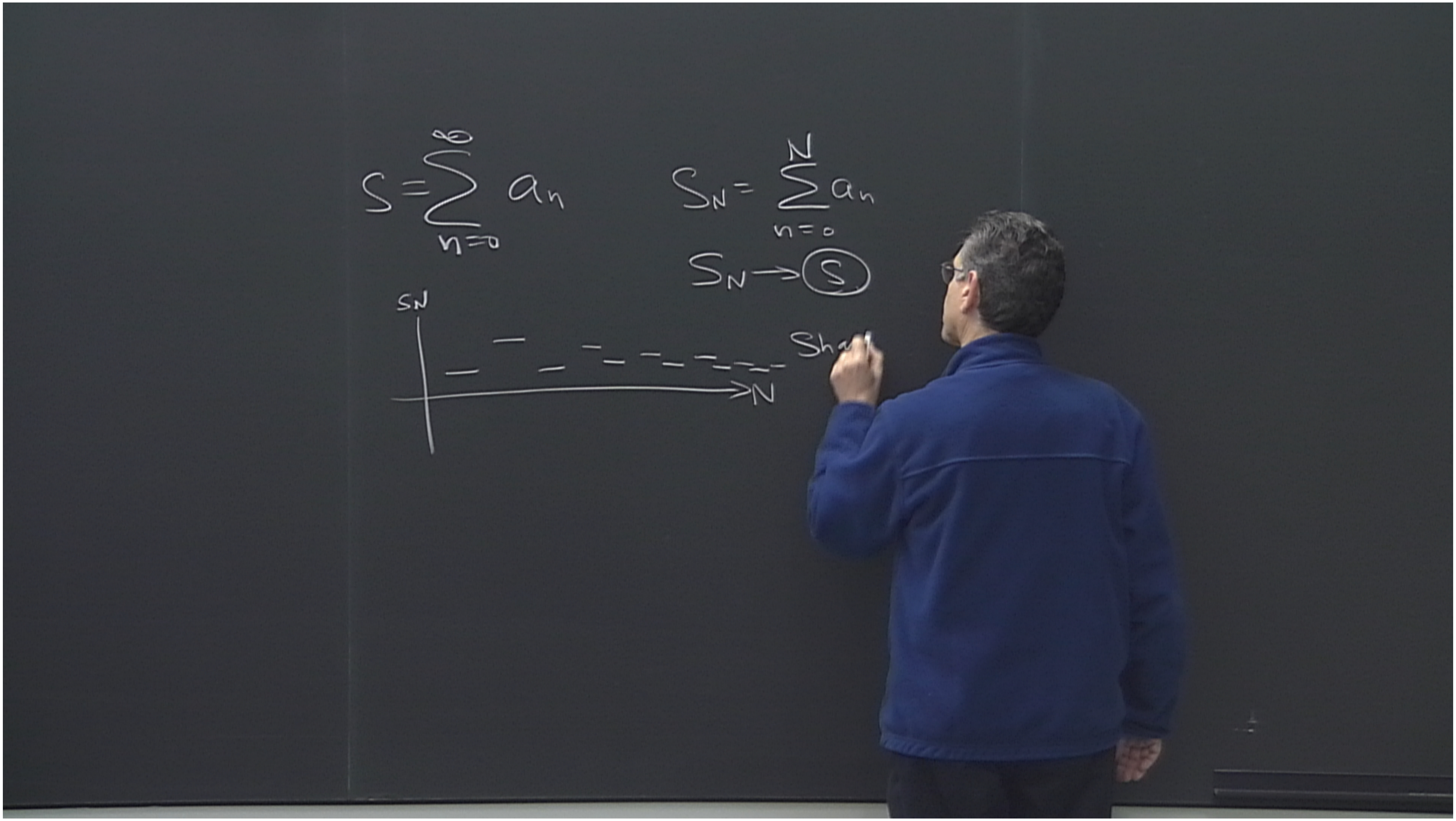
Title: Mathematical Physics - Lecture 4

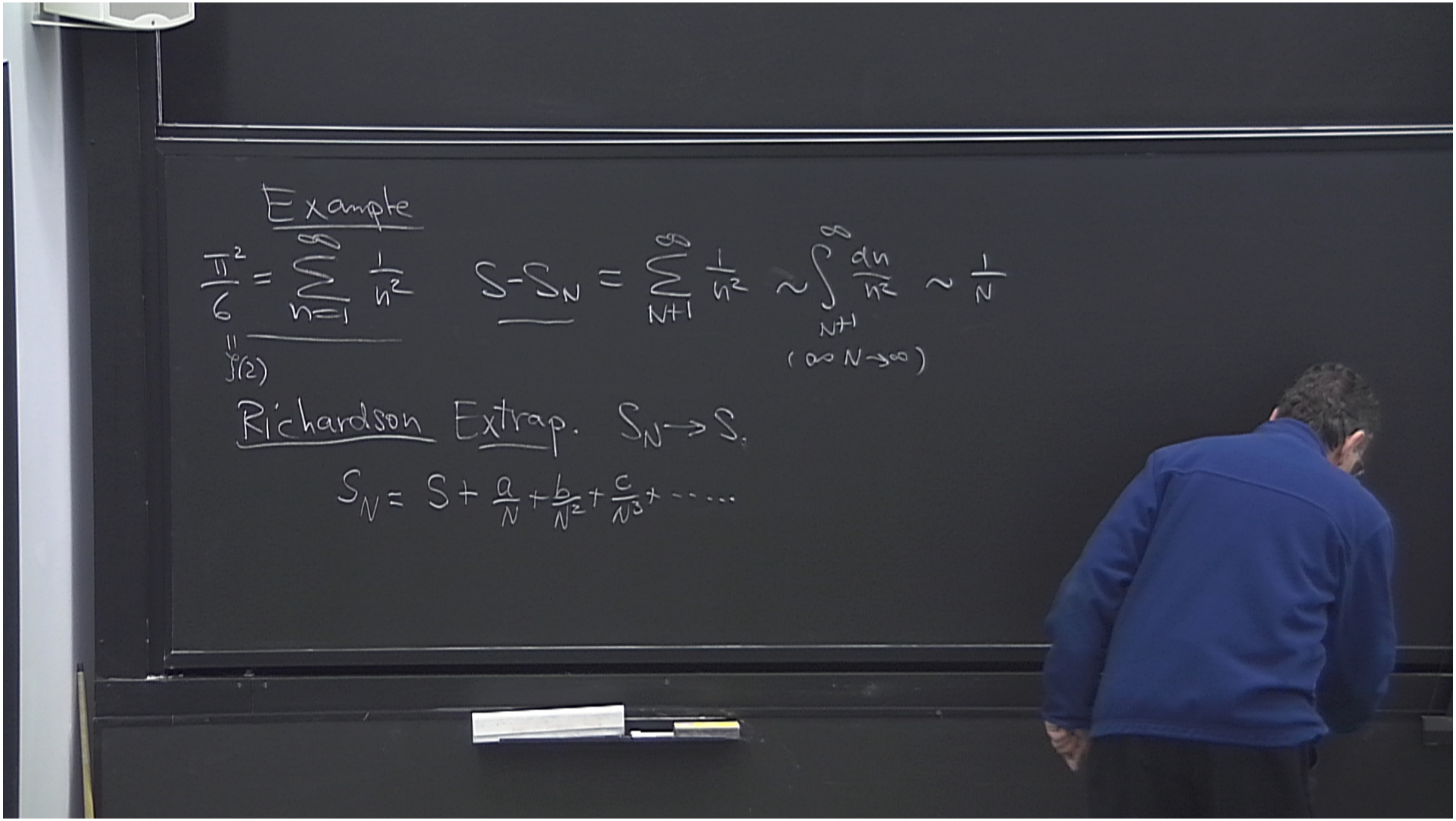
Date: Nov 24, 2011 09:00 AM

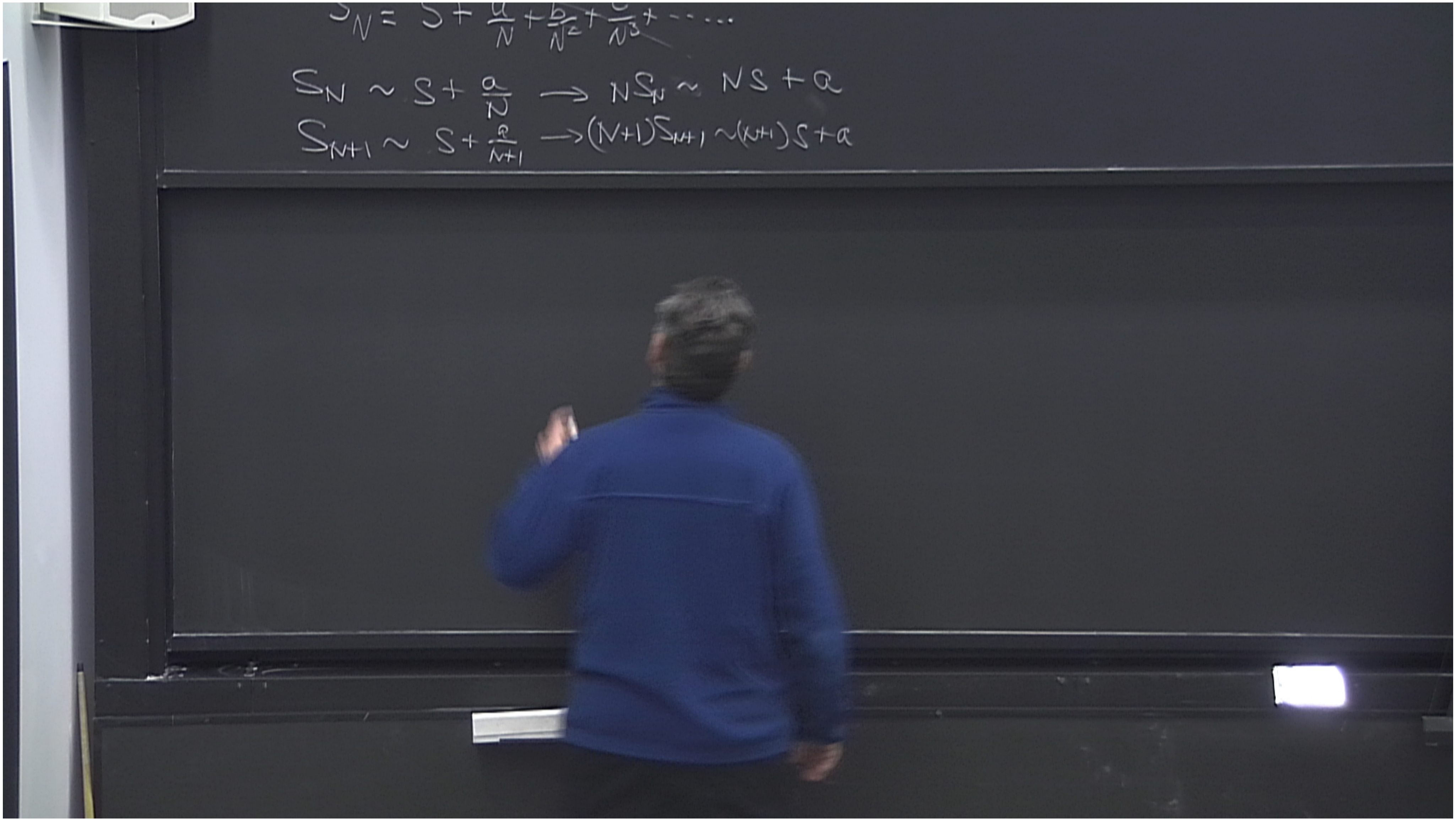
URL: http://pirsa.org/11110043

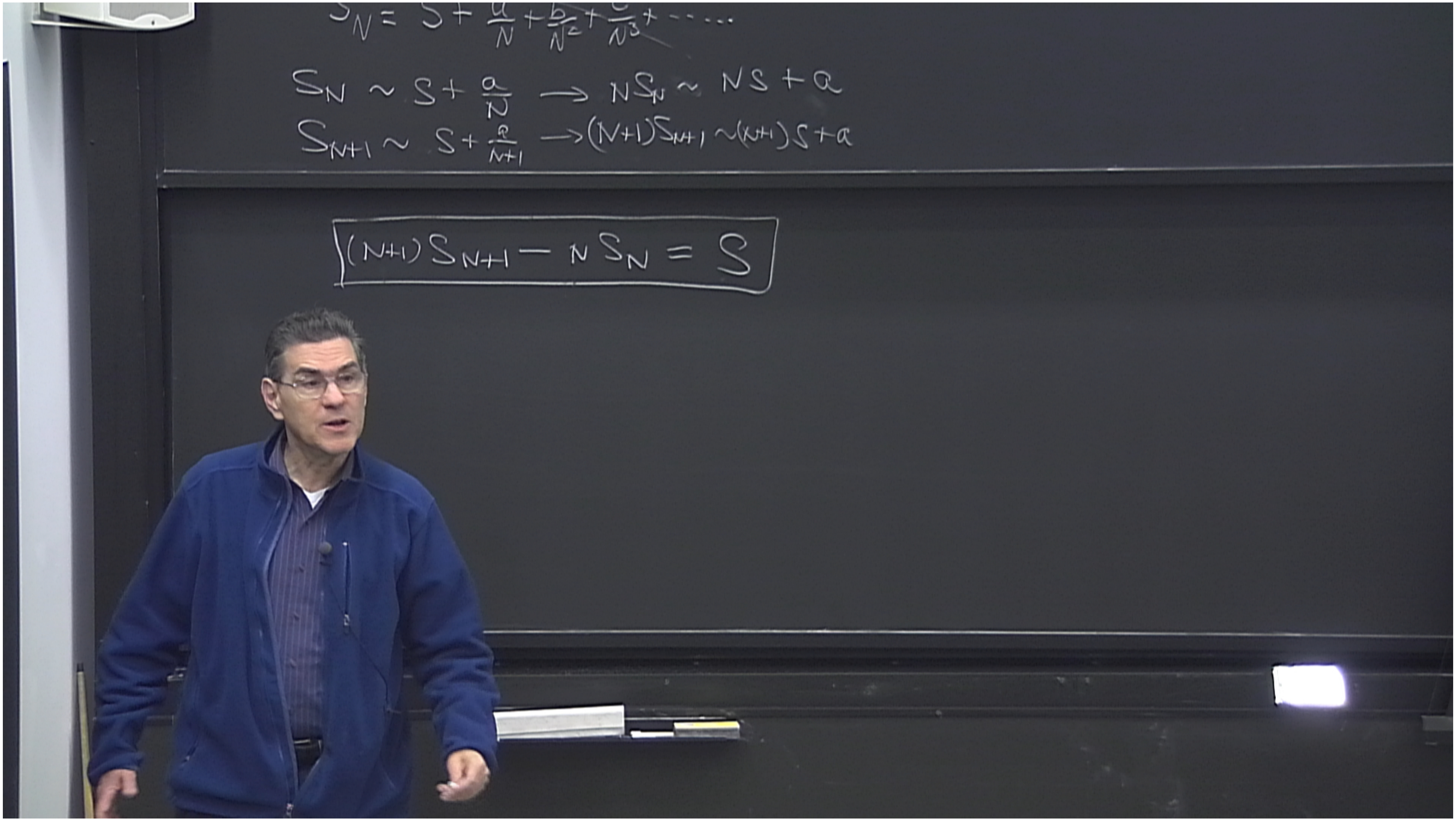
Abstract:

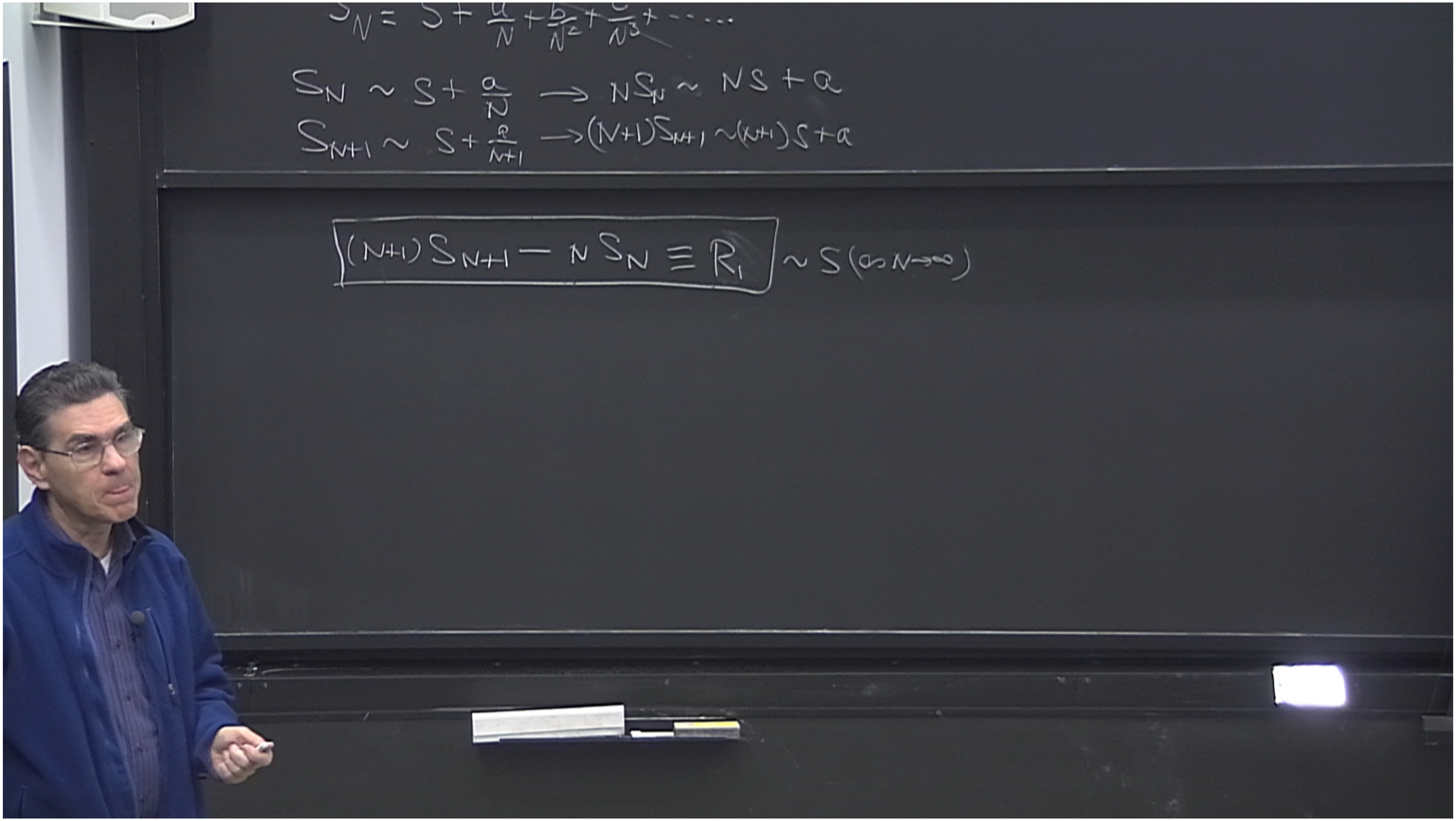


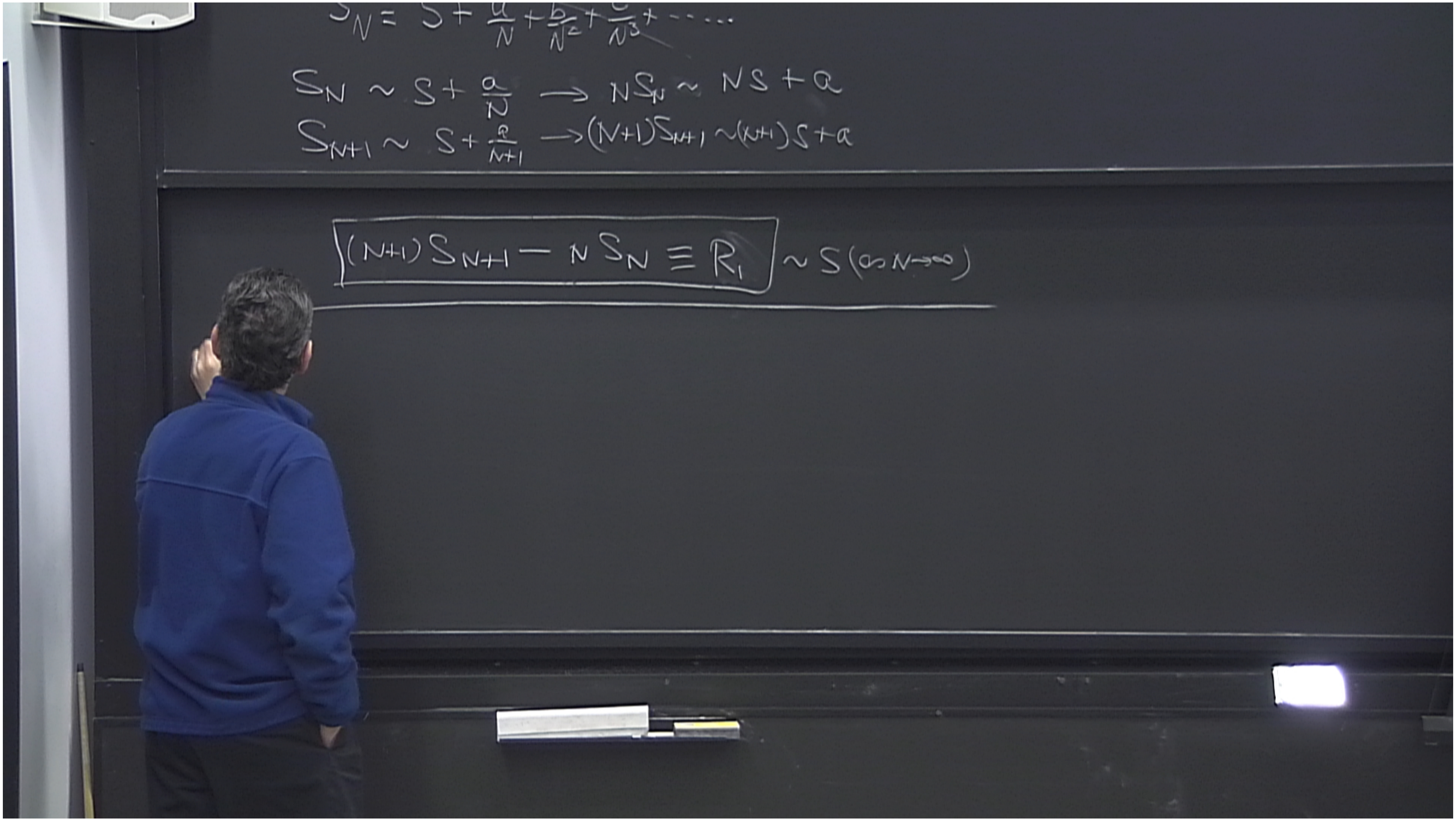
$S = \sum_{n=0}^{\infty} a_n$
$S_N = \sum_{n=0}^{N} a_n$
$S_N \to S$
S_N
N


Example
$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$
$= \zeta(2)$
$S - S_N = \sum_{N+1}^{\infty} \frac{1}{n^2} \sim \int_{N+1}^{\infty} \frac{dn}{n^2} \sim \frac{1}{N}$
$(\infty\ N \to \infty)$
Richardson Extrap. $S_N \to S$.
$S_N = S + \frac{a}{N} + \frac{b}{N^2} + \frac{c}{N^3} + \cdots$


$S_N = S + \frac{a}{N} + \frac{b}{N^2} + \frac{c}{N^3} + \cdots$
$S_N \sim S + \frac{a}{N} \longrightarrow N S_N \sim NS + a$
$S_{N+1} \sim S + \frac{a}{N+1} \longrightarrow (N+1) S_{N+1} \sim (N+1) S + a$


S_N = S + a/N + b/N^2 + c/N^3 + ...
S_N ~ S + a/N → N S_N ~ N S + a
S_{N+1} ~ S + a/(N+1) → (N+1) S_{N+1} ~ (N+1) S + a
(N+1) S_{N+1} − N S_N = S


S_N = S + a/N + b/N^2 + c/N^3 + ...
S_N ~ S + a/N → N S_N ~ NS + a
S_{N+1} ~ S + a/(N+1) → (N+1) S_{N+1} ~ (N+1) S + a
(N+1) S_{N+1} − N S_N ≡ R_1 ~ S (as N→∞)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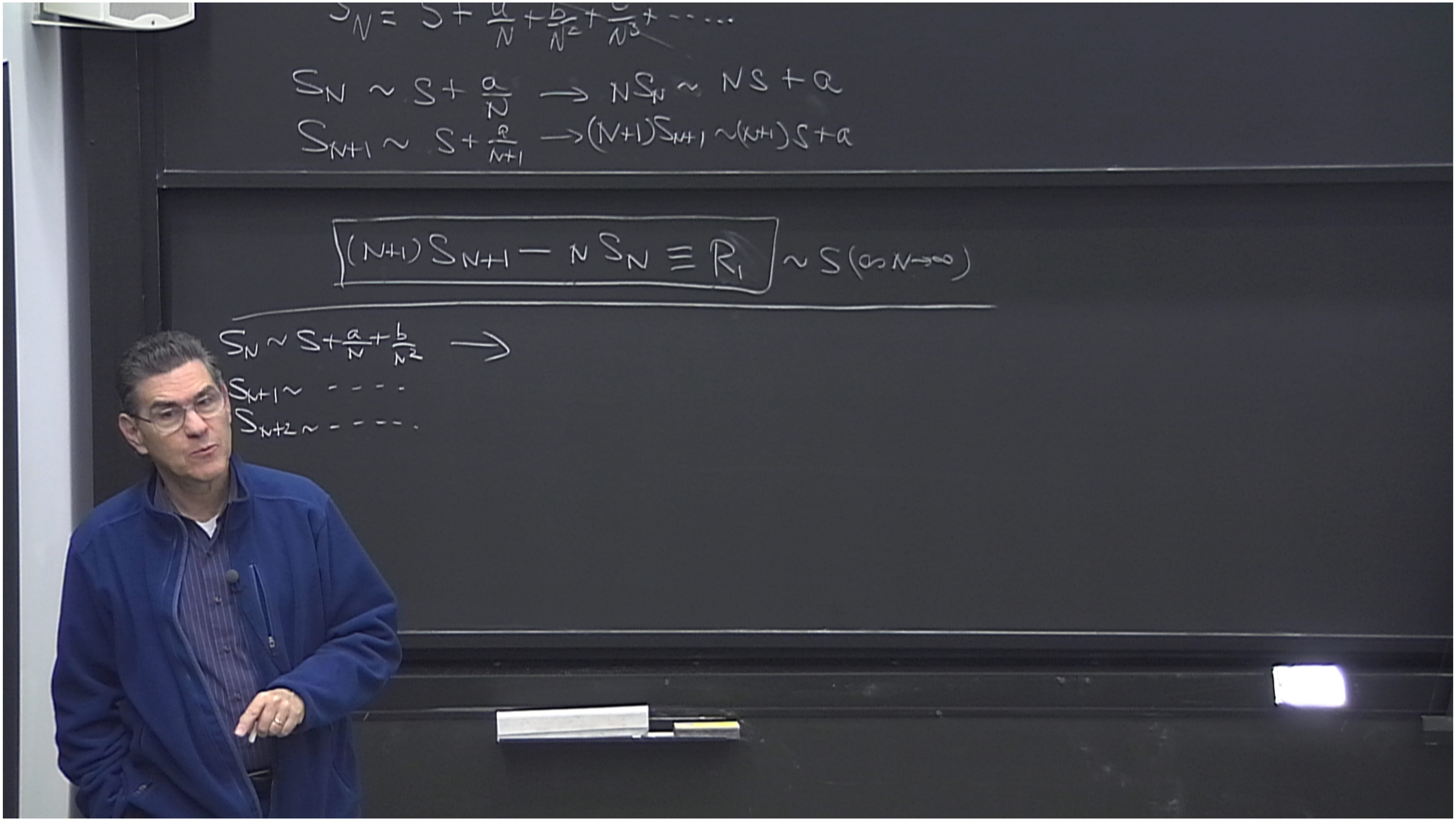
$(N+1) S_{N+1} - N S_N \equiv R_1 \sim S \ (\text{as } N \to \infty)$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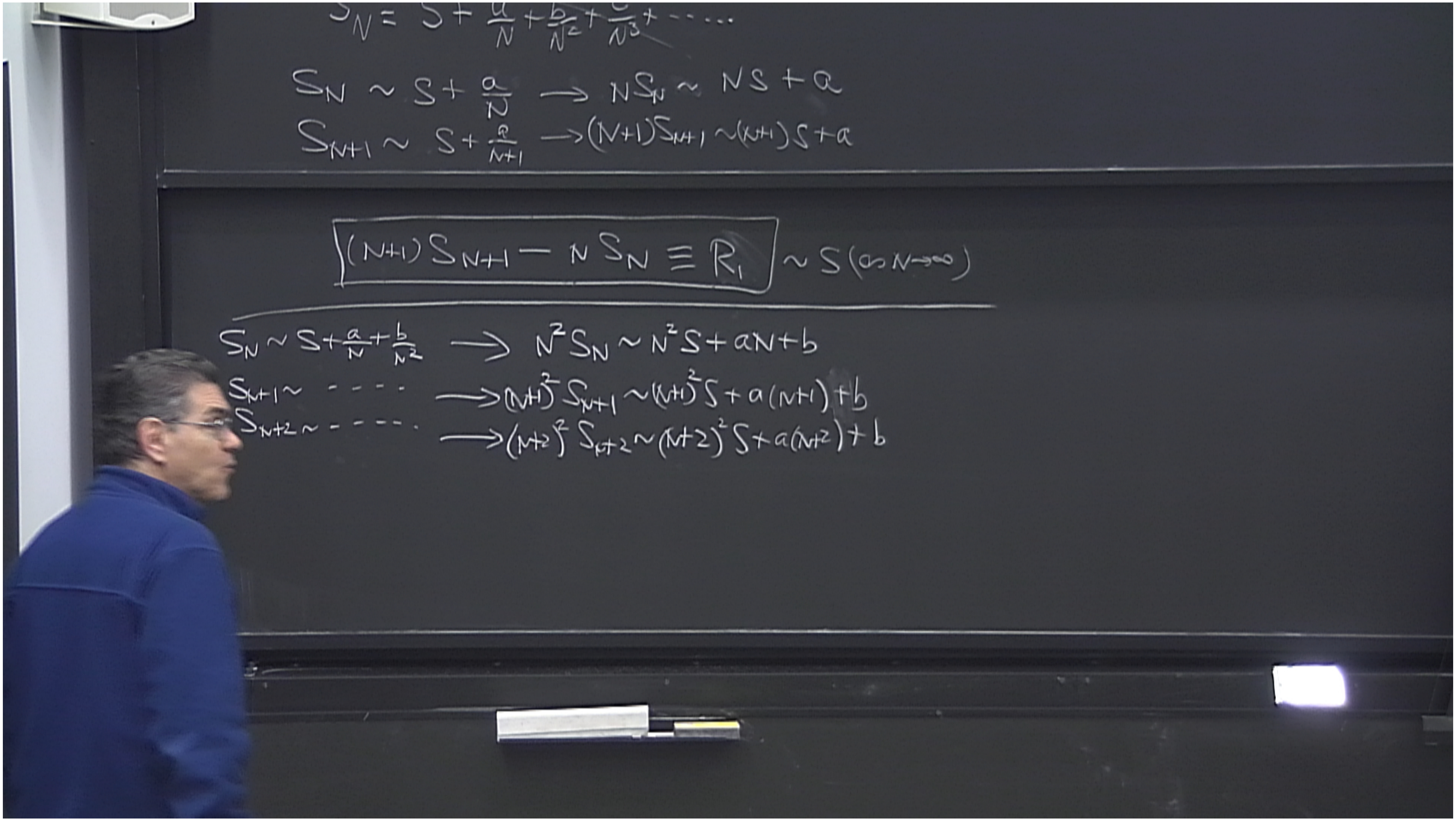
$(N+1) S_{N+1} - N S_N \equiv R_1 \sim S \text{ (as } N \to \infty)$
$S_N \sim S + \frac{a}{N} + \frac{b}{N^2} \longrightarrow$
$S_{N+1} \sim \cdots$
$S_{N+2} \sim \cdots$



$S_N = S + \frac{a}{N} + \frac{b}{N^2} + \frac{c}{N^3} + \cdots$
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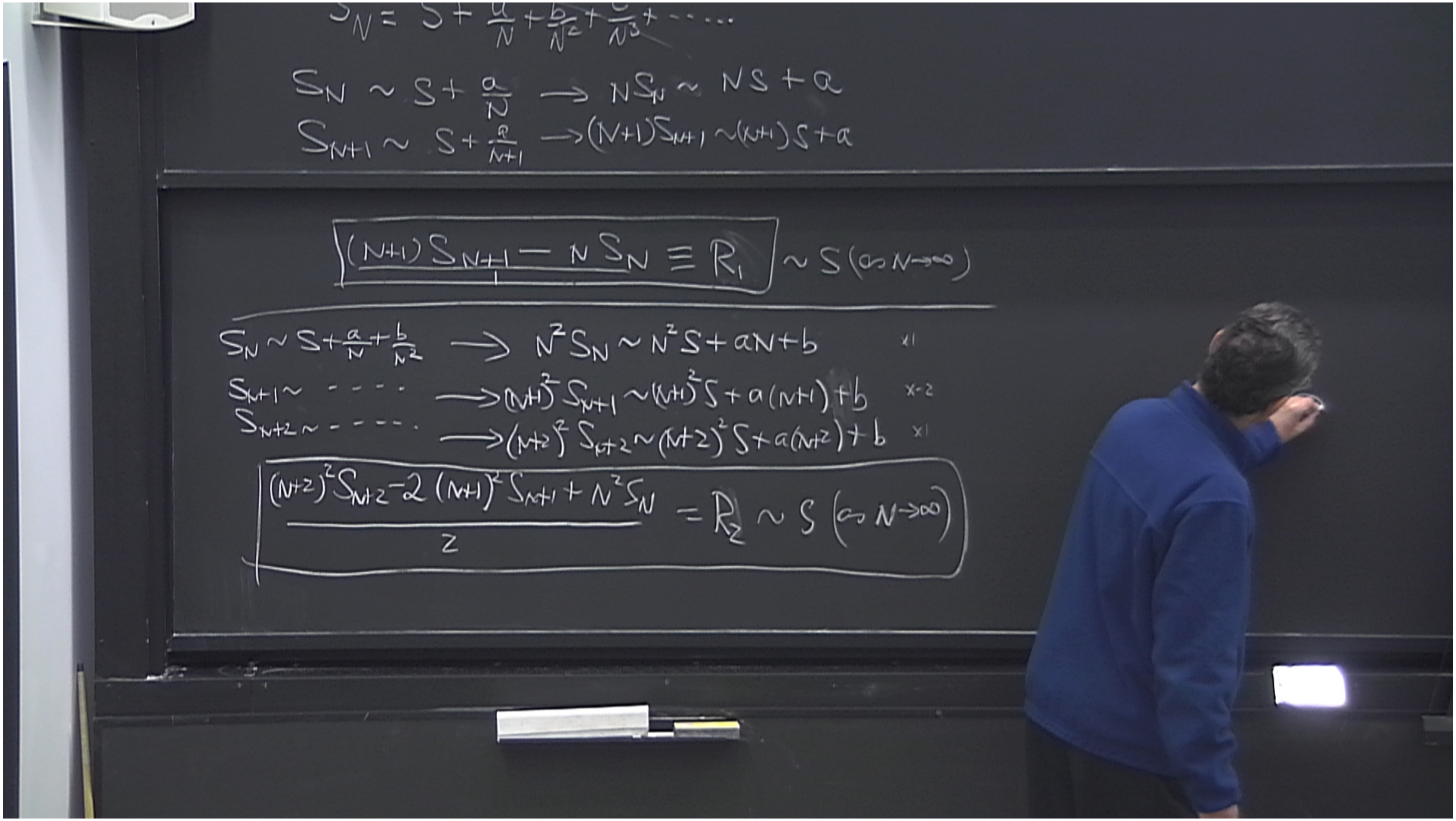
$S_N \sim S + \frac{a}{N} + \frac{b}{N^2} \longrightarrow N^2 S_N \sim N^2 S + aN + b$
$S_{N+1} \sim \cdots \longrightarrow (N+1)^2 S_{N+1} \sim (N+1)^2 S + a(N+1) + b$
$S_{N+2} \sim \cdots \longrightarrow (N+2)^2 S_{N+2} \sim (N+2)^2 S + a(N+2) + b$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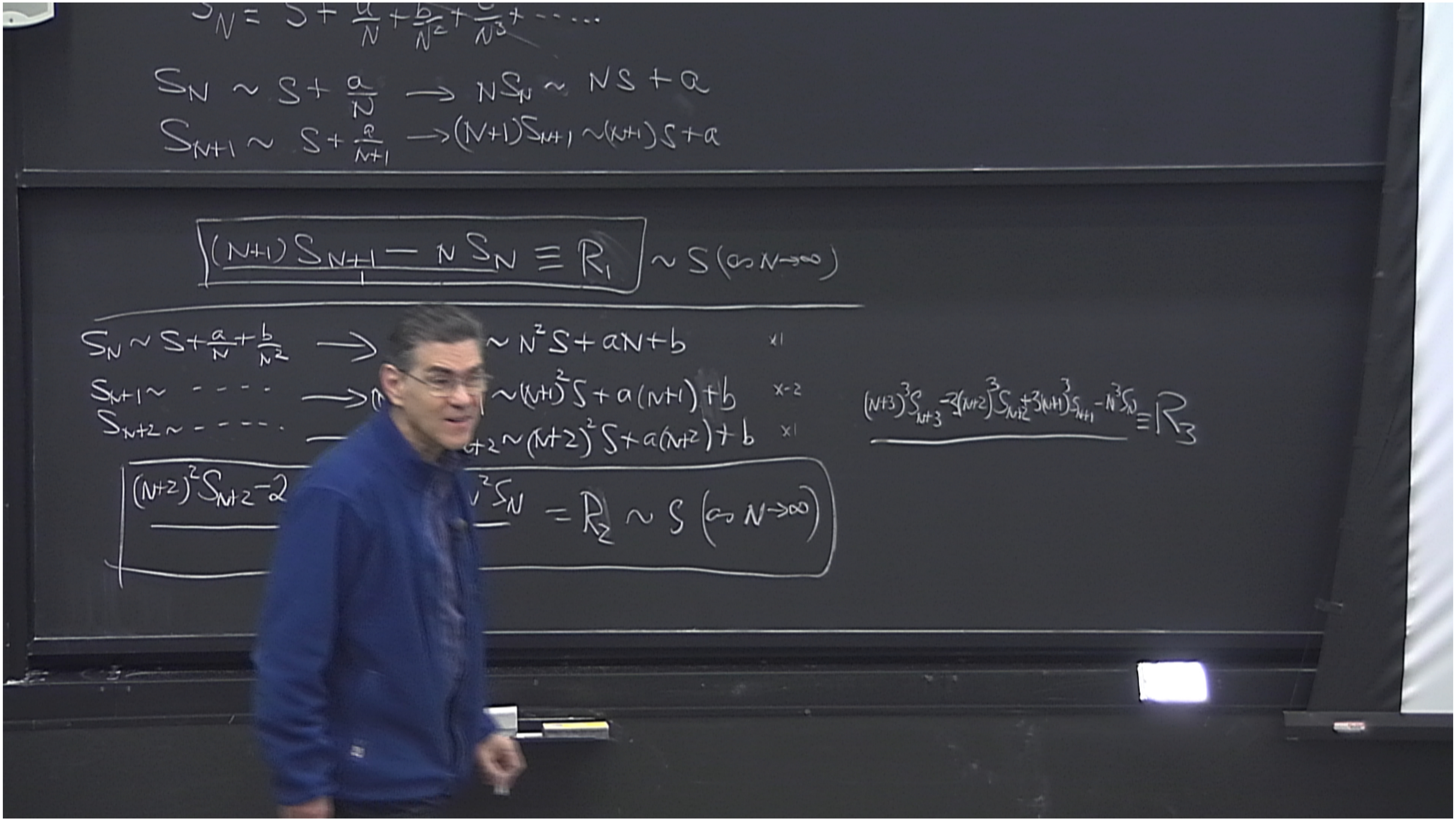
$(N+1) S_{N+1} - N S_N \equiv R_1 \sim S \; (\text{as } N \to \infty)$
$S_N \sim S + \frac{a}{N} + \frac{b}{N^2} \rightarrow N^2 S_N \sim N^2 S + aN + b \quad \times 1$
$S_{N+1} \sim \cdots \rightarrow (N+1)^2 S_{N+1} \sim (N+1)^2 S + a(N+1) + b \quad \times -2$
$S_{N+2} \sim \cdots \rightarrow (N+2)^2 S_{N+2} \sim (N+2)^2 S + a(N+2) + b \quad \times 1$
$\frac{(N+2)^2 S_{N+2} - 2 (N+1)^2 S_{N+1} + N^2 S_N}{2} = R_2 \sim S \; (\text{as } N \to \infty)$



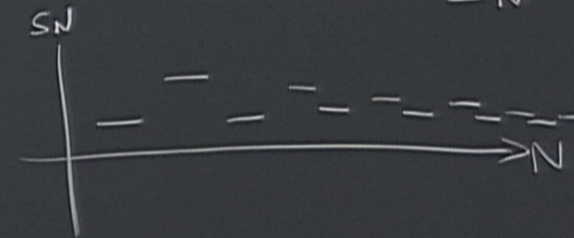
S_N ~ S + a/N → N S_N ~ N S + a
S_{N+1} ~ S + a/(N+1) → (N+1) S_{N+1} ~ (N+1) S + a
(N+1) S_{N+1} − N S_N ≡ R_1 ~ S (as N→∞)
S_N ~ S + a/N + b/N^2 → ~ N^2 S + aN + b ×1
S_{N+1} ~ ~ (N+1)^2 S + a(N+1) + b ×−2
S_{N+2} ~ ~ (N+2)^2 S + a(N+2) + b ×1
= R_2 ~ S (as N→∞)
(N+3)^3 S_{N+3} − 3(N+2)^3 S_{N+2} + 3(N+1)^3 S_{N+1} − N^3 S_N ≡ R_3

$f(n)$

$f'(n) \sim f(n+1) - f(n)$ as $n \to \infty$

$$S = \sum_{n=0}^{\infty} a_n \qquad S_N = \sum_{n=0}^{N} a_n$$

$$S_N \to S$$



Shanks.

$$S_N = S + \frac{a}{N} + \frac{b}{N^2} + \frac{c}{N^3} + \cdots$$

$$S_N \sim S + \frac{a}{N} \longrightarrow N S_N \sim N S + a$$

$$S_{N+1} \sim S + \frac{a}{N+1} \longrightarrow (N+1) S_{N+1} \sim (N+1) S + a$$

$$\boxed{(N+1) S_{N+1} - N S_N \equiv R_1} \sim S \ (\text{as } N \to \infty)$$

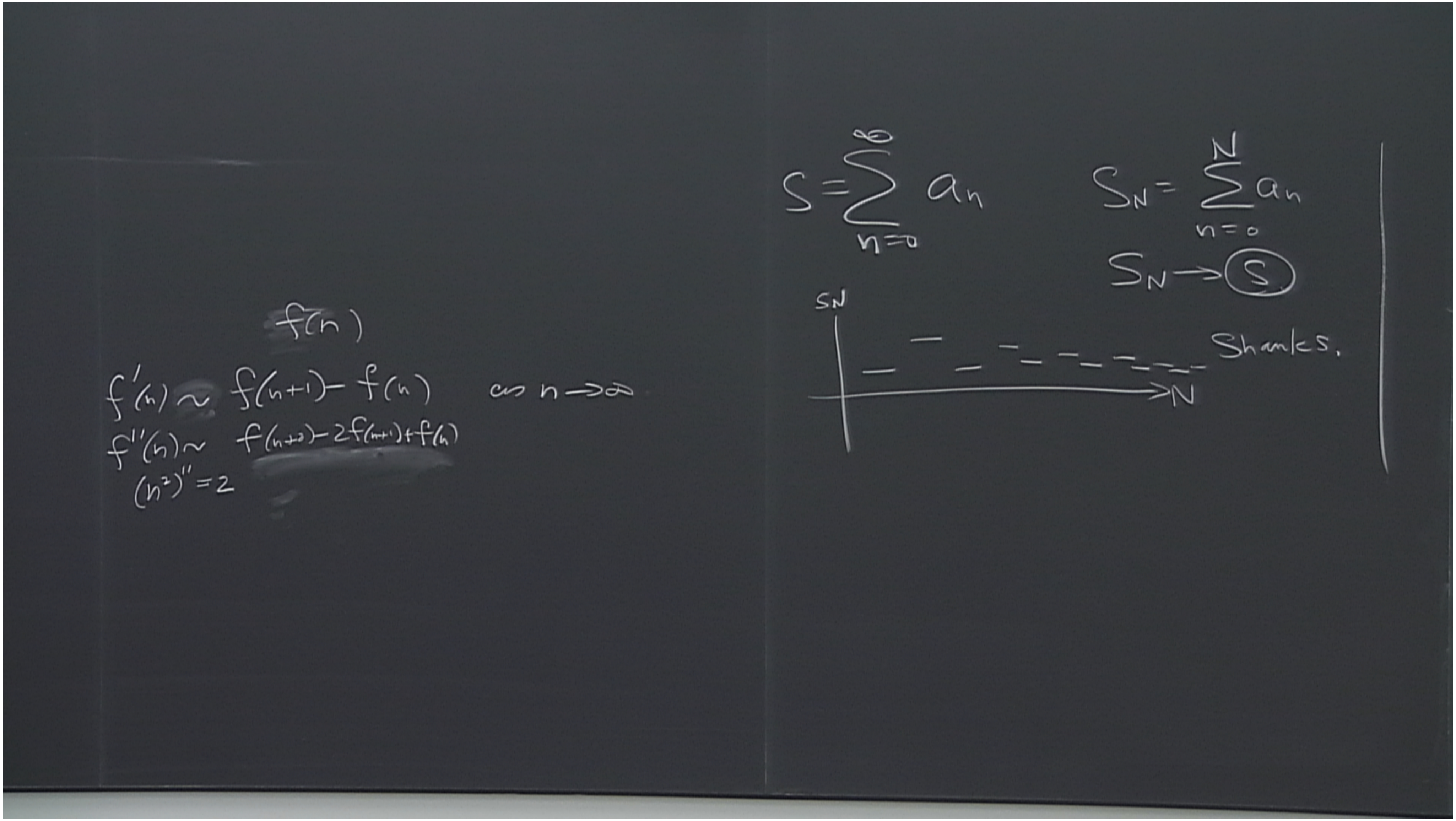
$$S_N \sim S + \frac{a}{N} + \frac{b}{N^2} \longrightarrow N^2 S_N \sim N^2 S + aN + b \qquad \times 1$$

$$S_{N+1} \sim \cdots\cdots \longrightarrow (N+1)^2 S_{N+1} \sim (N+1)^2 S + a(N+1) + b \qquad \times -2$$

$$S_{N+2} \sim \cdots\cdots \longrightarrow (N+2)^2 S_{N+2} \sim (N+2)^2 S + a(N+2) + b \qquad \times 1$$

$$\boxed{\frac{(N+2)^2 S_{N+2} - 2 (N+1)^2 S_{N+1} + N^2 S_N}{2} = R_2 \sim S \ (\text{as } N \to \infty)}$$

$$\frac{(N+3)^3 S_{N+3} - 3 (N+2)^3 S_{N+2} + 3 (N+1)^3 S_{N+1} - N^3 S_N}{6} \equiv R_3$$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$S_N \to S$
S_N
N
Shanks.
$f(n)$
$f'(n) \sim f(n+1) - f(n)$ as $n \to \infty$
$f''(n) \sim f(n+2) - 2f(n+1) + f(n)$
$(n^2)'' = 2$

Richardson extrapolation of 1+1/4+1/9+1/16+…=1.644 934 066 848

n	N=0	N=2	N=4	N=6
1	**1.000**	**1.625 000**	**1.644 965 278**	**1.644 935 185 185**
5	**1.464**	**1.644 167**	**1.644 935 811**	**1.644 934 060 147**
10	**1.550**	**1.644 809**	**1.644 934 195**	**1.644 934 066 526**
15	**1.580**	**1.644 893**	**1.644 934 090**	**1.644 934 066 812**
20	**1.596**	**1.644 916**	**1.644 934 073**	**1.644 934 066 842**
25	**1.606**	**1.644 924**	**1.644 934 069**	**1.644 934 066 847**

Richardson extrapolation of 1+1/4+1/9+1/16+…=1.644 934 066 848

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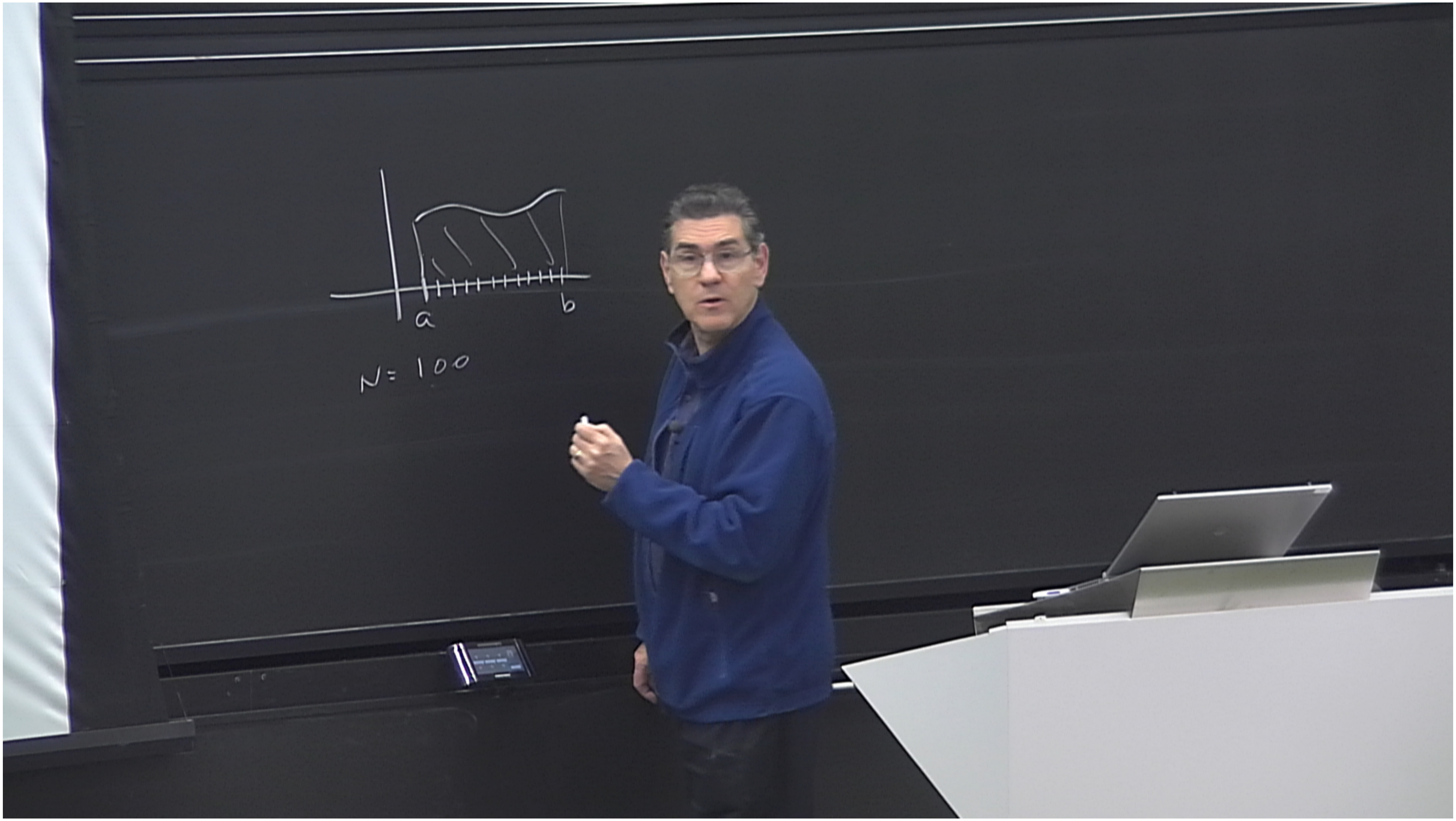
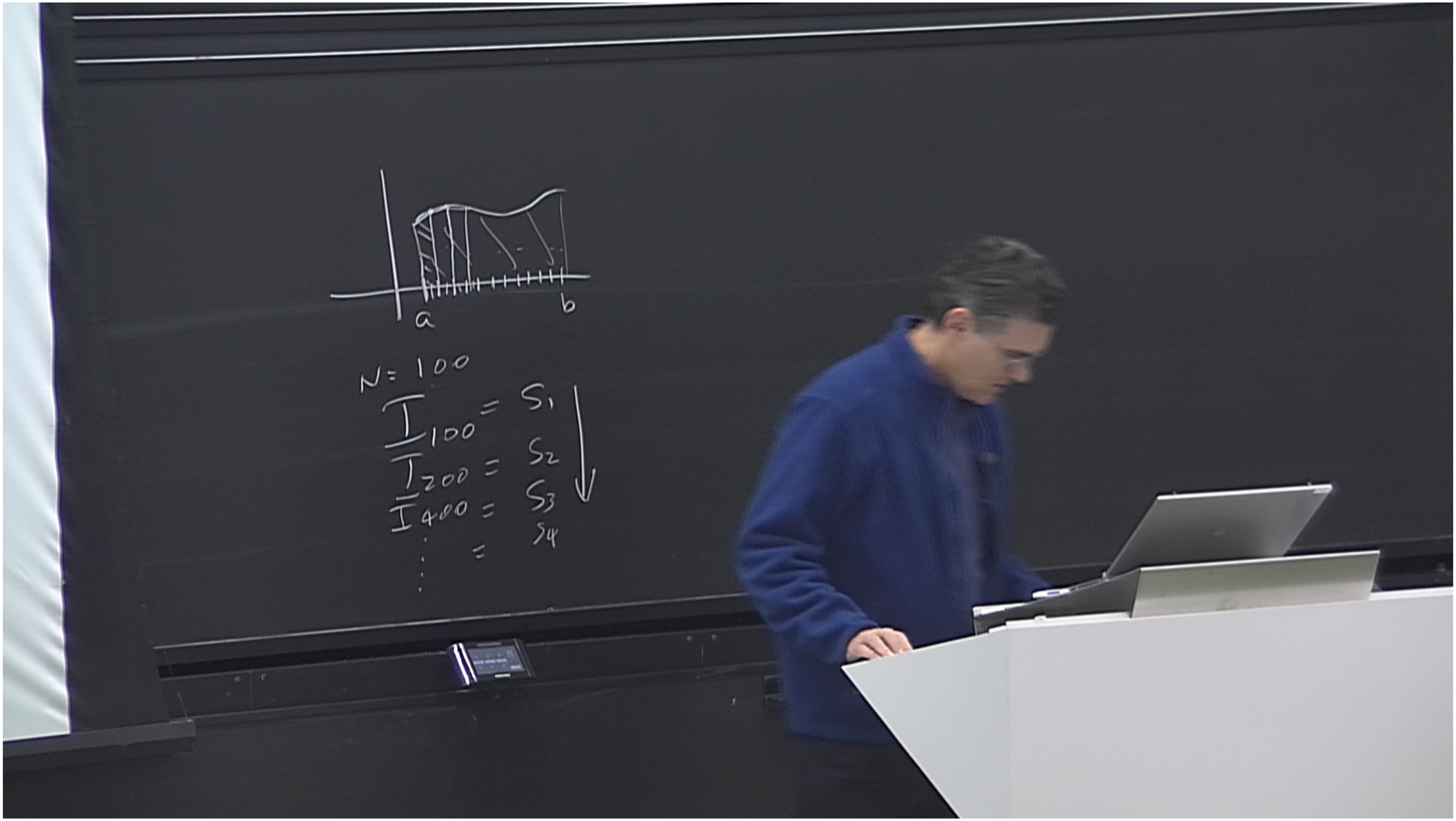
Application to numerical integration

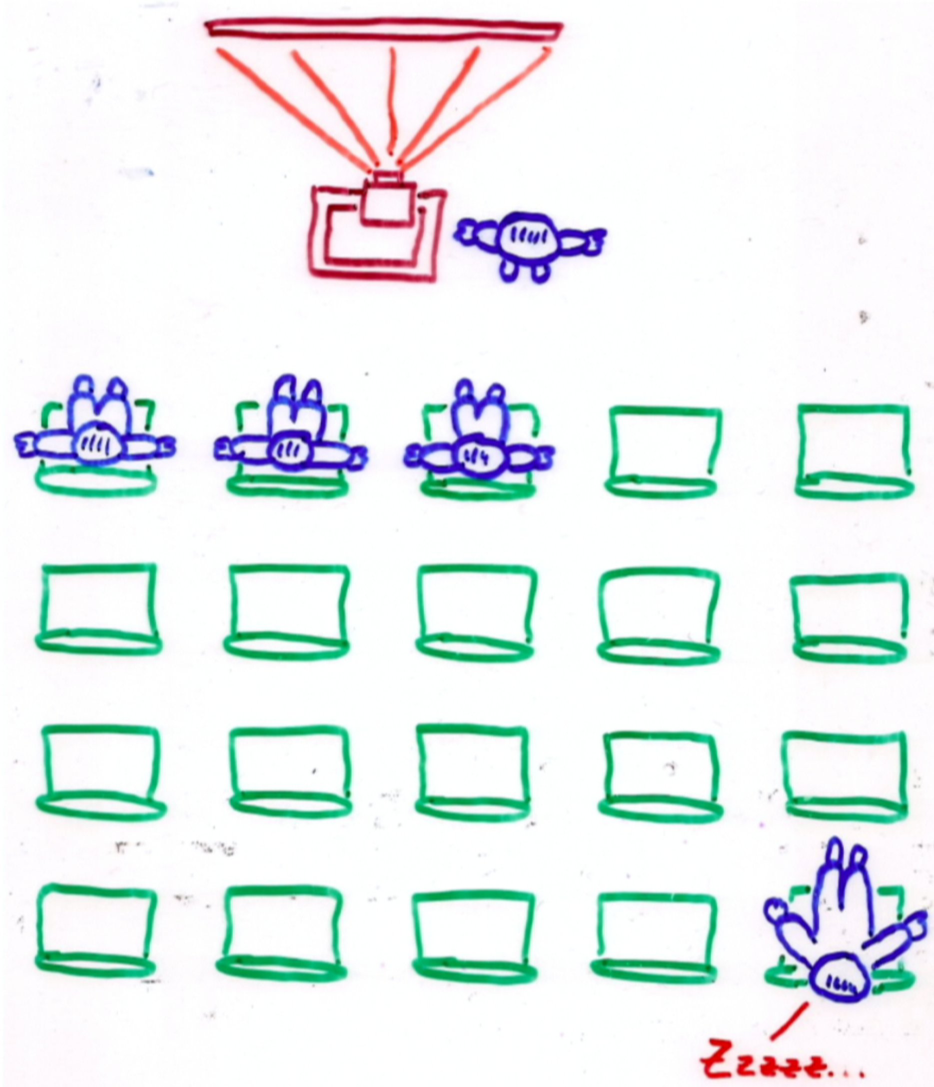
Simpson's rule is the first Richardson of the trapezoid rule

Romberg integration


a
b
N = 100


a
b
N = 100
I100 = S1
I200 = S2
I400 = S3
= S4

Overview of course:



Summing divergent series

Some examples

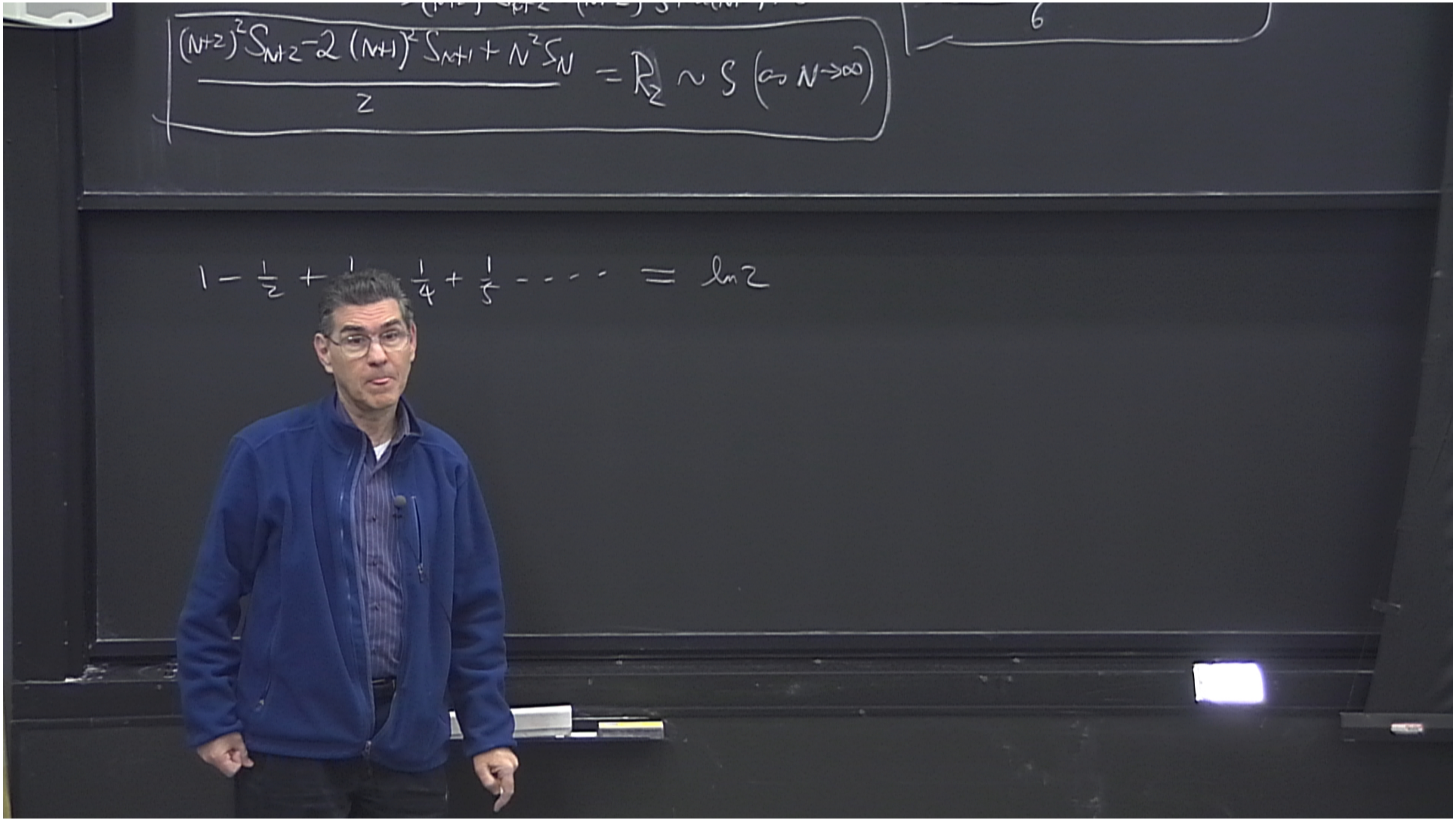
$$1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 + \ldots$$

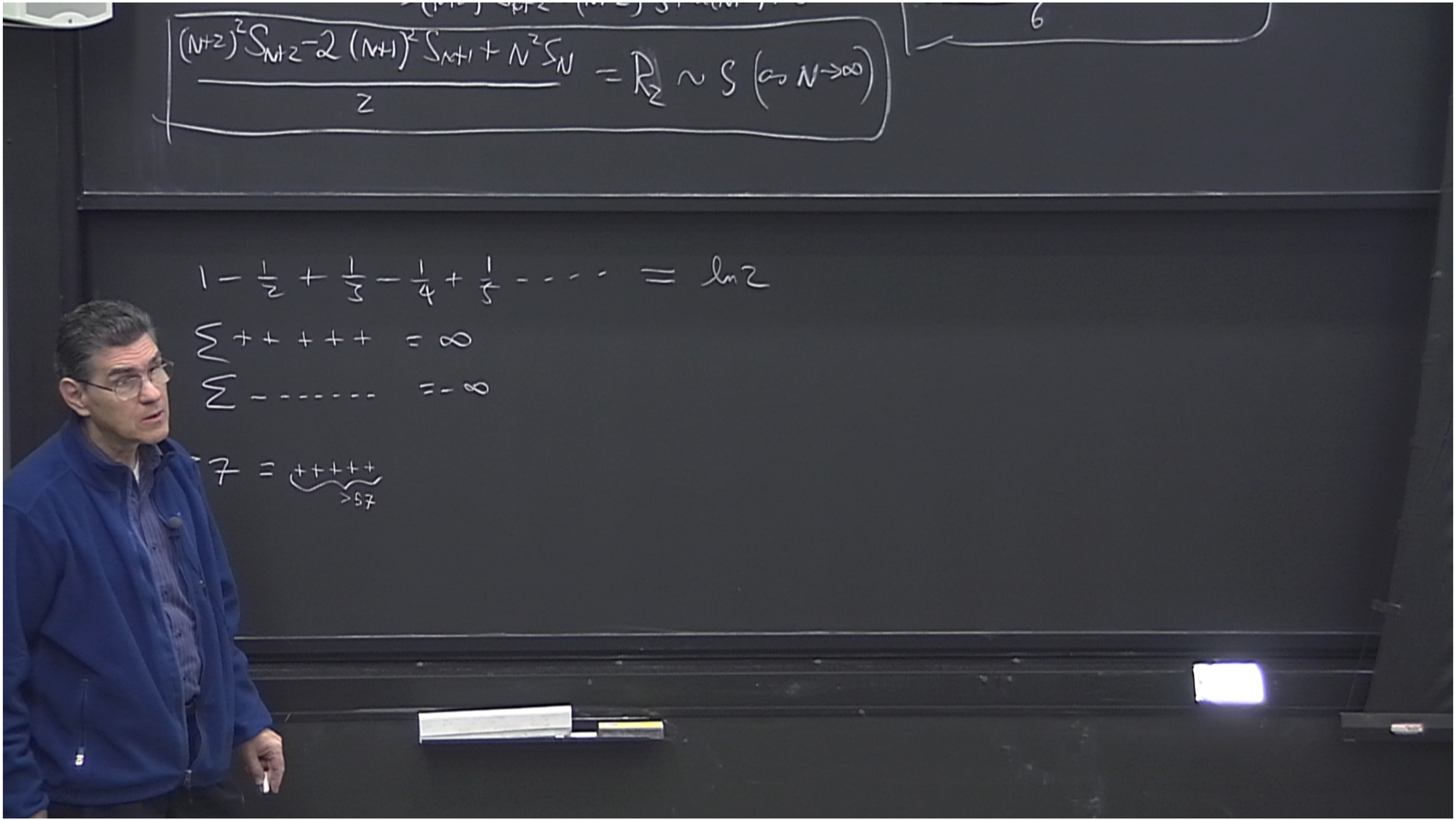
$$1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + \ldots$$

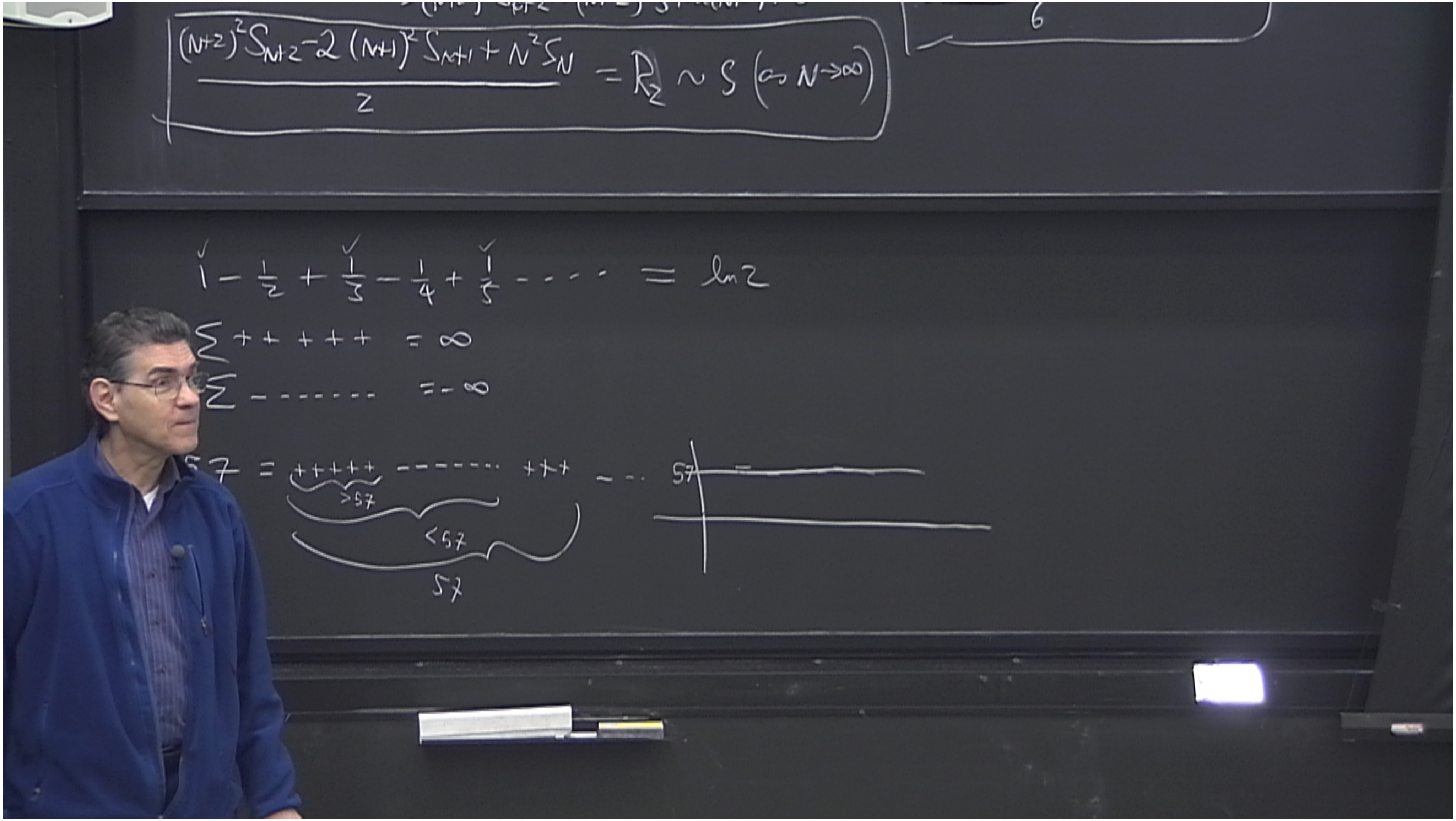
$$1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + \ldots$$

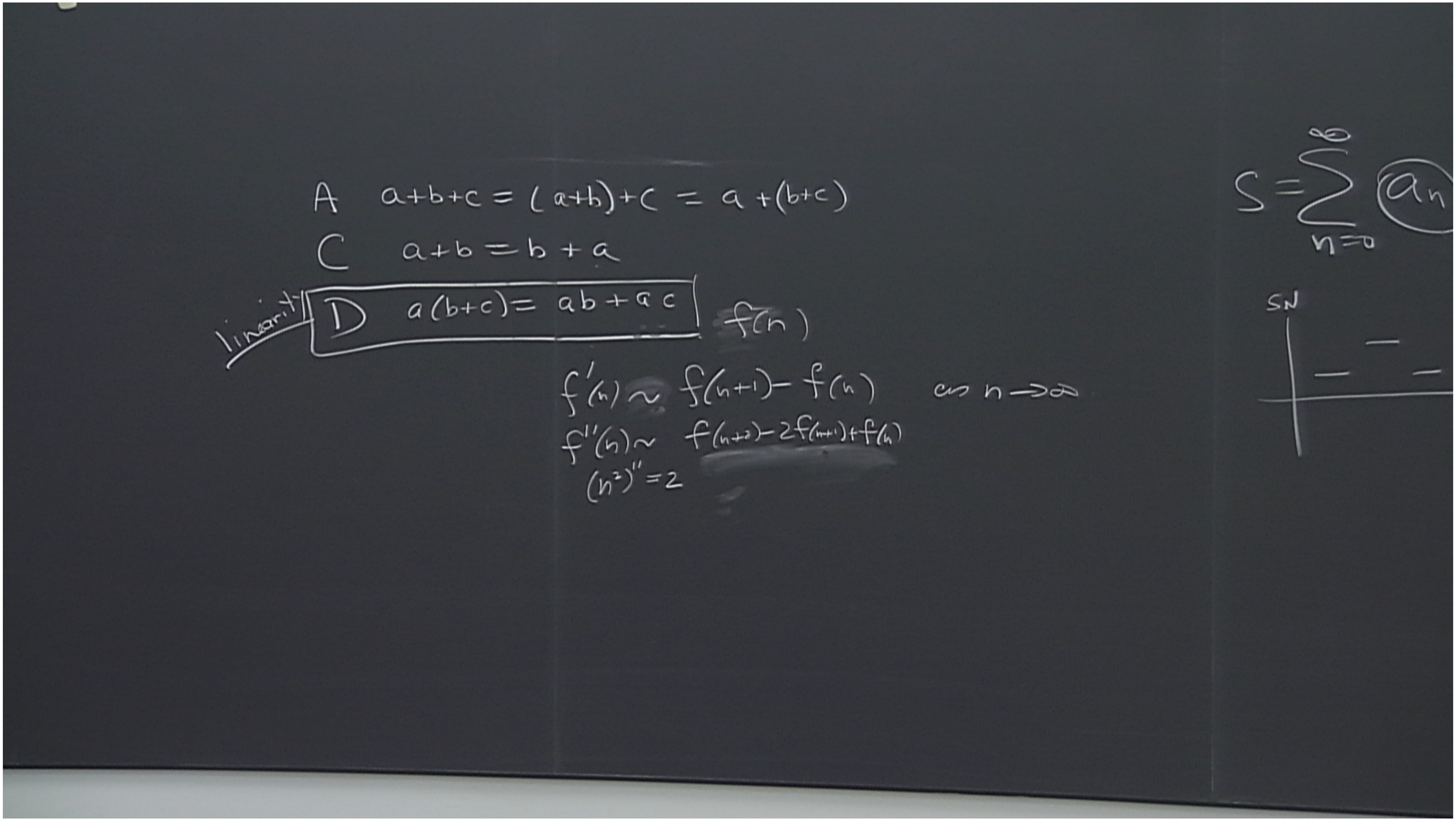
$$1 + 0 - 1 + 1 + 0 - 1 + 1 + 0 - 1 + \ldots$$

Divergent series are not bad! They are useful, and they converge faster than convergent series!


$\frac{(N+2)^2 S_{N+2} - 2(N+1)^2 S_{N+1} + N^2 S_N}{2} = R_2 \sim S$ (as $N \to \infty$)
$1 - \frac{1}{2} + \cdots \frac{1}{4} + \frac{1}{5} \cdots\cdots = \ln 2$


$\frac{(N+2)^2 S_{N+2} - 2(N+1)^2 S_{N+1} + N^2 S_N}{2} = R_2 \sim S$ (as $N \to \infty$)
6
$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - - - - = \ln 2$
$\sum + + + + + = \infty$
$\sum - - - - - - - = -\infty$
7 = + + + + +
>57


(N+2)² S_{N+2} − 2 (N+1)² S_{N+1} + N² S_N / 2 = R₂ ~ S (as N→∞)
1 − 1/2 + 1/3 − 1/4 + 1/5 − − − − = ln 2
Σ + + + + + = ∞
Σ − − − − − − − = − ∞
57 = + + + + + − − − − − − − + + + − − −
>57
<57
57


A $a+b+c=(a+b)+c=a+(b+c)$
C $a+b=b+a$
linearity
D $a(b+c)=ab+ac$
$f(n)$
$f'(n)\sim f(n+1)-f(n)$ as $n\to\infty$
$f''(n)\sim f(n+2)-2f(n+1)+f(n)$
$(n^2)''=2$
$S=\sum_{n=0}^{\infty} a_n$
S_N

Some examples

$$1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 + \ldots$$

$$1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + \ldots$$

$$1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + \ldots$$

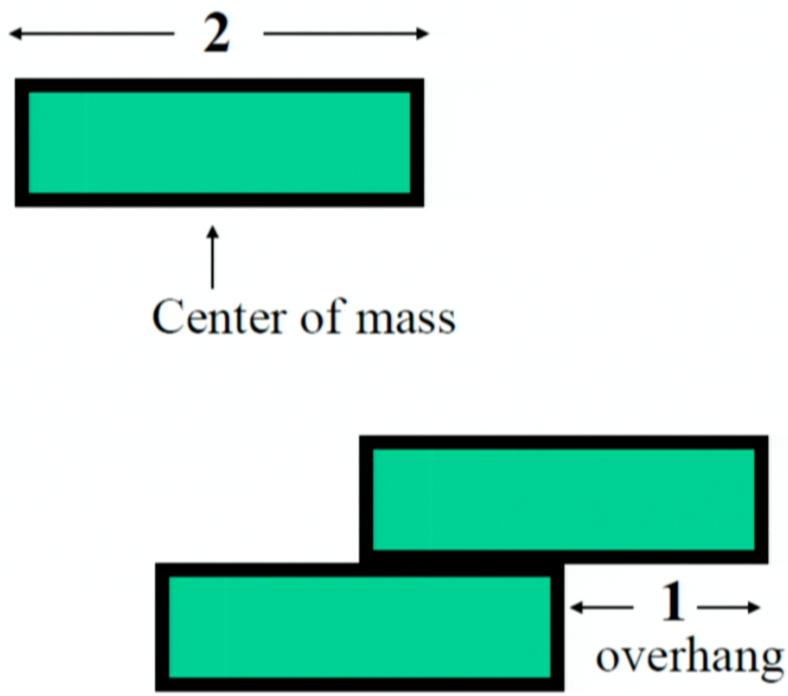
$$1 + 0 - 1 + 1 + 0 - 1 + 1 + 0 - 1 + \ldots$$

Divergent series are not bad! They are useful, and they converge faster than convergent series!

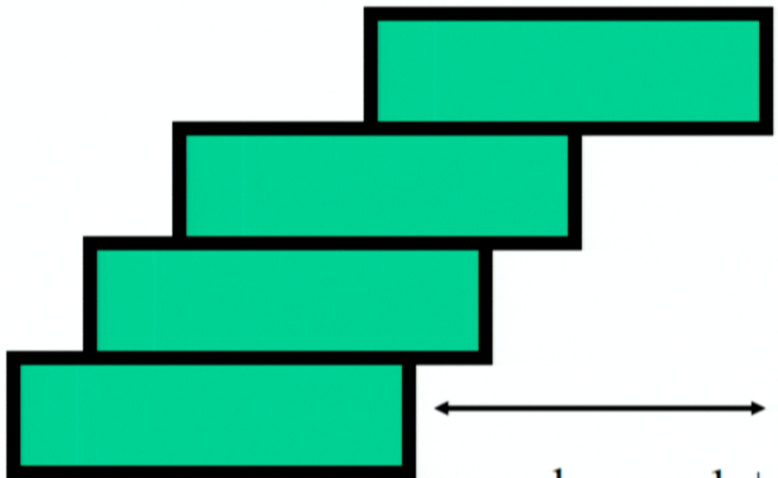
Some divergent series REALLY sum up to infinity--- this is perfectly OK!

$$1 + \frac{1}{2} + 1/3 + \frac{1}{4} + 1/5 + \ldots = \zeta(1)$$

Building a bridge with bricks



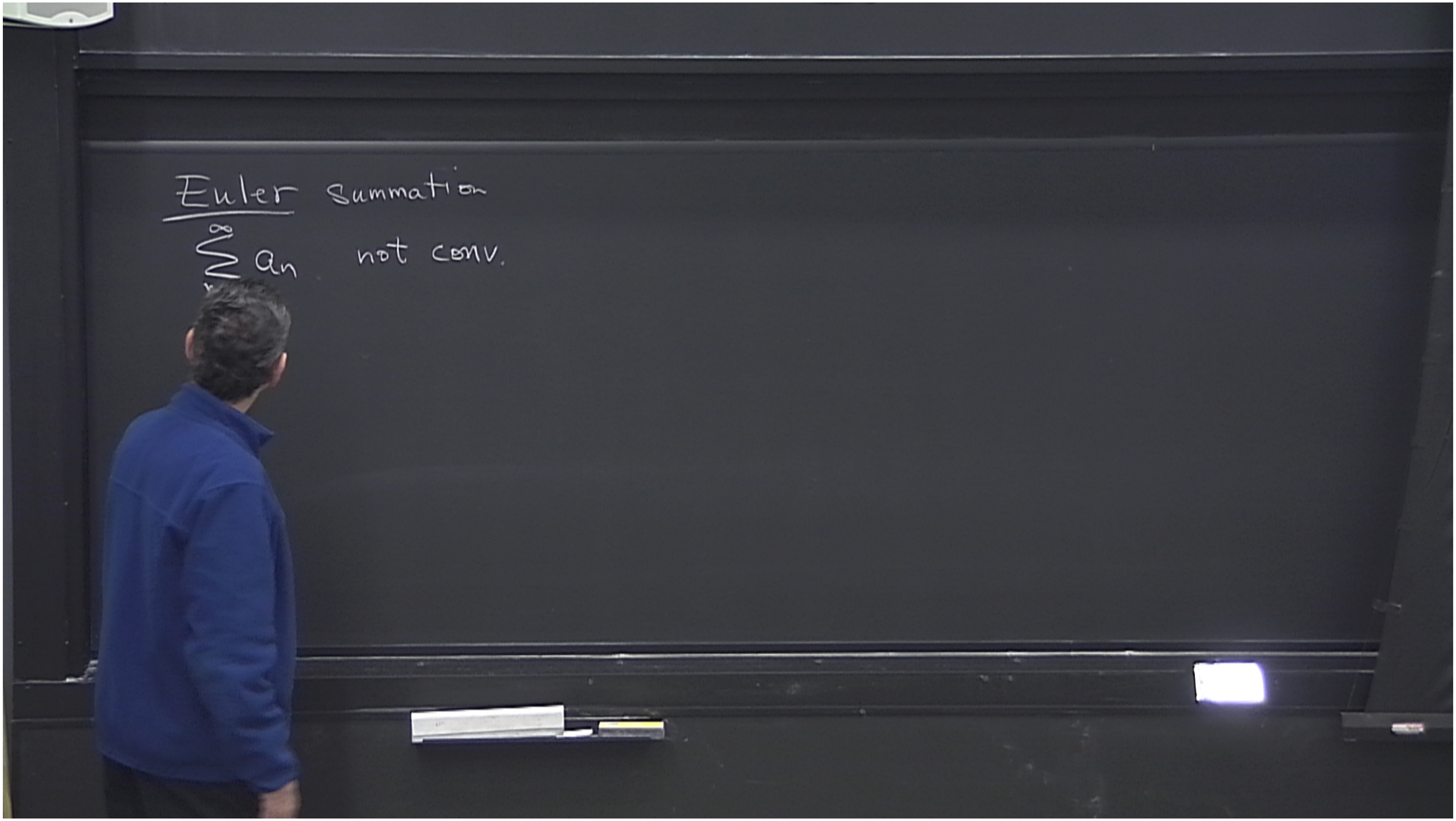
Building a bridge with bricks

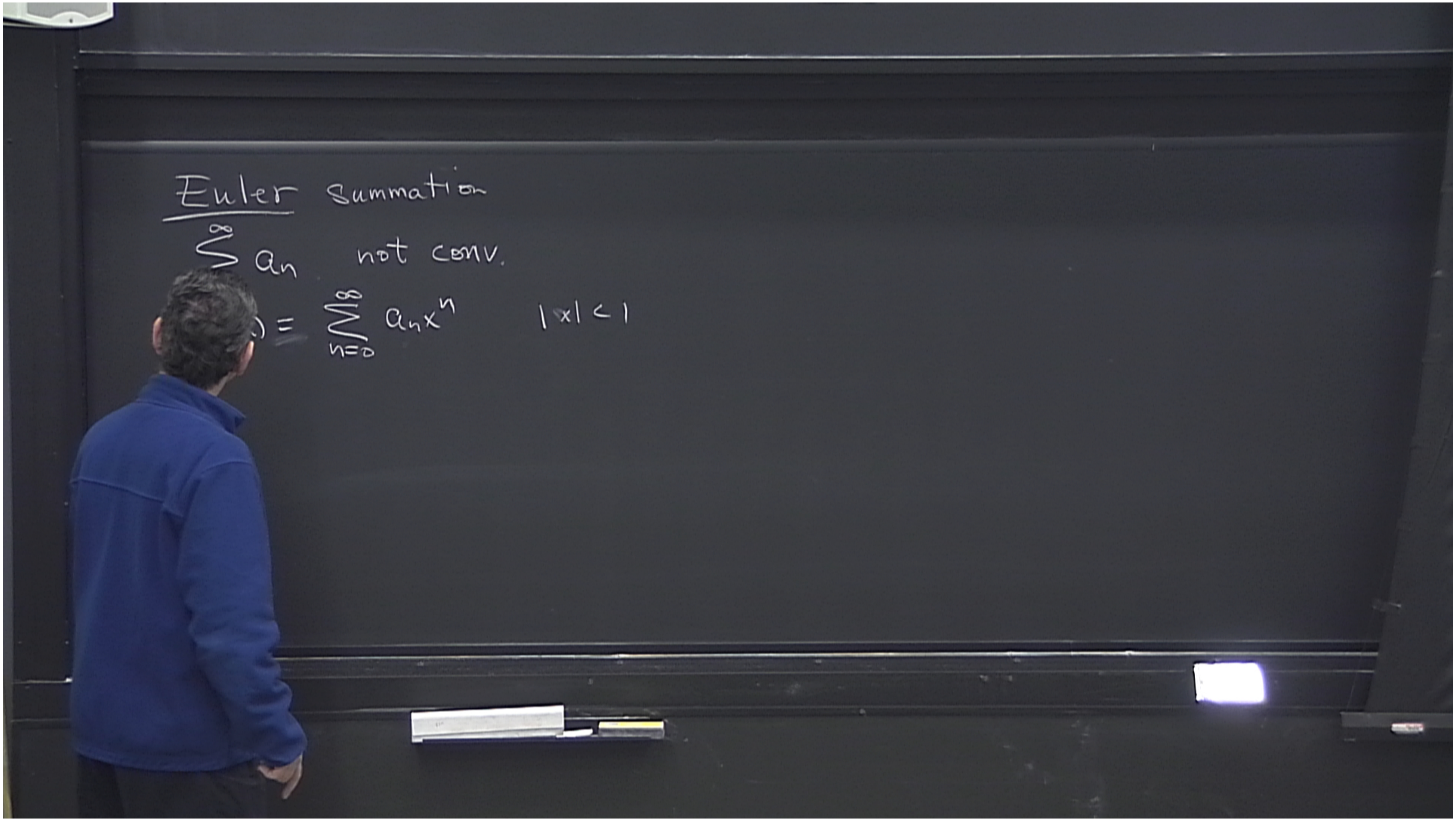


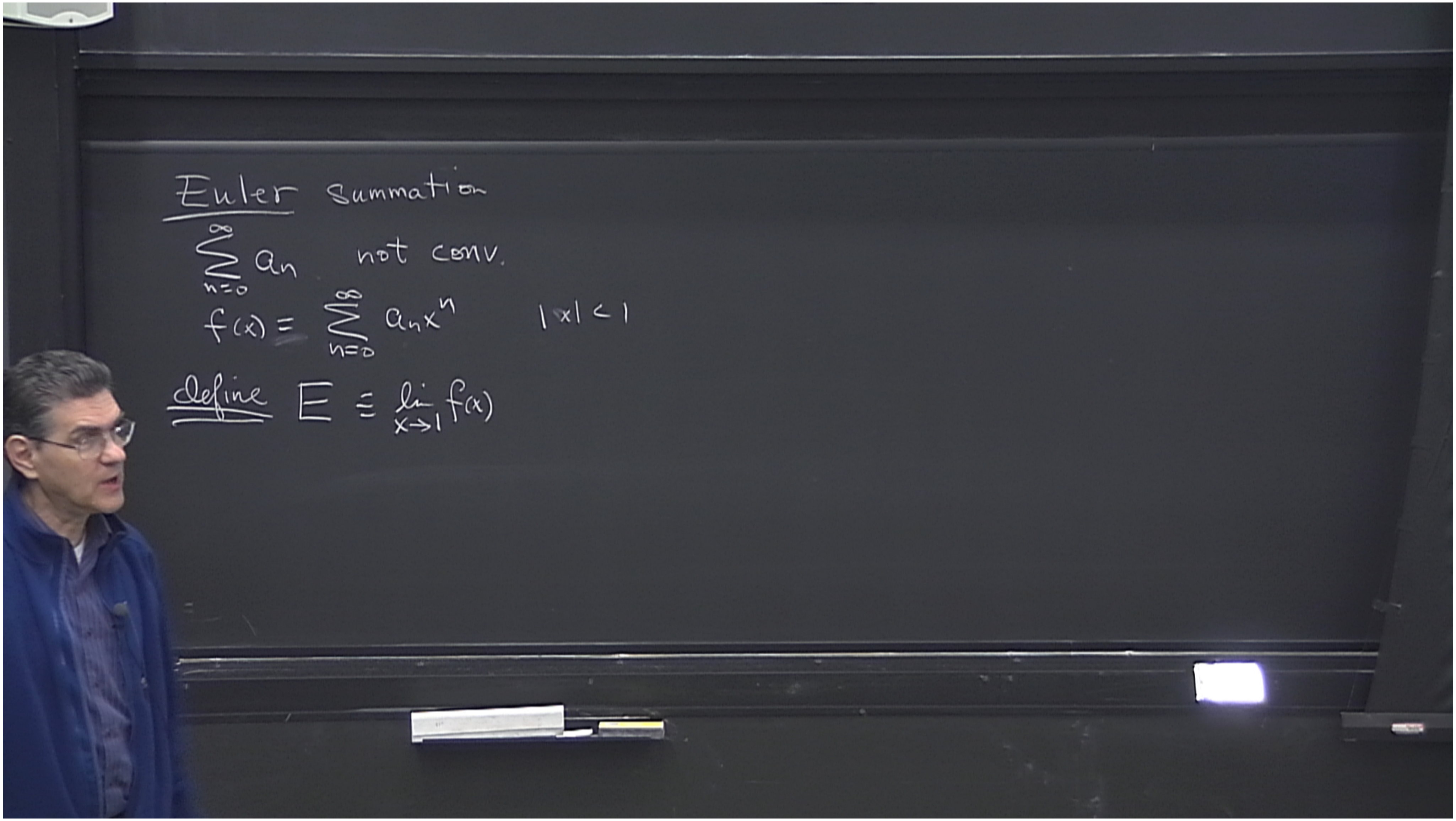
$$\text{overhang} = 1 + \frac{1}{2} + \frac{1}{3}$$

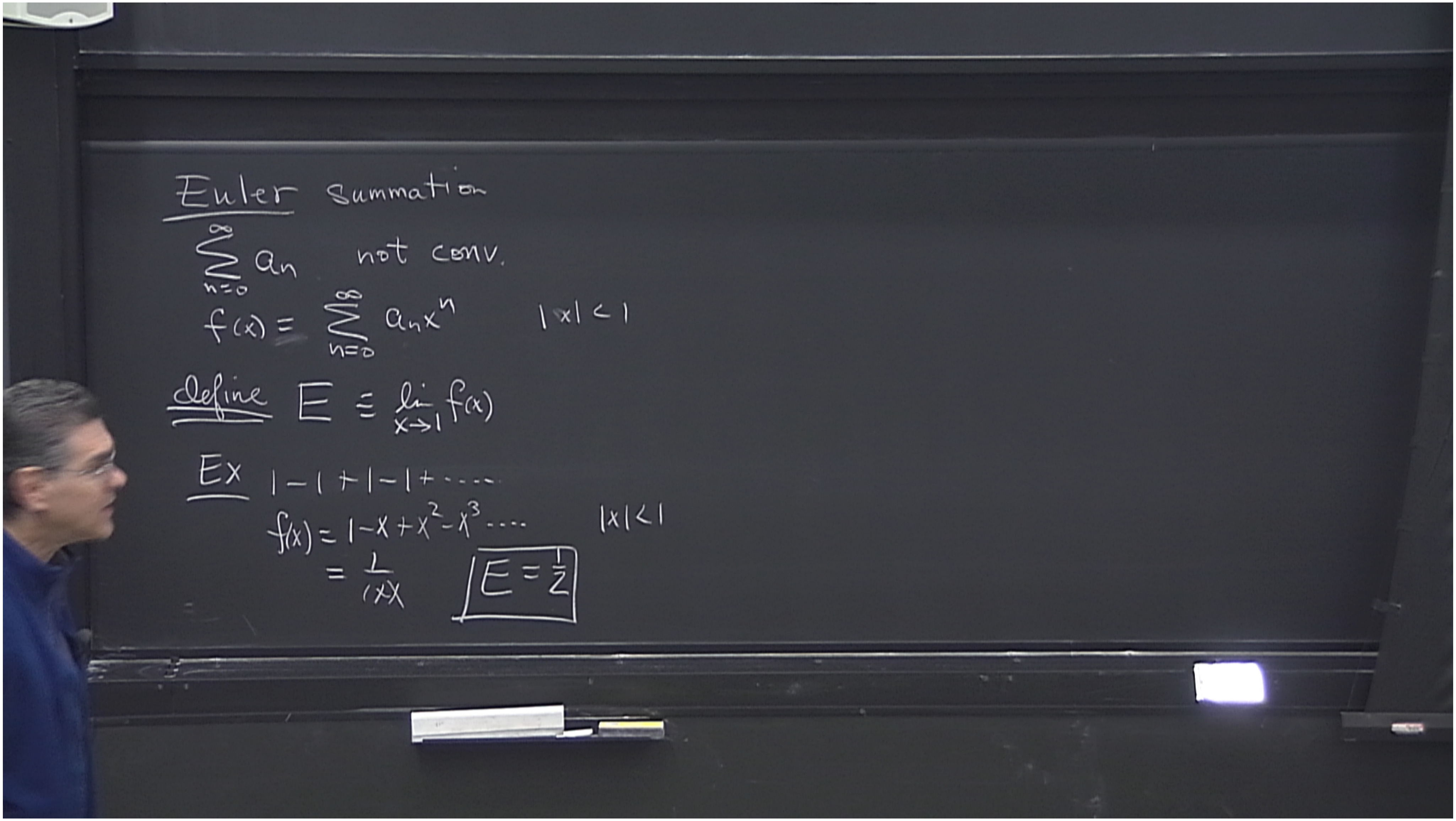
Techniques for summing series

- Note: addition is not infinitely commutative
- Euler summation
- Borel summation
- Generic summation methods
- Note: addition is not infinitely associative
- Zeta summation
- Continued functions


Euler summation
$\sum^{\infty} a_n$ not conv.


Euler summation
∑^∞ a_n not conv.
) = ∑_{n=0}^∞ a_n x^n |x| < 1


Euler summation
$\sum_{n=0}^{\infty} a_n$ not conv.
$f(x) = \sum_{n=0}^{\infty} a_n x^n$ $|x| < 1$
define $E \equiv \lim_{x \to 1} f(x)$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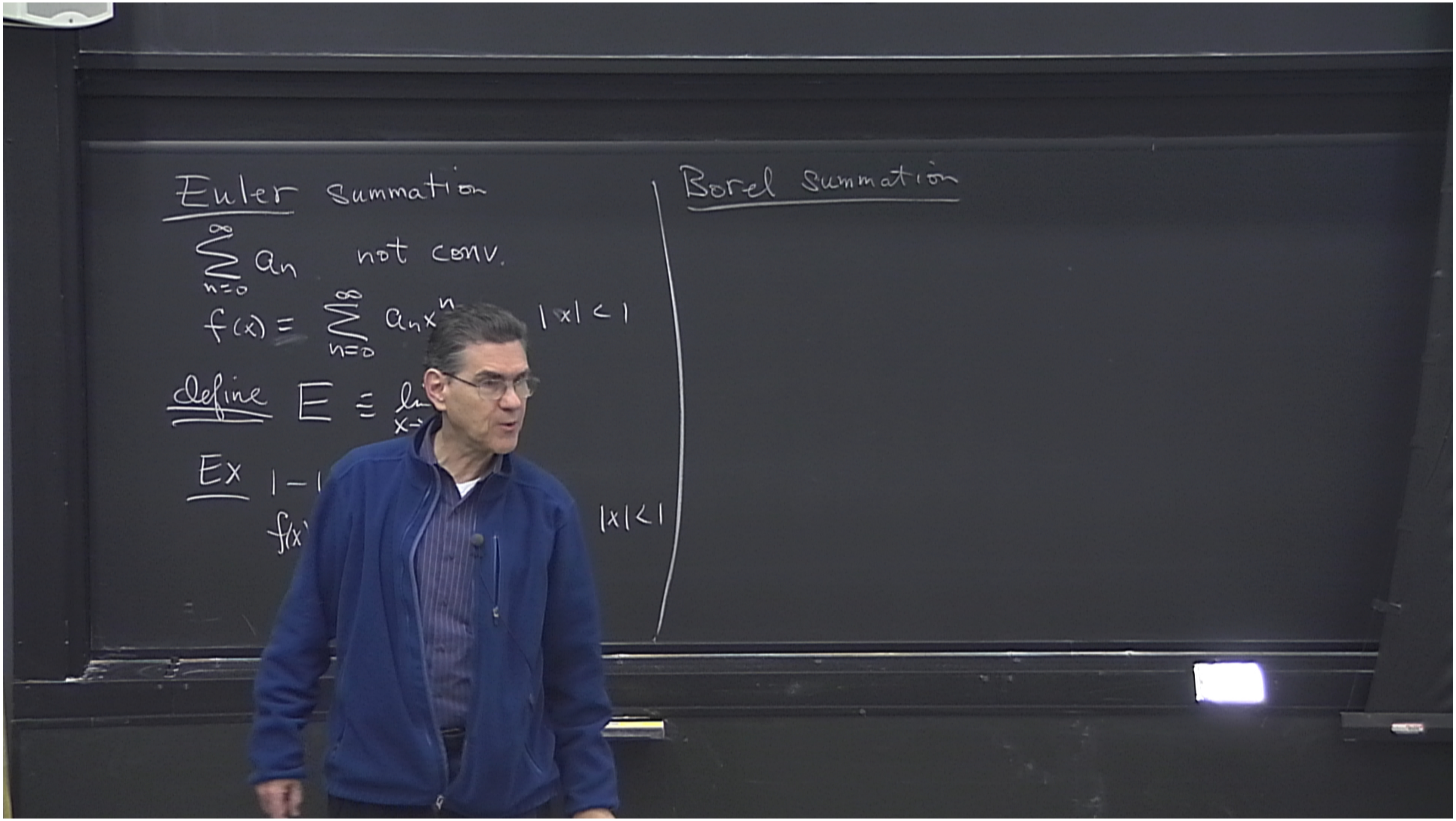
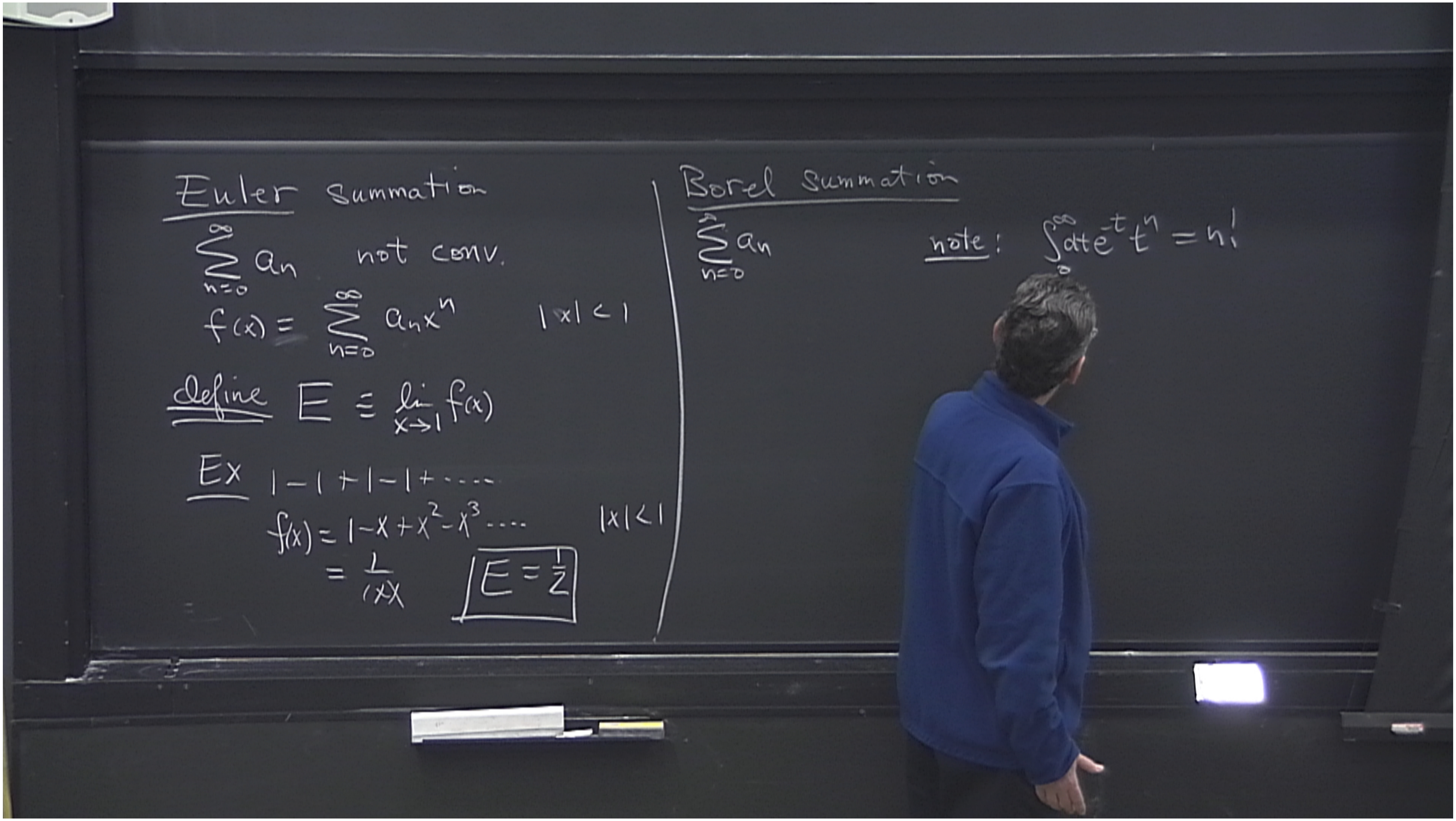
$f(x) = \sum_{n=0}^{\infty} a_n x^n$ $|x| < 1$
define $E \equiv \lim_{x \to 1} f(x)$
Ex $1 - 1 + 1 - 1 + \cdots$
$f(x) = 1 - x + x^2 - x^3 \cdots$ $|x| < 1$
$= \frac{1}{1+x}$
$E = \frac{1}{2}$


Euler summation
not conv.
|x| < 1
Define
Ex
|x| < 1
Borel summation


Euler summation
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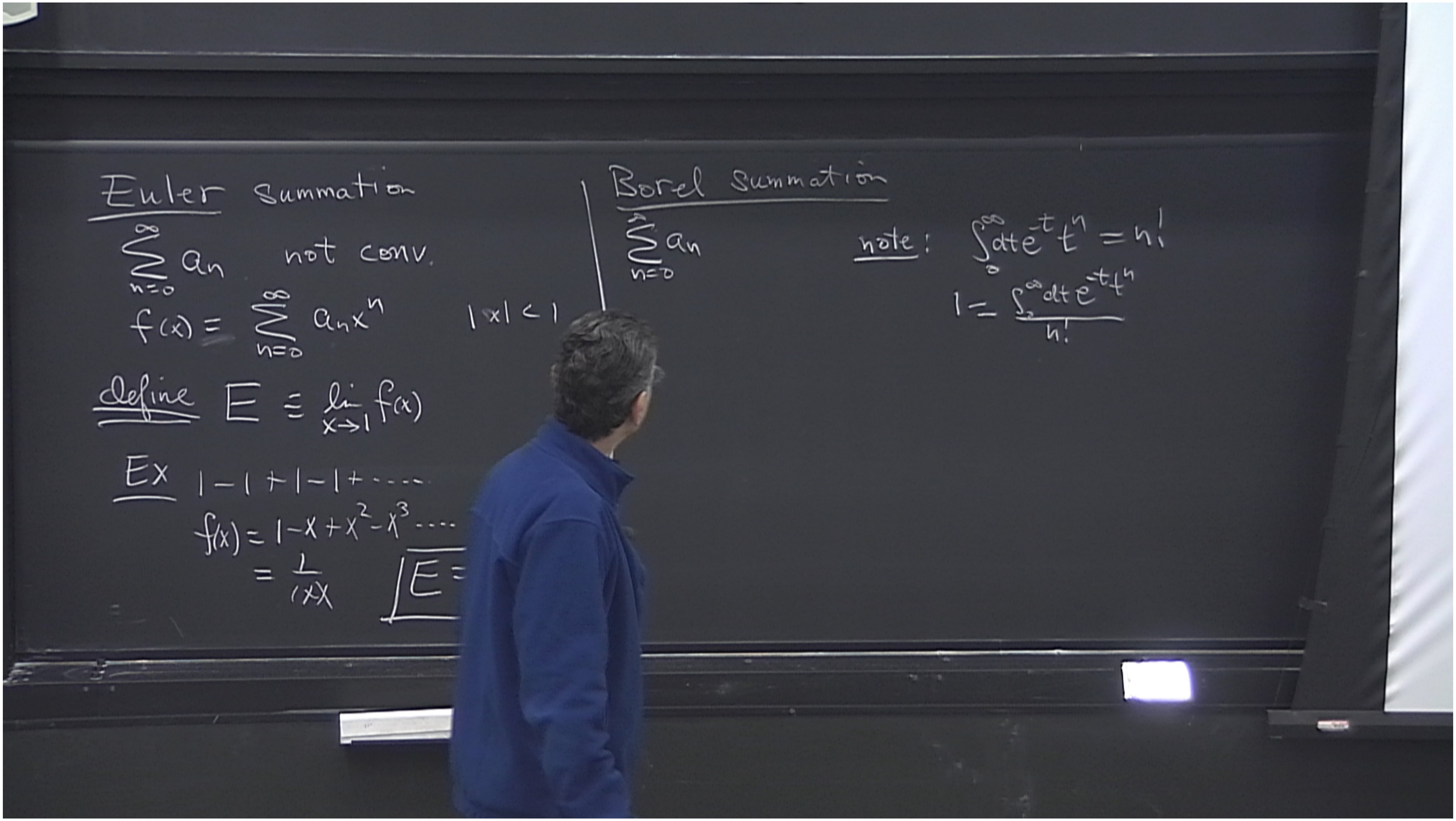
define $E \equiv \lim_{x \to 1} f(x)$
Ex $1 - 1 + 1 - 1 + \cdots$
$f(x) = 1 - x + x^2 - x^3 \cdots$ $|x| < 1$
$= \frac{1}{1+x}$
$E = \frac{1}{2}$
Borel summation
$\sum_{n=0}^{\infty} a_n$
note: $\int_0^\infty dt\, e^{-t} t^n = n!$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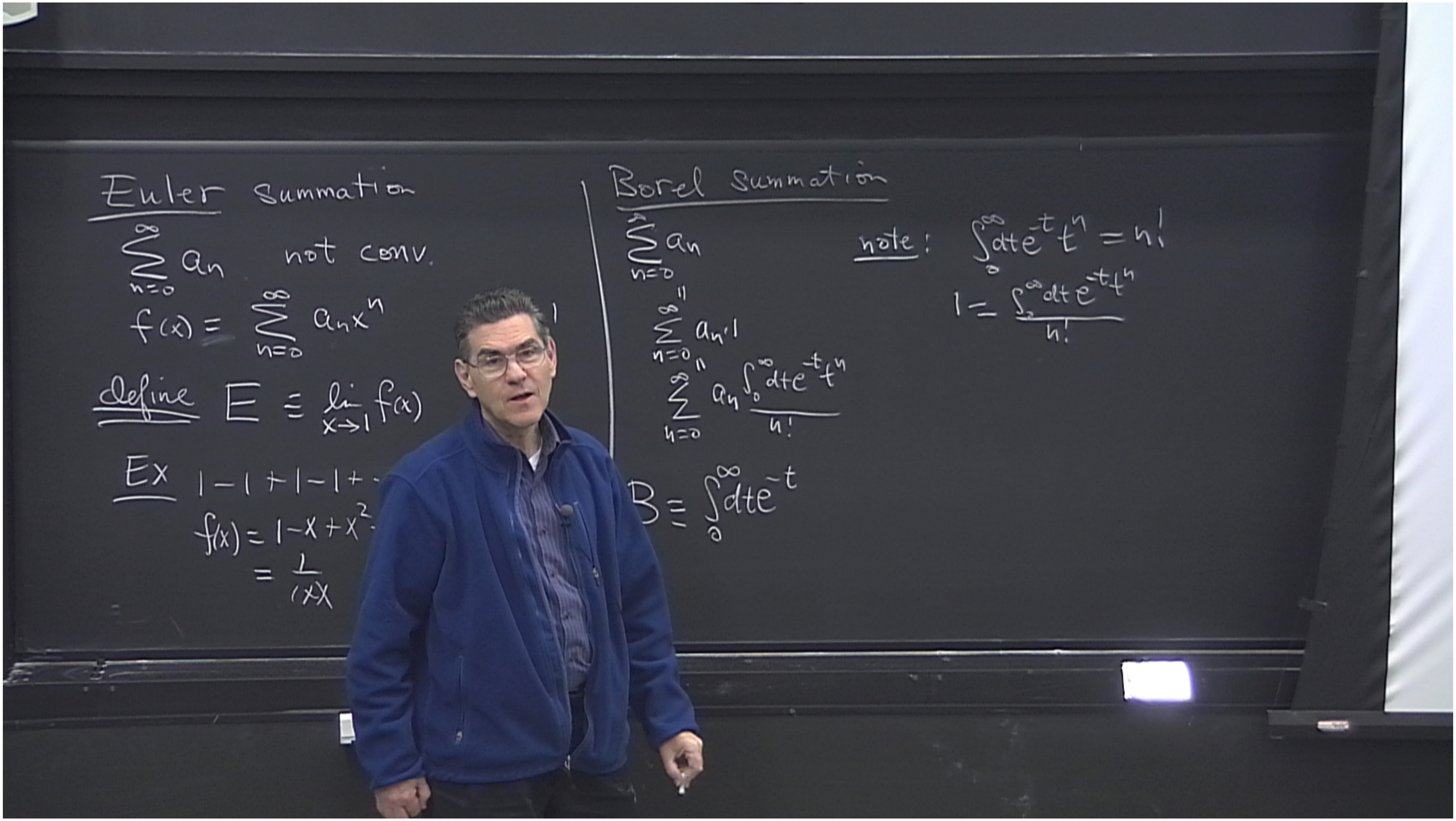
Ex $1 - 1 + 1 - 1 + \cdots$
$f(x) = 1 - x + x^2 - x^3 \cdots$
$= \frac{1}{1+x}$
$E =$
Borel summation
$\sum_{n=0}^{\infty} a_n$
note: $\int_0^{\infty} dt\, e^{-t} t^n = n!$
$1 = \frac{\int_0^{\infty} dt\, e^{-t} t^n}{n!}$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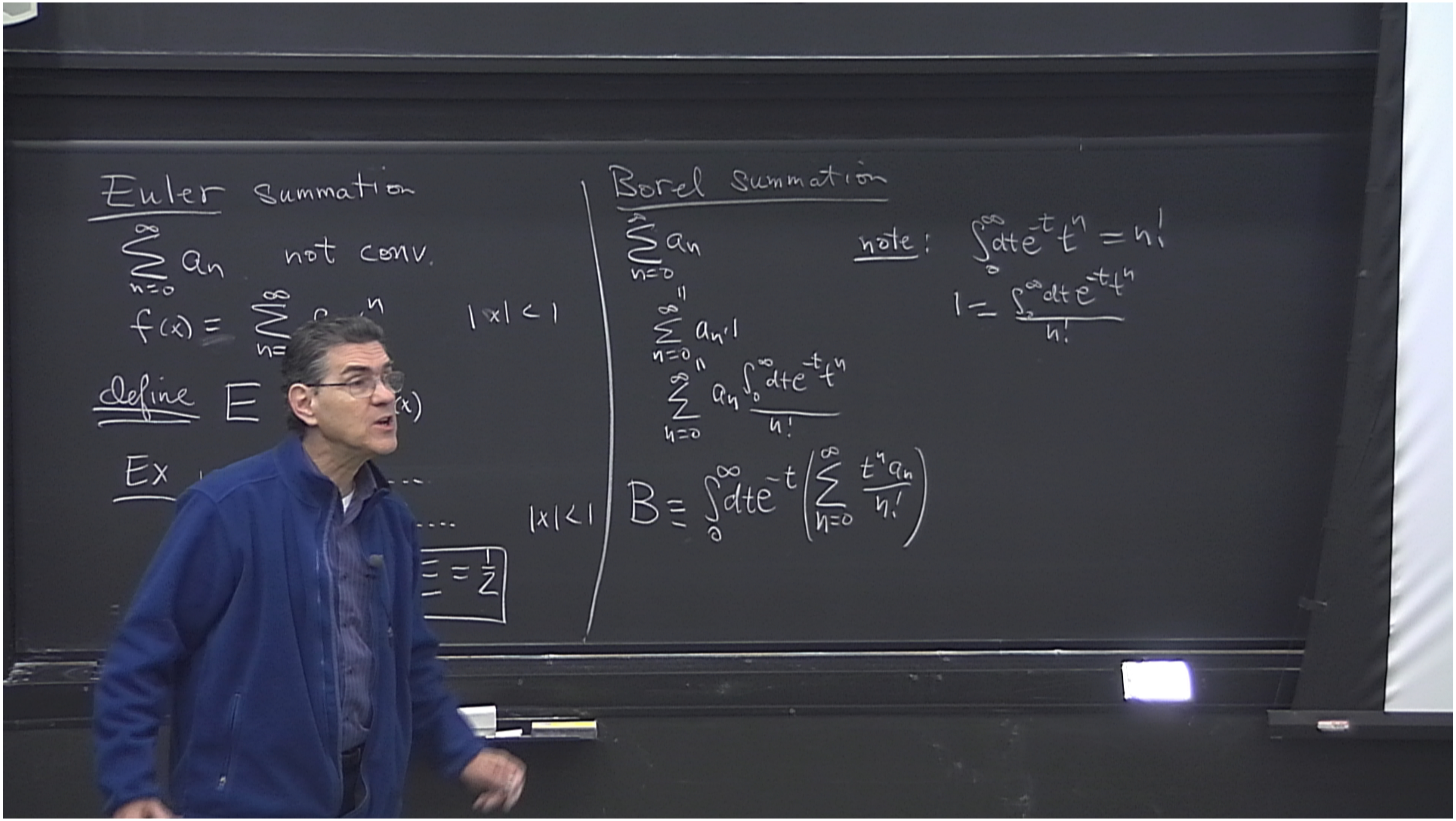
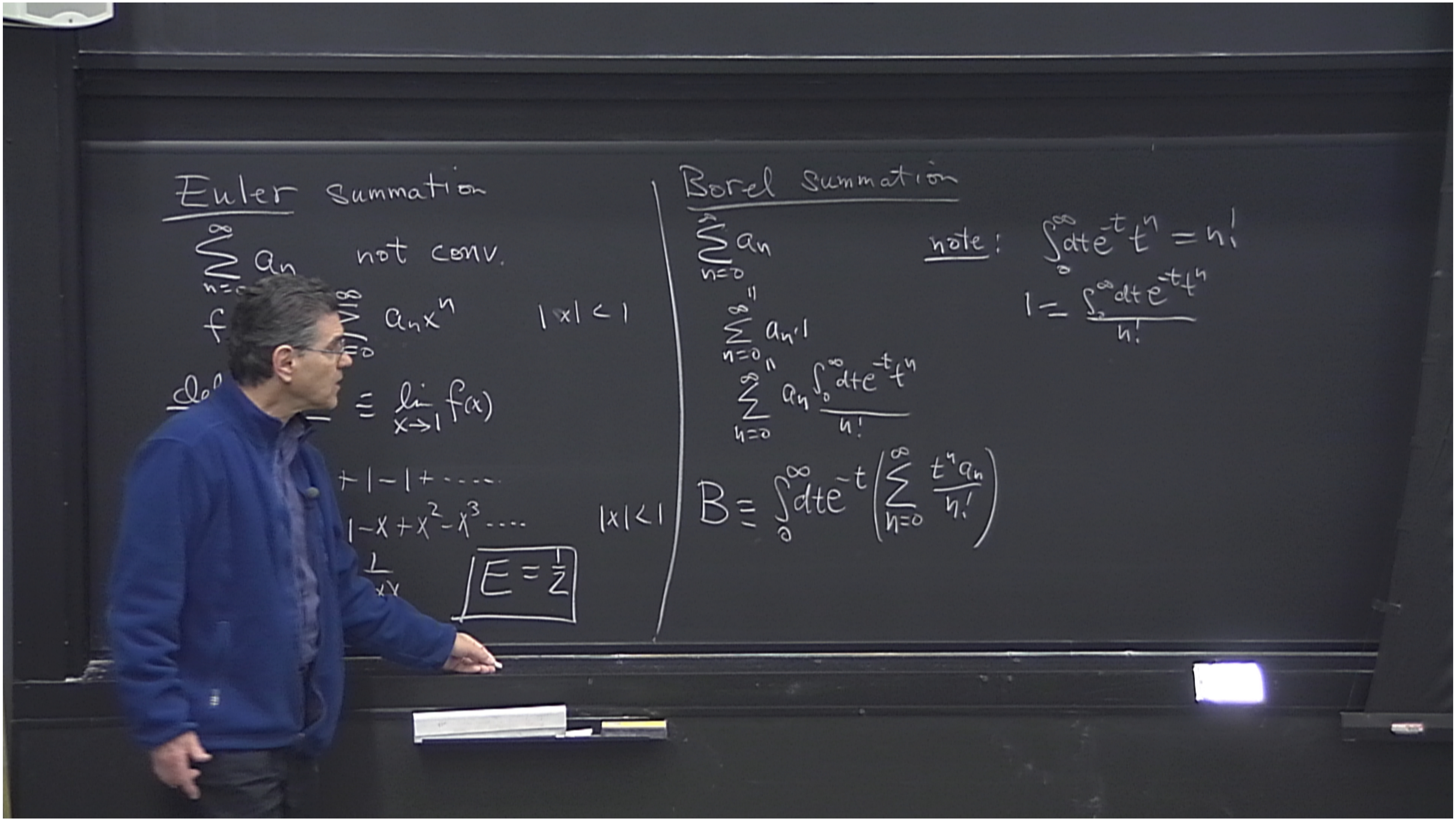
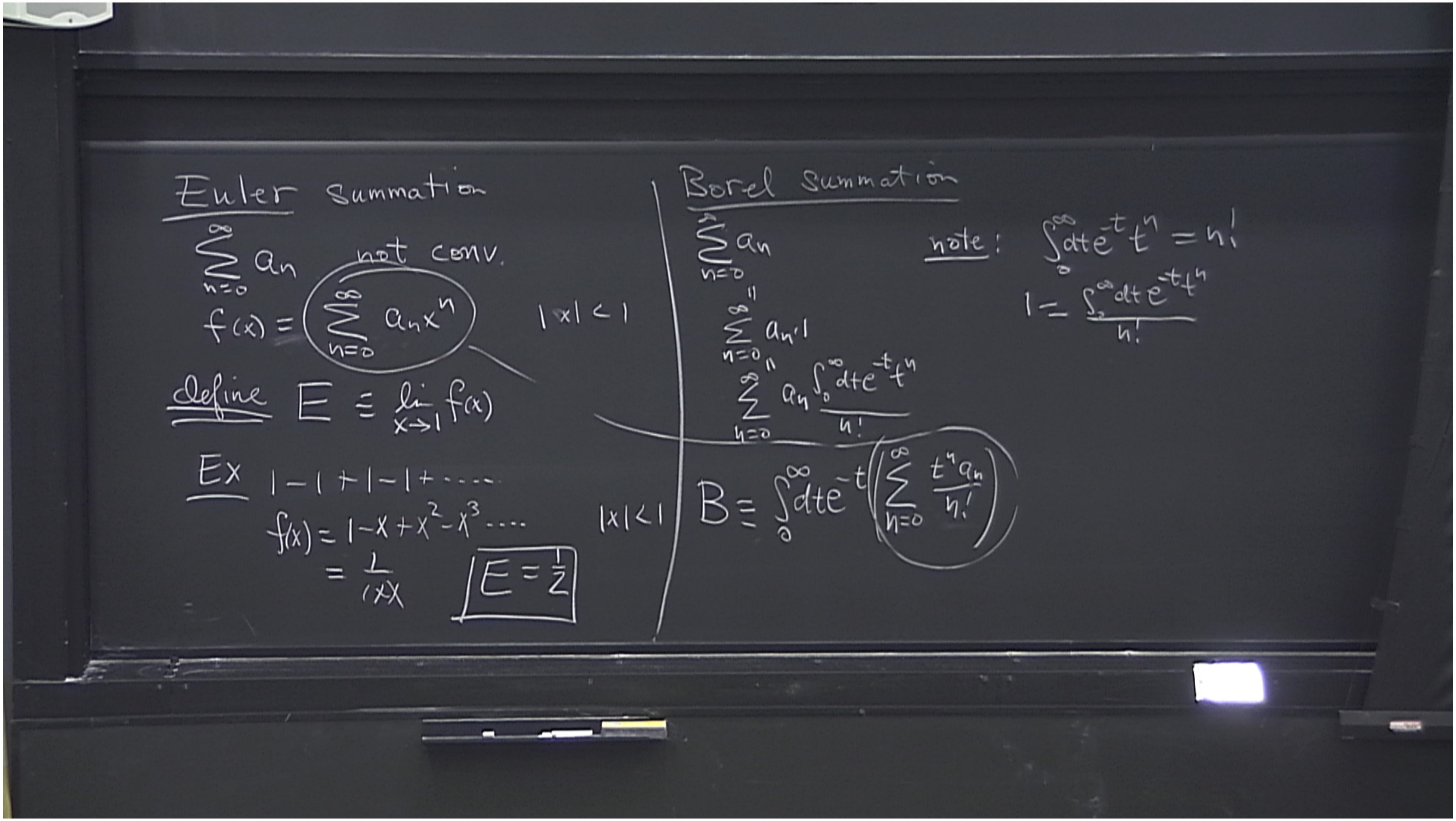
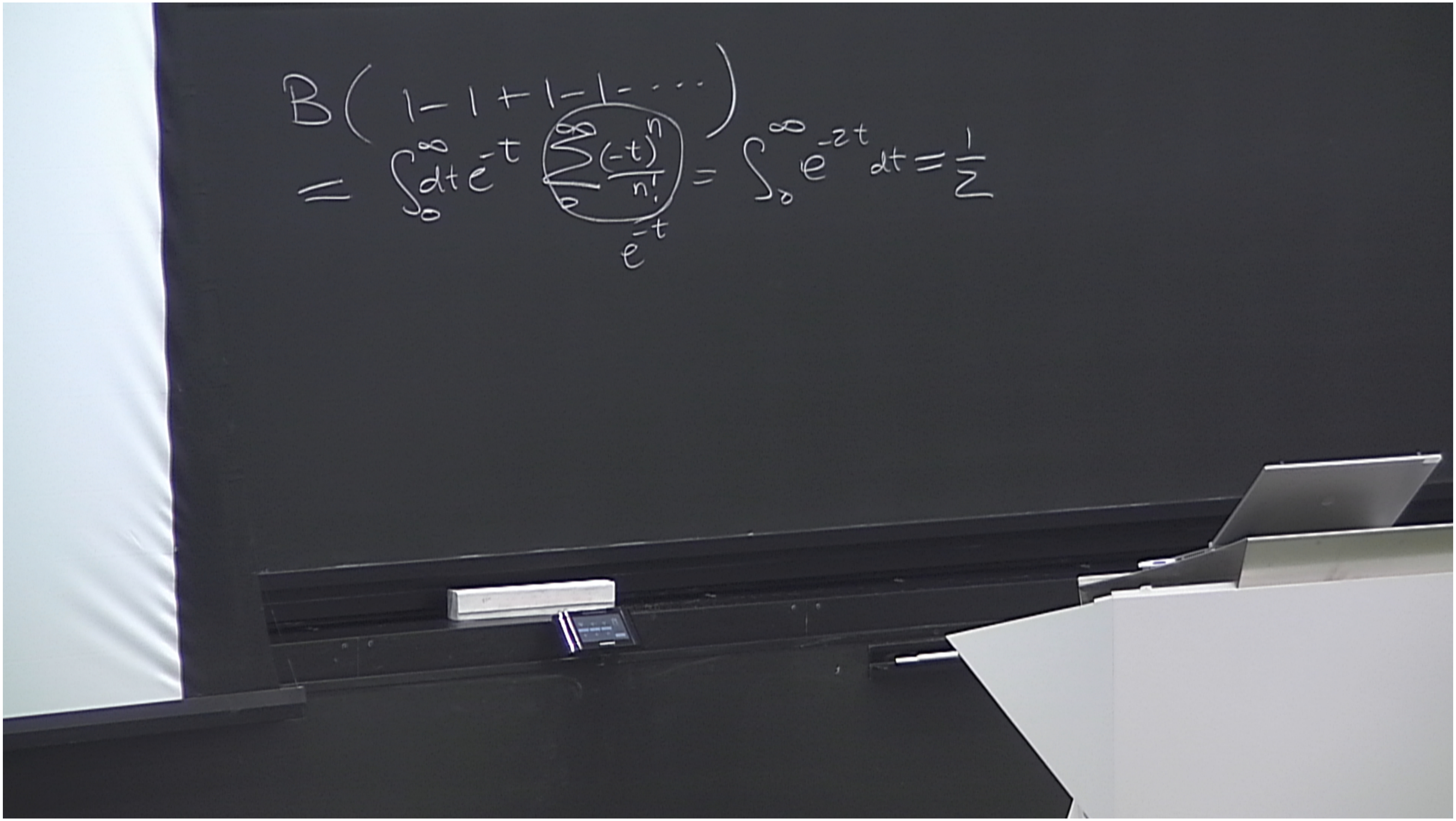
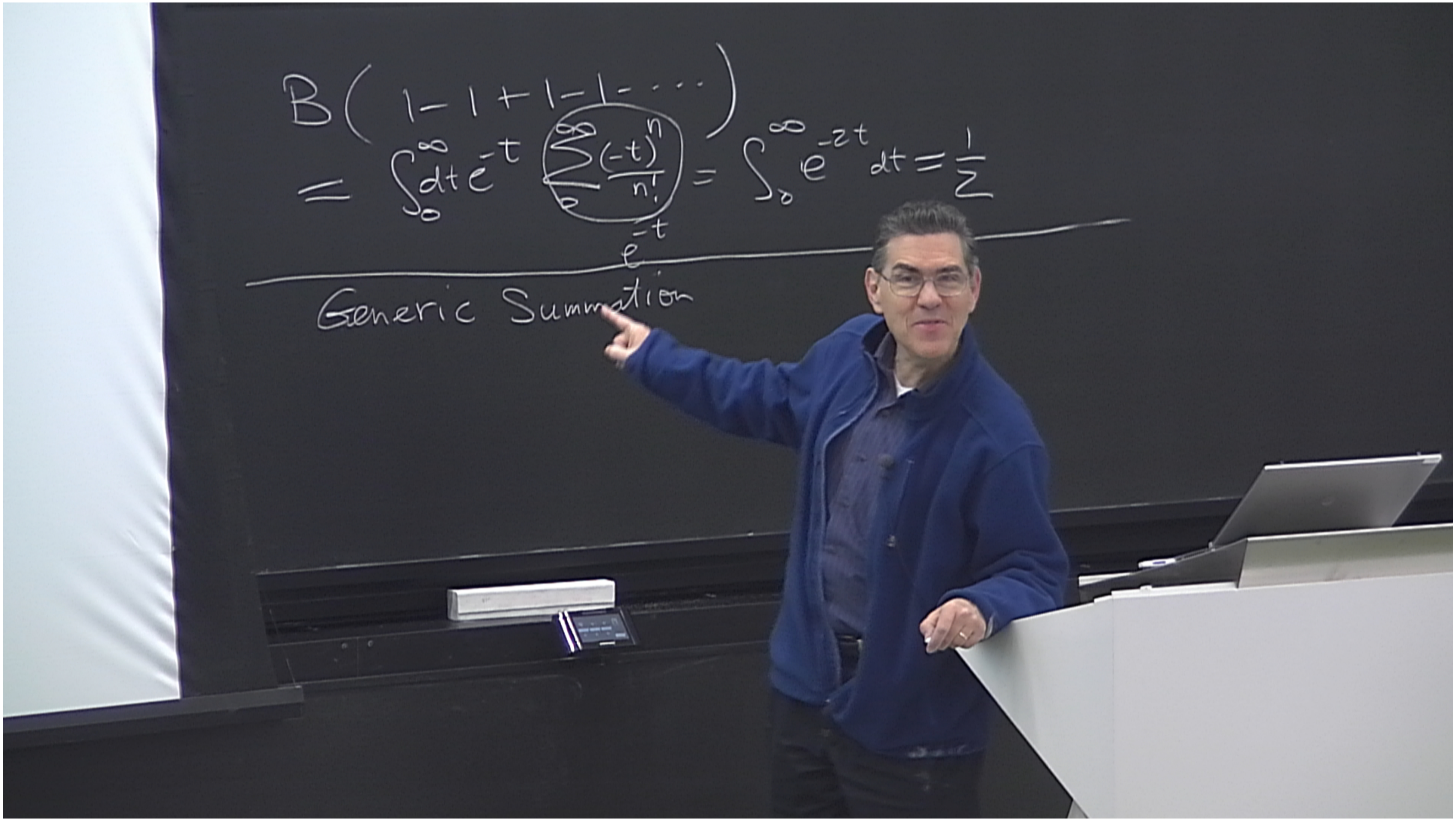
define $E \equiv \lim_{x \to 1} f(x)$
Ex $1-1+1-1+\cdots$
$f(x) = 1 - x + x^2 \cdots$
Borel summation
$\sum_{n=0}^{\infty} a_n$
$\sum_{n=0}^{\infty} a_n \cdot 1$
$\sum_{n=0}^{\infty} a_n \frac{\int_0^{\infty} dt\, e^{-t} t^n}{n!}$
$B \equiv \int_0^{\infty} dt\, e^{-t}$
note: $\int_0^{\infty} dt\, e^{-t} t^n = n!$
$1 = \frac{\int_0^{\infty} dt\, e^{-t} t^n}{n!}$


Euler summation
$\sum_{n=0}^{\infty} a_n$ not conv.
$|x| < 1$
define
Ex
$|x|<1$
Borel summation
$\sum_{n=0}^{\infty} a_n$
$= \sum_{n=0}^{\infty} a_n \cdot 1$
$= \sum_{n=0}^{\infty} a_n \frac{\int_0^\infty dt\, e^{-t} t^n}{n!}$
$B \equiv \int_0^\infty dt\, e^{-t} \left(\sum_{n=0}^{\infty} \frac{t^n a_n}{n!} \right)$
note: $\int_0^\infty dt\, e^{-t} t^n = n!$
$1 = \frac{\int_0^\infty dt\, e^{-t} t^n}{n!}$

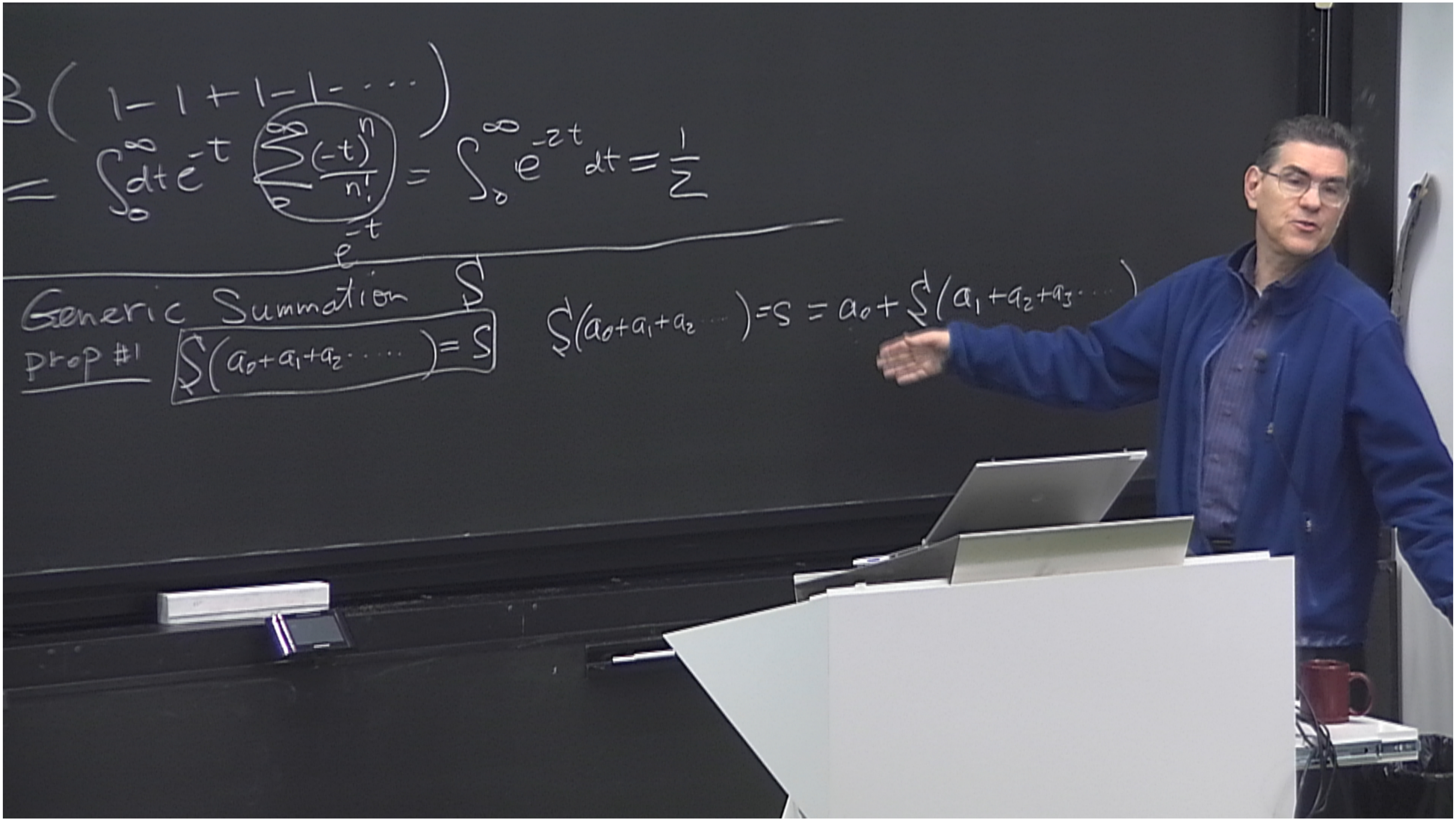

Euler summation
not conv.
|x| < 1
|x| < 1
E = 1/2
Borel summation
note:
B =

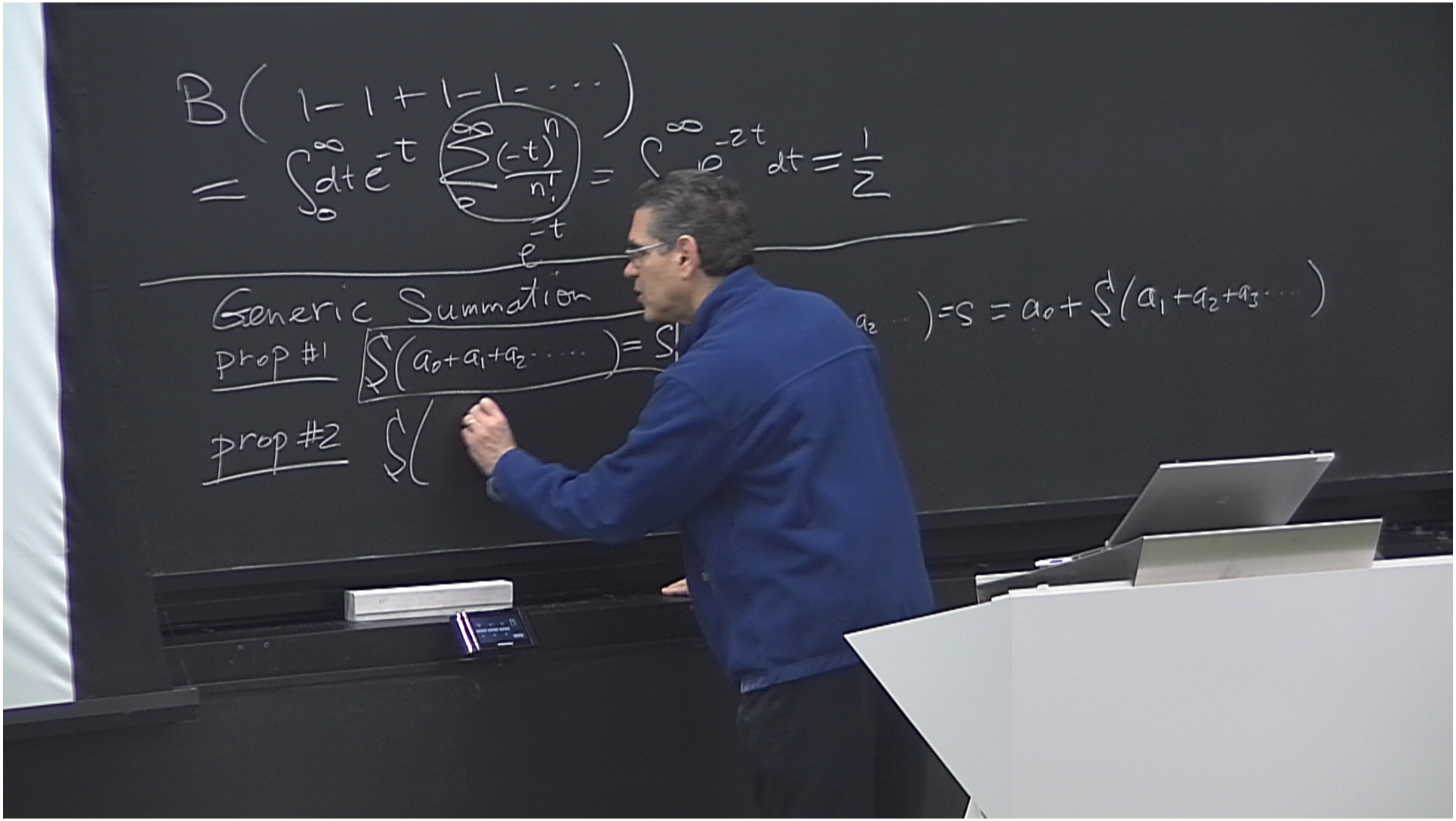

Euler summation
$\sum_{n=0}^{\infty} a_n$ not conv.
$f(x) = \sum_{n=0}^{\infty} a_n x^n$ $|x| < 1$
Define $E \equiv \lim_{x \to 1} f(x)$
Ex $1 - 1 + 1 - 1 + \cdots$
$f(x) = 1 - x + x^2 - x^3 \cdots$ $|x| < 1$
$= \frac{1}{1+x}$
$E = \frac{1}{2}$
Borel summation
$\sum_{n=0}^{\infty} a_n$
$\sum_{n=0}^{\infty} a_n \cdot 1$
$\sum_{n=0}^{\infty} a_n \frac{\int_0^\infty dt\, e^{-t} t^n}{n!}$
$B \equiv \int_0^\infty dt\, e^{-t} \left(\sum_{n=0}^{\infty} \frac{t^n a_n}{n!} \right)$
note: $\int_0^\infty dt\, e^{-t} t^n = n!$
$1 = \frac{\int_0^\infty dt\, e^{-t} t^n}{n!}$

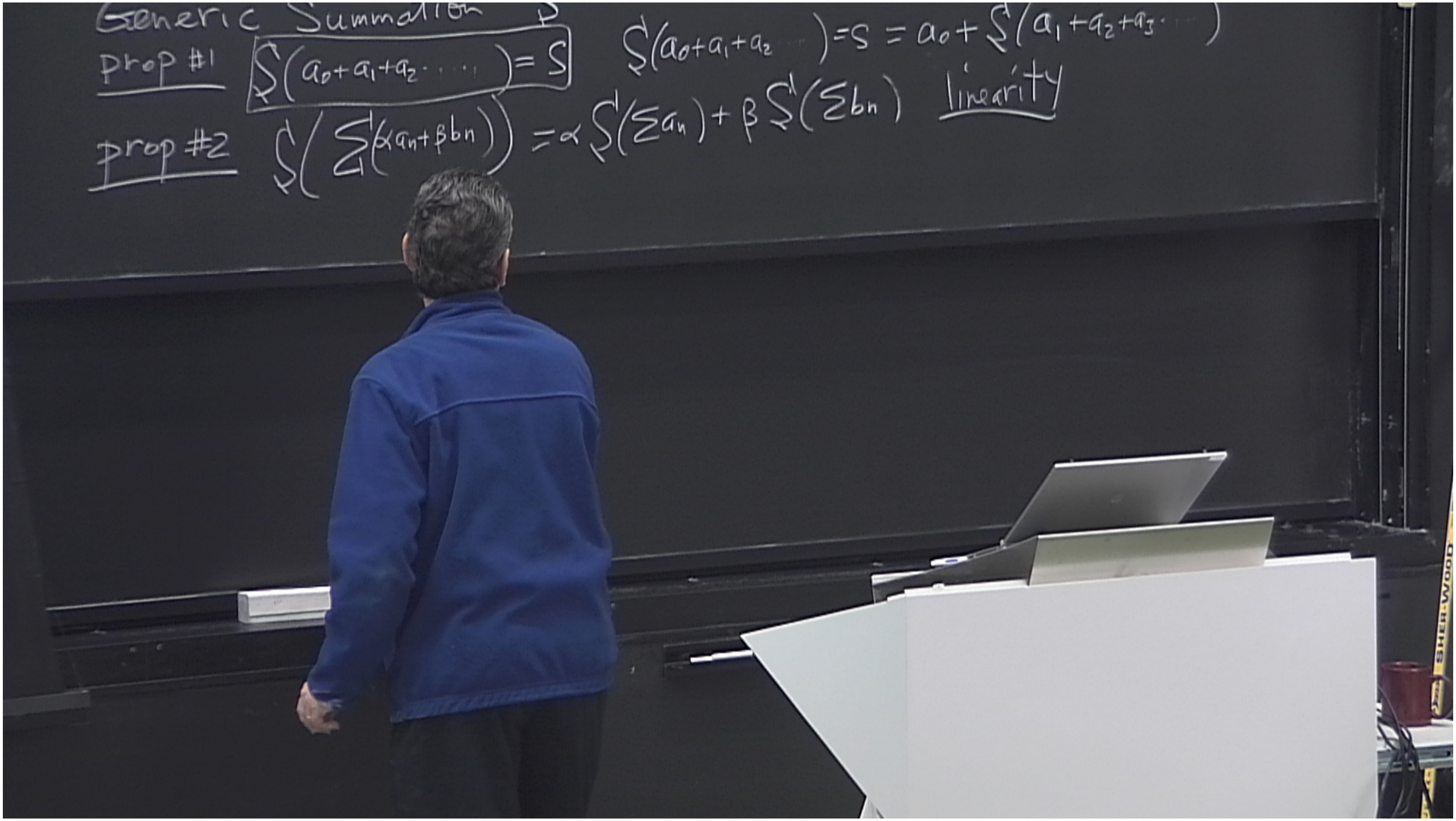

$B(1-1+1-1-\cdots)$
$= \int_0^\infty dt\, e^{-t} \sum_0^\infty \frac{(-t)^n}{n!} = \int_0^\infty e^{-2t} dt = \frac{1}{2}$
e^{-t}

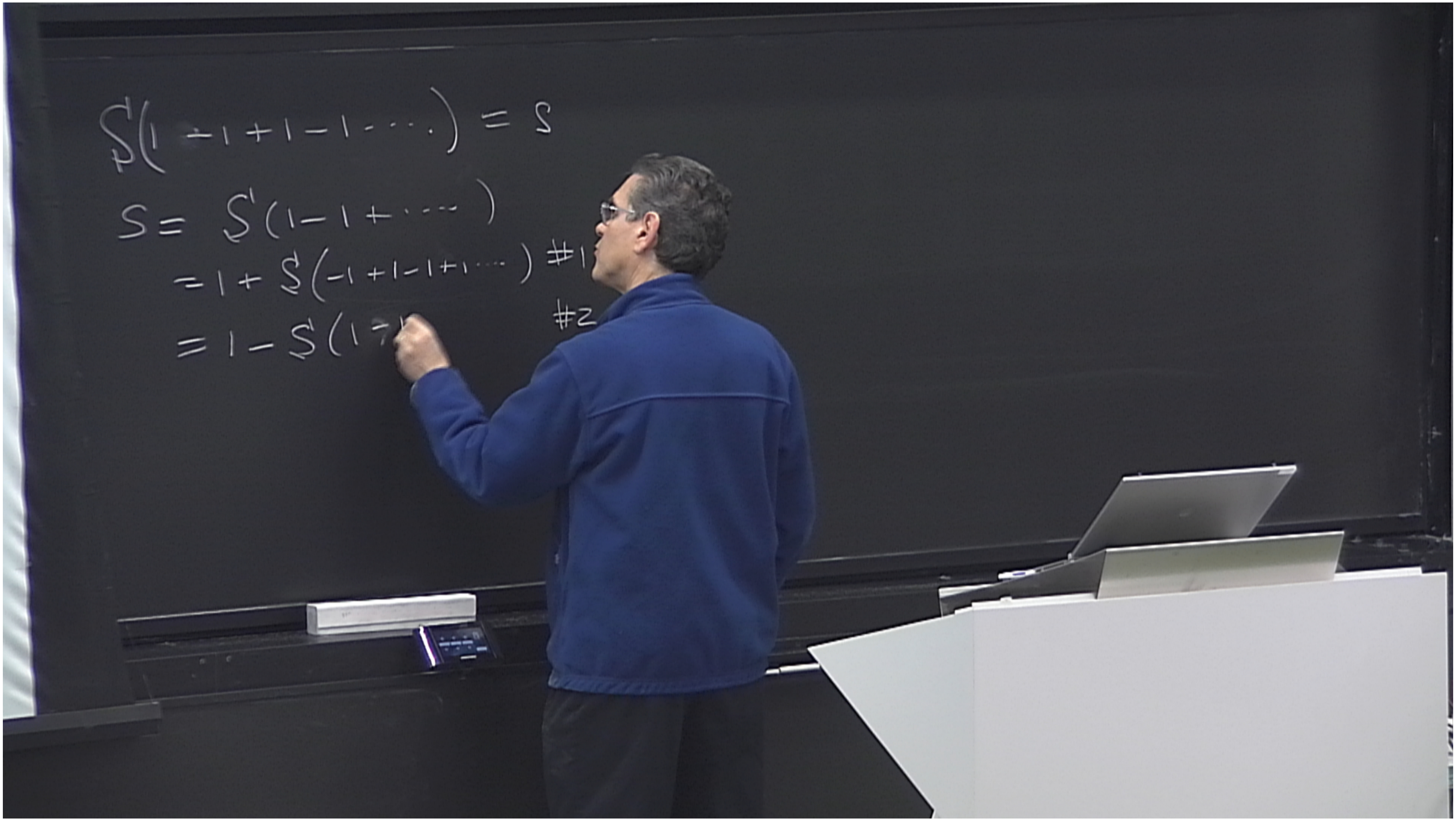


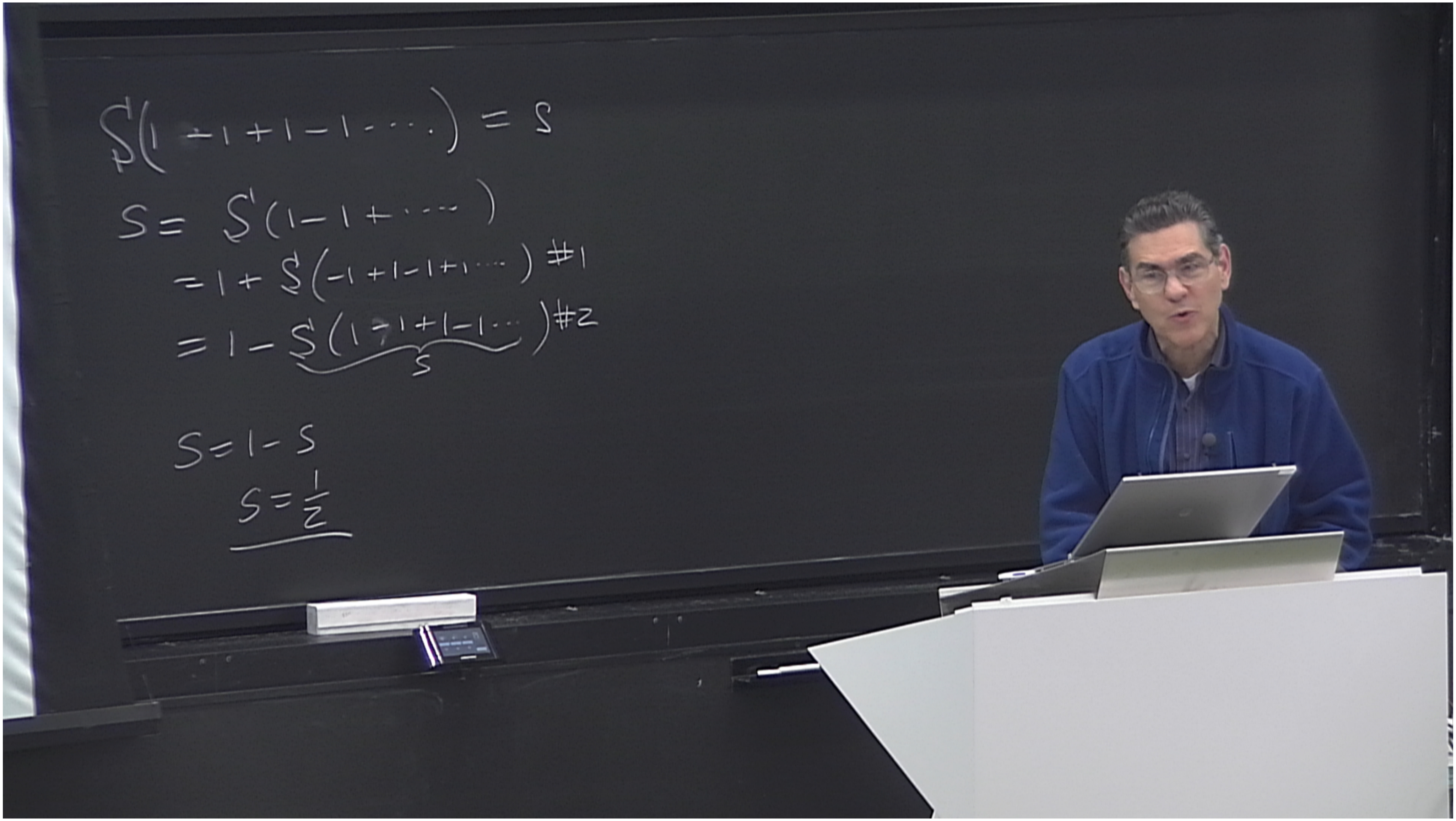
$B(1-1+1-1-\cdots)$
$= \int_0^\infty dt\, e^{-t} \sum_0^\infty \frac{(-t)^n}{n!} = \int_0^\infty e^{-2t}\, dt = \frac{1}{2}$
e^{-t}
Generic Summation

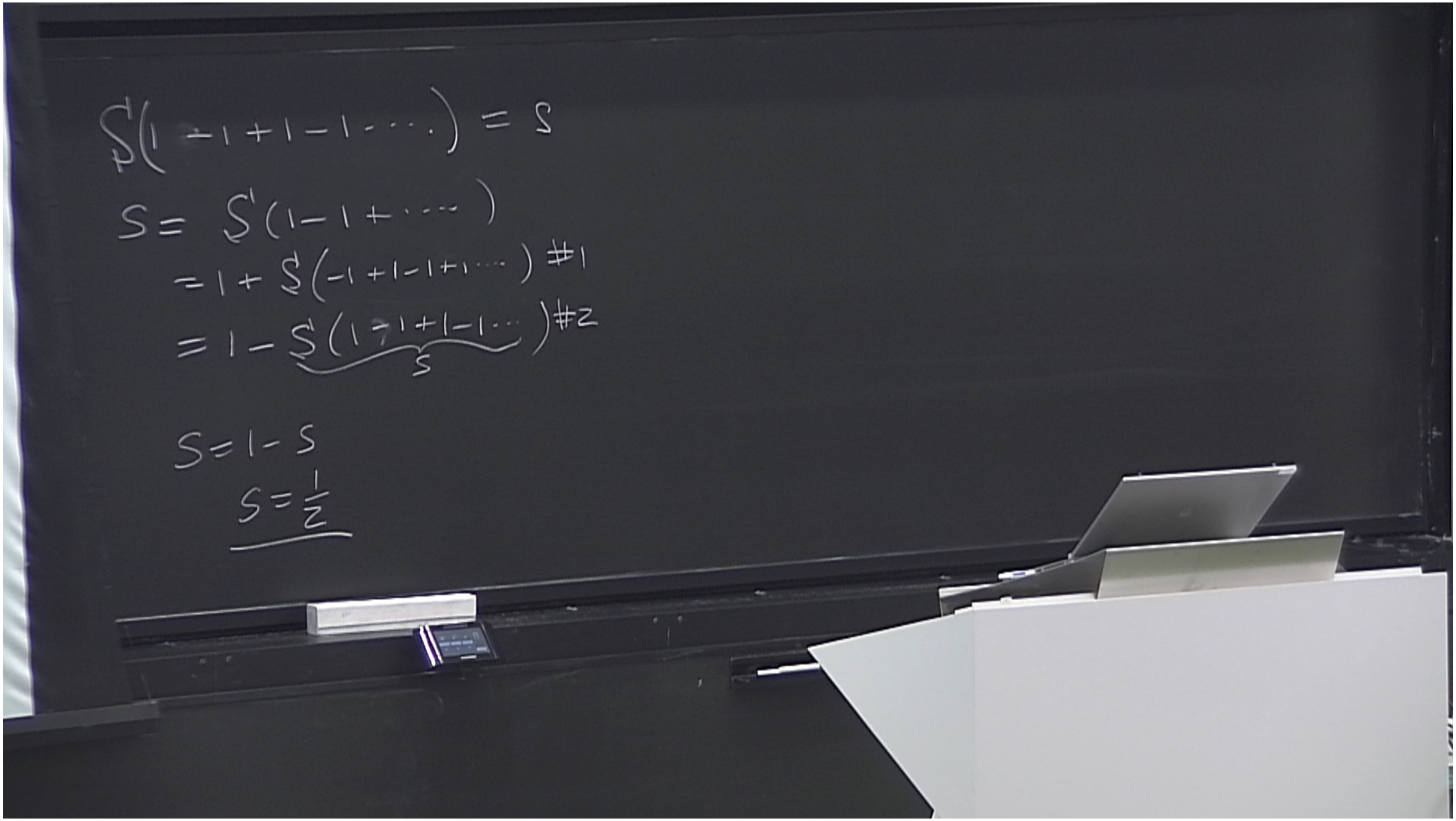

Generic Summation
prop #1

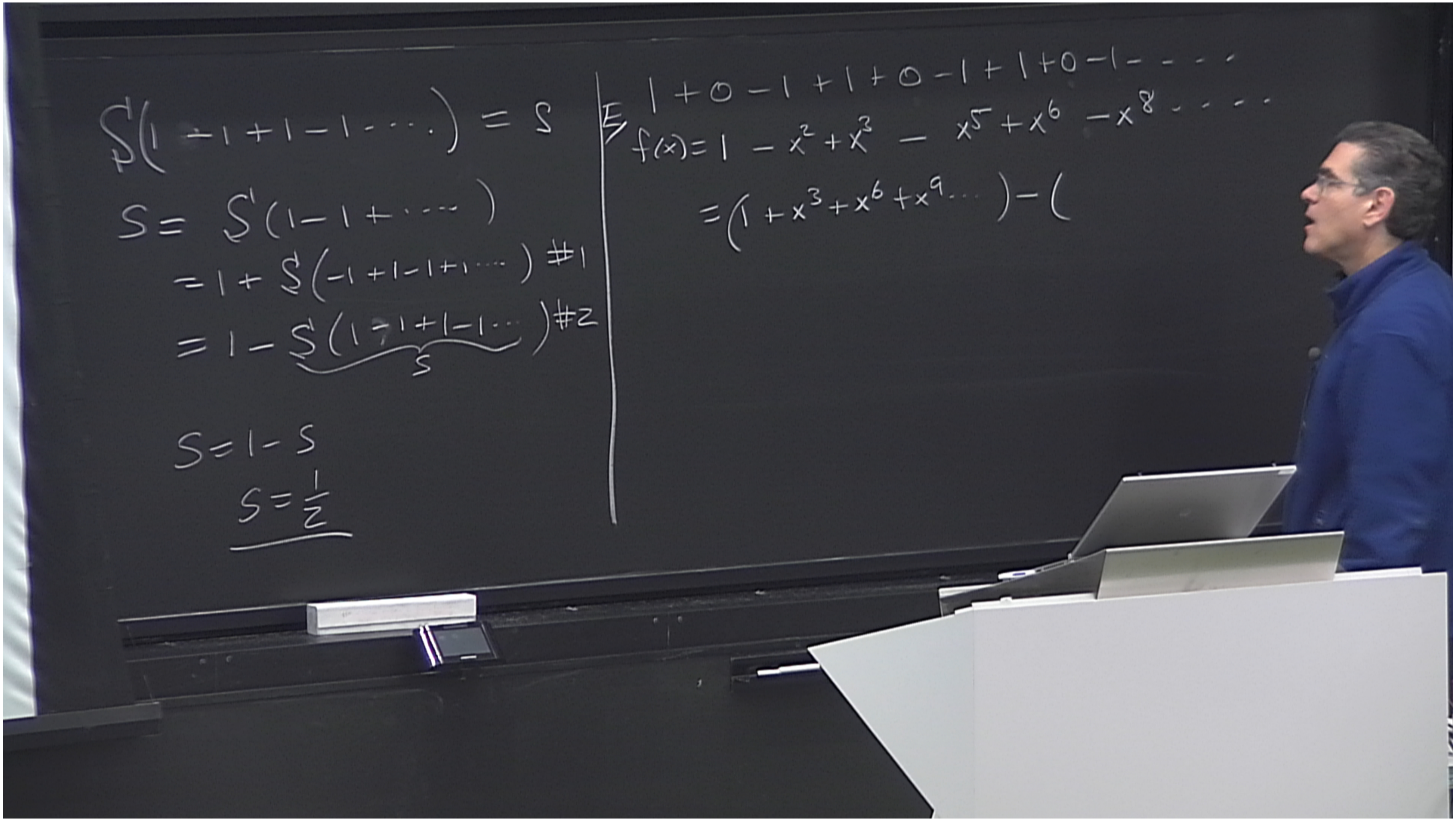

B(1-1+1-1-...)
Generic Summation
prop #1
prop #2

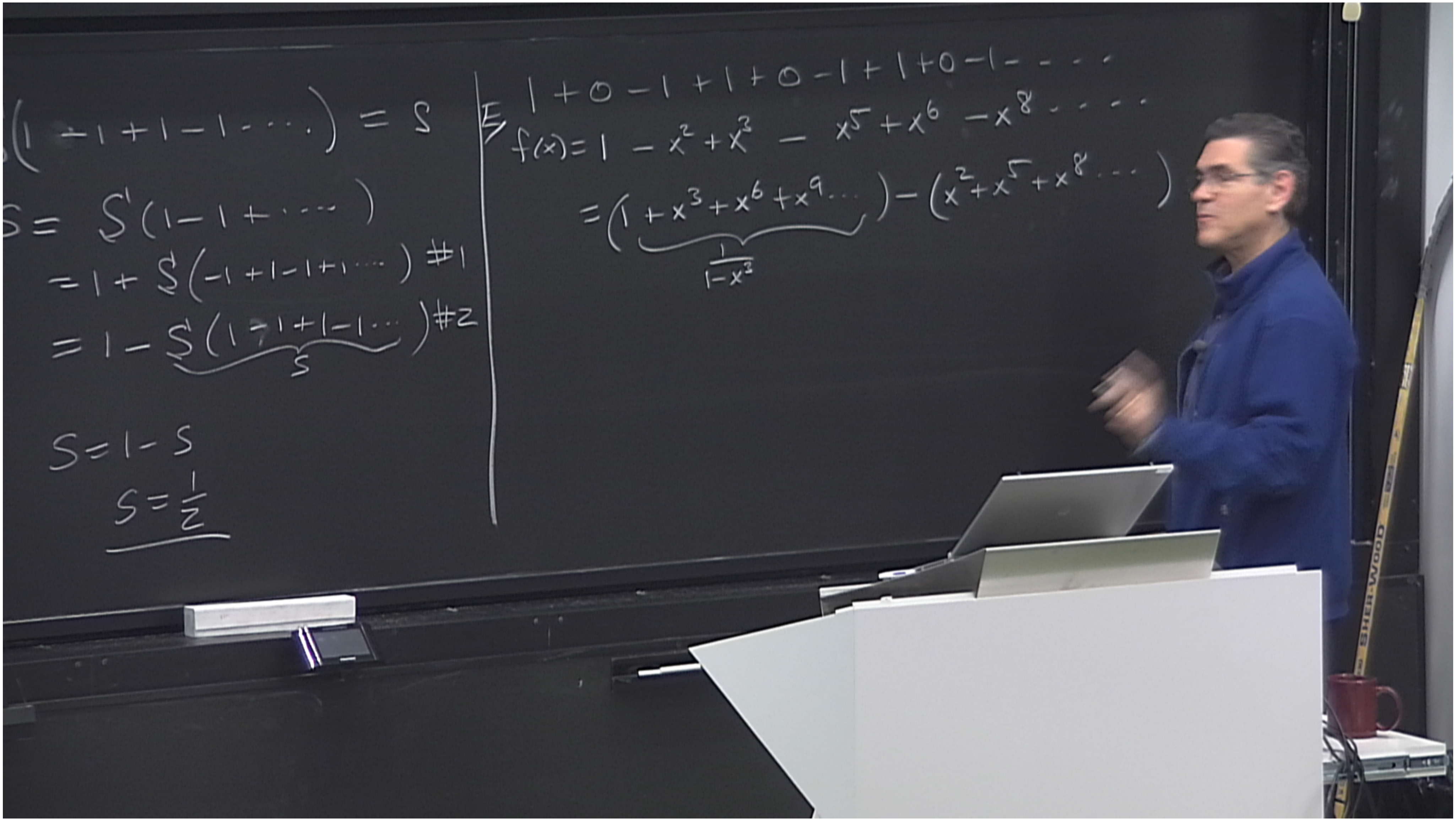

Generic Summation S
prop #1 S(a0+a1+a2 ....)= S
S(a0+a1+a2 ...)=S = a0+S(a1+a2+a3 ...)
prop #2 S(Σ(αan+βbn)) = α S(Σan) + β S(Σbn)
linearity


S(1 −1 +1 −1 ···) = S
S = S(1−1+···)
= 1 + S(−1+1−1+1···) #1
#2
= 1 − S(1

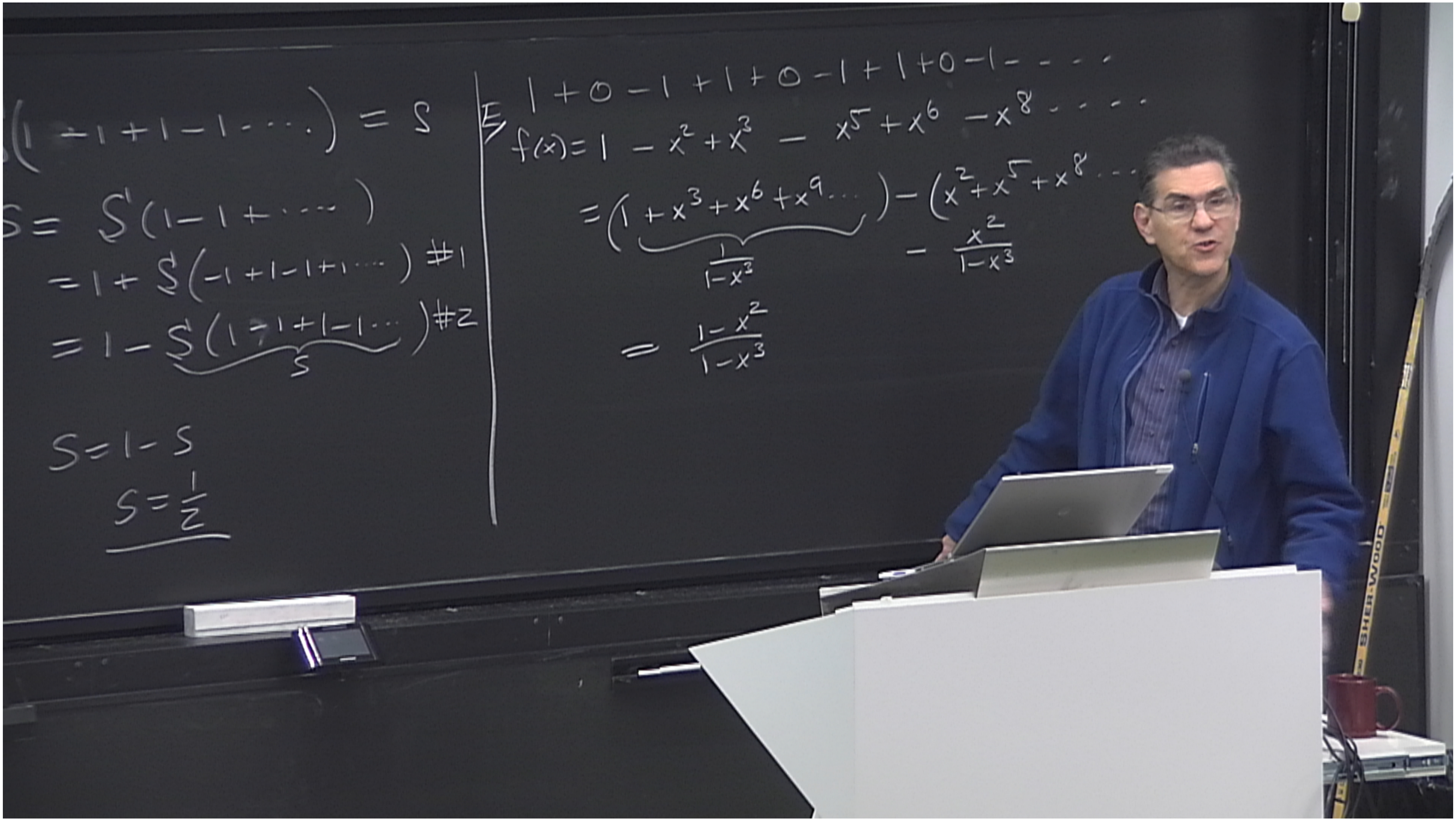

S(1 − 1 + 1 − 1 · · · ·) = s
s = S(1 − 1 + · · ·)
= 1 + S(−1 + 1 − 1 + 1 · · ·) #1
= 1 − S(1 − 1 + 1 − 1 · · ·) #2
s
s = 1 − s
s = 1/2

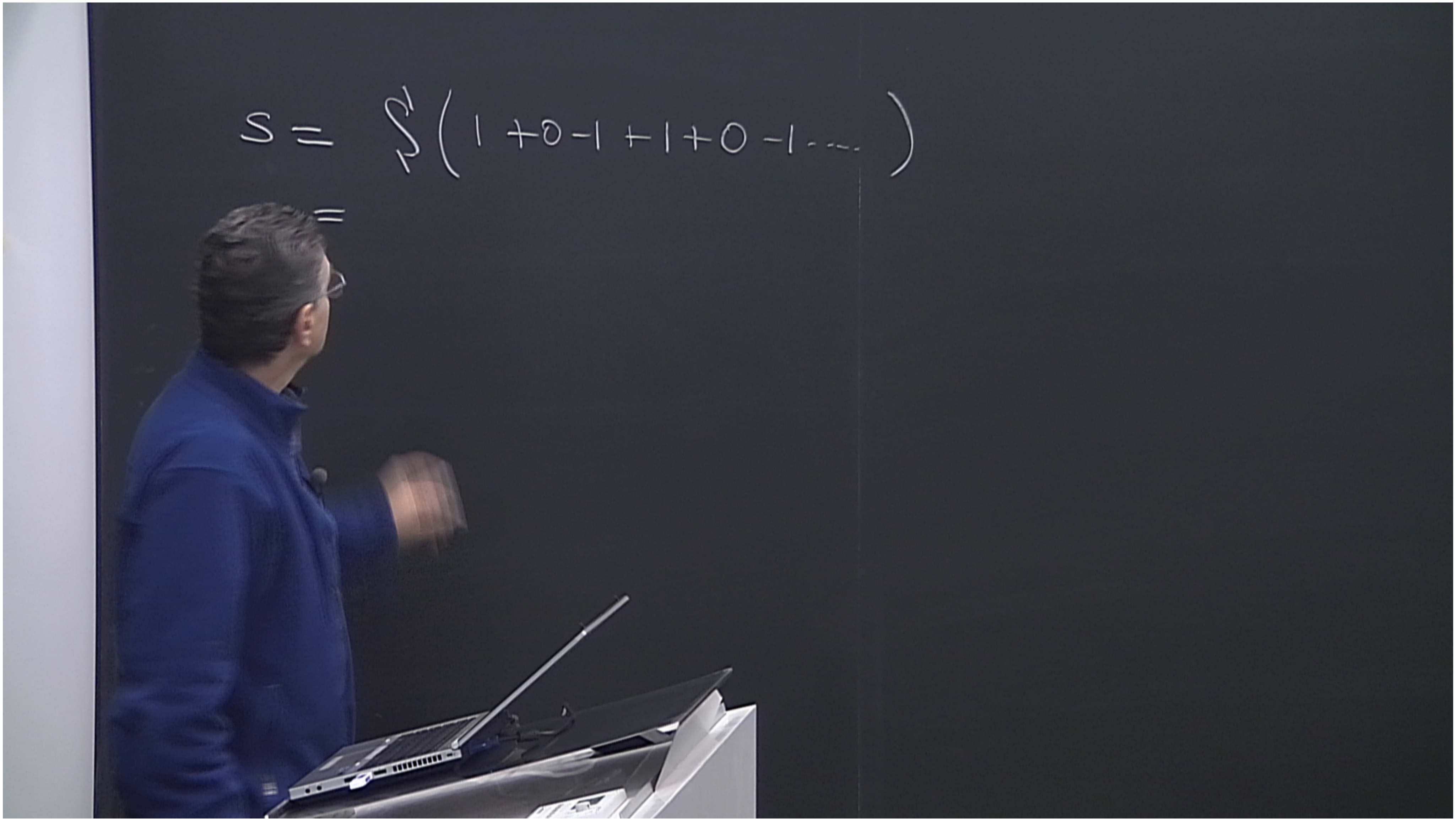

S(1 - 1 + 1 - 1 ....) = S
S = S(1 - 1 + ...)
= 1 + S(-1 + 1 - 1 + 1 ....) #1
= 1 - S(1 - 1 + 1 - 1 ...) #2
S
S = 1 - S
S = 1/2

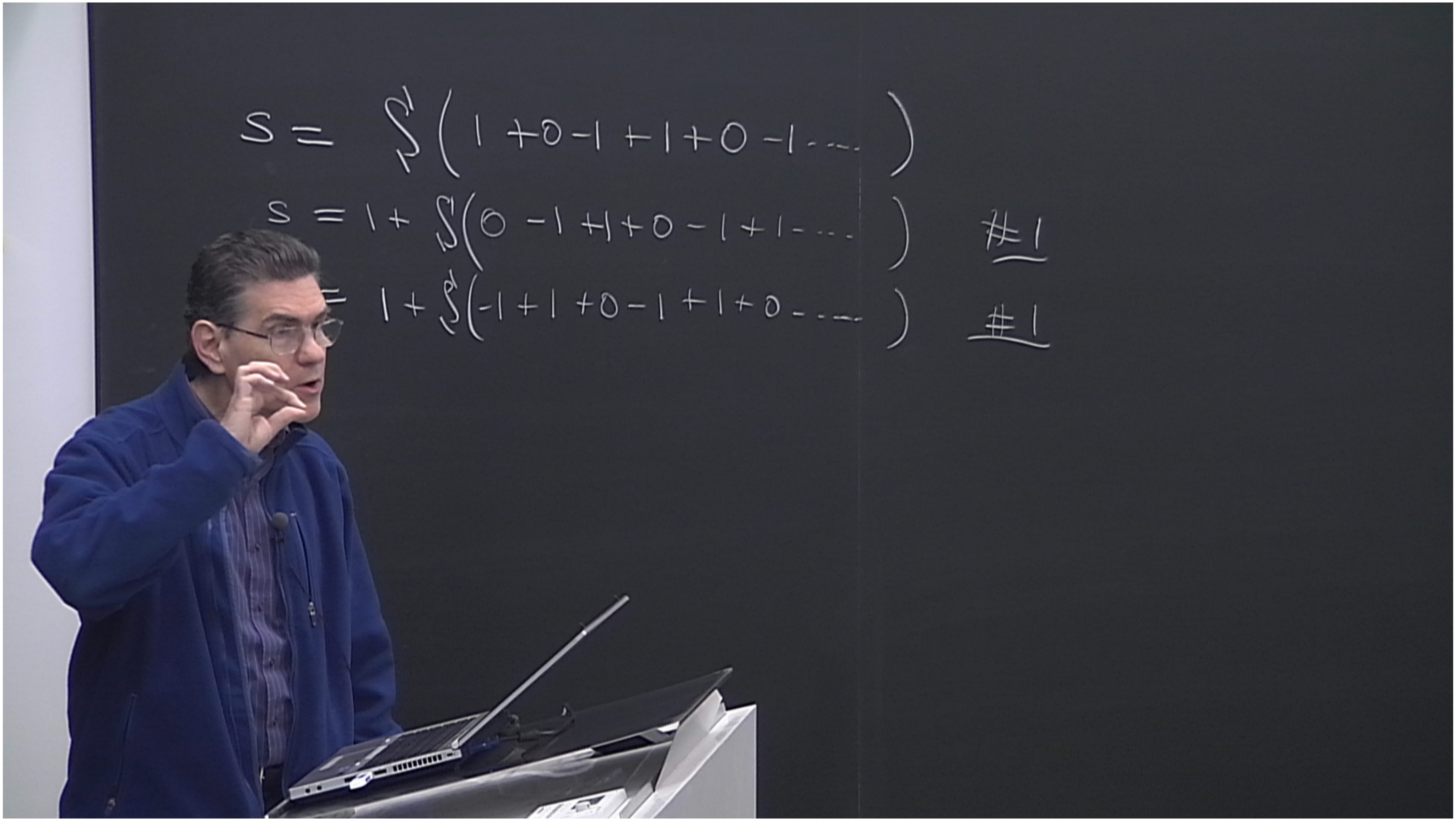

S(1 − 1 + 1 − 1 ....) = S
S = S(1 − 1 + ...)
= 1 + S(−1 + 1 − 1 + 1 ....) #1
= 1 − S(1 − 1 + 1 − 1 ...) #2
S
S = 1 − S
S = 1/2
E
1 + 0 − 1 + 1 + 0 − 1 + 1 + 0 − 1 − ....
f(x) = 1 − x^2 + x^3 − x^5 + x^6 − x^8 ....
= (1 + x^3 + x^6 + x^9 ...) − (

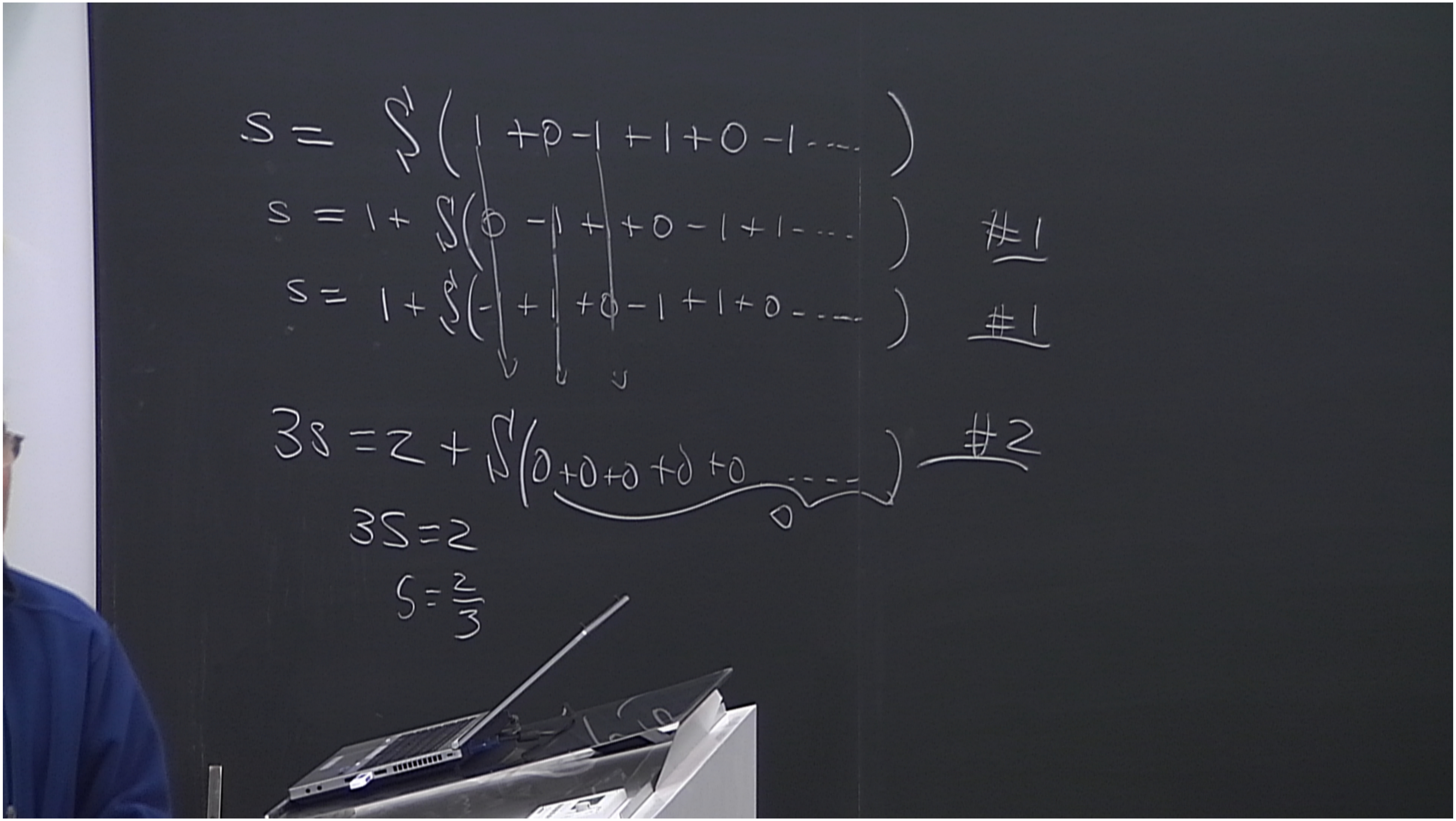


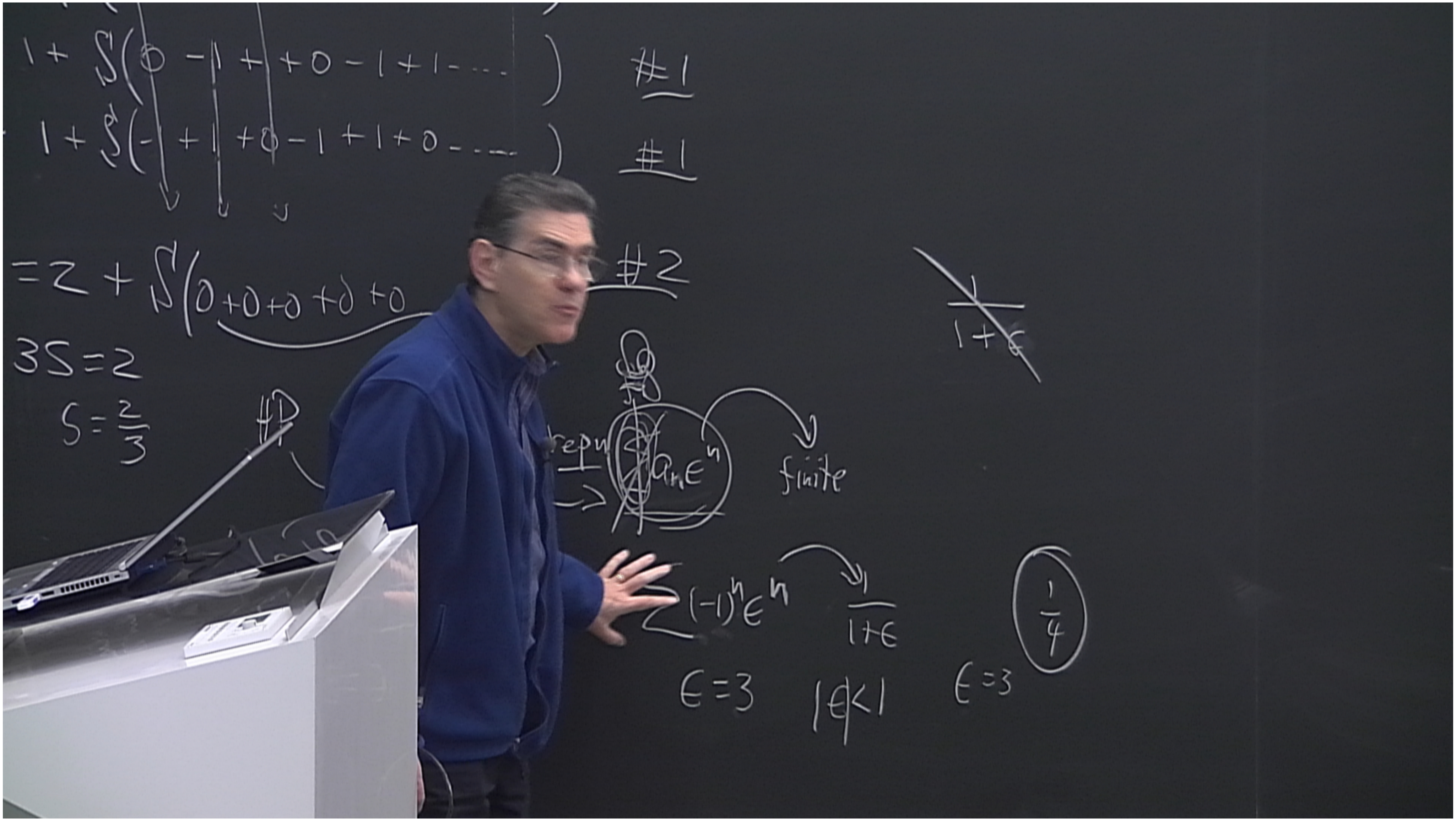
1 + 0 − 1 + 1 + 0 − 1 + 1 + 0 − 1 − . . . .
f(x) = 1 − x² + x³ − x⁵ + x⁶ − x⁸ . . . .
#1
#2
S = 1 − S
S = 1/2


(1 − 1 + 1 − 1 · · ·) = S
S = S(1 − 1 + · · ·)
= 1 + S(−1 + 1 − 1 + 1 · · ·) #1
= 1 − S(1 − 1 + 1 − 1 · · ·) #2
S
S = 1 − S
S = 1/2
E 1 + 0 − 1 + 1 + 0 − 1 + 1 + 0 − 1 − · · ·
f(x) = 1 − x^2 + x^3 − x^5 + x^6 − x^8 · · ·
= (1 + x^3 + x^6 + x^9 · · ·) − (x^2 + x^5 + x^8 · · ·)
1/(1 − x^3)
− x^2/(1 − x^3)
= (1 − x^2)/(1 − x^3)
SHER-WOOD


S =
(1 + 0 − 1 + 1 + 0 − 1 .....)
=


S = S(1 + 0 − 1 + 1 + 0 − 1 .....)
S = 1 + S(0 − 1 + 1 + 0 − 1 + 1 − ....)
#1
= 1 + S(−1 + 1 + 0 − 1 + 1 + 0 .....)
#1


S = S(1 + 0 − 1 + 1 + 0 − 1 ....)
S = 1 + S(0 − 1 + 1 + 0 − 1 + 1 ....) #1
S = 1 + S(−1 + 1 + 0 − 1 + 1 + 0 ....) #1
3S = 2 + S(0 + 0 + 0 + 0 + 0 ....) #2
0
3S = 2
S = 2/3


#1
#1
#2
3S=2
finite
ε=3
|ε|<1
ε=3