Title: Mathematical Physics - Lecture 1

Date: Nov 21, 2011 09:00 AM

URL: http://pirsa.org/11110040

Abstract:

How to solve hard problems

Carl Bender
Washington University

Perimeter Institute, 2011

Series arise when you use perturbation theory

Perturbation theory for a HARD PROBLEM:

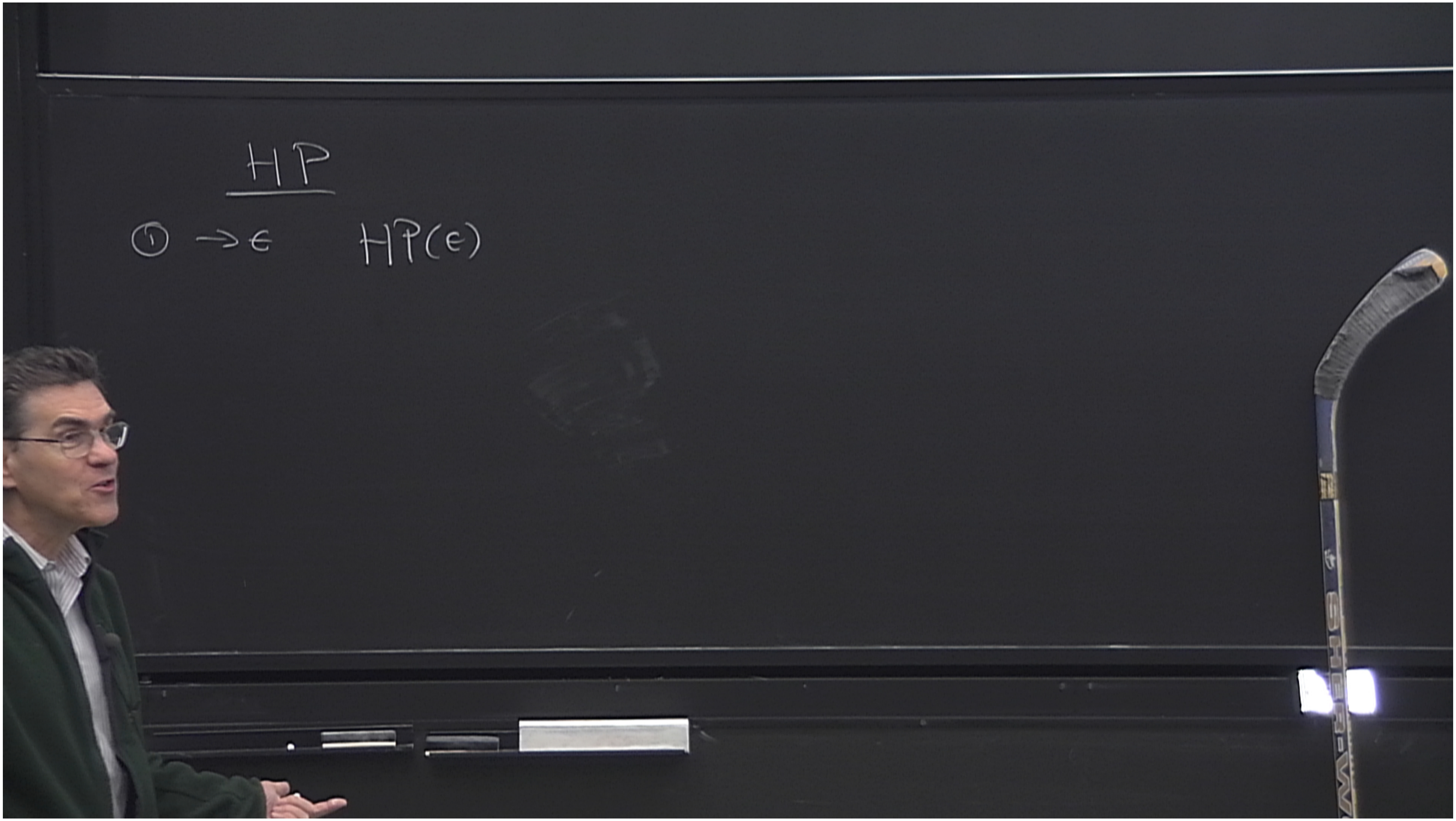
Step 1. Insert a small parameter ε:

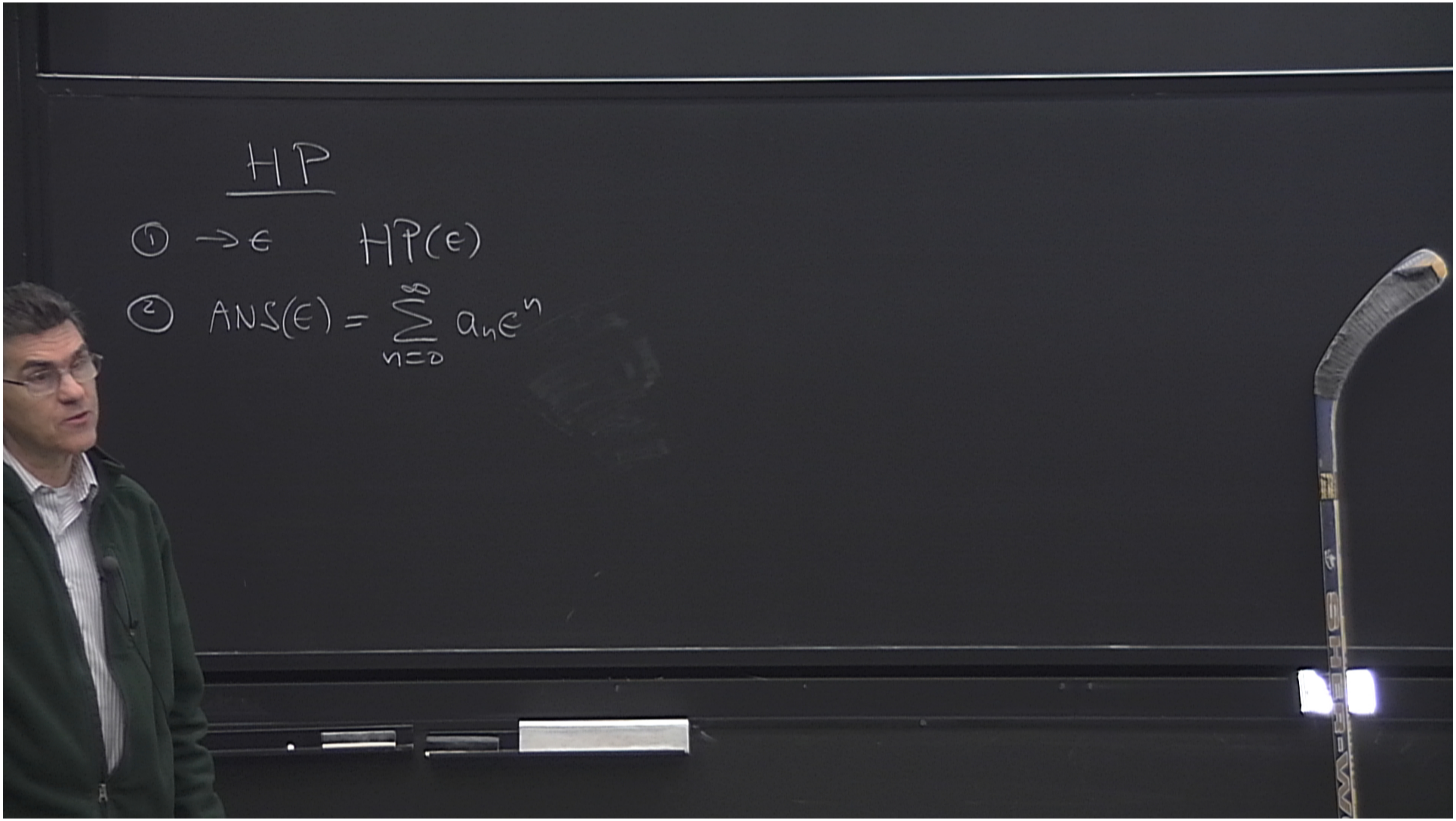
HARD PROBLEM(ε)

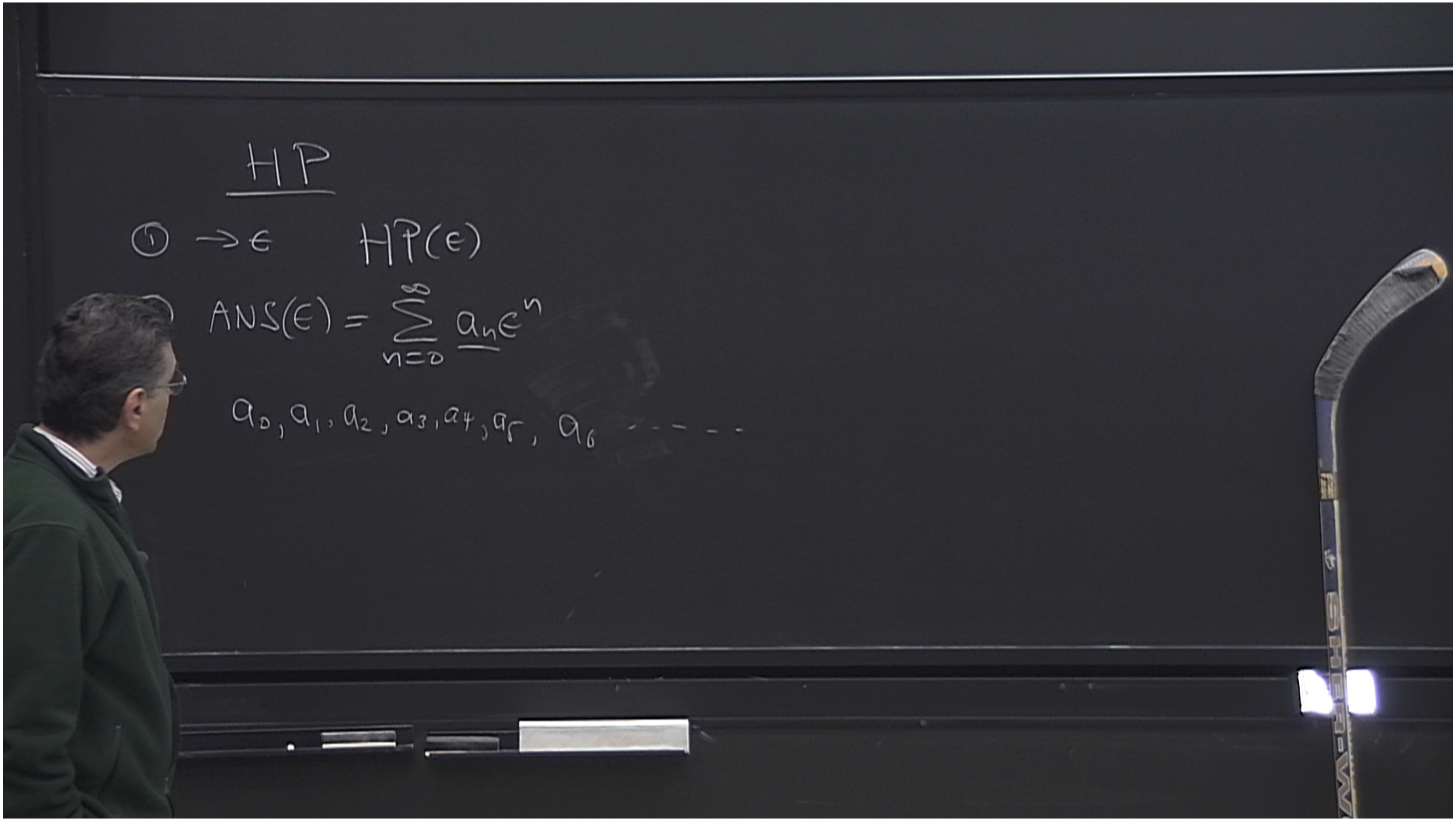
Step 2. Expand answer as a perturbation series in powers of ε:

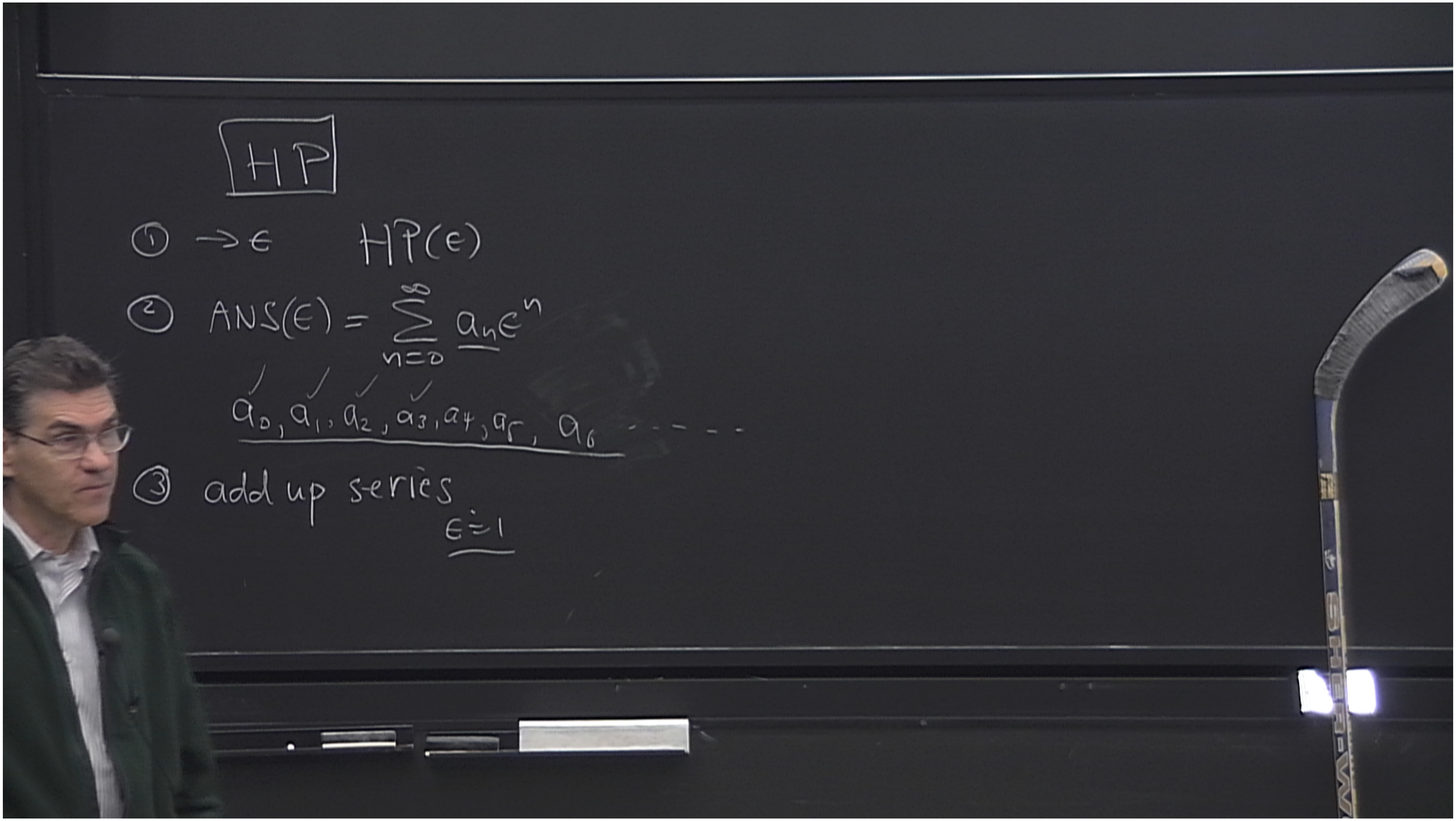
$$\text{ANSWER}(\varepsilon) = \sum_{n=0}^{\infty} a_n \varepsilon^n$$

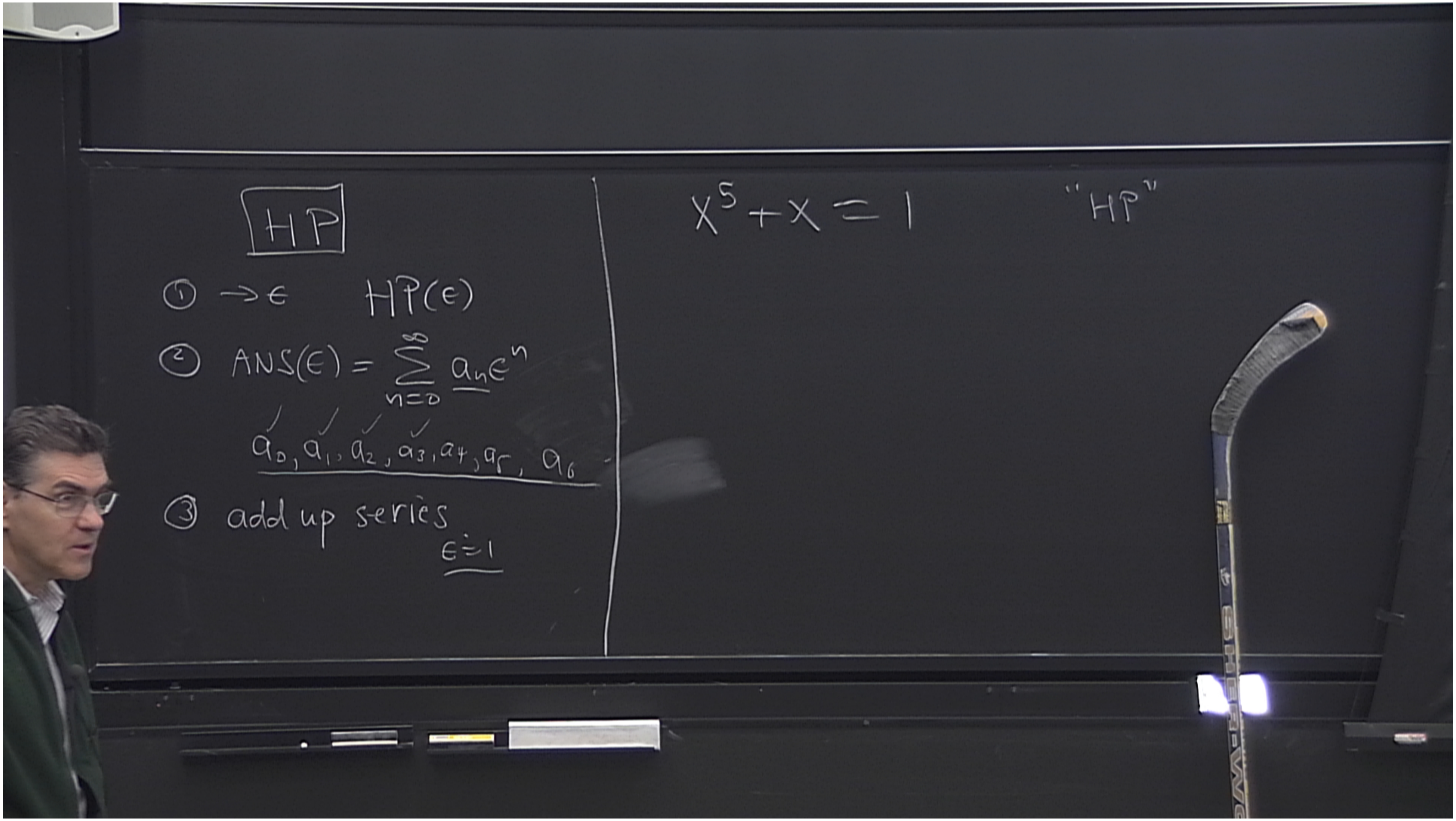
Step 3. Set ε=1 and sum the series – **this is not so easy!**

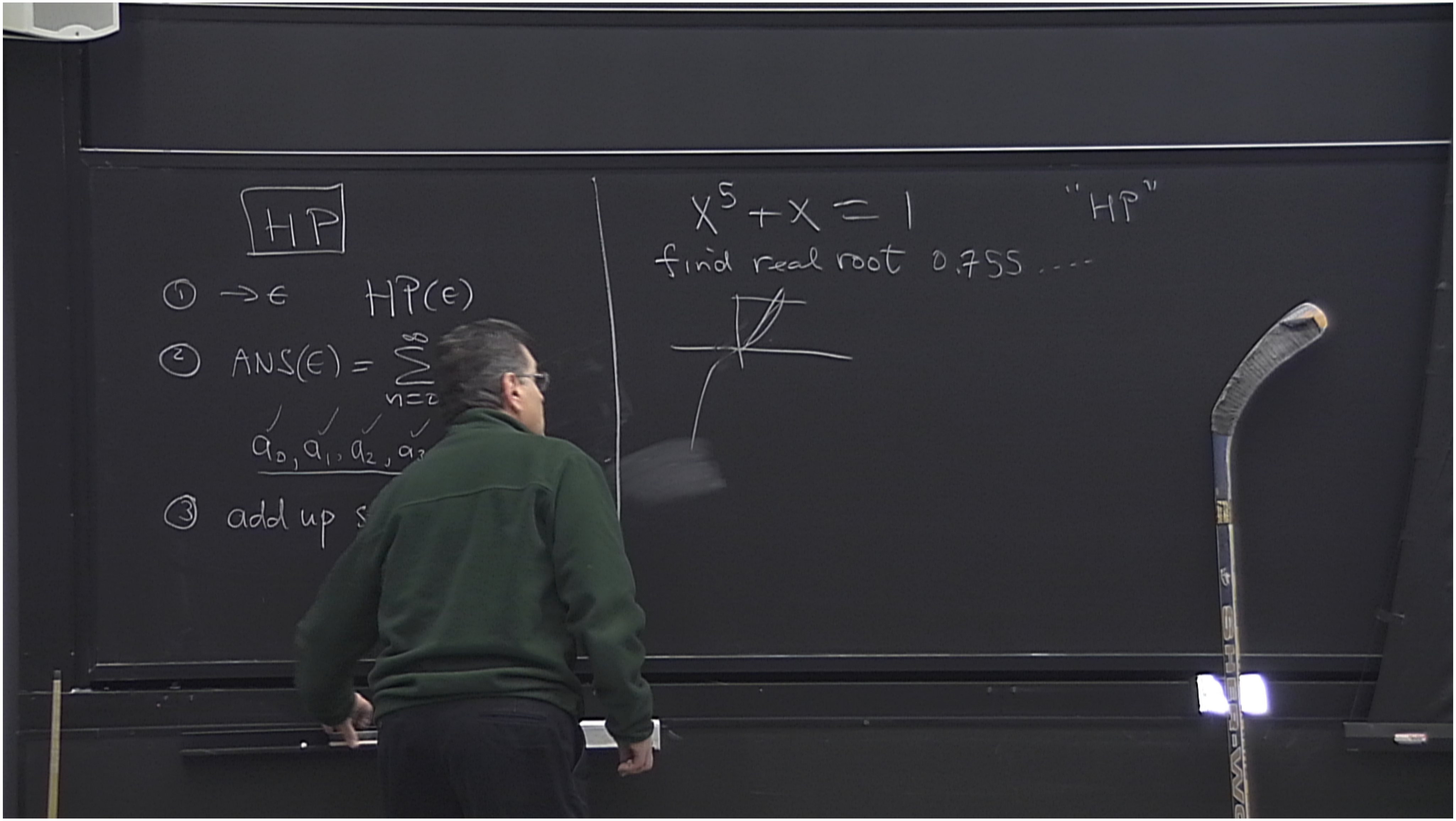

HP
① → ε HP(ε)
SHER-W

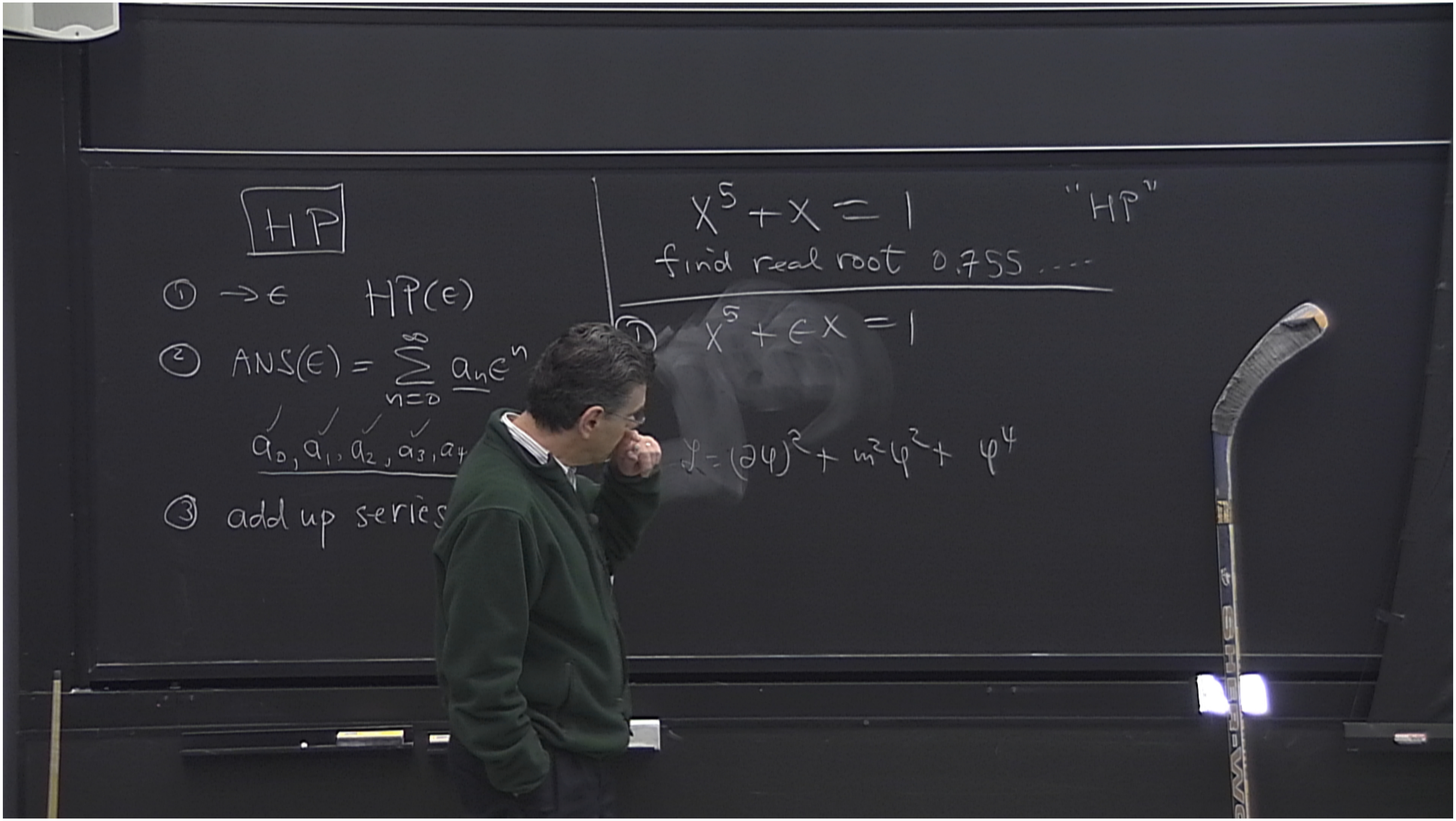

HP
① → ε   HP(ε)
② ANS(ε) = $\sum_{n=0}^{\infty} a_n \epsilon^n$

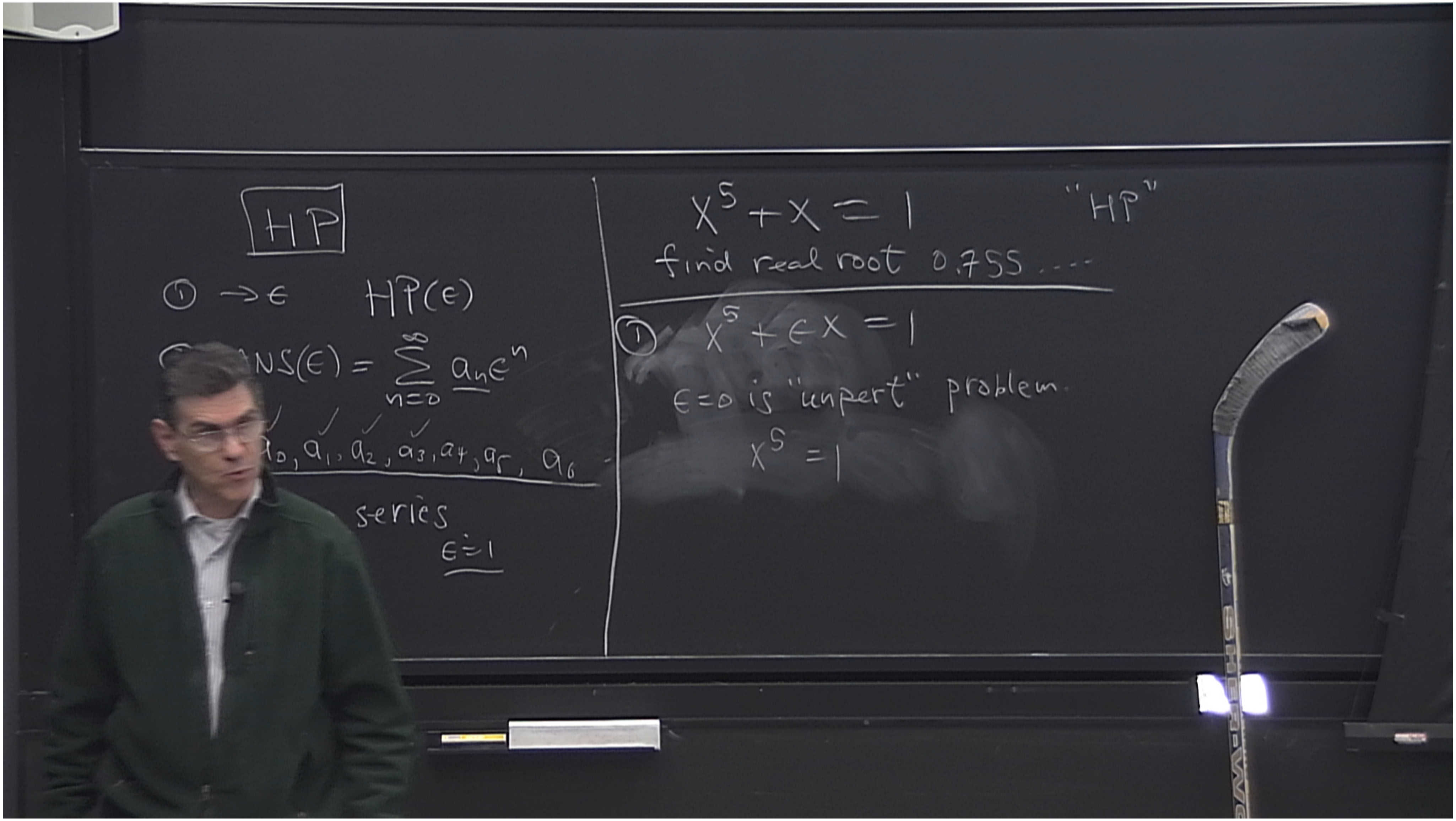

HP
① → ε HP(ε)
ANS(ε) = Σ_{n=0}^{∞} a_n ε^n
a_0, a_1, a_2, a_3, a_4, a_5, a_6 - - - - -

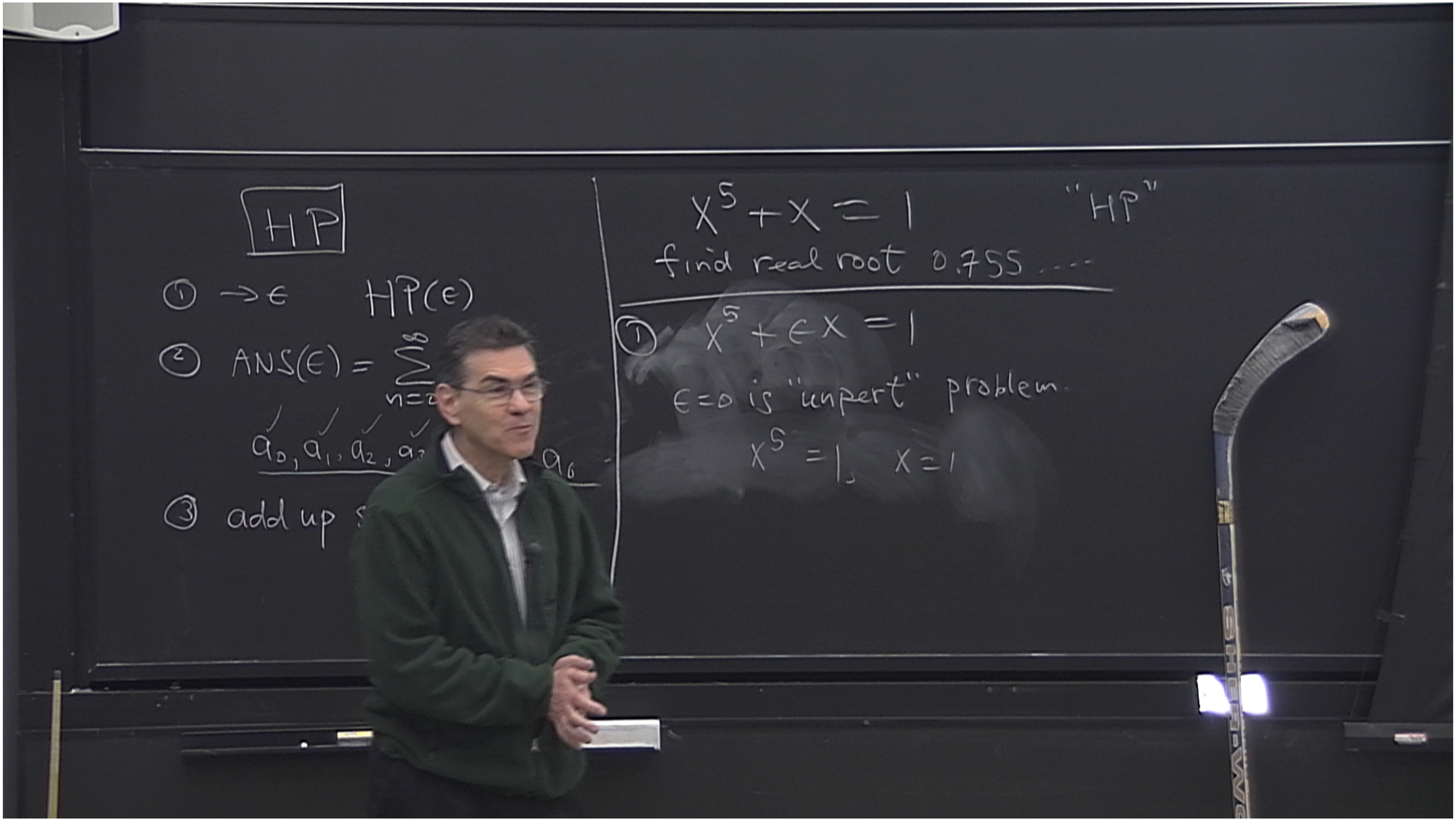

HP
① → ε   HP(ε)
② $ANS(\epsilon) = \sum_{n=0}^{\infty} a_n \epsilon^n$
$a_0, a_1, a_2, a_3, a_4, a_5, a_6$ - - - - -
③ add up series
$\epsilon \doteq 1$

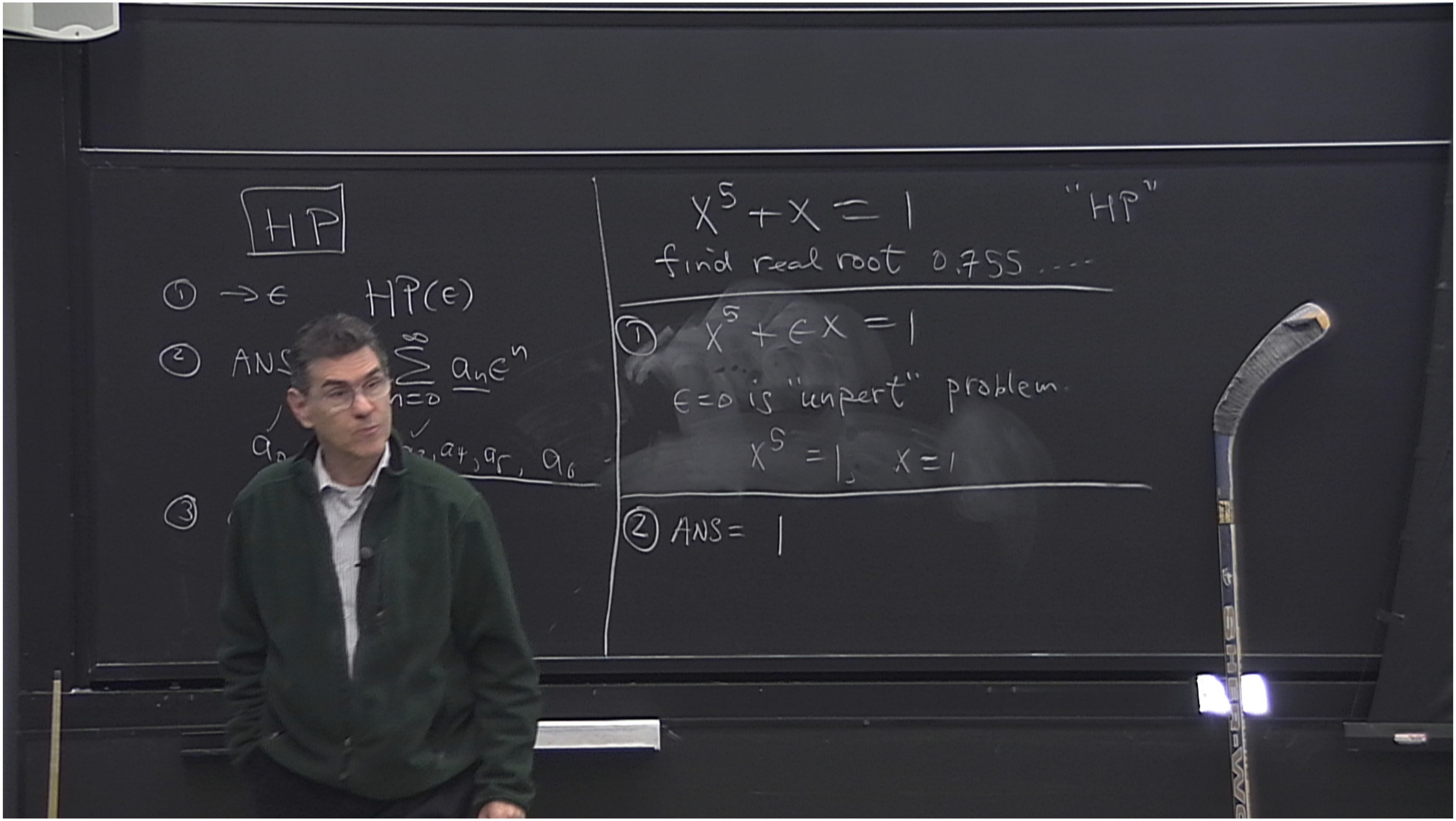

HP
① → ϵ HP(ϵ)
② $ANS(\epsilon) = \sum_{n=0}^{\infty} a_n \epsilon^n$
$a_0, a_1, a_2, a_3, a_4, a_5, a_6$
③ add up series
$\epsilon \doteq 1$
$x^5 + x = 1$
"HP"

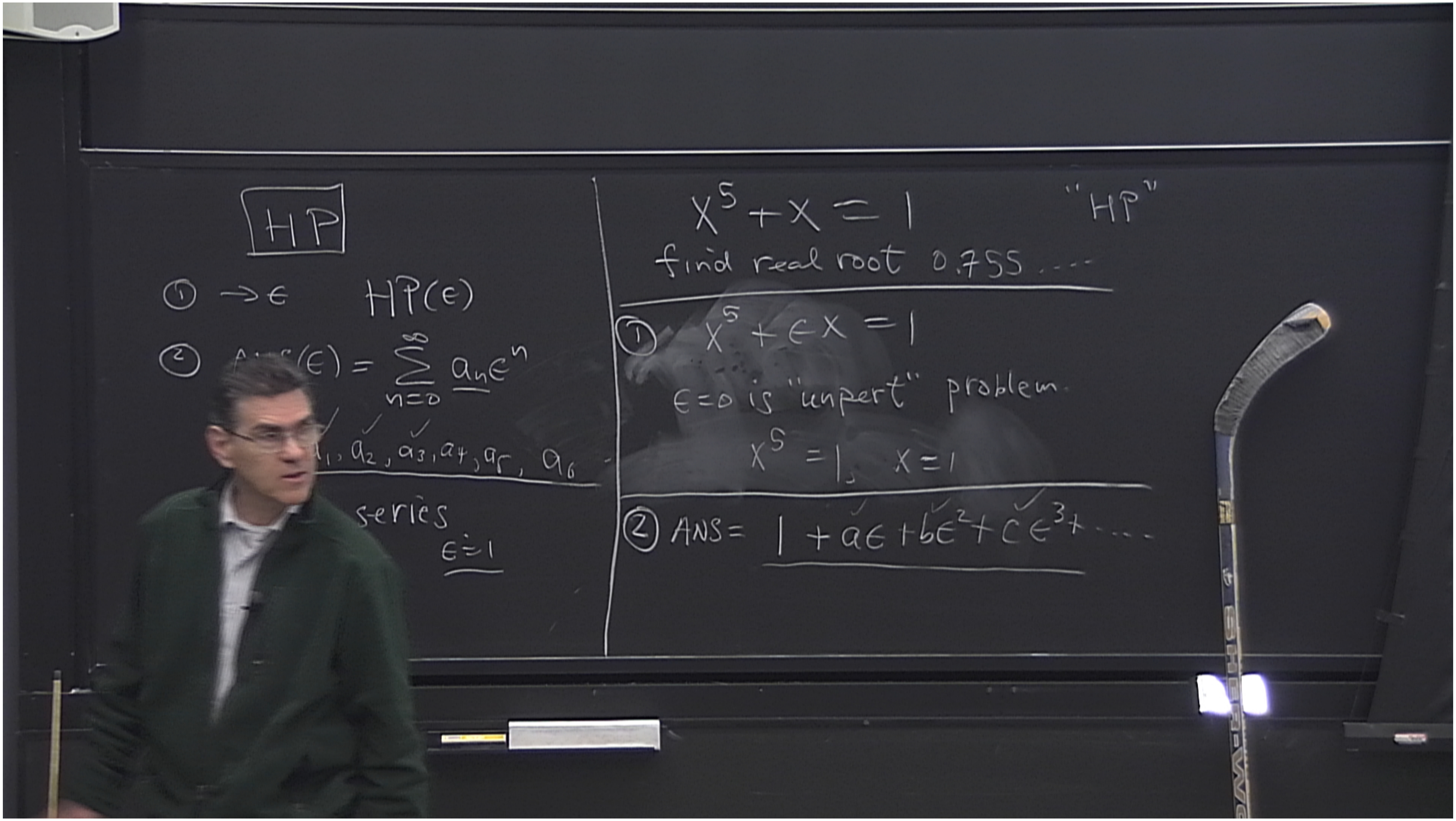

HP
① → ε HP(ε)
② ANS(ε) =
③ add up
x^5 + x = 1
"HP"
find real root 0.755 ....

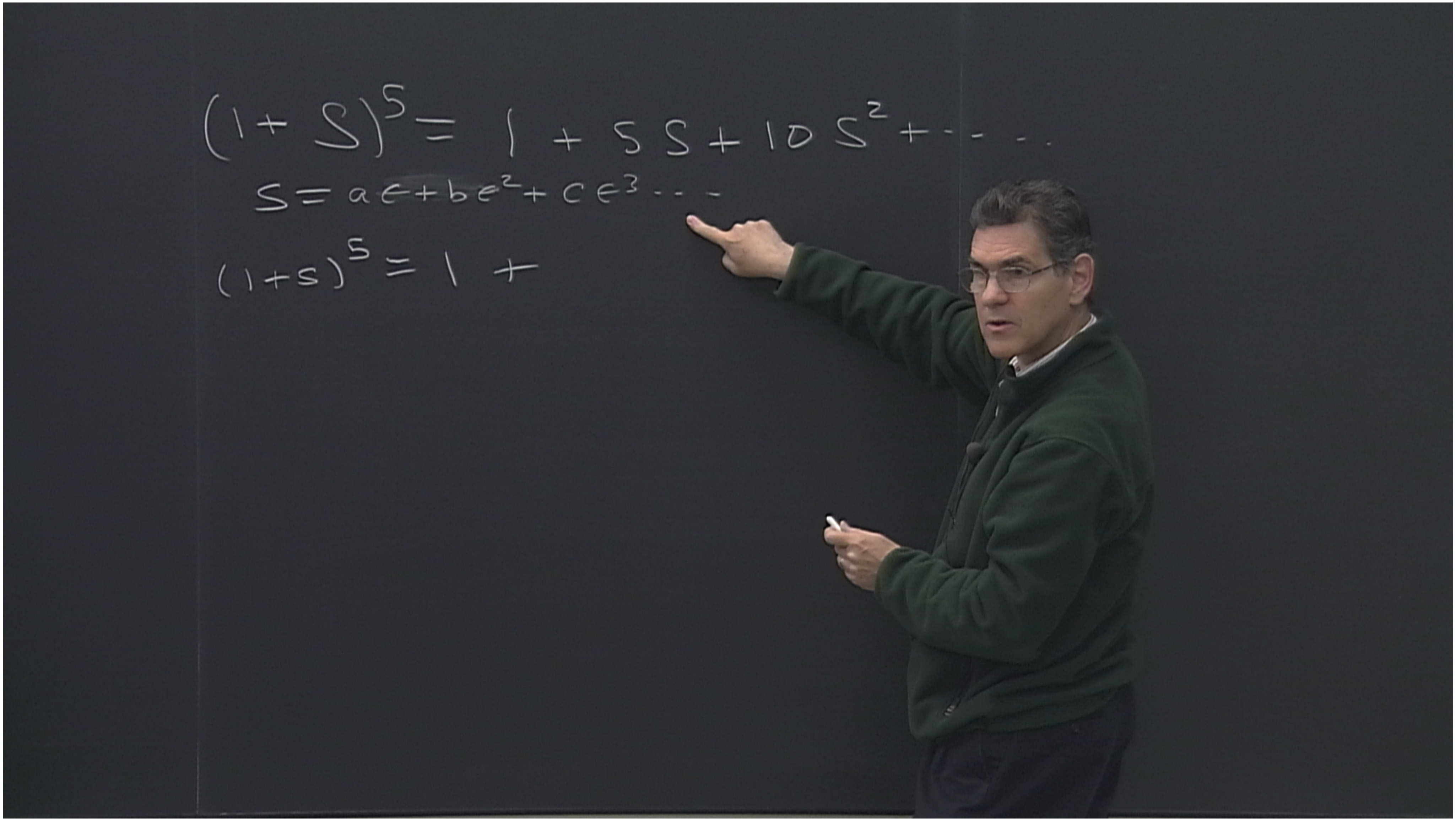

HP
① → ϵ HP(ϵ)
② ANS(ϵ) = Σ_{n=0}^∞ a_n ϵ^n
a_0, a_1, a_2, a_3, a_4
③ add up series
x^5 + x = 1
"HP"
find real root 0.755 ....
① x^5 + ϵx = 1
ℒ = (∂φ)^2 + m^2φ^2 + φ^4


HP
① → ϵ HP(ϵ)
ANS(ϵ) = Σ a_n ϵ^n
n=0
a₀, a₁, a₂, a₃, a₄, a₅, a₆
series
ϵ = 1
x^5 + x = 1 "HP"
find real root 0.755 ....
① x^5 + ϵx = 1
ϵ = 0 is "unpert" problem.
x^5 = 1


HP
① → ϵ HP(ϵ)
② ANS(ϵ) = Σ
a0, a1, a2, a3 a6
③ add up
x^5 + x = 1 "HP"
find real root 0.755 ....
① x^5 + ϵx = 1
ϵ=0 is "unpert" problem.
x^5 = 1, x = 1


HP
① → ϵ HP(ϵ)
x^5 + x = 1
"HP"
find real root 0.755 ....
① x^5 + ϵx = 1
ϵ=0 is "unpert" problem.
x^5 = 1, x = 1
② ANS = 1


HP
① → ϵ HP(ϵ)
② ANS(ϵ) = Σ_{n=0}^{∞} a_n ϵ^n
a₁, a₂, a₃, a₄, a₅, a₆
series
ϵ ≐ 1
x⁵ + x = 1 "HP"
find real root 0.755....
① x⁵ + ϵx = 1
ϵ=0 is "unpert" problem.
x⁵ = 1, x = 1
② ANS = 1 + aϵ + bϵ² + cϵ³ + ...


(1+ S)^5 = 1 + 5 S + 10 S^2 + ...
S = aε + bε^2 + cε^3 . . .
(1+s)^5 = 1 +

$$(1+S)^5 = 1 + 5S + 10S^2 + \cdots$$

$$S = a\epsilon + b\epsilon^2 + c\epsilon^3 \cdots$$

$$(1+S)^5 = 1 + 5a\epsilon + 5b\epsilon^2 + 5c\epsilon^3 + \cdots$$
$$+10\left(a^2\epsilon^2 + 2ab\epsilon^3 + \cdots\right)$$
$$= 1 + 5a\epsilon + \epsilon^2\left(5b + 10a^2\right) + \epsilon^3\left(5c + 20ab\right)$$

$$(1+S)^5 = 1 + 5S + 10S^2 + \cdots$$

$$S = a\epsilon + b\epsilon^2 + c\epsilon^3 \cdots$$

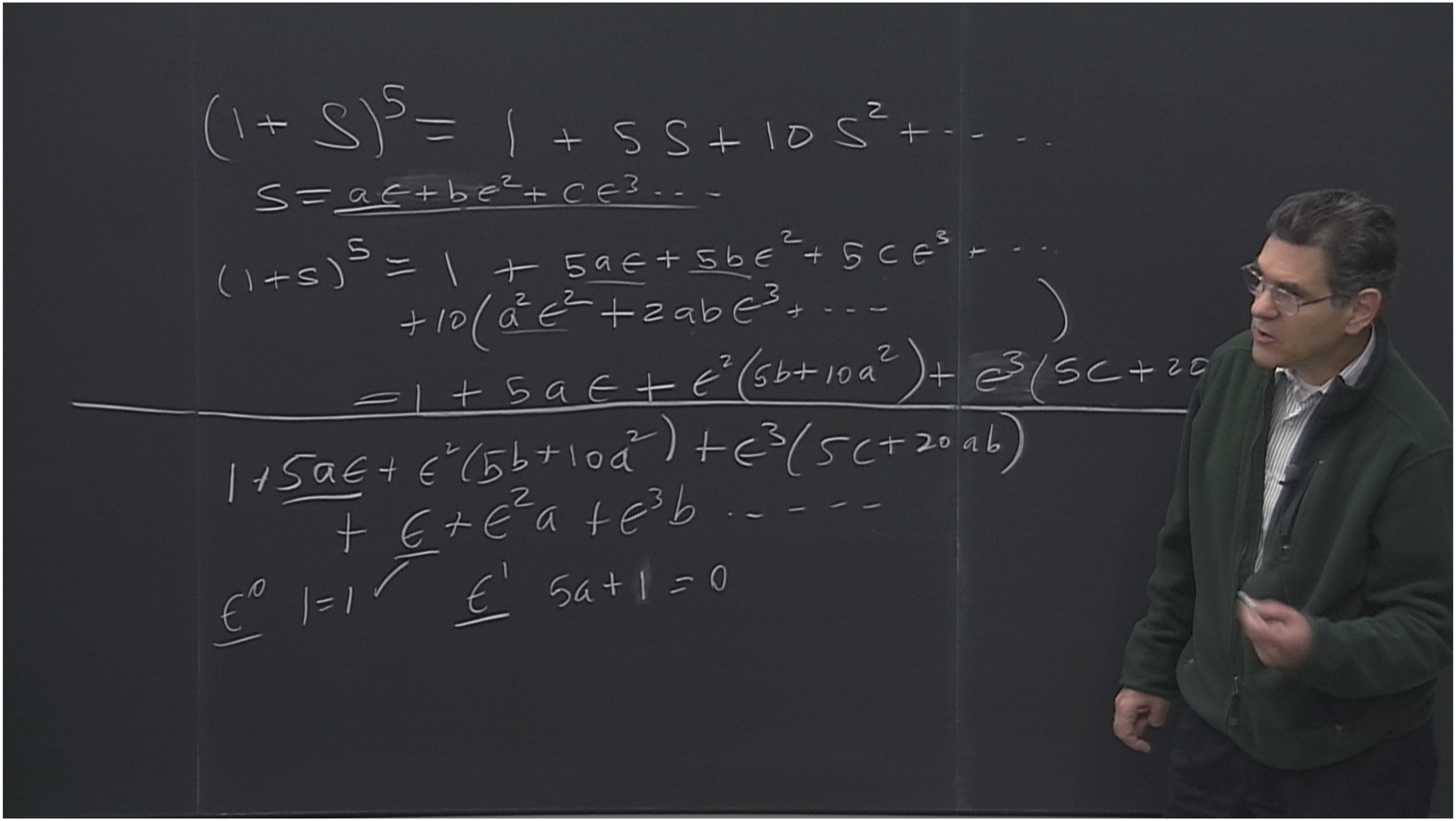
$$(1+S)^5 = 1 + 5a\epsilon + 5b\epsilon^2 + 5c\epsilon^3 + \cdots$$

$$+10\left(a^2\epsilon^2 + 2ab\epsilon^3 + \cdots\right)$$

$$= 1 + 5a\epsilon + \epsilon^2(5b + 10a^2) + \epsilon^3(5c + 20ab)$$

$$1 + 5a\epsilon + \epsilon^2(5b + 10a^2) + \epsilon^3(5c + 20ab)$$

$$+ \epsilon + \epsilon^2 a + \epsilon^3 b \cdots \quad = 1$$



(1+S)^5 = 1 + 5S + 10S^2 + ...
S = aε + bε^2 + cε^3 ...
(1+s)^5 = 1 + 5aε + 5bε^2 + 5cε^3 + ...
+10(a^2ε^2 + 2abε^3 + ...)
= 1 + 5aε + ε^2(5b+10a^2) + ε^3(5c + 2
1 + 5aε + ε^2(5b+10a^2) + ε^3(5c+20ab)
+ ε + ε^2a + ε^3b .....
ε^0 1=1 ✓
ε^1 5a+1=0

$$(1+S)^5 = 1 + 5S + 10S^2 + \cdots$$

$$S = a\epsilon + b\epsilon^2 + c\epsilon^3 \cdots$$

$$(1+s)^5 = 1 + 5a\epsilon + 5b\epsilon^2 + 5c\epsilon^3 + \cdots$$
$$+10(a^2\epsilon^2 + 2ab\epsilon^3 + \cdots)$$
$$= 1 + 5a\epsilon + \epsilon^2(5b + 10a^2) + \epsilon^3(5c + 20ab)$$

$$1 + 5a\epsilon + \epsilon^2(5b + 10a^2) + \epsilon^3(5c + 20ab)$$
$$+ \epsilon + \epsilon^2 a + \epsilon^3 b \cdots \quad = 1$$

ϵ^0: $1 = 1$ ✓

ϵ^1: $5a + 1 = 0$

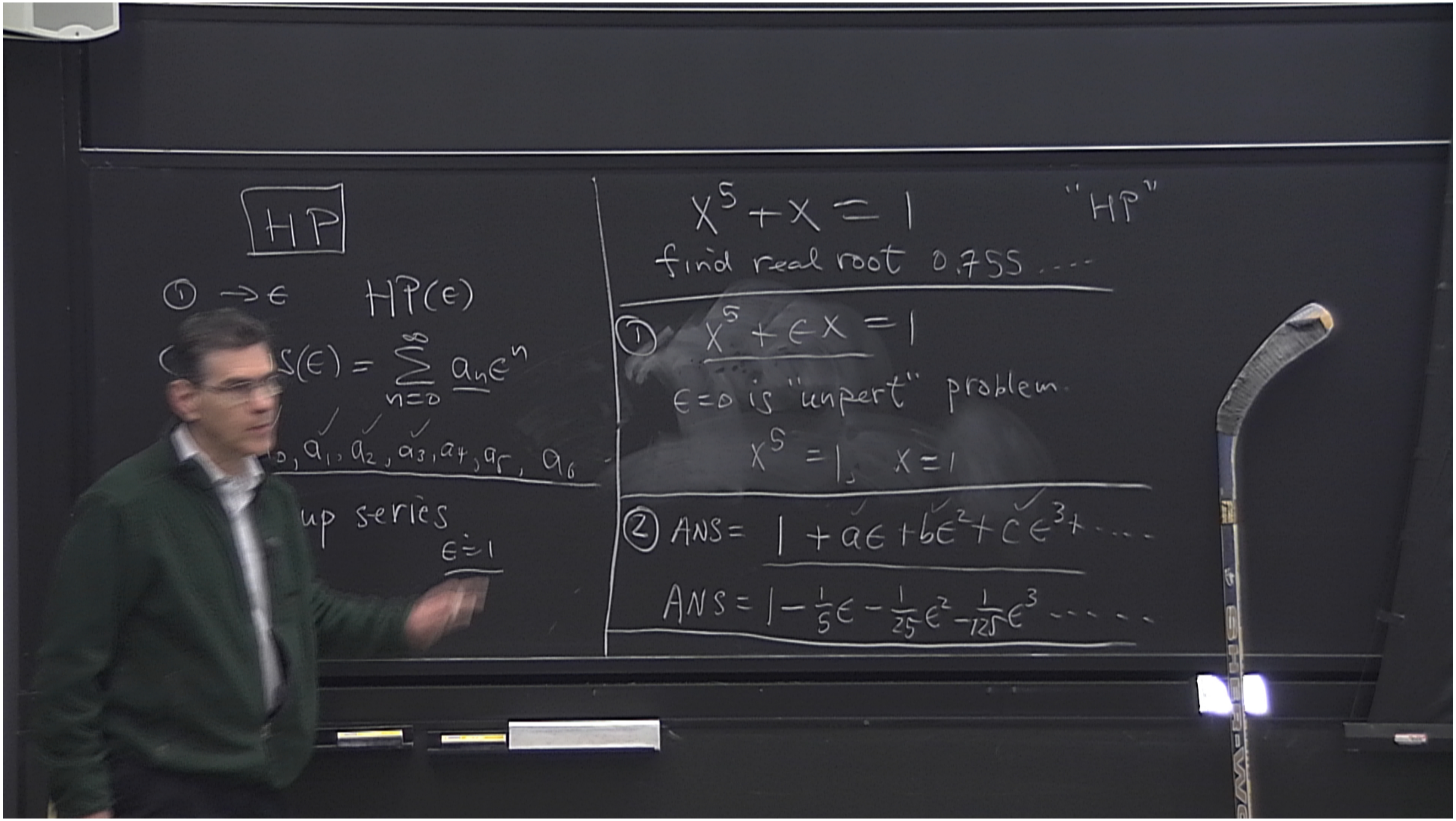
$a = -\frac{1}{5}$

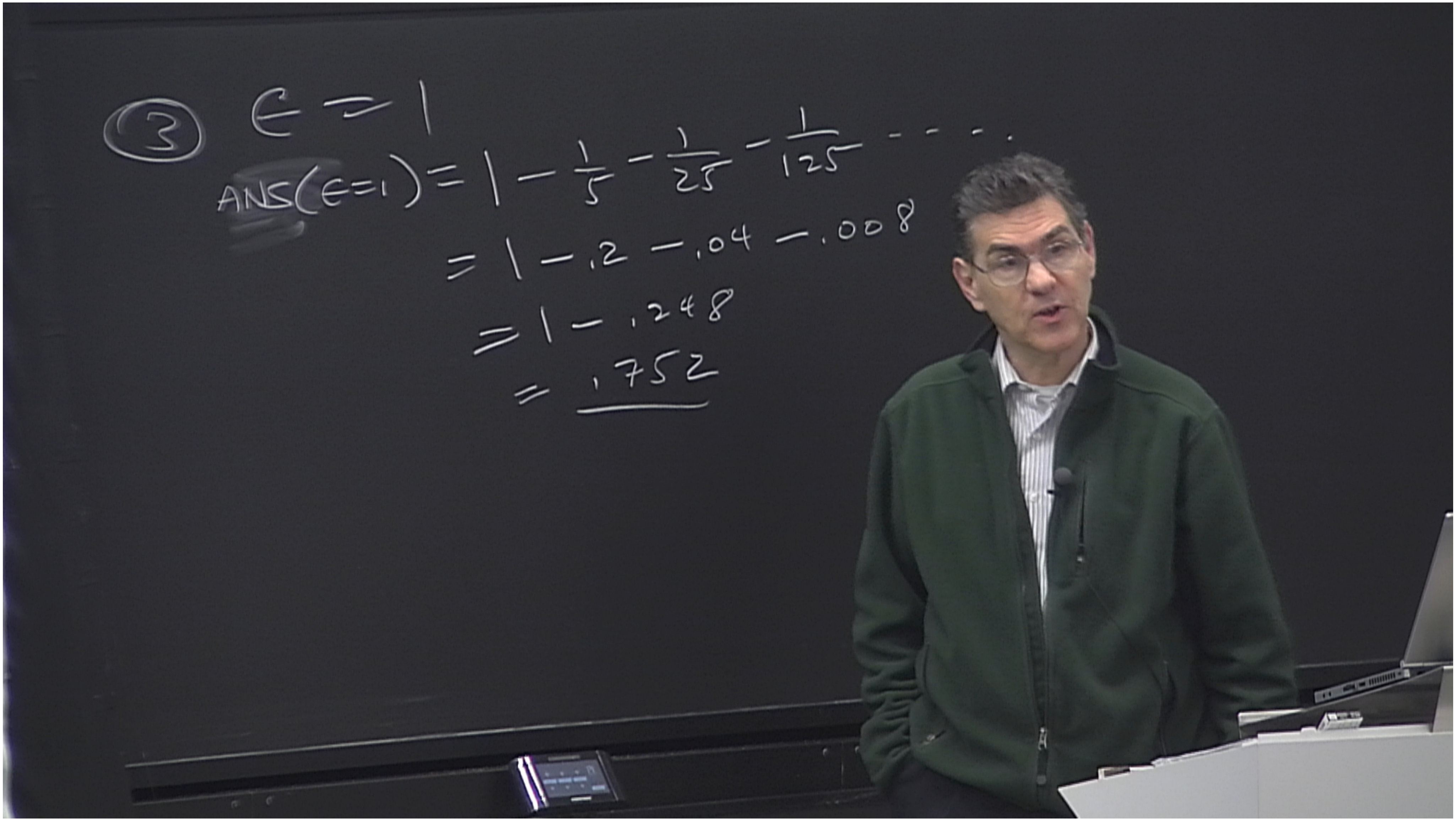
ϵ^2: $5b + 10a^2 + a = 0$

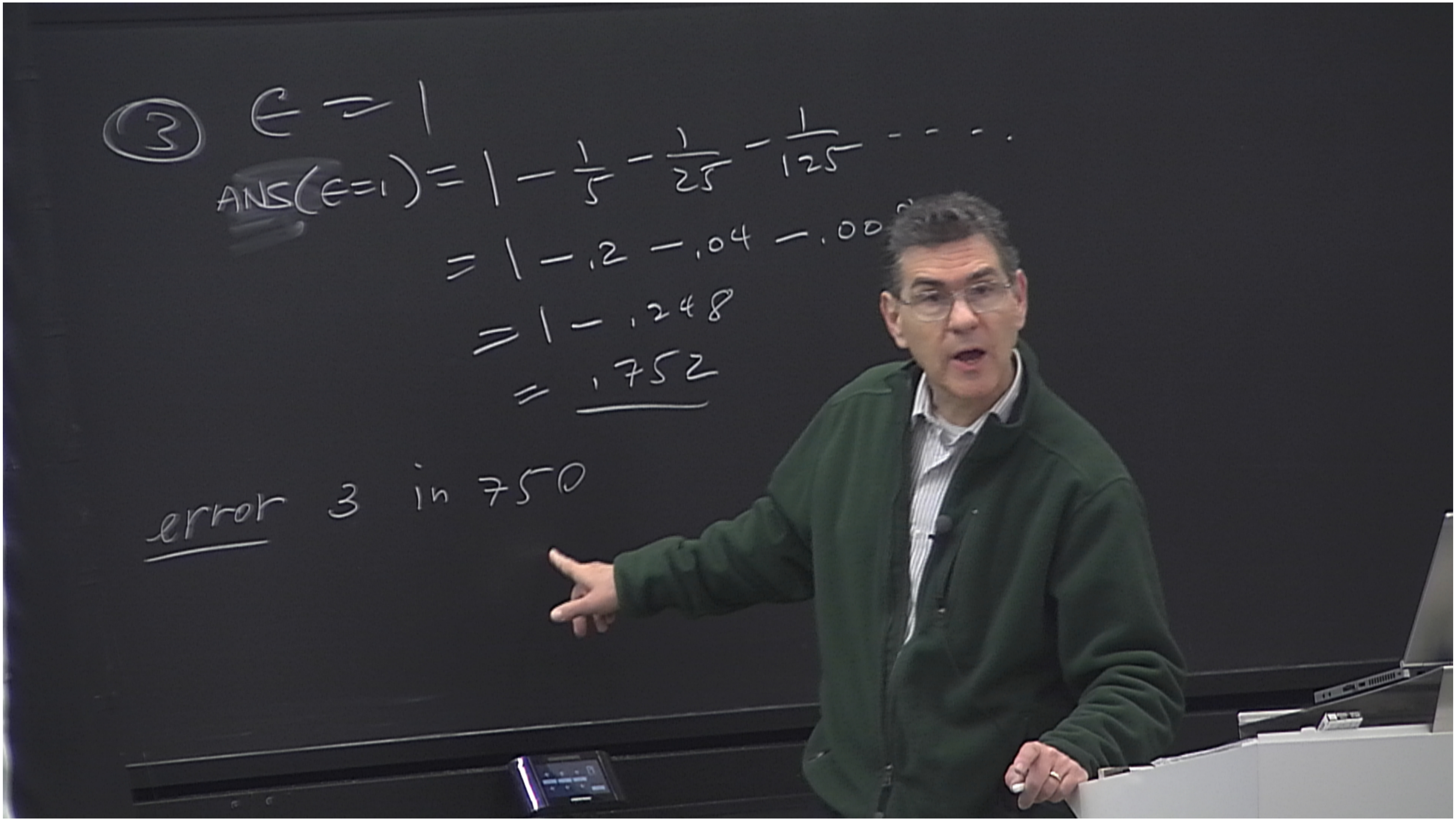
$5b + \frac{2}{5} - \frac{1}{5} = 0$

$5b + \frac{1}{5} = 0$

$b = -\frac{1}{25}$


HP
① → ϵ HP(ϵ)
S(ϵ) = Σ_{n=0}^{∞} a_n ϵ^n
a_1, a_2, a_3, a_4, a_5, a_6
up series
ϵ ≐ 1
x^5 + x = 1 "HP"
find real root 0.755 ....
① x^5 + ϵx = 1
ϵ=0 is "unpert" problem.
x^5 = 1, x = 1
② ANS = 1 + aϵ + bϵ^2 + cϵ^3 + ...
ANS = 1 − 1/5 ϵ − 1/25 ϵ^2 − 1/125 ϵ^3 ......


③ ε = 1
ANS(ε=1) = 1 − 1/5 − 1/25 − 1/125 − - - - .
= 1 − .2 − .04 − .008
= 1 − .248
= .752


③ ∈ = 1
ANS(∈=1) = 1 − 1/5 − 1/25 − 1/125 − . . . .
= 1 − .2 − .04 − .00
= 1 − .248
= .752
error 3 in 750

Simple example

HARD PROBLEM: Find the positive root of

$$x^5 + x = 1$$

ANSWER: $x = 0.75487767\cdots$

Step 1. Insert ε: $x^5 + \epsilon x = 1$ (Strong coupling)

Step 2. $x(\epsilon) = \sum_{n=0}^{\infty} a_n \epsilon^n$

$$a_0 = 1$$

Match powers of ε

$$
\begin{aligned}
5a_1 + 1 &= 0, \\
5a_2 + 10a_1^2 + a_1 &= 0, \\
5a_3 + 20a_1a_2 + a_2 + 10a_1^3 &= 0, \\
5a_4 + 20a_1a_3 + a_3 + 10a_2^2 + 30a_1^2a_2 + 5a_1^4 &= 0.
\end{aligned}
$$

$$a_1 = -\frac{1}{5}, \quad a_2 = -\frac{1}{25}, \quad a_3 = -\frac{1}{125},$$

$$a_4 = 0, \quad a_5 = \frac{21}{15625}, \quad a_6 = \frac{78}{78125}$$

The perturbation series…

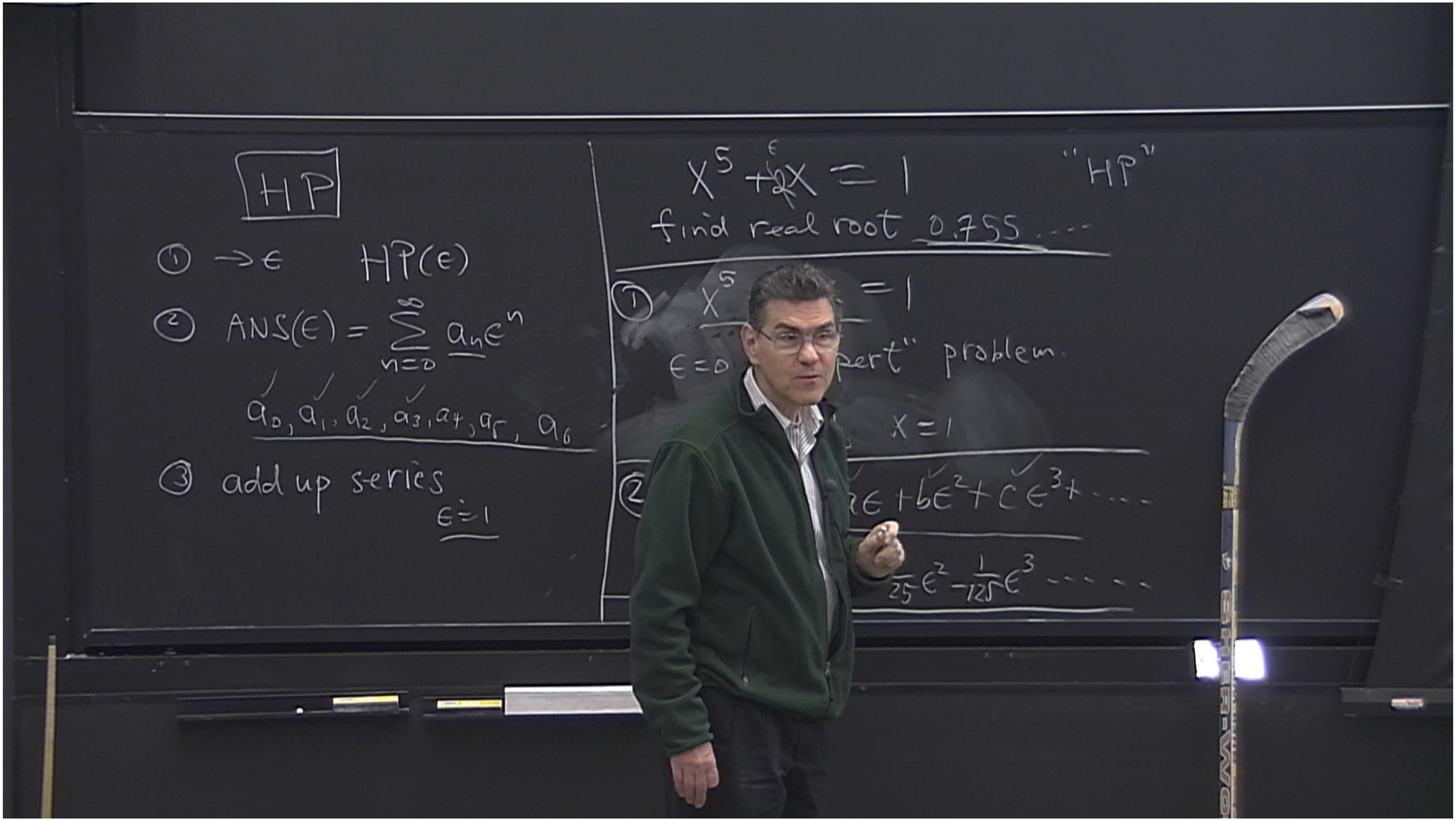
$$x(\epsilon) \ = \ 1 - \frac{1}{5}\epsilon - \frac{1}{25}\epsilon^2 - \frac{1}{125}\epsilon^3 + \frac{21}{15625}\epsilon^5 + \frac{78}{78125}\epsilon^6 + \cdots$$

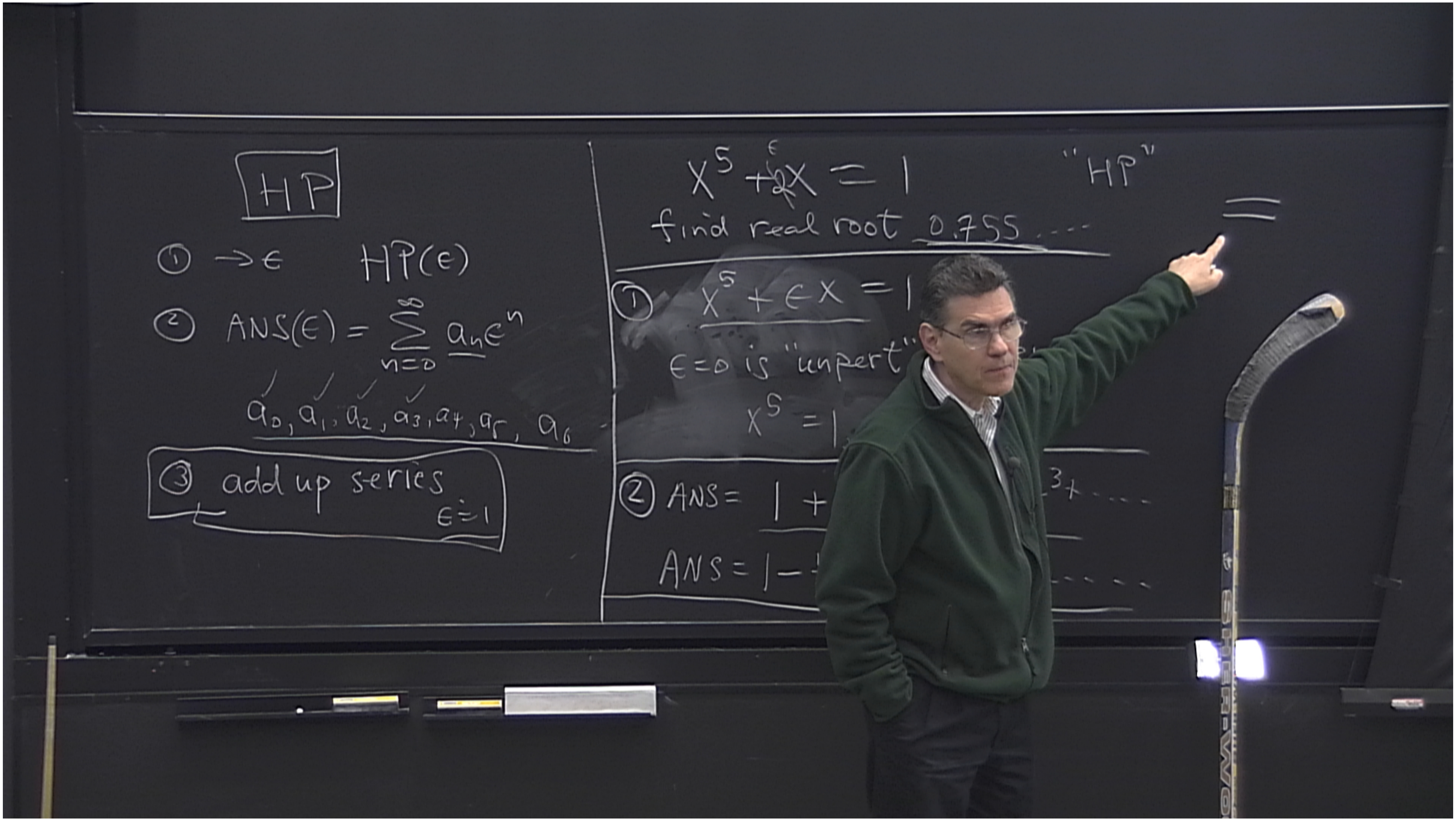
Step 3. Sum the series at $x = 1$

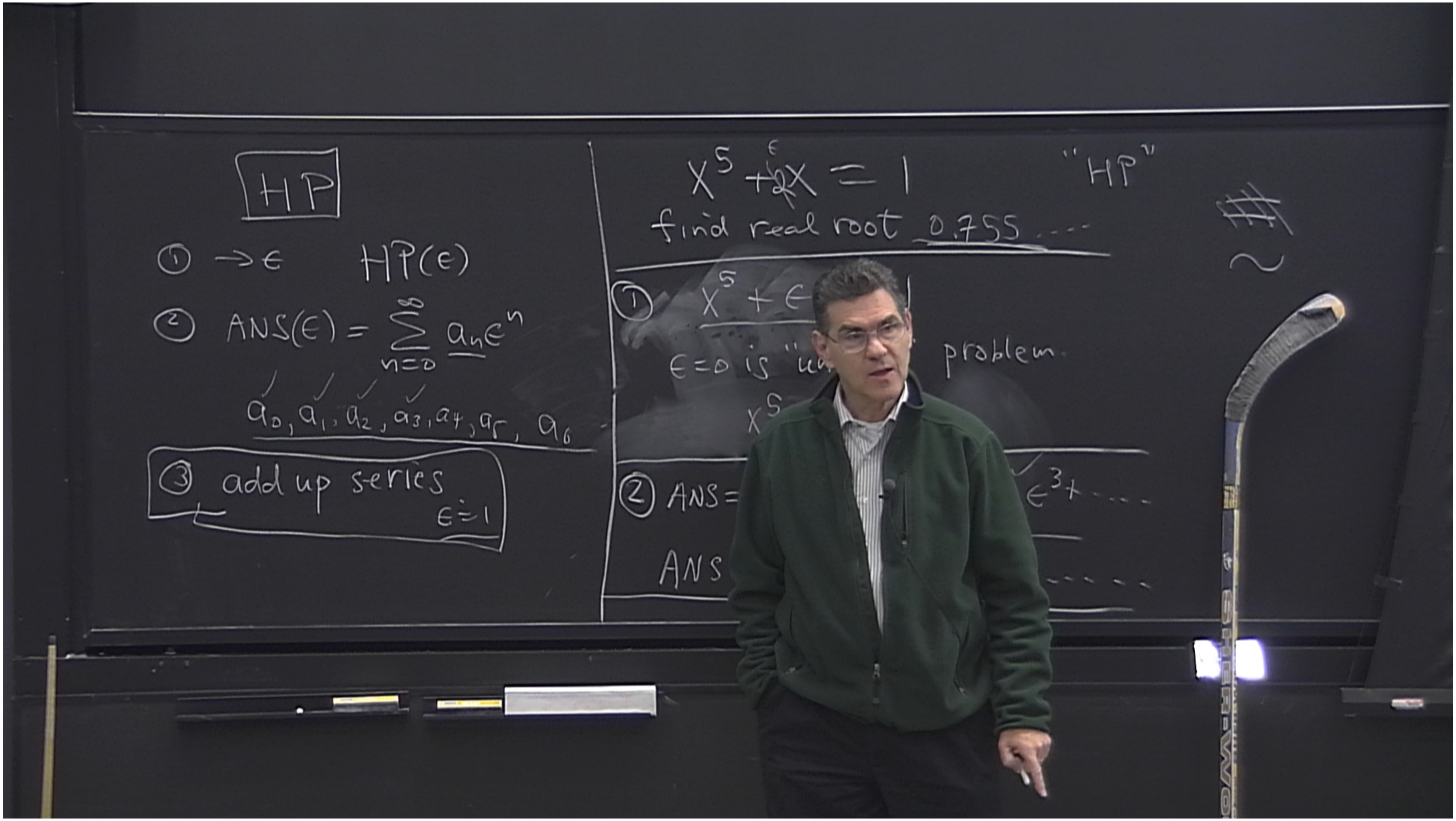
Radius of convergence of this series: 1.64938…

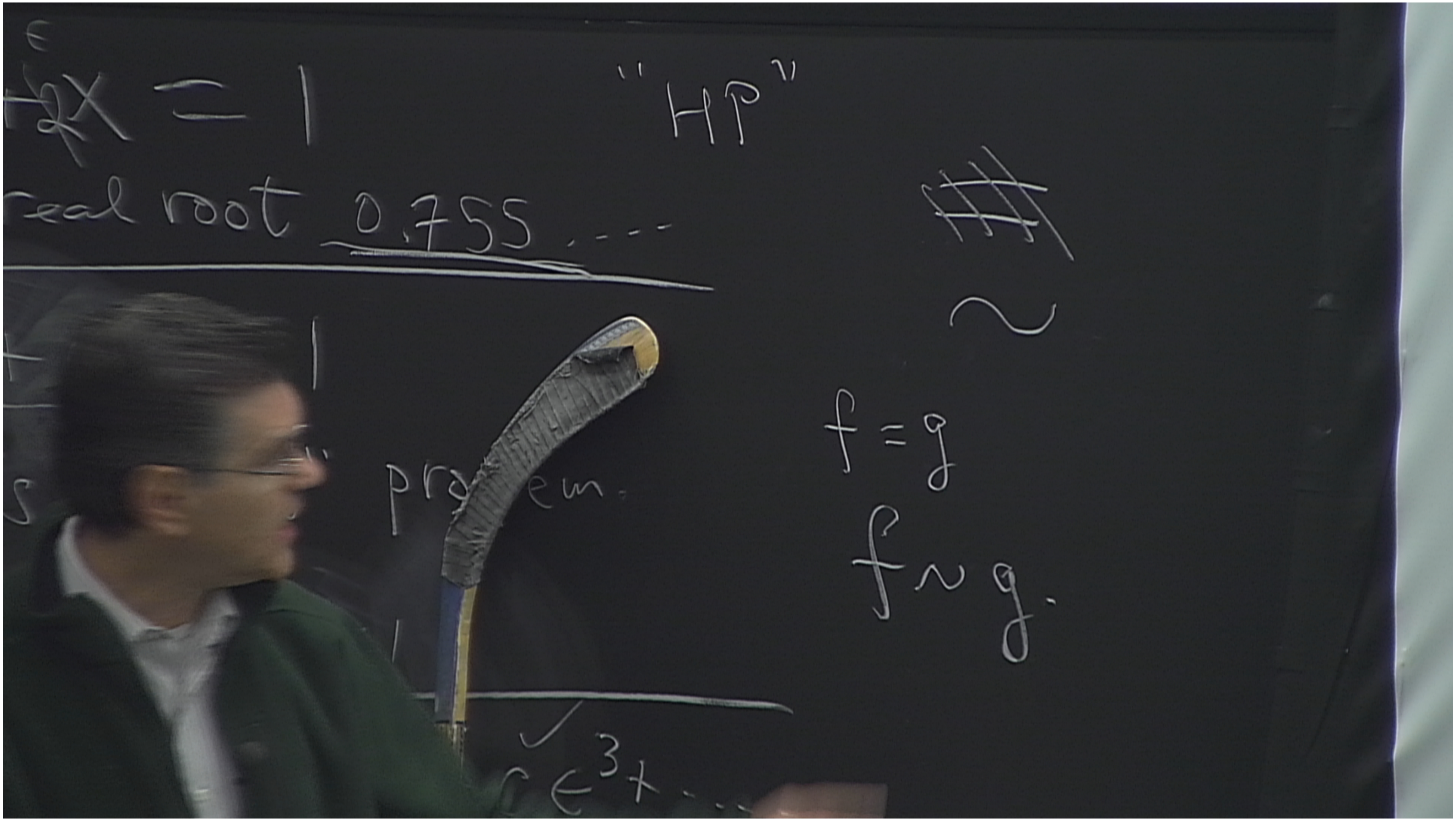
Sixth-order result $x(1) = 0.75434$

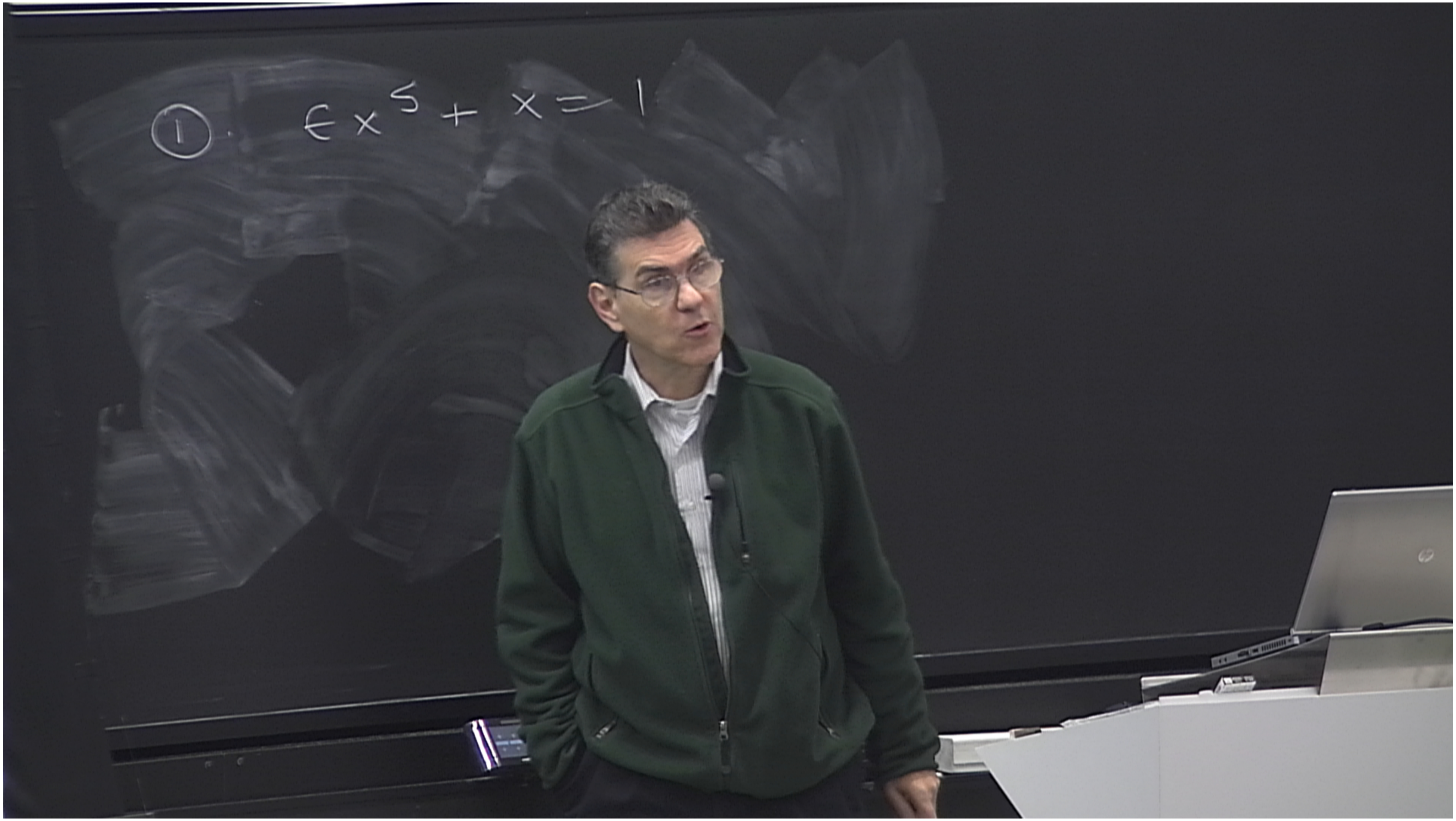
Exact answer $x = 0.75488$

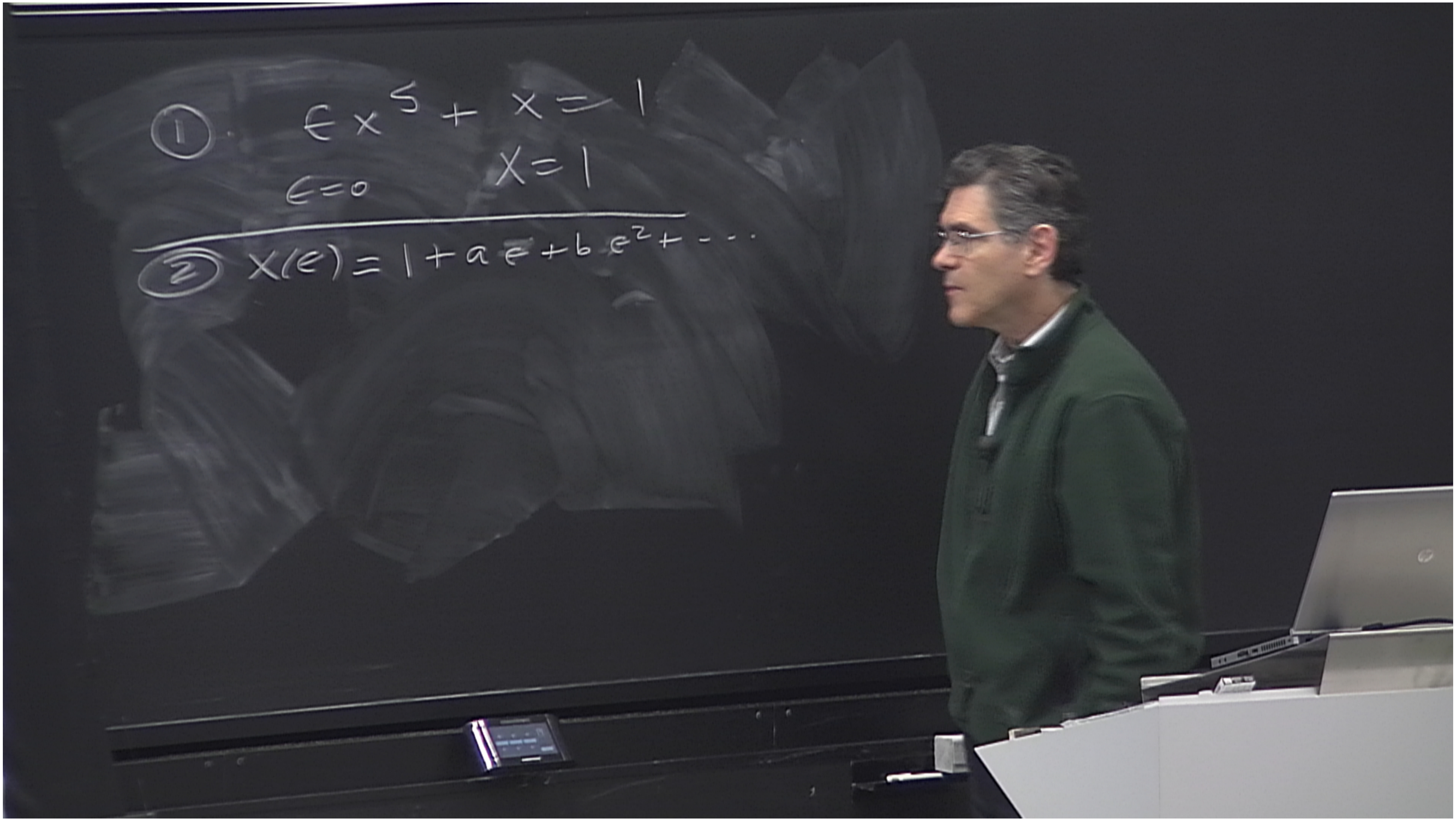

HP
① → ϵ HP(ϵ)
② ANS(ϵ) = Σ aₙϵⁿ
a₀, a₁, a₂, a₃, a₄, a₅, a₆
③ add up series
ϵ = 1
x⁵ + x = 1
"HP"
find real root 0.755 ....
problem.
x = 1

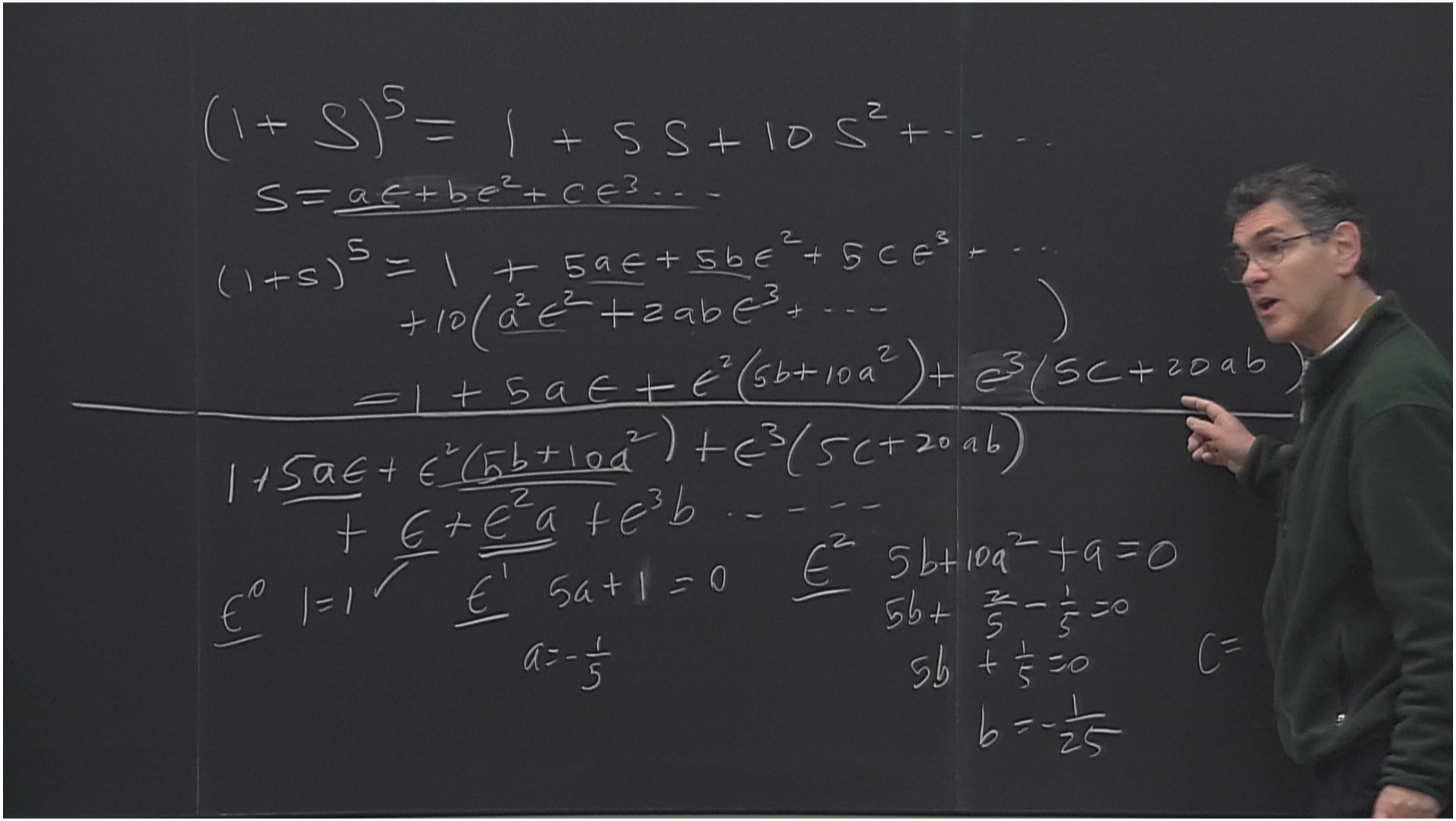

HP
① → ε   HP(ε)
② ANS(ε) = Σ_{n=0}^∞ a_n ε^n
a_0, a_1, a_2, a_3, a_4, a_5, a_6
③ add up series
ε = 1
find real root 0.755 ....
"HP"
ε=0 is "unpert"
② ANS = 1 +
ANS = 1 −

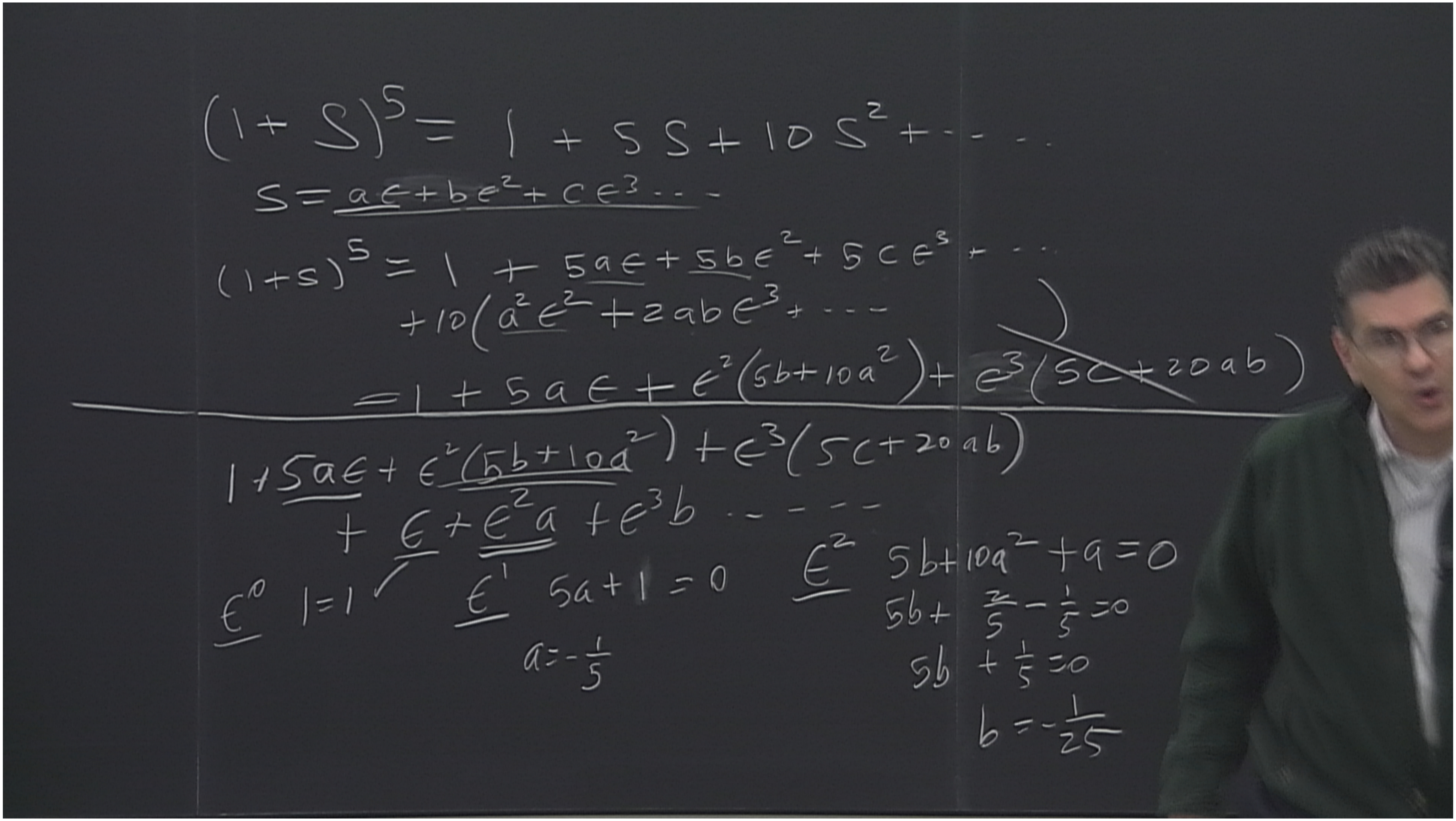

HP
① → ε HP(ε)
② ANS(ε) = Σ a_n ε^n n=0 ∞
a0, a1, a2, a3, a4, a5, a6
③ add up series ε ≐ 1
x^5 + εx = 1
"HP"
find real root 0.755 ....
problem.
ANS

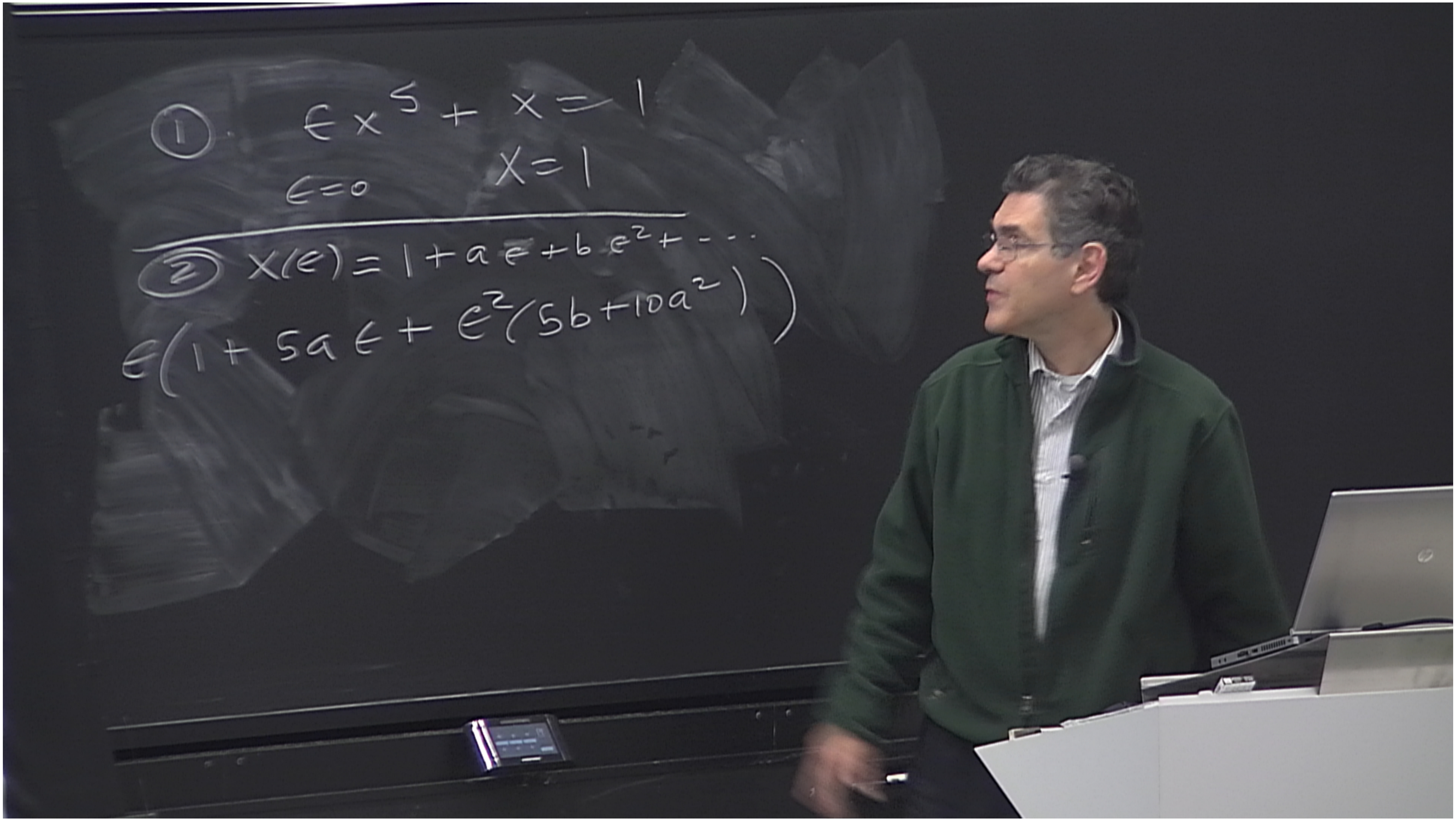

"HP"
real root 0.755 ....
f = g
f ~ g.

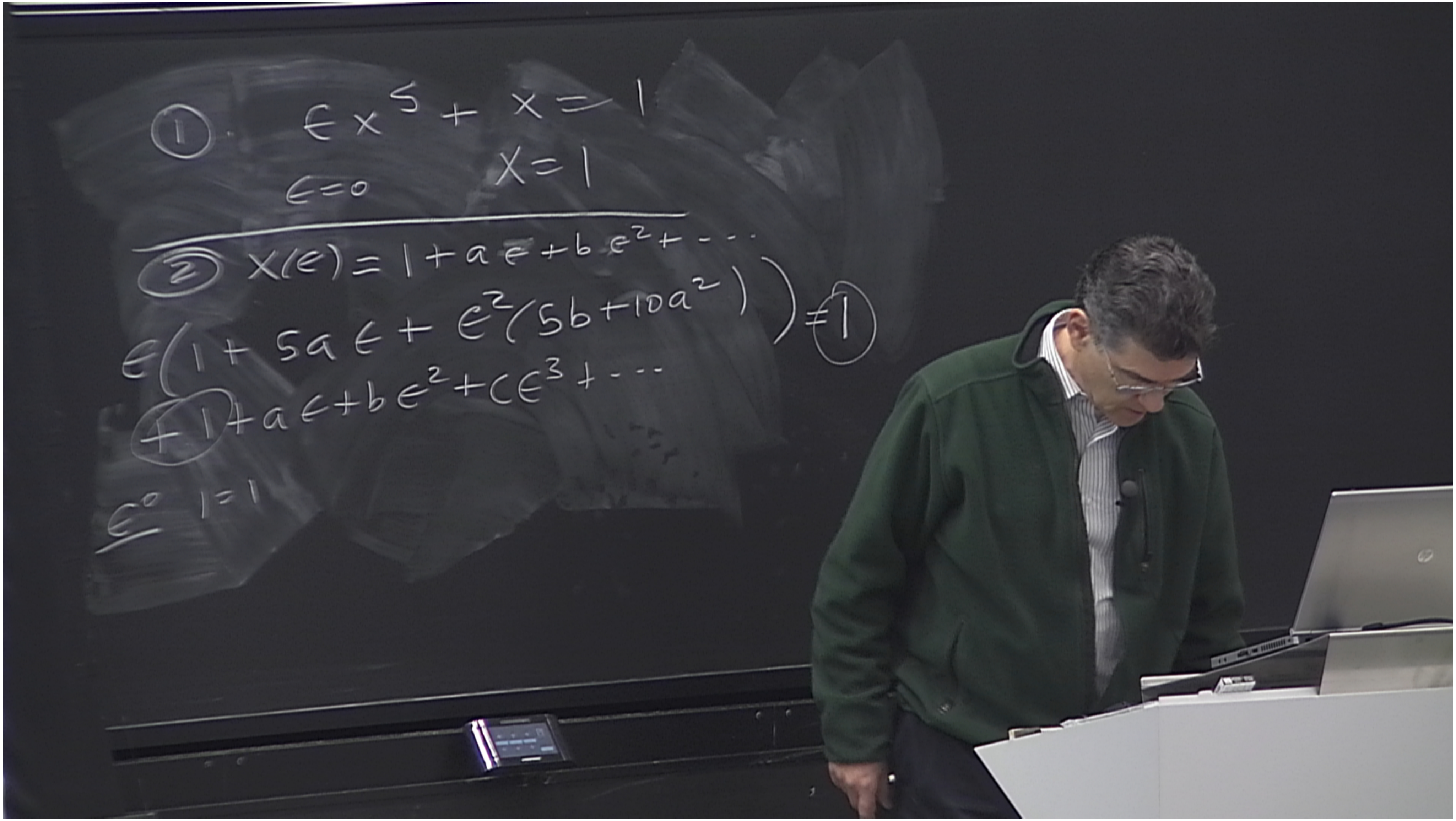

1. $\epsilon x^5 + x = 1$

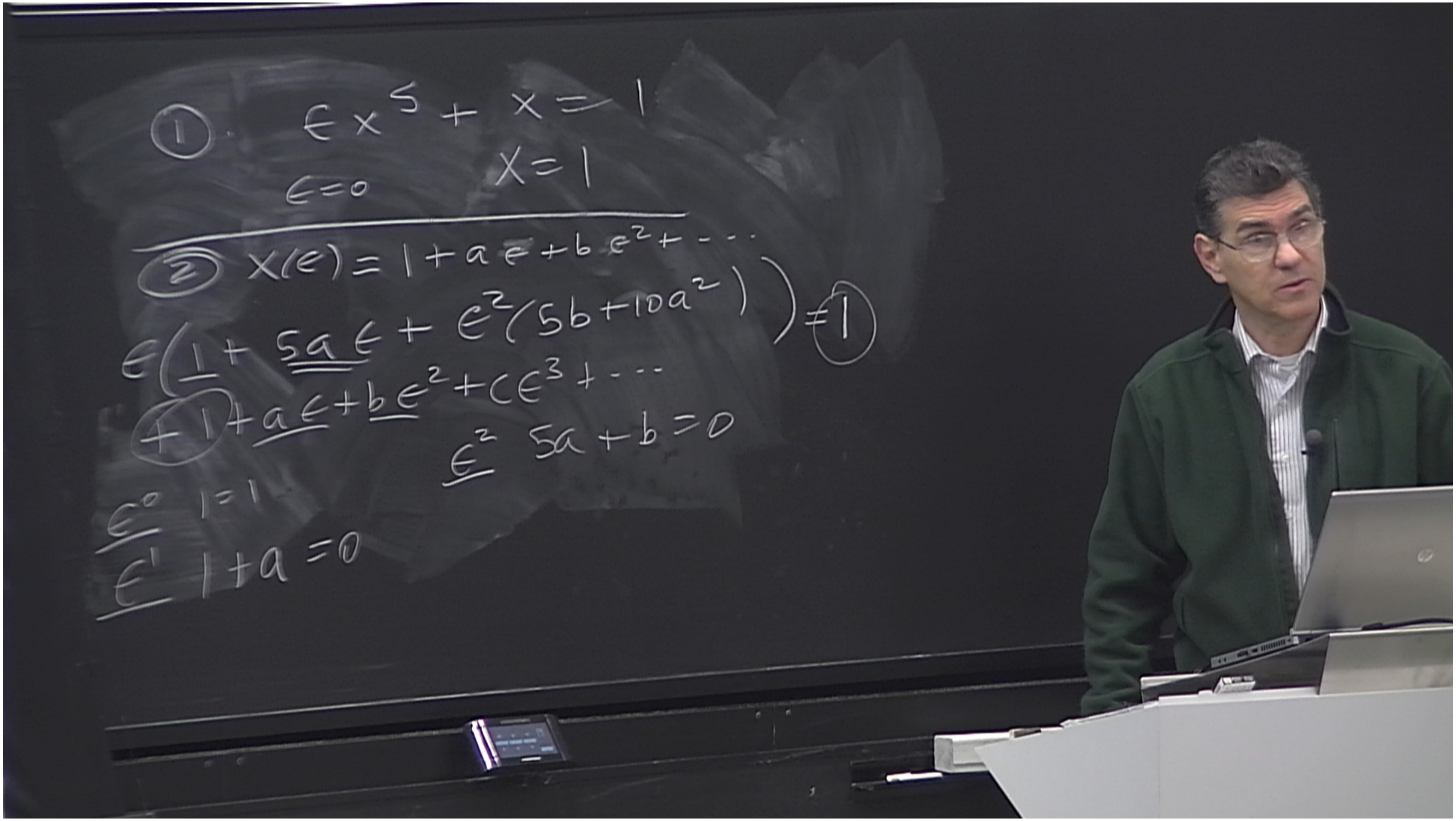

① $\epsilon x^5 + x = 1$
$\epsilon = 0$ $X = 1$
② $x(\epsilon) = 1 + a\epsilon + b\epsilon^2 + \dots$

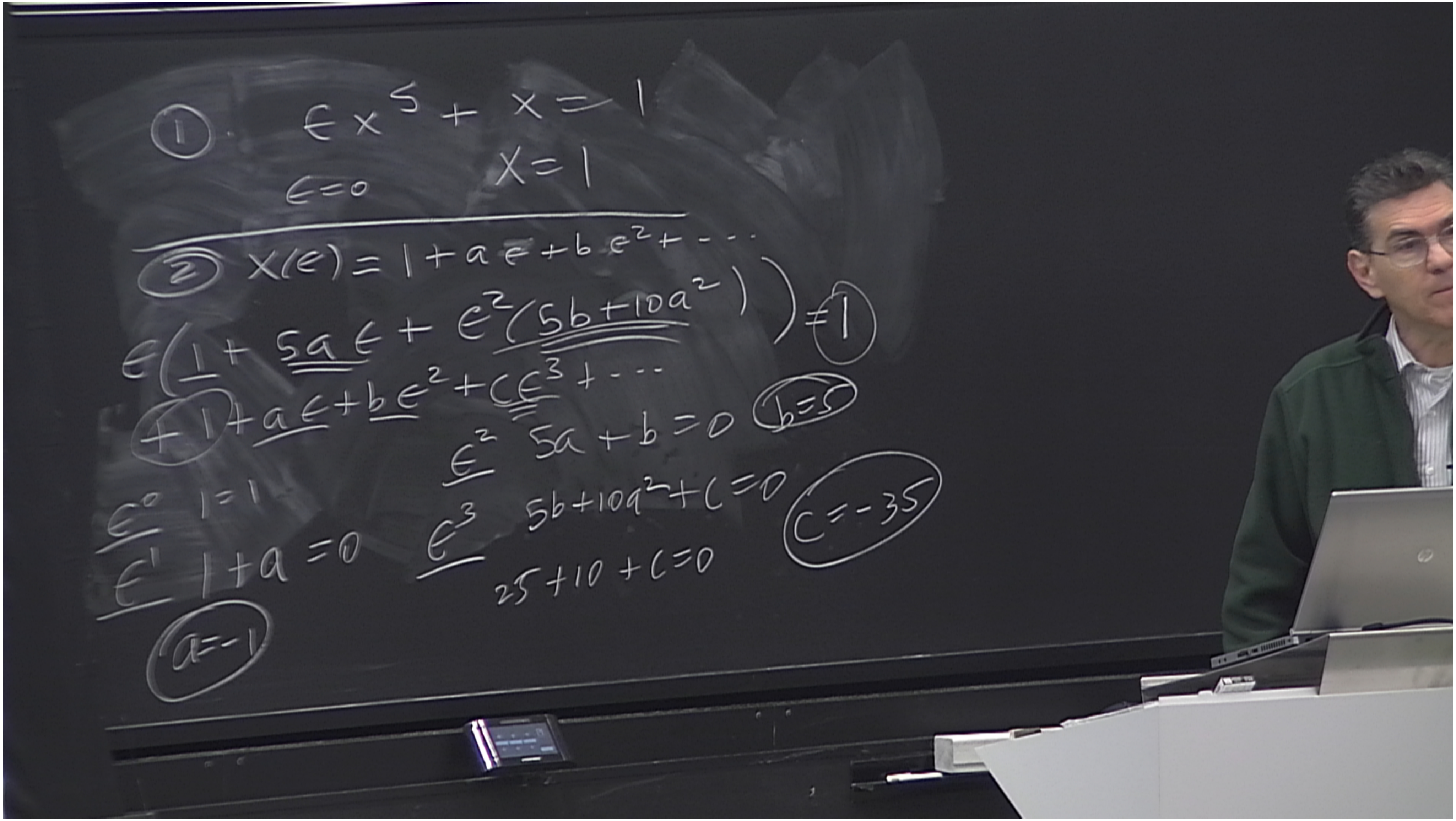

(1+ S)^5 = 1 + 5S + 10S^2 + ...
S = aє + bє^2 + cє^3 . . . .
(1+s)^5 = 1 + 5aє + 5bє^2 + 5cє^3 + . . .
+10(a^2є^2 + 2abє^3 + . . .)
= 1 + 5aє + є^2(5b+10a^2) + є^3(5c + 20ab)
1 + 5aє + є^2(5b+10a^2) + є^3(5c+20ab)
+ є + є^2a + є^3b . . . . .
є^0 1=1 ✓
є^1 5a+1 = 0
a = -1/5
є^2 5b+10a^2 + a = 0
5b + 2/5 - 1/5 = 0
5b + 1/5 = 0
b = -1/25
c =


(1+ S)^5 = 1 + 5S + 10S^2 + ...
S = aε + bε^2 + cε^3 ....
(1+s)^5 = 1 + 5aε + 5bε^2 + 5cε^3 + ...
+10(a^2ε^2 + 2abε^3 + ...)
= 1 + 5aε + ε^2(5b+10a^2) + ε^3(5c+20ab)
1 + 5aε + ε^2(5b+10a^2) + ε^3(5c+20ab)
+ ε + ε^2a + ε^3b .....
ε^0 1=1
ε^1 5a+1 = 0
a = -1/5
ε^2 5b+10a^2 + a = 0
5b + 2/5 - 1/5 = 0
5b + 1/5 = 0
b = -1/25


① εx^5 + x = 1
ε=0 x=1
② x(ε) = 1 + aε + bε^2 + . . .
ε(1 + 5aε + ε^2(5b+10a^2))


① $\epsilon x^5 + x = 1$
$\epsilon = 0$ $x = 1$
② $x(\epsilon) = 1 + a\epsilon + b\epsilon^2 + \cdots$
$\epsilon(1 + 5a\epsilon + \epsilon^2(5b + 10a^2)) = 1$
$+1 + a\epsilon + b\epsilon^2 + c\epsilon^3 + \cdots$
ϵ^0 $1 = 1$


① εx^5 + x = 1
ε=0 x=1
② x(ε) = 1 + aε + bε^2 + ...
ε(1 + 5aε + ε^2(5b + 10a^2)) + 1
+ 1 + aε + bε^2 + cε^3 + ...
ε^2 5a + b = 0
ε^0 1=1
ε^1 1 + a = 0


① εx^5 + x = 1
ε=0 x=1
② x(ε) = 1 + aε + bε^2 + ...
ε(1 + 5aε + ε^2(5b+10a^2))) = 1
+1 + aε + bε^2 + cε^3 + ...
b=5
ε^2 5a + b = 0
ε^0 1=1
ε^3 5b+10a^2+c=0
c=-35
ε^1 1+a=0
25+10+c=0
a=-1

Conclusion:

We need to learn how to sum a series!

Outline of Course

- Beginning
- Middle
- End
- (applause)

Accelerating the convergence of a convergent series…

1. Richardson extrapolation
2. Shanks transformation
3. Fourier series

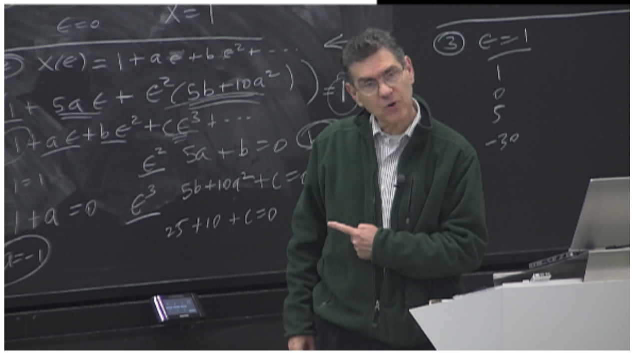
How to sum a divergent series…

1. Euler summation
2. Borel summation
3. Continued fractions and Pade theory

Asymptotics and perturbation theory…

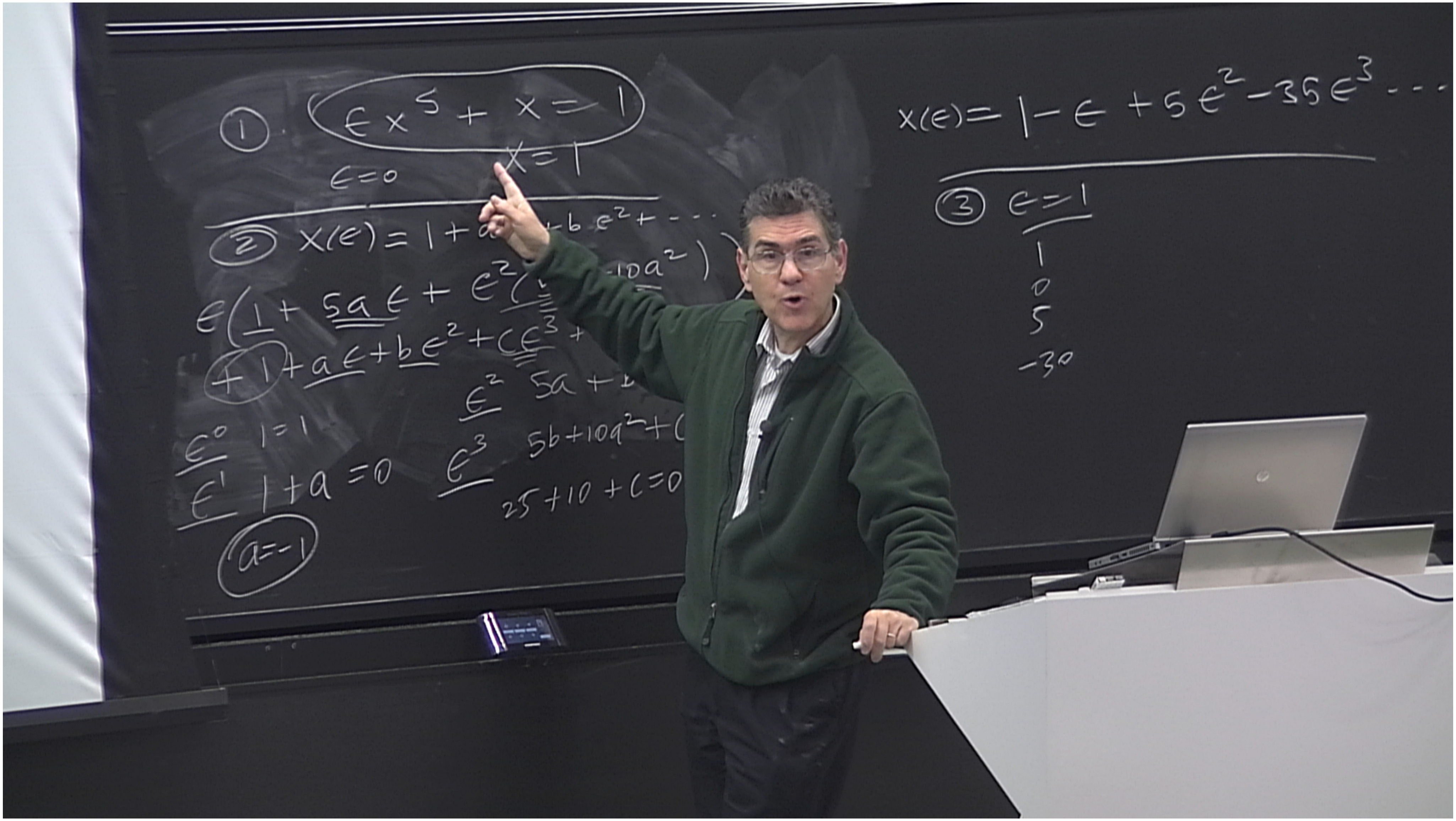
1. Boundary layer theory
2. WKB theory
3. And lots more...

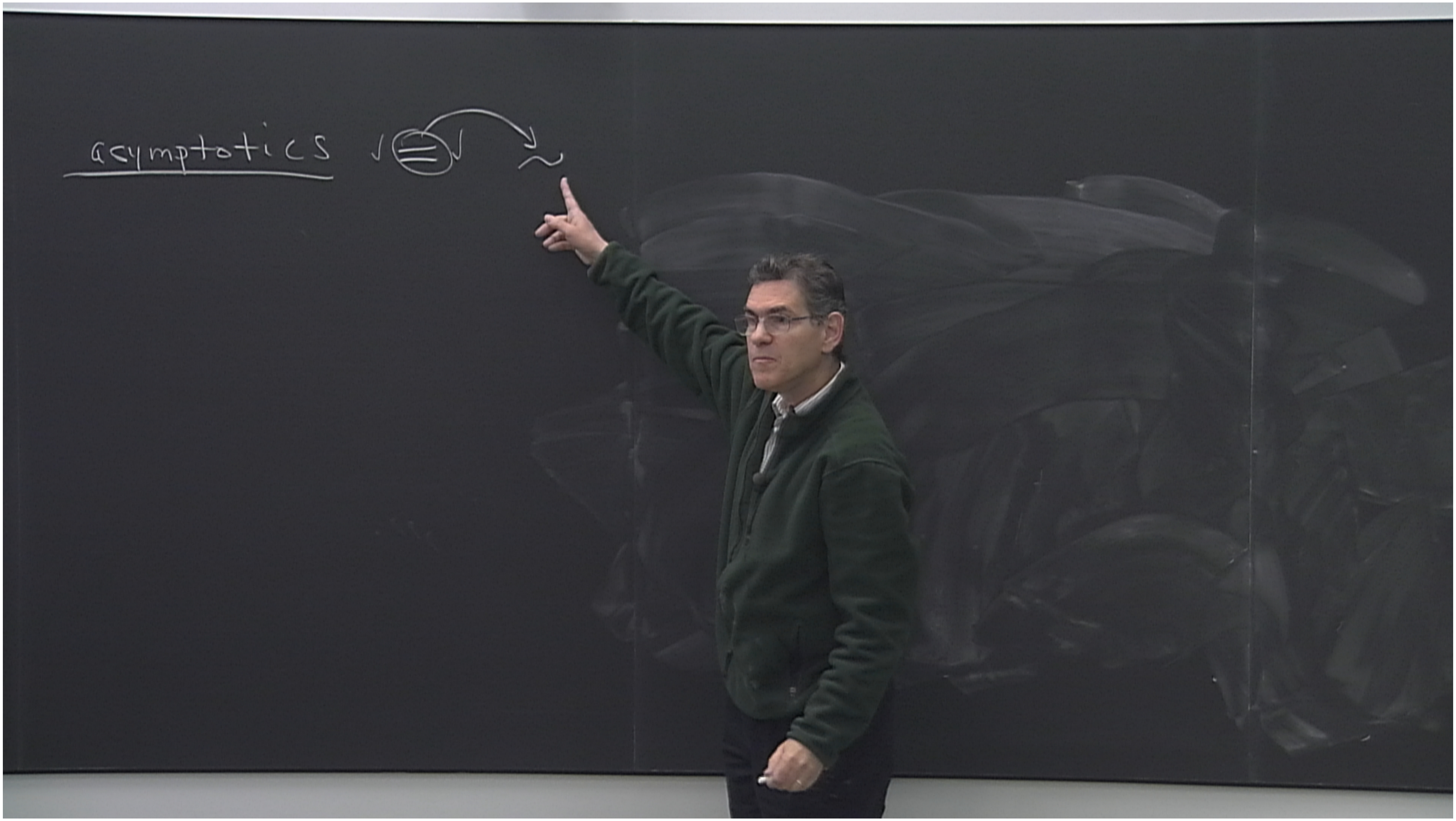
Shanks transformation

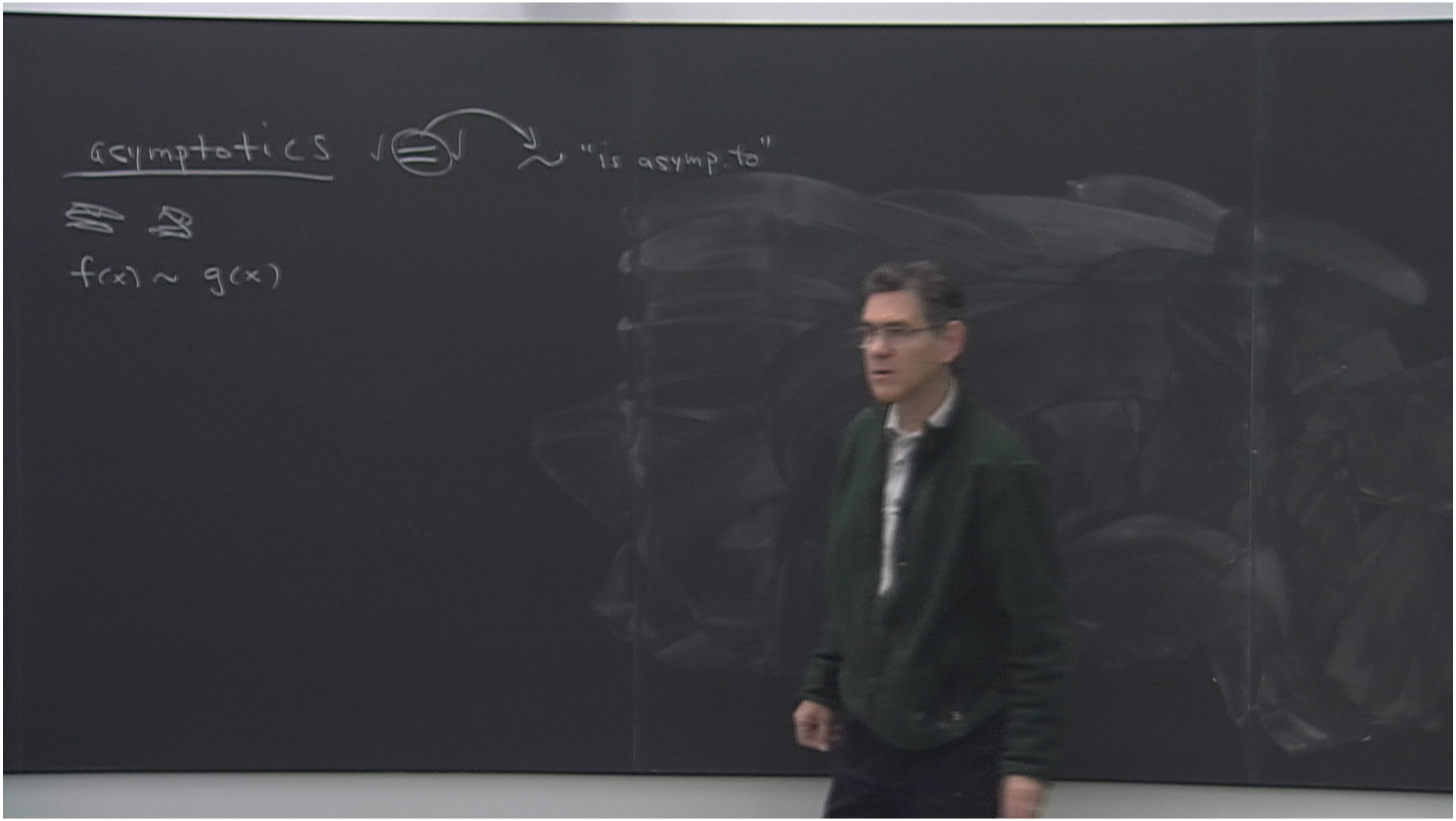


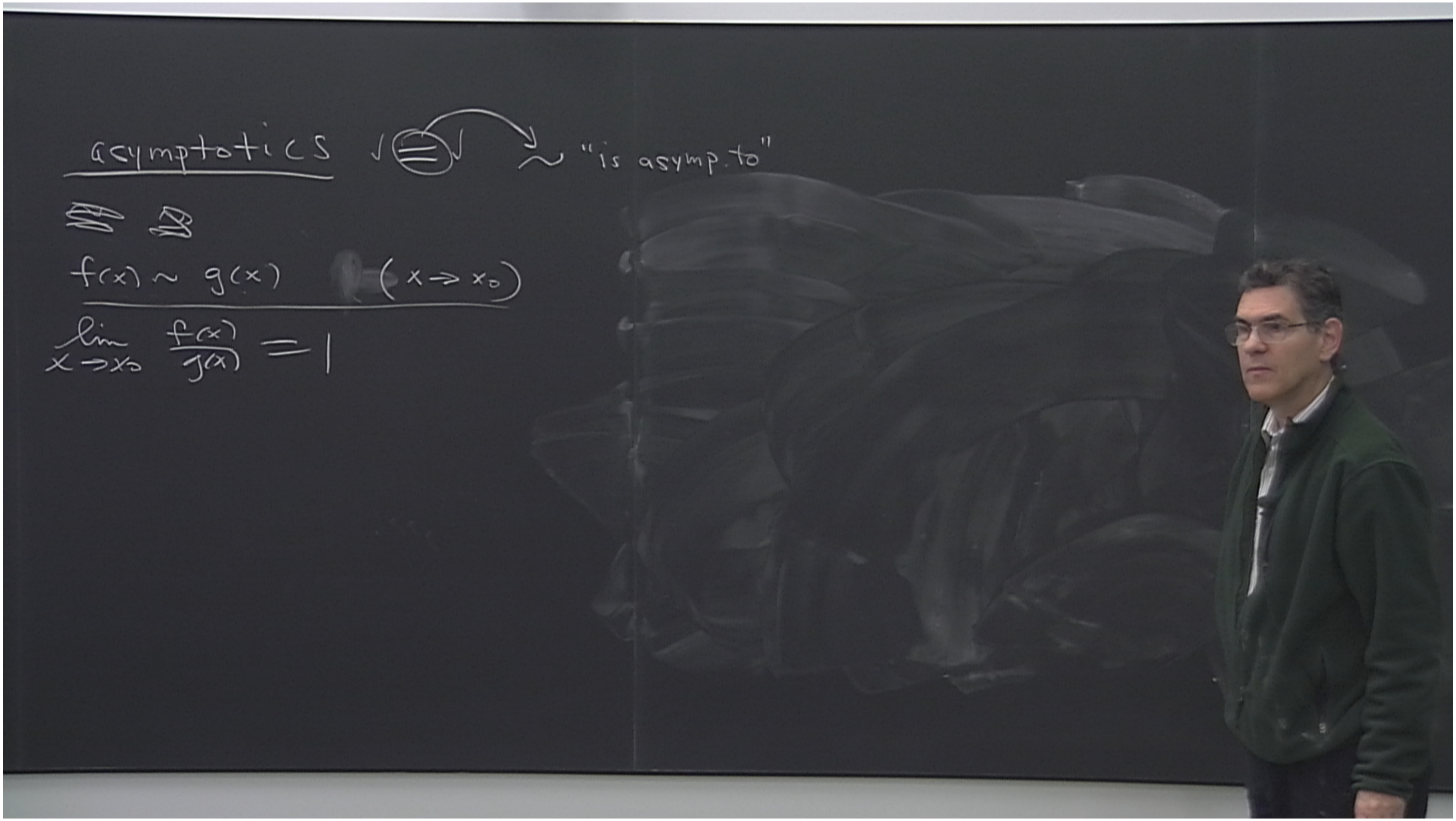
Asymptotics and perturbation theory…

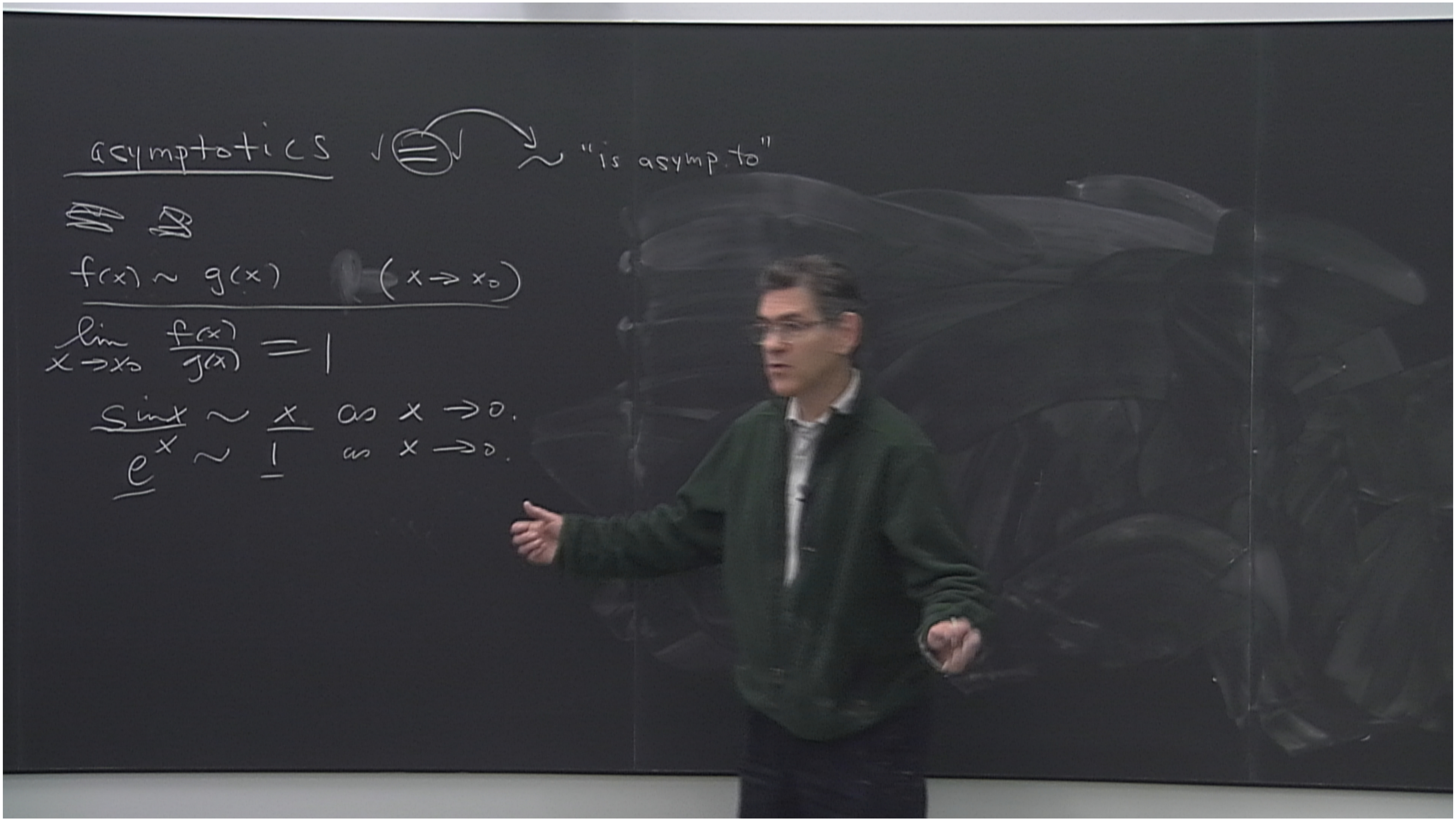
1. Boundary layer theory
2. WKB theory
3. And lots more...

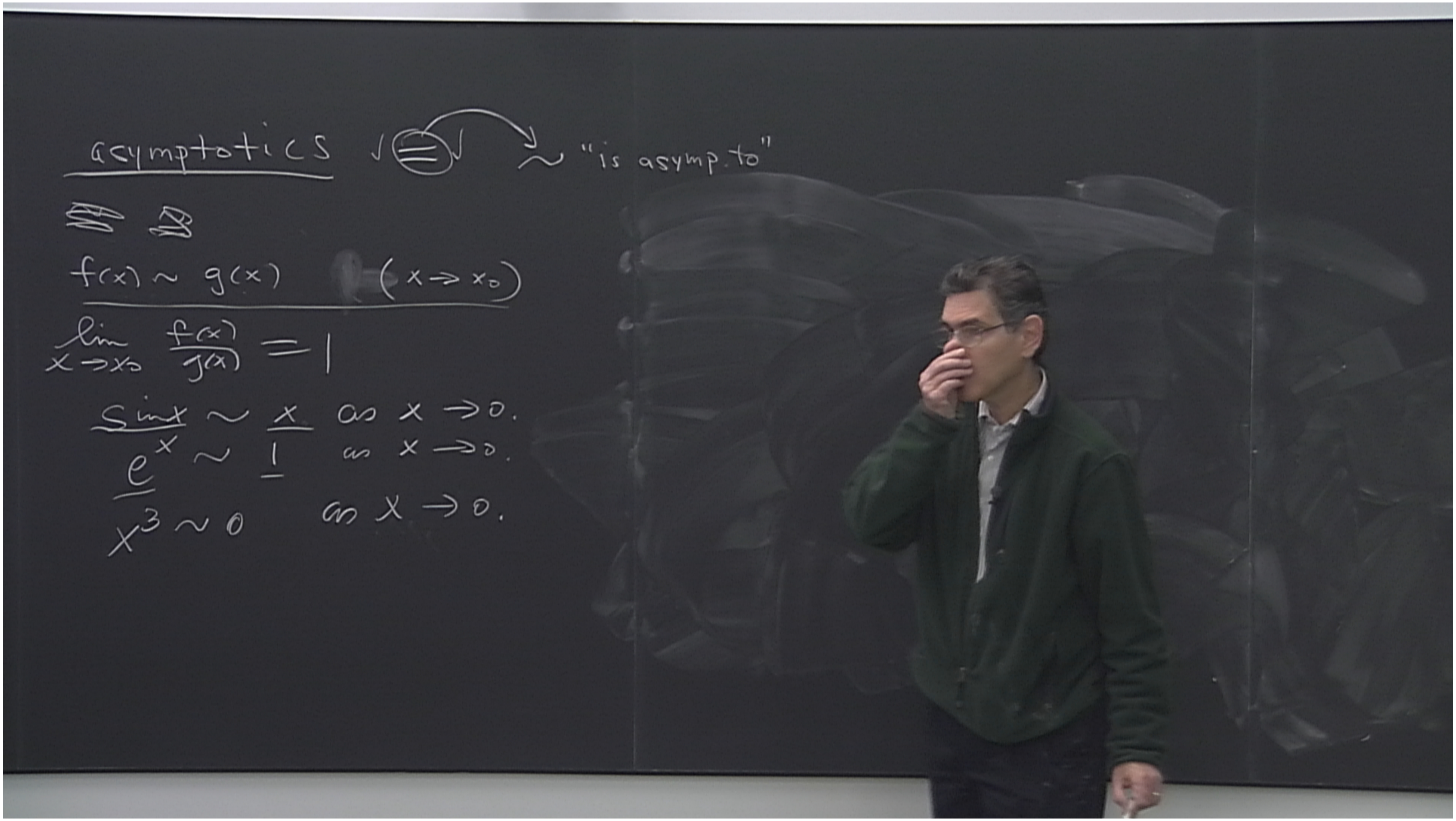

① εx⁵ + x = 1
ε=0   x=1
② x(ε) = 1 + aε + bε² + . . .
ε⁰ 1=1
ε¹ 1+a=0
a=-1
ε³ 5b+10a²+c
25+10+c=0
x(ε) = 1 − ε + 5ε² − 35ε³ . . .
③ ε=1
1
0
5
-30

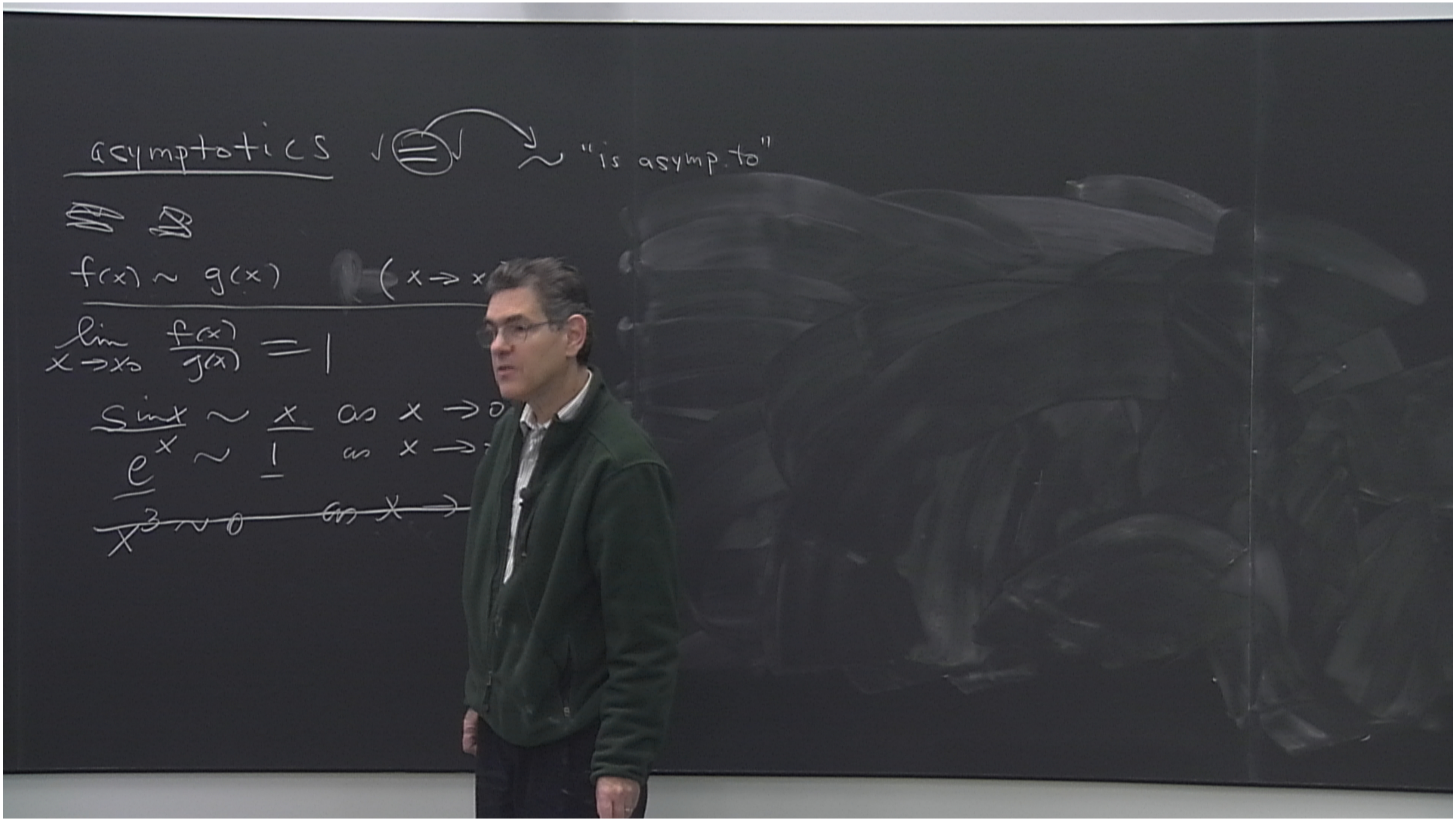

asymptotics

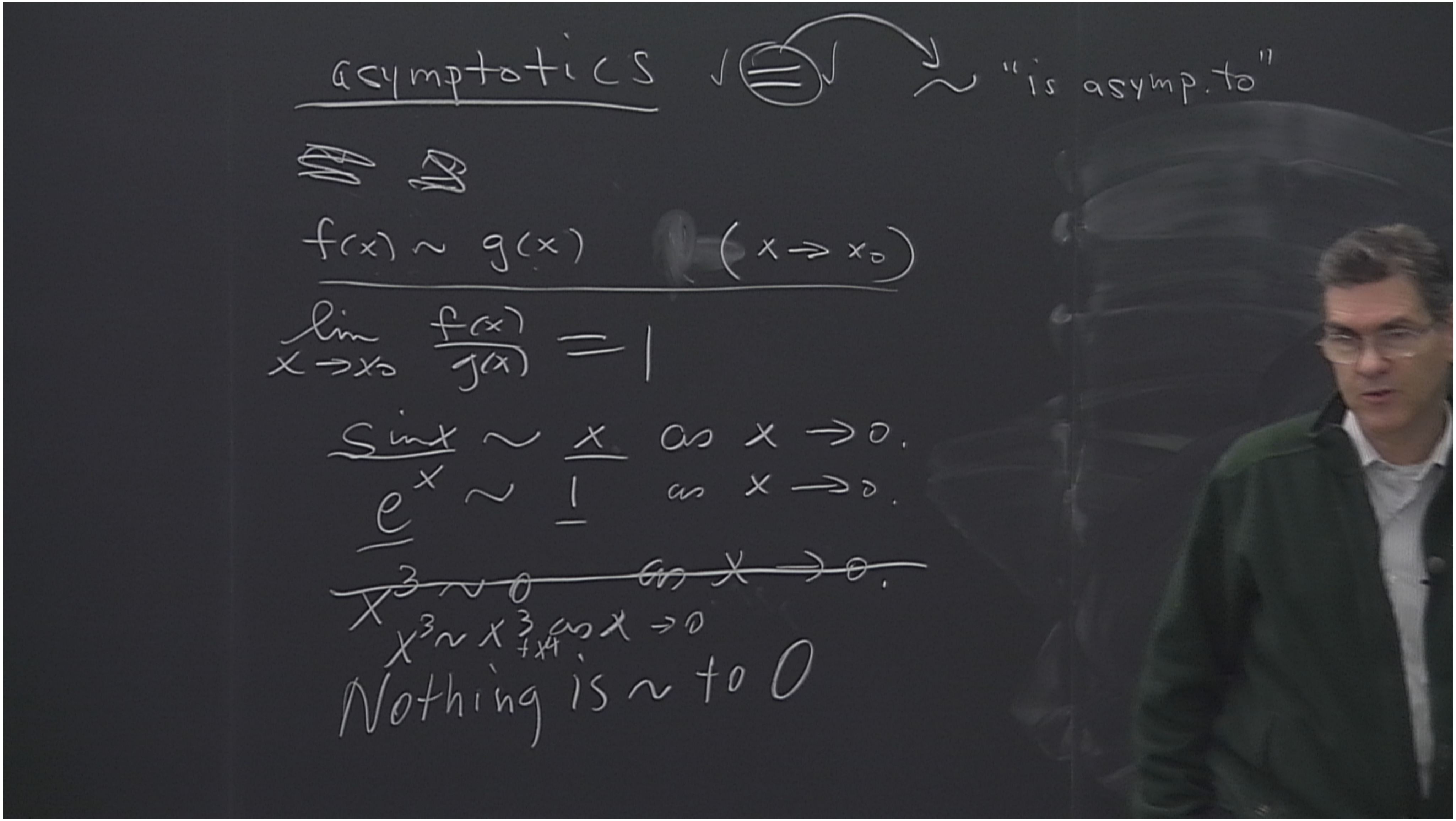

asymptotics
~ "is asymp. to"
f(x) ~ g(x)

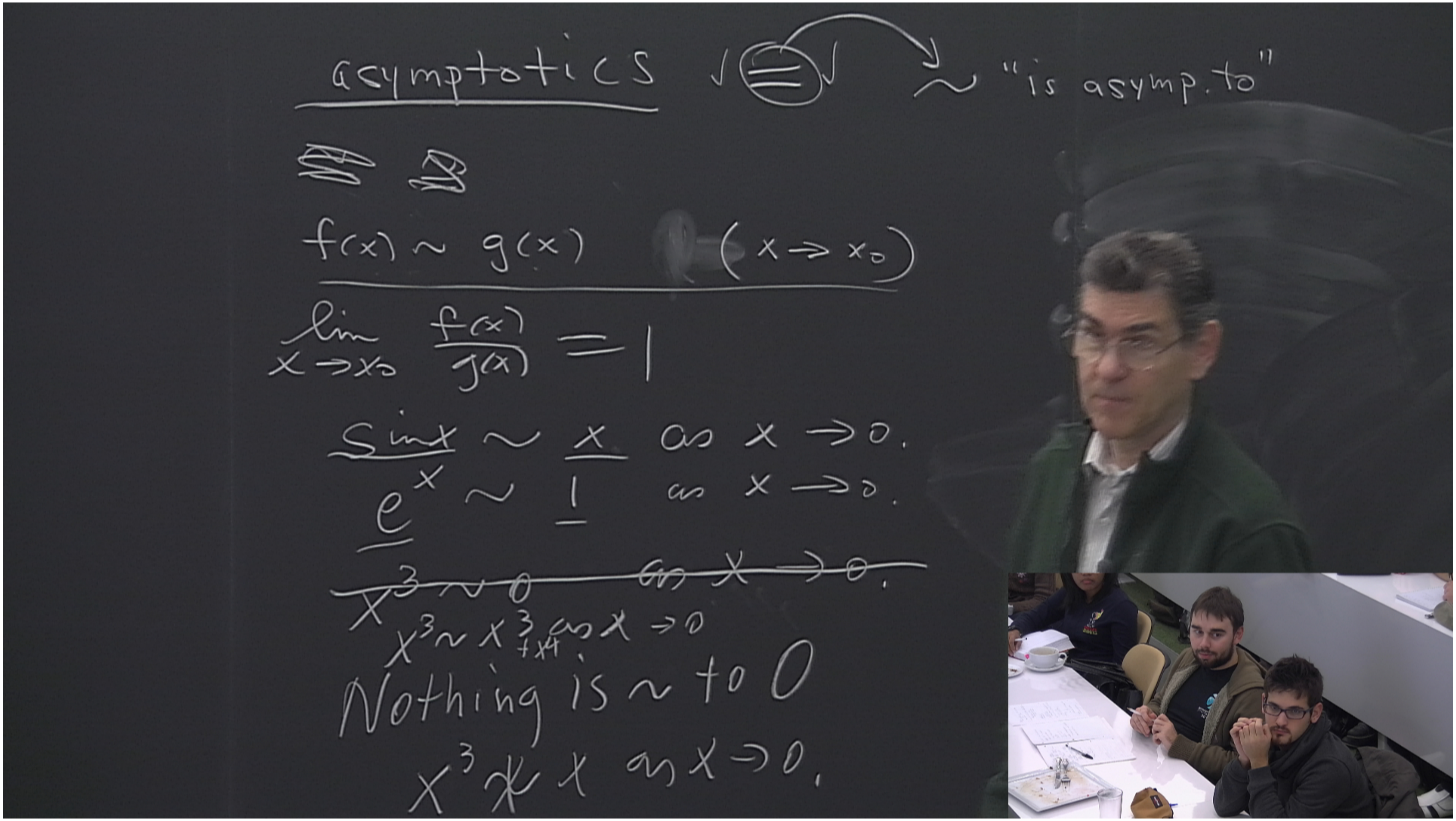

asymptotics
~ "is asymp. to"
f(x) ~ g(x) (x → x0)
lim x→x0 f(x)/g(x) = 1

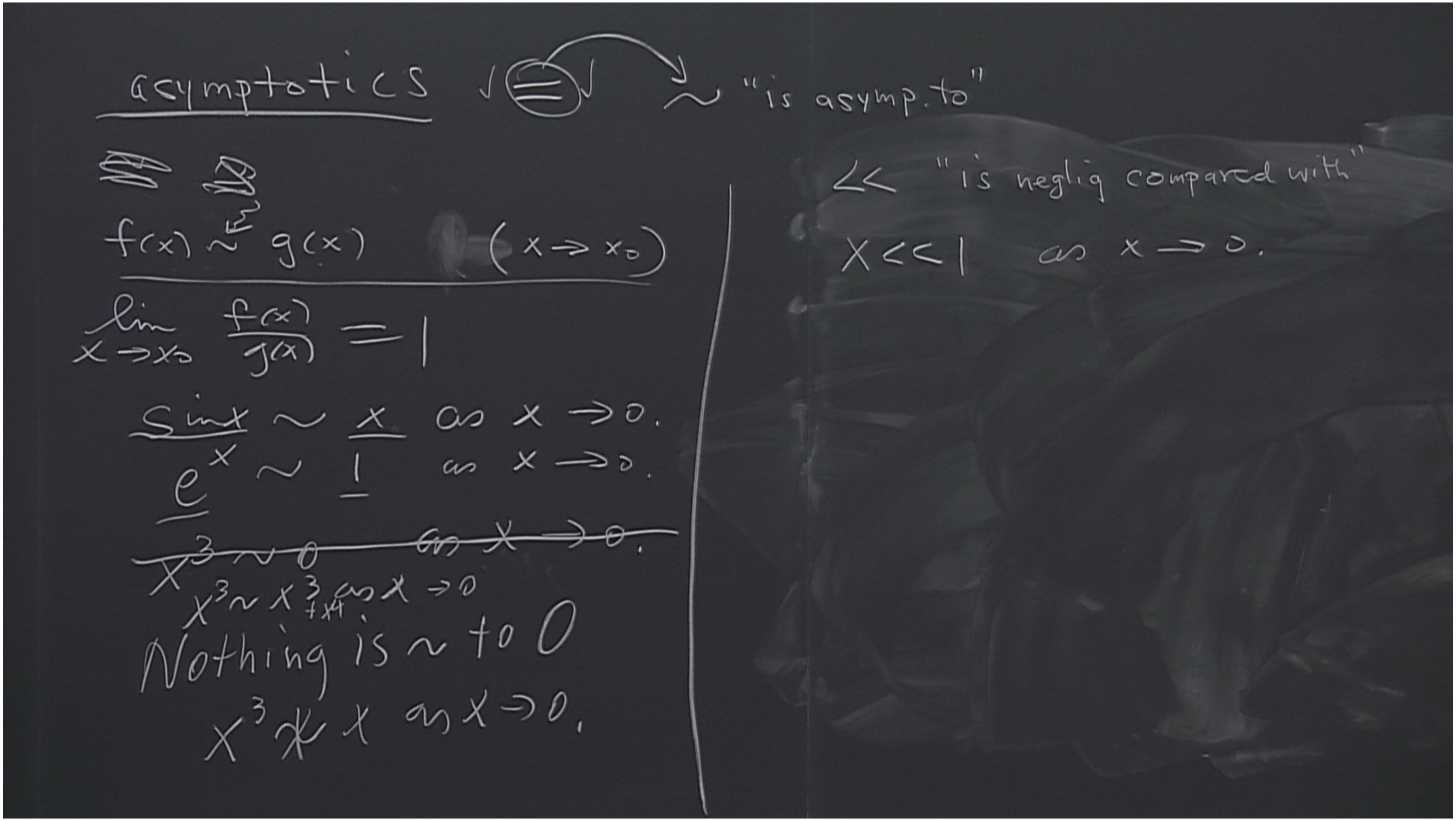

asymptotics
~ "is asymp. to"
f(x) ~ g(x) (x → x₀)
lim f(x)/g(x) = 1
sin x ~ x as x → 0.
e^x ~ 1 as x → 0.

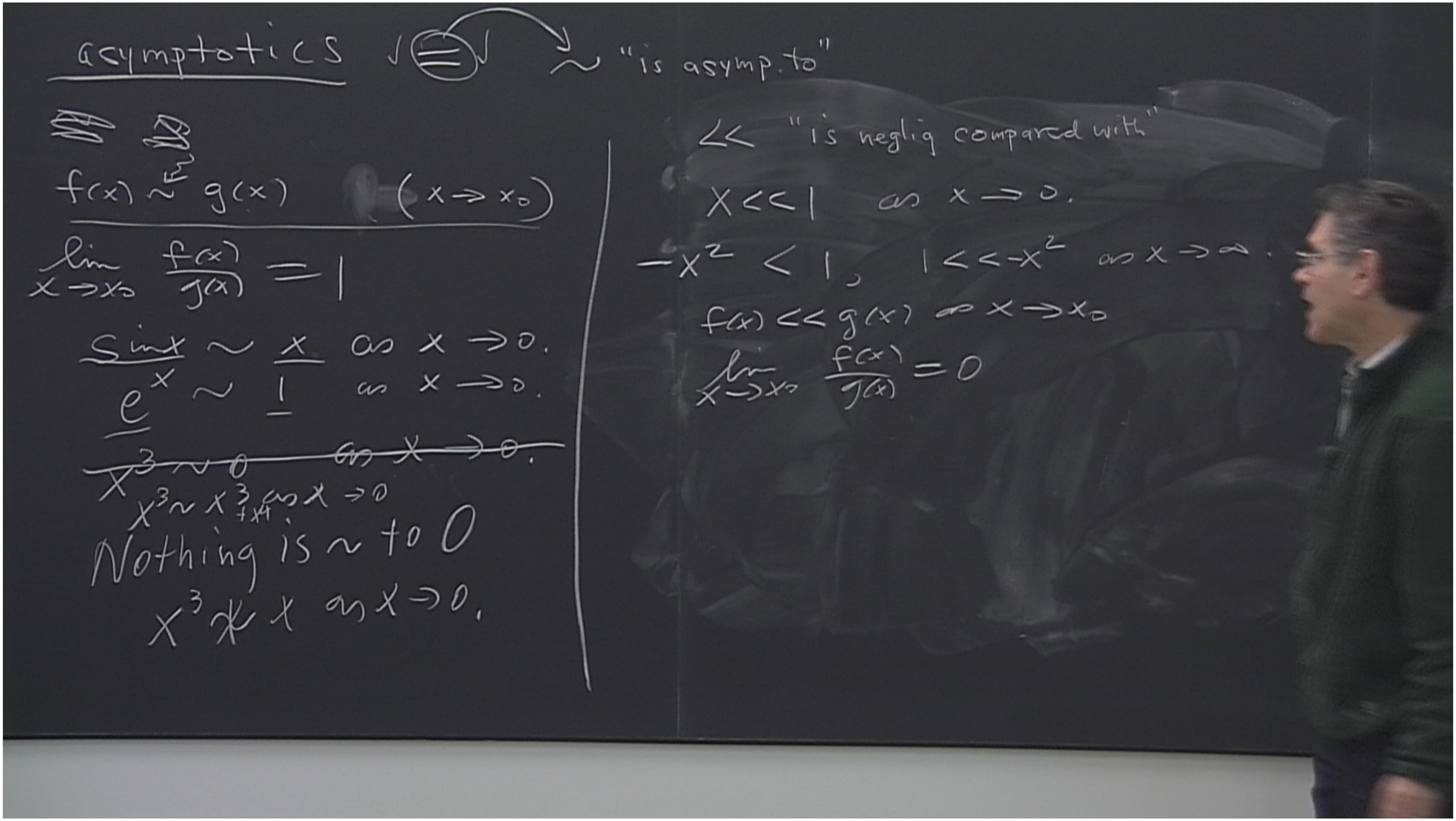

asymptotics
~ "is asymp. to"
f(x) ~ g(x) (x → x0)
lim x→x0 f(x)/g(x) = 1
sin x ~ x as x → 0.
e^x ~ 1 as x → 0.
x^3 ~ 0 as x → 0.

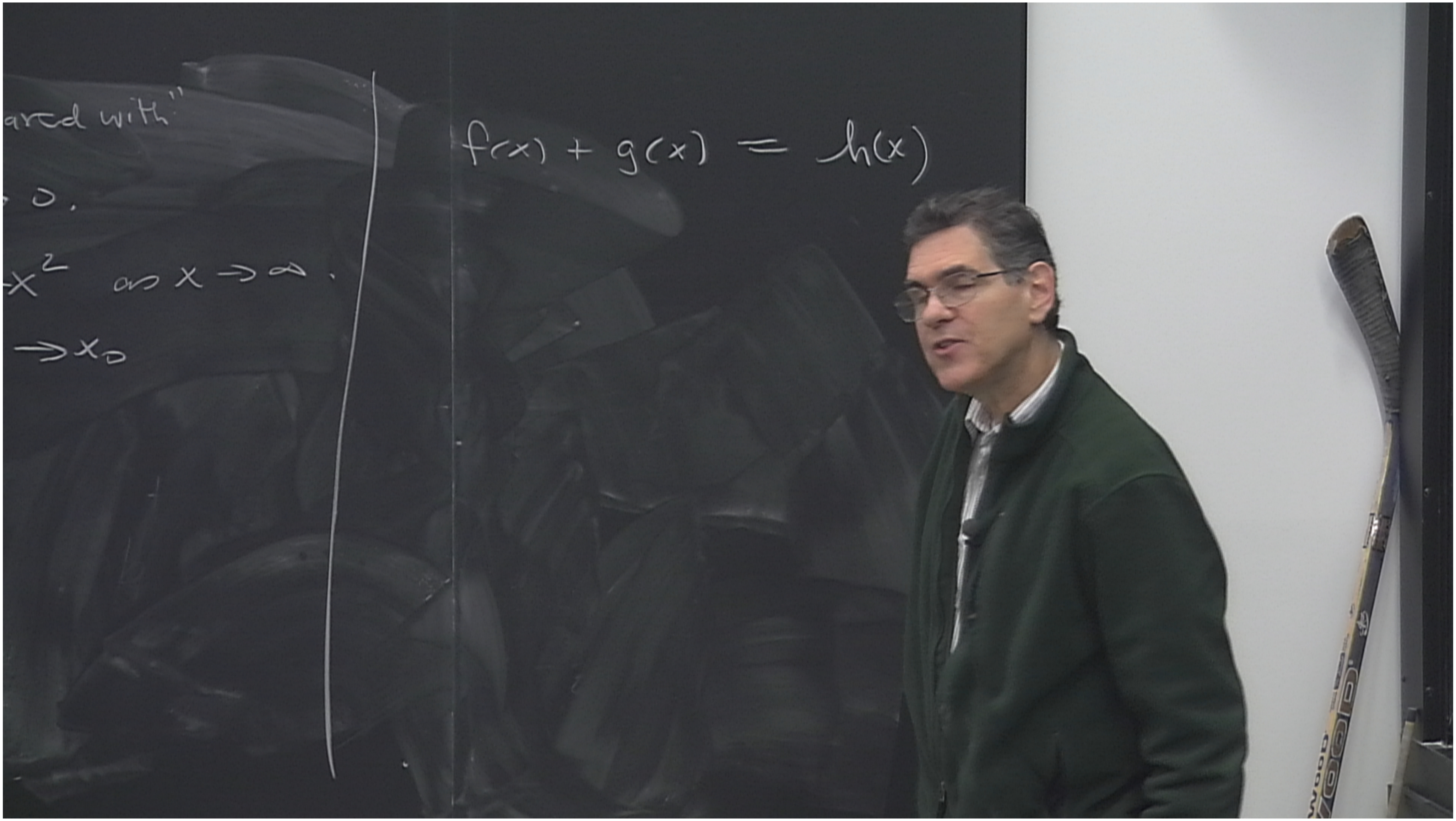

asymptotics
"is asymp. to"
f(x) ~ g(x)
sin x ~ x as x → 0

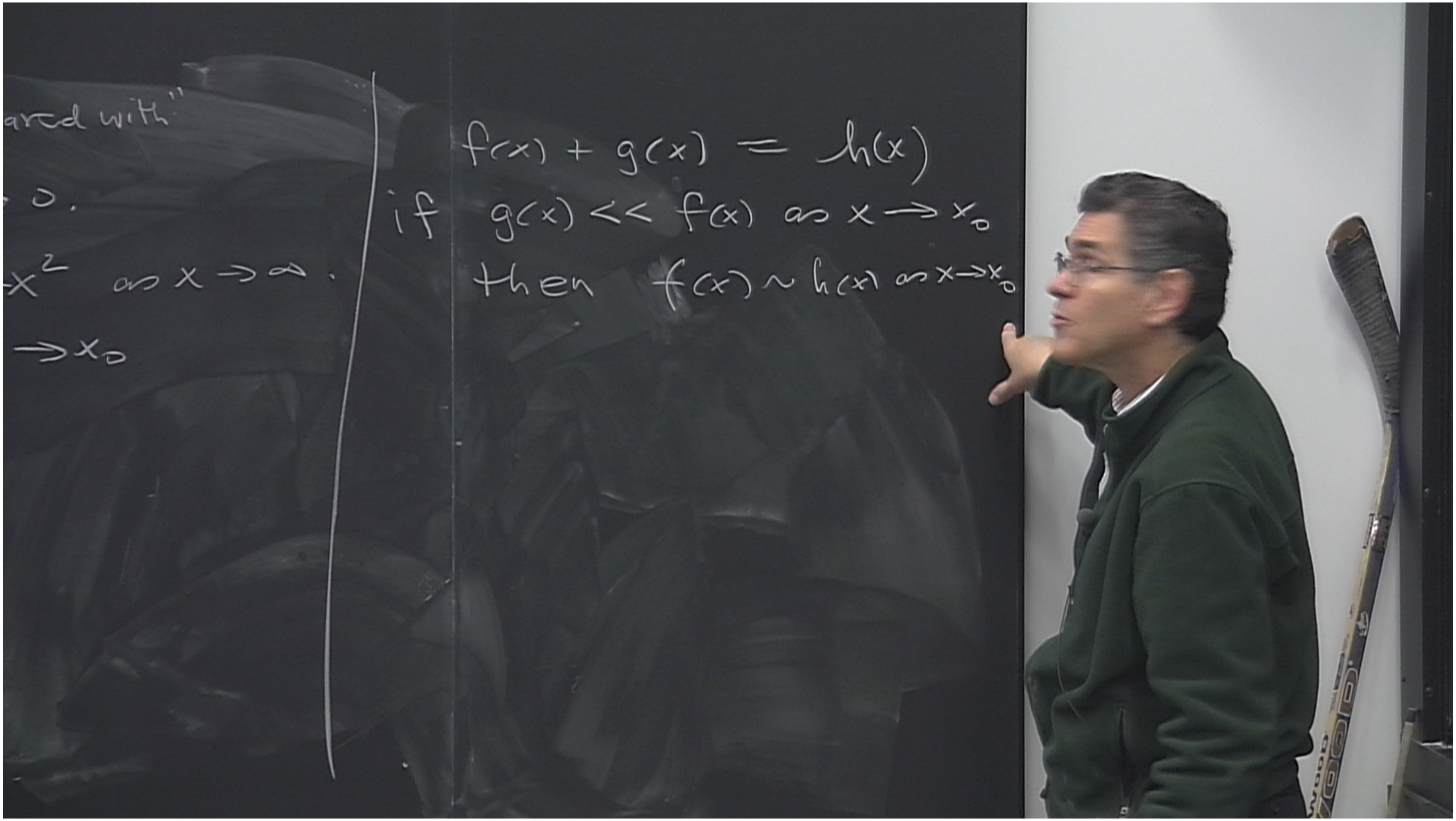

asymptotics
~ "is asymp. to"
f(x) ~ g(x) (x → x0)
lim x→x0 f(x)/g(x) = 1
Sin x ~ x as x → 0.
e^x ~ 1 as x → 0.
x^3 ~ 0 as x → 0.
x^3 ~ x^3 + x^4 as x → 0
Nothing is ~ to 0

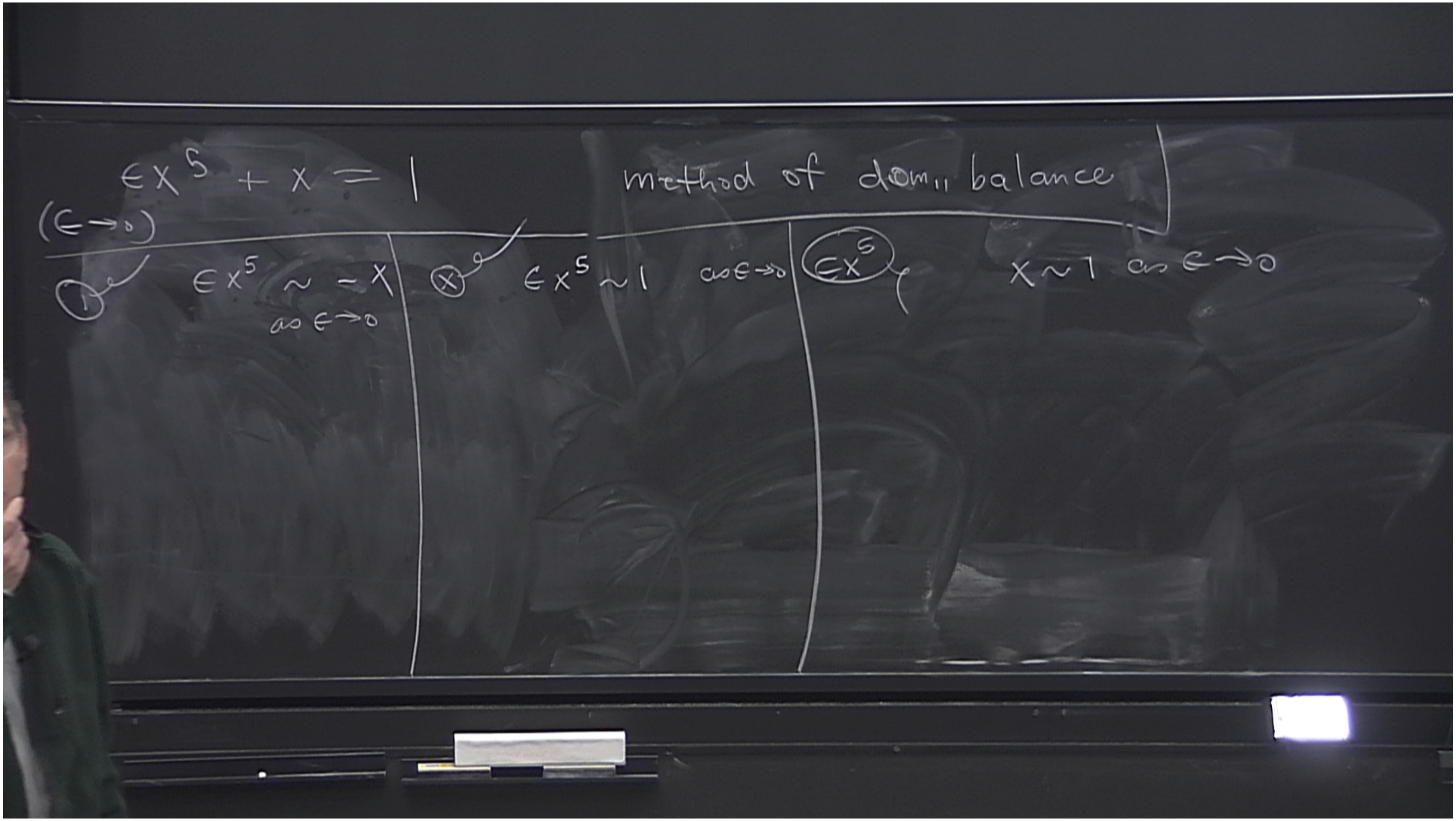

asymptotics
~ "is asymp. to"
f(x) ~ g(x) (x → x0)
lim x→x0 f(x)/g(x) = 1
sin x ~ x as x → 0.
e^x ~ 1 as x → 0.
x^3 ~ 0 as x → 0.
x^3 ~ x^3 + x^4 as x → 0
Nothing is ~ to 0
x^3 ≁ x as x → 0.

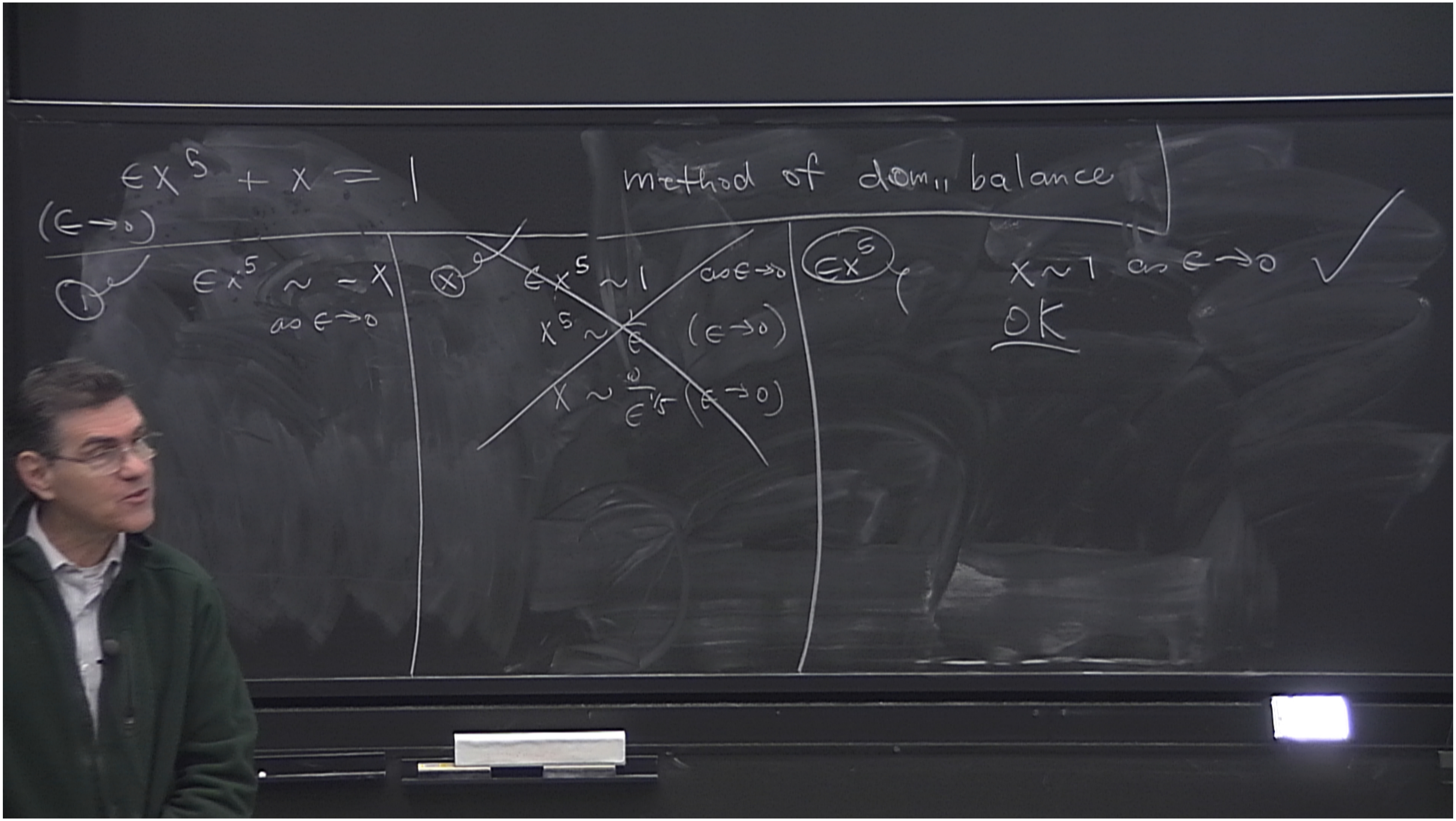

asymptotics
~ "is asymp. to"
f(x) ~ g(x) (x → x₀)
lim x→x₀ f(x)/g(x) = 1
sin x ~ x as x → 0.
e^x ~ 1 as x → 0.
x³ ~ 0 as x → 0.
x³ ~ x³ + x⁴ as x → 0
Nothing is ~ to 0
x³ ≁ x as x → 0.
<< "is neglig compared with"
x << 1 as x → 0.

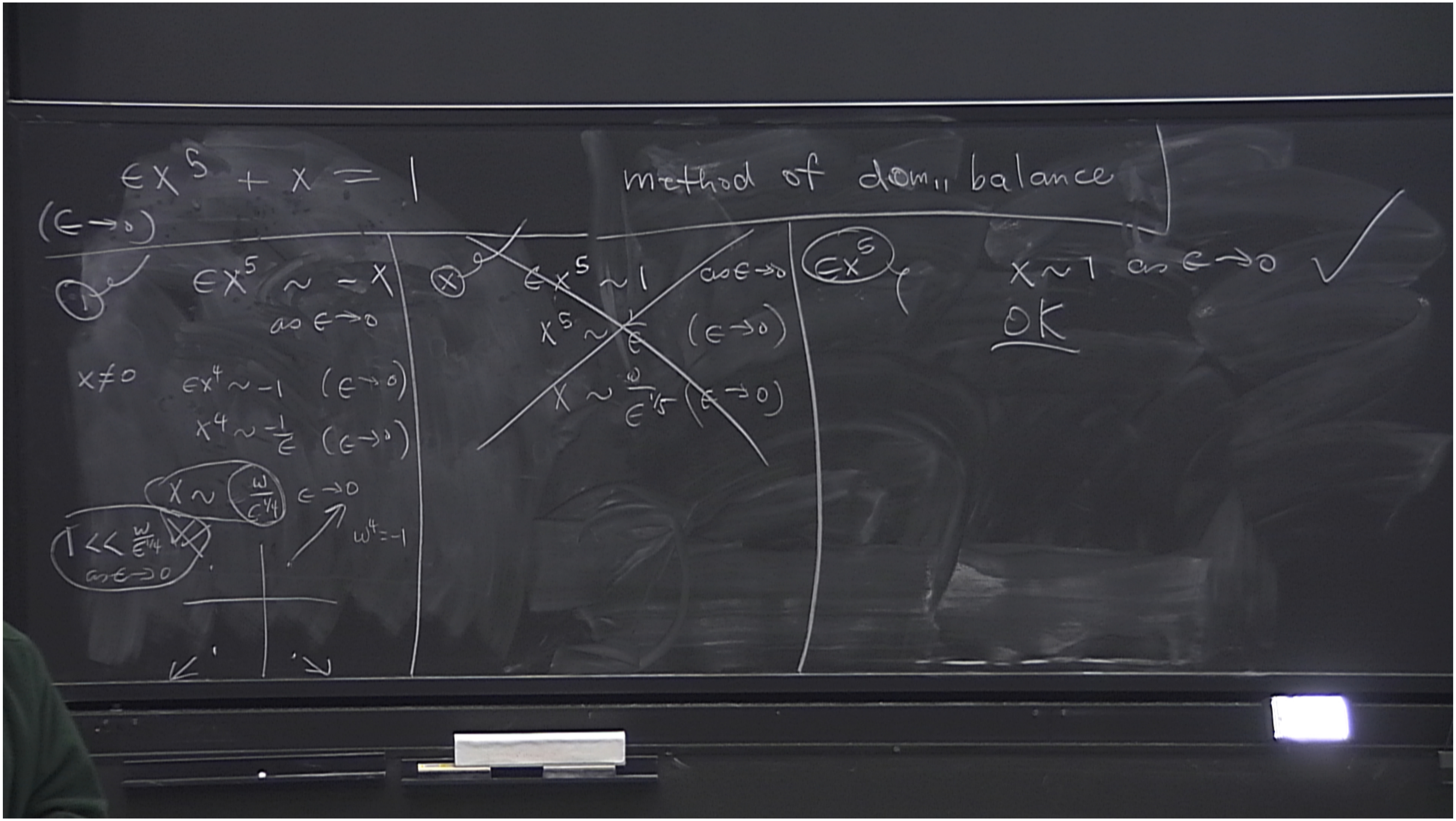

asymptotics
~ "is asymp. to"
<< "is neglig compared with"
f(x) ~ g(x) (x → x0)
lim x→x0 f(x)/g(x) = 1
Sin x ~ x as x → 0.
e^x ~ 1 as x → 0.
x^3 ~ x^3 + x^4 as x → 0
Nothing is ~ to 0
x^3 ≁ x as x → 0.
x << 1 as x → 0.
x^2 < 1, 1 << x^2 as x → ∞.
f(x) << g(x) as x → x0
lim x→x0 f(x)/g(x) = 0


ared with"
f(x) + g(x) = h(x)
x^2 as x → ∞.
→ x_0


f(x) + g(x) = h(x)
if g(x) << f(x) as x → x0
then f(x) ~ h(x) as x → x0
x2 as x → ∞.
→ x0


εx5 + x = 1
(ε→0)
method of dom. balance
εx5 ~ -x as ε→0
εx5 ~ 1 as ε→0
x ~ 1 as ε→0


εx5 + x = 1
(ε→0)
method of dom. balance
εx5 ~ -x
as ε→0
εx5 ~ 1 as ε→0
x5 ~ 1/ε (ε→0)
x ~ ω/ε1/5 (ε→0)
εx5
x ~ 1 as ε→0
OK



εx^5 + x = 1
(ε→0)
method of dom. balance
εx^5 ~ −x
as ε→0
x≠0
εx^4 ~ −1 (ε→0)
x^4 ~ −1/ε (ε→0)
x ~ ω/ε^{1/4} ε→0
1 << ω/ε^{1/4}
as ε→0
ω^4 = −1
εx^5 ~ 1 as ε→0
x^5 ~ 1/ε (ε→0)
x ~ ω/ε^{1/5} (ε→0)
εx^5
x ~ 1 as ε→0
OK