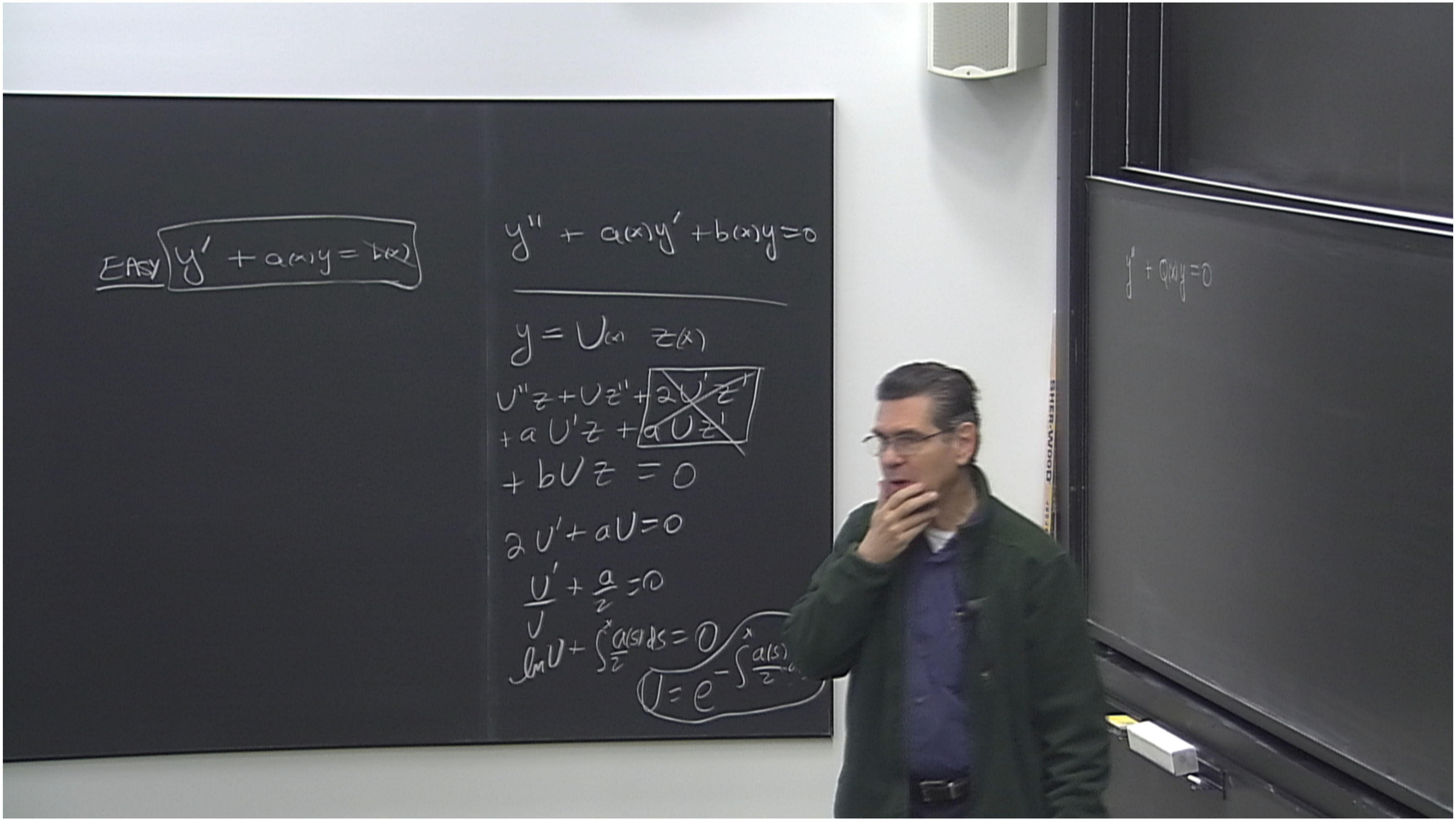
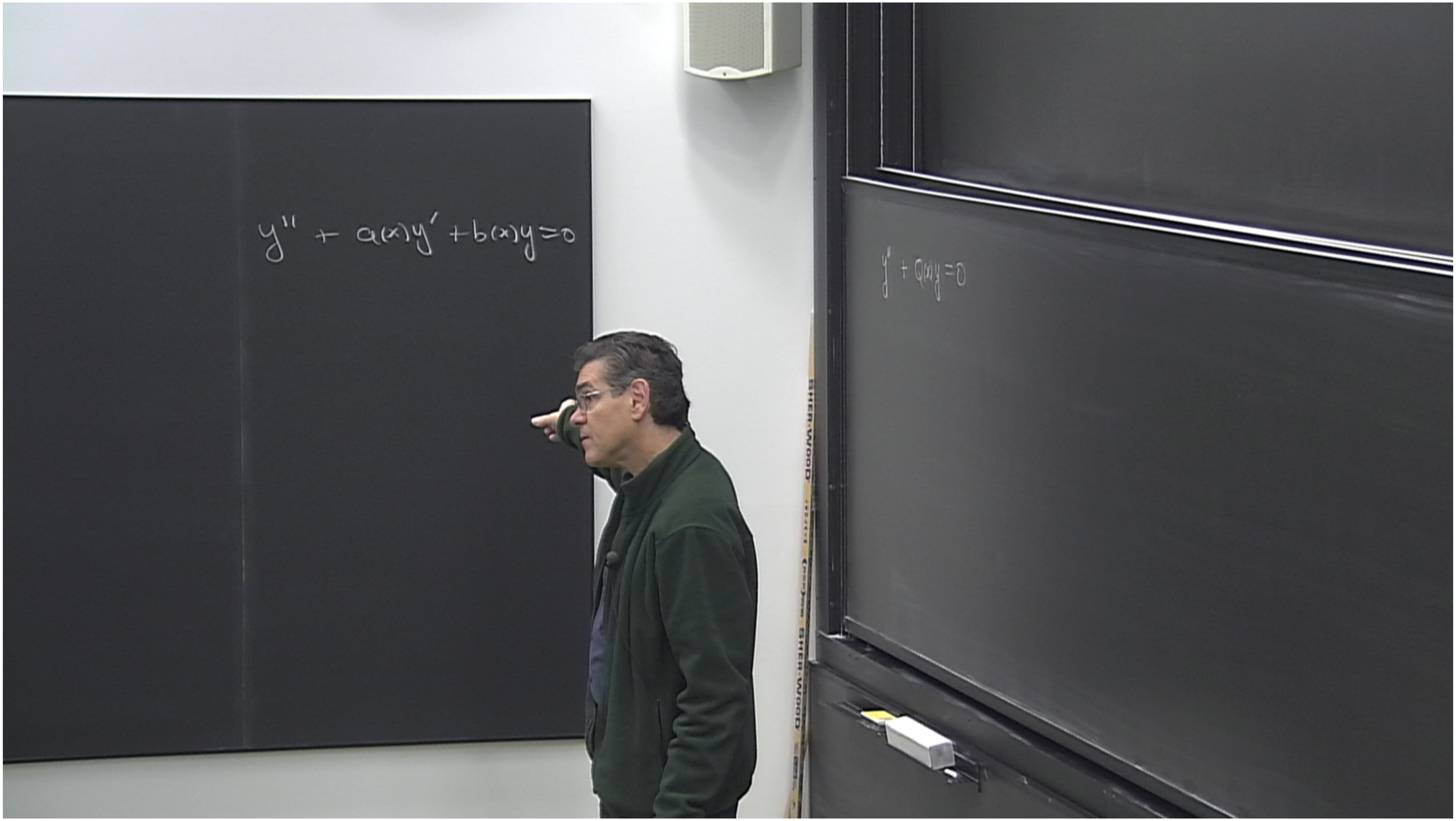
Title: Mathematical Physics - Lecture 2

Date: Nov 22, 2011 09:00 AM

URL: http://pirsa.org/11110041

Abstract:


EASY y' + a(x)y = b(x)
y'' + a(x)y' + b(x)y = 0
y = U(x) z(x)
U''z + Uz'' + 2U'z'
+ aU'z + aUz'
+ bUz = 0
2U' + aU = 0
U'/U + a/2 = 0
ln U + ∫^x a(s)/2 ds = 0
U = e^{-∫^x a(s)/2 ds}
y'' + Q(x)y = 0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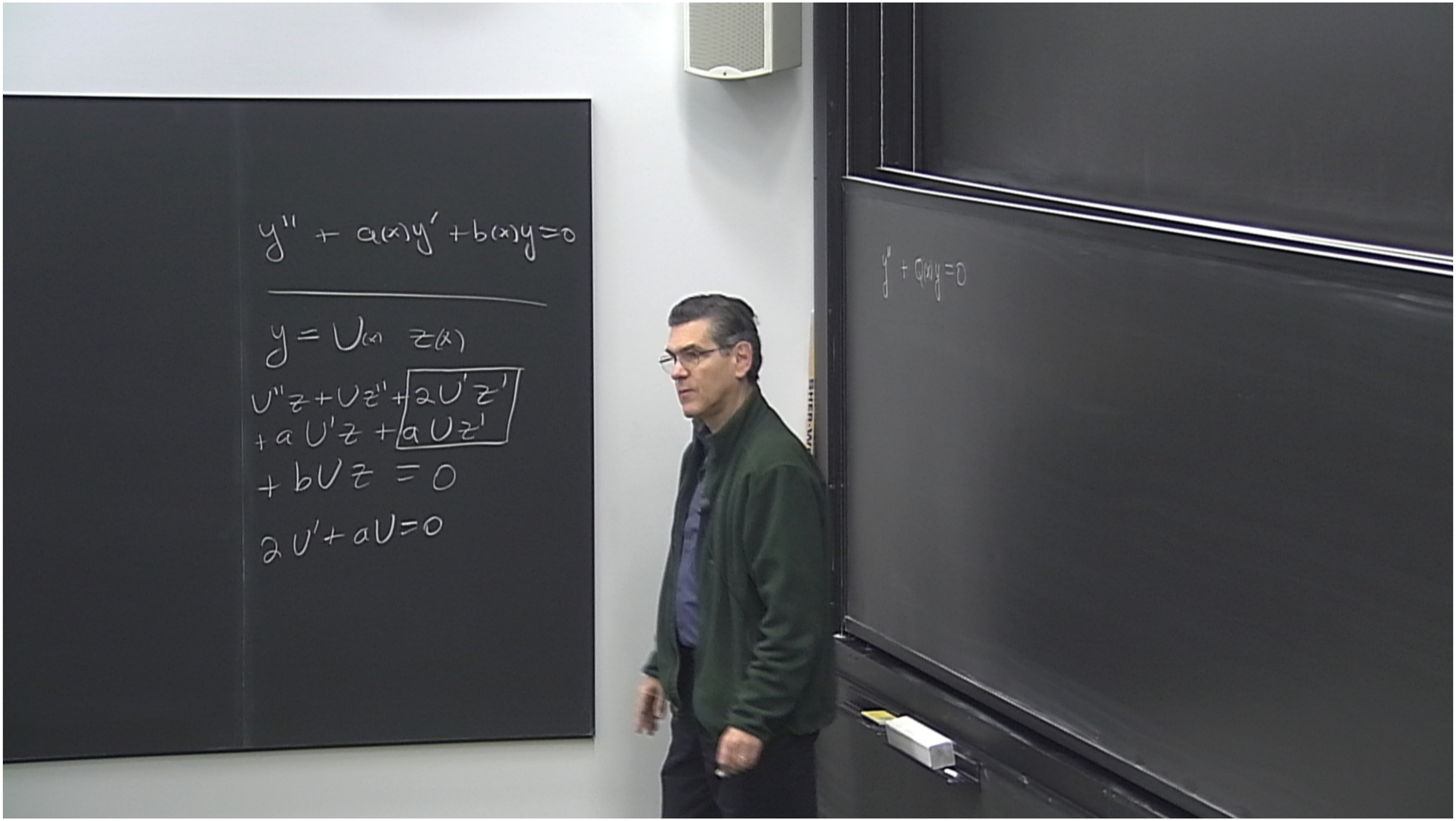
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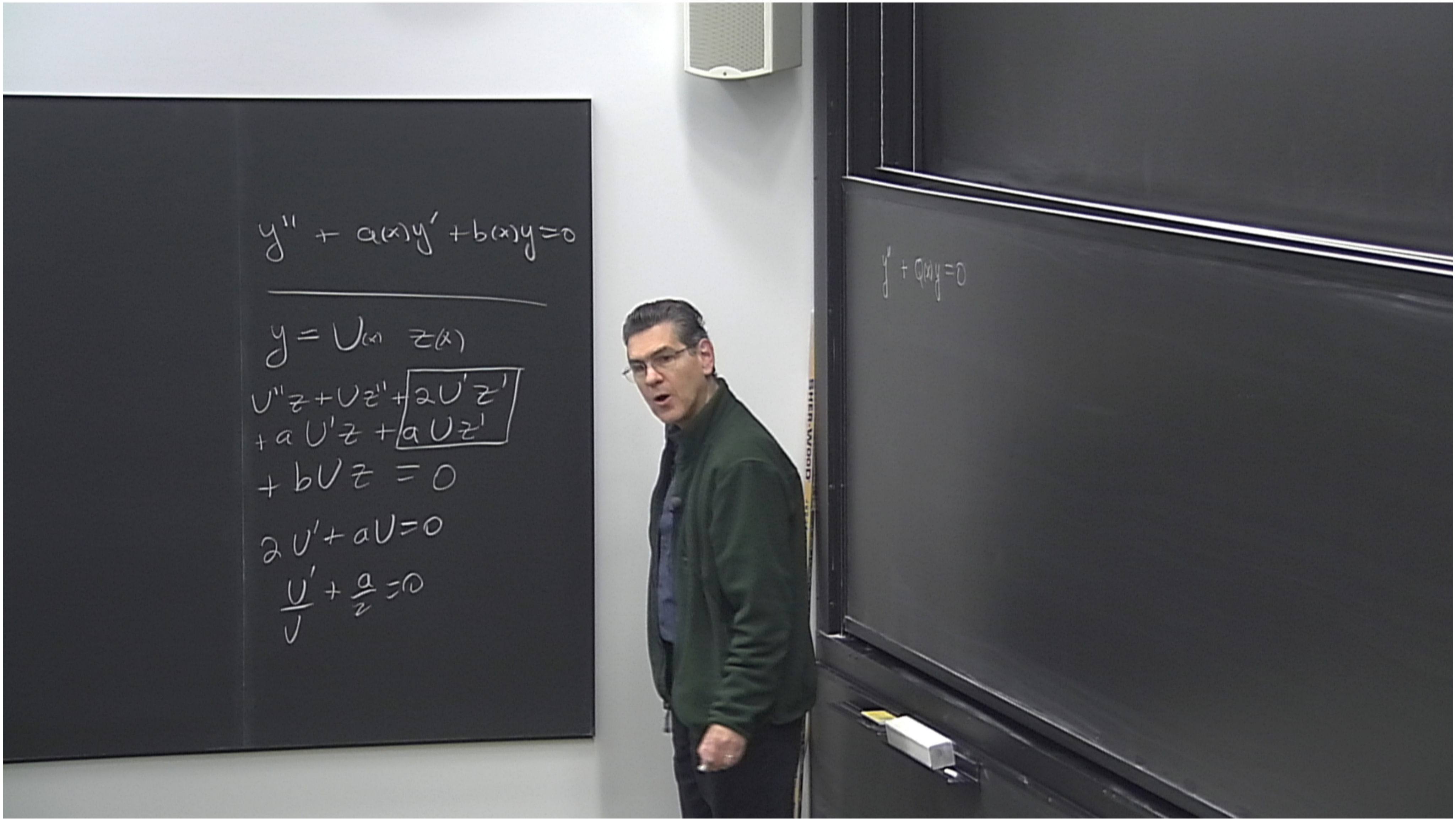
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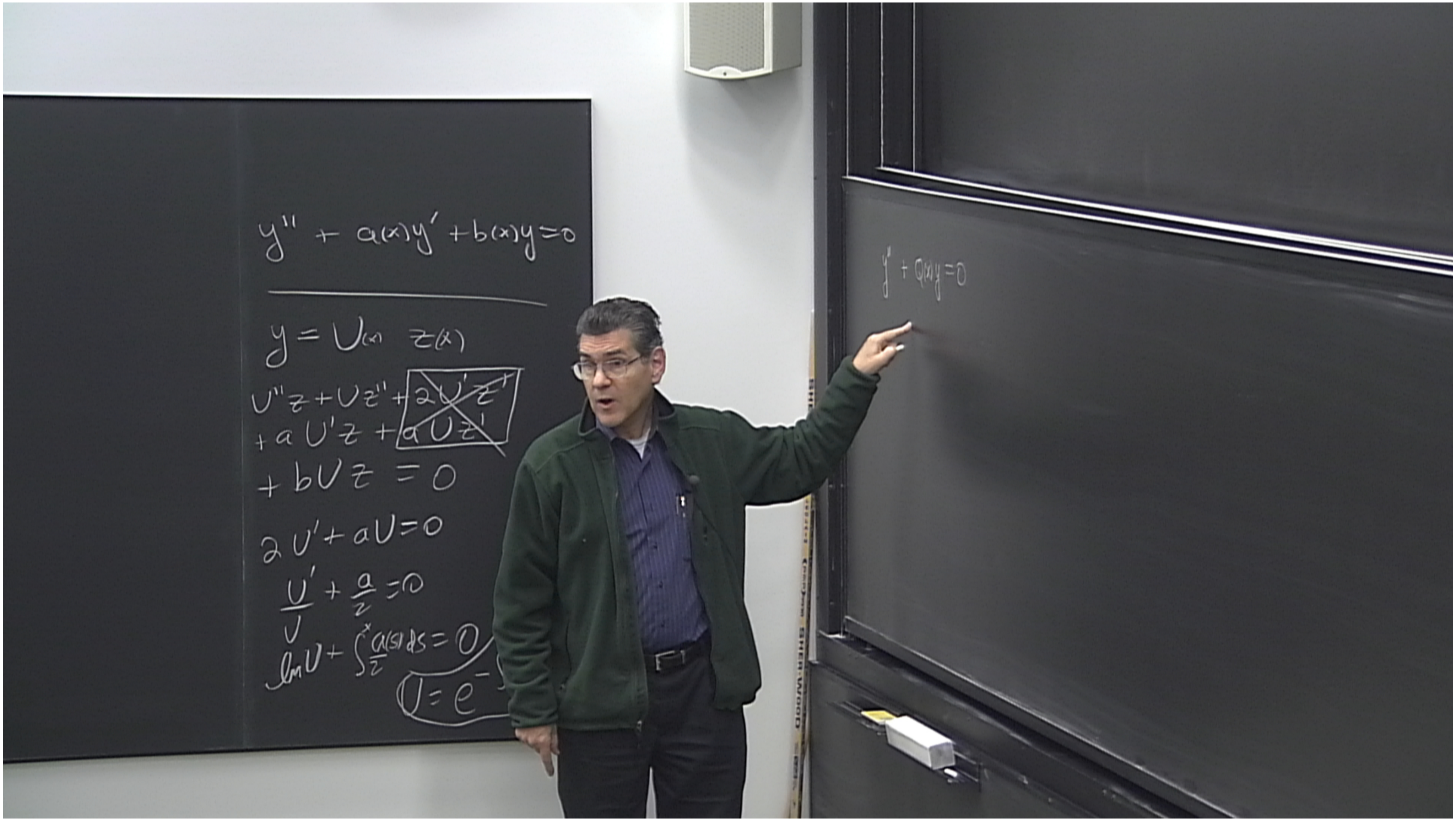
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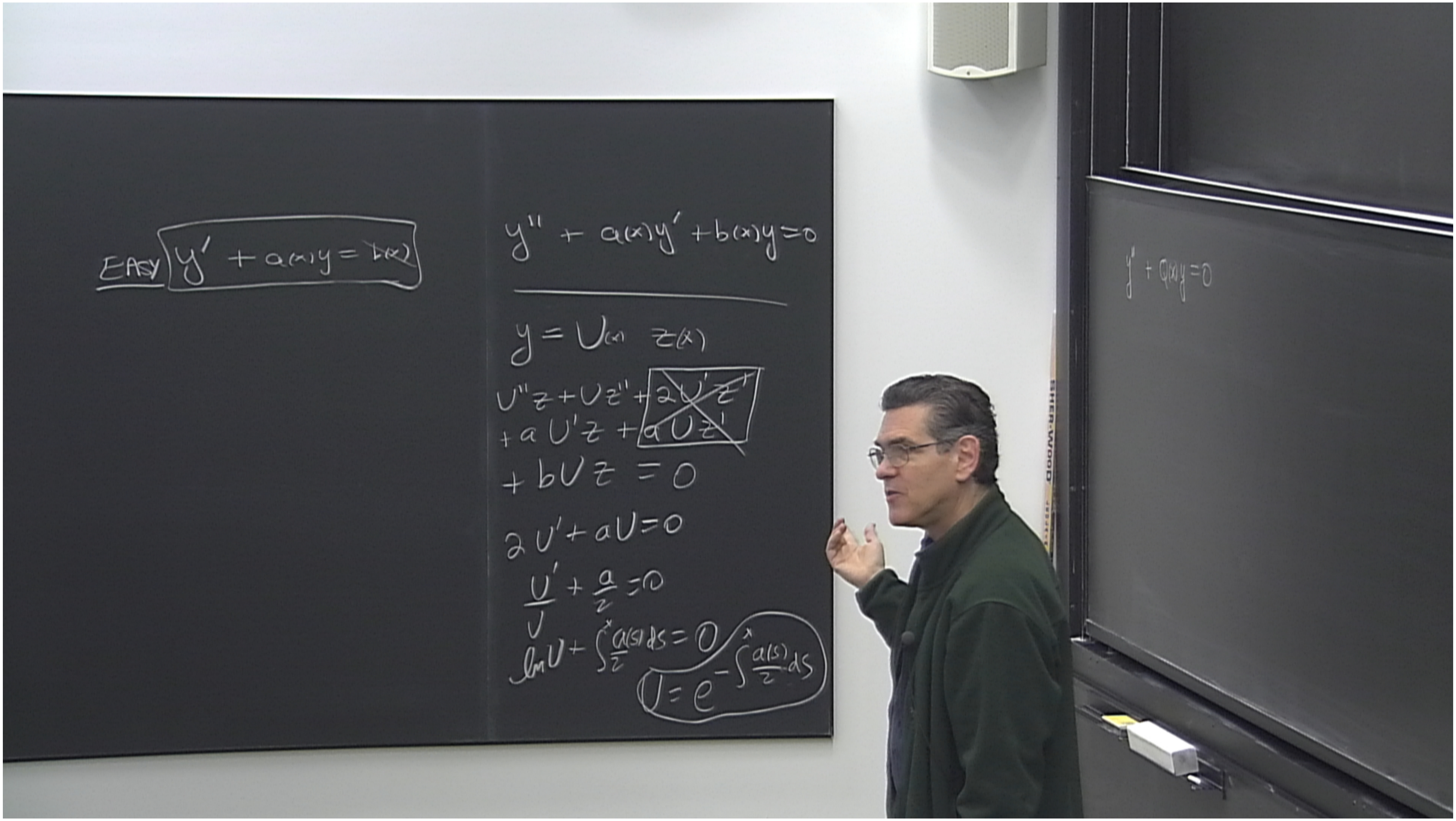
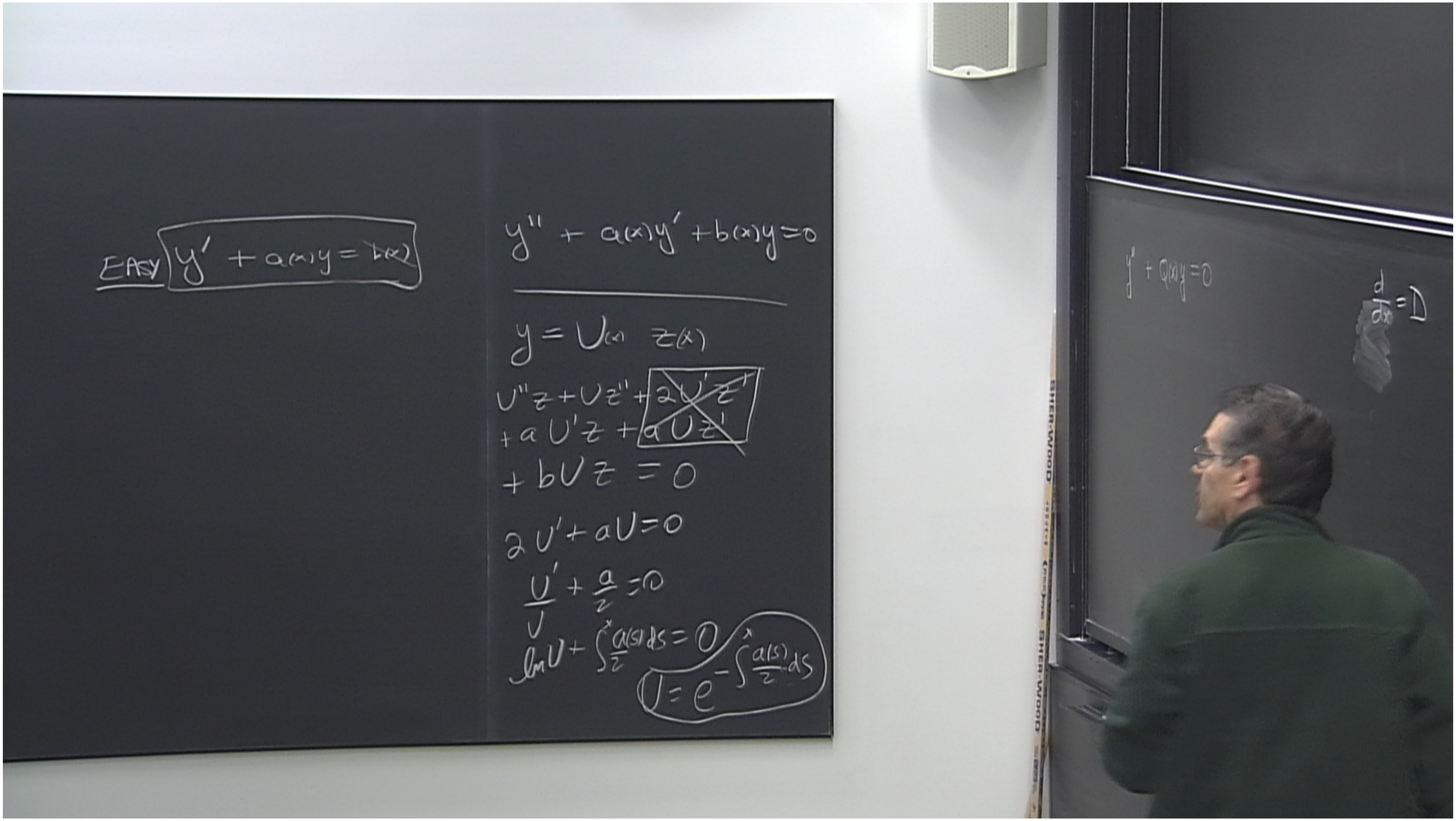

EASY $y' + a(x)y = b(x)$
$y'' + a(x)y' + b(x)y = 0$
$y = U(x)\, z(x)$
$U''z + Uz'' + 2U'z'$
$+ aU'z + aUz'$
$+ bUz = 0$
$2U' + aU = 0$
$\frac{U'}{U} + \frac{a}{2} = 0$
$\ln U + \int^x \frac{a(s)}{2} ds = 0$
$U = e^{-\int^x \frac{a(s)}{2} ds}$
$y'' + Q(x)y = 0$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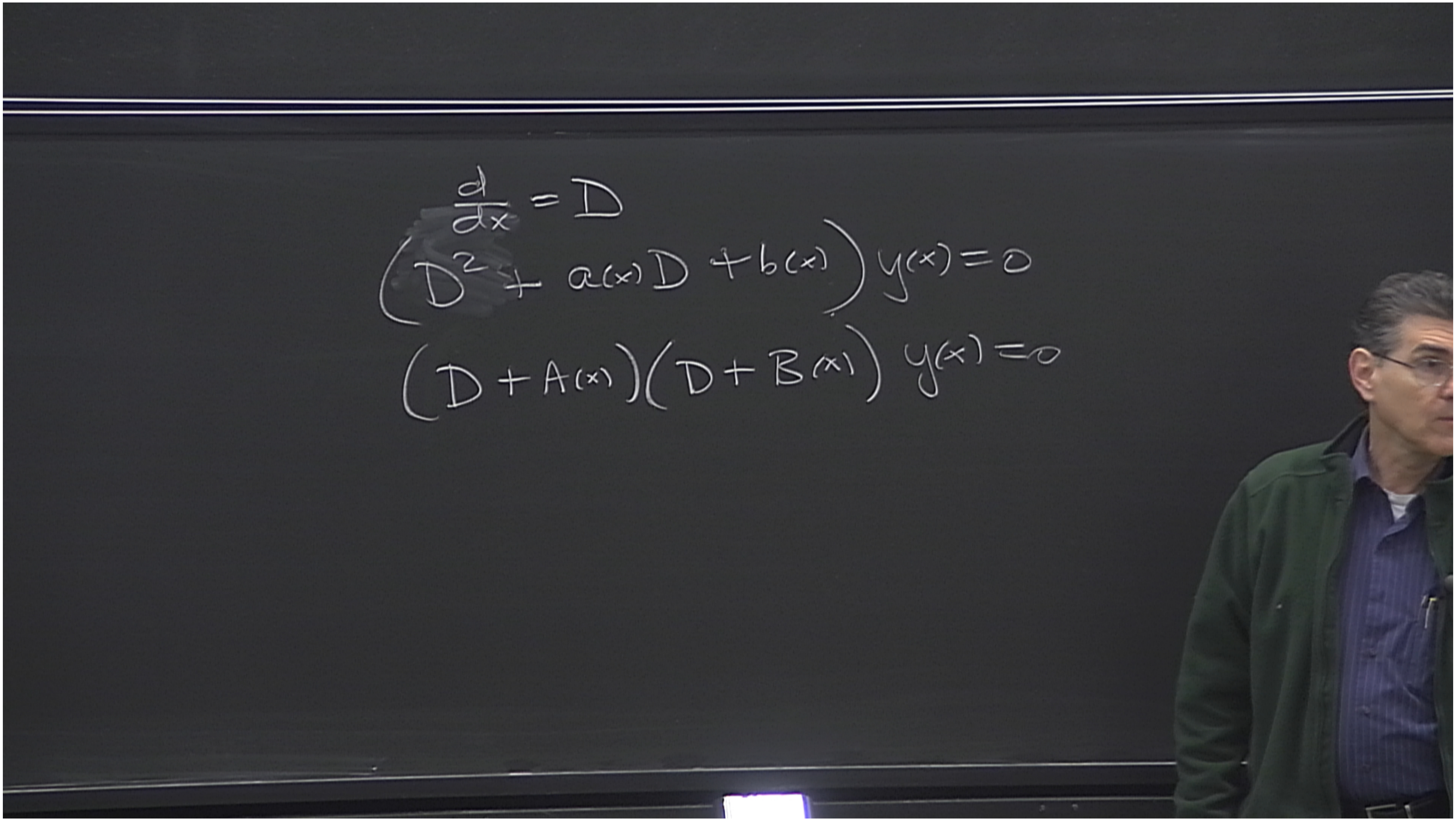
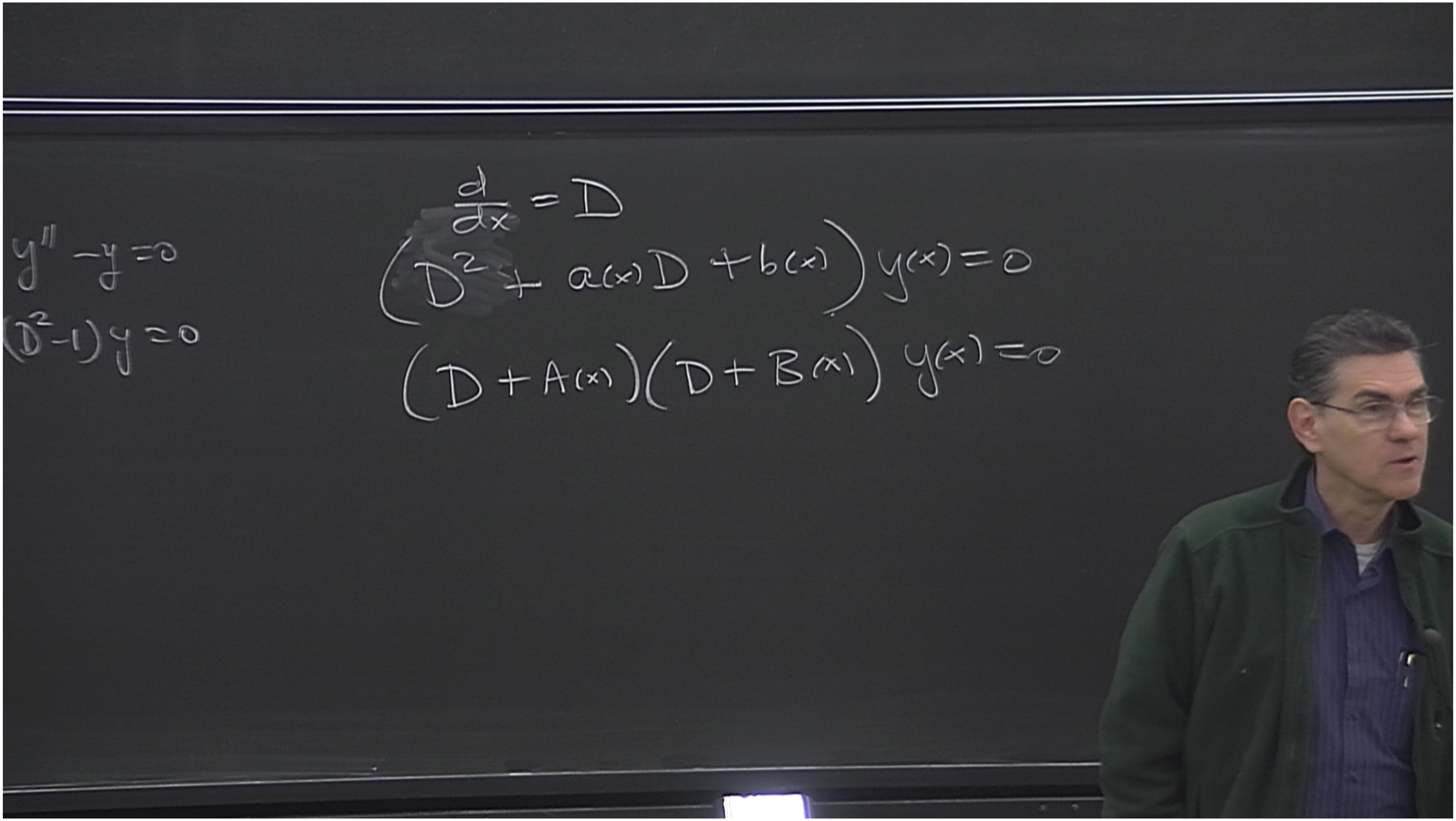
2U' + aU = 0
U'/U + a/2 = 0
ln U + ∫ a(s)/2 ds = 0
U = e^{-∫ a(s)/2 ds}
y'' + Q(x)y = 0
d/dx = D


$\frac{d}{dx} = D$
$\left(D^2 + a(x)D + b(x)\right) y(x) = 0$
$\left(D + A(x)\right)\left(D + B(x)\right) y(x) = 0$


y'' − y = 0
(D² − 1) y = 0
d/dx = D
(D² + a(x) D + b(x)) y(x) = 0
(D + A(x))(D + B(x)) y(x) = 0

$y = 0$

$y'' - y = 0$

$(D^2 - 1) y = 0$

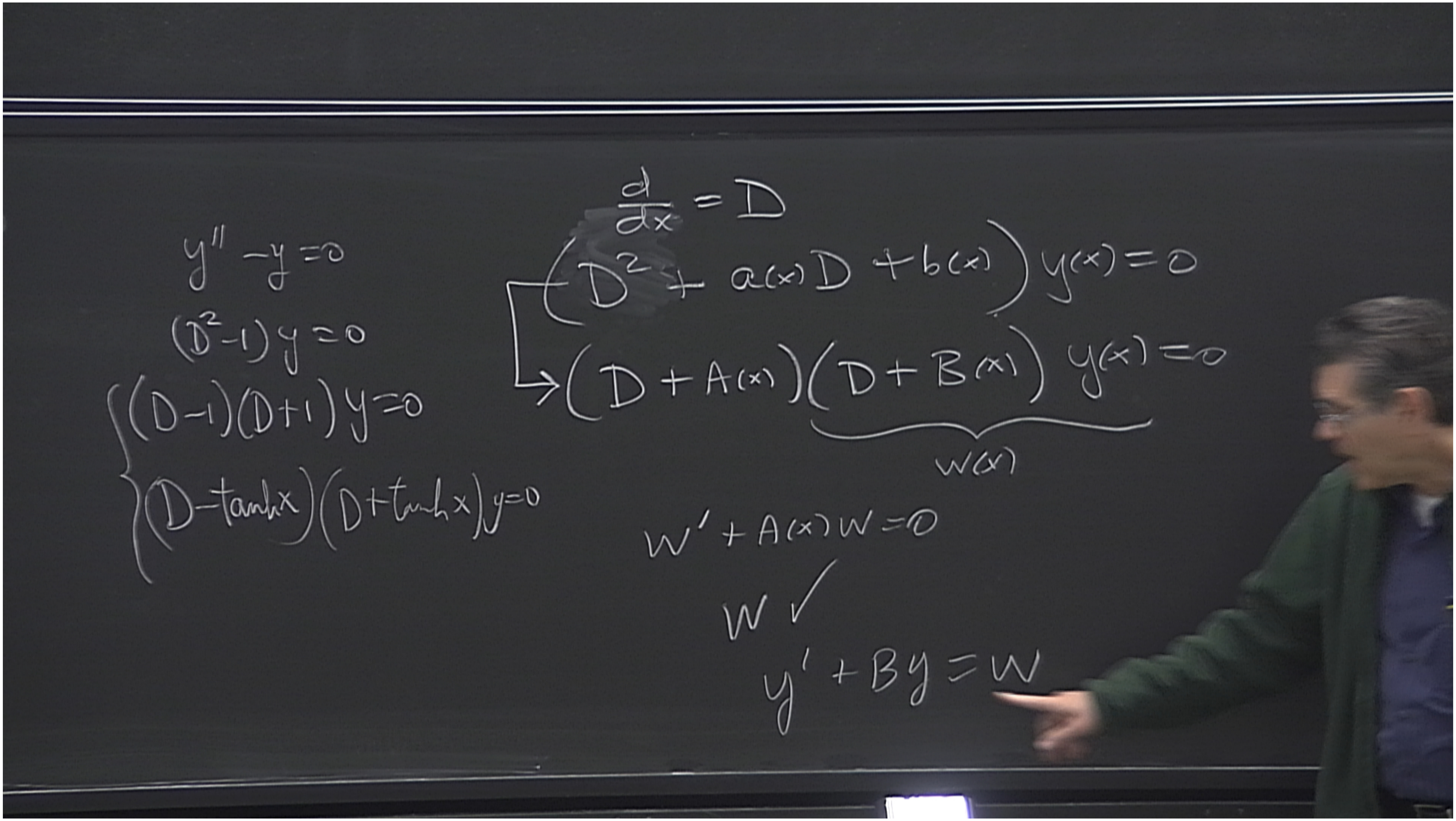
$(D-1)(D+1) y = 0$

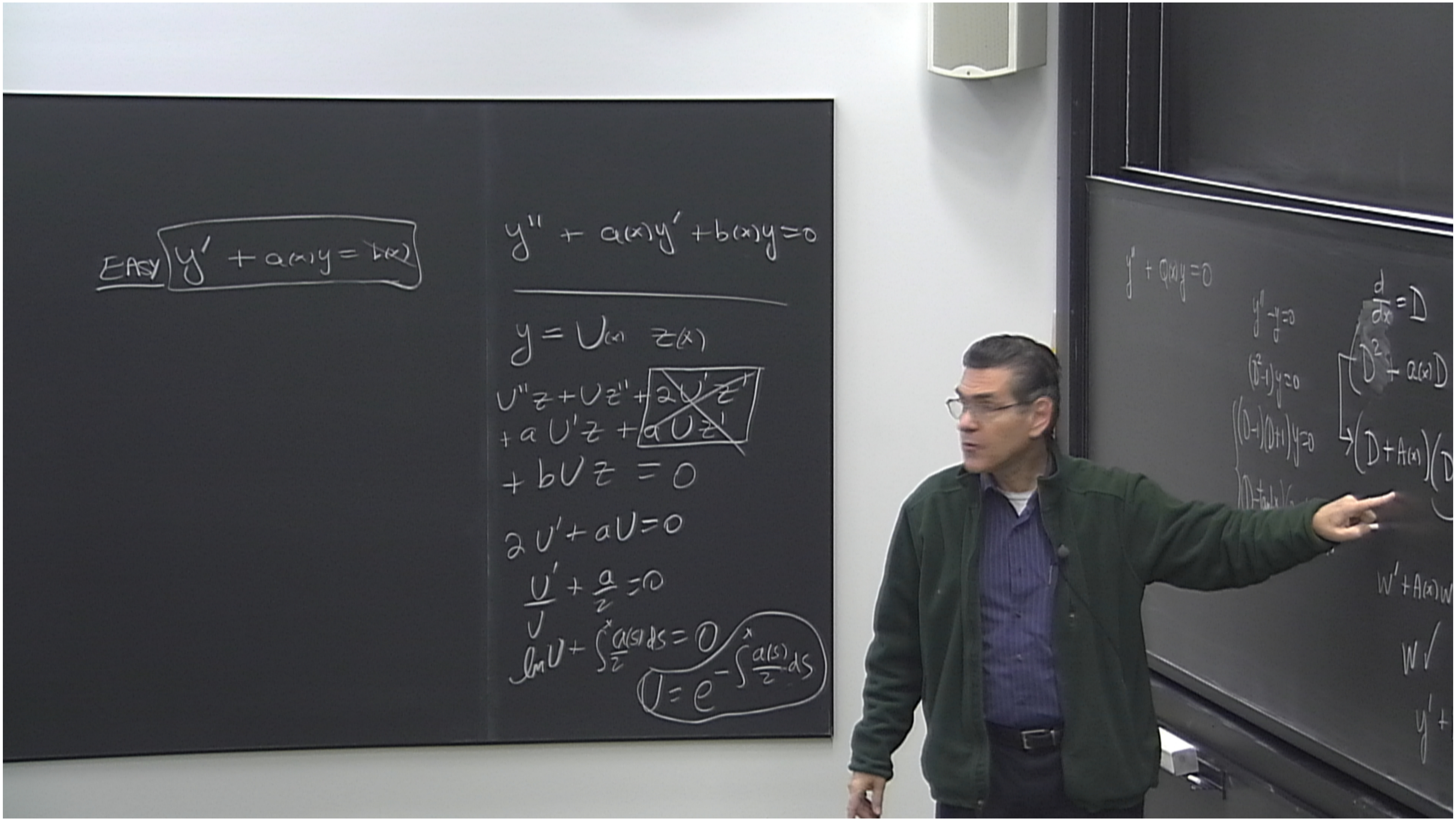
$(D - \tanh x)(D + \tanh x) y = 0$

$\frac{d}{dx} = D$

$\left(D^2 + a(x) D + b(x)\right) y(x) = 0$

$\left(D + A(x)\right)\left(D + B(x)\right) y(x) = 0$


d/dx = D
y'' − y = 0
(D² − 1) y = 0
(D − 1)(D + 1) y = 0
(D − tanh x)(D + tanh x) y = 0
(D² + a(x) D + b(x)) y(x) = 0
(D + A(x))(D + B(x)) y(x) = 0
W(x)
W' + A(x) W = 0
W ✓
y' + By = W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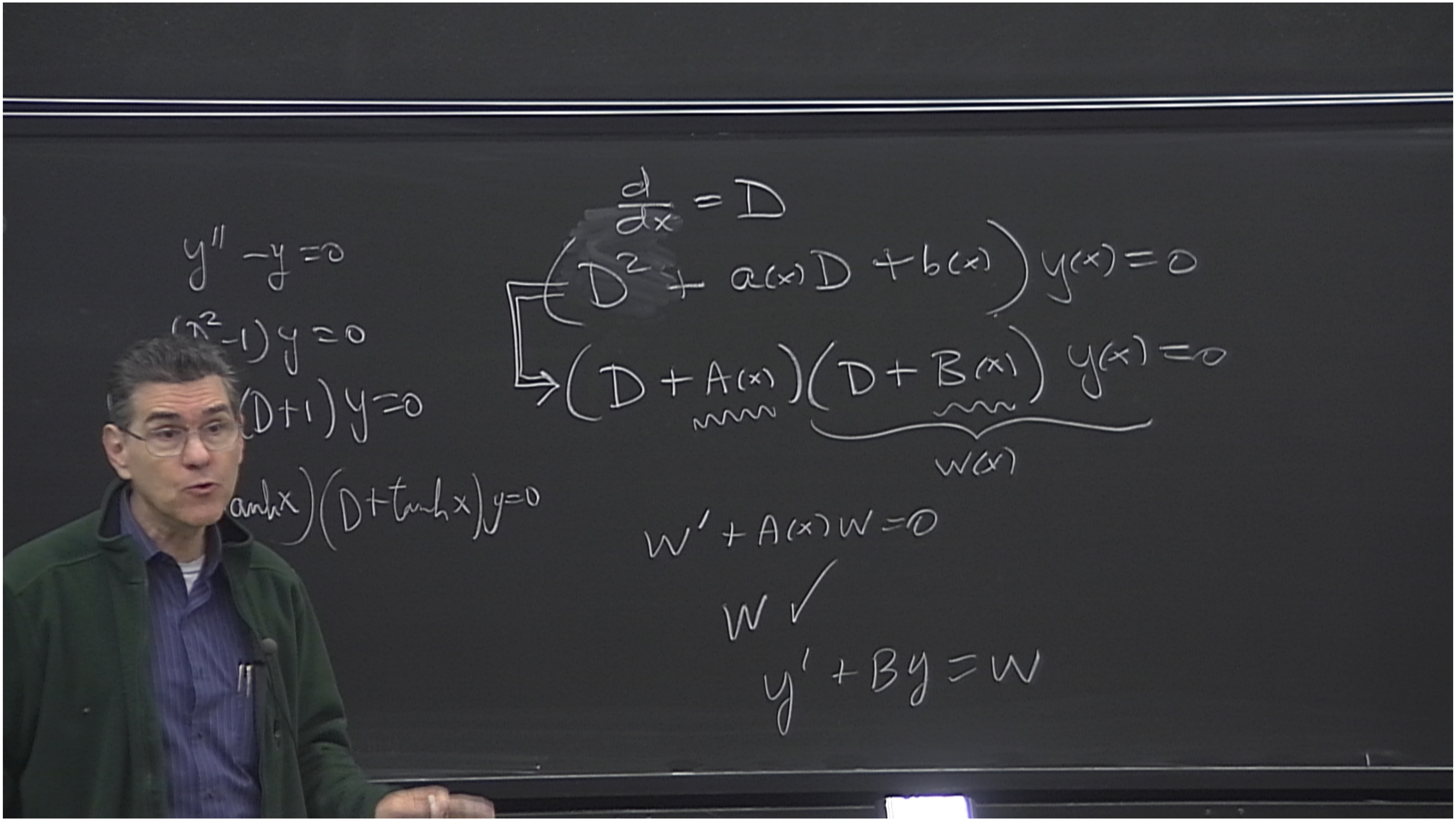
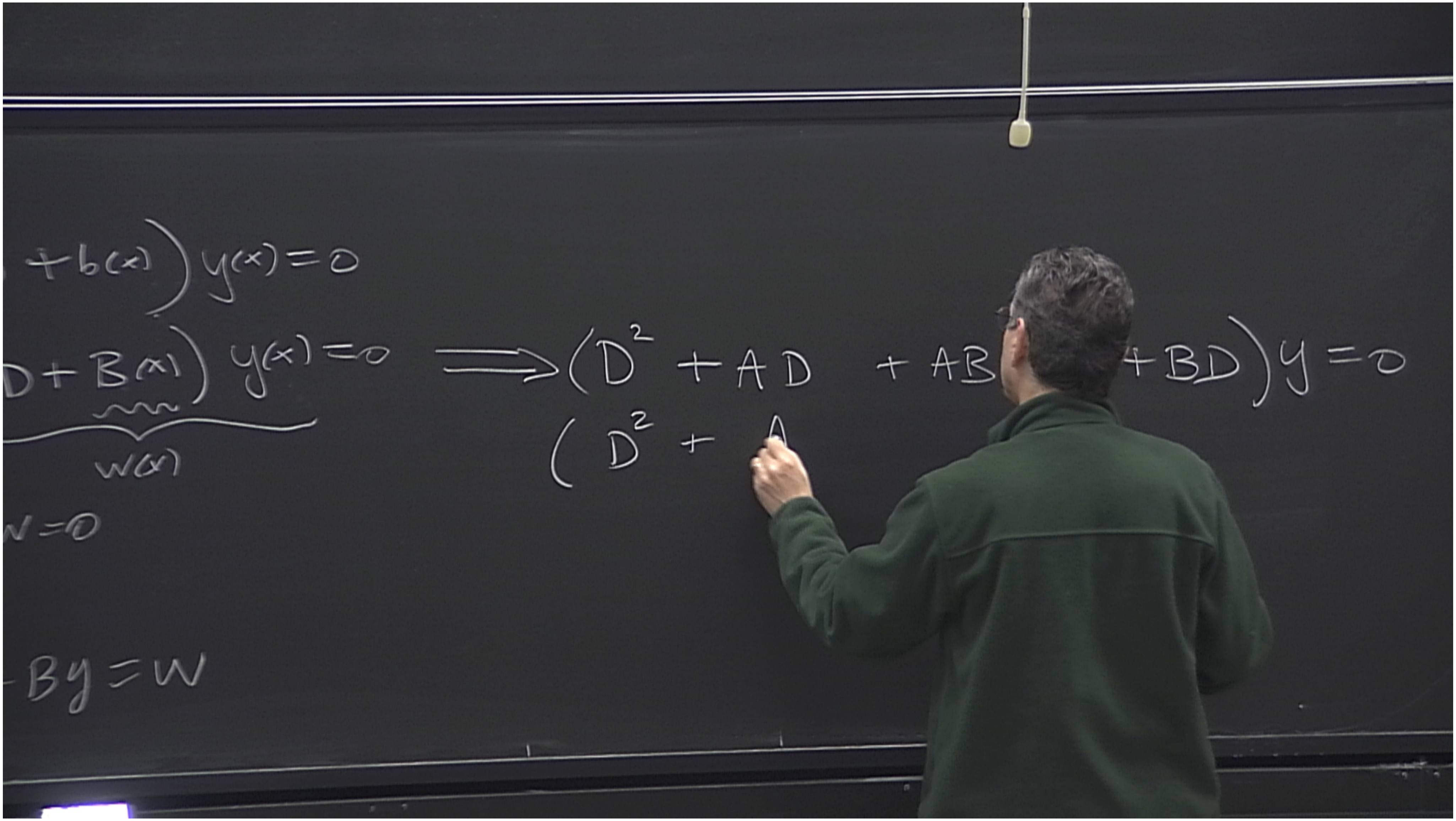
+ bUz = 0
2U' + aU = 0
U'/U + a/2 = 0
ln U + ∫ a(s)/2 ds = 0
U = e^(−∫ a(s)/2 ds)
y'' + Q(x)y = 0
y'' − y = 0
(D² − 1)y = 0
(D − 1)(D + 1)y = 0
d/dx = D
D² + a(x)D
(D + A(x))(D
W' + A(x)W
W ✓
y' +


d/dx = D
y'' − y = 0
(D² − 1) y = 0
(D+1) y = 0
(D² + a(x) D + b(x)) y(x) = 0
(D + A(x)) (D + B(x)) y(x) = 0
W(x)
W' + A(x) W = 0
W ✓
y' + By = W


+b(x)) y(x) = 0
D + B(x)) y(x) = 0
⟹ (D² + A D + AB + BD) y = 0
(D² + A
W(x)
W = 0
By = W

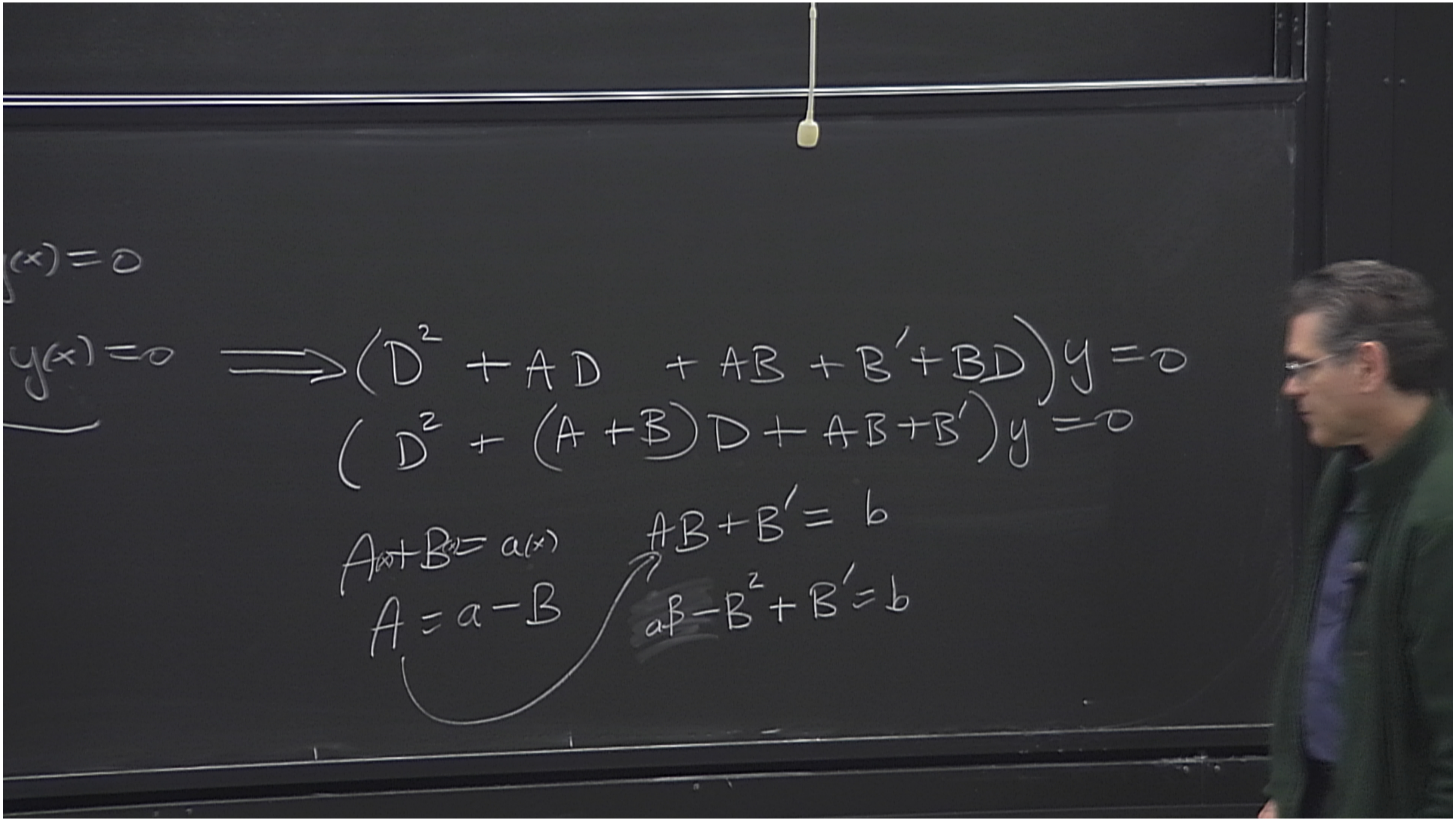
$$+b(x)\Big)y(x)=0$$

$$D+B(x)\Big)\,y(x)=0 \Longrightarrow \left(D^2+AD\quad+AB+B'+BD\right)y=0$$

$$\underbrace{\qquad\qquad}_{W(x)} \qquad \left(D^2+(A+B)D+AB+B'\right)y=0$$

$$W=0 \qquad A(x)+B(x)=a(x) \qquad AB+B'=b$$

$$-By=W$$


y(x) = 0
y(x) = 0
⟹ (D² + AD + AB + B′ + BD) y = 0
(D² + (A + B) D + AB + B′) y = 0
AB + B′ = b
A(x) + B(x) = a(x)
A = a − B
aB − B² + B′ = b

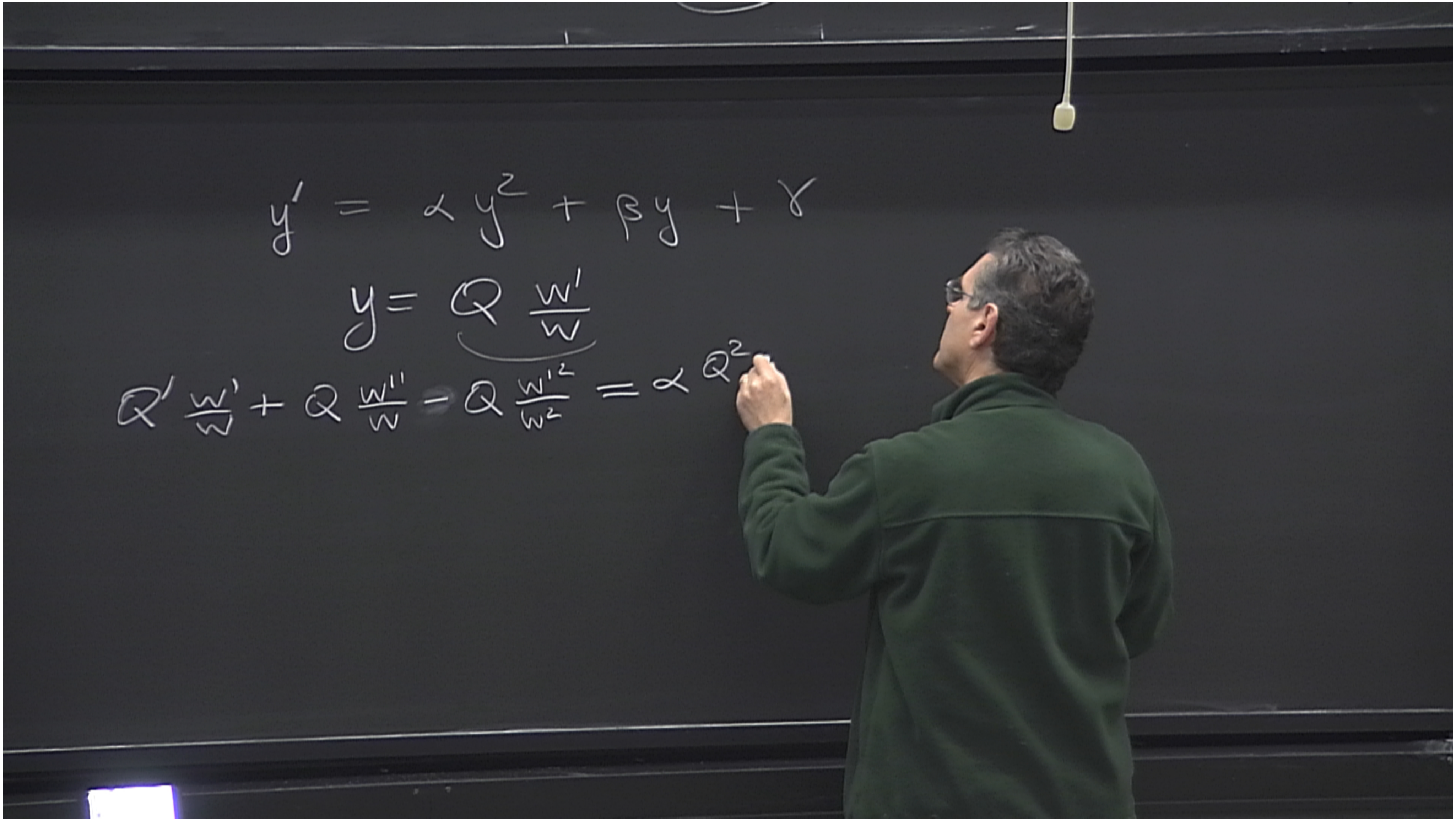
$$y(x) = 0$$

$$y(x) = 0 \implies (D^2 + AD + AB + B' + BD)\,y = 0$$

$$(D^2 + (A+B)D + AB + B')\,y = 0$$

$$A(x) + B(x) = a(x) \qquad AB + B' = b$$

$$A = a - B \qquad aB - B^2 + B' = b$$


$y' = \alpha y^2 + \beta y + \gamma$
$y = Q \frac{w'}{w}$
$Q' \frac{w'}{w} + Q \frac{w''}{w} - Q \frac{w'^2}{w^2} = \alpha Q^2$

$$y' = \alpha y^2 + \beta y + \gamma$$

$$y = Q \frac{W'}{W}$$

$$Q' \frac{W'}{W} + Q \frac{W''}{W} - Q \frac{W'^2}{W^2} = \alpha Q^2 \frac{W'^2}{W^2} + \beta Q \frac{W'}{W} + \gamma$$

$$y' = \alpha y^2 + \beta y + \gamma$$

$$y = Q \frac{w'}{w}$$

$$Q' \frac{w'}{w} + Q \frac{w''}{w} - Q \frac{w'^2}{w^2} = \alpha Q^2 \frac{w'^2}{w^2} + \beta Q \frac{w'}{w} + \gamma$$

$$-Q = \alpha Q^2$$

$$\boxed{Q = -\frac{1}{\alpha}}$$

$$y' = \alpha y^2 + \beta y + \gamma$$

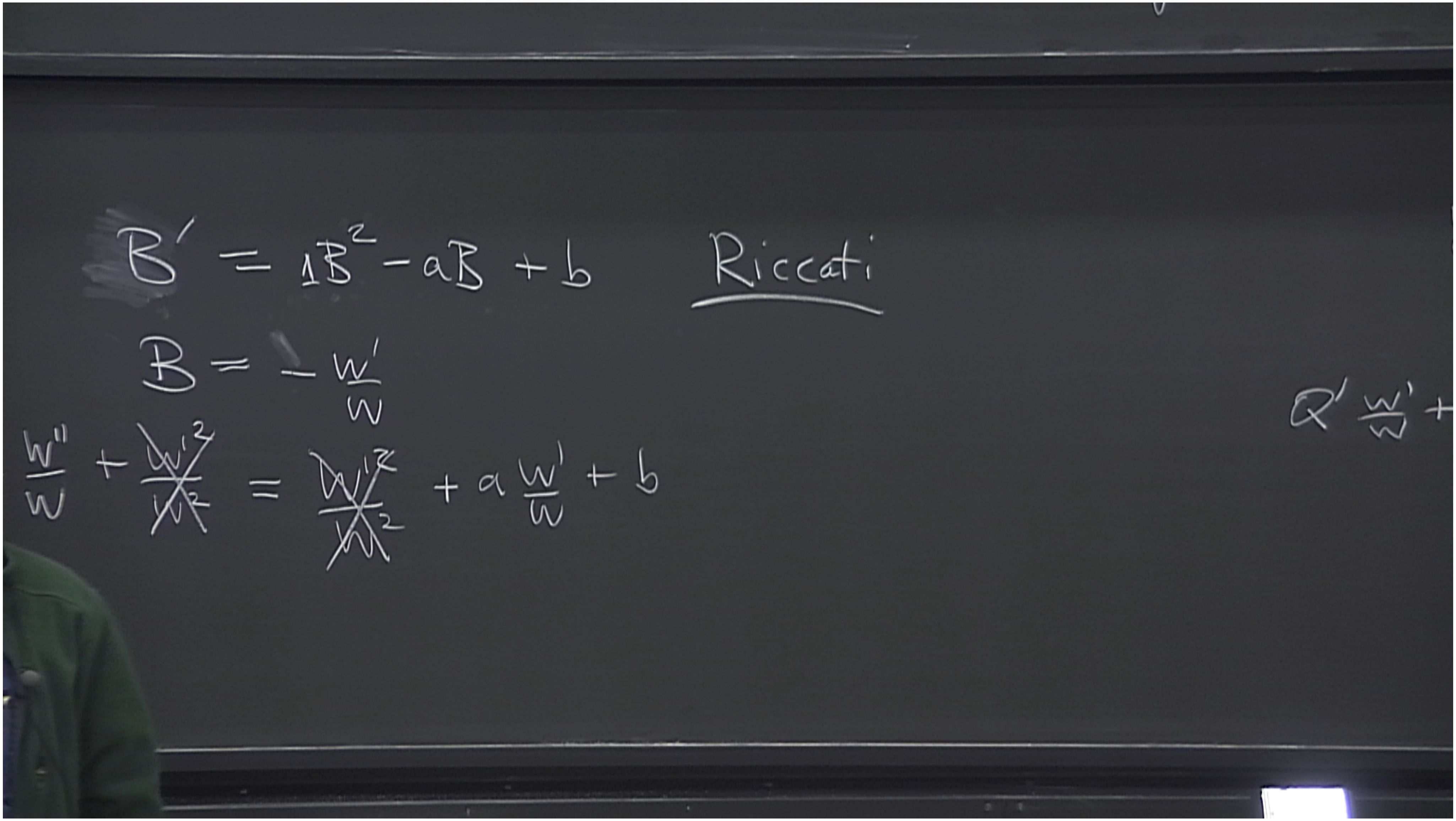
$$y = Q \frac{w'}{w}$$

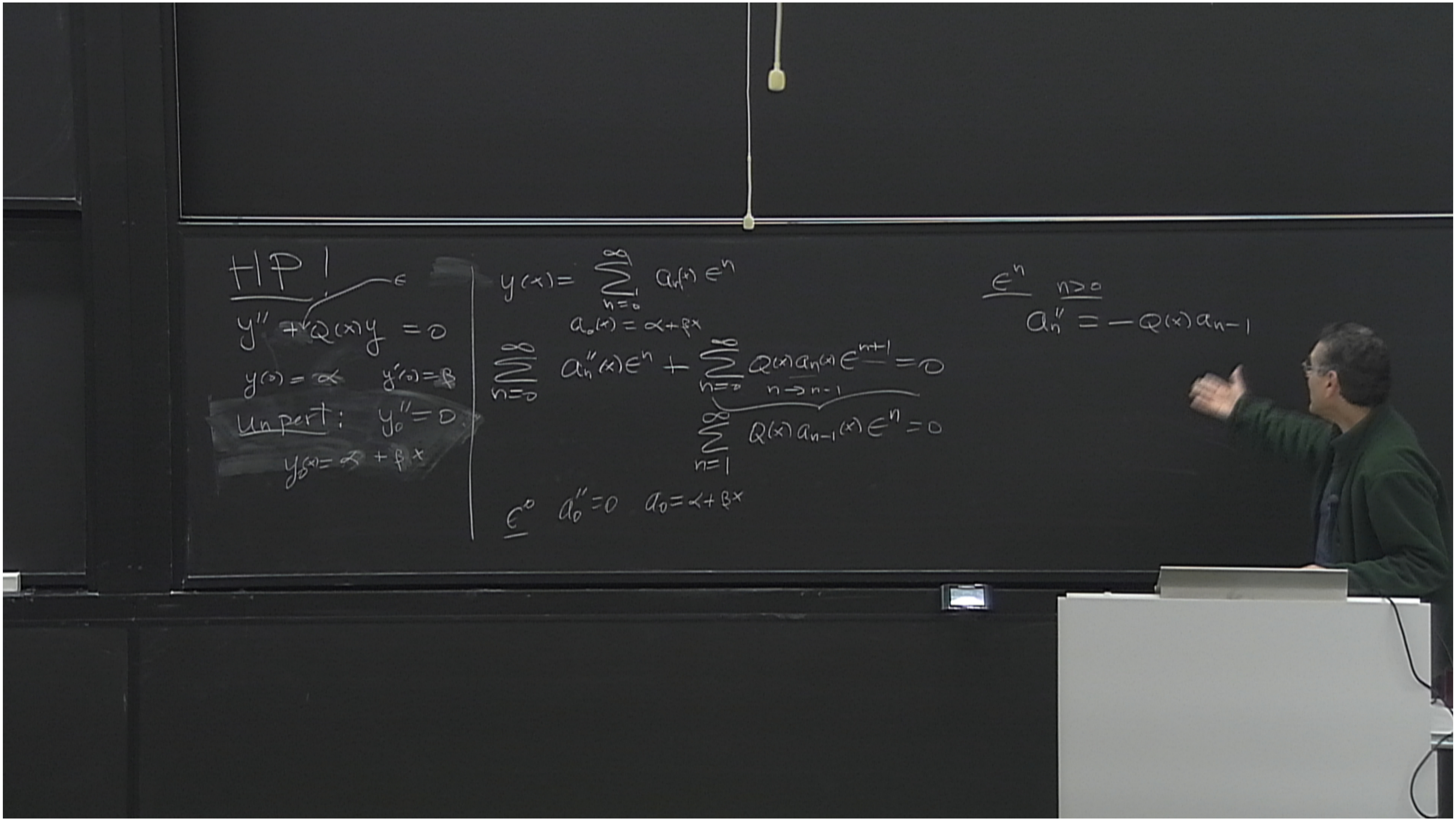
$$Q' \frac{w'}{w} + Q \frac{w''}{w} - Q \frac{w'^2}{w^2} = \alpha Q^2 \frac{w'^2}{w^2} + \beta Q \frac{w'}{w} + \gamma$$

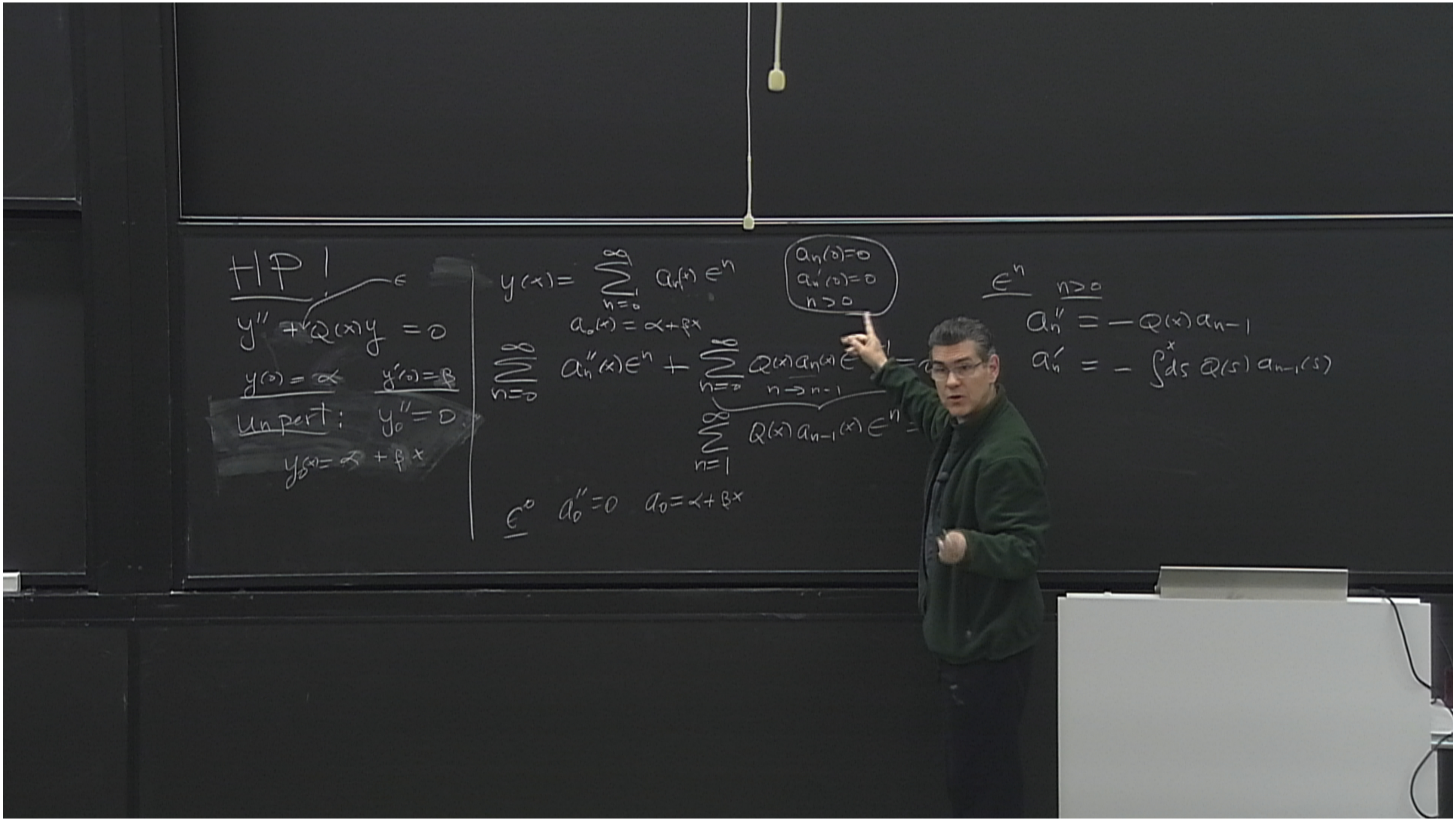
$$-Q = \alpha Q^2$$

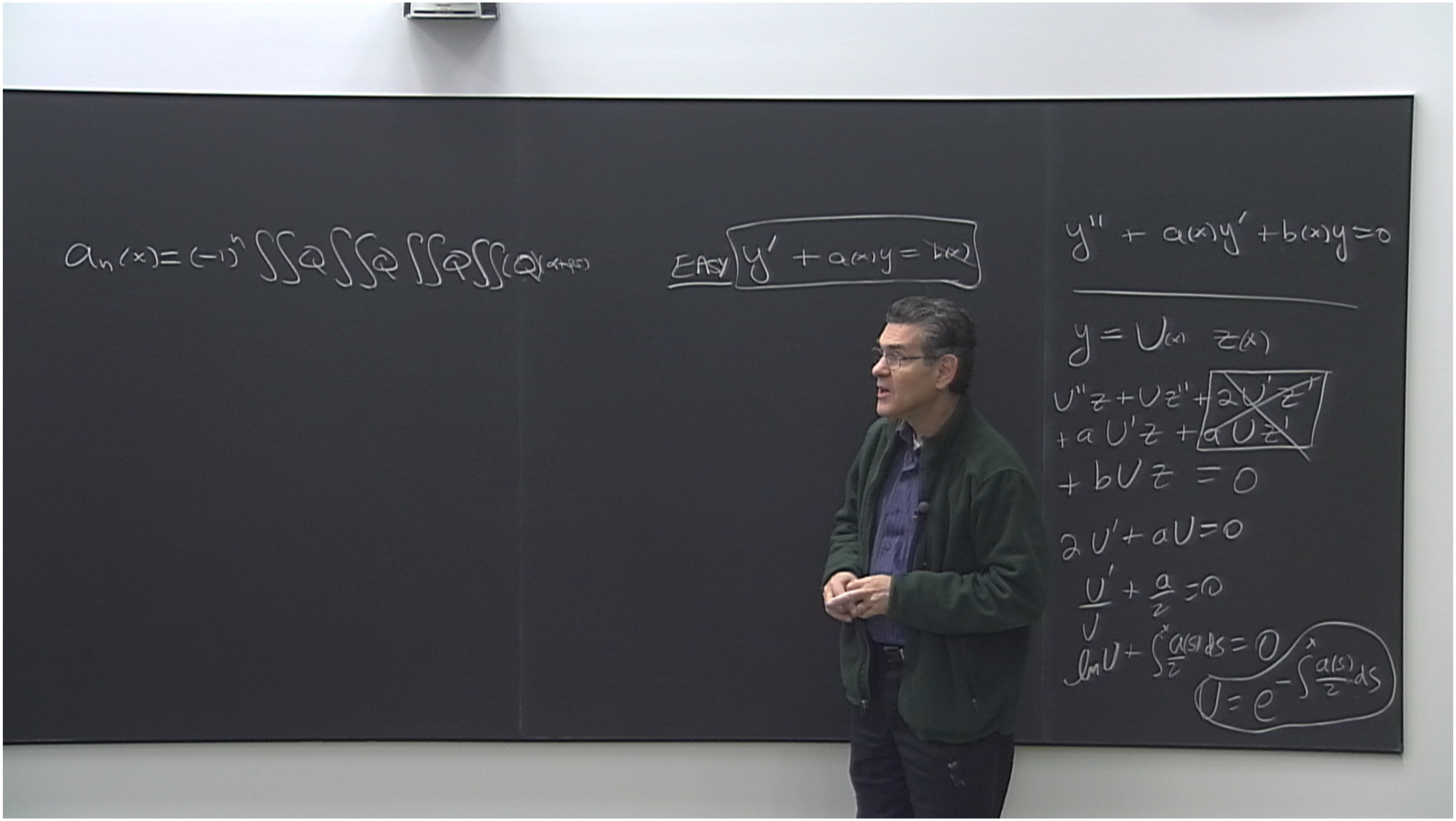
$$Q = -\frac{1}{\alpha}$$

$$Q' w' + Q w'' = \beta Q w' + \gamma w$$


$B' = 1B^2 - aB + b$
Riccati
$B = -\frac{w'}{w}$
$\frac{w''}{w} + \frac{w'^2}{w^2} = \frac{w'^2}{w^2} + a\frac{w'}{w} + b$
$Q' \frac{w'}{w} +$


HP!
y'' + ε Q(x) y = 0
y(0) = α   y'(0) = β
Unpert: y₀'' = 0
y₀(x) = α + βx
y(x) = Σ_{n=0}^∞ a_n(x) ε^n
a₀(x) = α + βx
Σ_{n=0}^∞ a_n''(x) ε^n + Σ_{n=0}^∞ Q(x) a_n(x) ε^{n+1} = 0
n → n-1
Σ_{n=1}^∞ Q(x) a_{n-1}(x) ε^n = 0
ε⁰ a₀'' = 0 a₀ = α + βx
ε^n n>0
a_n'' = − Q(x) a_{n-1}


HP!
$y'' + \epsilon Q(x) y = 0$
$y(0) = \alpha$ $y'(0) = \beta$
Unpert: $y_0'' = 0$
$y_0(x) = \alpha + \beta x$
$y(x) = \sum_{n=0}^{\infty} a_n(x) \epsilon^n$
$a_0(x) = \alpha + \beta x$
$a_n(0) = 0$
$a_n'(0) = 0$
$n > 0$
$\sum_{n=0}^{\infty} a_n''(x) \epsilon^n + \sum_{n=0}^{\infty} Q(x) a_n(x) \epsilon^{n+1}$
$n \to n-1$
$\sum_{n=1}^{\infty} Q(x) a_{n-1}(x) \epsilon^n$
ϵ^0 $a_0'' = 0$ $a_0 = \alpha + \beta x$
ϵ^n $n > 0$
$a_n'' = -Q(x) a_{n-1}$
$a_n' = -\int^x ds\, Q(s)\, a_{n-1}(s)$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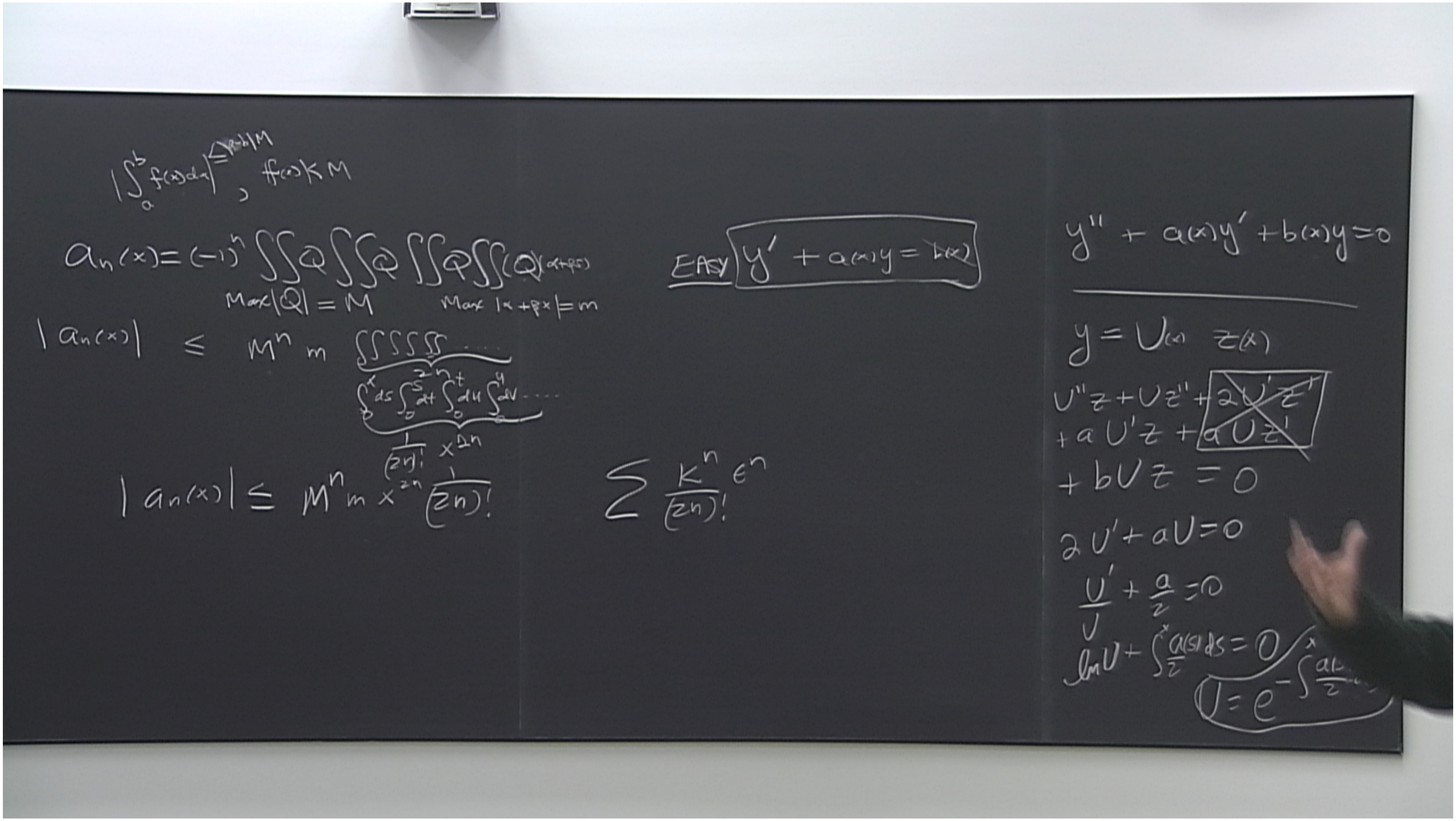
2U' + aU = 0
U'/U + a/2 = 0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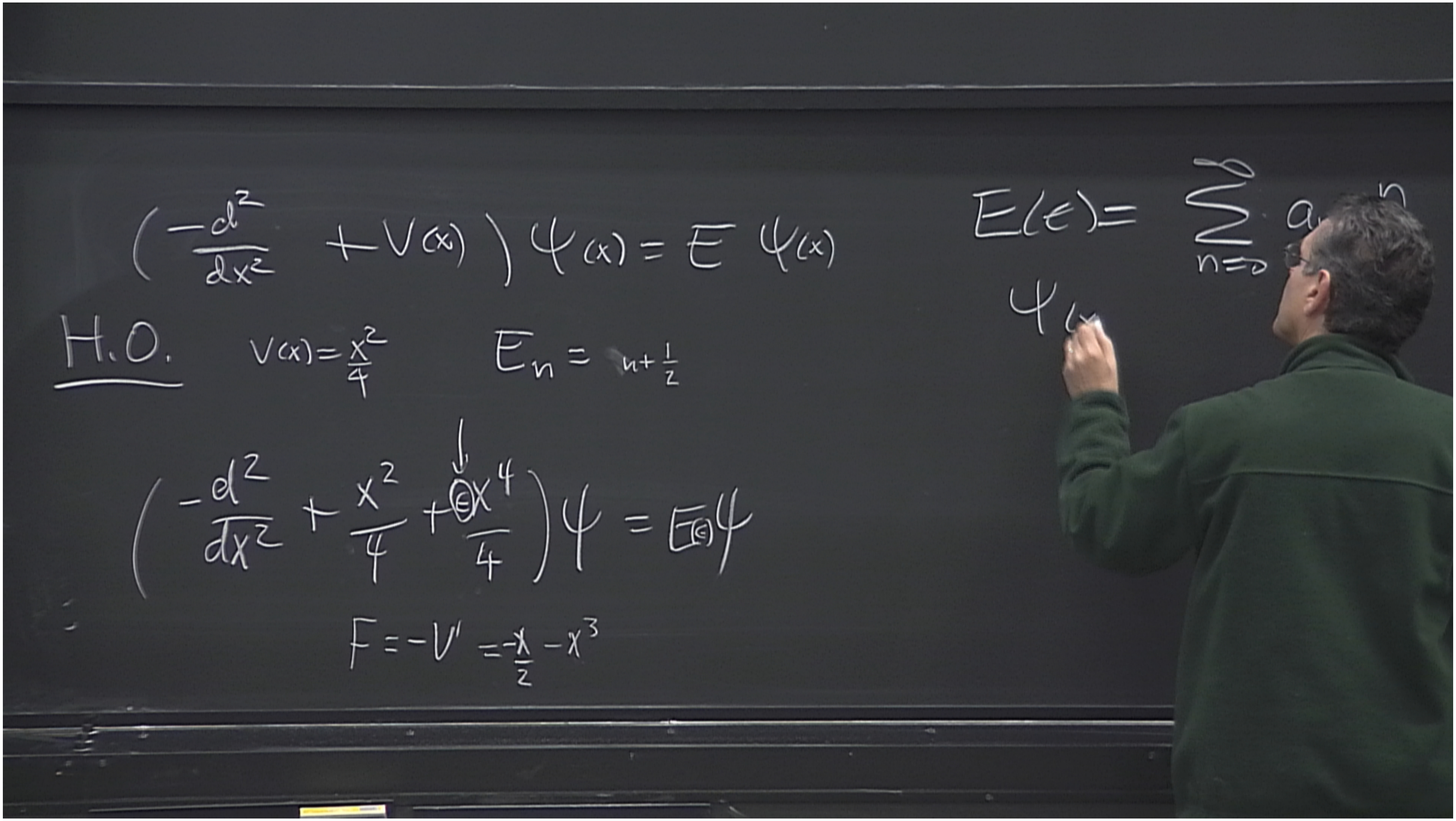
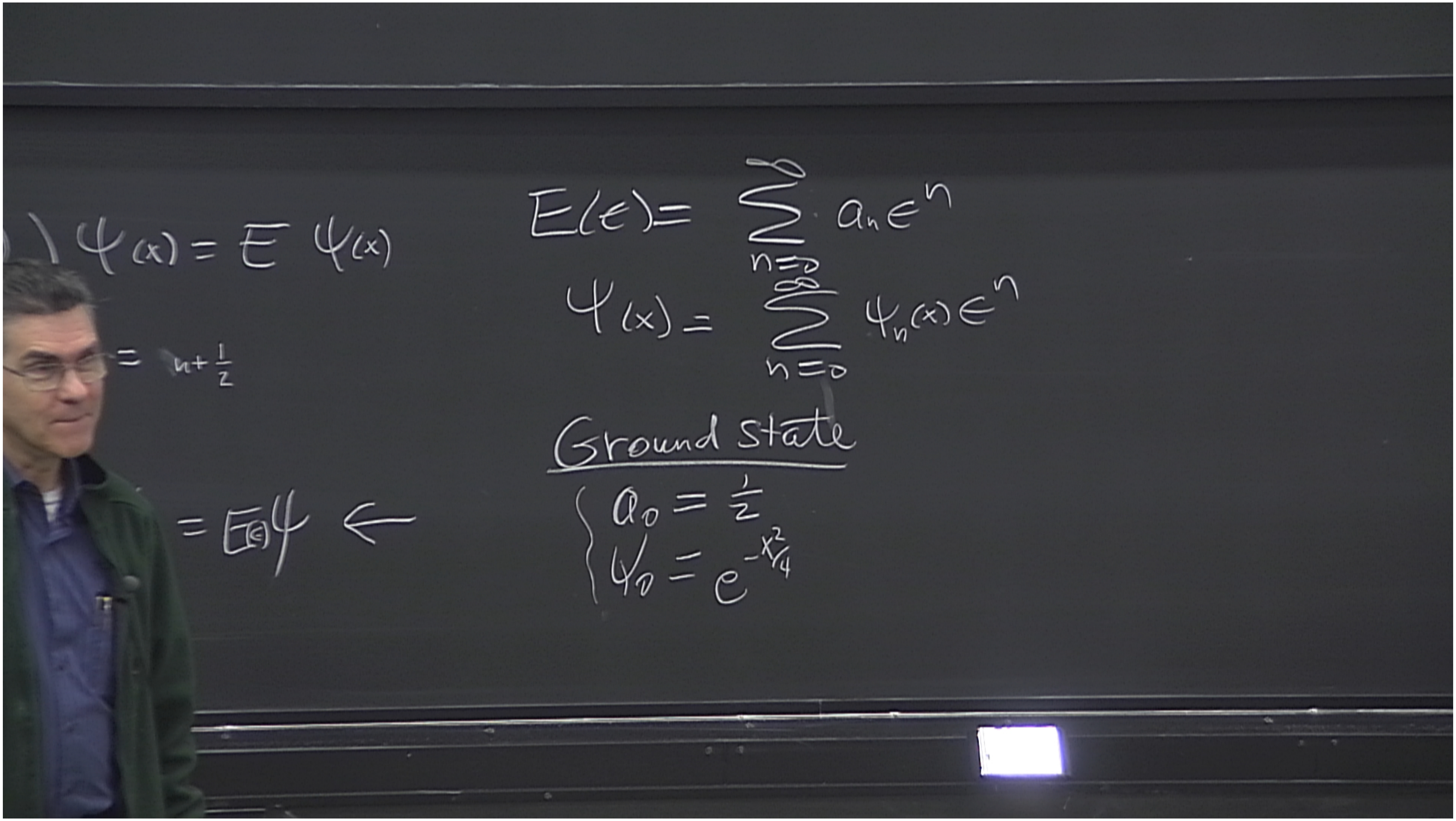
+ aU'z + aUz'
+ bUz = 0
2U' + aU = 0
Max |Q| = M
Max |α + βx| = m



(−d²/dx² + V(x)) ψ(x) = E ψ(x)
H.O.
V(x) = x²/4
E_n = n + 1/2
(−d²/dx² + x²/4 + ε x⁴/4) ψ = E(ε) ψ
F = −V′ = −x/2 − x³
E(ε) = Σ n=0 ∞ a
ψ


$\psi(x) = E\,\psi(x)$
$= n + \frac{1}{2}$
$= E(\epsilon)\psi \leftarrow$
$E(\epsilon) = \sum_{n=0}^{\infty} a_n \epsilon^n$
$\psi(x) = \sum_{n=0}^{\infty} \psi_n(x) \epsilon^n$
Ground state
$a_0 = \frac{1}{2}$
$\psi_0 = e^{-x^2/4}$

$$E_{\text{ground state}} = \frac{1}{2} + \frac{3}{4}\epsilon - \frac{21}{8}\epsilon^2 + \frac{333}{16}\epsilon^3 + \cdots$$

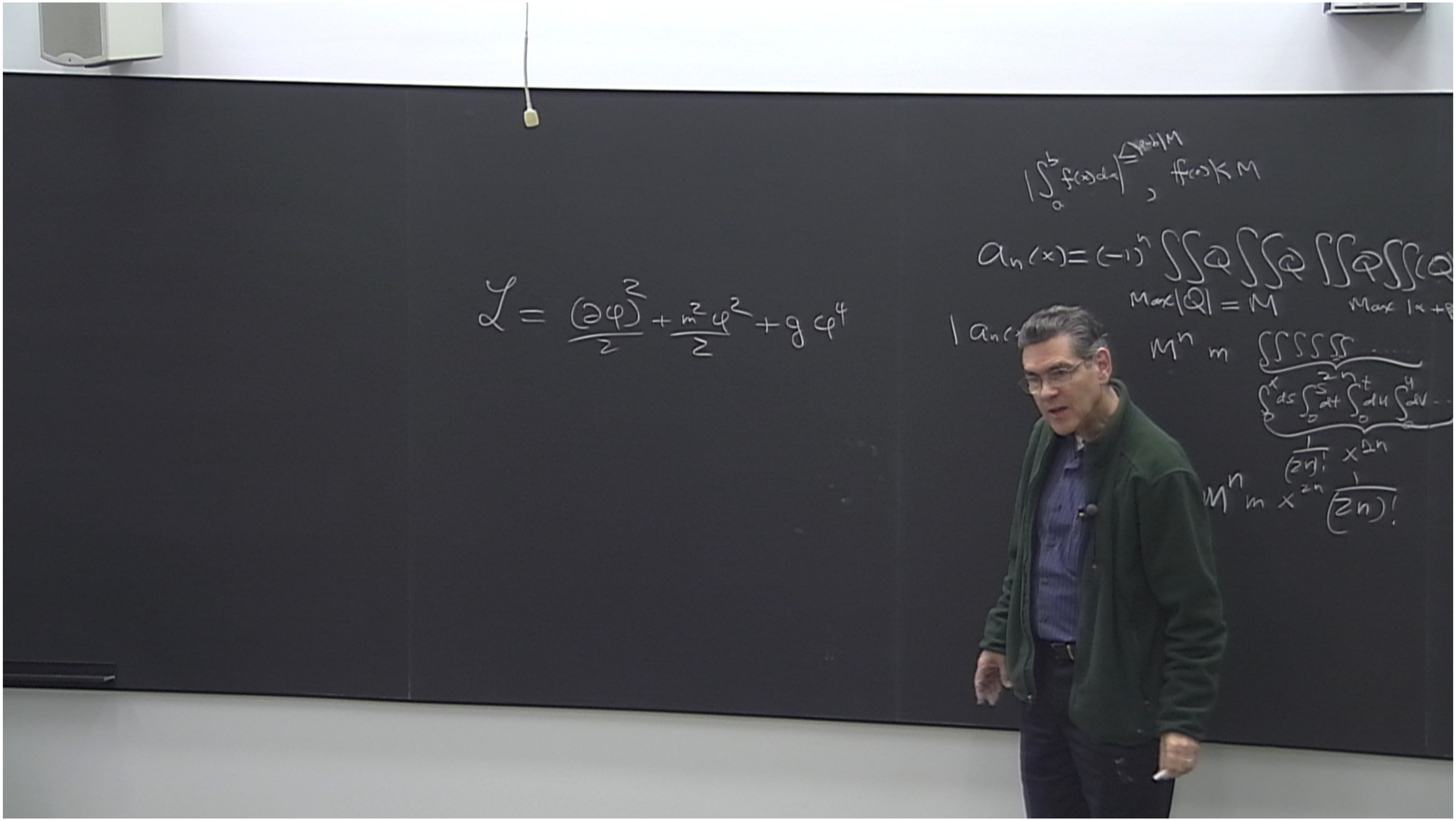
$$(\text{as } n \to \infty) \quad a_n \sim (-1)^{n+1} \frac{\sqrt{6}}{\pi^{3/2}} \, 3^n \, \Gamma(n + \tfrac{1}{2})$$

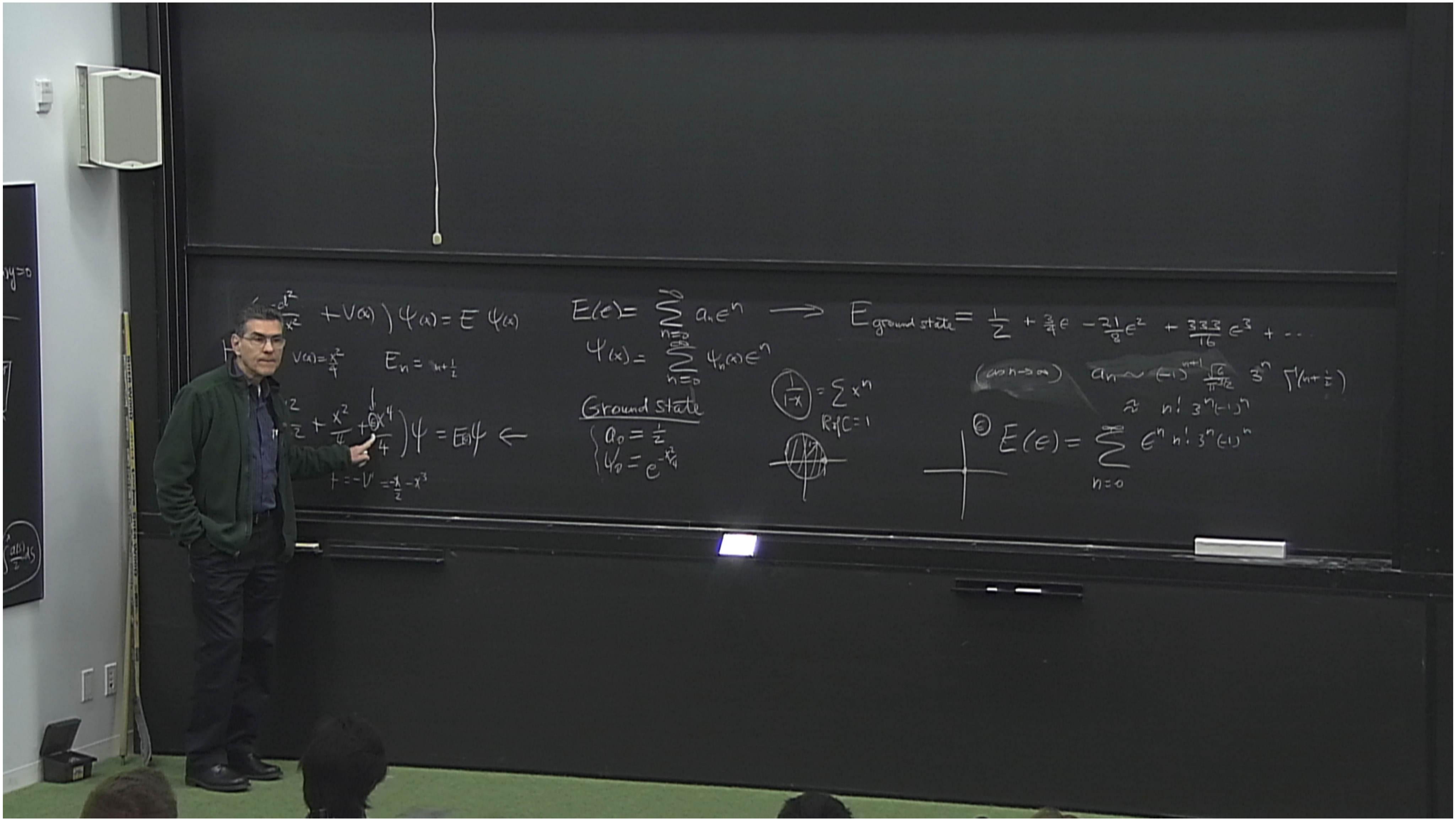
$$E_{\text{ground state}} = \frac{1}{2} + \frac{3}{4}\epsilon - \frac{21}{8}\epsilon^2 + \frac{333}{16}\epsilon^3 + \cdots$$

$$(\text{as } n \to \infty) \quad a_n \sim (-1)^{n+1} \frac{\sqrt{6}}{\pi^{3/2}} 3^n \, \Gamma\!\left(n + \tfrac{1}{2}\right)$$

$$\approx n! \, 3^n (-1)^n$$

$$E(\epsilon) = \sum_{n=0}^{\infty} \epsilon^n \, n! \, 3^n (-1)^n$$


$\mathcal{L} = \frac{(\partial\varphi)^2}{2} + \frac{m^2\varphi^2}{2} + g\varphi^4$
Max|Q| = M


Ground state
$E_{\text{ground state}} = \frac{1}{2} + \frac{3}{4}\epsilon - \frac{21}{8}\epsilon^2 + \frac{333}{16}\epsilon^3 + \ldots$
$E(\epsilon) = \sum_{n=0}^{\infty} a_n \epsilon^n$

$$H = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} + \epsilon \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$a \qquad b$$

$$H = \begin{pmatrix} a & \epsilon c \\ \epsilon c & b \end{pmatrix}$$

$$\det \begin{pmatrix} a-E & \epsilon c \\ \epsilon c & b-E \end{pmatrix} = 0$$

$$E^2 - (a+b)E + ab - \epsilon^2 c^2 = 0$$

$$H = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} + \epsilon \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$a \qquad b$$

$$H = \begin{pmatrix} a & \epsilon c \\ \epsilon c & b \end{pmatrix}$$

$$\det \begin{pmatrix} a-E & \epsilon c \\ \epsilon c & b-E \end{pmatrix} = 0$$

$$E^2 - (a+b)E + ab - \epsilon^2 c^2 = 0$$

$$E_\pm = \frac{a+b \pm \sqrt{(a+b)^2 - 4ab + 4\epsilon^2 c^2}}{2}$$

$$H = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} + \epsilon \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$a \qquad b$$

$$H = \begin{pmatrix} a & \epsilon c \\ \epsilon c & b \end{pmatrix}$$

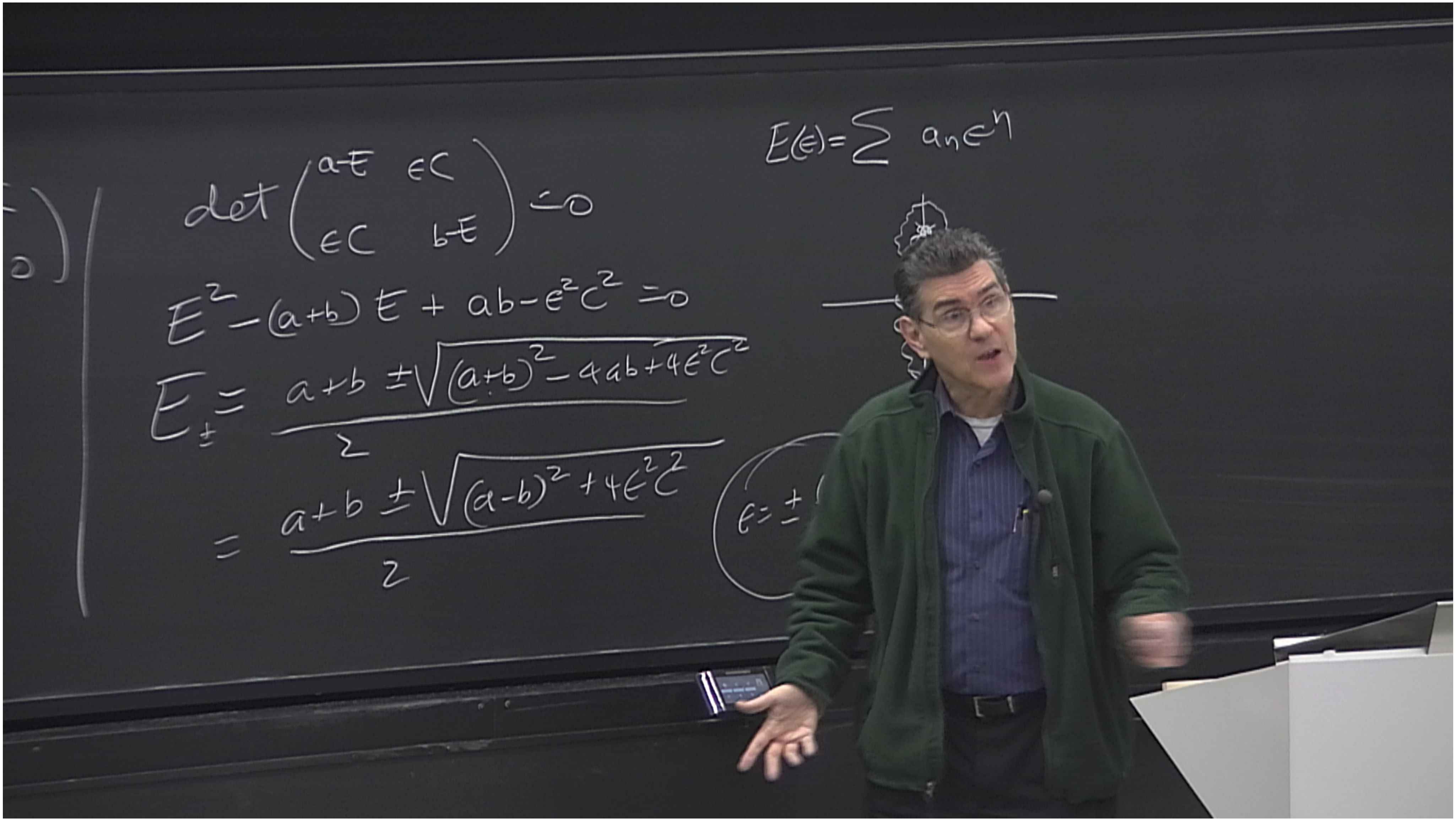
$$\det \begin{pmatrix} a-E & \epsilon c \\ \epsilon c & b-E \end{pmatrix} = 0$$

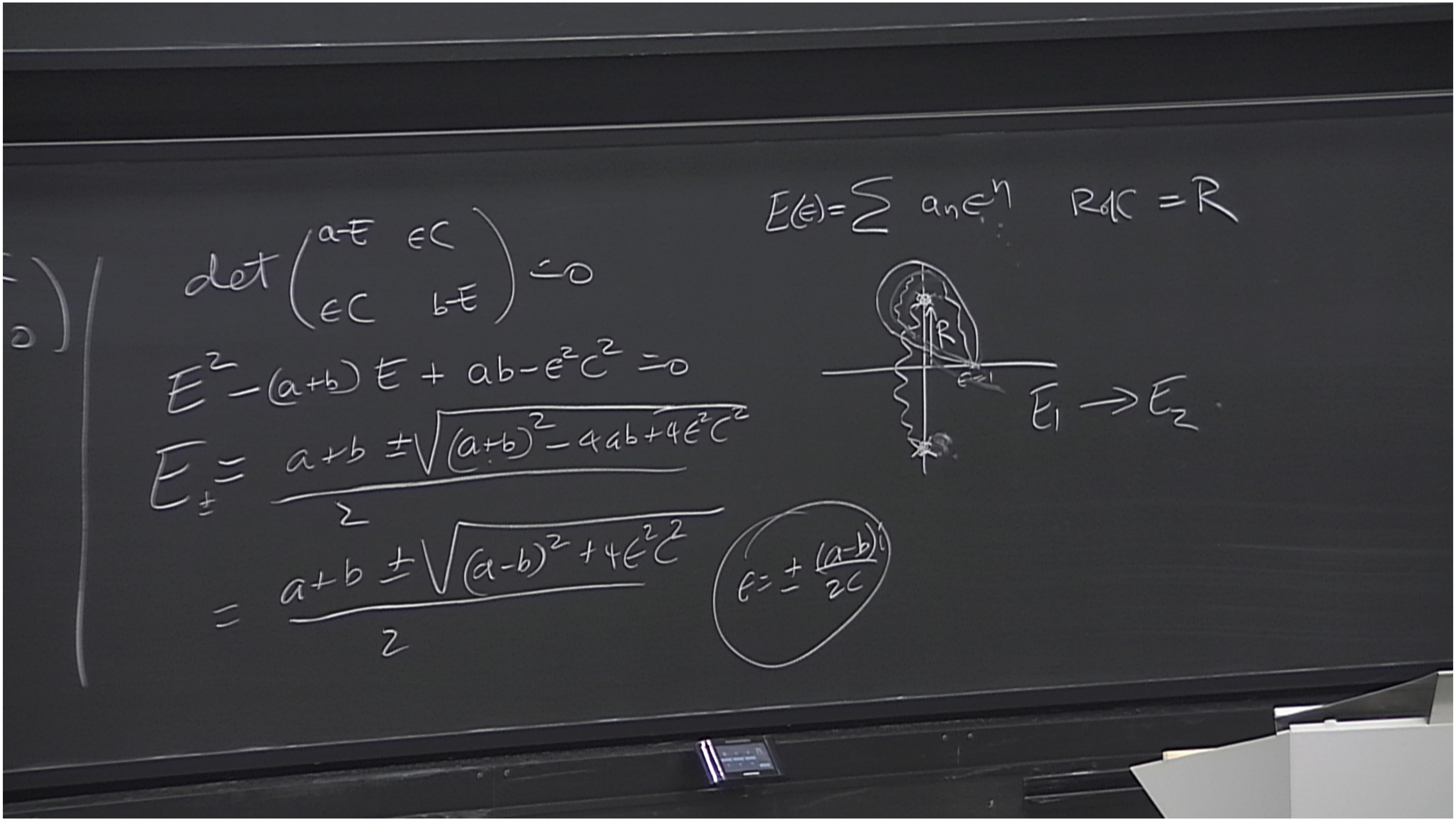
$$E^2 - (a+b)E + ab - \epsilon^2 c^2 = 0$$

$$E_\pm = \frac{a+b \pm \sqrt{(a+b)^2 - 4ab + 4\epsilon^2 c^2}}{2}$$

$$= \frac{a+b \pm \sqrt{(a-b)^2 + 4\epsilon^2 c^2}}{2}$$

$\sum a_n$


det (a-E εC ; εC b-E) = 0
E(ε) = Σ a_n ε^n
E² - (a+b) E + ab - ε²C² = 0
E± = (a+b ± √((a+b)² - 4ab + 4ε²C²)) / 2
= (a+b ± √((a-b)² + 4ε²C²)) / 2


E(ε) = Σ a_n ε^n
RoC = R
det (a−E εC / εC b−E) = 0
E² − (a+b) E + ab − ε²C² = 0
E± = (a+b ± √((a+b)² − 4ab + 4ε²C²)) / 2
= (a+b ± √((a−b)² + 4ε²C²)) / 2
ε = ± (a−b)i / 2C
R
E₁ → E₂