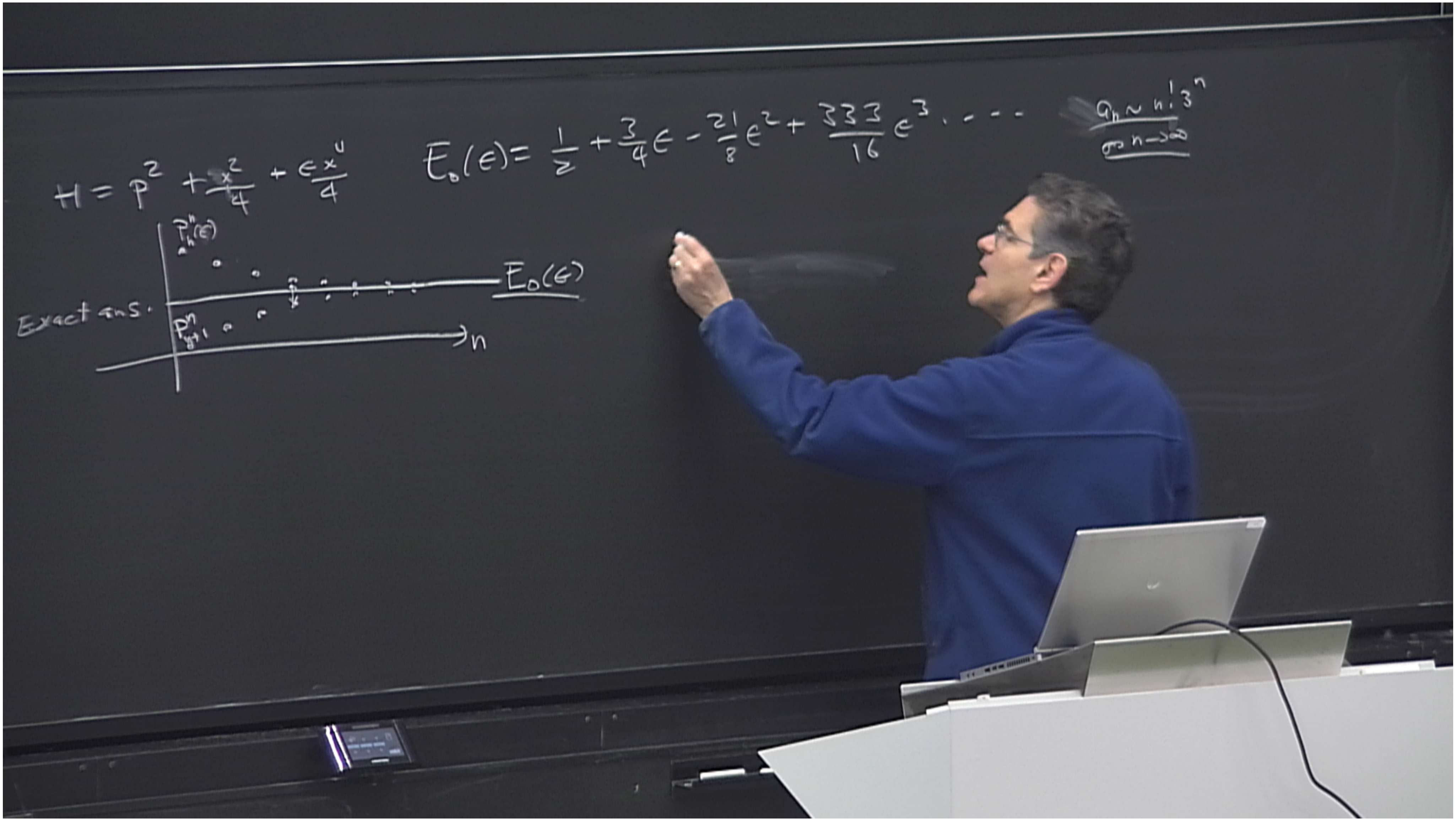

Σ a_n x^n
b_0 , b_0 / (1 - b_1 x) , ...
p+q+1 terms


main
Padé
sequence
p+q+1 terms
p=1 p=2

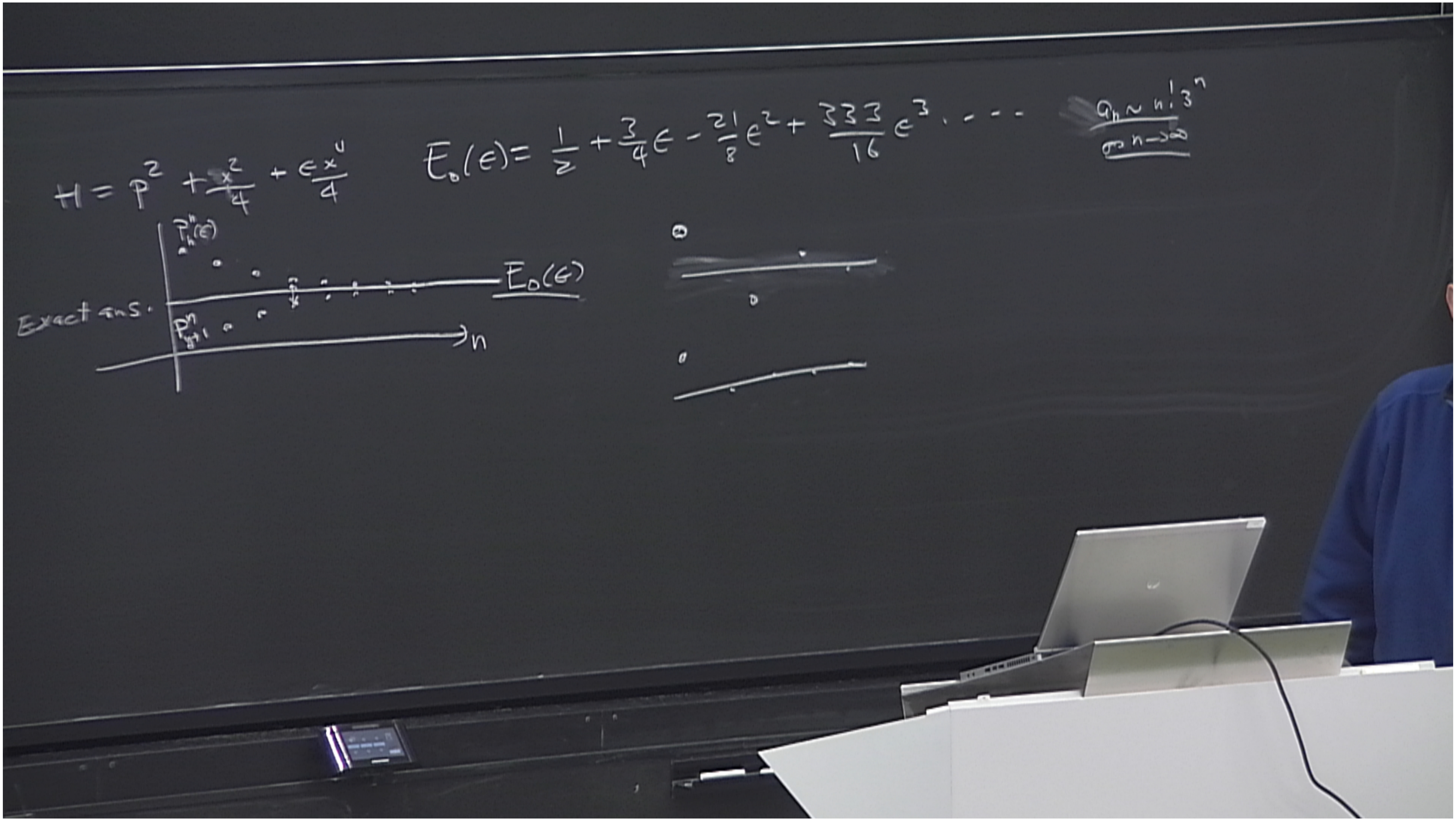

$H = p^2 + \frac{x^2}{4} + \frac{\epsilon x^4}{4}$
$E_0(\epsilon) = \frac{1}{2} + \frac{3}{4}\epsilon - \frac{21}{8}\epsilon^2 + \frac{333}{16}\epsilon^3 \cdots$
n



$H = p^2 + \frac{x^2}{4} + \frac{\epsilon x^4}{4}$
$E_0(\epsilon) = \frac{1}{2} + \frac{3}{4}\epsilon - \frac{21}{8}\epsilon^2 + \frac{333}{16}\epsilon^3 \cdots$
$a_n \sim n!\, 3^n$



$H = p^2 + \frac{x^2}{4} + \frac{\epsilon x^4}{4}$
$E_0(\epsilon) = \frac{1}{2} + \frac{3}{4}\epsilon - \frac{21}{8}\epsilon^2 + \frac{333}{16}\epsilon^3 \cdots$
$a_n \sim n!\,3^n$
$n \to \infty$
$P^n_n(\epsilon)$
$E_0(\epsilon)$
Exact ans.
P^n_{n+1}
n


$H = p^2 + \frac{x^2}{4} + \epsilon \frac{x^4}{4}$
$E_0(\epsilon) = \frac{1}{2} + \frac{3}{4}\epsilon - \frac{21}{8}\epsilon^2 + \frac{333}{16}\epsilon^3 \cdots$
$a_n \sim n!\,3^n$ as $n \to \infty$
Exact ans.
$E_0(\epsilon)$
n

Pade for exp(x)

Table 8.9 A comparison of the convergence rates of the Taylor and Padé approximants to e^x at $x = 1$

The actual values of the approximants and their relative errors [relative error = (approximant − exact value)/exact value] are listed. Observe that (*a*) the Taylor approximants are monotone increasing while the Padé approximants have errors with the sign pattern $++--++--\cdots$, which will be explained in Sec. 8.5; and (*b*) each Padé approximant is constructed from and contains the same information as the Taylor approximant listed on the same line, but its relative error is noticeably smaller. It will be shown in Sec. 8.5 that the error in the Padé approximants is smaller than the error in the Taylor approximants by a factor proportional to 2^n as $n \to +\infty$

			Relative errors	
n	Taylor series $\sum_{k=0}^{n} \frac{1}{k!}$	Padé approximant $P_M^N(1)$	Taylor series	Padé approximant
0	1.0	$P_0^0 = 1$		
1	2.0	$P_1^0 = \infty$	−2.64 (−1)	∞
2	2.5	$P_1^1 = 3$	−8.03 (−2)	1.04 (−1)
3	2.666 67	$P_2^1 = 2.666\ 67$	−1.90 (−2)	−1.90 (−2)
4	2.708 33	$P_2^2 = 2.714\ 29$	−3.66 (−3)	−1.47 (−3)
5	2.716 67	$P_3^2 = 2.718\ 75$	−5.94 (−4)	1.72 (−4)
6	2.718 06	$P_3^3 = 2.718\ 31$	−8.32 (−5)	1.03 (−5)
7	2.718 25	$P_4^3 = 2.718\ 28$	−1.02 (−5)	−8.31 (−7)
8	2.718 28	$P_4^4 = 2.718\ 28$	−1.13 (−6)	−4.05 (−8)
9	2.718 28	$P_5^4 = 2.718\ 28$	−1.11 (−7)	2.48 (−9)
10	2.718 28	$P_5^5 = 2.718\ 28$	−1.00 (−8)	1.02 (−10)
11	2.718 28	$P_6^5 = 2.718\ 28$	−8.32 (−10)	−5.02 (−12)
12	2.718 28	$P_6^6 = 2.718\ 28$	−6.36 (−11)	−1.77 (−13)
13	2.718 28	$P_7^6 = 2.718\ 28$	−4.52 (−12)	7.31 (−15)
14	2.718 28	$P_7^7 = 2.718\ 28$	−3.00 (−13)	2.27 (−16)
15	2.718 28	$P_8^7 = 2.718\ 28$	−1.87 (−14)	−8.03 (−18)
16	2.718 28	$P_8^8 = 2.718\ 28$	−1.09 (−15)	−2.22 (−19)
17	2.718 28	$P_9^8 = 2.718\ 28$	−6.06 (−17)	6.89 (−21)
18	2.718 28	$P_9^9 = 2.718\ 28$	−3.18 (−18)	1.71 (−22)
19	2.718 28	$P_{10}^9 = 2.718\ 28$	−1.59 (−19)	−4.74 (−24)
20	2.718 28	$P_{10}^{10} = 2.718\ 28$	−7.54 (−21)	−1.07 (−25)

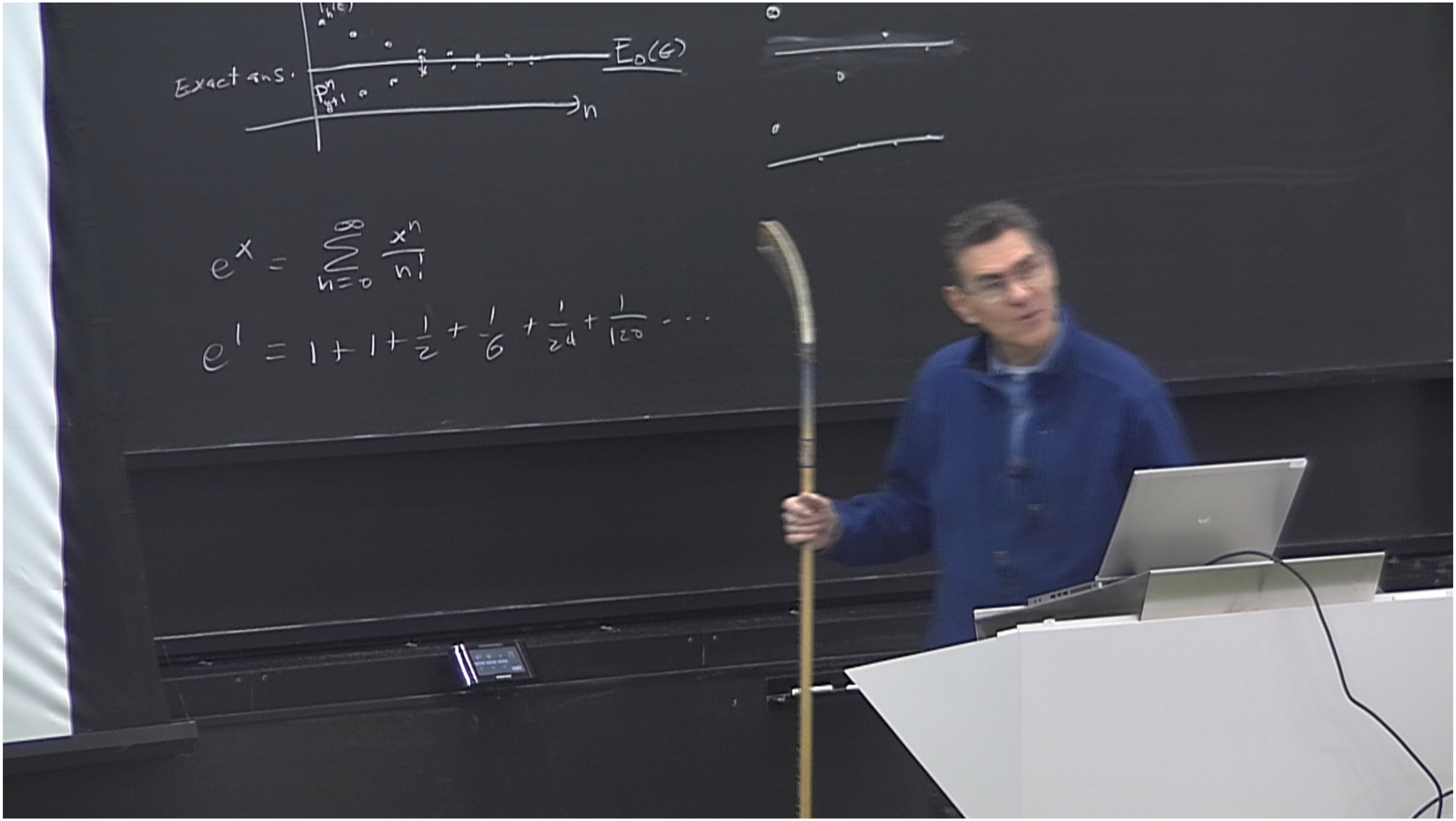

Exact ans.
$E_0(G)$
$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$
$e^1 = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} \cdots$

Table 8.10 Same as in Table 8.9 except that the approximants are evaluated at $x = 5$

n	Taylor series $\sum_{k=0}^{n} \frac{5^k}{k!}$	Padé approximants $P_M^N(5)$	Relative errors: Taylor series	Relative errors: Padé approximants
7	128.619	$P_4^3 =$ 71.385	$-1.33\ (-1)$	$-5.19\ (-1)$
8	138.307	$P_4^4 =$ 128.619	$-6.81\ (-2)$	$-1.33\ (-1)$
9	143.689	$P_5^4 =$ 158.621	$-3.18\ (-2)$	$6.88\ (-2)$
10	146.381	$P_5^5 =$ 149.697	$-1.37\ (-2)$	$8.65\ (-3)$
11	147.604	$P_6^5 =$ 148.001	$-5.45\ (-3)$	$-2.78\ (-3)$
12	148.114	$P_6^6 =$ 148.362	$-2.02\ (-3)$	$-3.43\ (-4)$
13	148.310	$P_7^6 =$ 148.427	$-6.98\ (-4)$	$9.05\ (-5)$
14	148.380	$P_7^7 =$ 148.415	$-2.26\ (-4)$	$1.03\ (-5)$
15	148.403	$P_8^7 =$ 148.413	$-6.90\ (-5)$	$-2.28\ (-6)$
16	148.410	$P_8^8 =$ 148.413	$-1.99\ (-5)$	$-2.41\ (-7)$
17	148.412	$P_9^8 =$ 148.143	$-5.42\ (-6)$	$4.56\ (-8)$
18	148.413	$P_9^9 =$ 148.413	$-1.40\ (-6)$	$4.48\ (-9)$
19	148.413	$P_{10}^9 =$ 148.413	$-3.45\ (-7)$	$-7.43\ (-10)$
20	148.413	$P_{10}^{10} =$ 148.413	$-8.11\ (-8)$	$-6.81\ (-11)$
21	148.413	$P_{11}^{10} =$ 148.413	$-1.82\ (-8)$	$1.00\ (-11)$
22	148.413	$P_{11}^{11} =$ 148.413	$-3.91\ (-9)$	$8.58\ (-13)$
23	148.413	$P_{12}^{11} =$ 148.413	$-8.07\ (-10)$	$-1.13\ (-13)$
24	148.413	$P_{12}^{12} =$ 148.413	$-1.60\ (-10)$	$-9.13\ (-15)$
25	148.413	$P_{13}^{12} =$ 148.413	$-3.05\ (-11)$	$1.09\ (-15)$
26	148.413	$P_{13}^{13} =$ 148.413	$-5.60\ (-12)$	$8.29\ (-17)$

Gamma(x)

Table 8.12 Padé summation of the Taylor series (5.4.4) for $1/\Gamma(x)$

The entries in the table were computed as follows. The truncated Taylor series for $1/[x\Gamma(x)]$ was used to construct the Padé approximants $P_N^N(x)$ and $P_{N+1}^N(x)$; the resulting Padé approximants were then inverted and multiplied by x. It is clear from the above table that the entries in the three columns are approaching their correct limits: $\Gamma(1)=1$, $\Gamma(2)=1$, and $\Gamma(4)=6$. Although Padé summation enhances the convergence of the Taylor series, the effect is not dramatic: the Padé sequence requires about half the number of terms that the Taylor series requires to achieve 1 percent accuracy (see Table 5.2). The slow convergence of the Taylor series is not due to the presence of singularities [$1/\Gamma(x)$ is entire], but rather to the enormous disparity of behavior of $1/\Gamma(x)$ for positive and negative x. Padé approximants improve the convergence of series by mimicking the singularities of the function to be represented; since $1/\Gamma(x)$ has no singularities, the benefits of Padé summation are marginal

$1/[xP_N^N(x)]$

N	$x=1$	$x=2$	$x=4$
1	0.787 279 606 6	0.369 614 402 3	0.176 506 593 3
2	0.993 980 121 1	0.983 777 381 1	−1.076 767 507 8
3	1.002 208 342 9	1.148 824 428 2	−0.501 000 407 3
4	0.999 993 179 2	0.996 095 931 1	1.662 143 630 6
5	0.999 992 416 9	0.995 977 024 8	1.648 553 231 8
6	1.000 000 023 8	1.000 087 294 7	9.332 982 540 2
7	0.999 999 998 8	0.999 982 355 2	4.271 068 387 3
8	1.000 000 000 0	0.999 999 507 8	5.891 861 437 4
9	1.000 000 000 0	1.000 000 001 6	6.016 004 972 1
10	1.000 000 000 0	1.000 000 000 1	6.001 556 368 7

$1/[xP_{N+1}^N(x)]$

N	$x=1$	$x=2$	$x=4$
1	0.938 583 793 1	0.620 521 015 8	0.519 926 684 9
2	1.012 480 430 2	4.152 436 398 7	0.085 466 832 0
3	1.000 317 915 9	1.035 343 862 7	−1.821 772 429 8
4	0.999 989 665 5	0.995 106 544 4	1.465 422 290 2
5	0.999 999 639 8	0.999 508 399 5	3.822 815 515 8
6	1.000 000 004 4	1.000 026 327 2	7.035 967 528 8
7	1.000 000 000 0	1.000 001 311 6	6.095 643 764 4
8	1.000 000 000 0	0.999 999 880 7	5.960 840 282 9
9	1.000 000 000 0	1.000 000 001 1	6.002 325 742 5
10	1.000 000 000 0	1.000 000 000 0	6.002 625 162 0

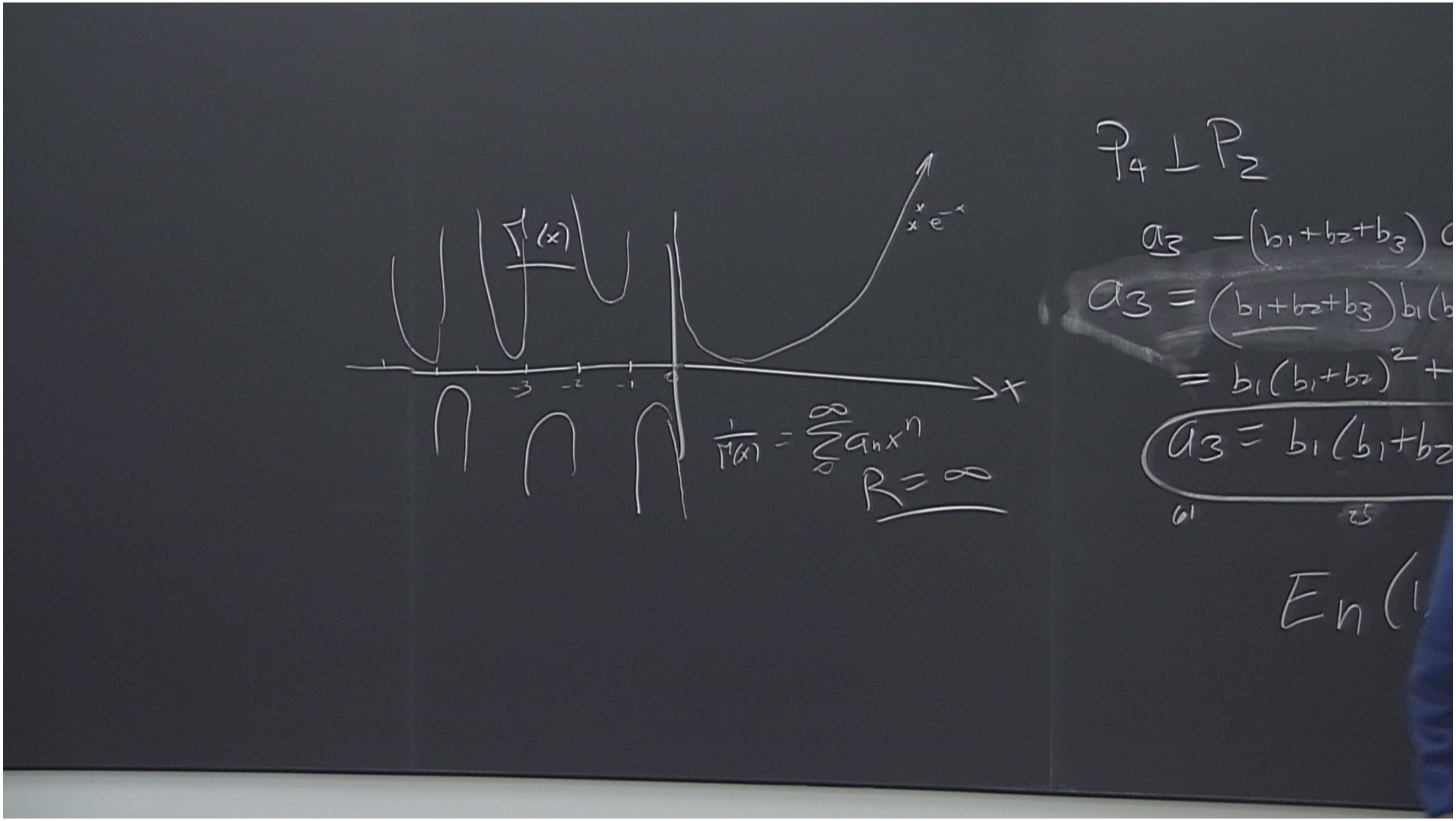

Γ(x)
x^x e^{-x}
1/Γ(x) = Σ a_n x^n
R = ∞
P4 ⊥ P2
a3 −(b1+b2+b3)
a3 = (b1+b2+b3) b1(b
= b1(b1+b2)^2 +
a3 = b1(b1+b2
En(

Table 8.13 Padé summation of the Taylor series for $z^{-1} \ln(1+z)$ about $z=0$

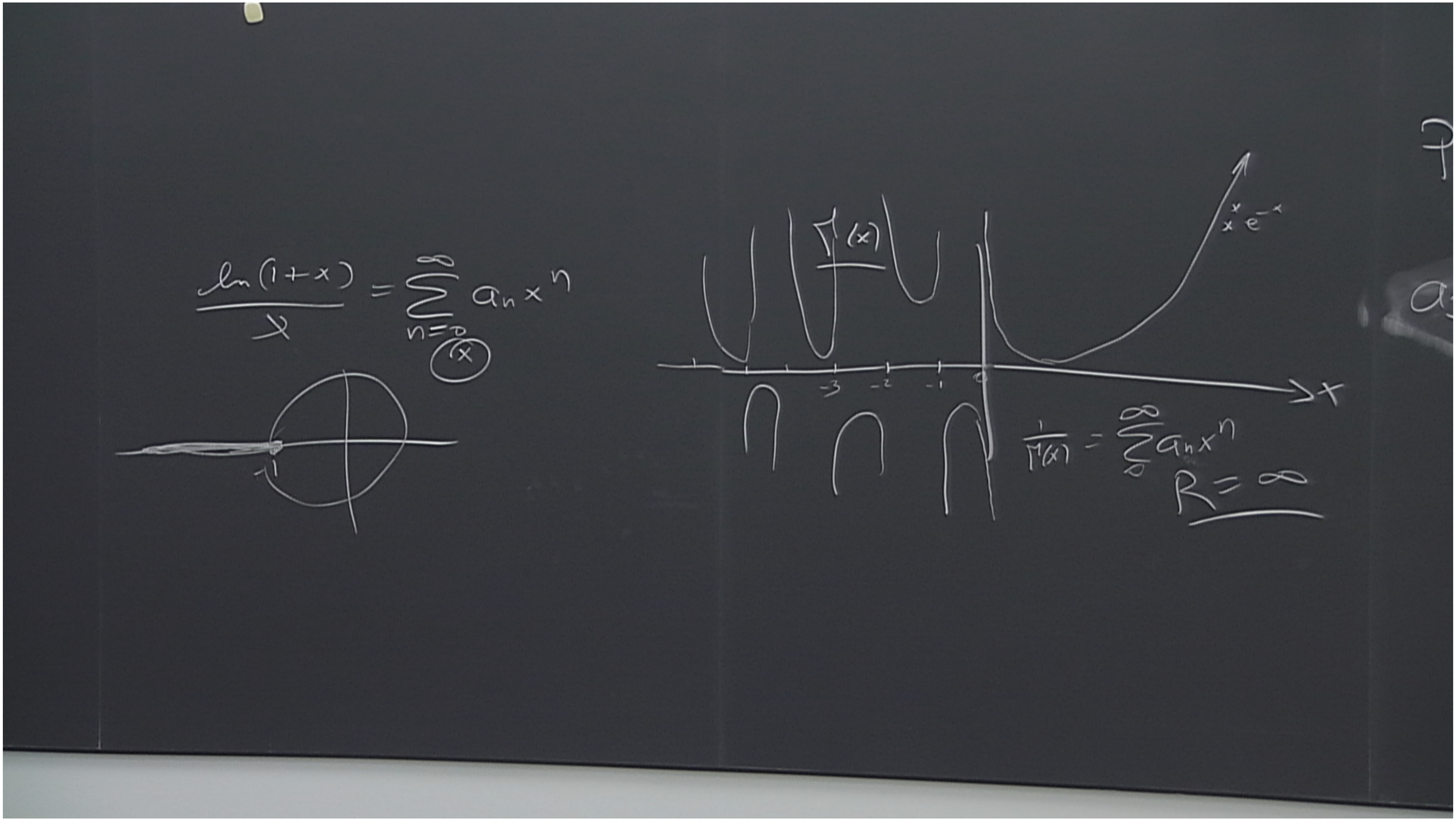
In Sec. 8.5, it will be shown that the sequences of Padé approximants $P_N^N(z)$ and $P_{N+1}^N(z)$ converge rapidly, even beyond the circle of convergence of the Taylor series $|z| < 1$. Observe that for real positive x the Padé approximants $P_N^N(x)$ monotonically decrease and $P_{N+1}^N(x)$ monotonically increase with N to the common limit $\ln(1+x)/x$. Thus, for any N, these Padé approximants supply upper and lower bounds on $\ln(1+x)/x$

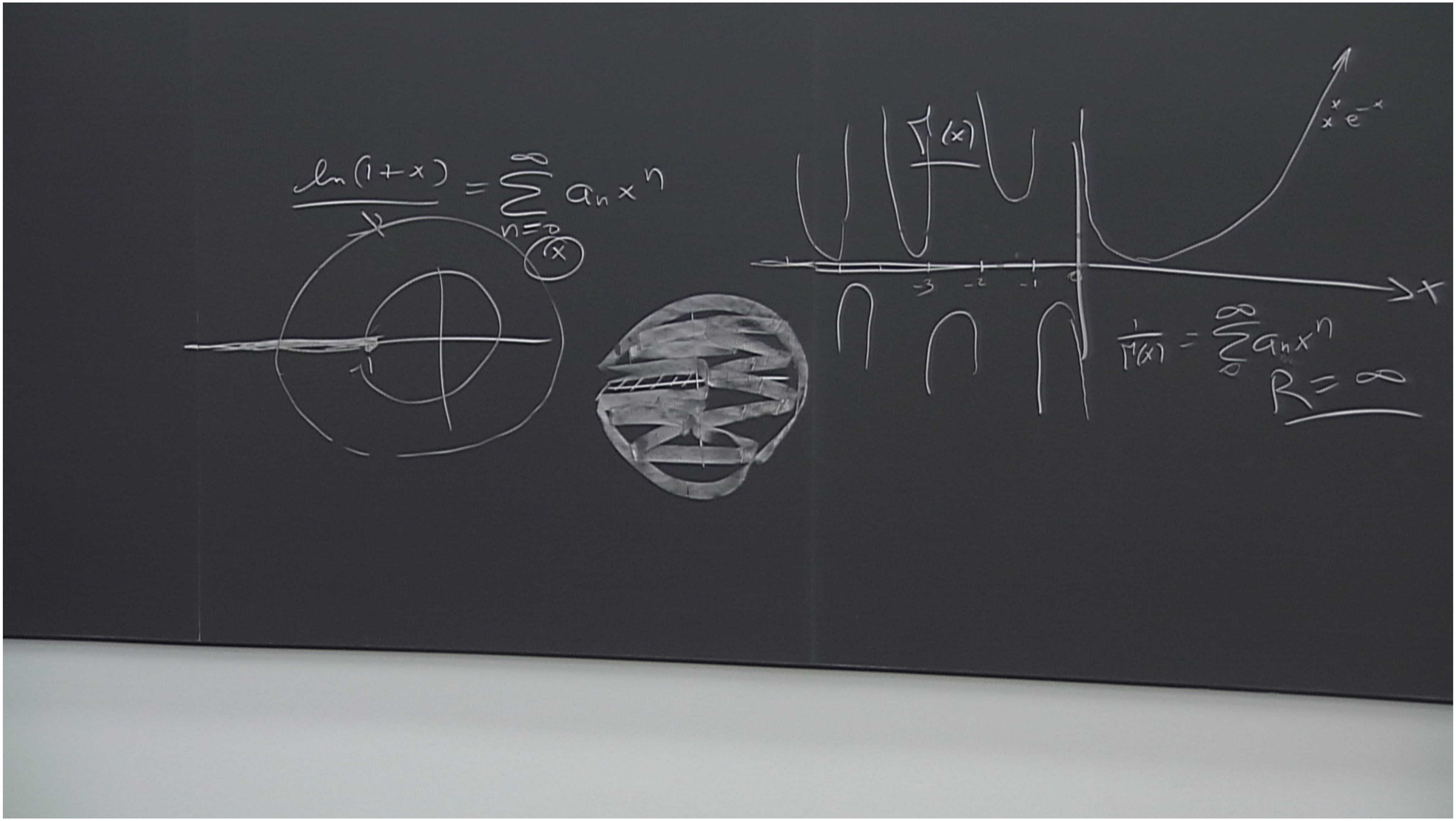
$P_N^N(x)$

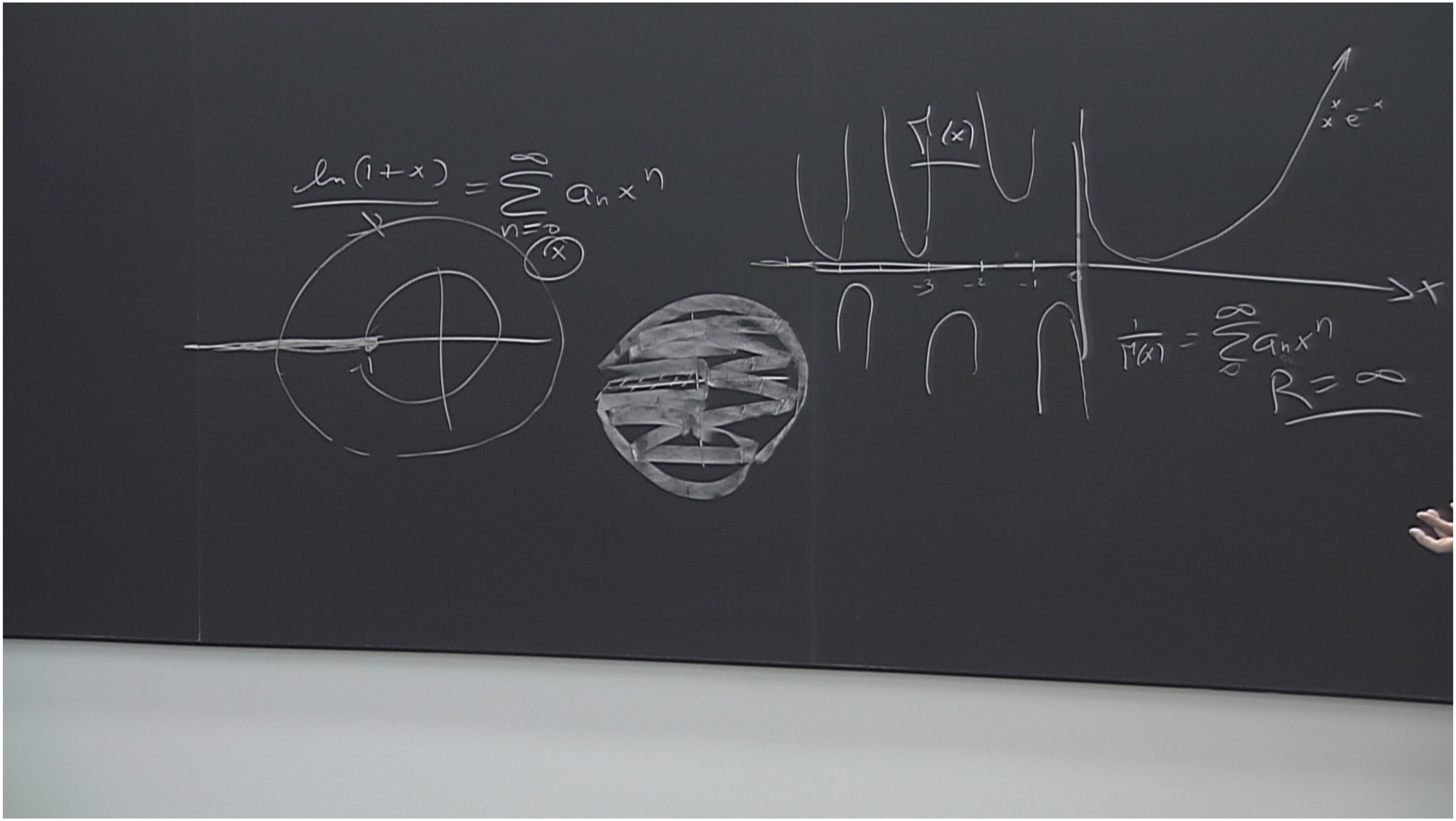
N	$x = 0.5$	$x = 1$	$x = 2$
1	0.812 500 000 0	0.700 000 000 0	0.571 428 571 4
2	0.810 945 273 6	0.693 333 333 3	0.550 724 637 7
3	0.810 930 365 3	0.693 152 454 8	0.549 402 823 0
4	0.810 930 217 7	0.693 147 332 4	0.549 312 879 5
5	0.810 930 216 2	0.693 147 185 0	0.549 306 618 4
6	0.810 930 216 2	0.693 147 180 7	0.549 306 177 9
7	0.810 930 216 2	0.693 147 180 6	0.549 306 146 7
8	0.810 930 216 2	0.693 147 180 6	0.549 306 144 5

$P_{N+1}^N(x)$

N	$x = 0.5$	$x = 1$	$x = 2$
1	0.810 810 810 8	0.692 307 692 3	0.545 454 545 5
2	0.810 928 961 7	0.693 121 693 1	0.549 019 607 8
3	0.810 930 203 2	0.693 146 417 4	0.549 285 176 8
4	0.810 930 216 1	0.693 147 157 9	0.549 304 620 9
5	0.810 930 216 2	0.693 147 179 9	0.549 306 034 1
6	0.810 930 216 2	0.693 147 180 5	0.549 306 136 4
7	0.810 930 216 2	0.693 147 180 6	0.549 306 143 8
8	0.810 930 216 2	0.693 147 180 6	0.549 306 144 3


ln(1+x)/x = Σ_{n=0}^∞ a_n x^n
-1
Γ(x)
x^x e^{-x}
-3 -2 -1 0
x
1/Γ(x) = Σ_0^∞ a_n x^n
R = ∞


$\frac{\ln(1+x)}{x} = \sum_{n=0}^{\infty} a_n x^n$
$\Gamma(x)$
$x^x e^{-x}$
$\frac{1}{\Gamma(x)} = \sum_0^{\infty} a_n x^n$
$R = \infty$



ln(1+x) / x = Σ_{n=0}^∞ a_n x^n
x
-1
Γ(x)
x^x e^{-x}
-3 -2 -1 0
x
1/Γ(x) = Σ_0^∞ a_n x^n
R = ∞