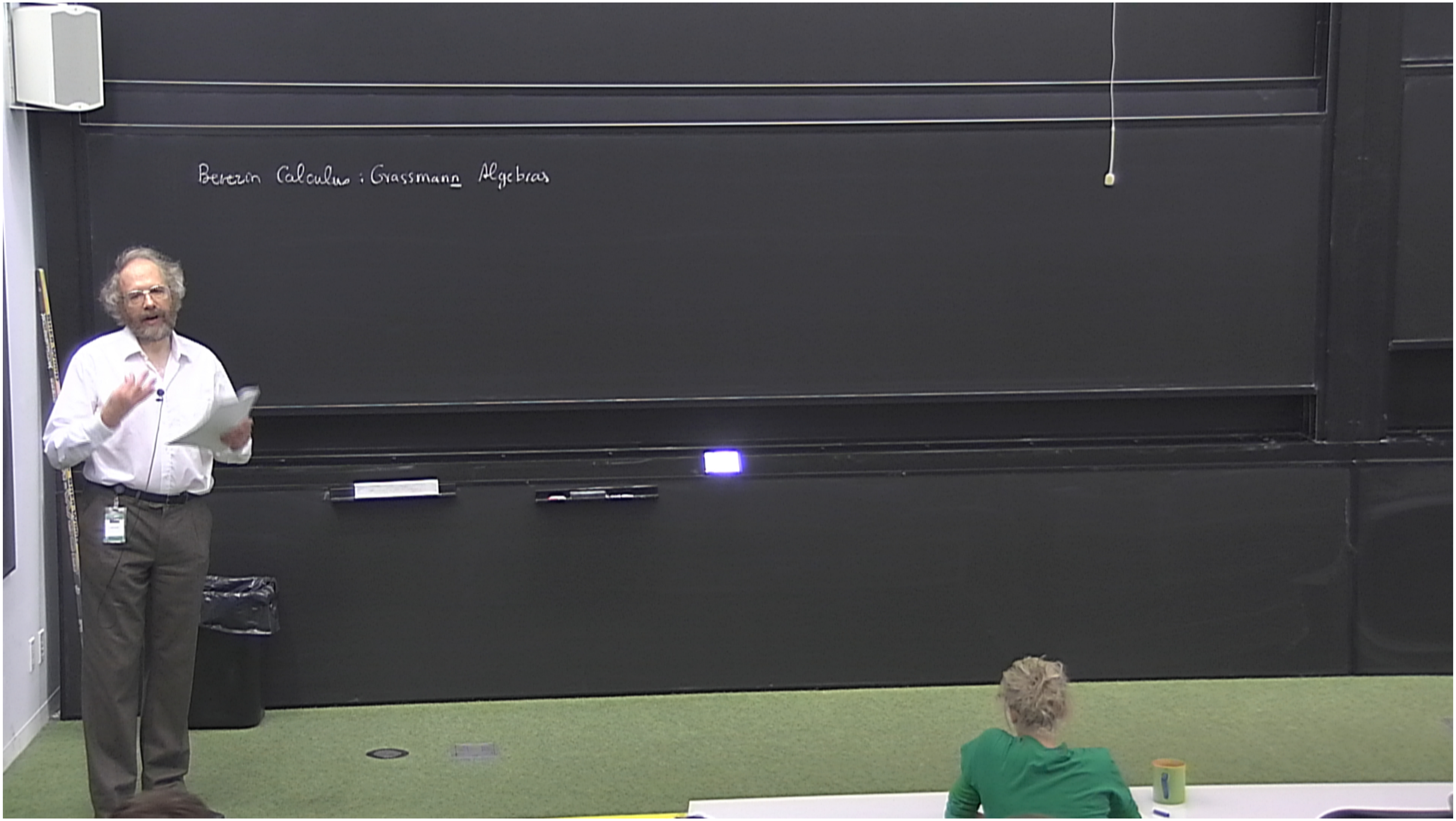
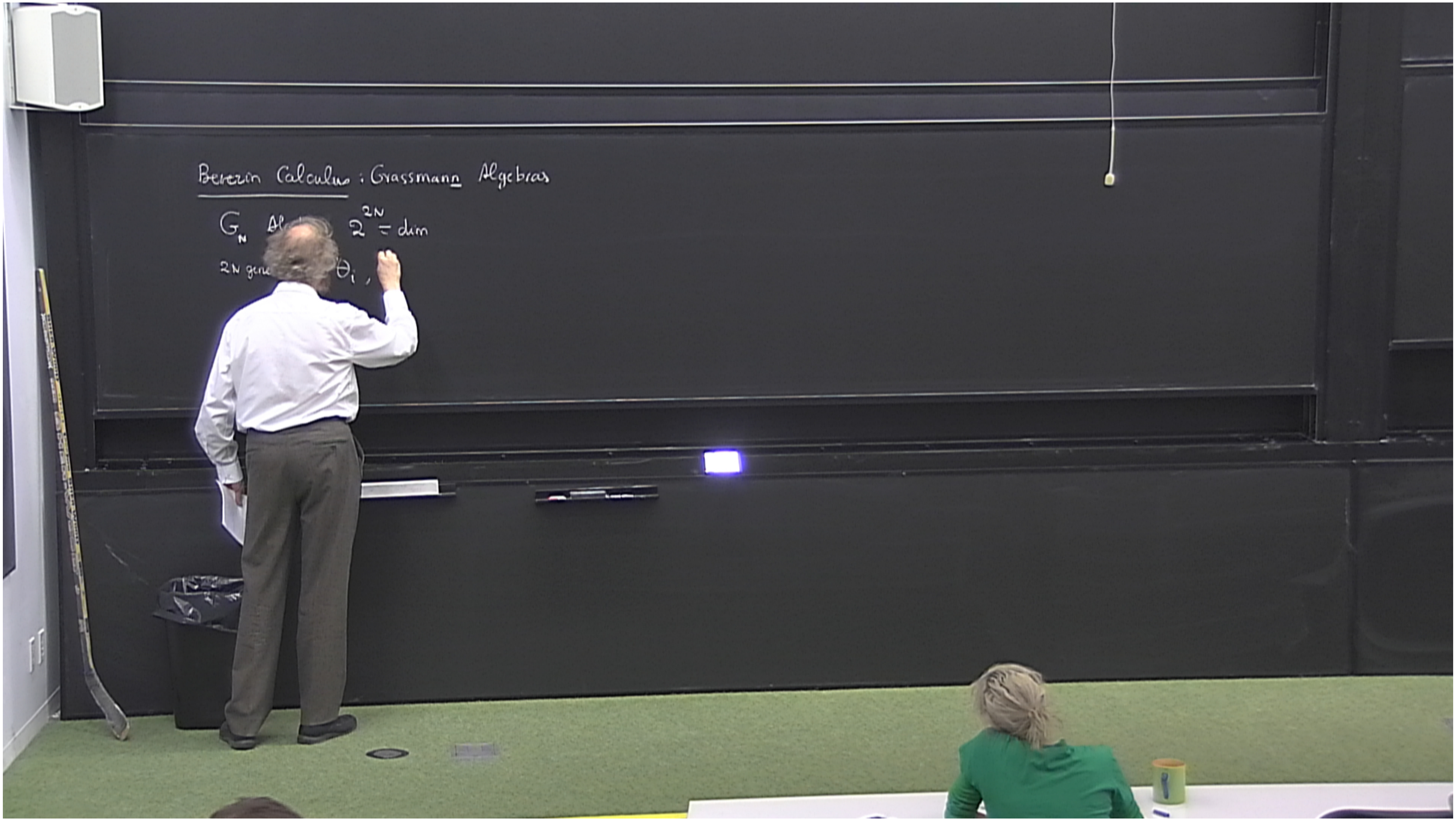
Title: Quantum Field Theory II - Lecture 11

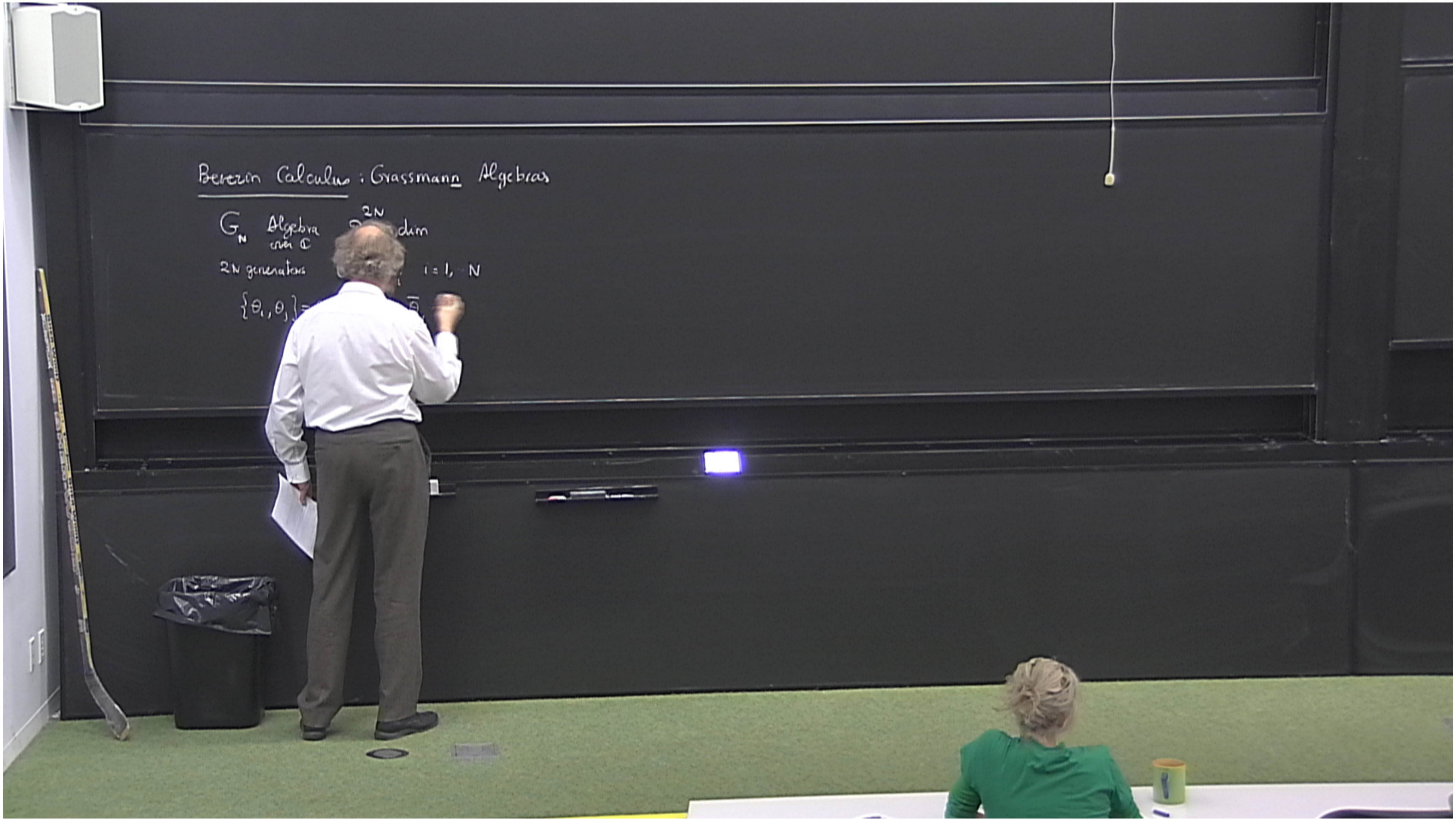
Date: Nov 14, 2011 09:00 AM

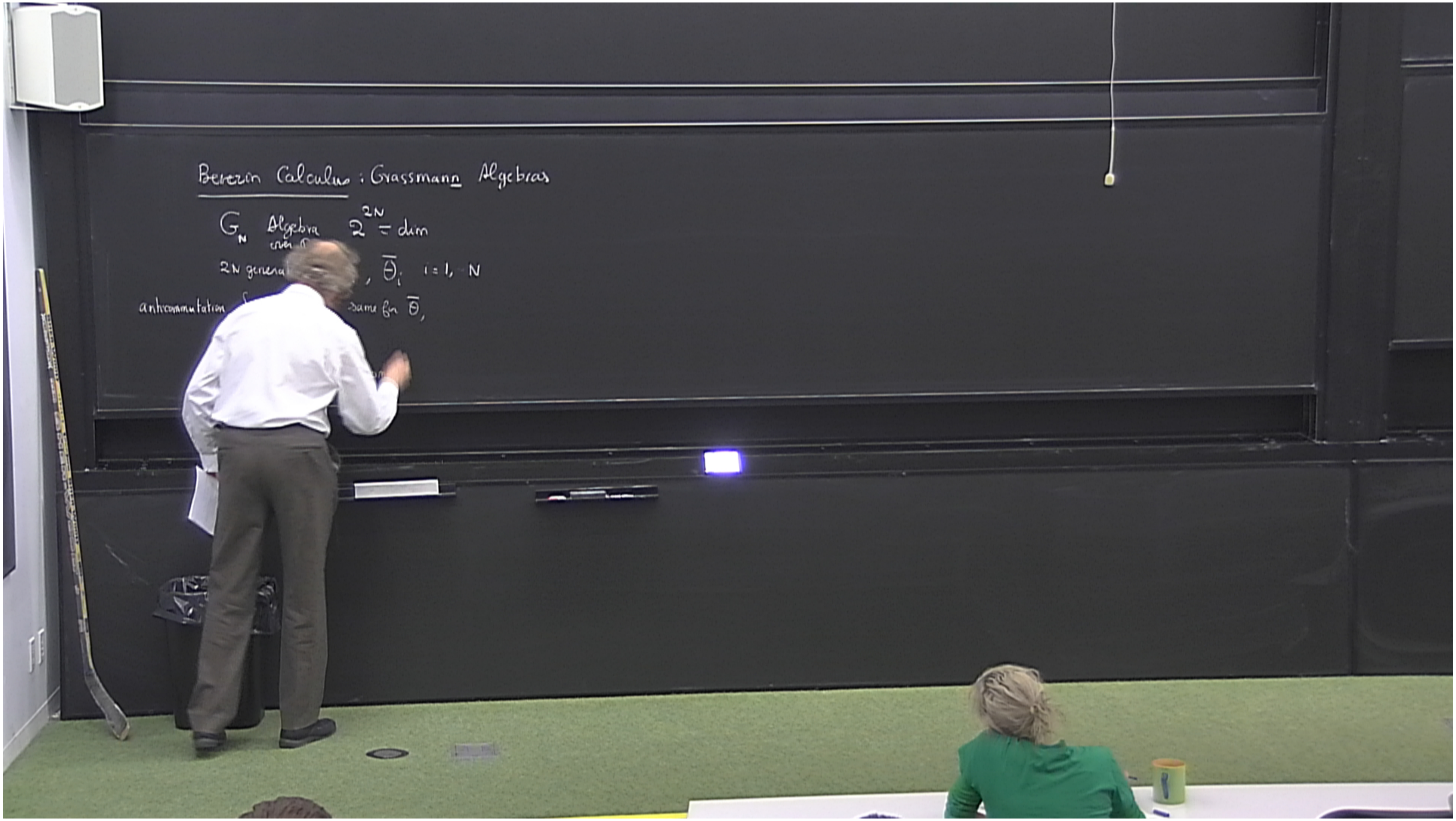
URL: http://pirsa.org/11110017

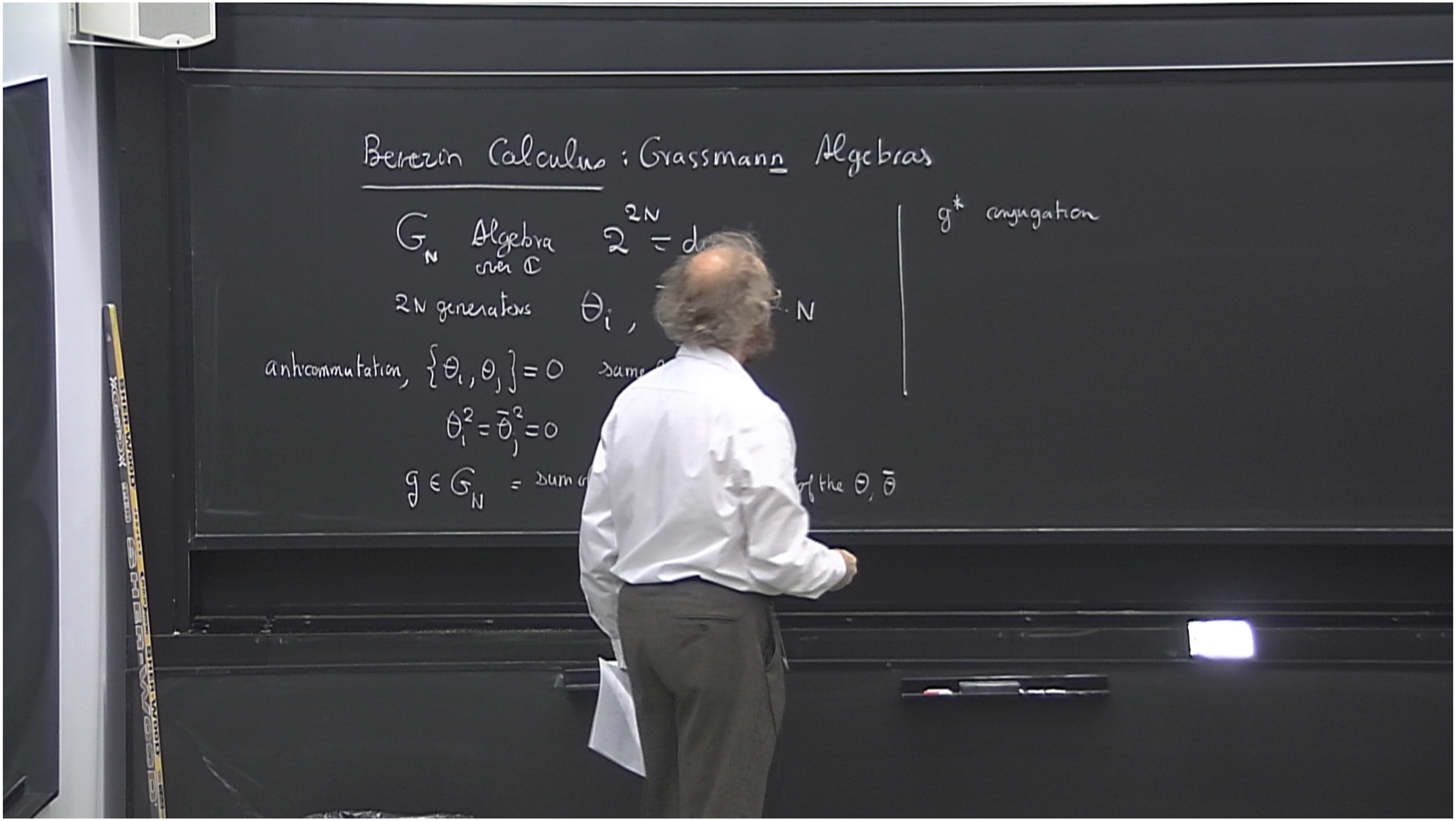
Abstract:

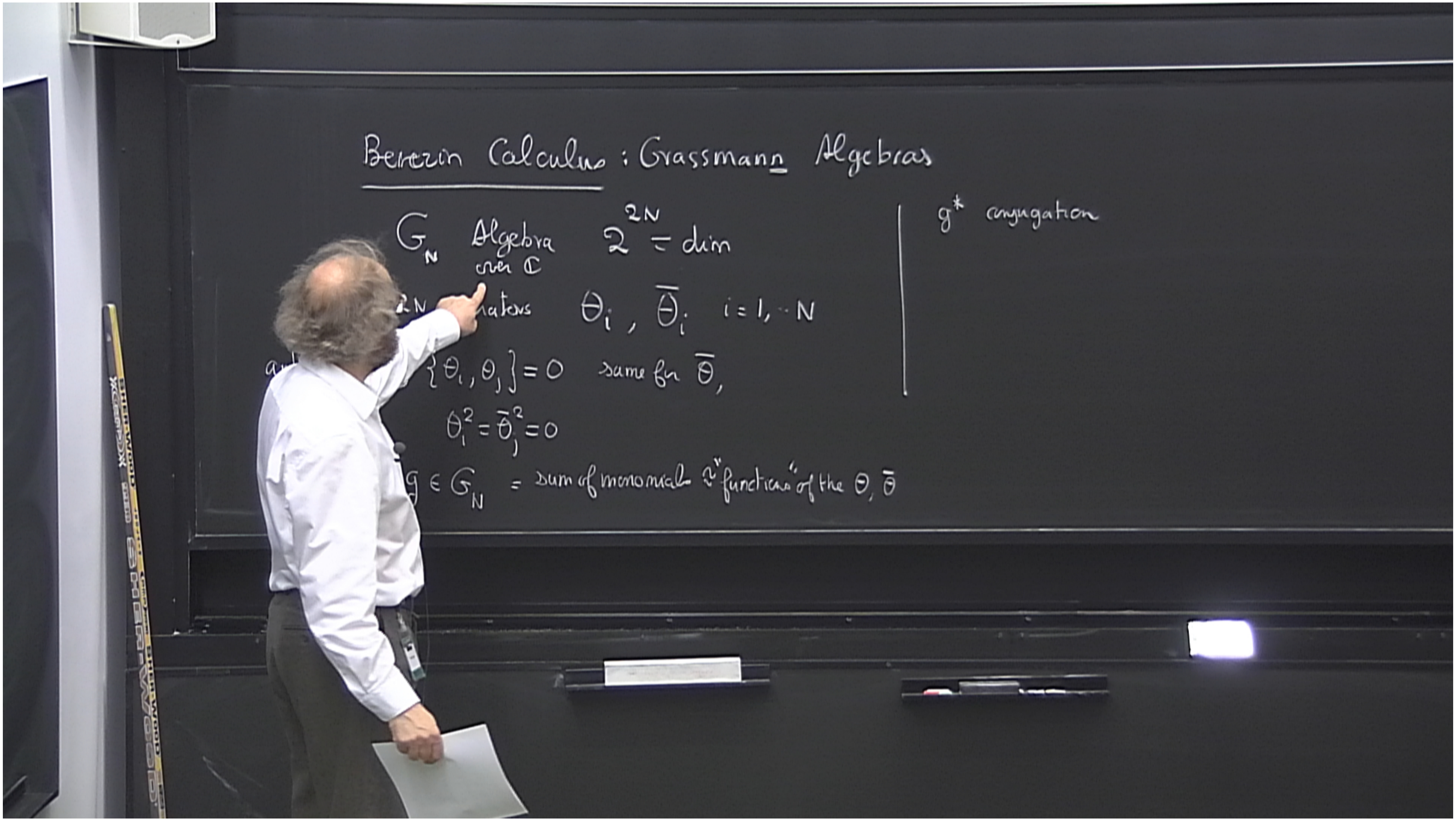

Berezin Calculus : Grassmann Algebras

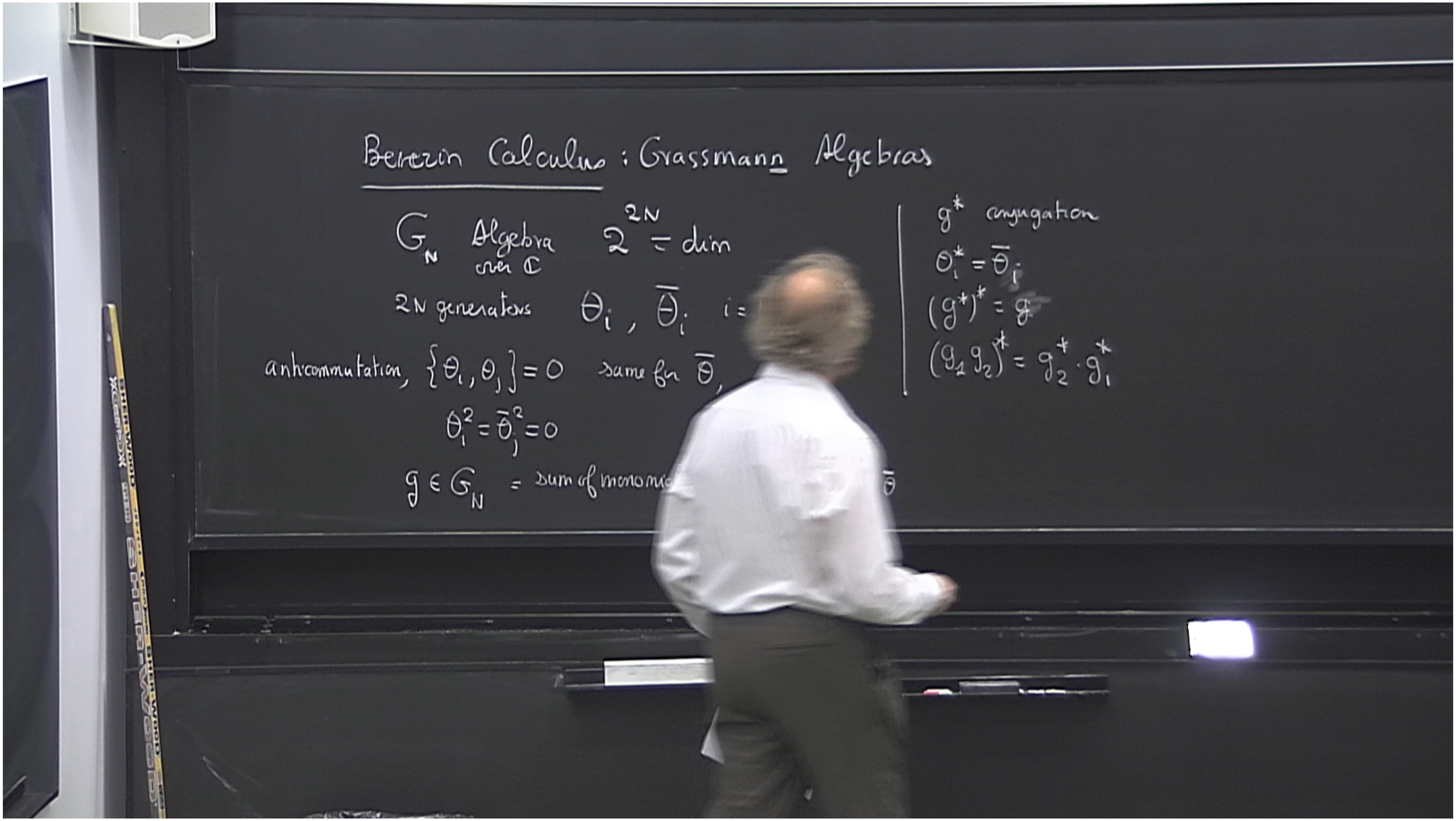

Berezin Calculus : Grassmann Algebras
2^2N = dim

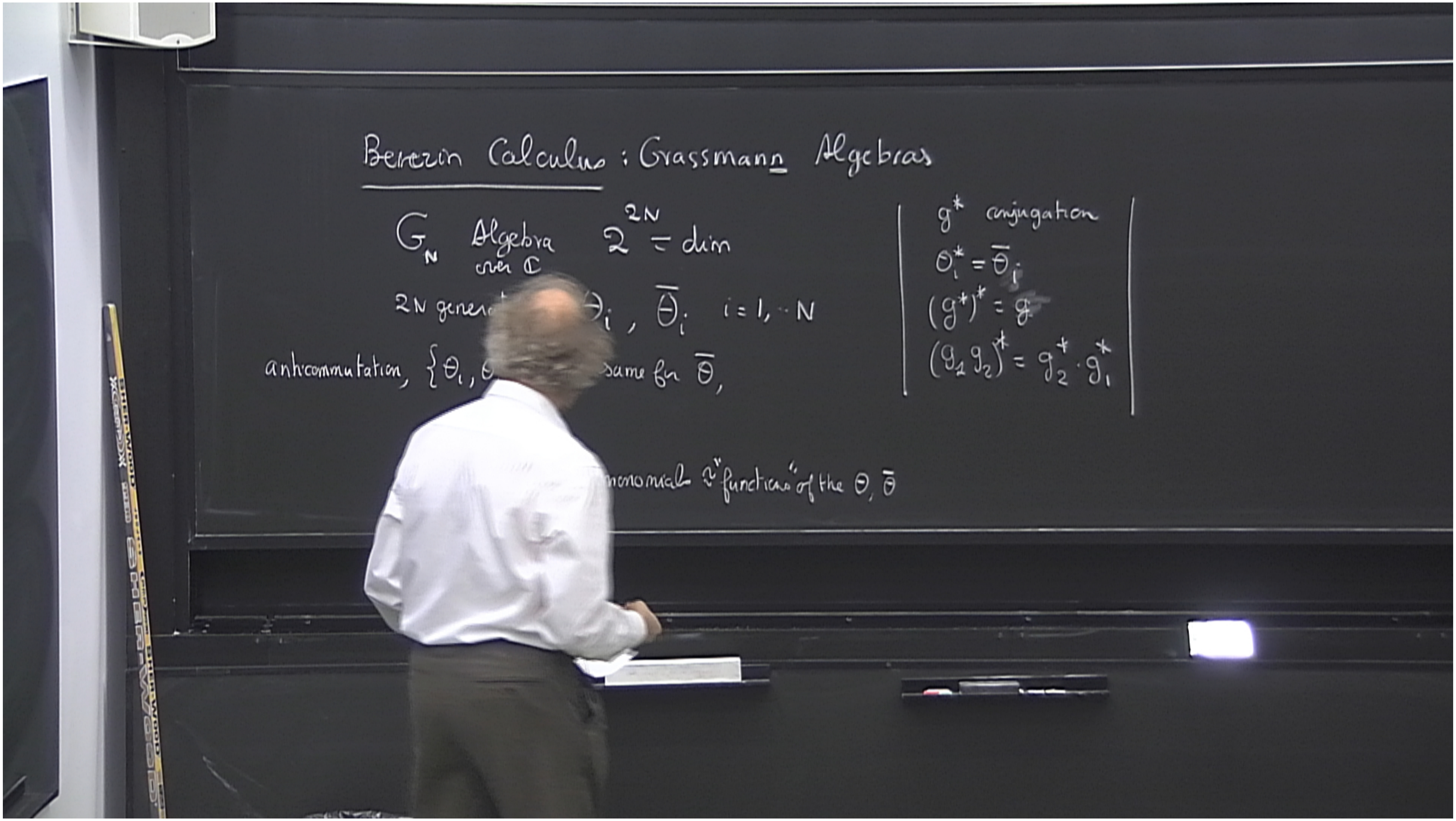

Berezin Calculus : Grassmann Algebras
G_N Algebra over ℂ
2N dim
2N generators
i = 1, N

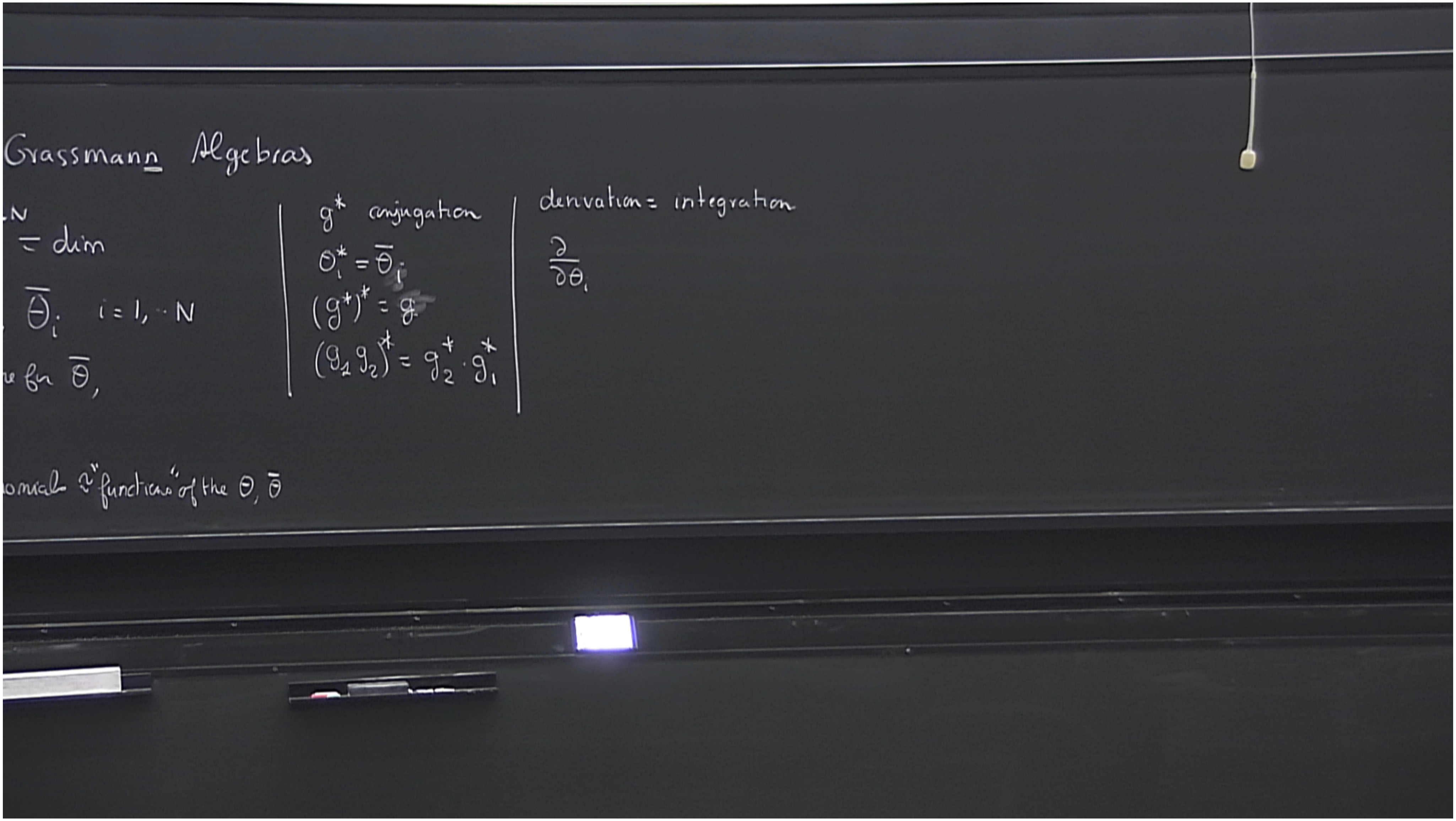

Berezin Calculus : Grassmann Algebras
G N Algebra 2 2N = dim
i = 1, .. N
antIcommutation
same for

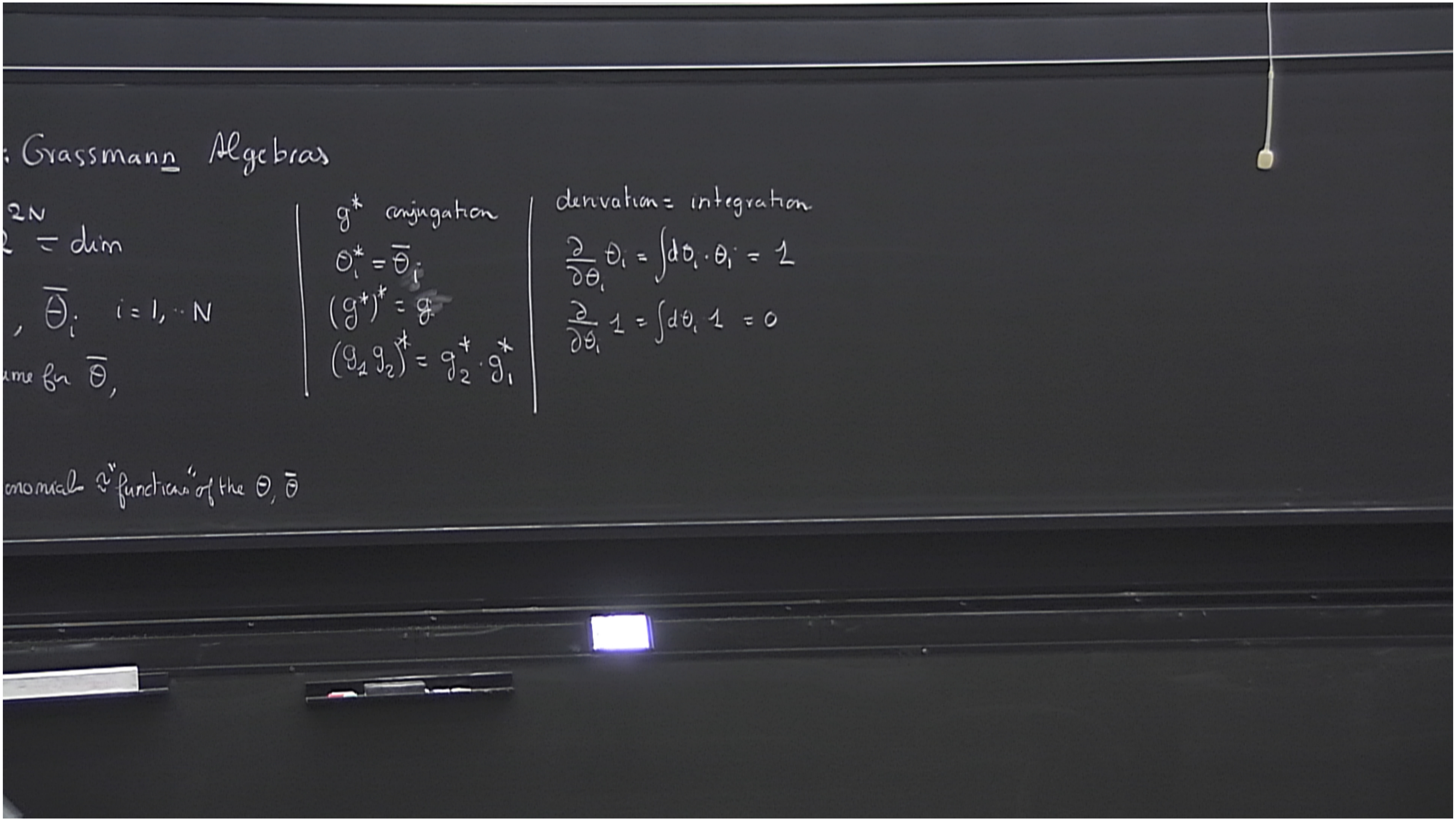

Berezin Calculus : Grassmann Algebras
G_N Algebra over ℂ
2^{2N} = d
2N generators θ_i ,
N
anticommutation, {θ_i , θ_j} = 0 same
θ_i^2 = θ̄_j^2 = 0
g ∈ G_N = sum
of the θ, θ̄
g* conjugation

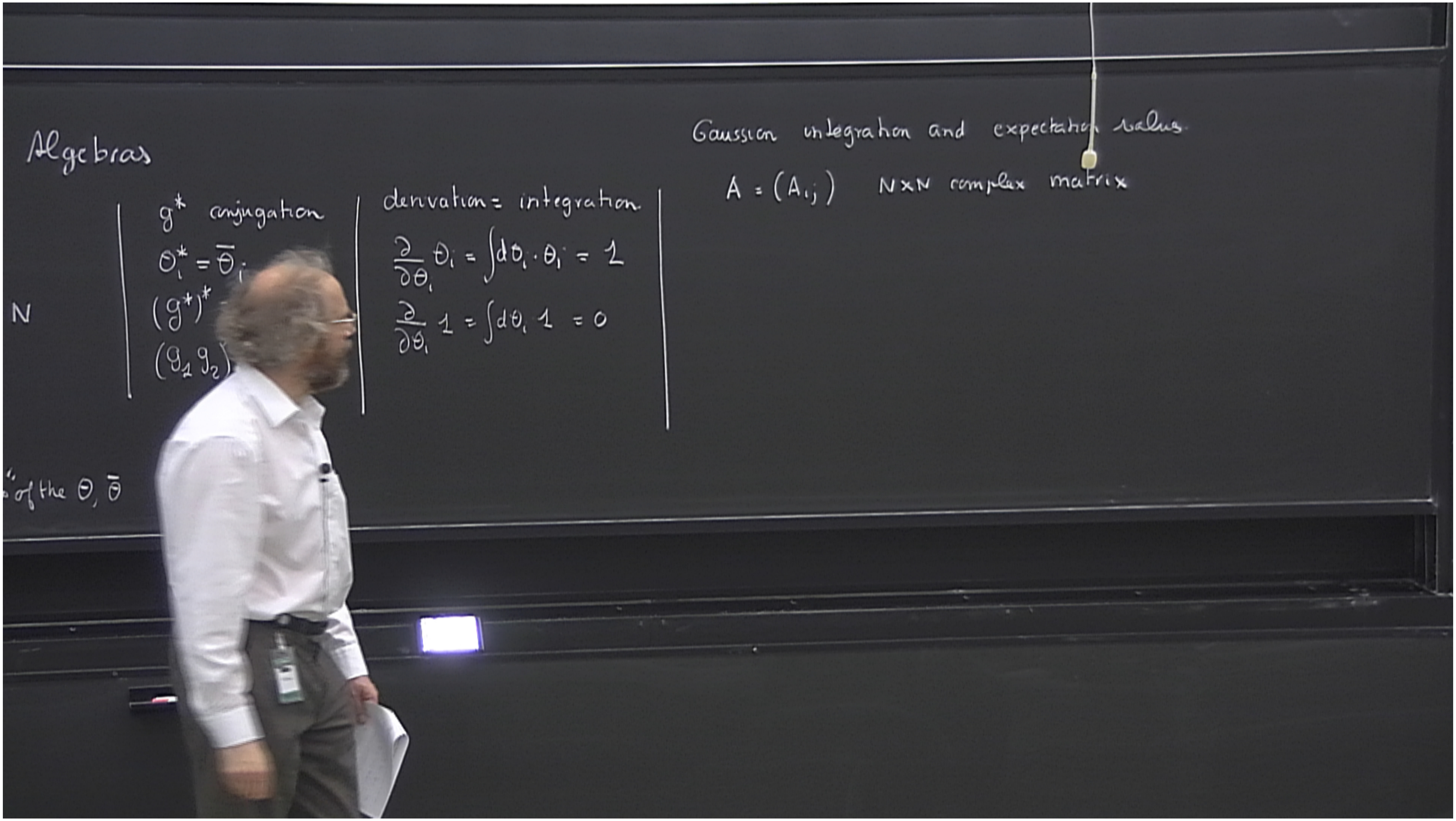

Berezin Calculus : Grassmann Algebras
G_N Algebra over ℂ 2^2N = dim
2N ...nators θ_i , θ̄_i i = 1, .. N
{θ_i , θ_j} = 0 same for θ̄,
θ_i^2 = θ̄_j^2 = 0
g ∈ G_N = sum of monomials "functions" of the θ, θ̄
g* conjugation

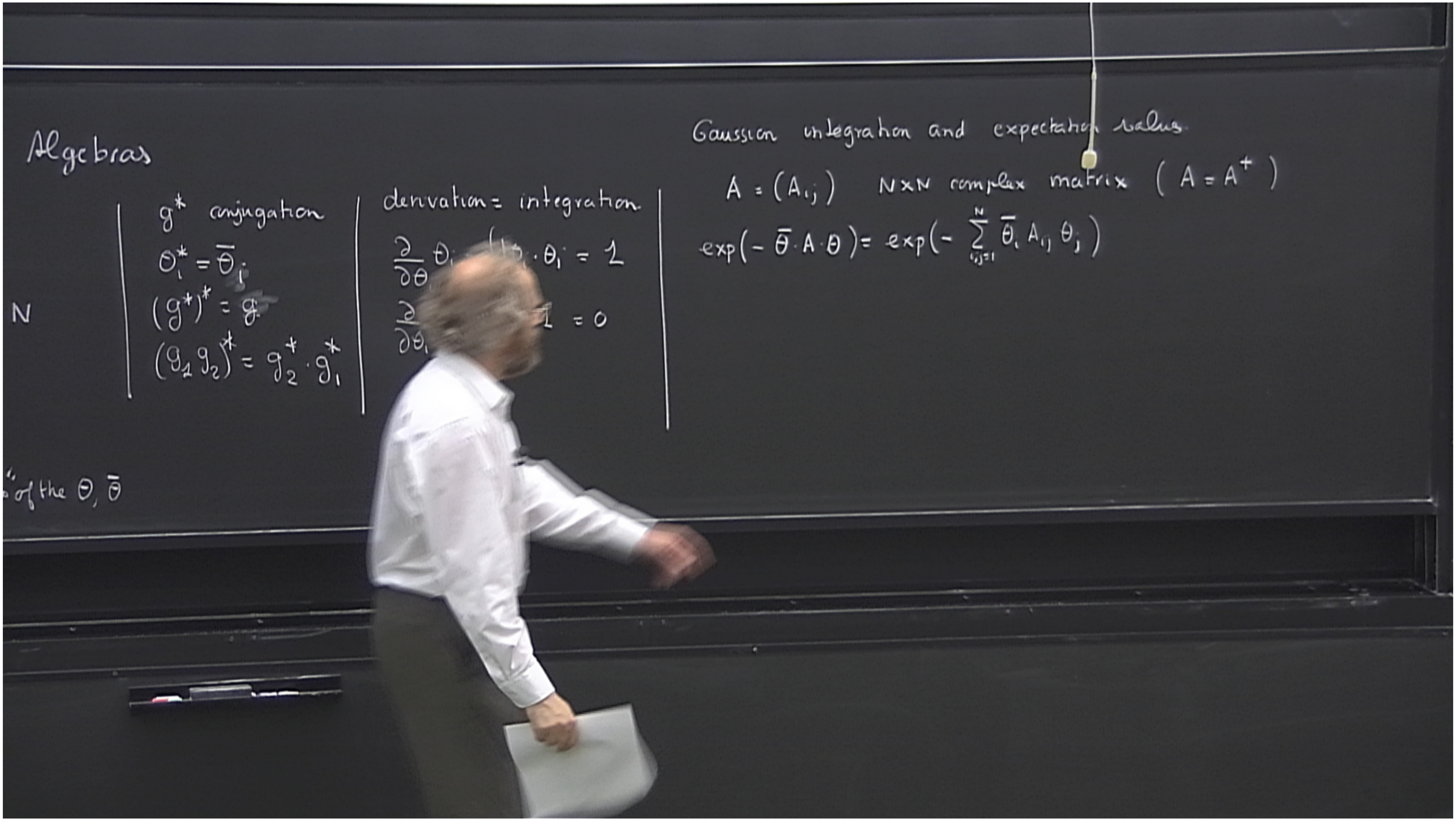

Berezin Calculus : Grassmann Algebras
G_N Algebra over $\mathbb{C}$
$2^{2N} = \dim$
2N generators θ_i , $\bar{\theta}_i$
anticommutation, $\{\theta_i, \theta_j\} = 0$ same for $\bar{\theta}$,
$\theta_i^2 = \bar{\theta}_j^2 = 0$
$g \in G_N$ = sum of
g^* conjugation
$\theta_i^* = \bar{\theta}_i$
$(g^*)^* = g$
$(g_1 g_2)^* = g_2^* \cdot g_1^*$

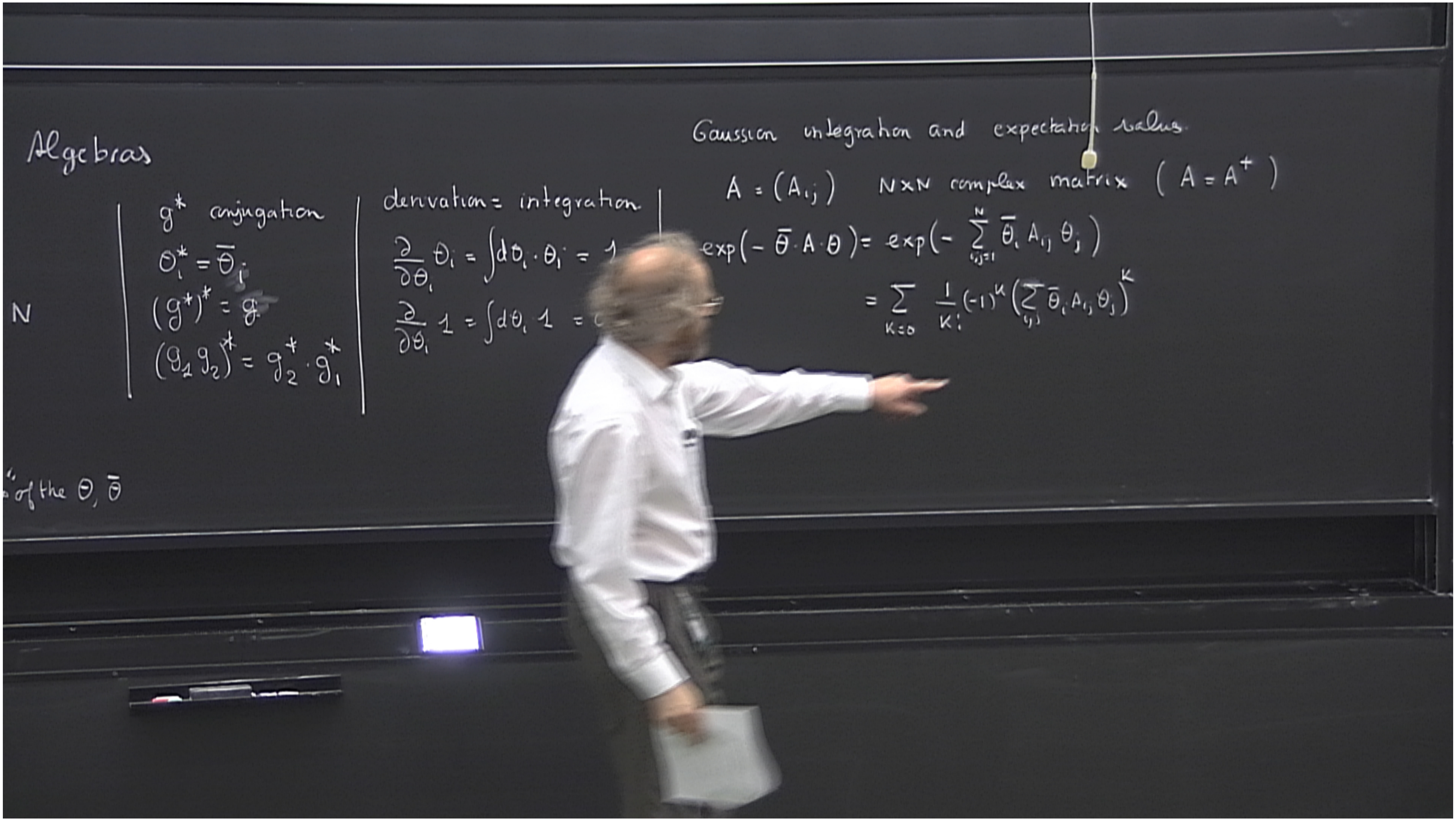

Berezin Calculus : Grassmann Algebras
G_N Algebra over ℂ 2^{2N} = dim
i = 1, ·· N
same for θ̄,
g* conjugation
θ_i* = θ̄_i
(g*)* = g
(g_1 g_2)* = g_2* · g_1*
"functions" of the θ, θ̄


Grassmann Algebras
N
= dim
$\bar{\theta}_i$ i = 1, · N
for $\bar{\theta}$,
g^* conjugation
$\theta_i^* = \bar{\theta}_i$
$(g^*)^* = g$
$(g_1 g_2)^* = g_2^* \cdot g_1^*$
derivation = integration
$\frac{\partial}{\partial \theta_i}$
"functions" of the θ, $\bar{\theta}$


Grassmann Algebras
2N
= dim
$\bar{\Theta}_i$ $i = 1, \dots N$
for $\bar{\Theta}$,
g^* conjugation
$\theta_i^* = \bar{\theta}_i$
$(g^*)^* = g$
$(g_1 g_2)^* = g_2^* \cdot g_1^*$
derivation = integration
$\frac{\partial}{\partial \theta_i} \theta_i = \int d\theta_i \cdot \theta_i = 1$
$\frac{\partial}{\partial \theta_i} 1 = \int d\theta_i \, 1 = 0$
2^N "functions" of the $\theta, \bar{\theta}$


Algebras
g* conjugation
θ*_i = θ̄_i
N
(g*)*
(g_1 g_2)
derivation = integration
∂/∂θ_i θ_i = ∫dθ_i · θ_i = 1
∂/∂θ_i 1 = ∫dθ_i 1 = 0
of the θ, θ̄
Gaussian integration and expectation values
A = (A_ij) N×N complex matrix


Algebras
g* conjugation
derivation = integration
N
of the Θ, Θ̄
Gaussian integration and expectation values
A = (A_ij)   N×N complex matrix   (A = A⁺)
exp(− Θ̄·A·Θ) = exp(− Σ_{i,j=1}^{N} Θ̄_i A_ij Θ_j)


Algebras
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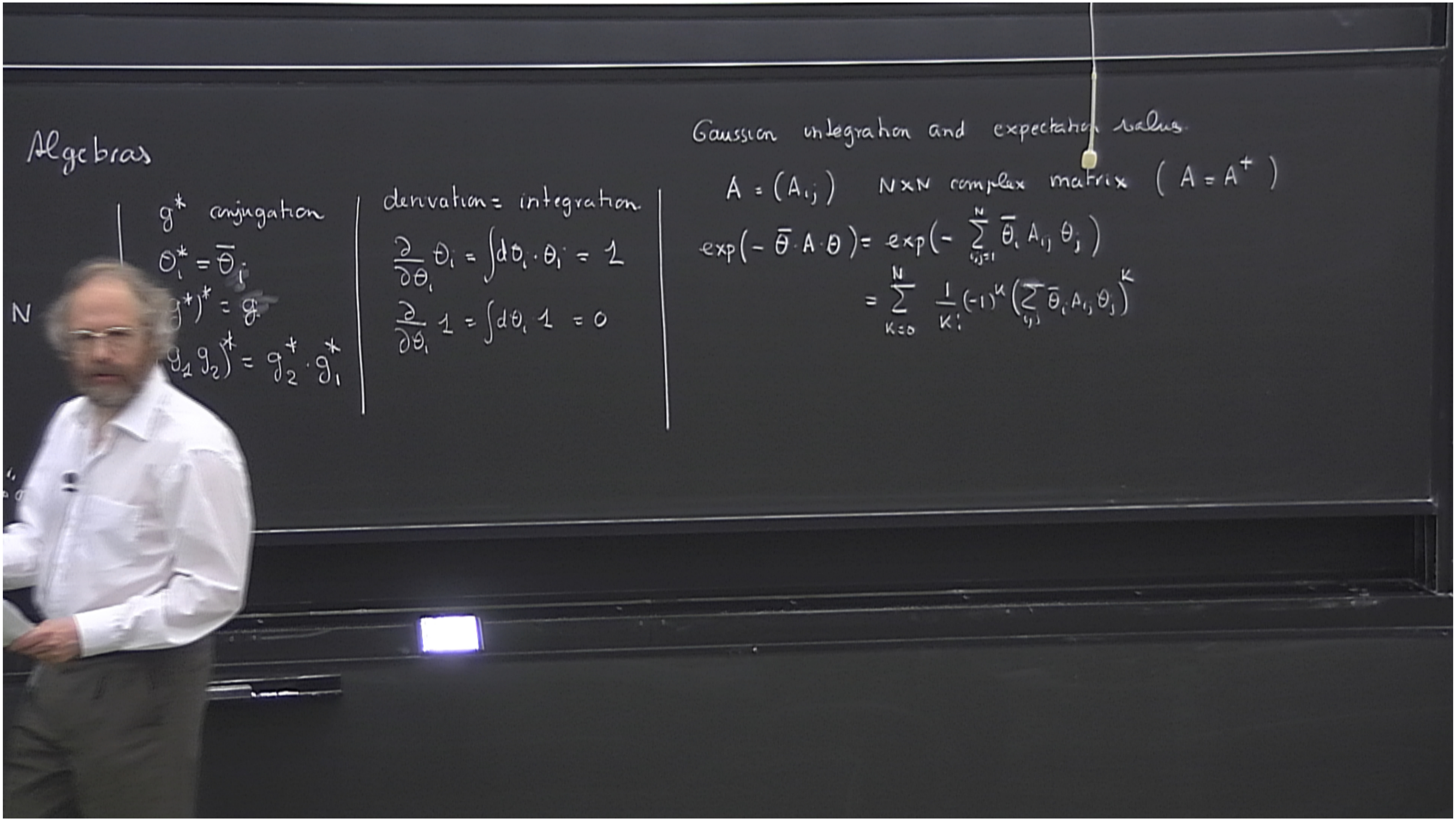
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$(g^*)^* = g$
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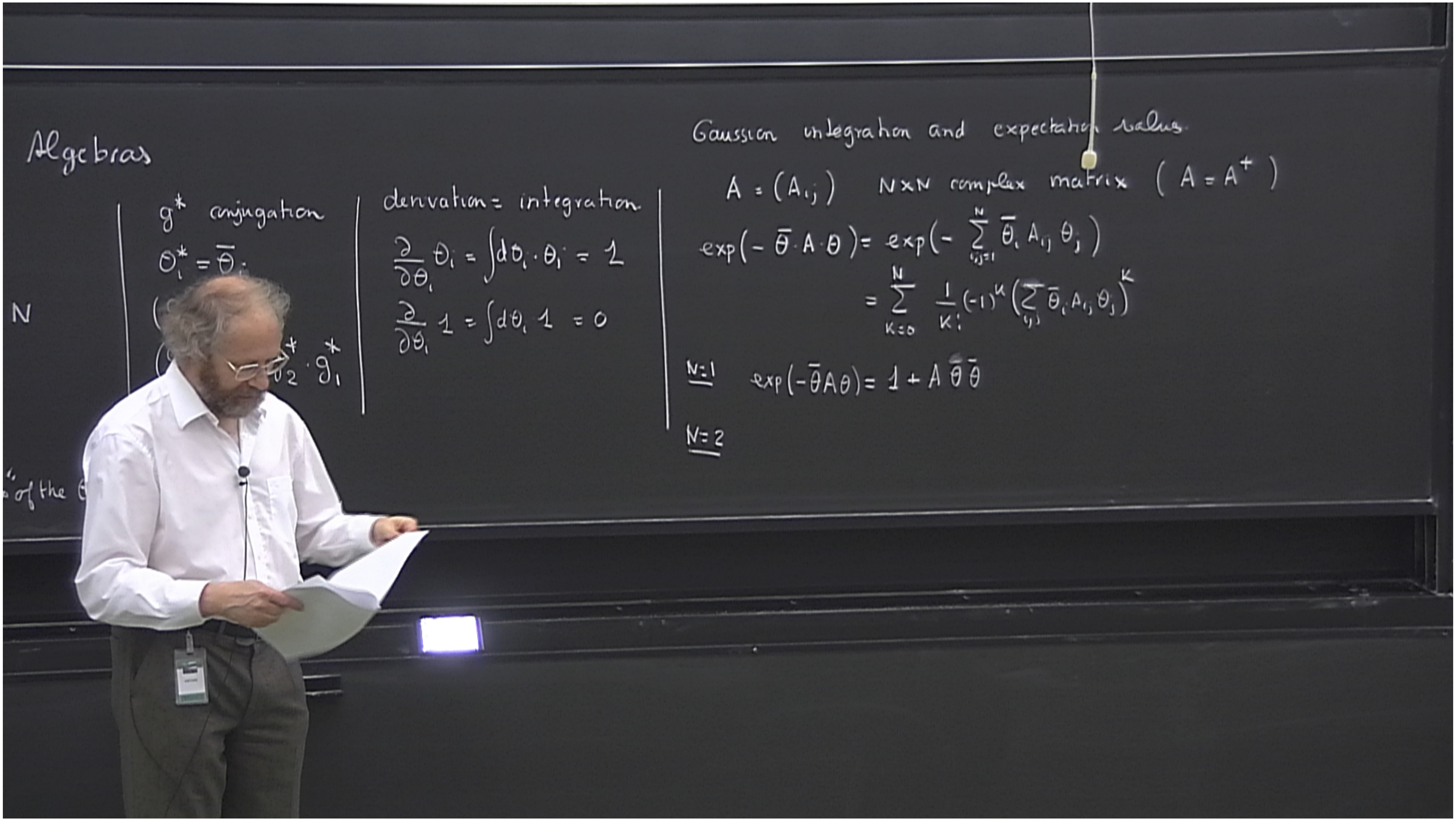
$A = (A_{ij})$ N×N complex matrix $(A = A^+)$
$\exp(-\bar{\theta} \cdot A \cdot \theta) = \exp(-\sum_{i,j=1}^{N} \bar{\theta}_i A_{ij} \theta_j)$
$= \sum_{k=0}^{N} \frac{1}{k!} (-1)^k \left(\sum_{i,j} \bar{\theta}_i A_{ij} \theta_j\right)^k$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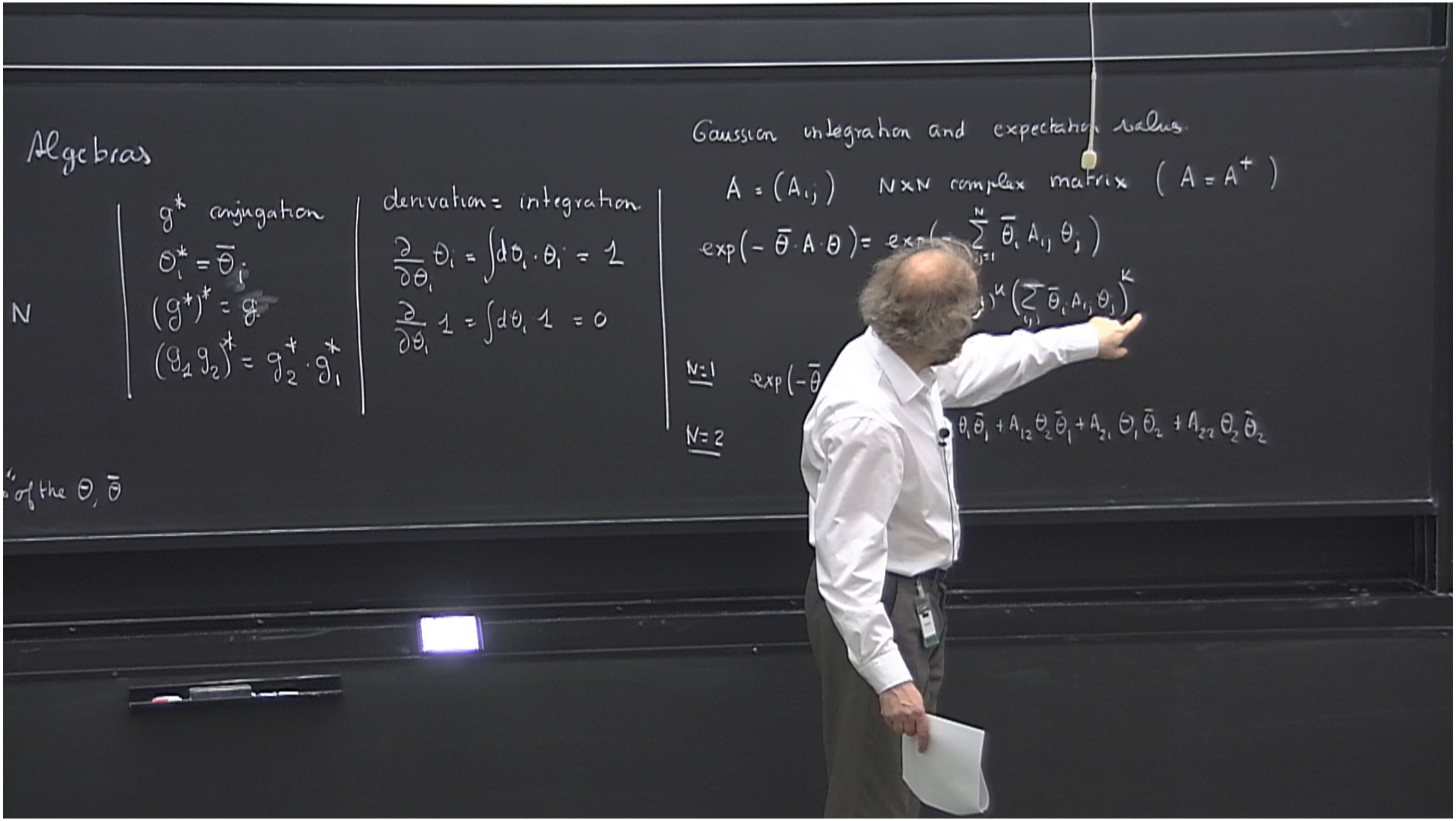
N=1 $\exp(-\bar{\theta} A \theta) = 1 + A \theta \bar{\theta}$
N=2


Algebras
g* conjugation
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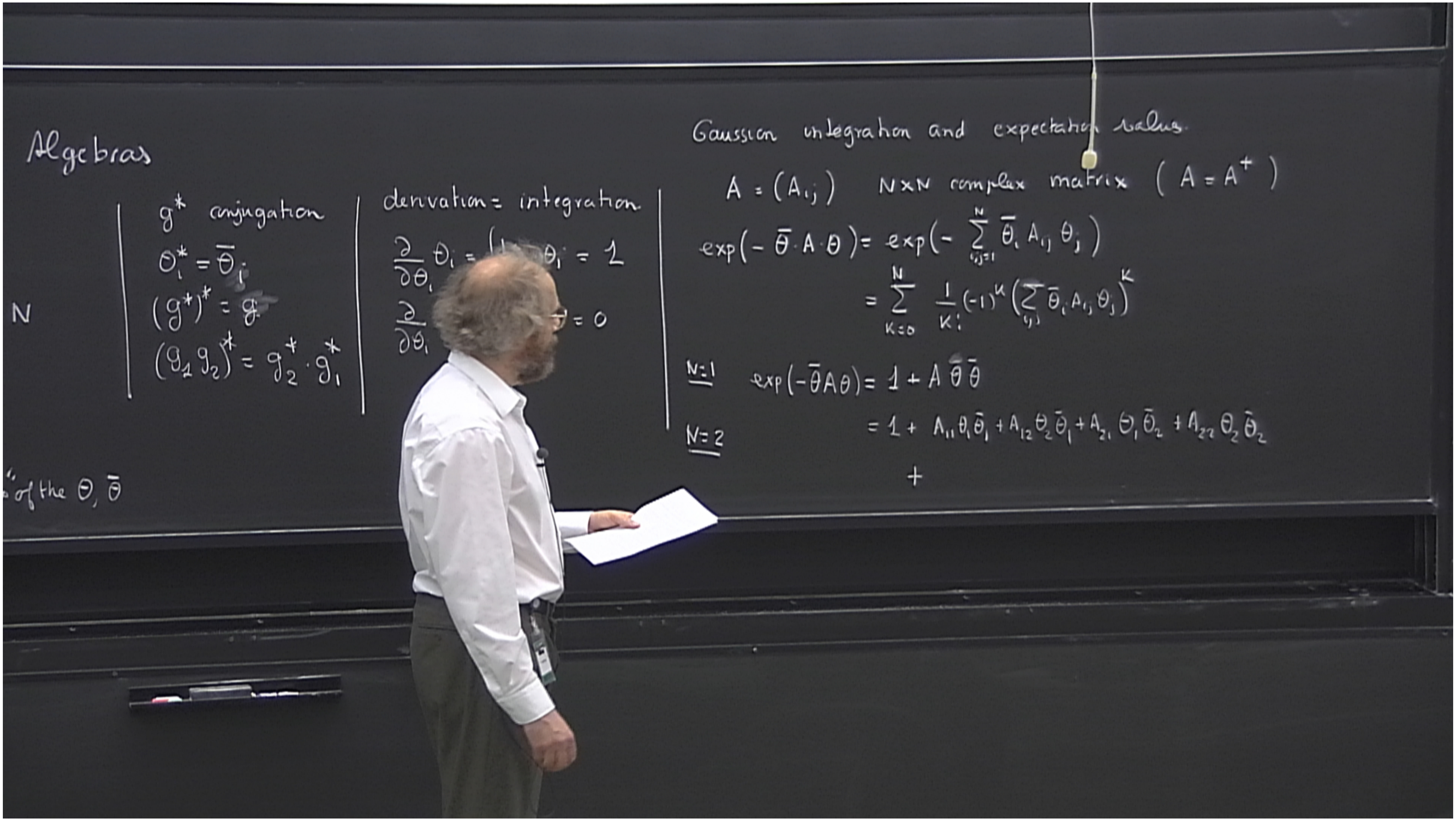
Gaussian integration and expectation values
A = (A_ij) N×N complex matrix (A = A^+)
N=1
N=2
of the Θ, Θ̄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$\theta_i^* = \bar{\theta}_i$
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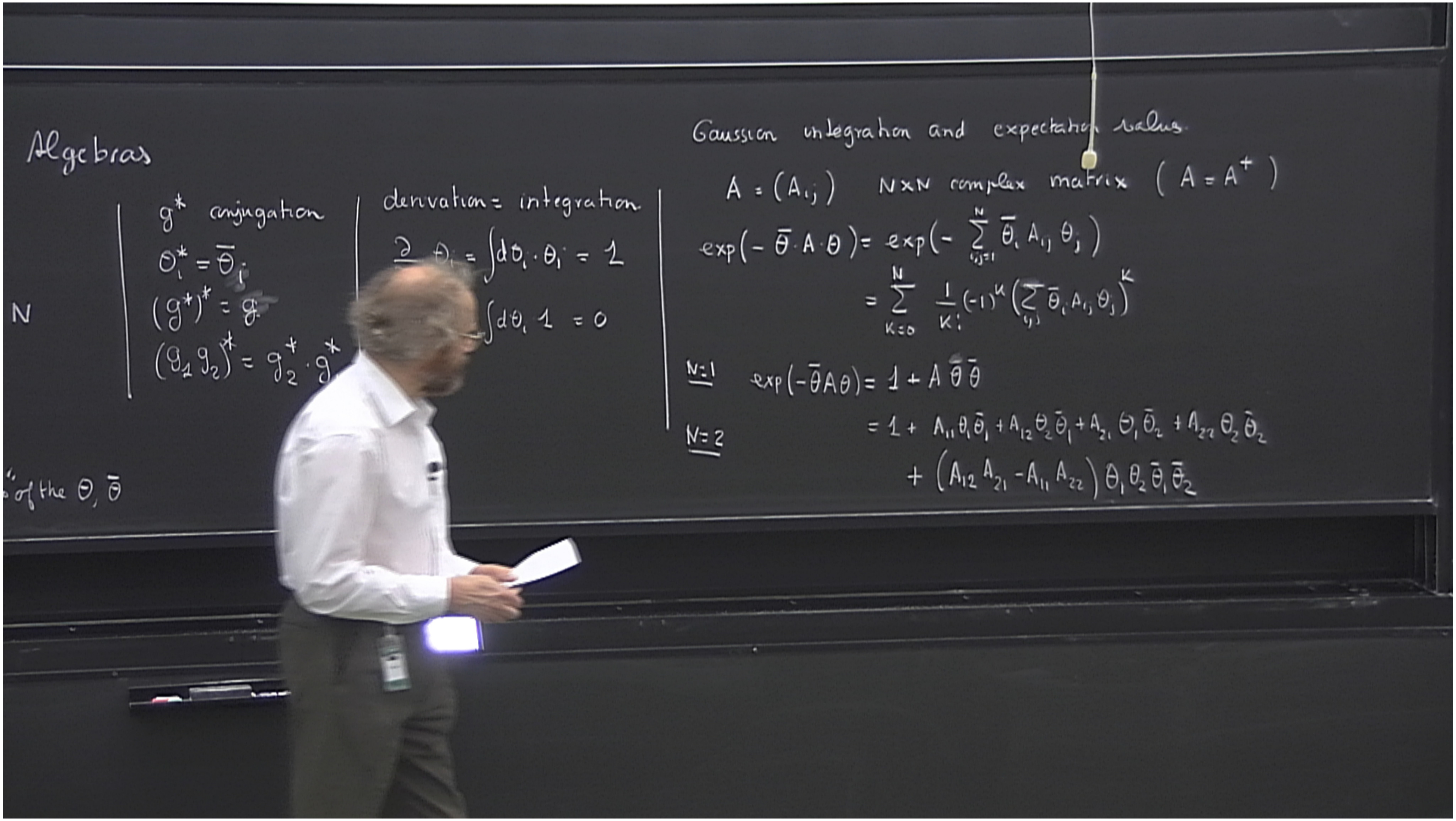
$= \sum_{k=0}^{N} \frac{1}{k!} (-1)^k (\sum_{i,j} \bar{\theta}_i A_{ij} \theta_j)^k$
N=1 $\exp(-\bar{\theta} A \theta) = 1 + A \theta \bar{\theta}$
N=2 $= 1 + A_{11} \theta_1 \bar{\theta}_1 + A_{12} \theta_2 \bar{\theta}_1 + A_{21} \theta_1 \bar{\theta}_2 + A_{22} \theta_2 \bar{\theta}_2$
+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$\theta_i^* = \bar{\theta}_i$
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$\frac{\partial}{\partial\theta_i}\theta_i = \int d\theta_i \cdot \theta_i = 1$
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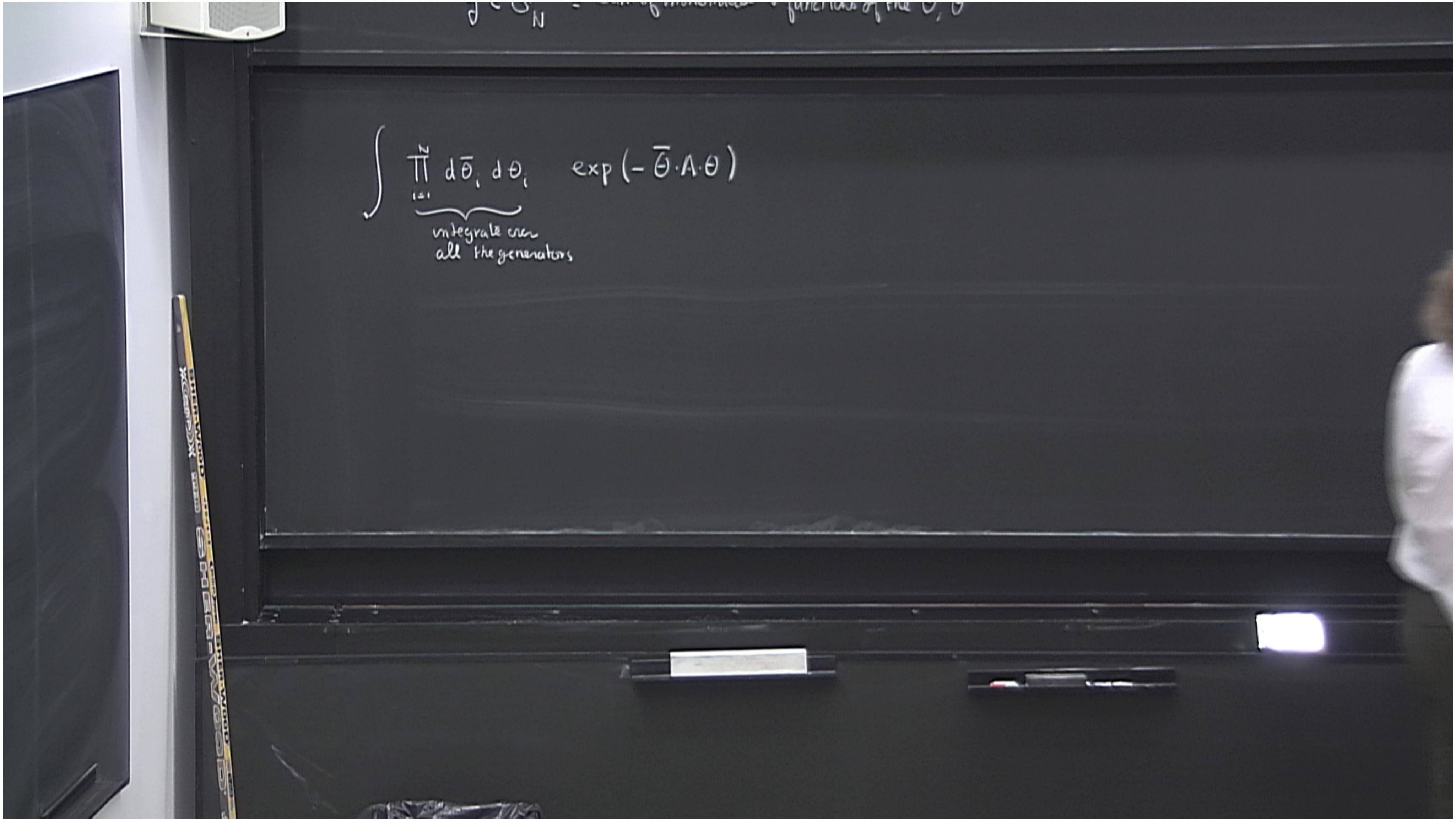
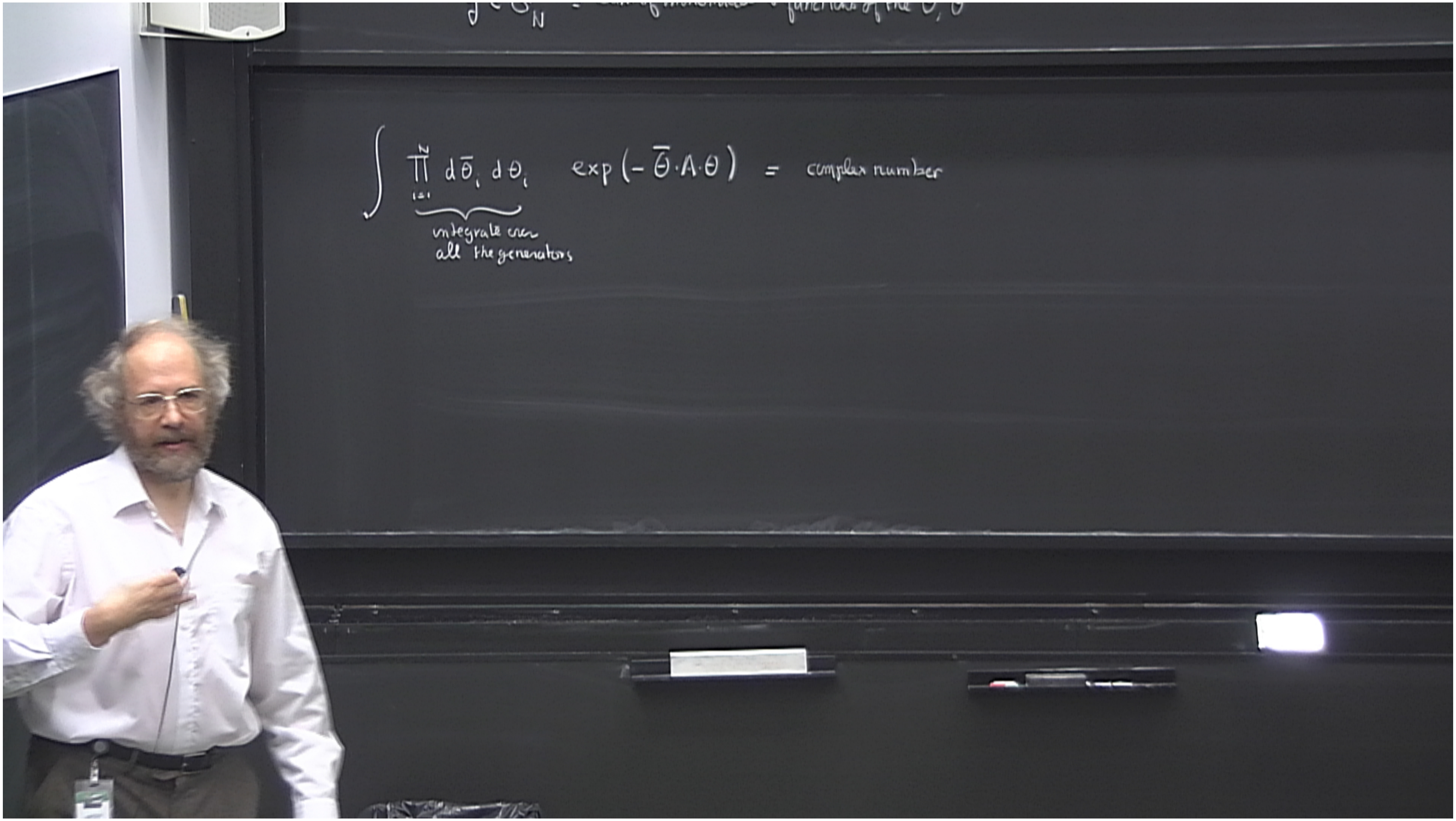
$\exp(-\bar{\theta} \cdot A \cdot \theta) = \exp(-\sum_{i,j=1}^{N} \bar{\theta}_i A_{ij} \theta_j)$
$= \sum_{k=0}^{N} \frac{1}{k!}(-1)^k \left(\sum_{i,j} \bar{\theta}_i A_{ij} \theta_j\right)^k$
N=1 $\exp(-\bar{\theta} A \theta) = 1 + A \theta \bar{\theta}$
N=2 $= 1 + A_{11}\theta_1\bar{\theta}_1 + A_{12}\theta_2\bar{\theta}_1 + A_{21}\theta_1\bar{\theta}_2 + A_{22}\theta_2\bar{\theta}_2$
$+ (A_{12}A_{21} - A_{11}A_{22})\theta_1\theta_2\bar{\theta}_1\bar{\theta}_2$

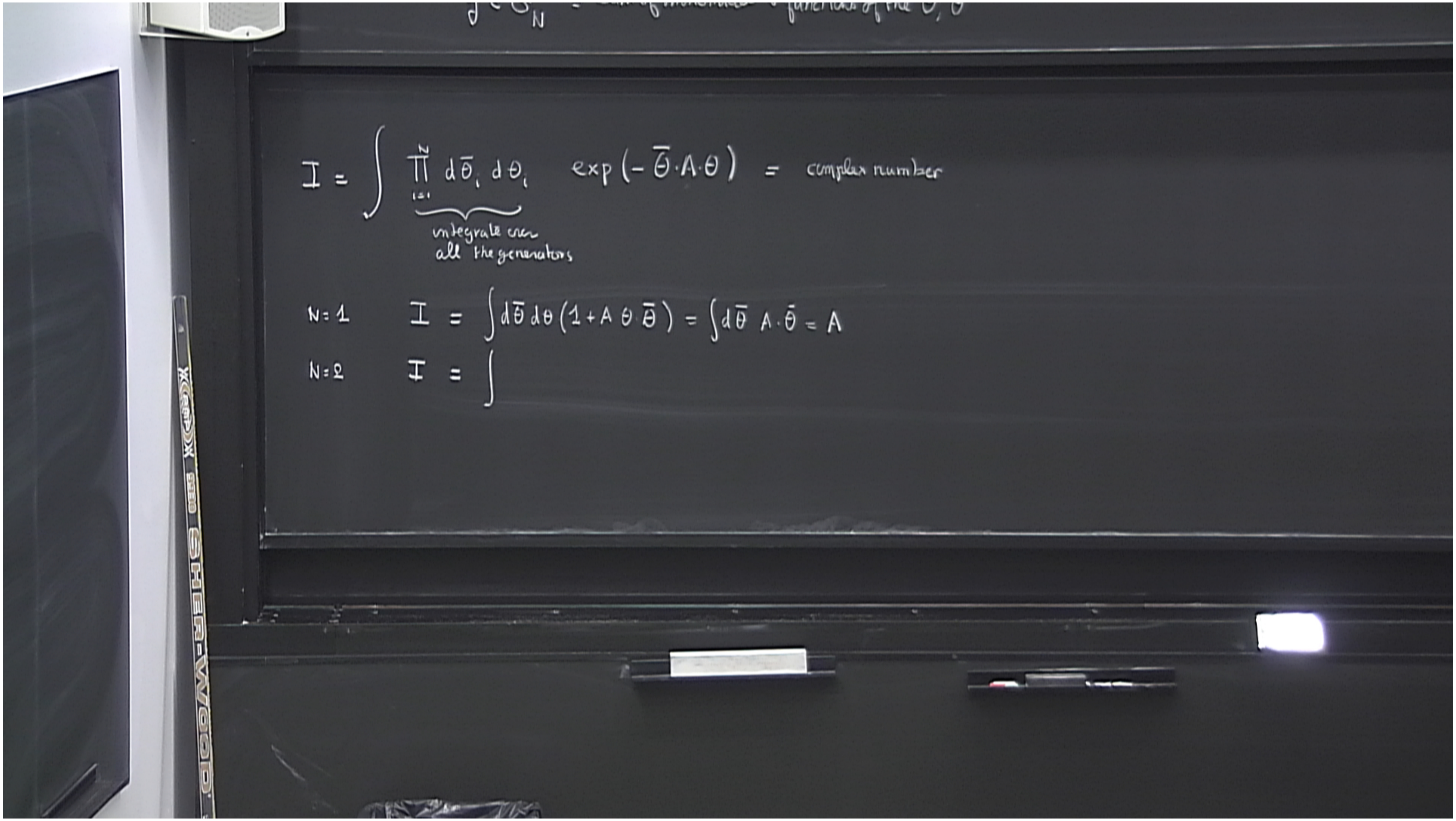

$\int \prod_{i=1}^{N} d\bar{\theta}_i \, d\theta_i \quad \exp(-\bar{\theta}\cdot A\cdot\theta)$
integrate over all the generators

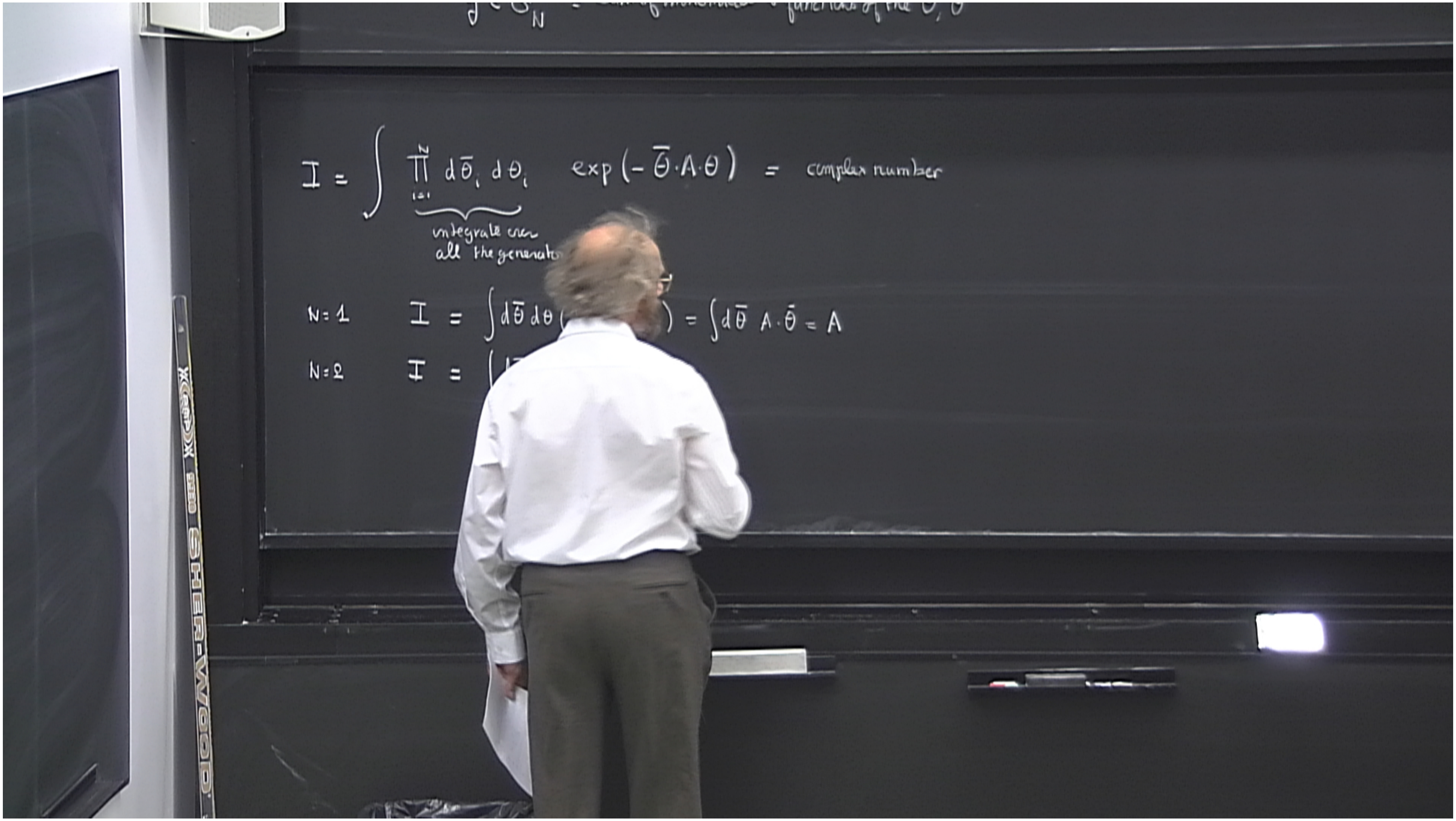

$\int \prod_{i=1}^{N} d\bar{\theta}_i \, d\theta_i \quad \exp(-\bar{\theta}\cdot A \cdot \theta) = $ complex number
integrate over
all the generators

$$I = \int \prod_{i=1}^{N} d\bar{\theta}_i \, d\theta_i \quad \exp\left(-\bar{\theta} \cdot A \cdot \theta\right) = \text{complex number}$$

(integrate over all the generators)

$N = 1 \qquad I =$


$I = \int \prod_{i=1}^{N} d\bar{\theta}_i \, d\theta_i \; \exp(-\bar{\theta}\cdot A\cdot\theta) =$ complex number
integrate over all the generators
$N=1 \quad I = \int d\bar{\theta}\, d\theta \,(1 + A\,\theta\,\bar{\theta}) = \int d\bar{\theta}\; A\cdot\bar{\theta} = A$
$N=2 \quad I = \int$


I = ∫ ∏ dθ̄ᵢ dθᵢ exp(− θ̄·A·θ) = complex number
integrate over
all the generato
N=1 I = ∫dθ̄ dθ ( ) = ∫dθ̄ A·θ̄ = A
N=2 I = ∫
SHER-WOOD

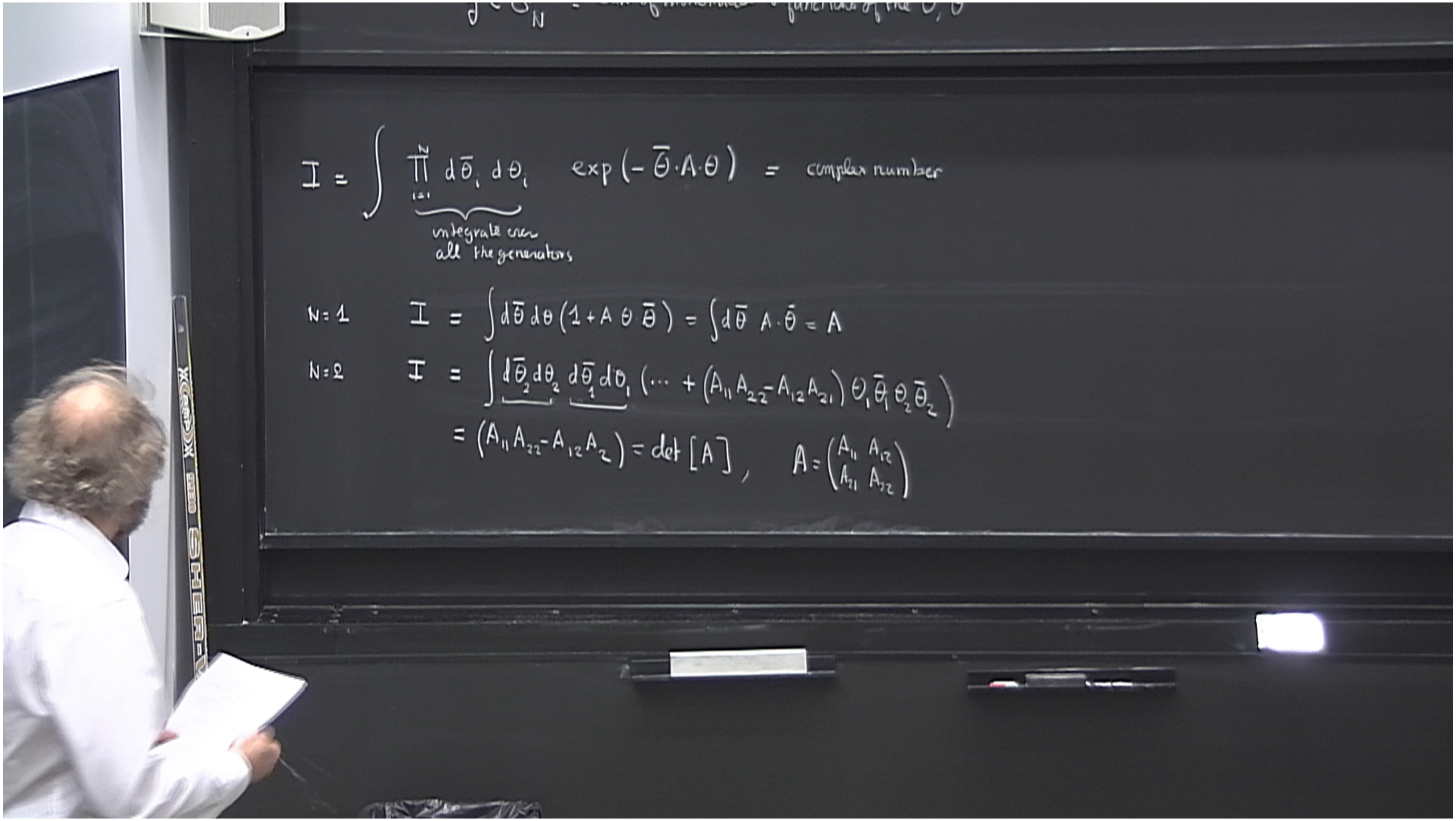
$$I = \int \prod_{i=1}^{N} d\bar{\theta}_i \, d\theta_i \quad \exp(-\bar{\theta}\cdot A \cdot \theta) = \text{complex number}$$

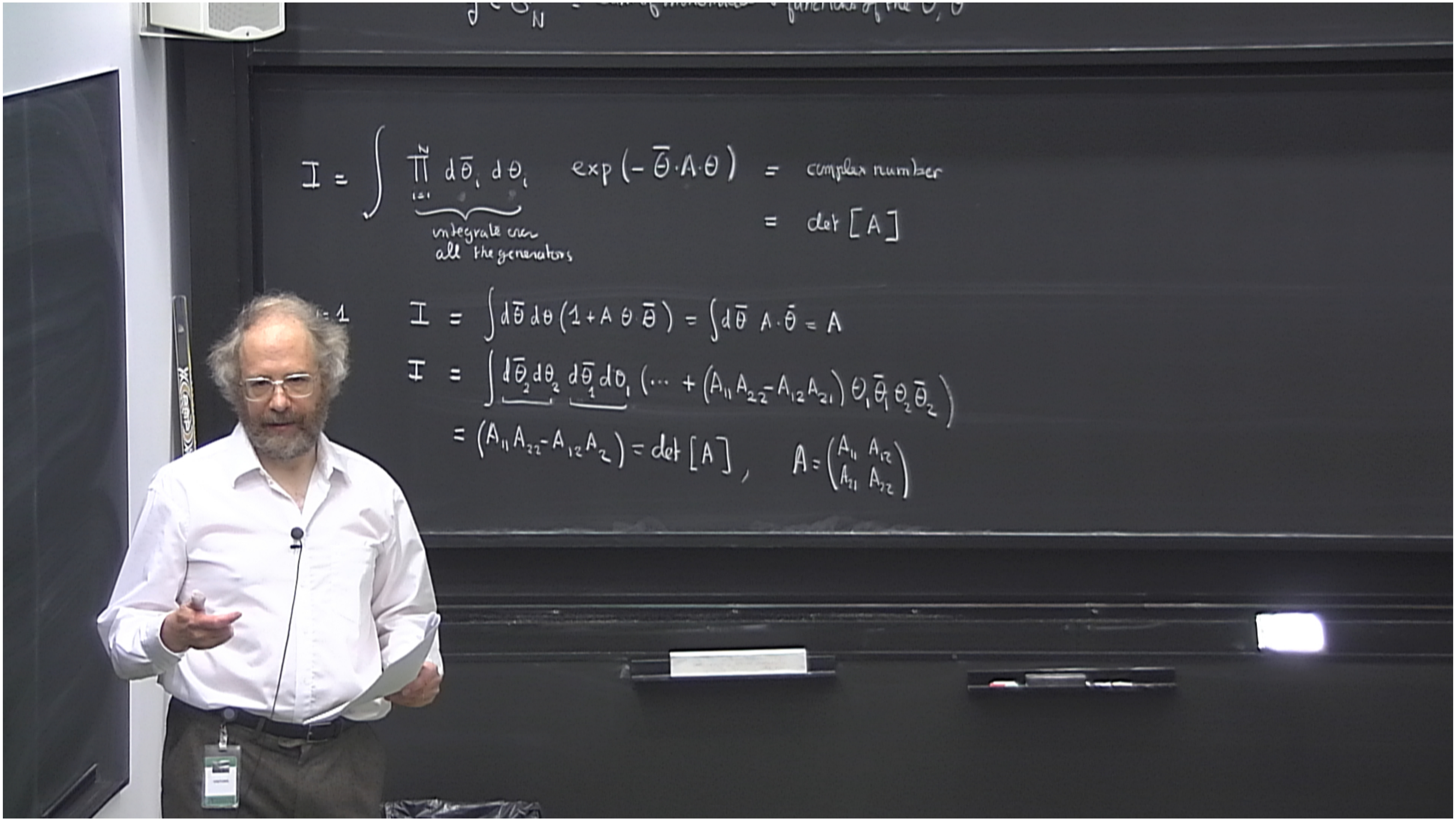
(integrate over all the generators)

$N=1$: $$I = \int d\bar{\theta}\, d\theta \,(1 + A\,\theta\,\bar{\theta}) = \int d\bar{\theta}\; A \cdot \bar{\theta} = A$$

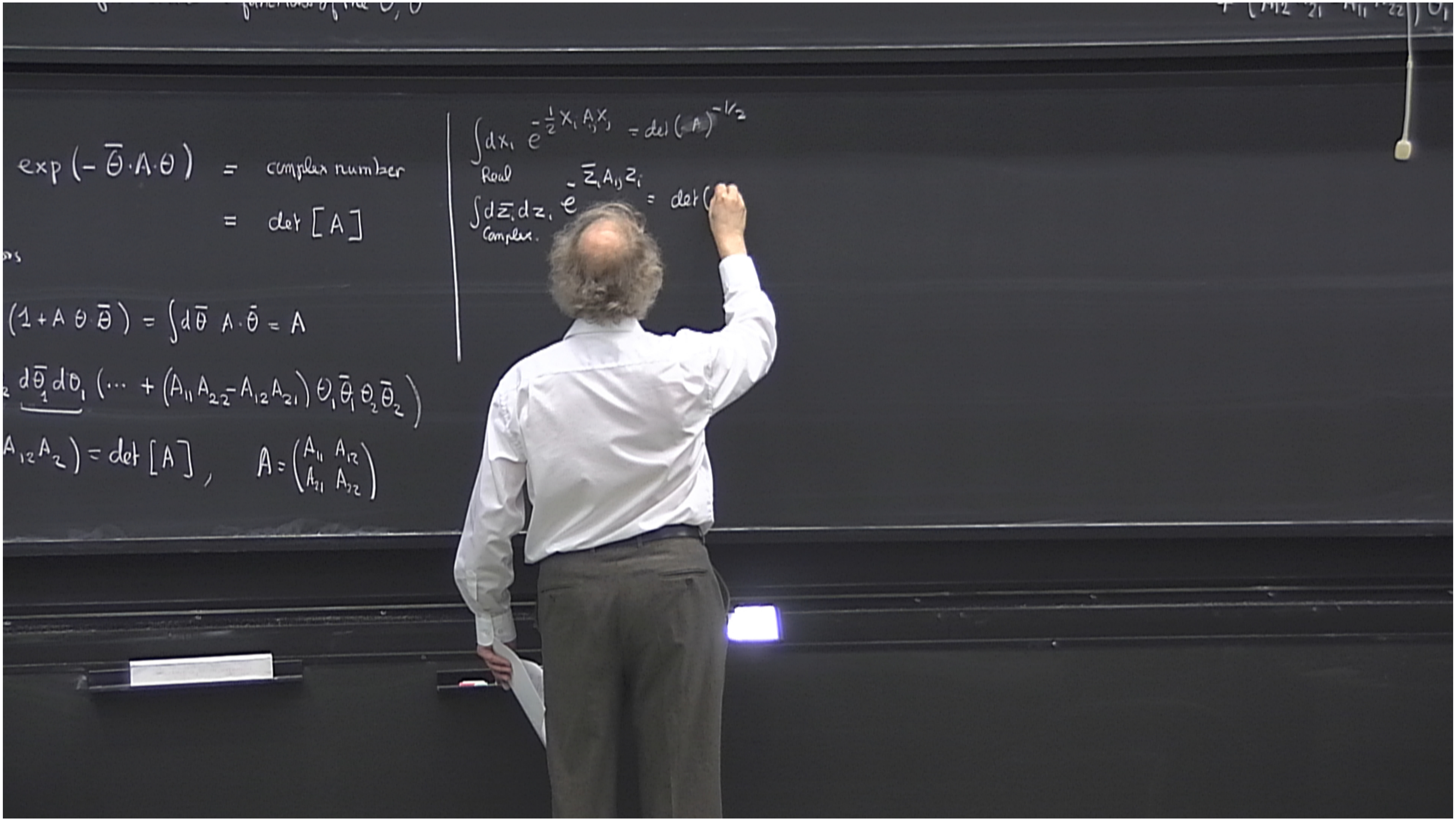
$N=2$: $$I = \int \underbrace{d\bar{\theta}_2 d\theta_2}\; \underbrace{d\bar{\theta}_1 d\theta_1}\, \left(\cdots + (A_{11}A_{22} - A_{12}A_{21})\, \theta_1 \bar{\theta}_1 \theta_2 \bar{\theta}_2 \right)$$

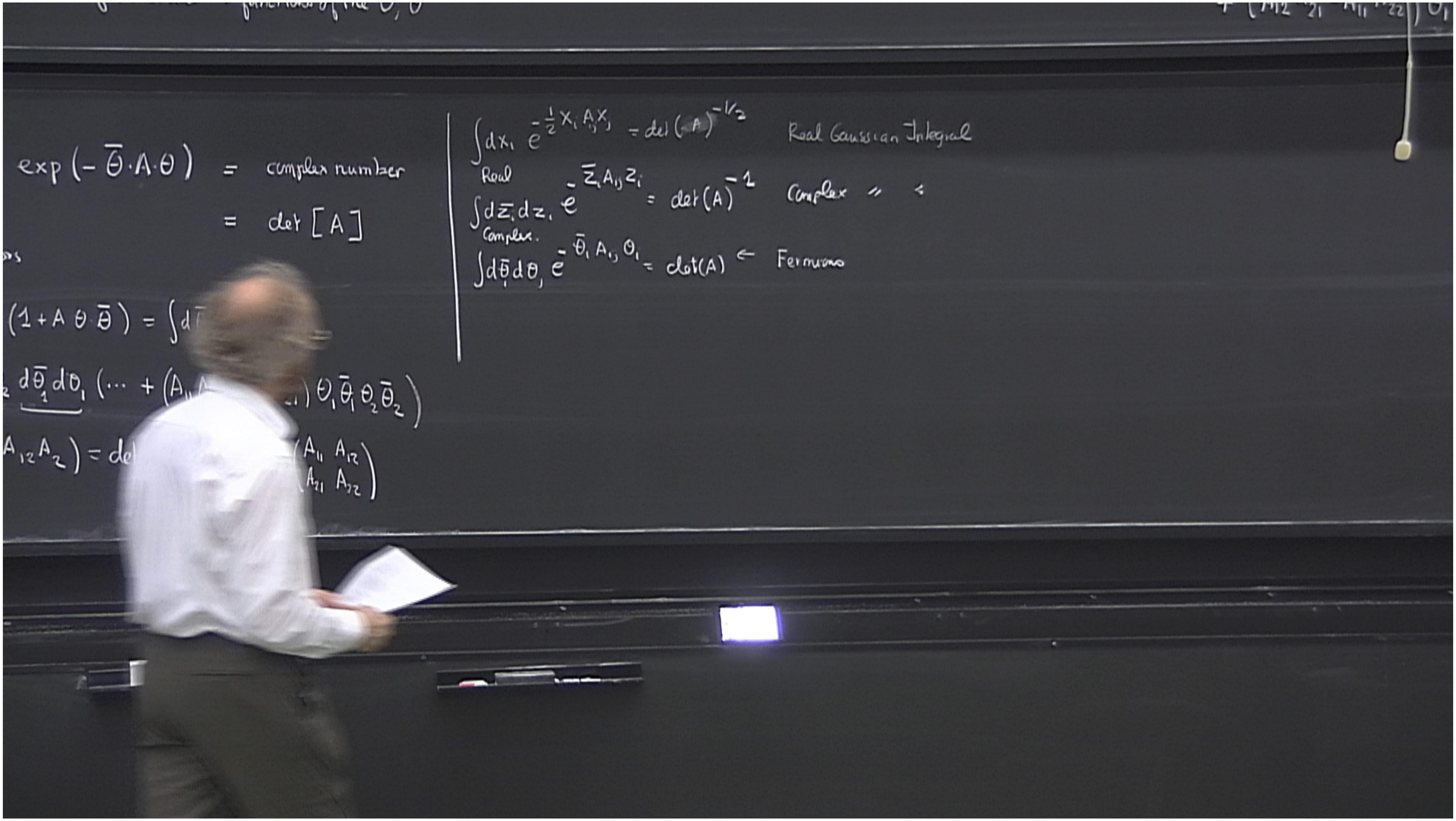
$$= (A_{11}A_{22} - A_{12}A_{21}) = \det[A] \qquad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

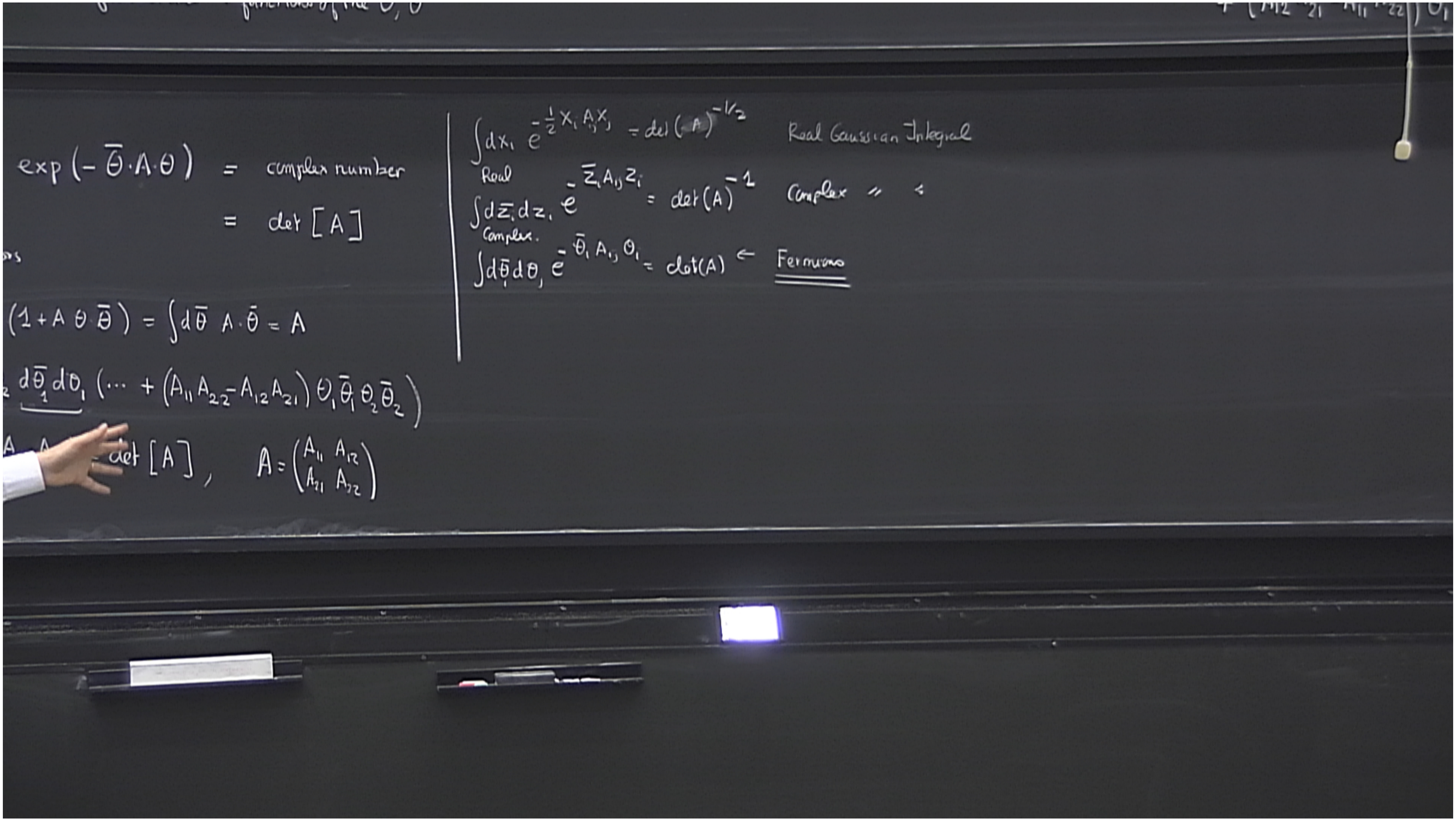

$I = \int \prod_{i=1}^{N} d\bar\theta_i \, d\theta_i \; \exp(-\bar\theta \cdot A \cdot \theta)$ = complex number
integrate over all the generators
$N=1 \quad I = \int d\bar\theta \, d\theta \, (1 + A\,\theta\,\bar\theta) = \int d\bar\theta \; A \cdot \bar\theta = A$
$N=2 \quad I = \int d\bar\theta_2 d\theta_2 \, d\bar\theta_1 d\theta_1 \left(\cdots + (A_{11}A_{22} - A_{12}A_{21}) \, \theta_1 \bar\theta_1 \theta_2 \bar\theta_2 \right)$
$= (A_{11}A_{22} - A_{12}A_{21}) = \det[A], \quad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$

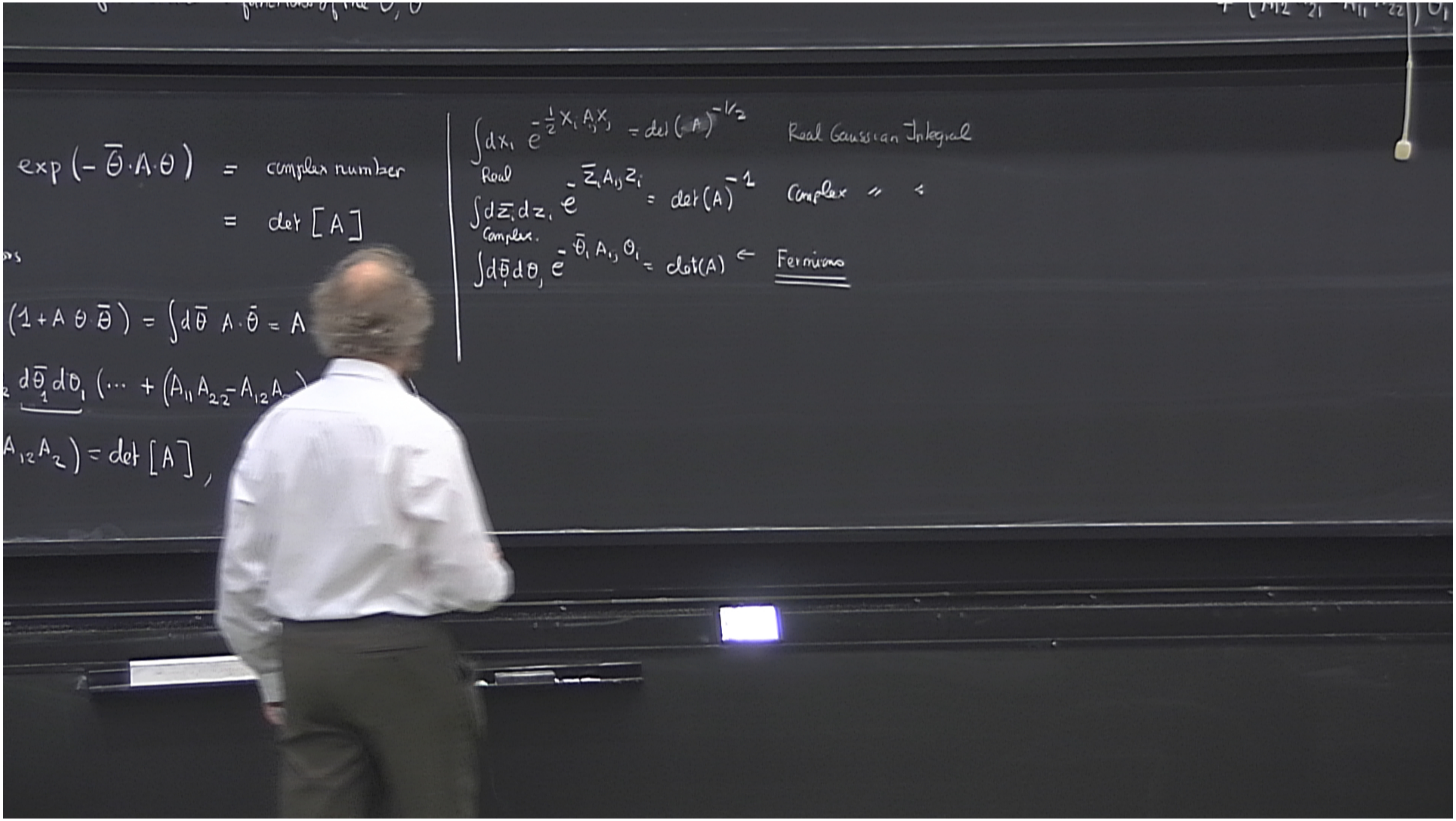


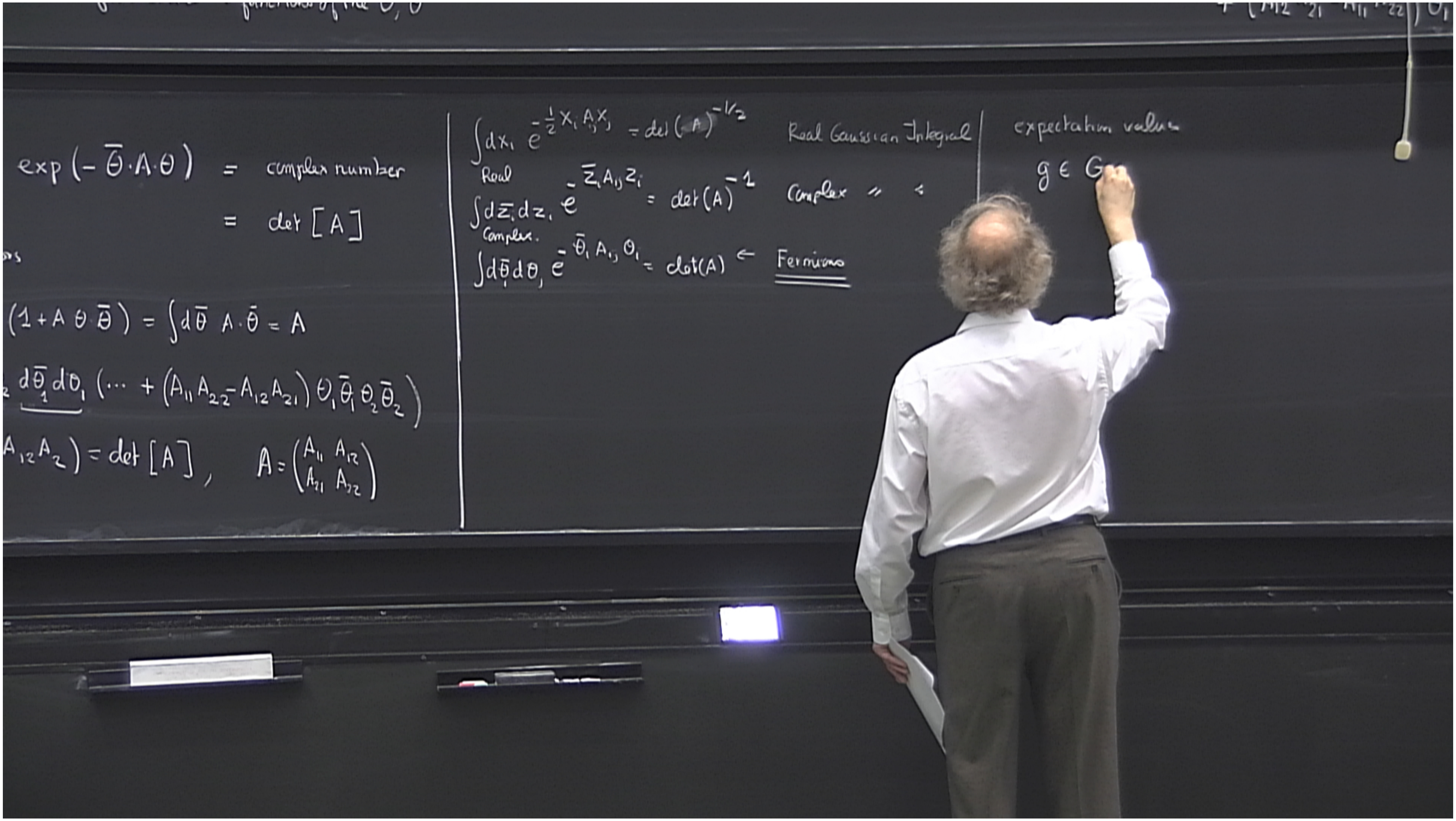
$I = \int \prod_{i=1}^{N} d\bar{\theta}_i \, d\theta_i \; \exp(-\bar{\theta}\cdot A\cdot\theta)$ = complex number
$= \det[A]$
integrate over all the generators
$I = \int d\bar{\theta}\, d\theta \,(1 + A\,\theta\,\bar{\theta}) = \int d\bar{\theta}\; A\cdot\bar{\theta} = A$
$I = \int d\bar{\theta}_2 d\theta_2 \, d\bar{\theta}_1 d\theta_1 \,(\cdots + (A_{11}A_{22} - A_{12}A_{21})\,\theta_1\bar{\theta}_1\theta_2\bar{\theta}_2)$
$= (A_{11}A_{22} - A_{12}A_{2}) = \det[A]$, $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$

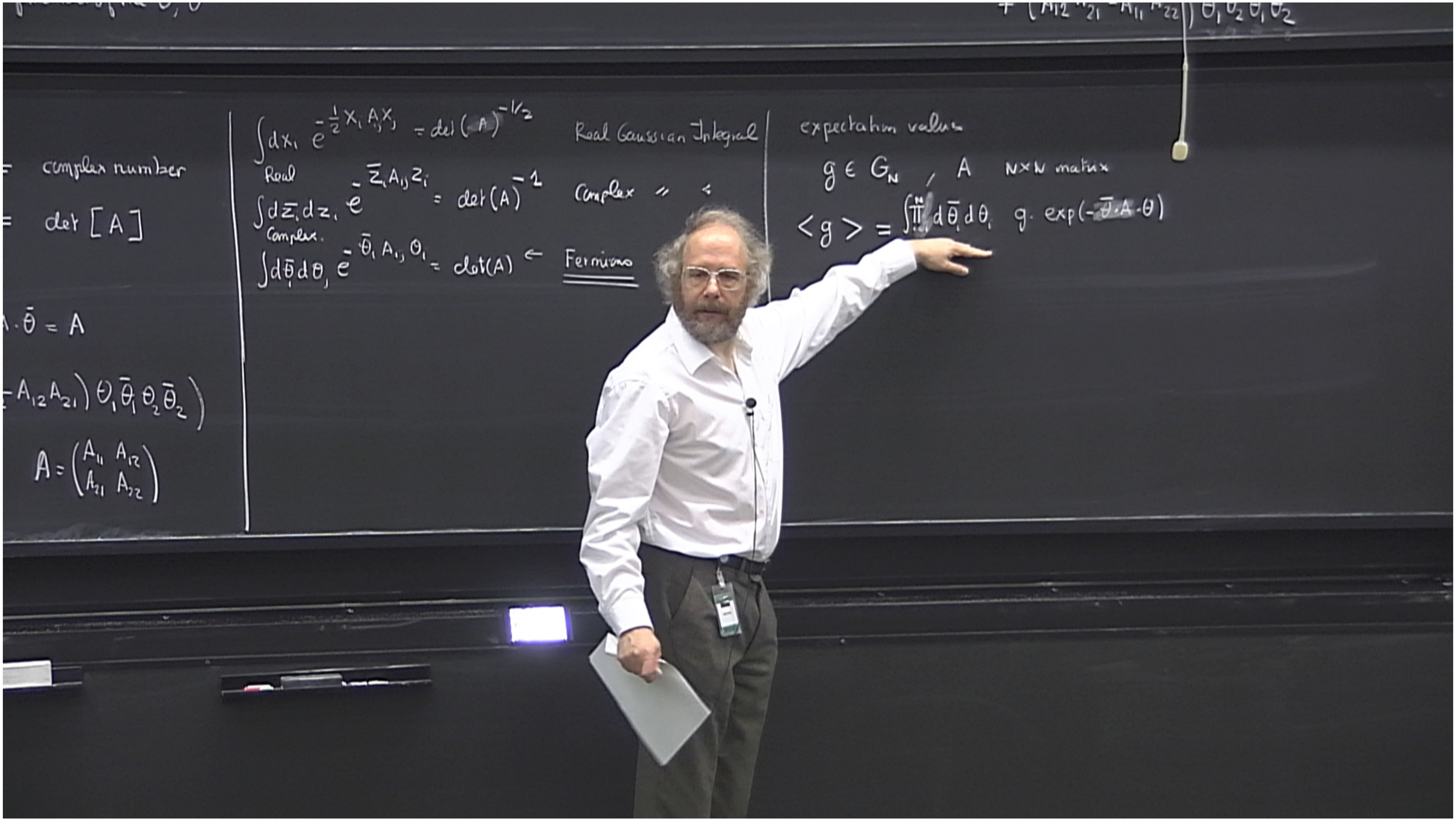

exp (− θ̄·A·θ) = complex number
= det [A]
(1 + A θ θ̄) = ∫dθ̄ A·θ̄ = A
dθ̄1 dθ1 (… + (A11 A22 − A12 A21) θ1 θ̄1 θ2 θ̄2)
A12 A2) = det [A], A = (A11 A12 / A21 A22)
∫dxi e−½ xi Aij xj = det (A)−½
Real
∫dz̄i dzi e−z̄i Aij zi = det (
Complex.


exp(− θ̄·A·θ) = complex number
= det [A]
∫dx_i e^{−½ x_i A_ij x_j} = det(A)^{−1/2} Real Gaussian Integral
Real
∫dz̄_i dz_i e^{−z̄_i A_ij z_i} = det(A)^{−1} Complex " "
Complex.
∫dθ̄_i dθ_i e^{−θ̄_i A_ij θ_i} = det(A) ← Fermions


exp(−θ̄·A·θ) = complex number
= det [A]
(1 + A θ θ̄) = ∫dθ̄ A·θ̄ = A
dθ̄₁ dθ₁ (… + (A₁₁A₂₂ − A₁₂A₂₁) θ₁θ̄₁θ₂θ̄₂)
det [A], A = (A₁₁ A₁₂ / A₂₁ A₂₂)
∫dxᵢ e^{−½ xᵢ Aᵢⱼ xⱼ} = det(A)^{−1/2}
Real Gaussian Integral
Real
∫dz̄ᵢ dzᵢ e^{−z̄ᵢ Aᵢⱼ zⱼ} = det(A)^{−1}
Complex
Complex.
∫dθ̄ᵢ dθⱼ e^{−θ̄ᵢ Aᵢⱼ θⱼ} = det(A) ←
Fermions


exp(− θ̄·A·θ) = complex number
= det [A]
(1 + A θ θ̄) = ∫dθ̄ A·θ̄ = A
dθ̄₁ dθ₁ (⋯ + (A₁₁A₂₂ − A₁₂A₂₁)
A₁₂A₂₁) = det [A],
∫dx e^{−½ xᵢAᵢⱼxⱼ} = det(A)^{−1/2}   Real Gaussian Integral
Real
∫dz̄ᵢdzᵢ e^{−z̄ᵢAᵢⱼzⱼ} = det(A)^{−1}   Complex " "
Complex.
∫dθ̄ᵢdθⱼ e^{−θ̄ᵢAᵢⱼθⱼ} = det(A) ← Fermions


$\exp(-\bar{\theta}\cdot A\cdot\theta)$ = complex number
= det [A]
$\int dx_i\, e^{-\frac{1}{2}x_i A_{ij} x_j} = \det(A)^{-1/2}$ Real Gaussian Integral
Real
$\int d\bar{z}_i dz_i\, e^{-\bar{z}_i A_{ij} z_j} = \det(A)^{-1}$ Complex
Complex
$\int d\bar{\theta}_i d\theta_i\, e^{-\bar{\theta}_i A_{ij}\theta_j} = \det(A)$ ← Fermions
expectation values
$g \in G$
$(1 + A\,\theta\,\bar{\theta}) = \int d\bar{\theta}\; A\cdot\bar{\theta} = A$
$d\bar{\theta}_1 d\theta_1\,(\cdots + (A_{11}A_{22} - A_{12}A_{21})\,\theta_1\bar{\theta}_1\theta_2\bar{\theta}_2)$
$A_{12}A_2) = \det[A], \quad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$


complex number
det [A]
Real Gaussian Integral
Real
Complex
Complex.
Fermions
expectation value
g ∈ G_N , A N×N matrix
<g> =

$+ (A_{12} A_{21} - A_{11} A_{22})\, \theta_1 \theta_2 \theta_1 \theta_2$

= complex number

= $\det [A]$

$A \cdot \bar\theta = A$

$- A_{12} A_{21}) \, \theta_1 \bar\theta_1 \theta_2 \bar\theta_2 \Big)$

$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$

$$\int dx_i \, e^{-\frac{1}{2} x_i A_{ij} x_j} = \det(A)^{-1/2}$$ Real Gaussian Integral

Real

$$\int d\bar z_i \, dz_i \, e^{-\bar z_i A_{ij} z_i} = \det(A)^{-1}$$ Complex " "

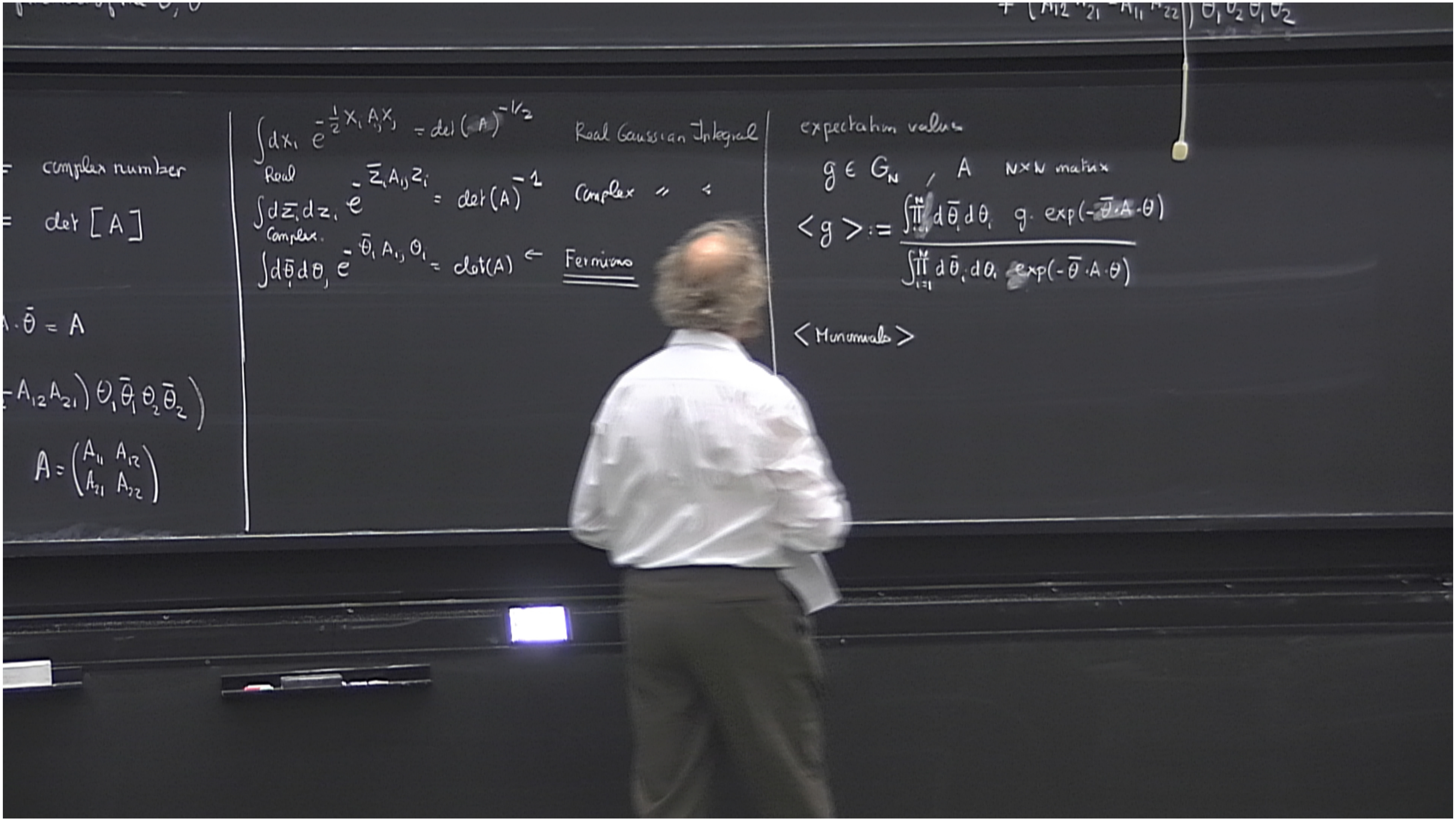
Complex.

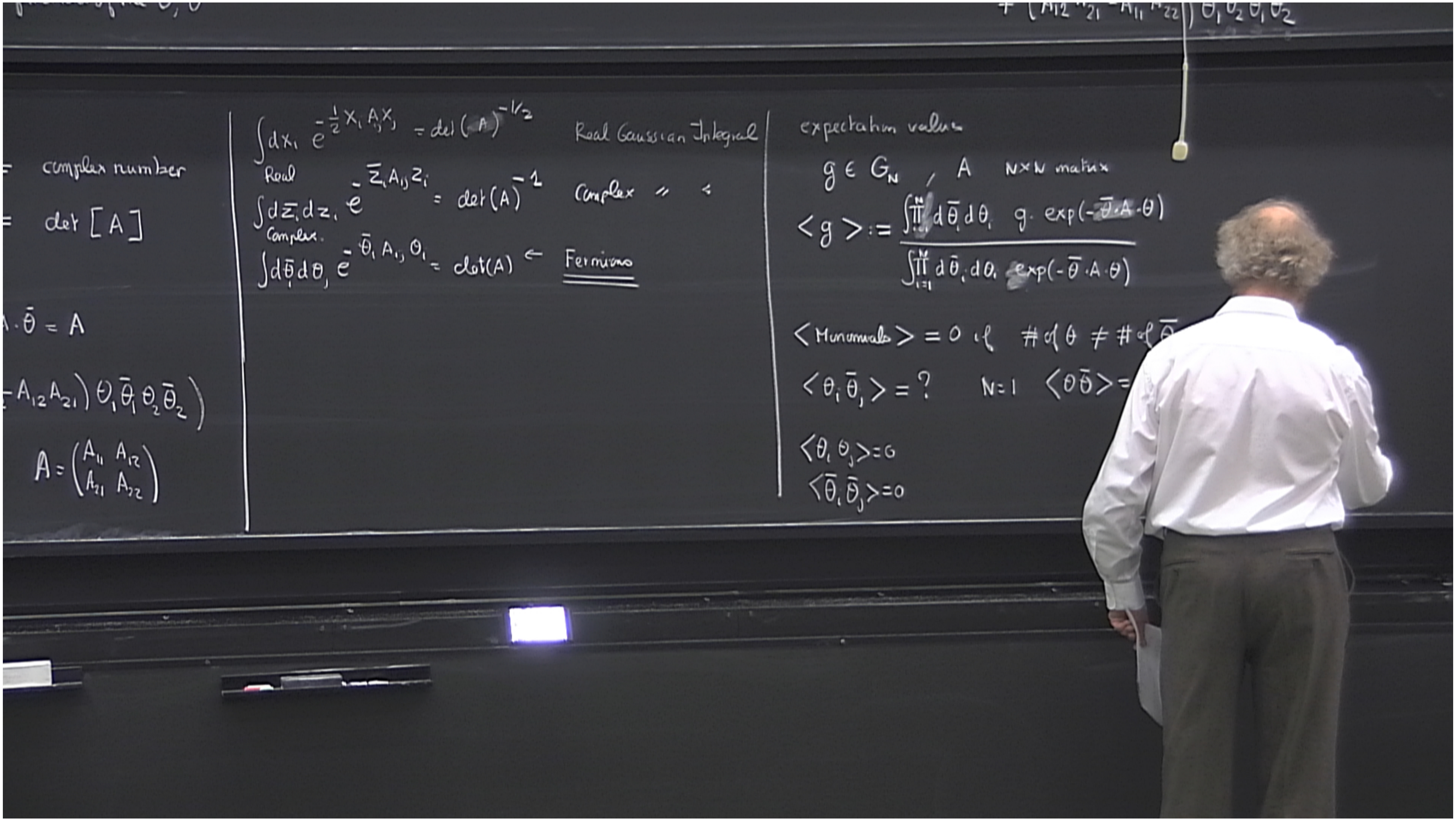
$$\int d\bar\theta_i \, d\theta_i \, e^{-\bar\theta_i A_{ij} \theta_i} = \det(A)$$ ← Fermions

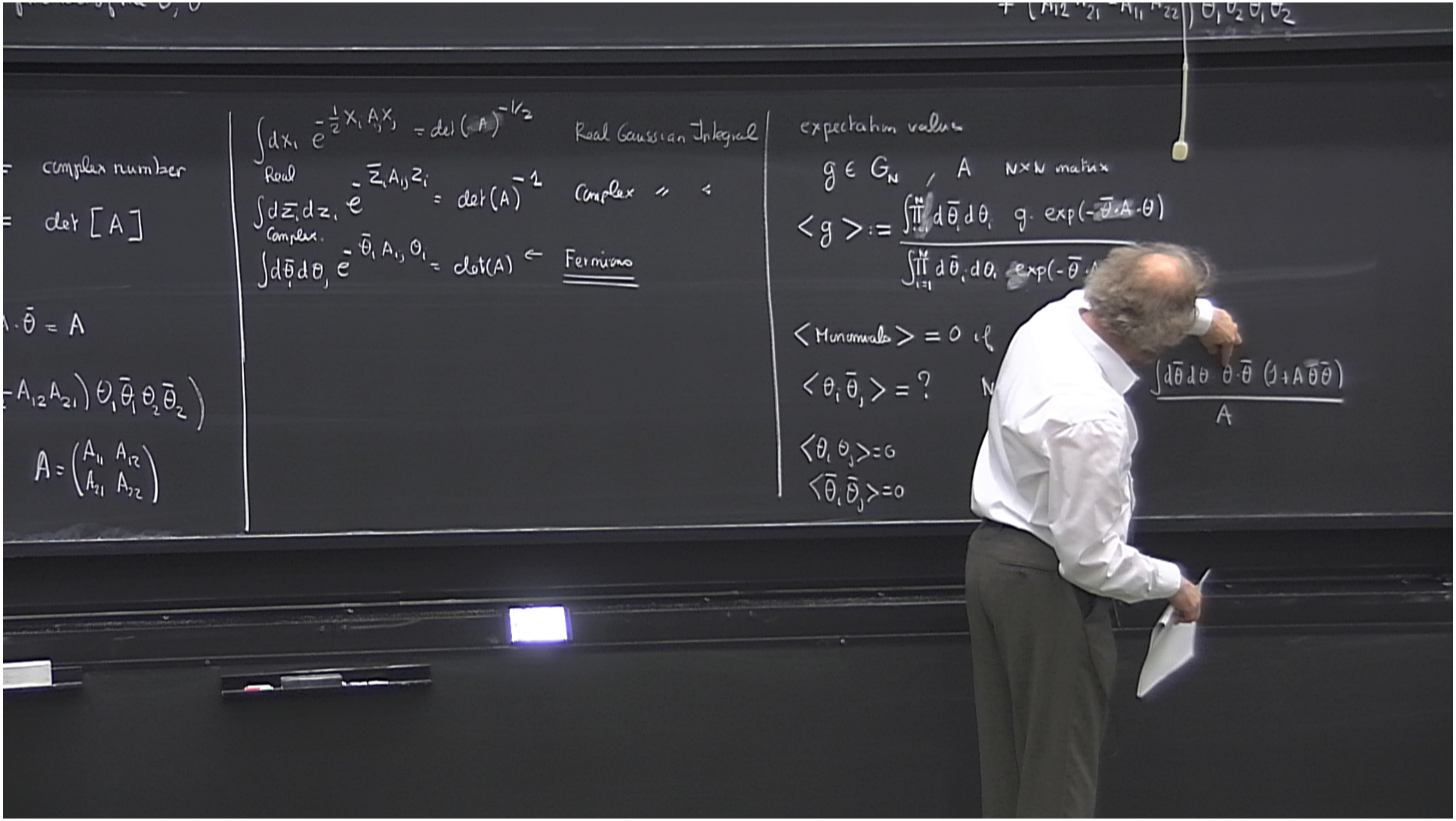
expectation value

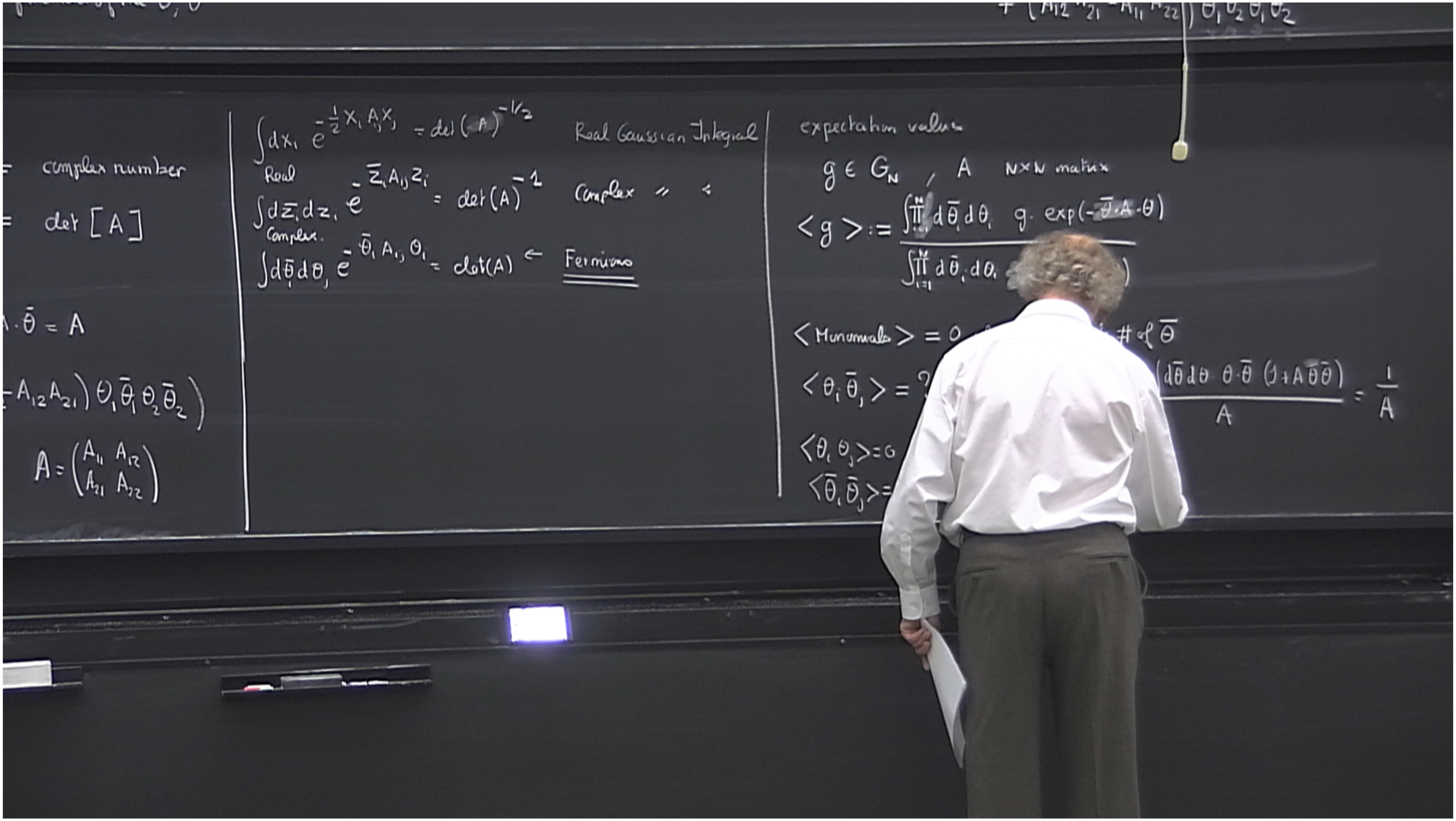
$g \in G_N$, A $N \times N$ matrix

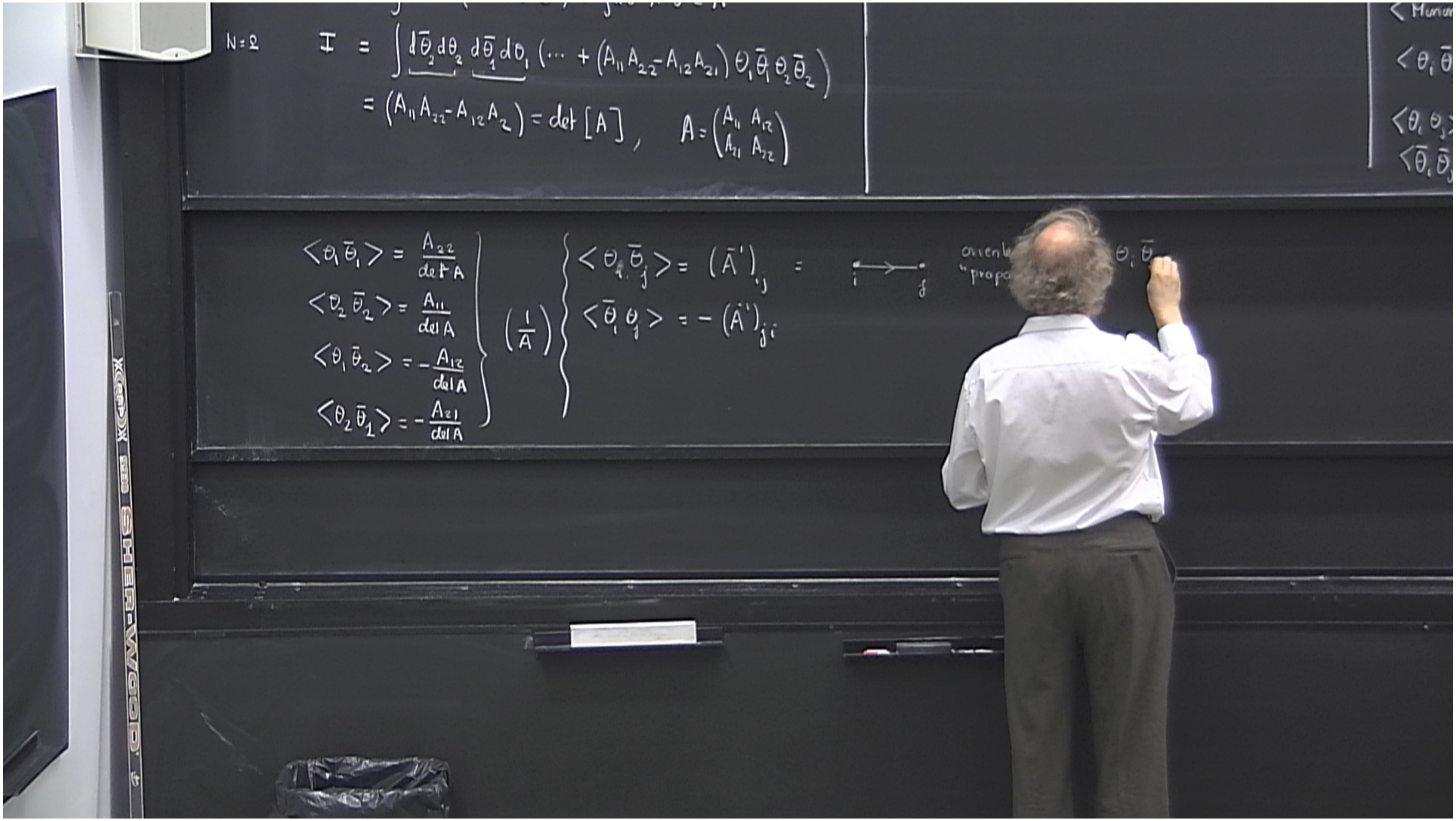
$$\langle g \rangle := \frac{\int \prod_{i=1}^{N} d\bar\theta_i \, d\theta_i \; g \cdot \exp(-\bar\theta \cdot A \cdot \theta)}{\int \prod_{i=1}^{N} d\bar\theta_i \, d\theta_i \; \exp(-\bar\theta \cdot A \cdot \theta)}$$

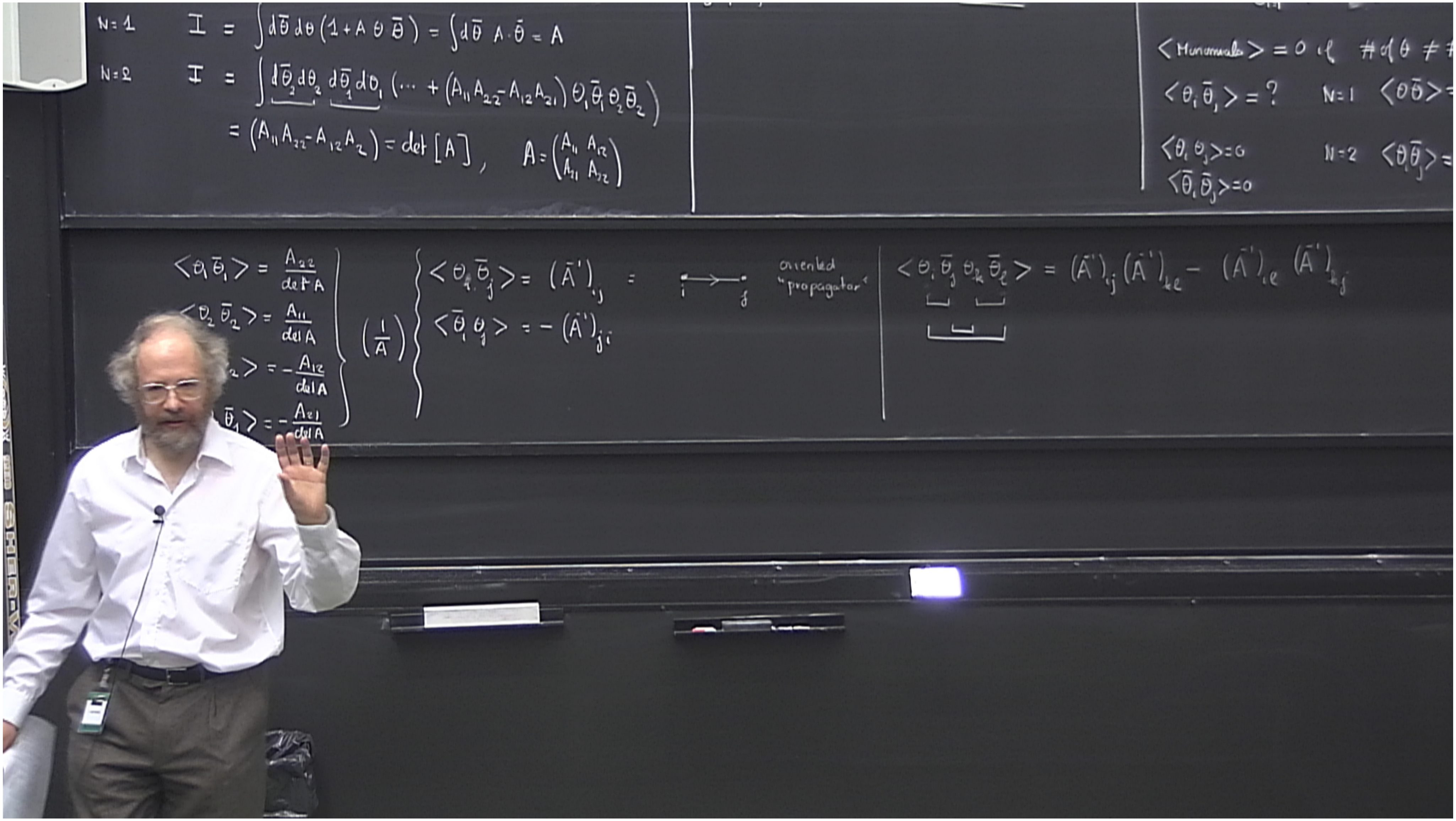

complex number
det [A]
$\int dx_i\, e^{-\frac{1}{2} x_i A_{ij} x_j} = \det(A)^{-1/2}$ Real Gaussian Integral
Real
$\int d\bar{z}_i dz_i\, e^{-\bar{z}_i A_{ij} z_j} = \det(A)^{-1}$ Complex
Complex
$\int d\bar{\theta}_i d\theta_i\, e^{-\bar{\theta}_i A_{ij} \theta_j} = \det(A)$ ← Fermions
$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$
expectation values
$g \in G_N$, A N×N matrix
$\langle g \rangle := \frac{\int \prod_{i=1}^{N} d\bar{\theta}_i d\theta_i\; g \cdot \exp(-\bar{\theta} \cdot A \cdot \theta)}{\int \prod_{i=1}^{N} d\bar{\theta}_i d\theta_i\; \exp(-\bar{\theta} \cdot A \cdot \theta)}$
< Monomials >

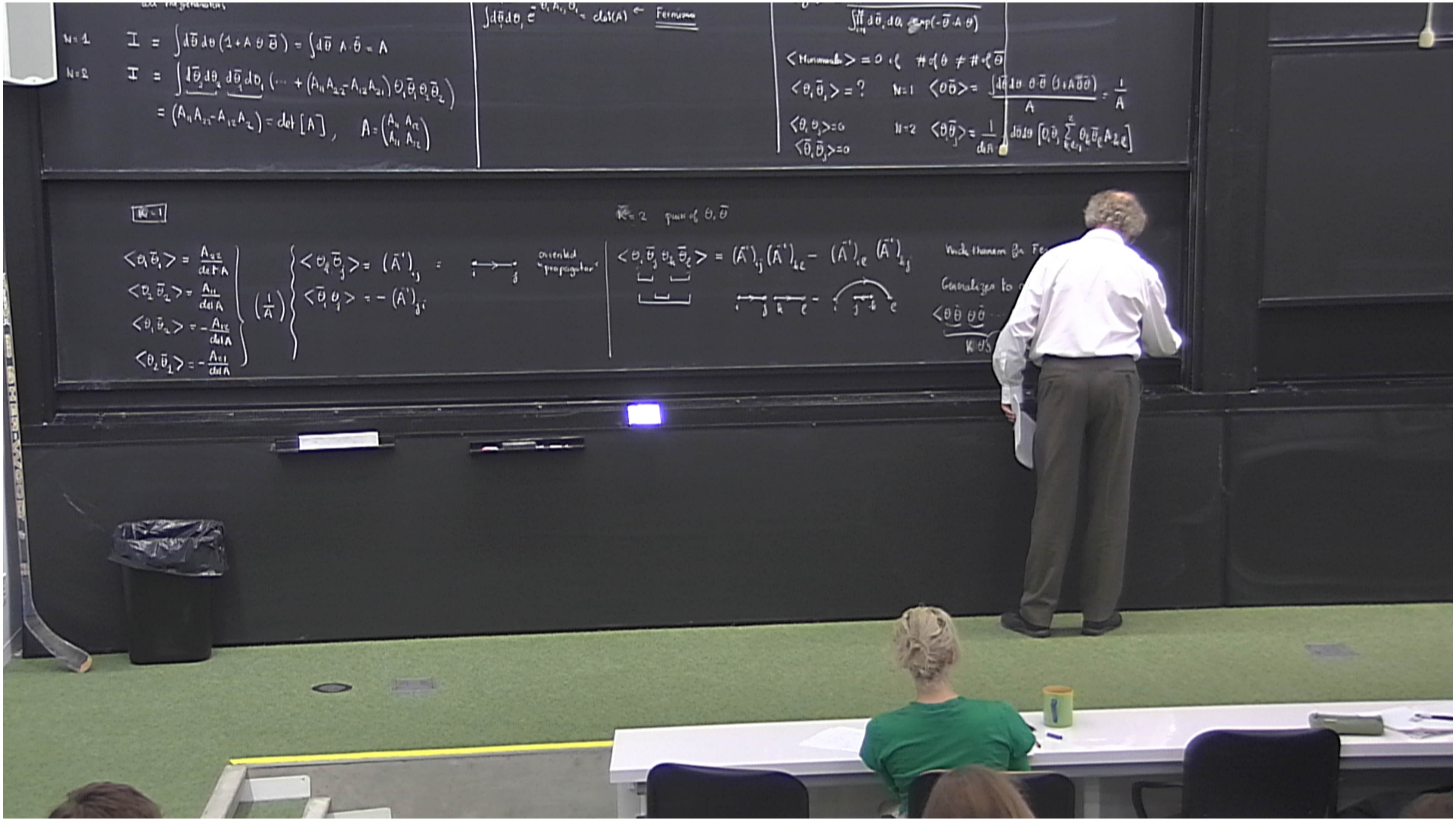

= complex number
= det [A]
$\cdot \bar\theta = A$
$A_{12}A_{21}) \theta_1 \bar\theta_1 \theta_2 \bar\theta_2)$
$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$
$\int dx_i \, e^{-\frac{1}{2} x_i A_{ij} x_j} = \det(A)^{-1/2}$ Real Gaussian Integral
Real
$\int d\bar z_i dz_i \, e^{-\bar z_i A_{ij} z_i} = \det(A)^{-1}$ Complex " "
Complex
$\int d\bar\theta_i d\theta_j \, e^{-\bar\theta_i A_{ij} \theta_i} = \det(A)$ ← Fermions
expectation value
$g \in G_N$, A N×N matrix
$\langle g \rangle := \frac{\int \prod_{i=1}^{N} d\bar\theta_i d\theta_i \; g \cdot \exp(-\bar\theta \cdot A \cdot \theta)}{\int \prod_{i=1}^{N} d\bar\theta_i d\theta_i \; \exp(-\bar\theta \cdot A \cdot \theta)}$
$\langle$ Monomials $\rangle = 0$ if # of $\theta \neq$ # of $\bar\theta$
$\langle \theta_i \bar\theta_j \rangle = ?$ N=1 $\langle \theta \bar\theta \rangle =$
$\langle \theta_i \theta_j \rangle = 0$
$\langle \bar\theta_i \bar\theta_j \rangle = 0$

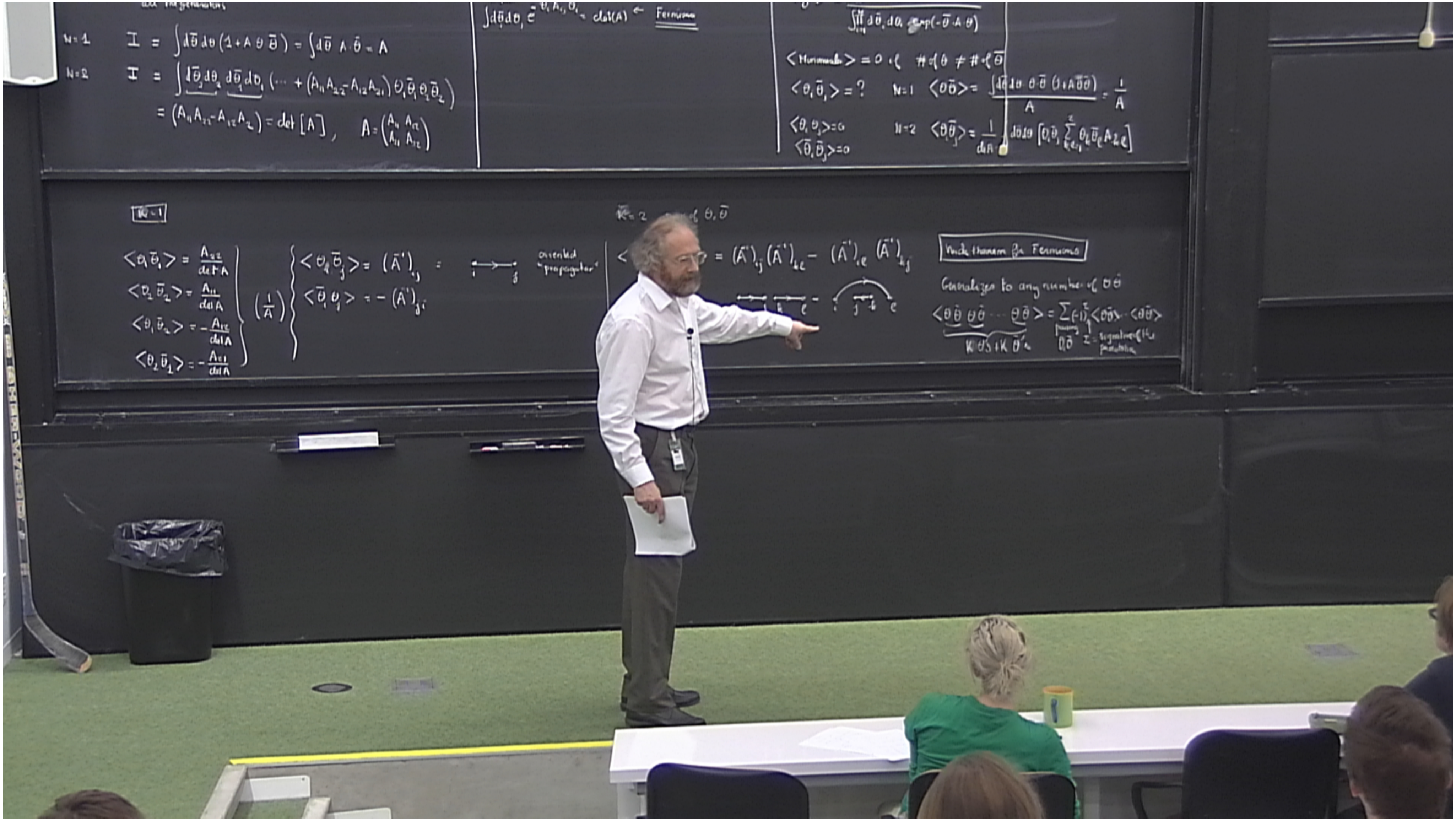

complex number
det [A]
Real Gaussian Integral
Real
Complex " "
Complex
Fermions
expectation value
g ∈ G_N , A N×N matrix
<Monomials> = 0 if
<θ_i θ̄_j> = ?
<θ_i θ_j>=0
<θ̄_i θ̄_j>=0

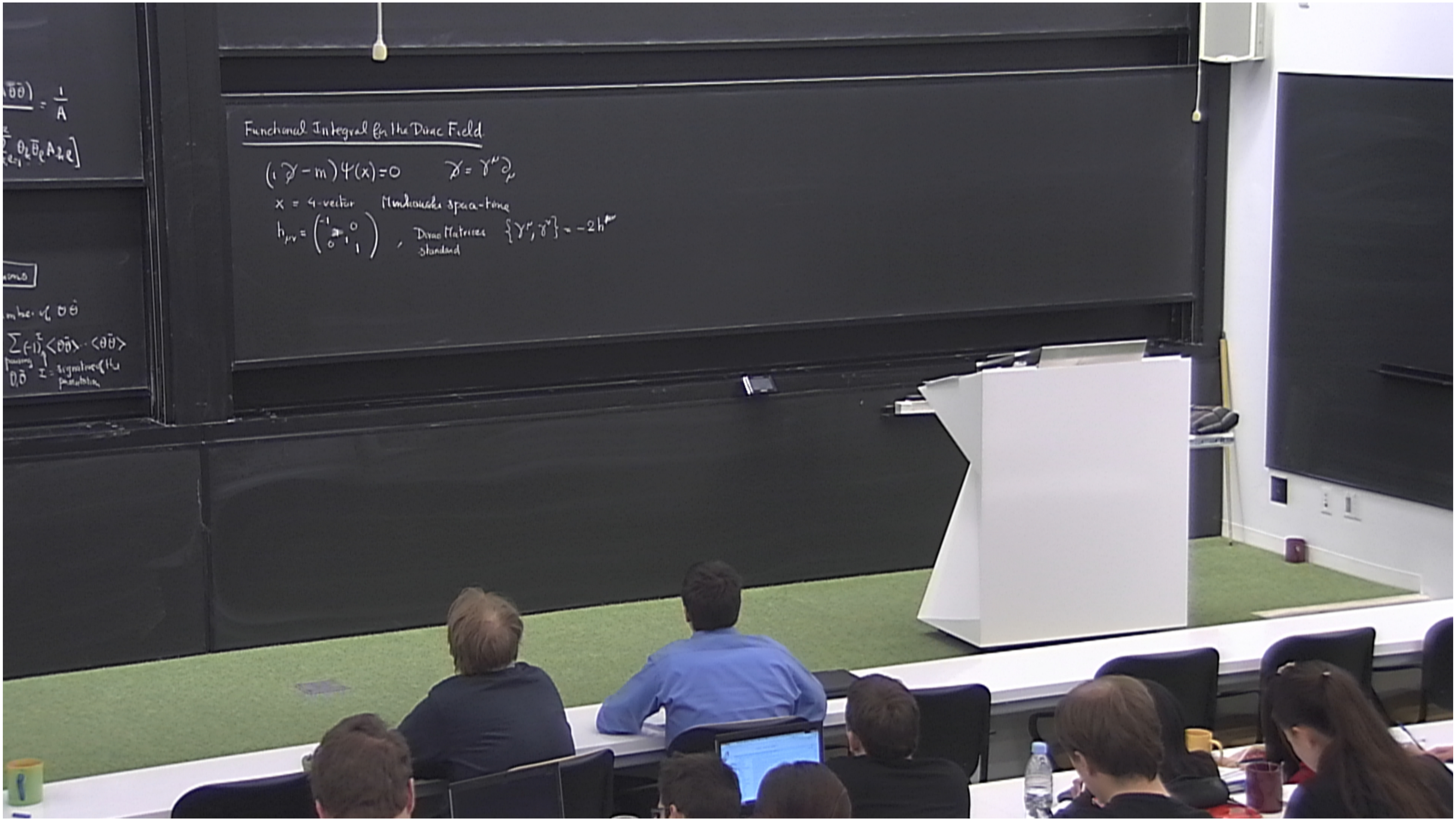

complex number
det [A]
Real Gaussian Integral
Real
Complex " "
Complex.
Fermions
expectation value
g ∈ G_N , A N×N matrix
<Monomials> = 0

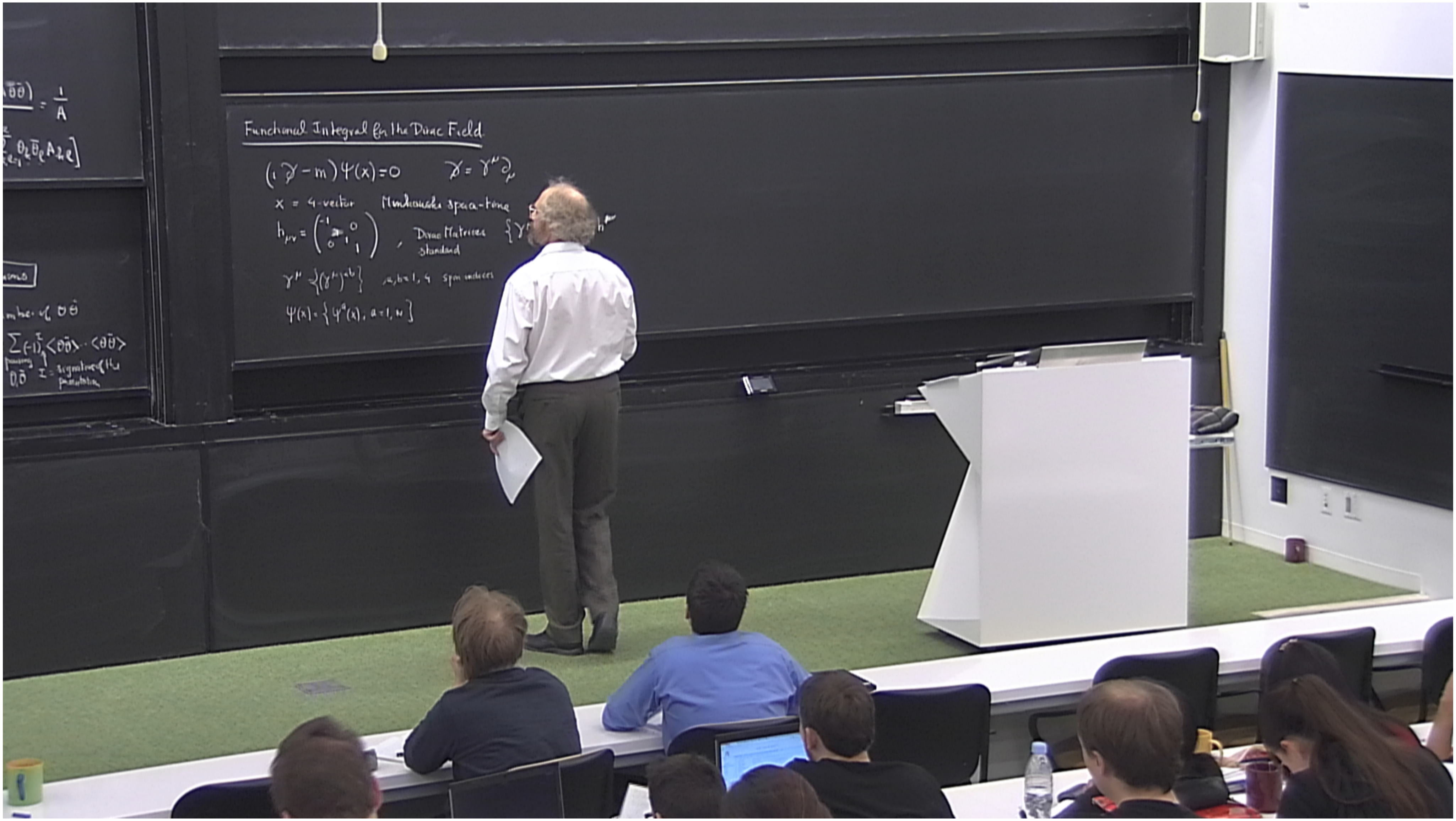

N=2
I = ∫ dθ̄2 dθ2 dθ̄1 dθ1 (… + (A11 A22 − A12 A21) θ1 θ̄1 θ2 θ̄2)
= (A11 A22 − A12 A21) = det [A], A = (A11 A12 / A21 A22)
⟨θ1 θ̄1⟩ = A22 / det A
⟨θ2 θ̄2⟩ = A11 / det A
⟨θ1 θ̄2⟩ = − A12 / det A
⟨θ2 θ̄1⟩ = − A21 / det A
(1/A)
⟨θi θ̄j⟩ = (A⁻¹)ij
⟨θ̄i θj⟩ = − (A⁻¹)ji
SHER-WOOD

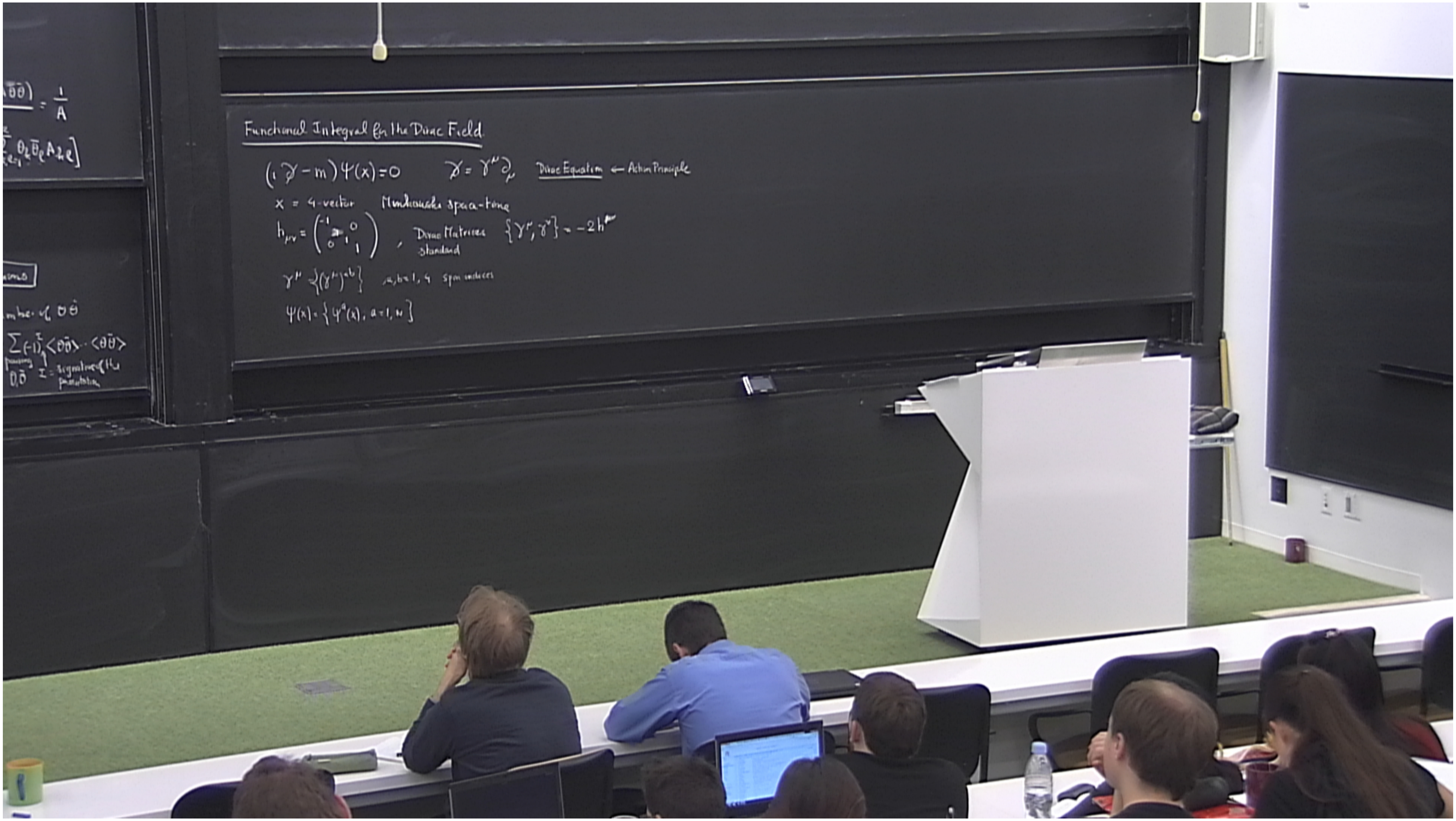

N=1
N=2
= det [A]
< Monomials > = 0 if
< θ₁ θ̄₁ > = ?
N=1
N=2
oriented
"propagator"

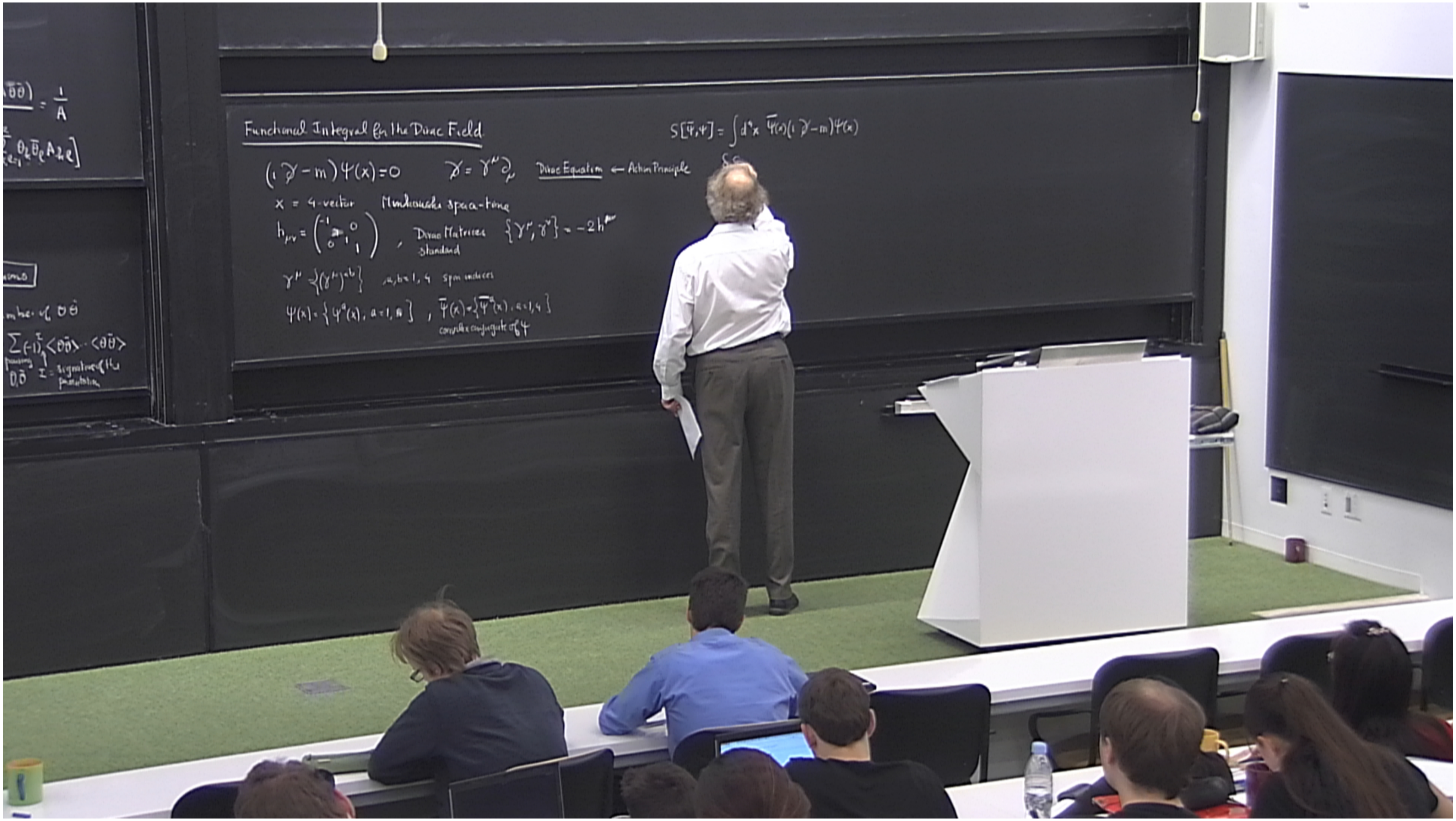

Fermions
oriented
"propagator"

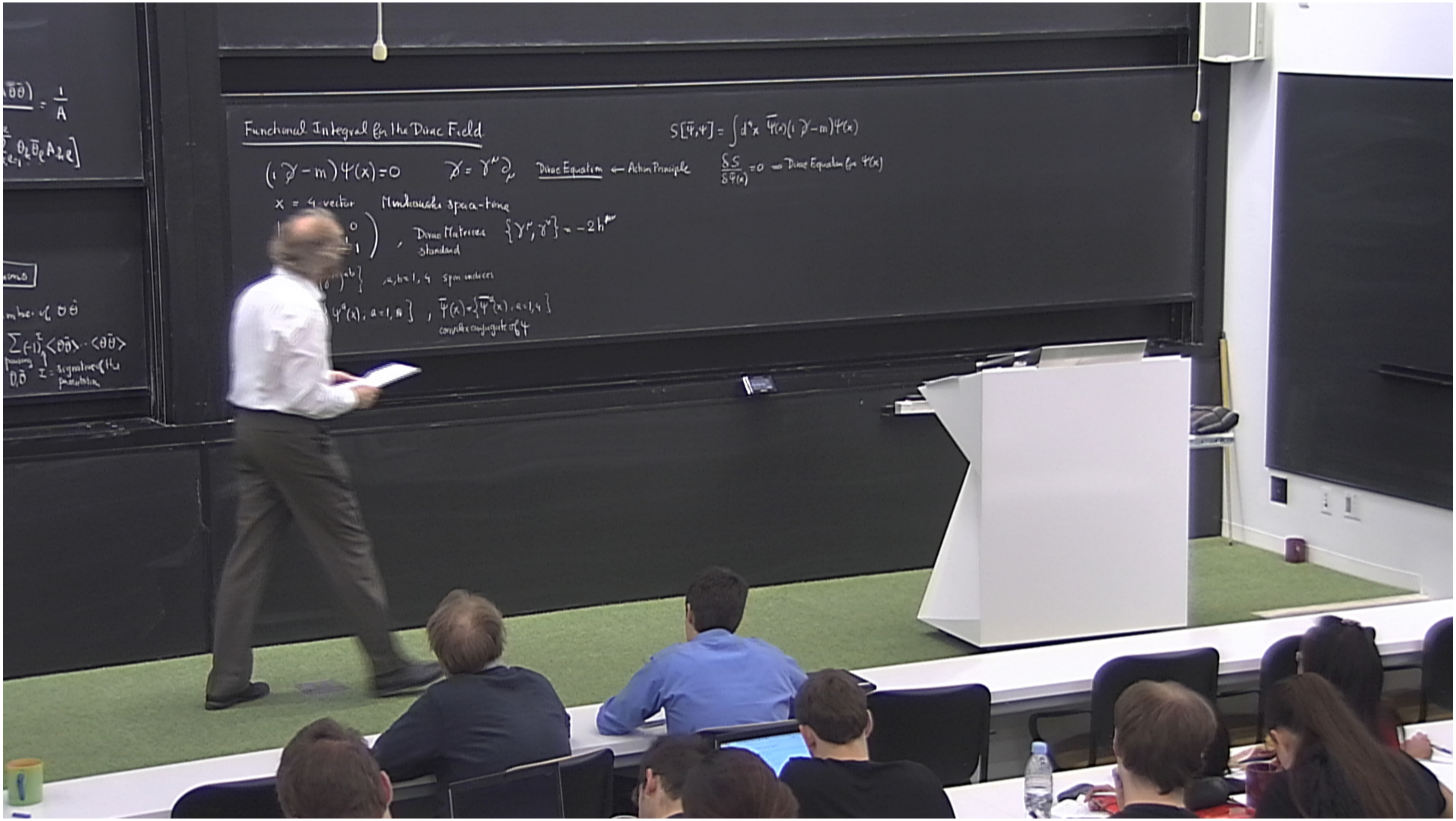


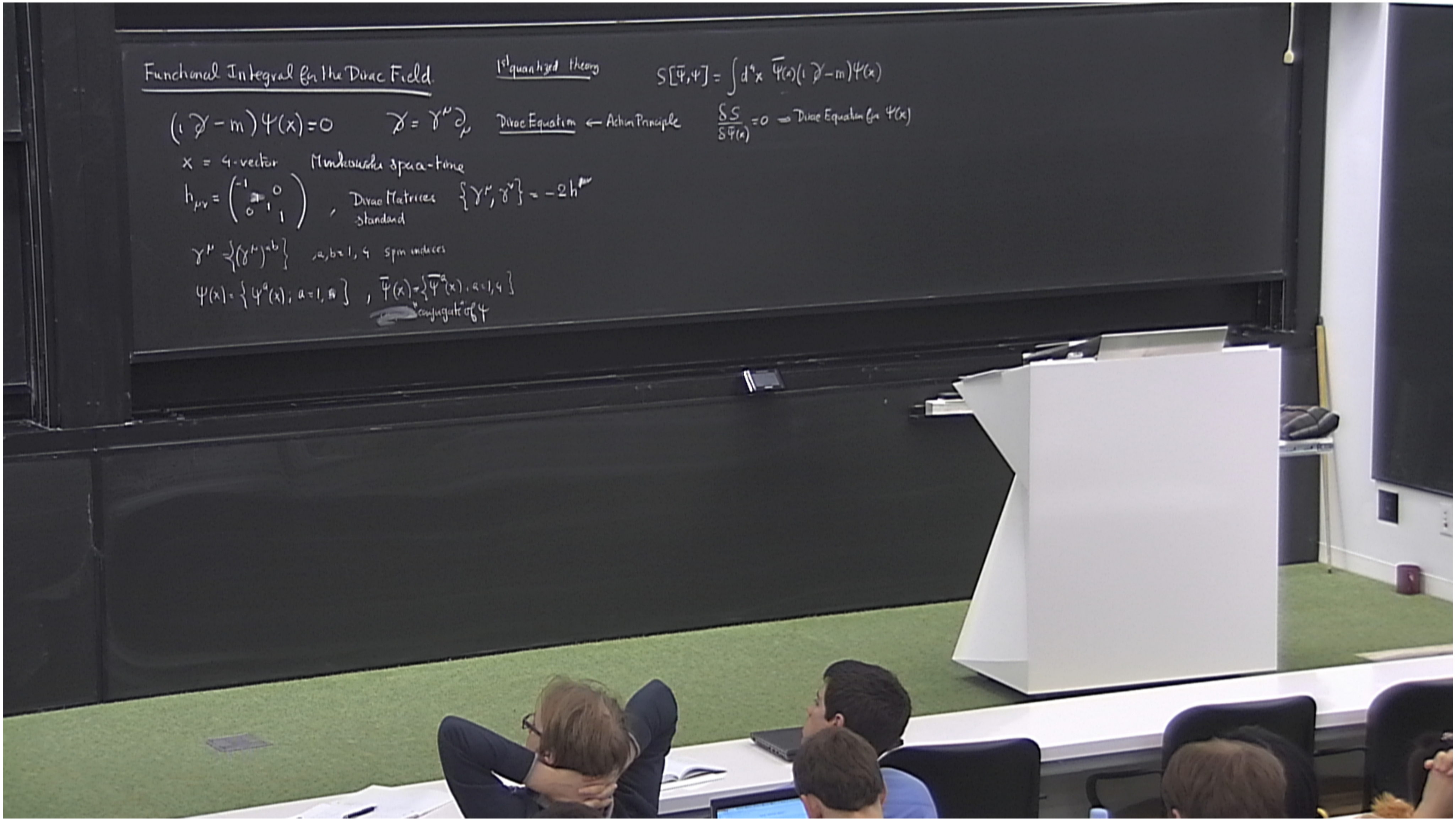

Functional Integral for the Dirac Field
x = 4-vector Minkowski space-time
Dirac Matrices
standard

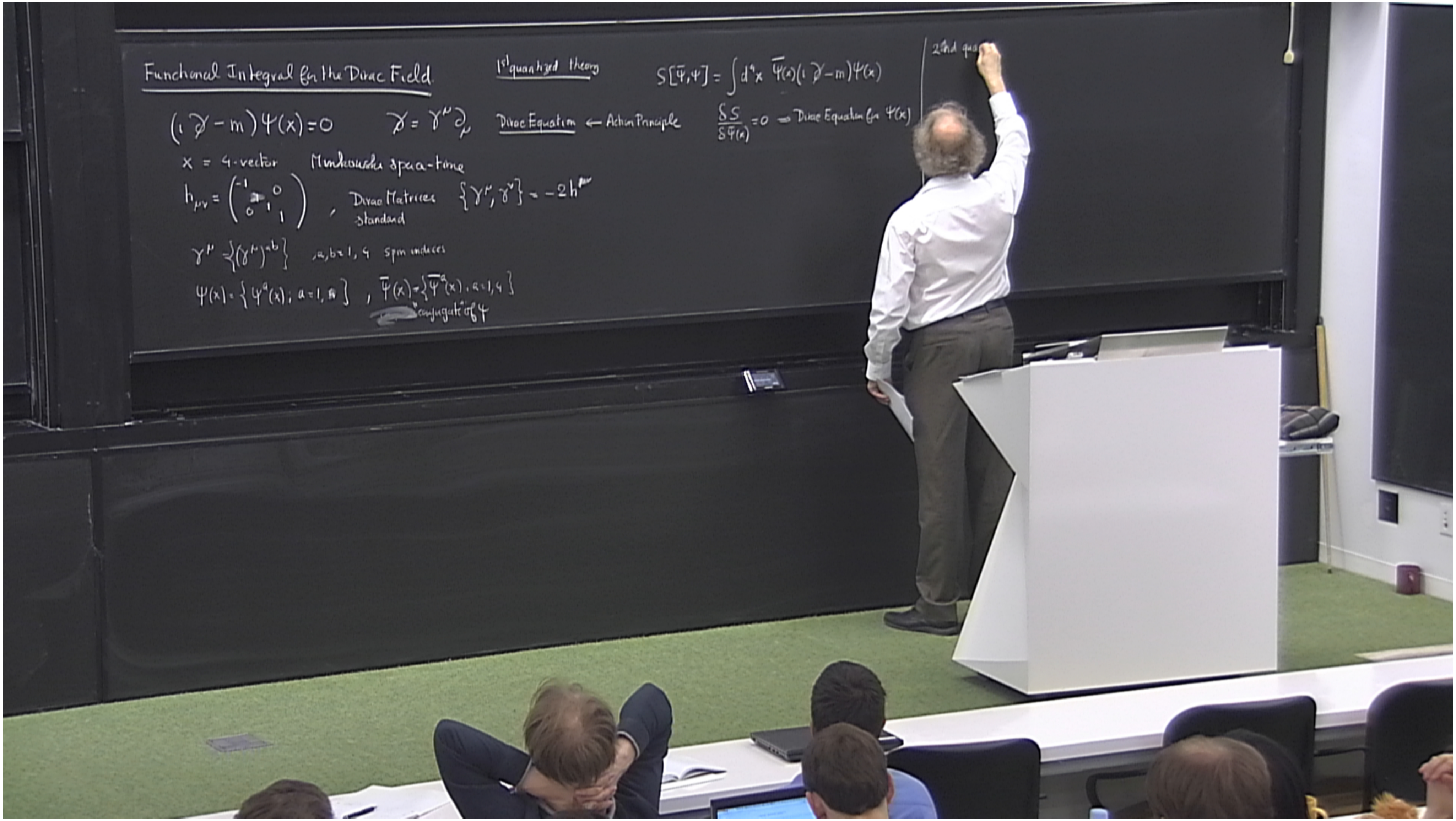

Functional Integral for the Dirac Field
x = 4-vector   Minkowski space-time
Dirac Matrices
standard
spin indices

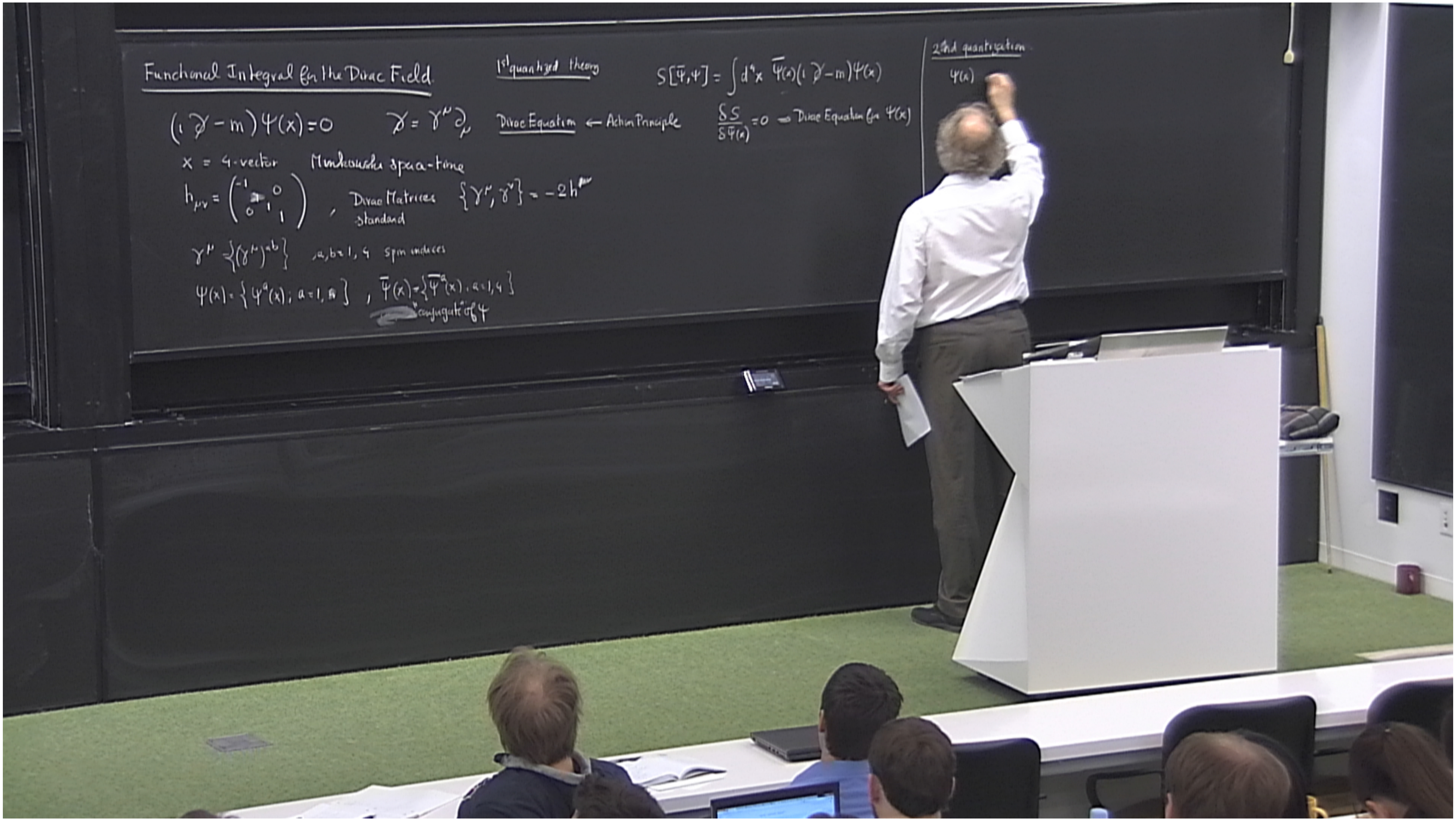

Functional Integral for the Dirac Field
Dirac Equation ← Action Principle
x = 4-vector
Dirac Matrices
standard
spin indices

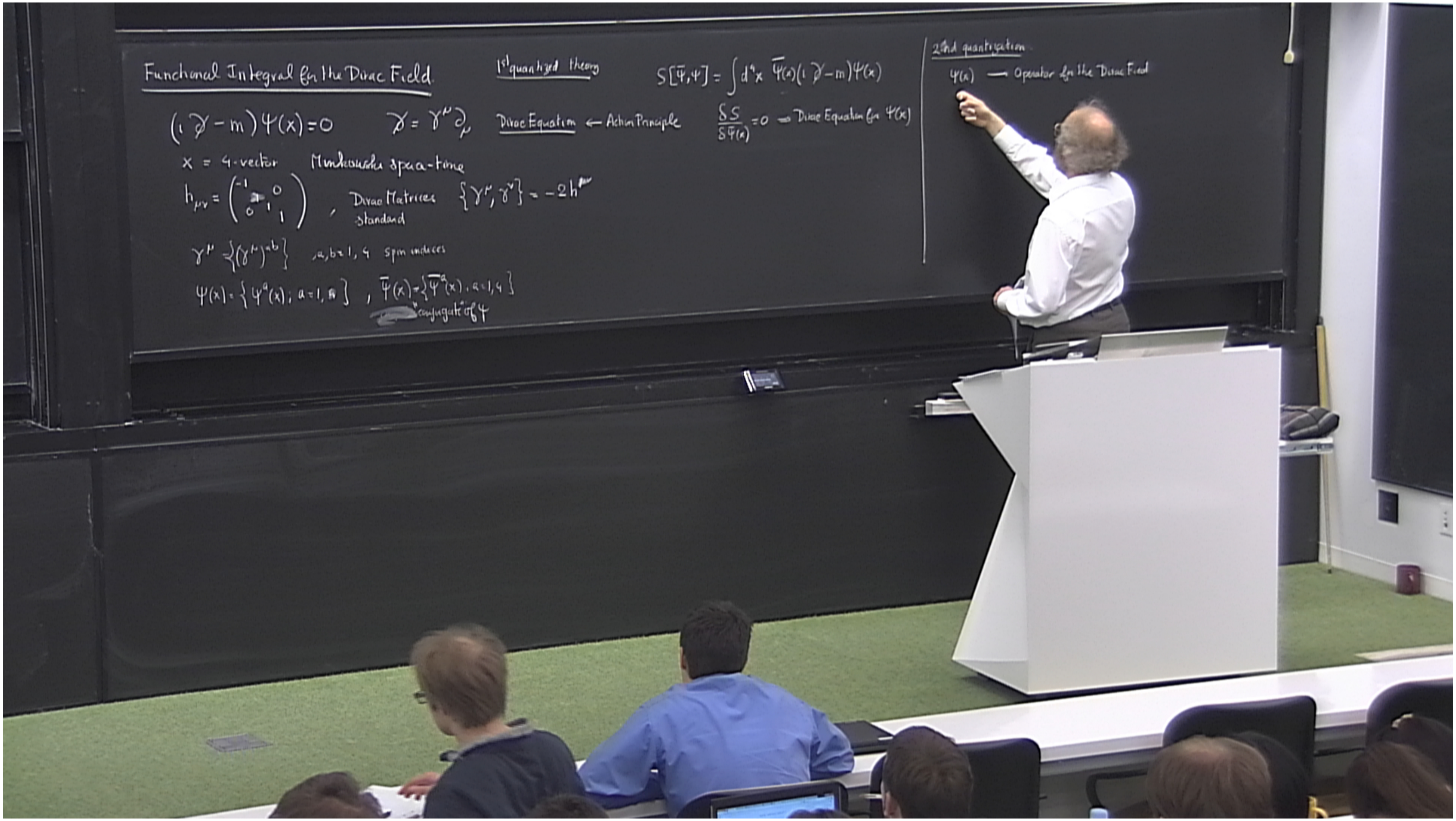

Functional Integral for the Dirac Field
S[Ψ̄,Ψ] = ∫d⁴x Ψ̄(x)(i ∂̸ − m)Ψ(x)
(i ∂̸ − m)Ψ(x) = 0
∂̸ = γ^μ ∂_μ
Dirac Equation ← Action Principle
x = 4-vector
Minkowski space-time
Dirac Matrices
standard
{γ^μ, γ^ν} = −2h^μν
spin indices
complex conjugate of Ψ

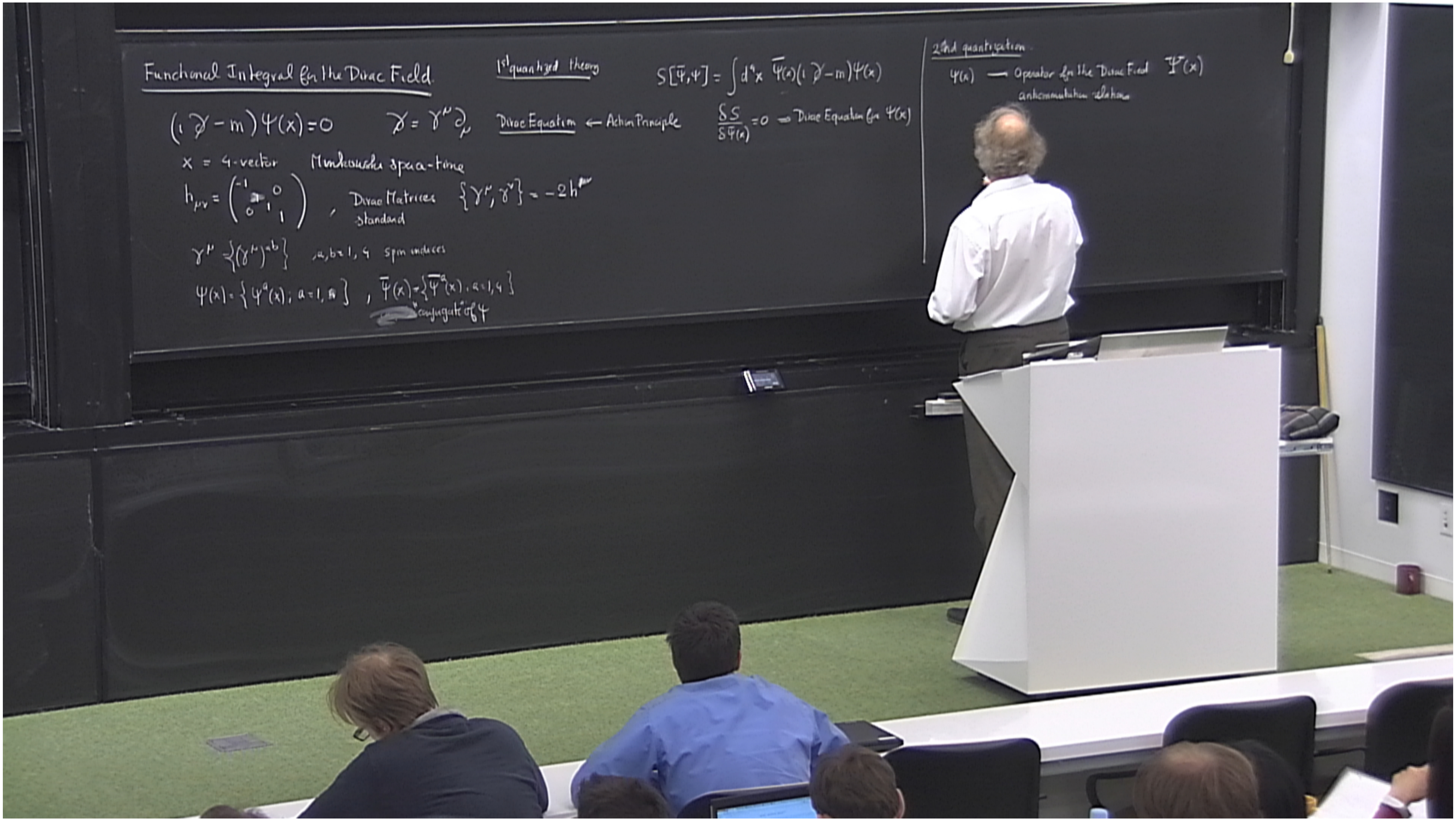

Functional Integral for the Dirac Field
Dirac Equation ← Action Principle
Minkowski space-time
Dirac Matrices
standard
spin indices
complex conjugate of ψ
Dirac Equation for ψ(x)

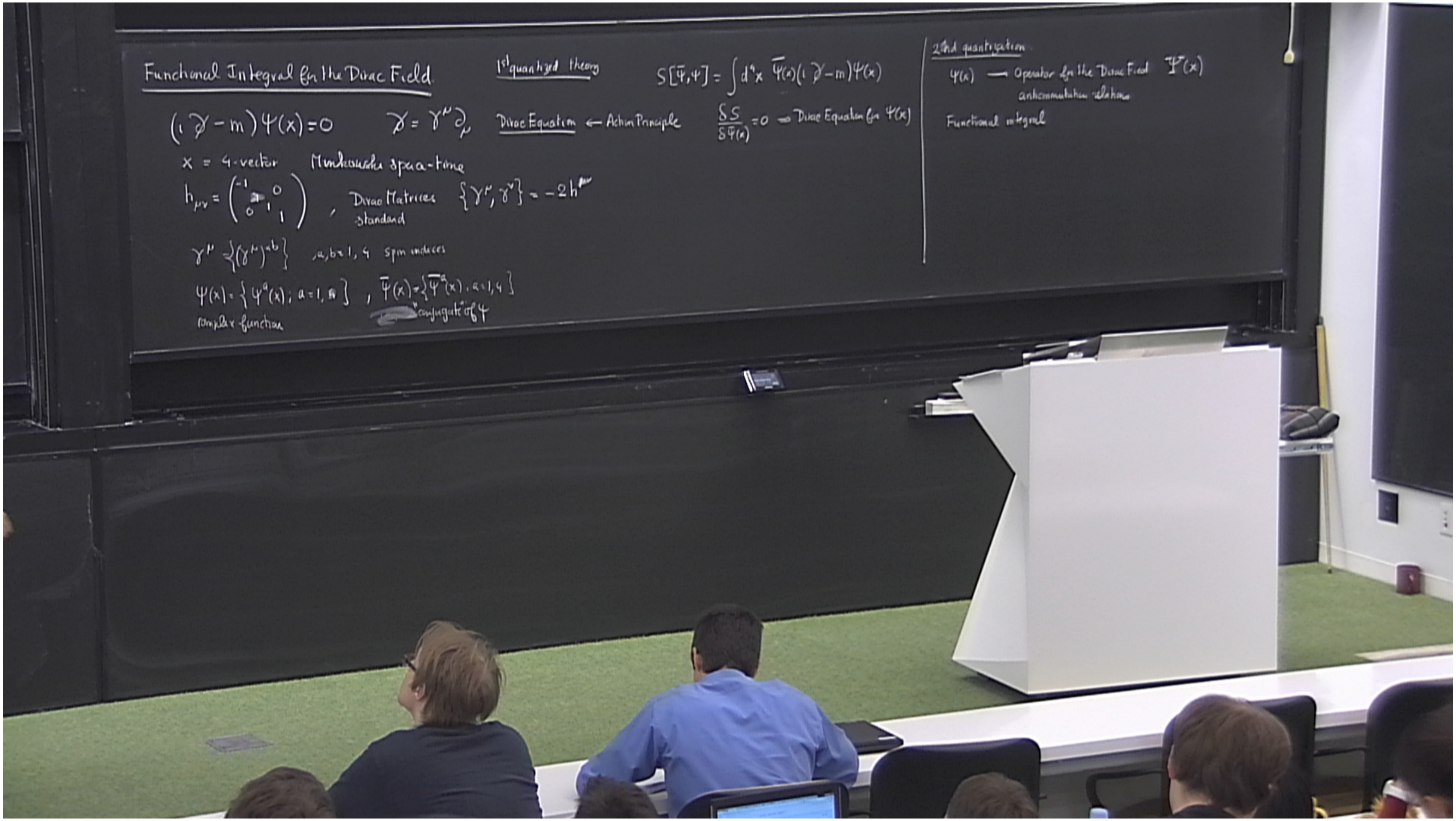

Functional Integral for the Dirac Field
1st quantized theory
S[Ψ̄,Ψ] = ∫d⁴x Ψ̄(x)(i ∂̸ − m)Ψ(x)
(i ∂̸ − m)Ψ(x) = 0
∂̸ = γ^μ ∂_μ
Dirac Equation ← Action Principle
δS/δΨ̄(x) = 0 ⇒ Dirac Equation for Ψ(x)
x = 4-vector
Minkowski space-time
Dirac Matrices
standard
{γ^μ, γ^ν} = −2h^μν
a,b = 1,4 spin indices
conjugate of Ψ

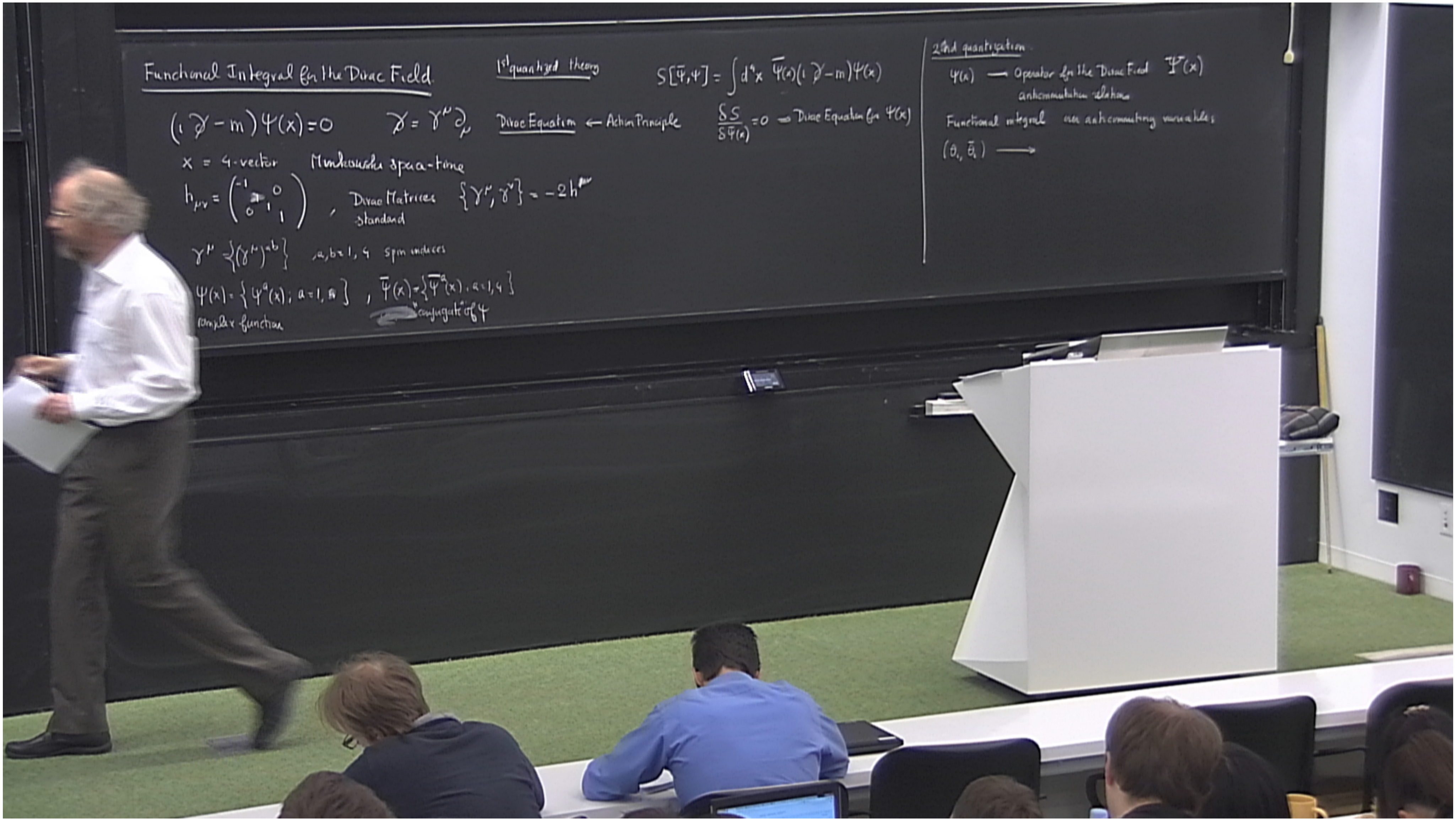

Functional Integral for the Dirac Field
1st quantized theory
Dirac Equation ← Action Principle
x = 4-vector
Dirac Matrices
standard
spin indices
conjugate of ψ

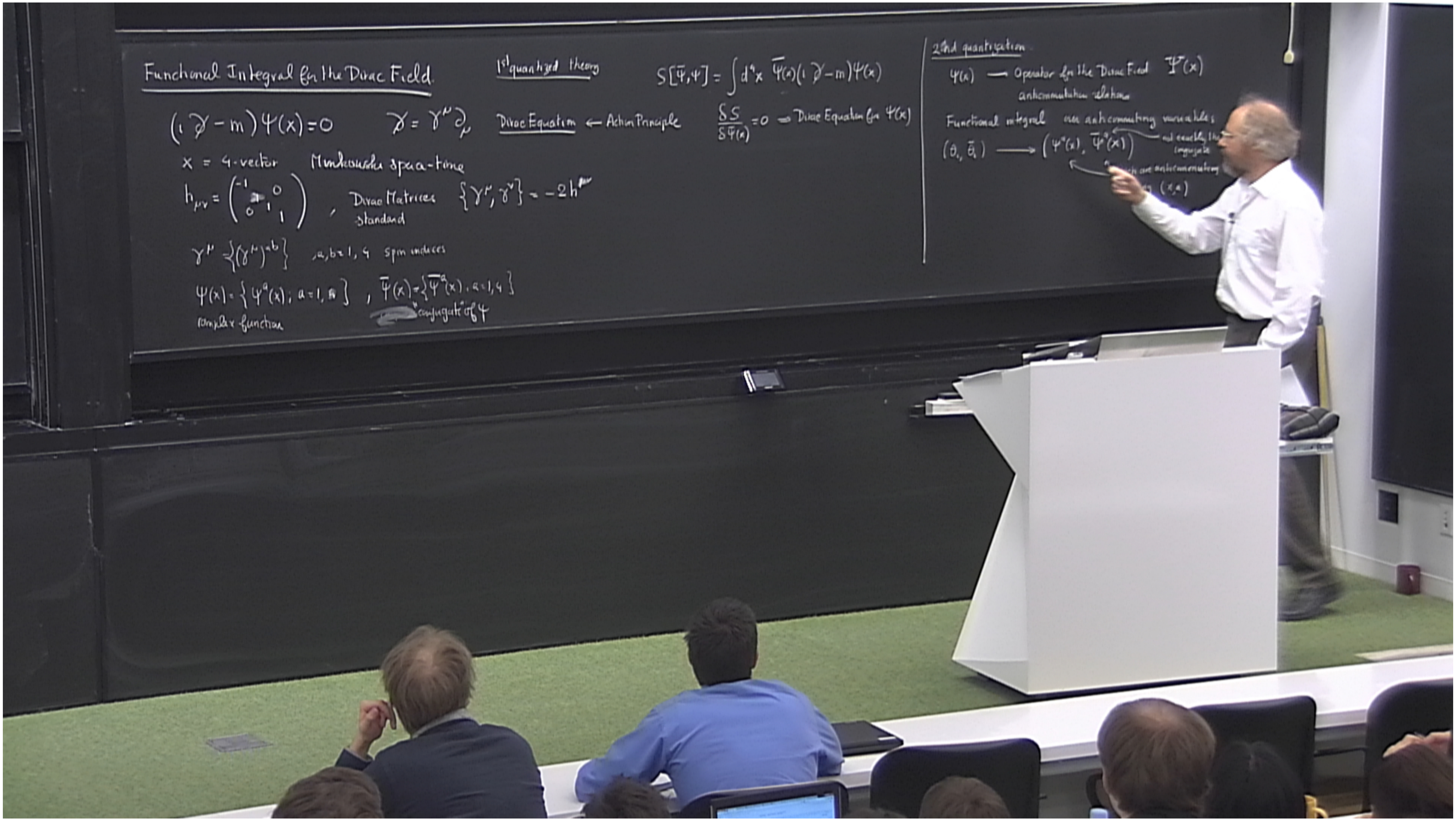

Functional Integral for the Dirac Field
1st quantized theory
2nd quantization
Dirac Equation ← Action Principle
x = 4-vector Minkowski space-time
Dirac Matrices
standard
spin indices
conjugate of ψ

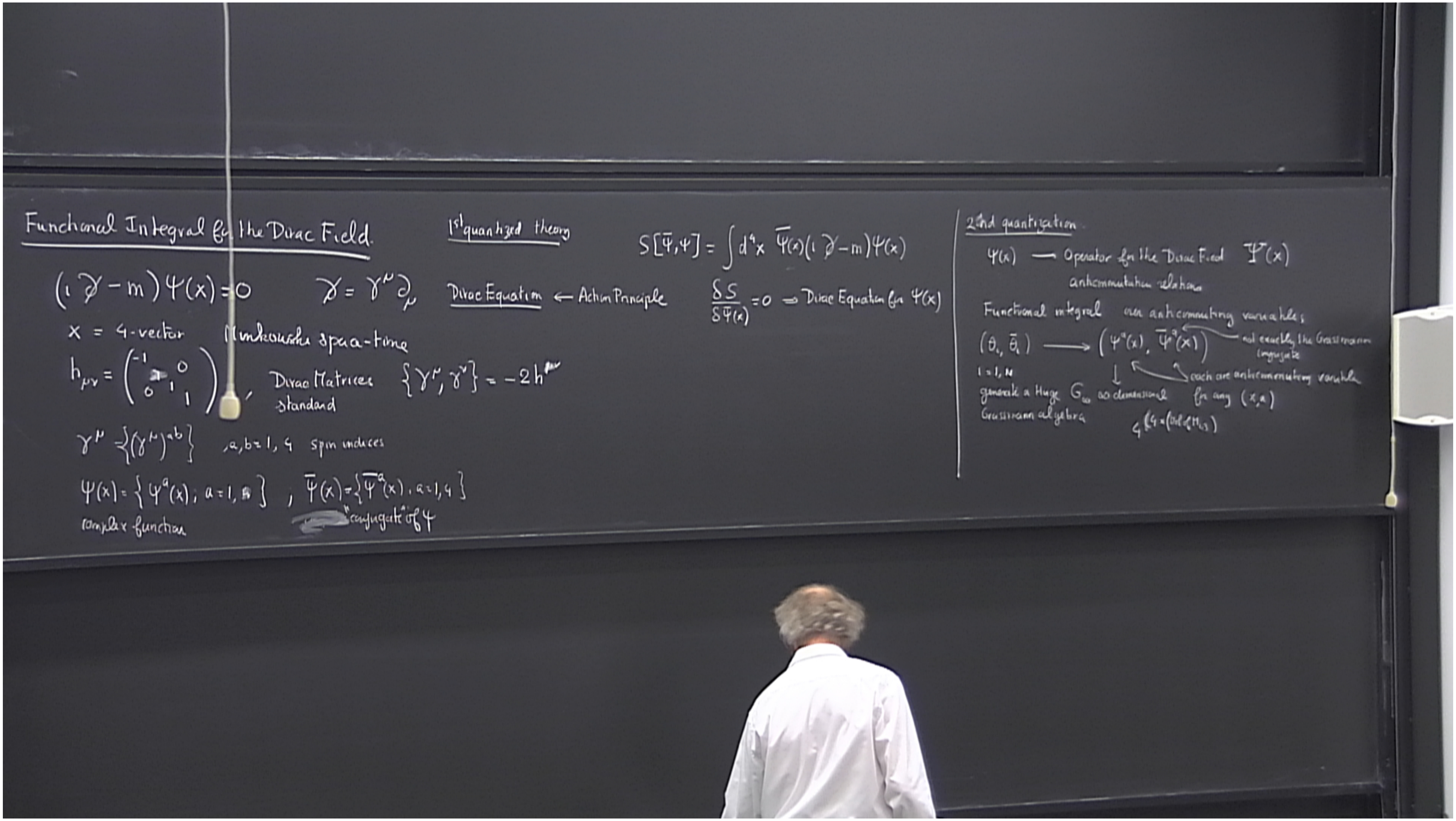

Functional Integral for the Dirac Field
1st quantized theory
Dirac Equation ← Action Principle
Dirac Equation for Ψ(x)
x = 4-vector
Minkowski space-time
Dirac Matrices
standard
spin indices
conjugate of Ψ
2nd quantization
Operator for the Dirac Field

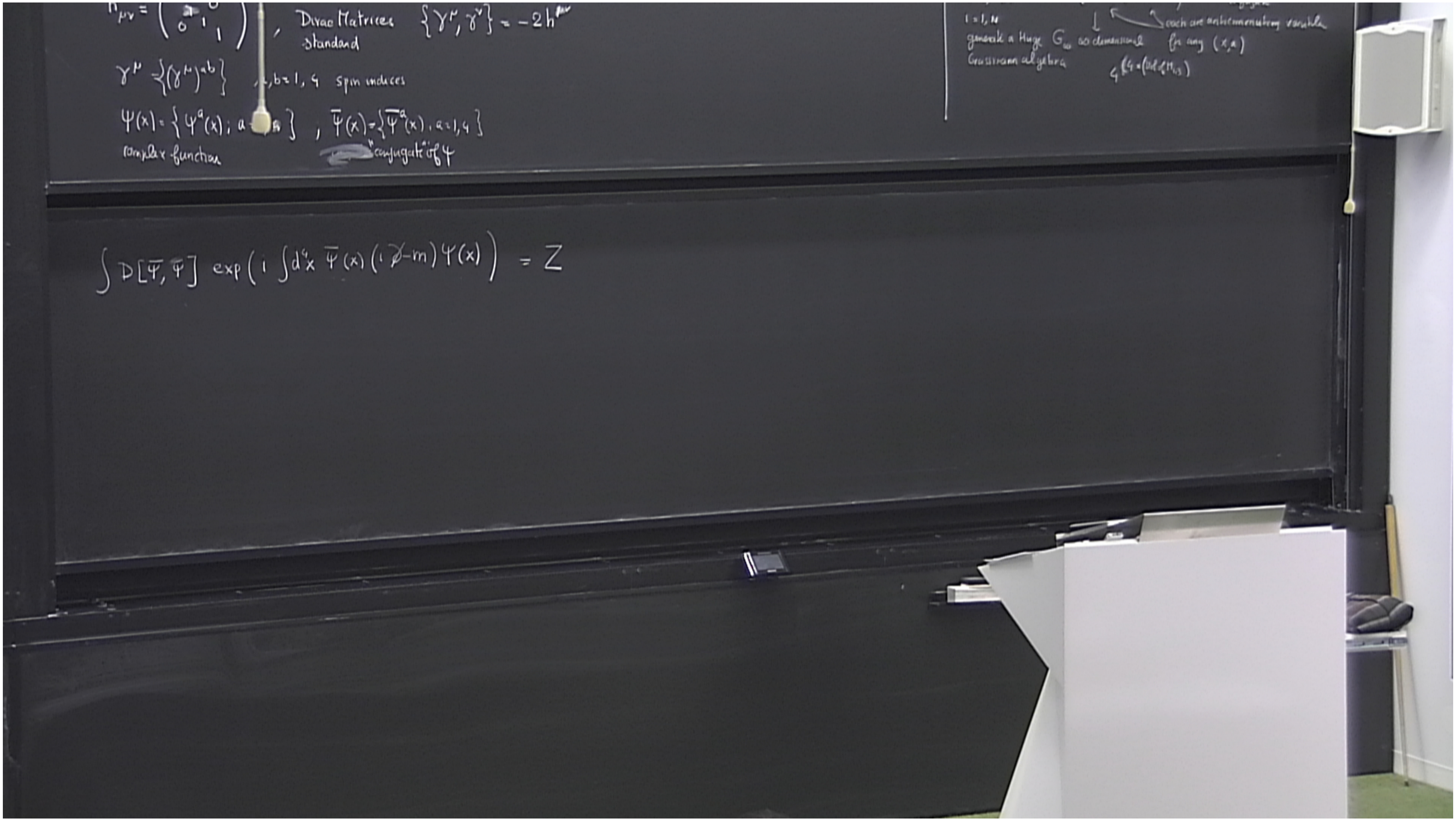

Functional Integral for the Dirac Field
1st quantized theory
Dirac Equation ← Action Principle
Dirac Equation for Ψ(x)
x = 4-vector
Minkowski space-time
Dirac Matrices
standard
spin indices
conjugate of Ψ
2nd quantization
Operator for the Dirac Field
anticommutation relations


Functional Integral for the Dirac Field
1st quantized theory
2nd quantization
Dirac Equation ← Action Principle
Dirac Matrices
standard
spin indices
complex function
conjugate of ψ
Operator for the Dirac Field
anticommutation relations
Functional integral
x = 4-vector Minkowski space-time


Functional Integral for the Dirac Field
1st quantized theory
$S[\bar\Psi,\Psi] = \int d^4x\, \bar\Psi(x)(i\partial\!\!\!/ - m)\Psi(x)$
$(i\partial\!\!\!/ - m)\Psi(x) = 0$
$\partial\!\!\!/ = \gamma^\mu \partial_\mu$
Dirac Equation ← Action Principle
$\frac{\delta S}{\delta \bar\Psi(x)} = 0$ ⇒ Dirac Equation for $\Psi(x)$
x = 4-vector
Minkowski space-time
Dirac Matrices
standard
$\{\gamma^\mu, \gamma^\nu\} = -2h^{\mu\nu}$
spin indices
complex function
conjugate of Ψ
2nd quantization
Operator for the Dirac Field
anticommutation relations
Functional integral over anticommuting variables


Functional Integral for the Dirac Field
1st quantized theory
Dirac Equation ← Action Principle
Dirac Equation for Ψ(x)
x = 4-vector
Dirac Matrices
standard
spin indices
complex function
2nd quantization
Operator for the Dirac Field
anticommutation relations
Functional integral over anticommuting variables


Functional Integral for the Dirac Field
1st quantized theory
$(i\not\partial - m)\psi(x) = 0$
$\not\partial = \gamma^\mu \partial_\mu$
Dirac Equation ← Action Principle
$S[\bar\psi,\psi] = \int d^4x\, \bar\psi(x)(i\not\partial - m)\psi(x)$
$\frac{\delta S}{\delta \bar\psi(x)} = 0$ ⇒ Dirac Equation for $\psi(x)$
x = 4-vector Minkowski space-time
Dirac Matrices
standard
$\{\gamma^\mu, \gamma^\nu\} = -2h^{\mu\nu}$
a,b = 1,4 spin indices
complex function
conjugate of ψ
2nd quantization
$\psi(x)$ — Operator for the Dirac Field $\Psi(x)$
anticommutation relations
Functional integral over anticommuting variables
$(\theta_i, \bar\theta_i) \longrightarrow (\psi^a(x), \bar\psi^a(x))$
not exactly the Grassmann conjugate
each are anticommuting variable for any (x,a)
i = 1, N
generate a Huge G_∞ ∞ dimensional Grassmann algebra


Dirac Matrices {γ^μ, γ^ν} = -2η^μν
standard
spin indices
complex function
conjugate of Ψ
each are anticommuting variable
Grassmann algebra
∫D[Ψ̄, Ψ] exp(i ∫d⁴x Ψ̄(x)(i∂̸-m)Ψ(x)) = Z

$\eta_{\mu\nu} = \begin{pmatrix} \ & & \\ 0 & 1 & \\ & & 1 \end{pmatrix}$, Dirac Matrices $\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu}$

standard

$\gamma^\mu = \{(\gamma^\mu)^{ab}\}$ a,b = 1, 4 spin indices

$\Psi(x) = \{\Psi^a(x);\ a = \ldots\}$, $\bar{\Psi}(x) = \{\bar{\Psi}^a(x),\ a = 1,4\}$

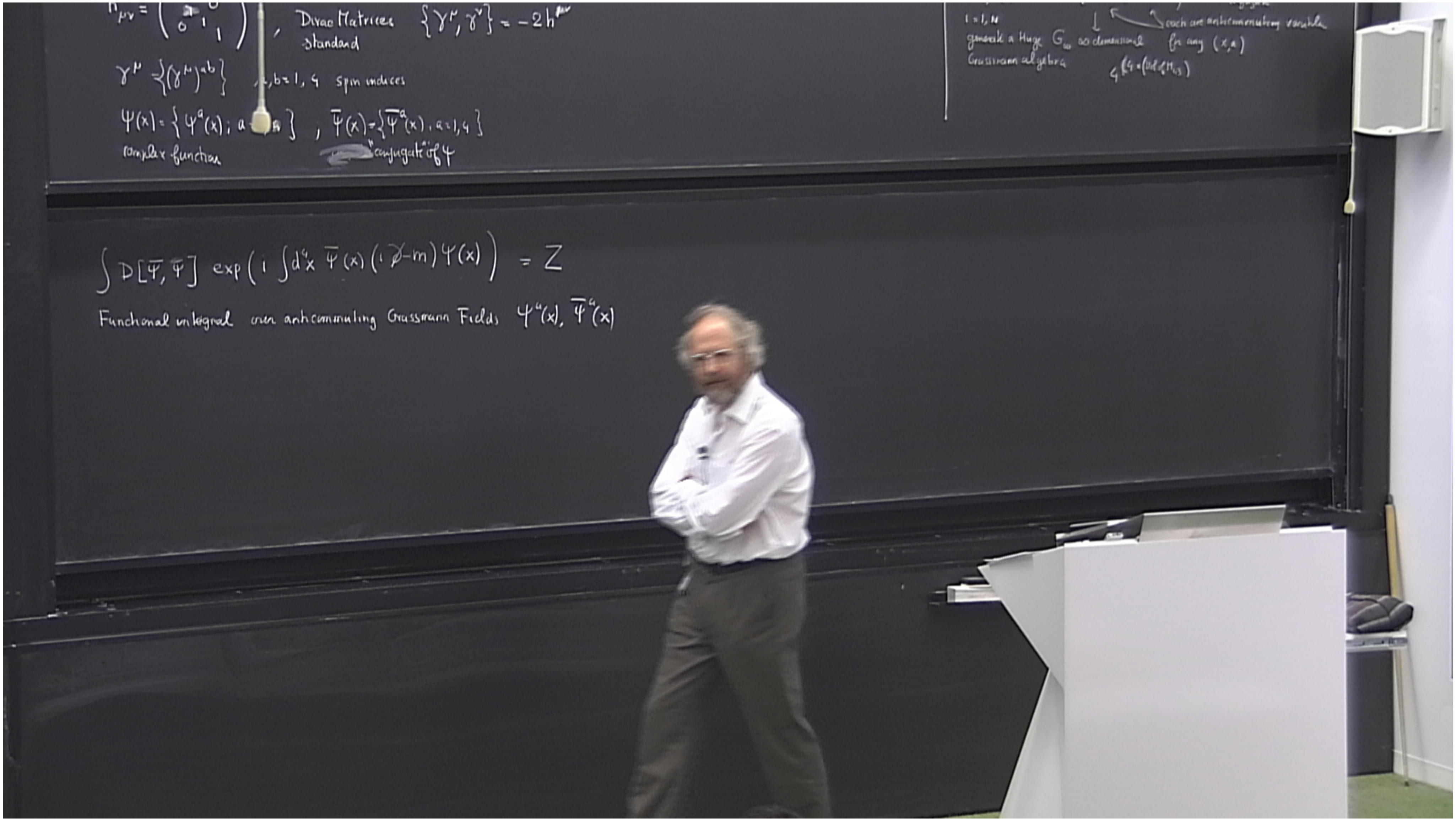
complex function "conjugate" of Ψ

i = 1, N — each are anticommuting variable

generate a Huge G_∞ ∞ dimensional Grassmann algebra — for any (x, a)

$$\int D[\bar{\Psi}, \Psi]\ \exp\left(i \int d^4x\ \bar{\Psi}(x)\,(i\not{\partial} - m)\,\Psi(x)\right) = Z$$

Functional integral over anticommuting Grassmann Fields $\Psi^a(x)$, $\bar{\Psi}^a(x)$


Dirac Matrices $\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu}$
standard
$\gamma^\mu = \{(\gamma^\mu)^{ab}\}$ a,b = 1, 4 spin indices
$\Psi(x) = \{\Psi^a(x);\ a = 1,4\}$, $\bar\Psi(x) = \{\bar\Psi^a(x),\ a = 1,4\}$
complex function
"conjugate" of Ψ
$\int D[\bar\Psi, \Psi]\ \exp\left(i \int d^4x\ \bar\Psi(x)(i\not\partial - m)\Psi(x)\right) = Z$
Functional integral over anticommuting Grassmann Fields $\Psi^a(x)$, $\bar\Psi^a(x)$

, Dirac Matrices $\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu}$

standard

$\gamma^\mu = \{(\gamma^\mu)^{ab}\}$ a, b = 1, 4 spin indices

$\psi(x) = \{\psi^a(x);\ a = 1,4\}$, $\bar\psi(x) = \{\bar\psi^a(x),\ a = 1,4\}$

complex function "conjugate" of ψ

i = 1, N

generate a Huge G_∞ ∞ dimensional Grassmann algebra

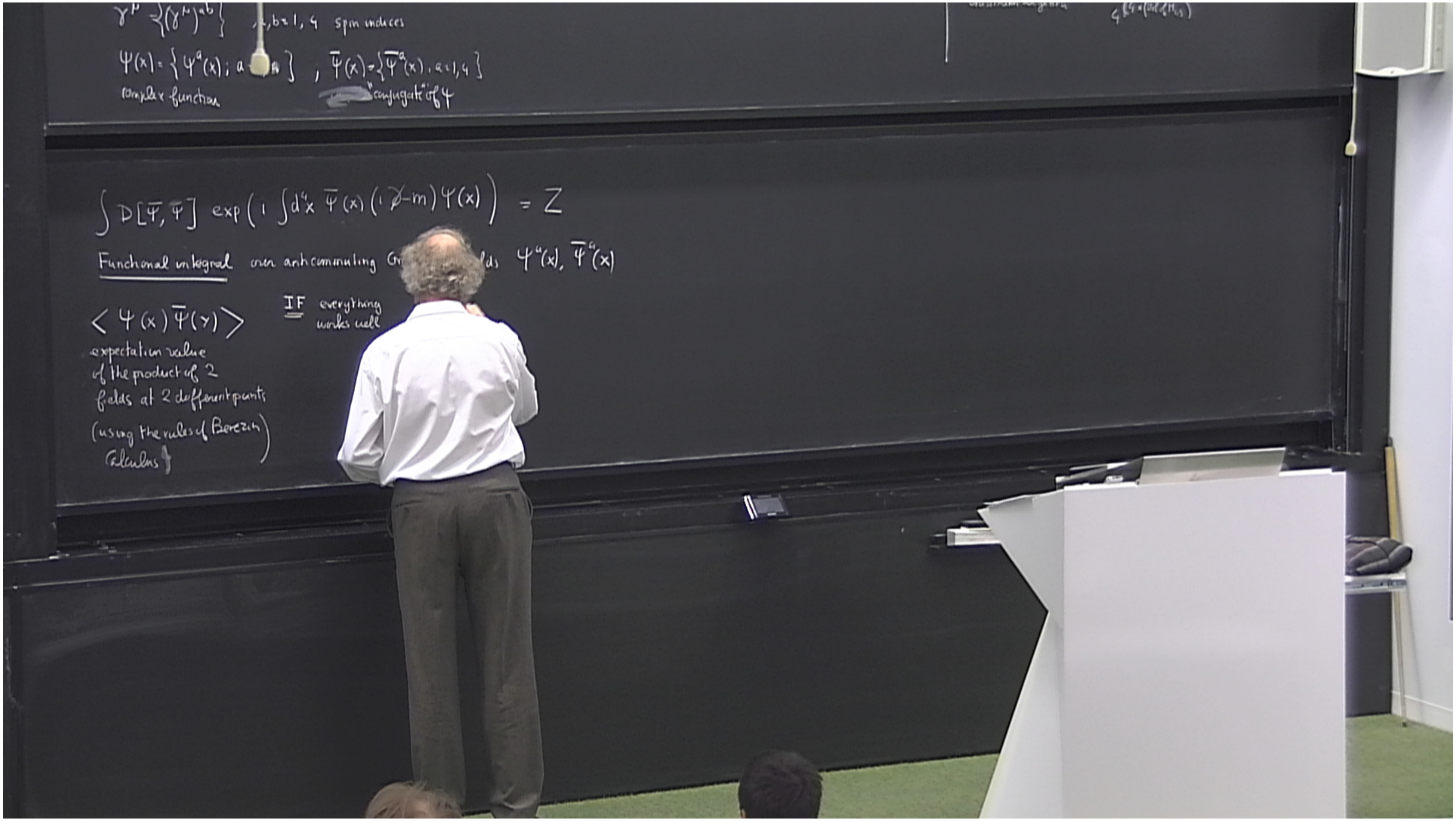
each are anticommuting variable for any (x, a)

$$\int D[\bar\psi, \psi]\ \exp\left(i \int d^4x\ \bar\psi(x)\,(i\partial\!\!\!/ - m)\,\psi(x)\right) = Z$$

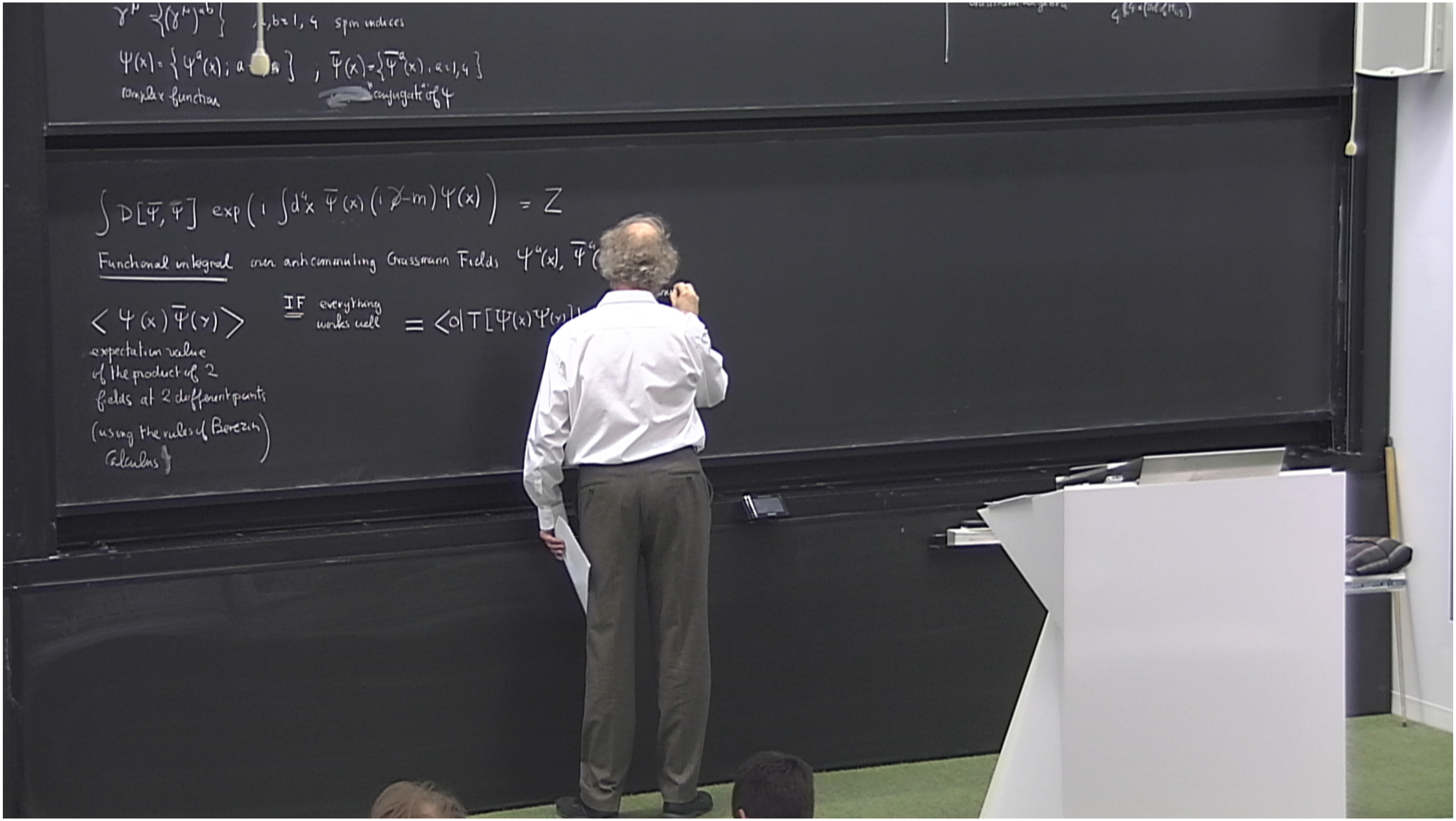
Functional integral over anticommuting Grassmann Fields $\psi^a(x)$, $\bar\psi^a(x)$

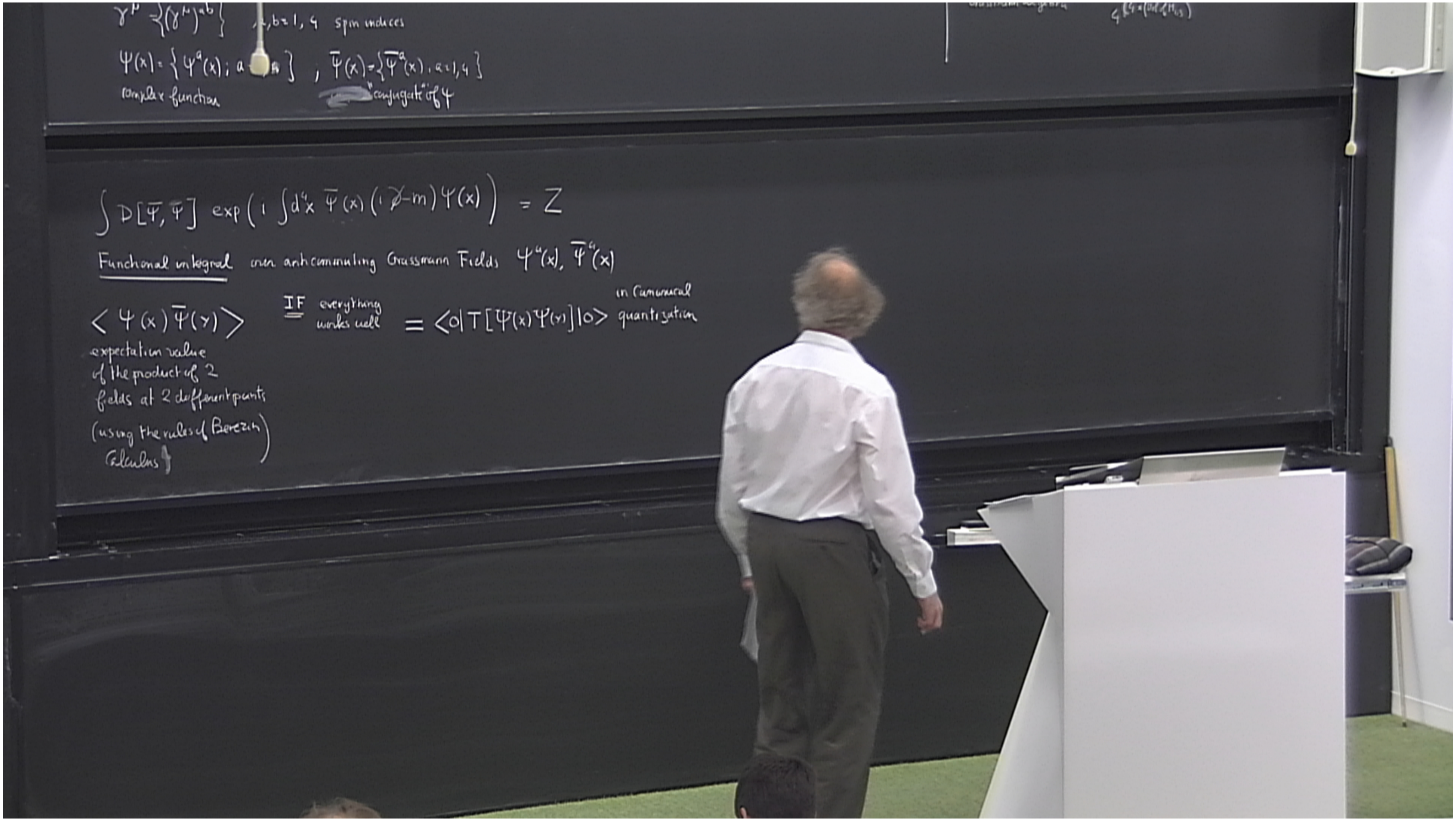
$$\langle \psi(x)\,\bar\psi(y) \rangle$$

expectation value of the product of 2 fields

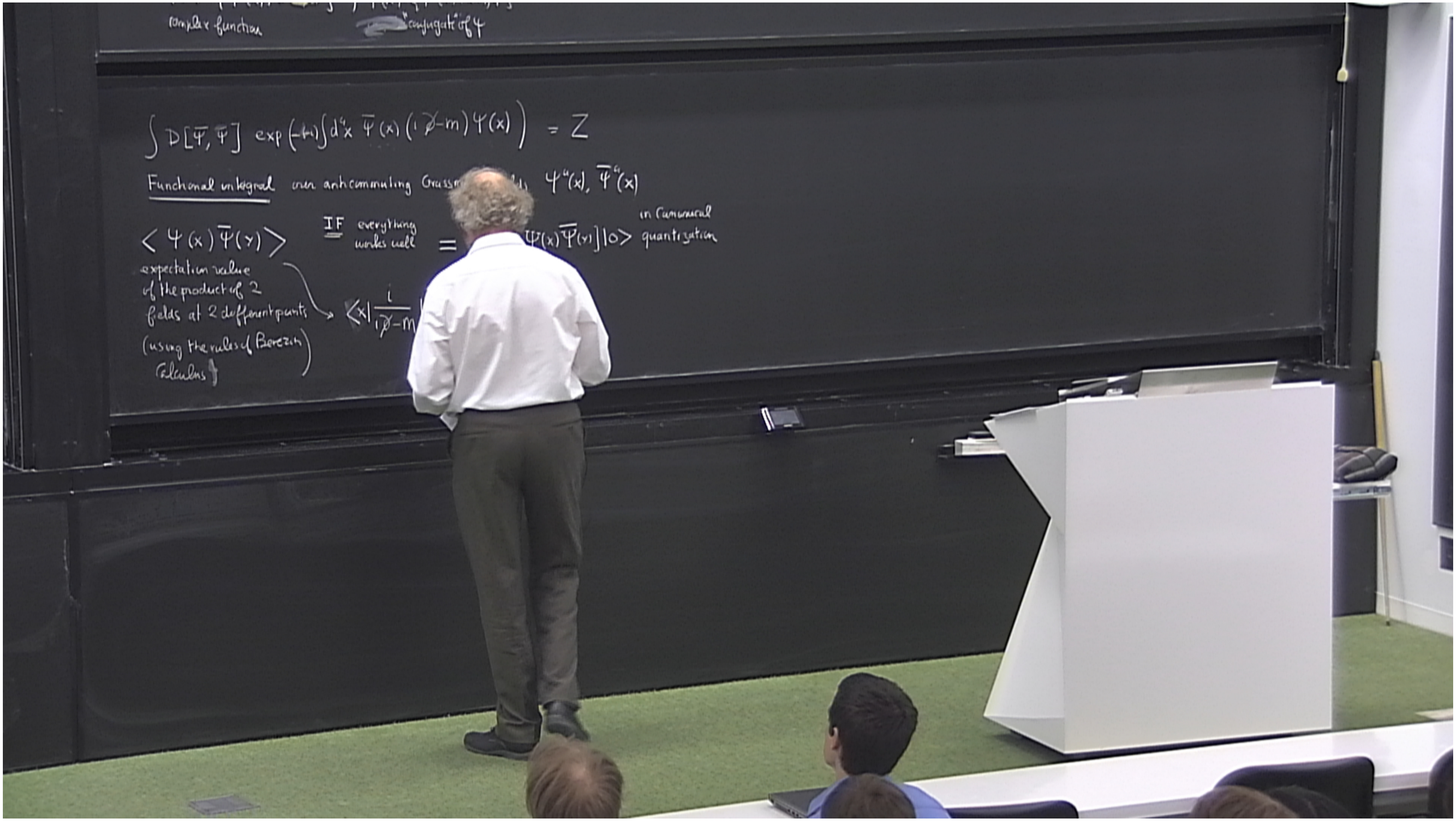


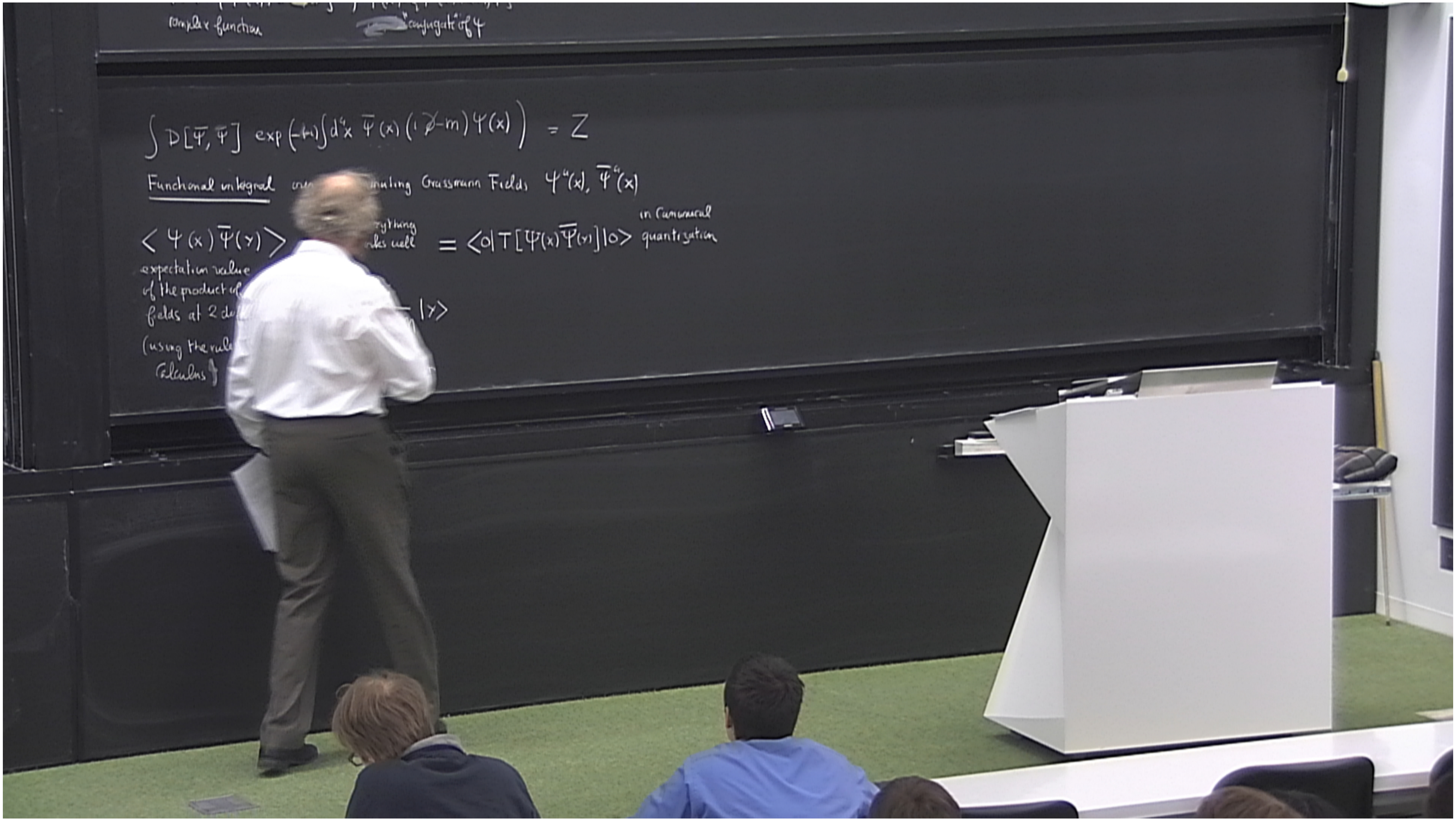
a,b=1,4 spin indices
complex function
"conjugate" of Ψ
Functional integral over anticommuting Gr... ...lds
IF everything works well
expectation value of the product of 2 fields at 2 different points
(using the rules of Berezin Calculus)

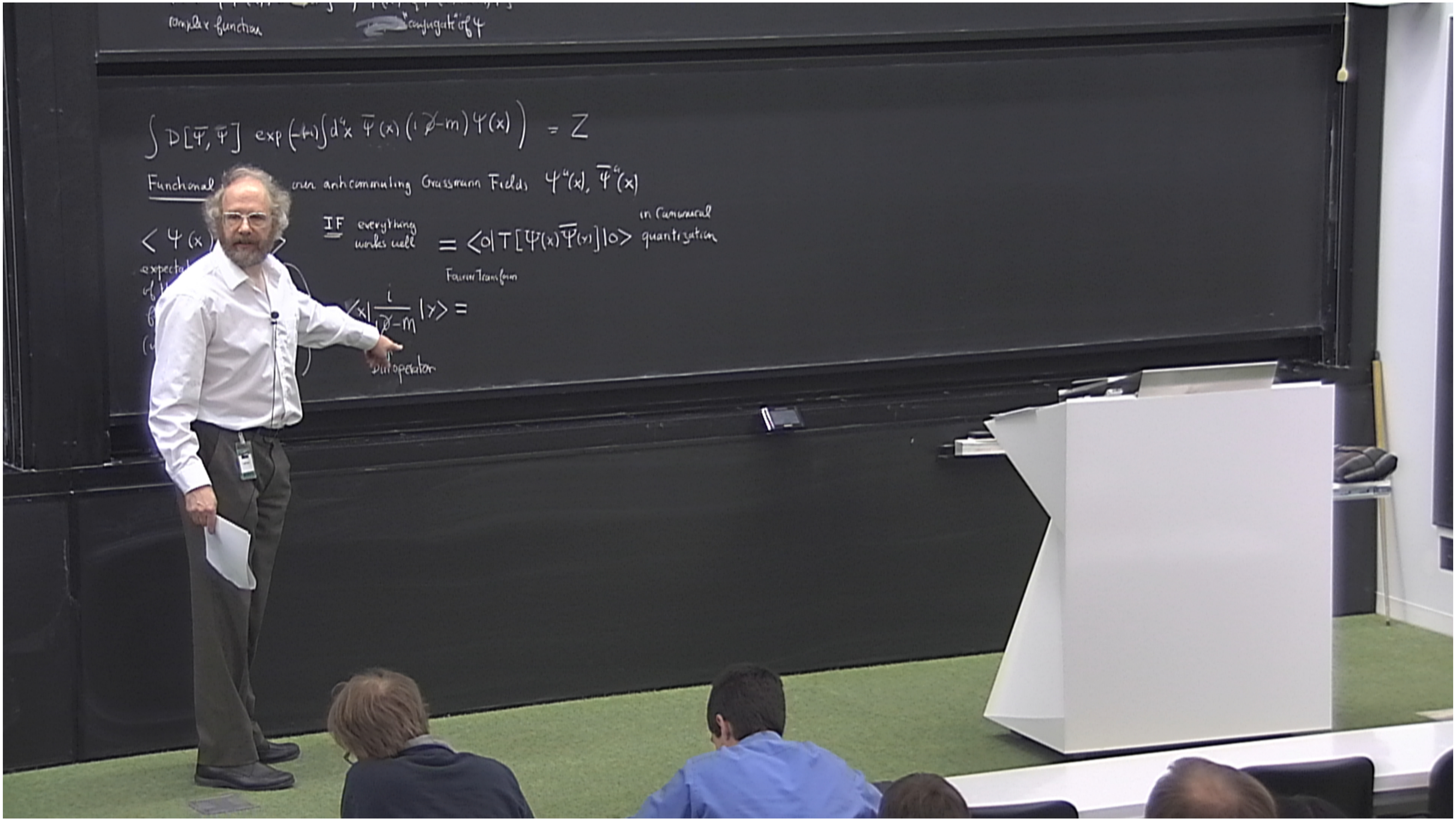

a,b=1,4 spin indices
complex function
"conjugate" of Ψ
∫D[Ψ̄, Ψ] exp(i∫d⁴x Ψ̄(x)(i∂̸−m)Ψ(x)) = Z
Functional integral over anticommuting Grassmann Fields
⟨Ψ(x)Ψ̄(y)⟩
IF everything works well
= ⟨0|T[Ψ(x)Ψ(y)]
expectation value of the product of 2 fields at 2 different points
(using the rules of Berezin Calculus)

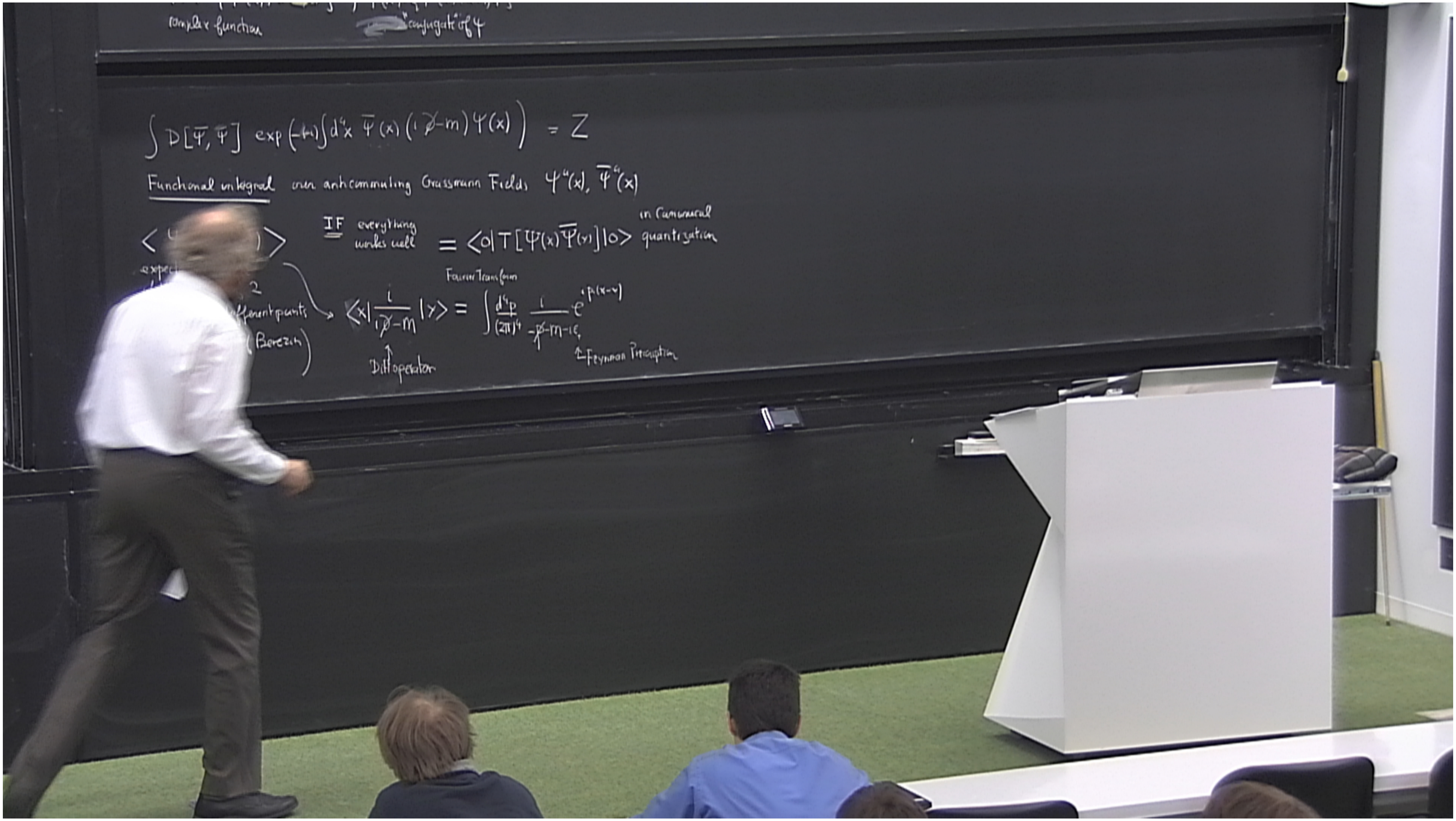


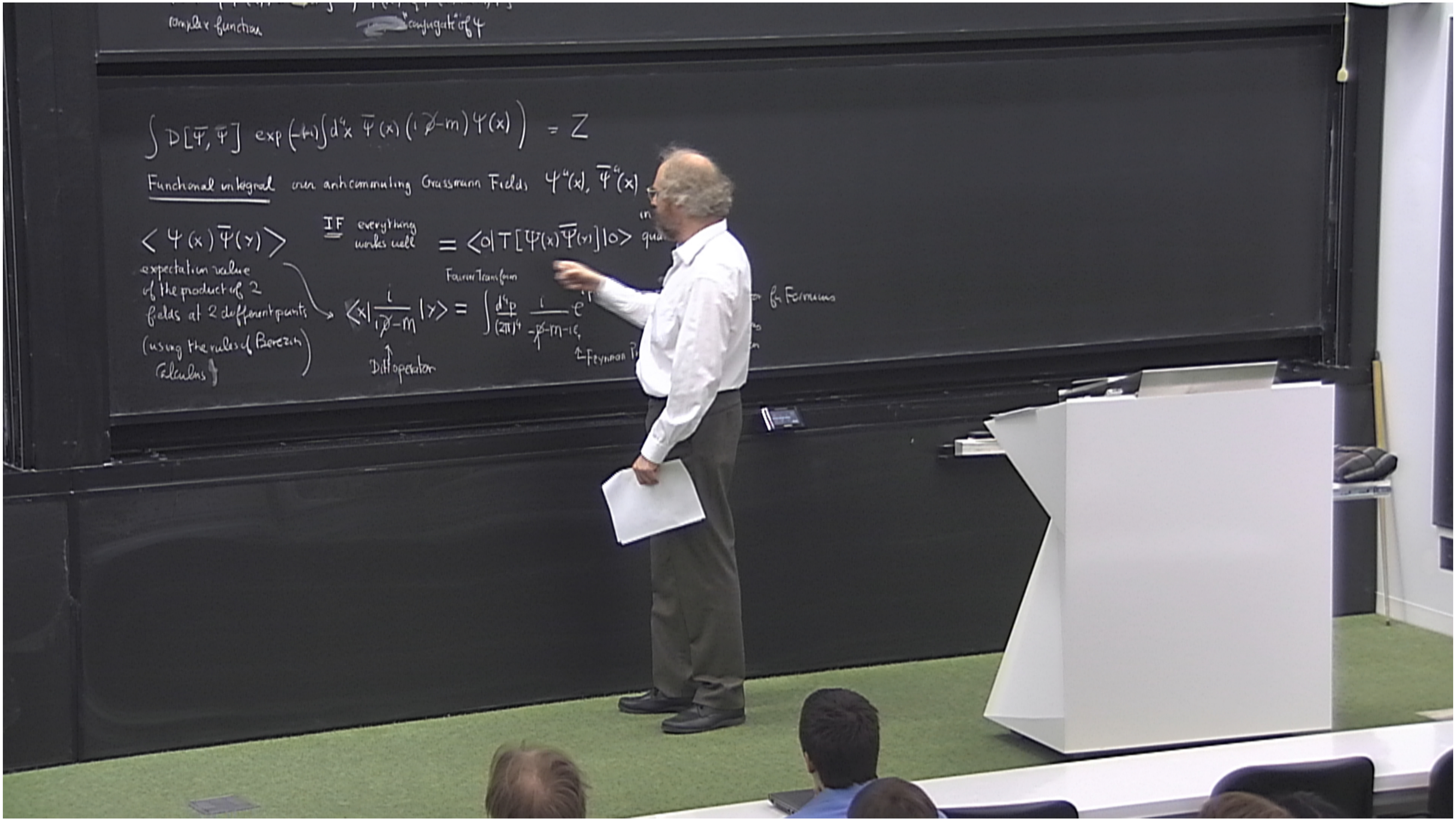
a,b = 1, 4 spin indices
complex function
"conjugate" of Ψ
∫ D[Ψ̄, Ψ] exp(i ∫d⁴x Ψ̄(x) (i∂̸ − m) Ψ(x)) = Z
Functional integral over anticommuting Grassmann Fields Ψ^a(x), Ψ̄^a(x)
< Ψ(x) Ψ̄(y) > IF everything works well = <0| T[Ψ(x)Ψ(y)] |0> in Canonical quantization
expectation value of the product of 2 fields at 2 different points
(using the rules of Berezin Calculus)

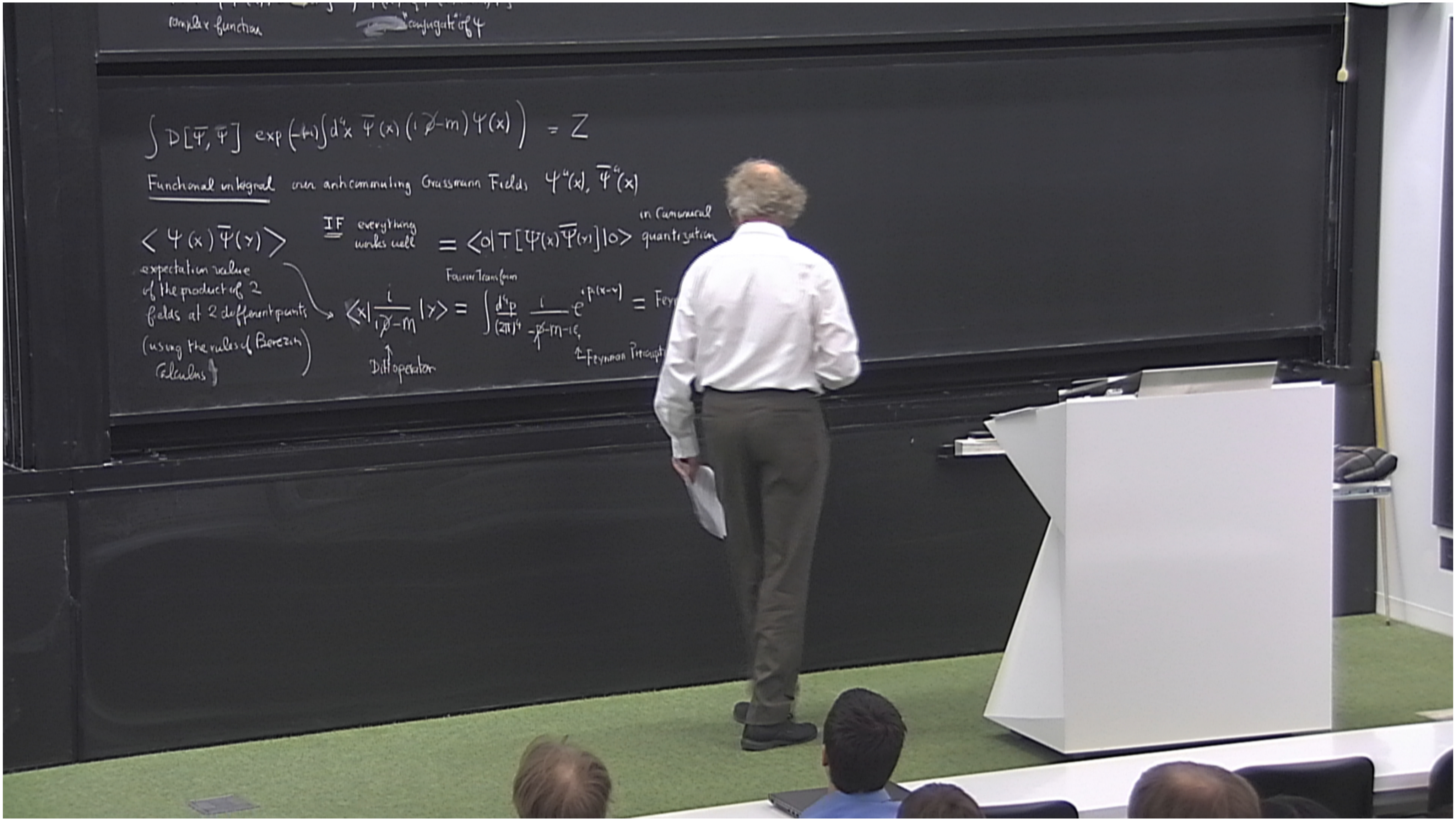

complex function
"conjugate" of ψ
∫ D[Ψ̄, Ψ] exp(−i ∫ d⁴x Ψ̄(x) (i∂̸ − m) Ψ(x)) = Z
Functional integral over anticommuting Grassm
ψ^α(x), Ψ̄^α(x)
⟨ Ψ(x) Ψ̄(y) ⟩
IF everything works well
Ψ(x)Ψ̄(y)]|0⟩
in Canonical quantization
expectation value of the product of 2 fields at 2 different points
(using the rules of Berezin Calculus)

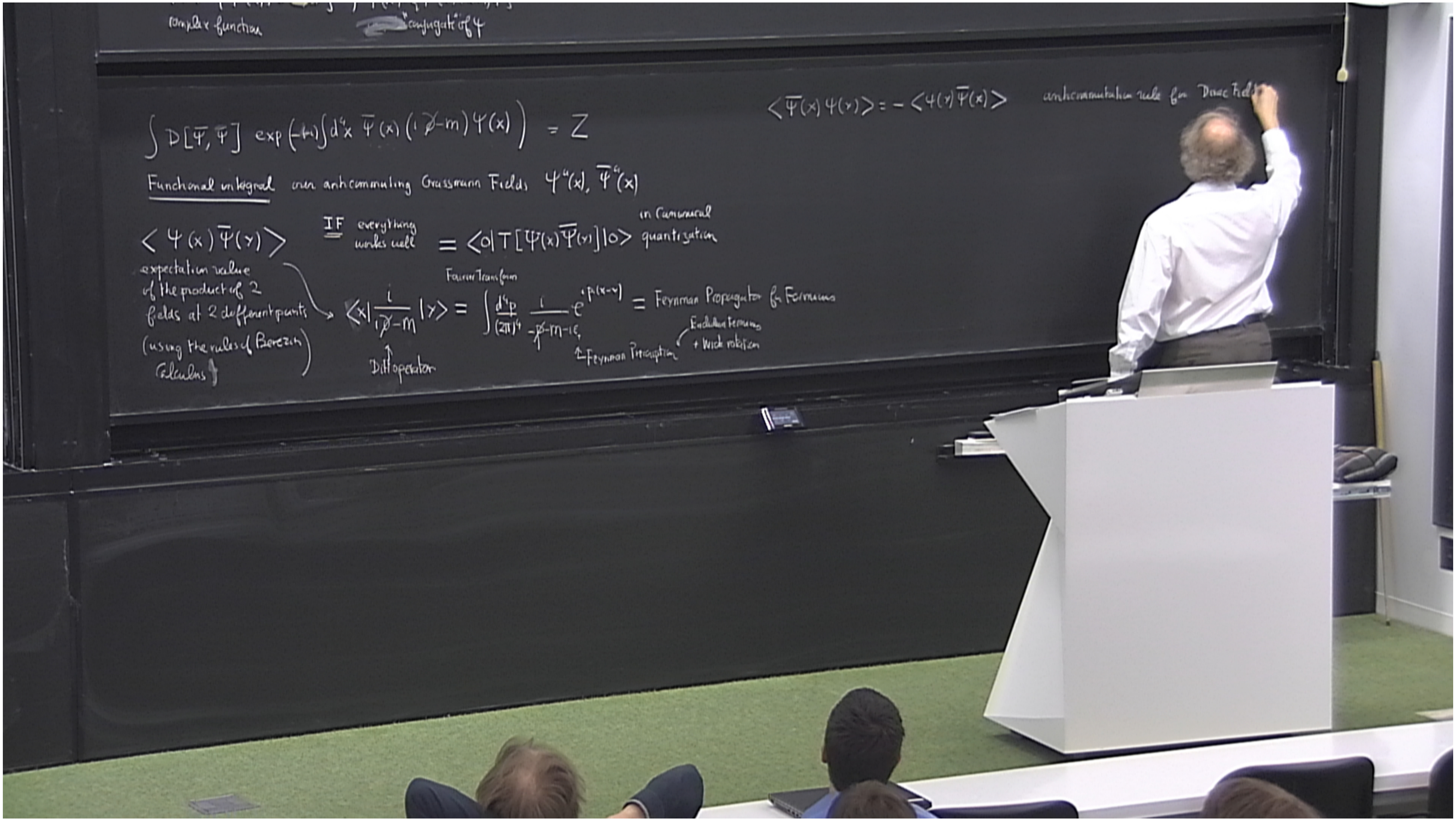

complex function
"conjugate" of Ψ
∫ D[Ψ̄, Ψ] exp(−i ∫ d⁴x Ψ̄(x)(i∂̸ − m)Ψ(x)) = Z
Functional integral
Grassmann Fields Ψ^α(x), Ψ̄^α(x)
in Canonical quantization
<Ψ(x) Ψ̄(y)>
= <0|T[Ψ(x)Ψ̄(y)]|0>
expectation value
of the product of
fields at 2
(using the rule
Calculus

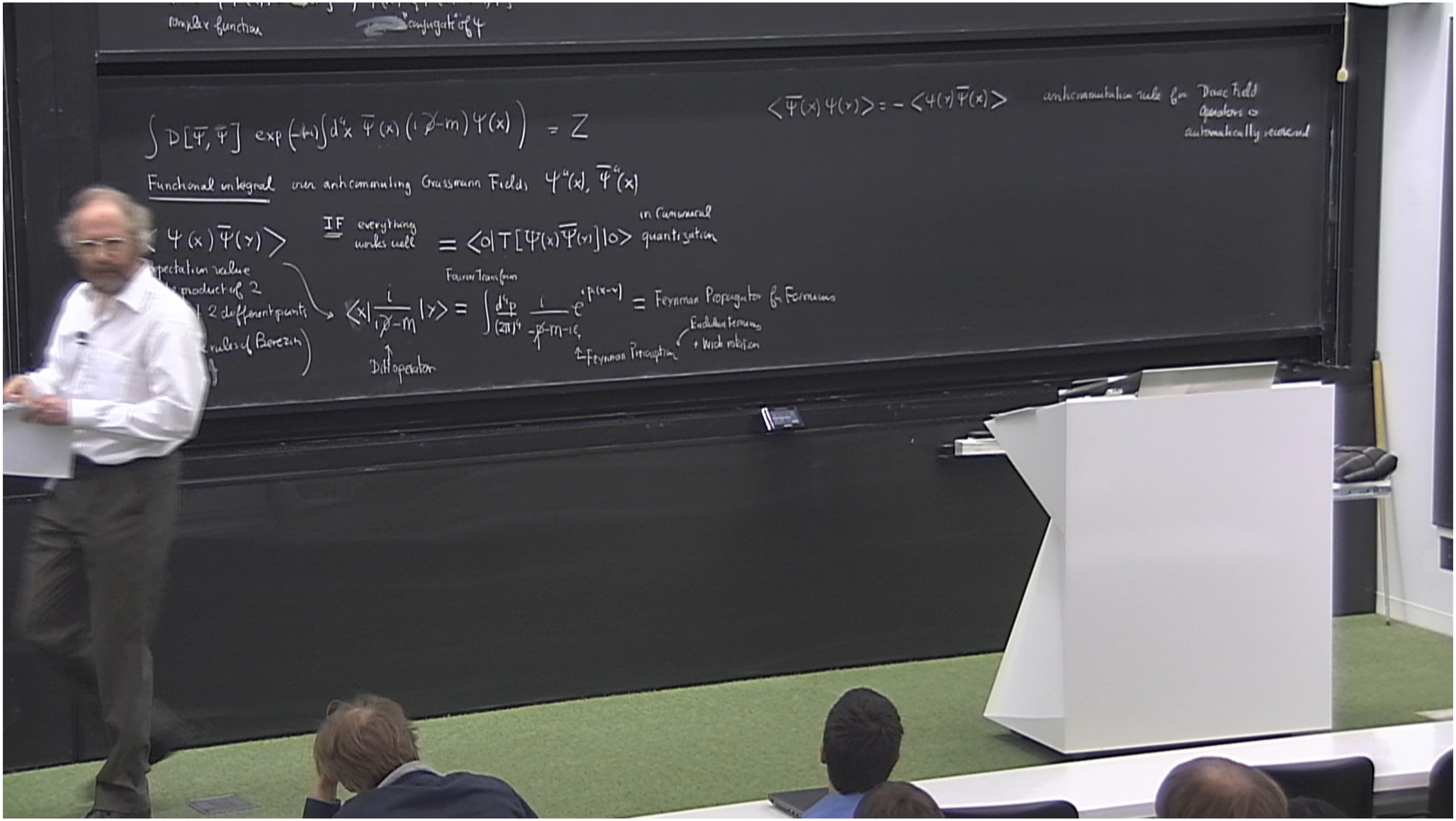

complex function
"conjugate" of ψ
∫ D[Ψ̄, Ψ] exp(-i∫d⁴x Ψ̄(x)(i∂̸-m)Ψ(x)) = Z
Functional over anticommuting Grassmann Fields ψ^α(x), Ψ̄^α(x)
IF everything works well
= <0|T[Ψ(x)Ψ̄(y)]|0>
in Canonical quantization
Fourier Transform
<x| i/(i∂̸-m) |y> =
Dirac operator

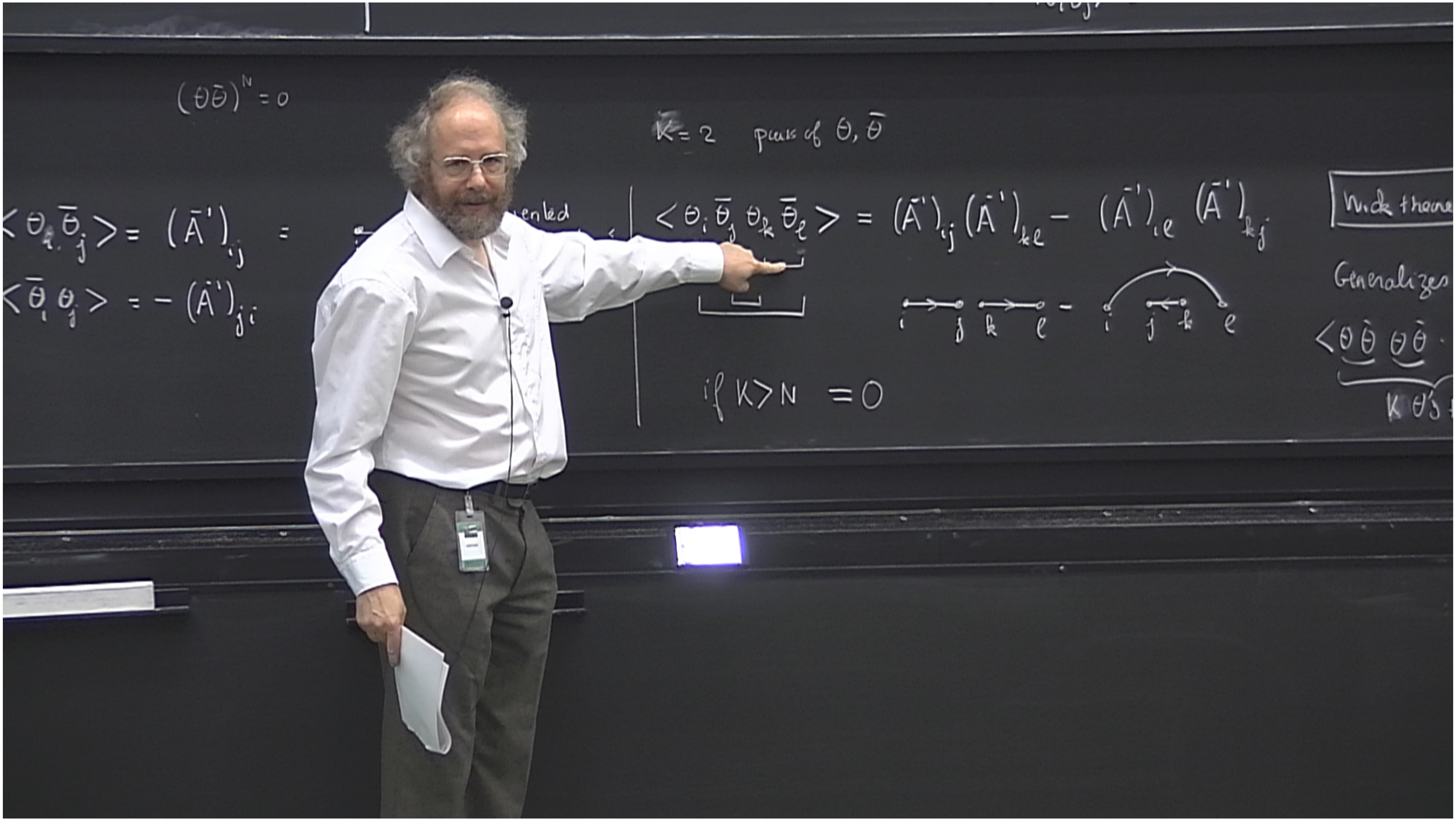

complex function
"conjugate" of ψ
∫ D[Ψ̄, Ψ] exp(−i∫d⁴x Ψ̄(x) (i∂̸−m) Ψ(x)) = Z
Functional integral over anticommuting Grassmann Fields ψ^α(x), Ψ̄^α(x)
IF everything works well = <0|T[Ψ(x)Ψ̄(y)]|0> in Canonical quantization
Fourier Transform
<x| i/(i∂̸−m) |y> = ∫ d⁴p/(2π)⁴ i/(−p̸−m−iε) e^{ip(x−y)}
Diff operator
Feynman Propagation
(Berezin)

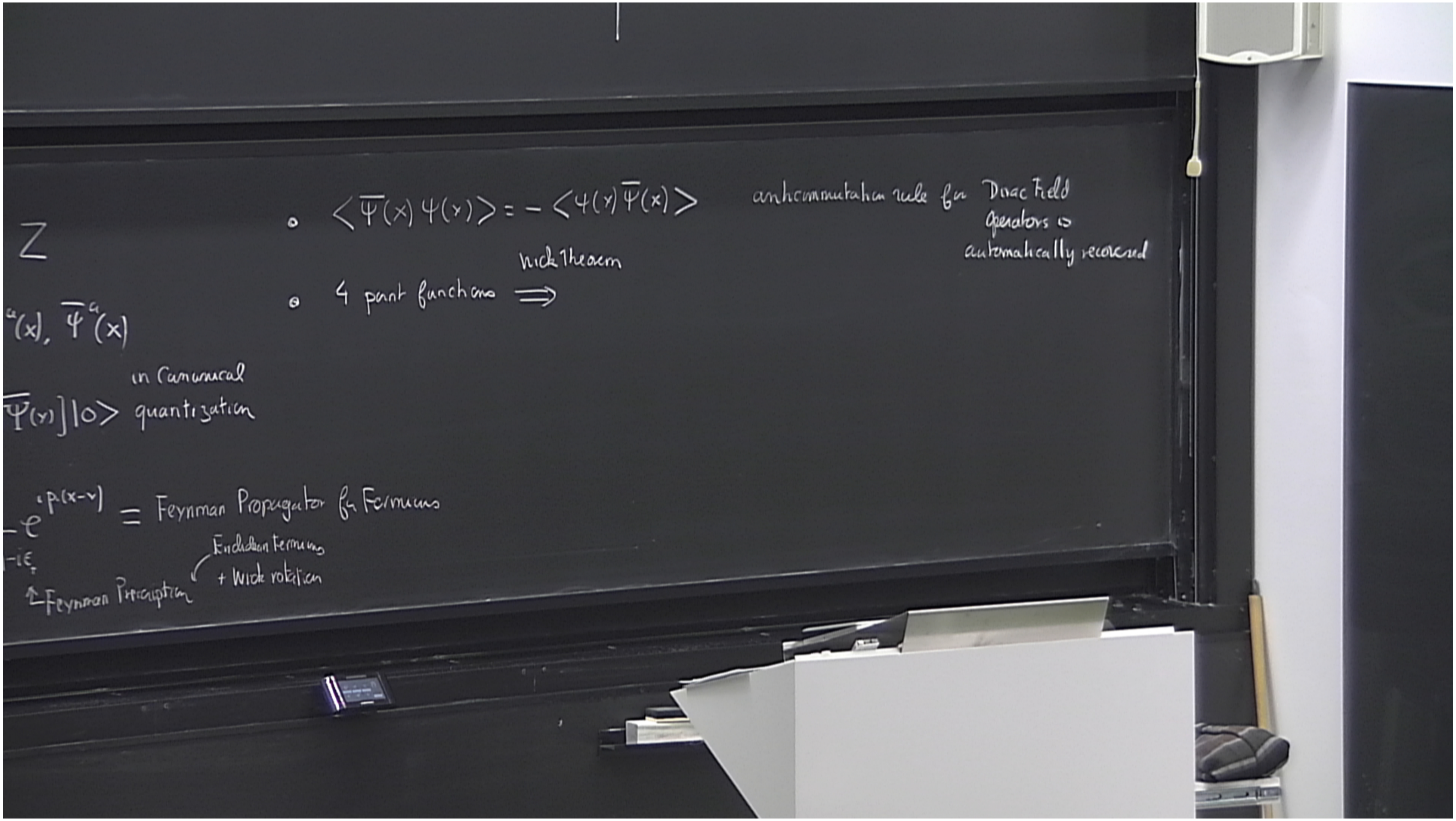

complex function
"conjugate" of Ψ
$\int D[\bar\Psi,\Psi]\ \exp\left(-i\int d^4x\ \bar\Psi(x)(i\partial\!\!\!/-m)\Psi(x)\right) = Z$
Functional integral over anticommuting Grassmann Fields $\Psi^a(x)$, $\bar\Psi^a(x)$
$\langle \Psi(x)\bar\Psi(y)\rangle$
IF everything works well
$= \langle 0|T[\Psi(x)\bar\Psi(y)]|0\rangle$
expectation value of the product of 2 fields at 2 different points
(using the rules of Berezin Calculus)
Fourier Transform
$\langle x|\frac{i}{i\partial\!\!\!/-m}|y\rangle = \int\frac{d^4p}{(2\pi)^4}\frac{i}{-p\!\!\!/-m-i\epsilon}$
Diff operator
Feynman
for Fermions

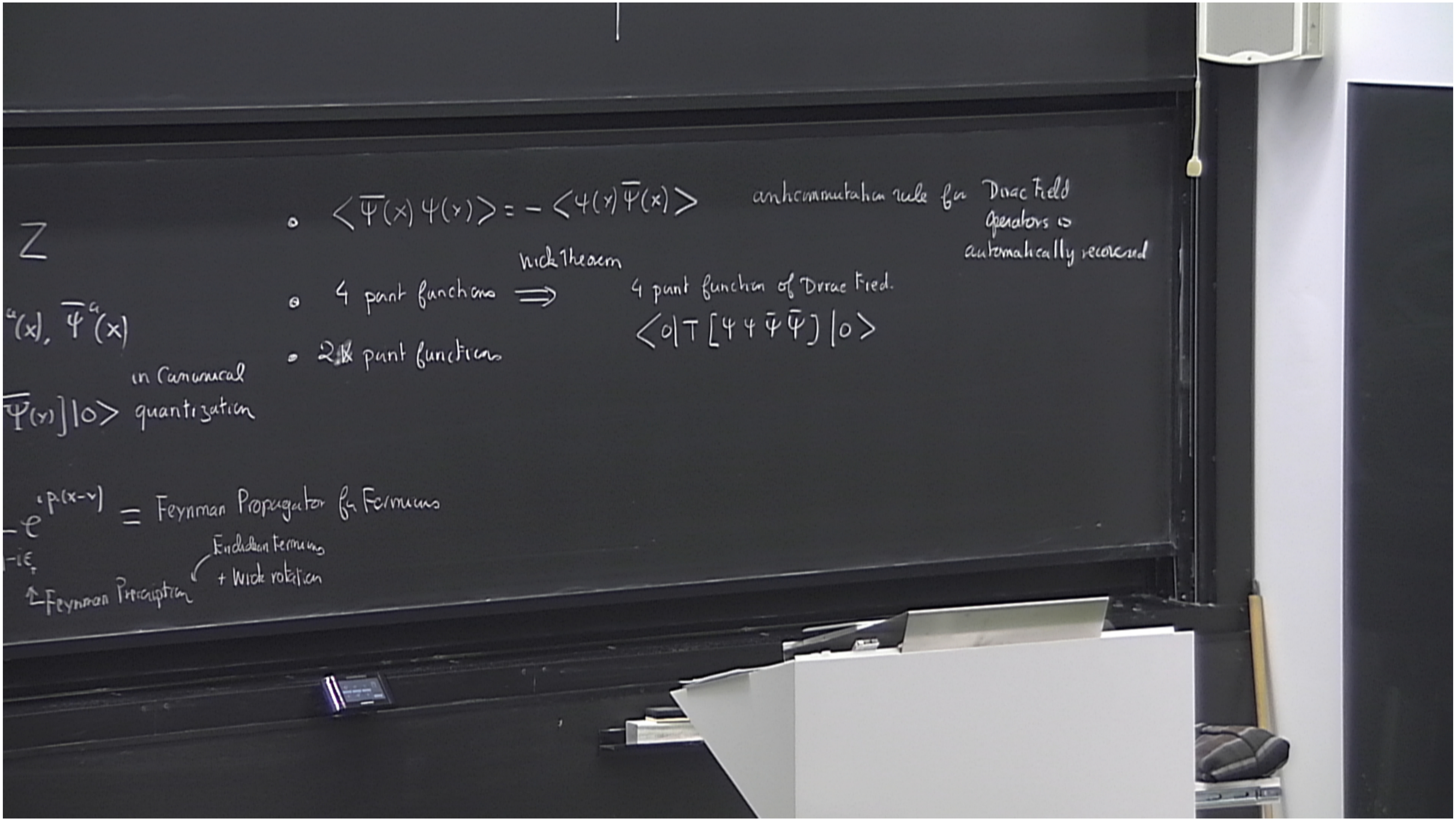

complex function
"conjugate" of Ψ
Functional integral over anticommuting Grassmann Fields
expectation value of the product of 2 fields at 2 different points (using the rules of Berezin Calculus)
IF everything works well
in Canonical quantization
Fourier Transform
Diff operator
Feynman Propagator

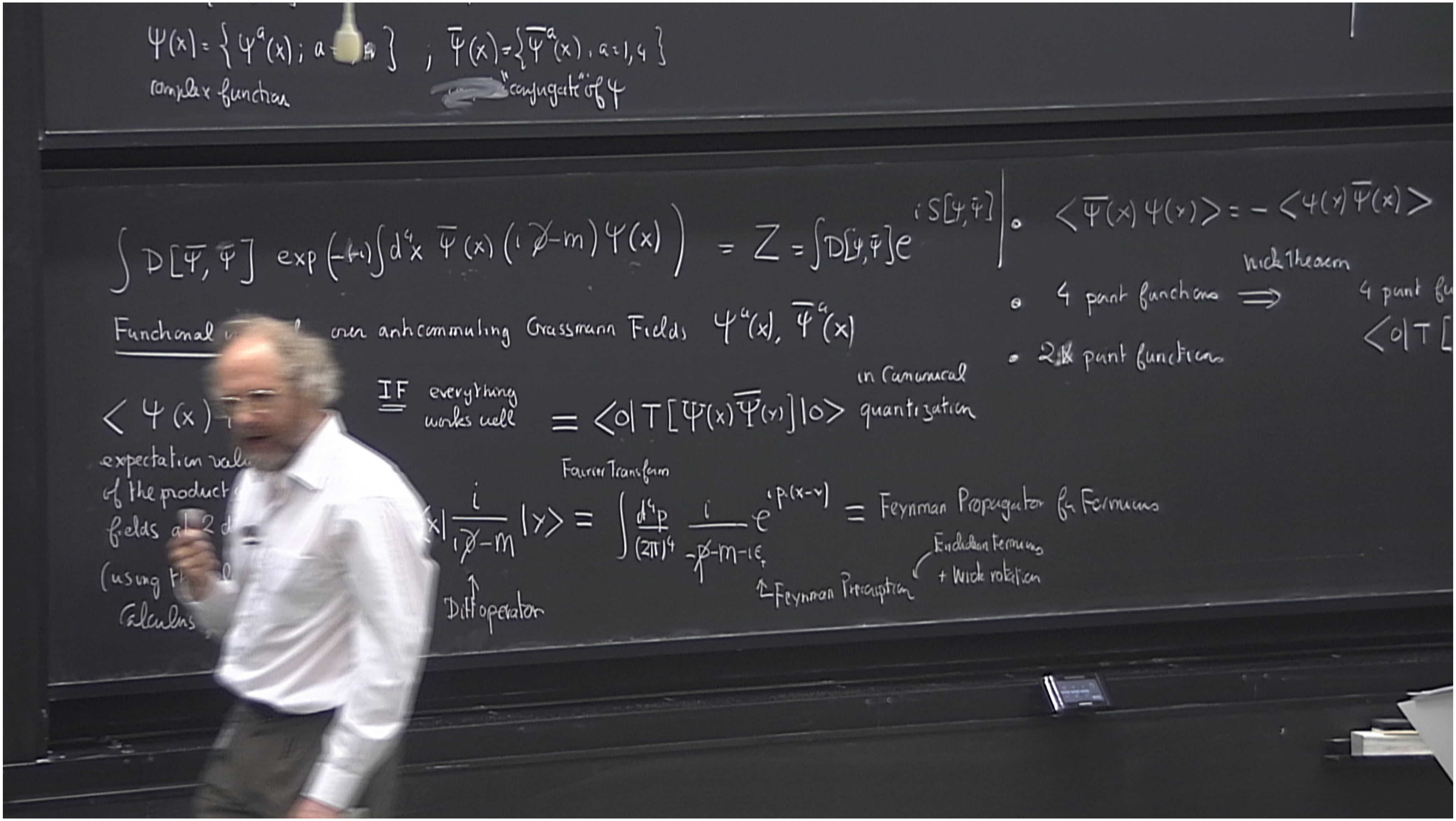

complex function
"conjugate" of ψ
⟨ψ̄(x) ψ(y)⟩ = − ⟨ψ(y) ψ̄(x)⟩
anticommutation rule for Dirac Fields
∫ D[ψ̄, ψ] exp(−i∫d⁴x ψ̄(x)(i∂̸−m)ψ(x)) = Z
Functional integral over anticommuting Grassmann Fields ψᵃ(x), ψ̄ᵃ(x)
⟨ψ(x) ψ̄(y)⟩
IF everything works well = ⟨0|T[ψ(x)ψ̄(y)]|0⟩ in Canonical quantization
expectation value of the product of 2 fields at 2 different points
(using the rules of Berezin Calculus)
Fourier Transform
⟨x| i/(i∂̸−m) |y⟩ = ∫ d⁴p/(2π)⁴ i/(−p̸−m−iε) e^{ip(x−y)} = Feynman Propagator for Fermions
Diff operator
Feynman Prescription
Euclidean Fermions + Wick rotation

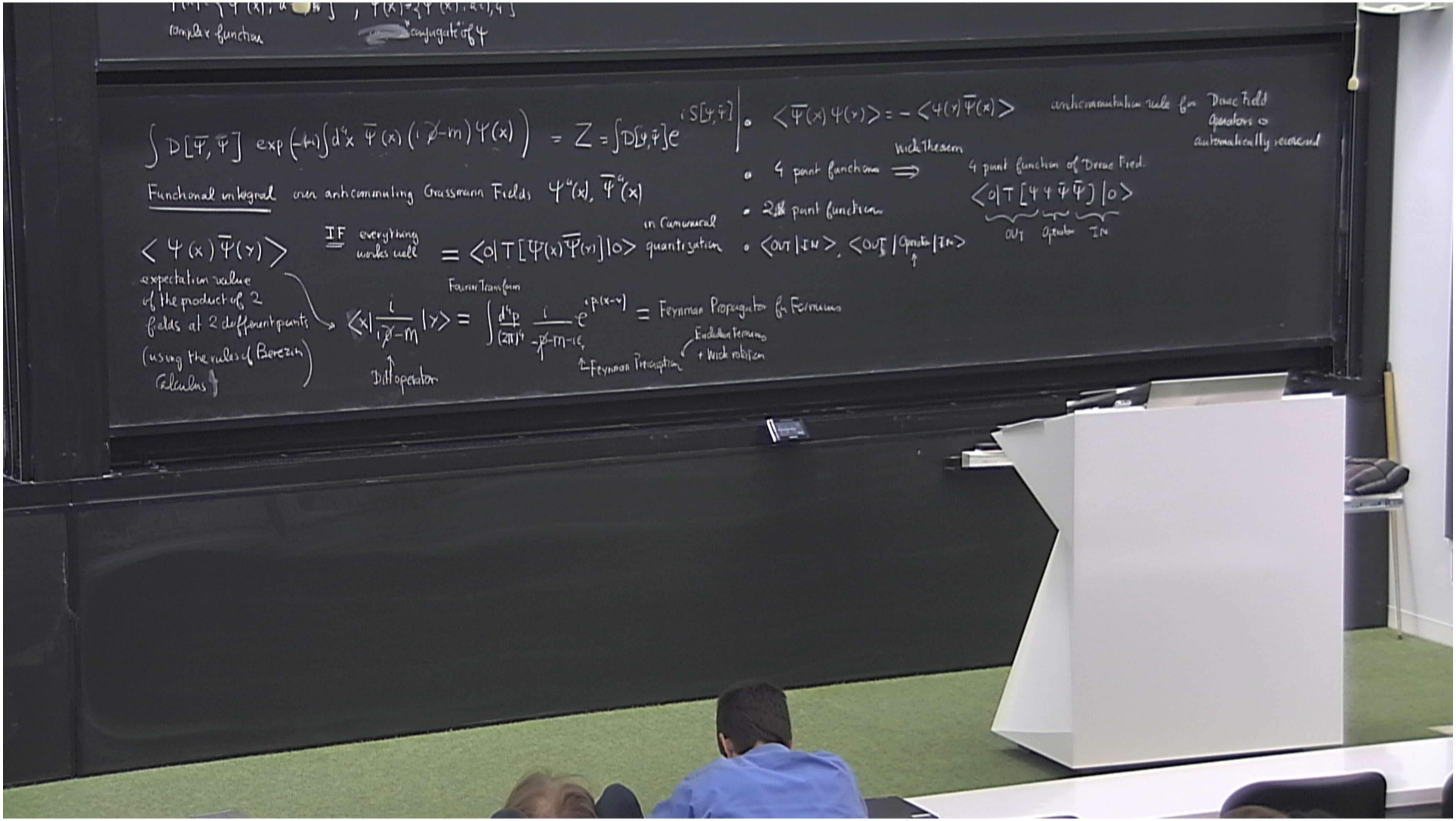

complex function
conjugate of ψ
= Z
Functional integral over anticommuting Grassmann Fields
IF everything works well
in Canonical quantization
Fourier Transform
= Feynman Propagator for Fermions
Euclidean Fermions
+ Wick rotation
Feynman Propagator
Diff operator
rules of Berezin
anticommutation rule for Dirac Field
Operators

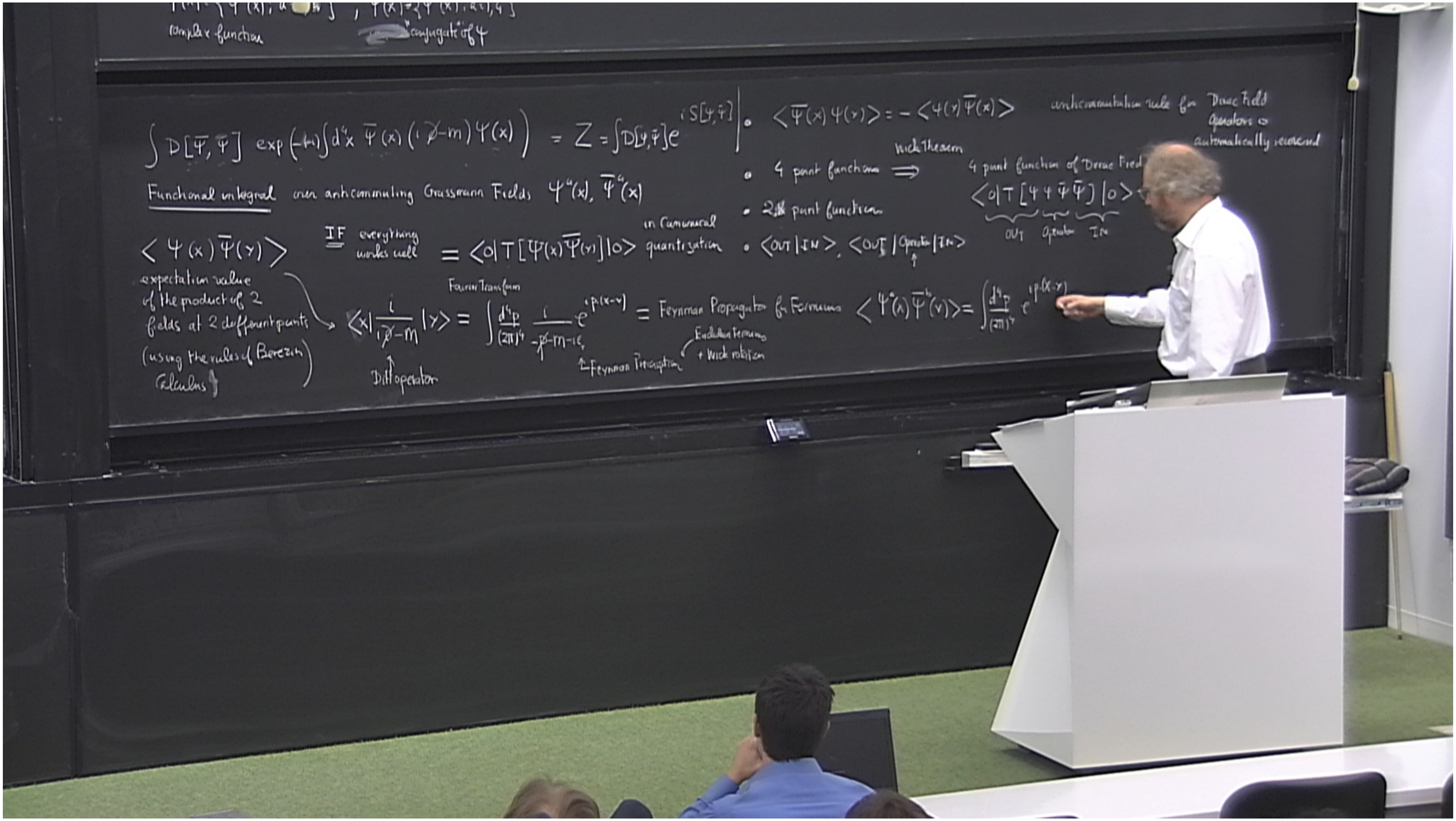

$(\theta\bar{\theta})^N = 0$
$\langle \theta_i \bar{\theta}_j \rangle = (\bar{A}^{-1})_{ij}$
$\langle \bar{\theta}_i \theta_j \rangle = -(\bar{A}^{-1})_{ji}$
K = 2 pairs of $\theta, \bar{\theta}$
$\langle \theta_i \bar{\theta}_j \theta_k \bar{\theta}_l \rangle = (\bar{A}^{-1})_{ij} (\bar{A}^{-1})_{kl} - (\bar{A}^{-1})_{il} (\bar{A}^{-1})_{kj}$
if K > N = 0
Generalizes

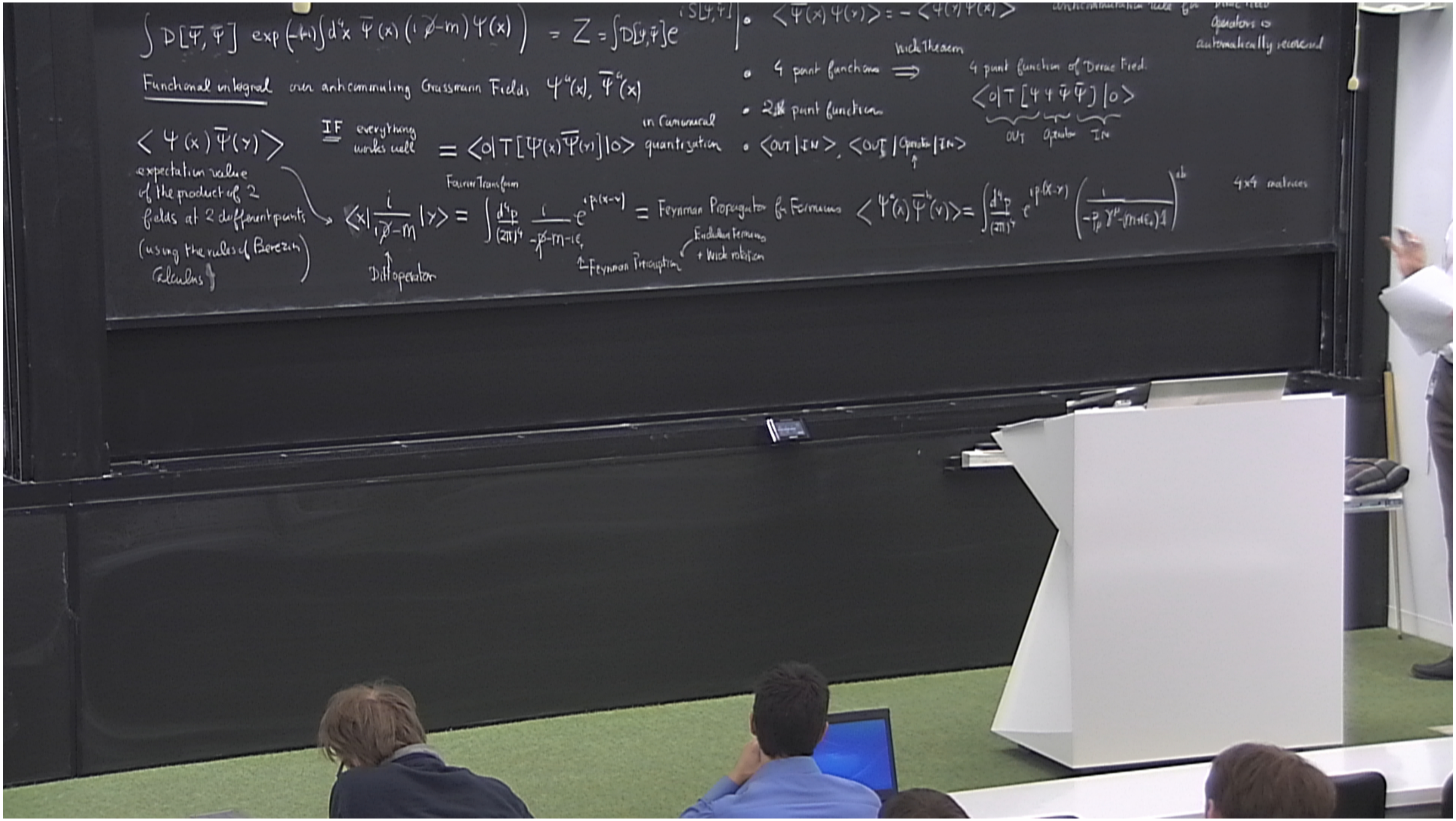

Z
< Ψ̄(x) Ψ(y) > = − < Ψ(y) Ψ̄(x) >
anticommutation rule for Dirac Field Operators is automatically recovered
Wick Theorem
4 point functions ⟹
in Canonical quantization
quantization
= Feynman Propagator for Fermions
Euclidean fermions
+ Wick rotation
Feynman Prescription

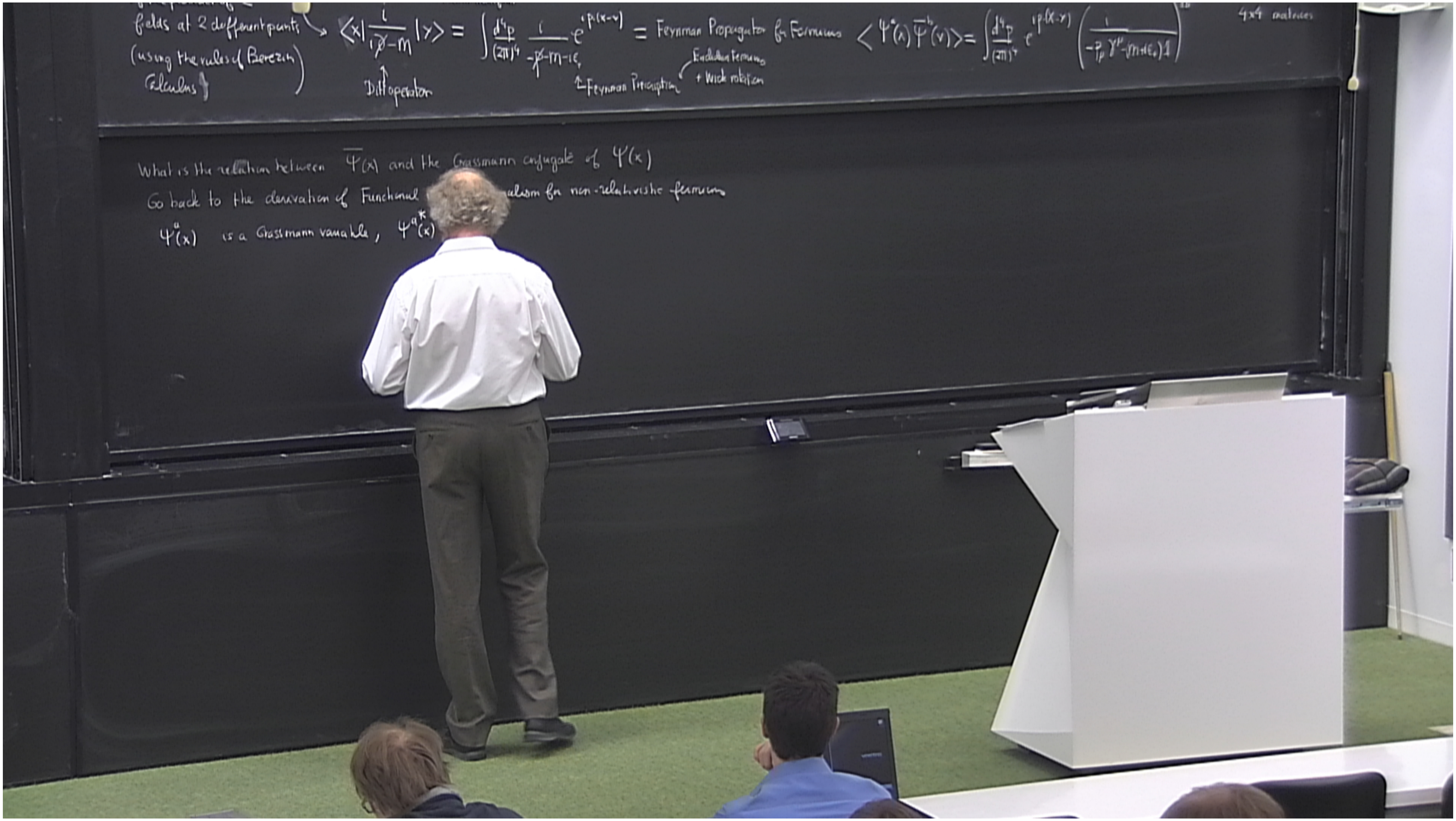

⟨Ψ̄(x) Ψ(y)⟩ = − ⟨Ψ(y) Ψ̄(x)⟩
anticommutation rule for Dirac Field Operators is automatically recovered
4 point functions ⟹ 4 point function of Dirac Field
Wick Theorem
⟨0| T [Ψ Ψ Ψ̄ Ψ̄] |0⟩
2k point functions
in Canonical quantization
= Feynman Propagator for Fermions
Euclidean Fermions + Wick rotation
Feynman Prescription

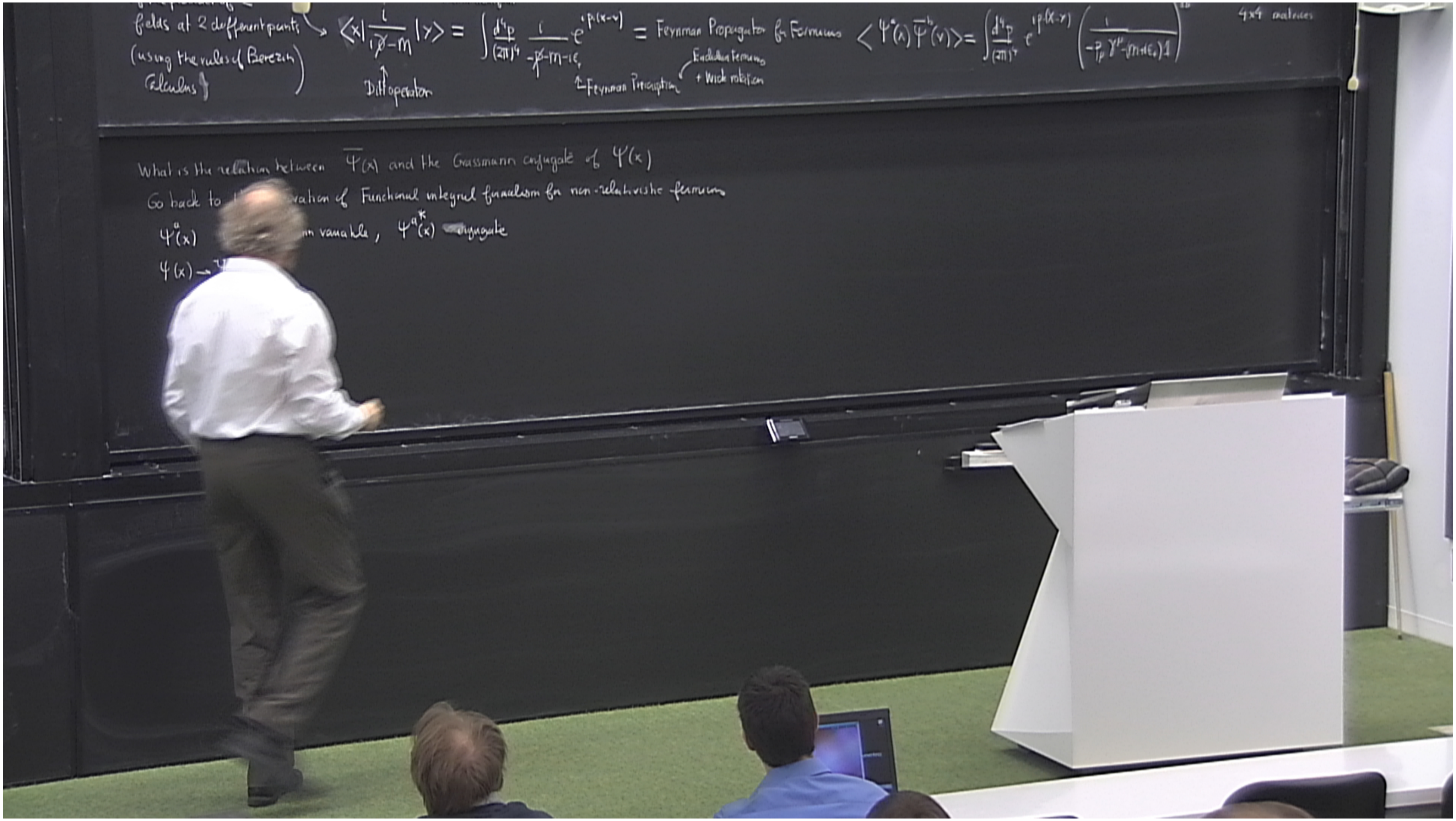

$\Psi(x) = \{ \Psi^a(x) ; a = \ldots \}$; $\bar\Psi(x) = \{ \bar\Psi^a(x) , a = 1,4 \}$
complex function
"conjugate" of Ψ
$\int D[\bar\Psi, \Psi] \exp\left(-(-i)\int d^4x \, \bar\Psi(x) (i\not\partial - m) \Psi(x)\right) = Z = \int D[\Psi,\bar\Psi] e^{iS[\Psi,\bar\Psi]}$
Functional … over anticommuting Grassmann Fields $\Psi^a(x)$, $\bar\Psi^a(x)$
IF everything works well
$= \langle 0 | T[\Psi(x)\bar\Psi(y)] | 0 \rangle$
in Canonical quantization
expectation value
of the product
fields at 2
(using
Calculus
Fourier Transform
$\frac{i}{i\not\partial - m} | y \rangle = \int \frac{d^4p}{(2\pi)^4} \frac{i}{-\not p - m - i\epsilon} e^{ip\cdot(x-y)}$ = Feynman Propagator for Fermions
Diff operator
Feynman Prescription
Euclidean Fermions + Wick rotation
$\langle \bar\Psi(x) \Psi(y) \rangle = - \langle \Psi(y) \bar\Psi(x) \rangle$
Wick Theorem
4 point functions ⟹
4 point fu
$\langle 0 | T [$
2k point functions

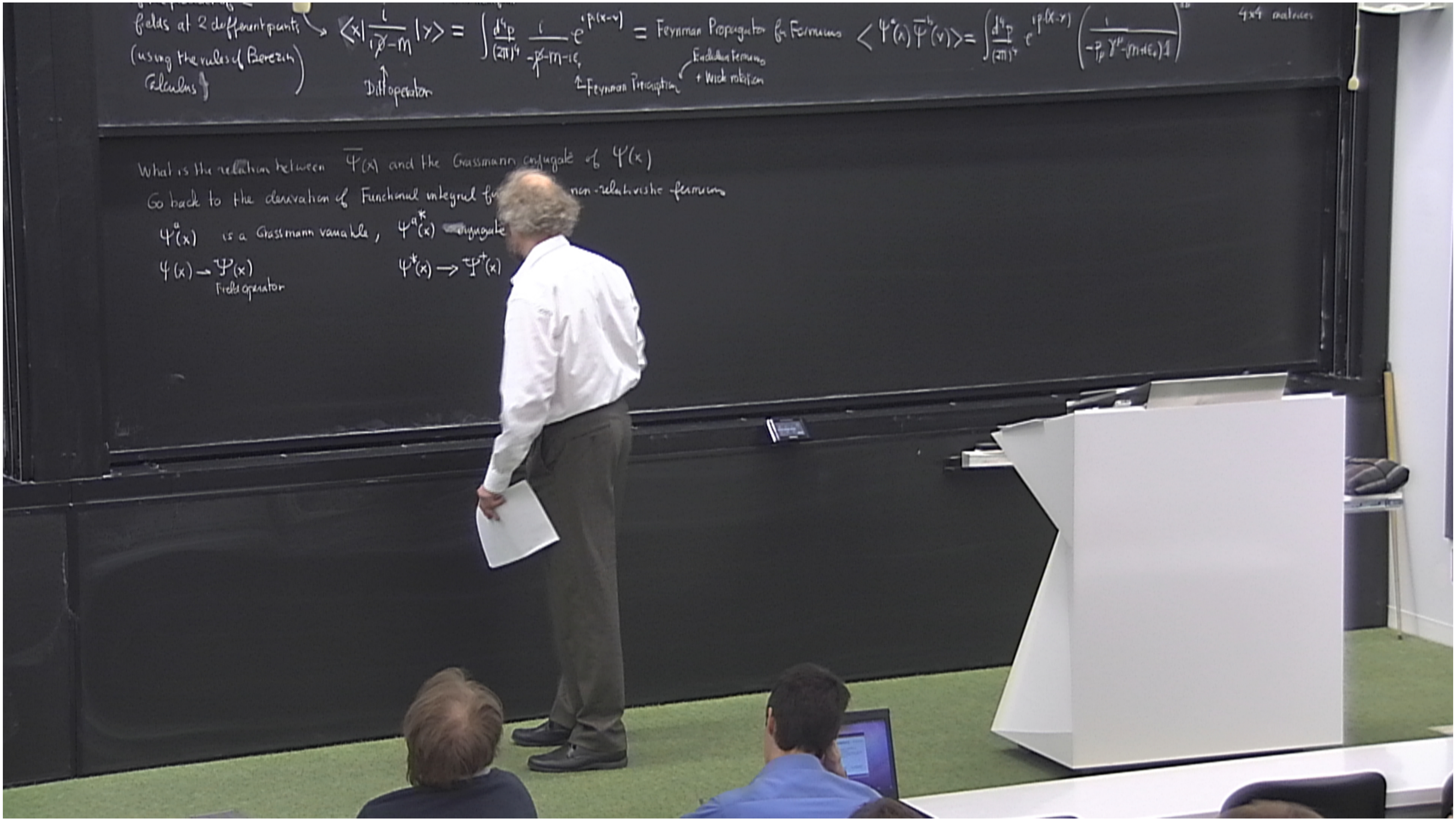

complex function
conjugate of Ψ
Functional integral over anticommuting Grassmann Fields
⟨Ψ(x) Ψ̄(y)⟩
IF everything works well
= ⟨0|T[Ψ(x)Ψ̄(y)]|0⟩
in Canonical quantization
expectation value of the product of 2 fields at 2 different points
(using the rules of Berezin Calculus)
Fourier Transform
= Feynman Propagator for Fermions
Diff operator
Feynman Prescription
Euclidean Fermions + Wick rotation
anticommutation rule for Dirac Field operators automatically recovered
Wick Theorem
4 point functions
4 point function of Dirac Field
⟨0|T[ΨΨΨ̄Ψ̄]|0⟩
OUT Operator IN
2n point functions
⟨OUT|IN⟩, ⟨OUT|Operator|IN⟩


complex function
conjugate of ψ
∫ D[Ψ̄, Ψ] exp(−i ∫ d⁴x Ψ̄(x)(i∂̸ − m)Ψ(x)) = Z = ∫ D[ψ, ψ̄] e^{iS[ψ, ψ̄]}
Functional integral over anticommuting Grassmann Fields ψ^a(x), ψ̄^a(x)
⟨ψ(x) ψ̄(y)⟩ IF everything works well = ⟨0| T[ψ(x) ψ̄(y)] |0⟩ in Canonical quantization
expectation value of the product of 2 fields at 2 different points
(using the rules of Berezin Calculus)
Fourier Transform
⟨x| i/(i∂̸ − m) |y⟩ = ∫ d⁴p/(2π)⁴ i/(−p̸ − m − iε) e^{ip(x−y)} = Feynman Propagator for Fermions
Diff operator
Feynman Prescription
Euclidean Fermions + Wick rotation
⟨Ψ̄(x) Ψ(y)⟩ = − ⟨Ψ(y) Ψ̄(x)⟩ anticommutation rule for Dirac Field Operators automatically recovered
4 point functions ⟹ Wick Theorem
4 point function of Dirac Field
⟨0| T[ψ ψ ψ̄ ψ̄] |0⟩
OUT Operator IN
2n point functions
⟨OUT | IN⟩, ⟨OUT | Operator | IN⟩
⟨ψ^a(x) ψ̄^b(y)⟩ = ∫ d⁴p/(2π)⁴ e^{ip(x−y)}

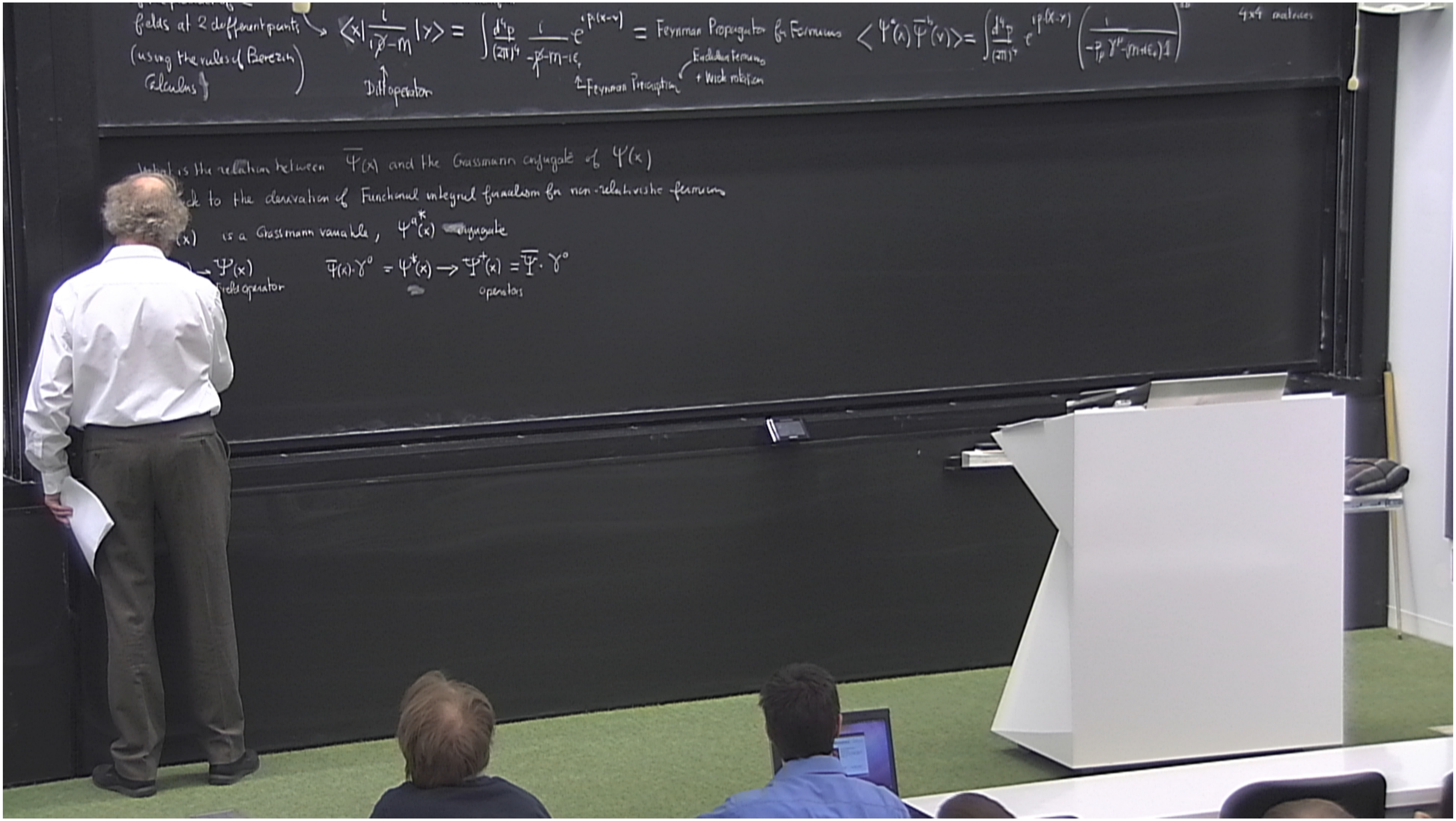

Functional integral over anticommuting Grassmann Fields
IF everything works well
in Canonical quantization
expectation value of the product of 2 fields at 2 different points
(using the rules of Berezin Calculus)
Fourier Transform
Diff operator
Feynman Propagator for Fermions
Feynman Prescription
Euclidean Fermions + Wick rotation
Wick Theorem
4 point functions
2 pt point functions
4 point function of Dirac Field
Operator
4x4 matrices
automatically reversed

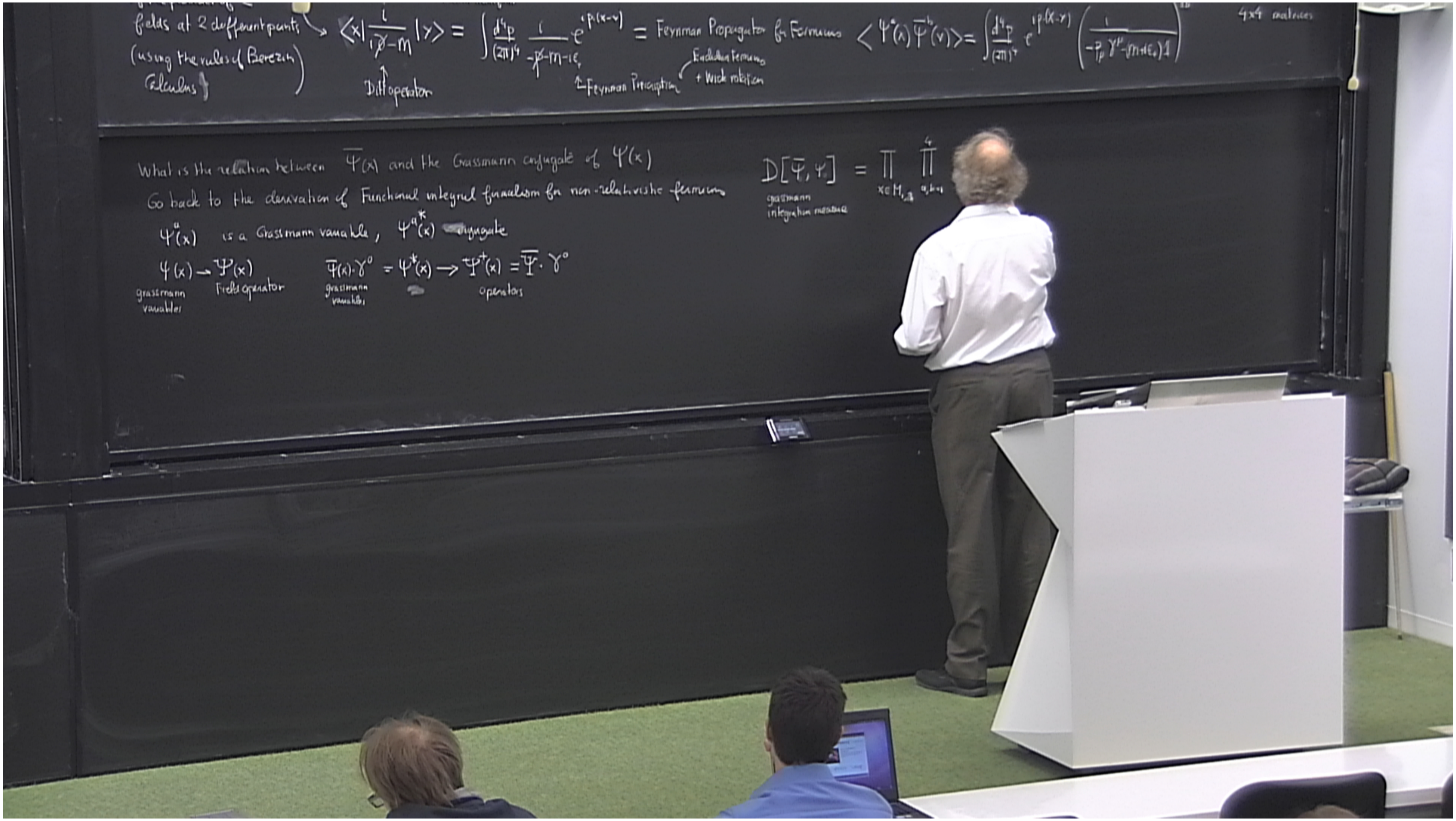

fields at 2 different points
(using the rules of Berezin Calculus)
Diff operator
Feynman Propagator for Fermions
Feynman Prescription
Euclidean Fermions + Wick rotation
4x4 matrices
What is the relation between Ψ̄(x) and the Grassmann conjugate of Ψ(x)
Go back to the derivation of Functional
for non-relativistic fermions
Ψ^a(x) is a Grassmann variable, Ψ^a*(x)

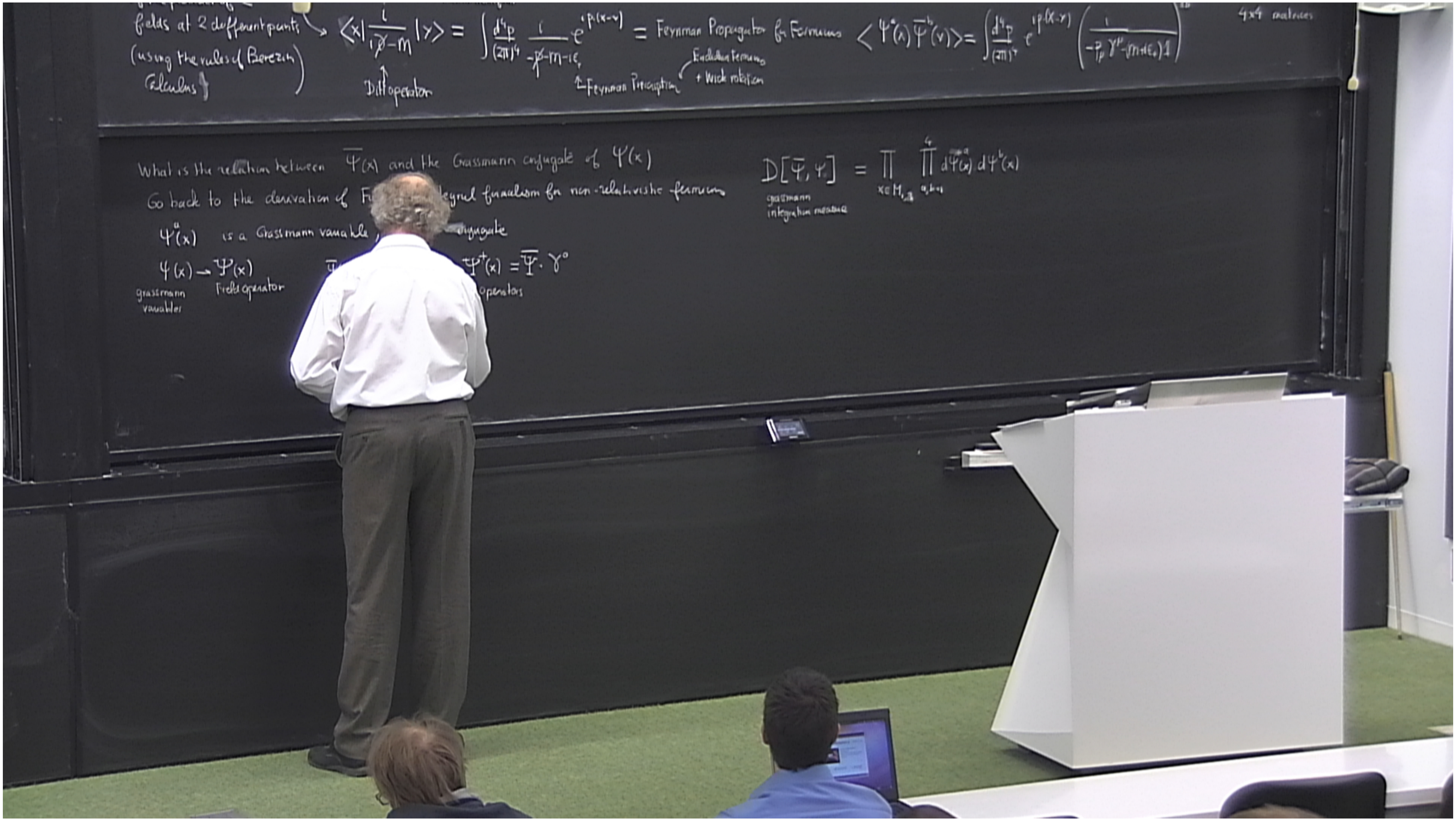

fields at 2 different points
(using the rules of Berezin Calculus)
Feynman Propagator for Fermions
Feynman Prescription
Euclidean Fermions + Wick rotation
4x4 matrices
What is the relation between $\overline{\Psi}(x)$ and the Grassmann conjugate of $\Psi(x)$
Go back to
Functional integral formalism for non-relativistic fermions

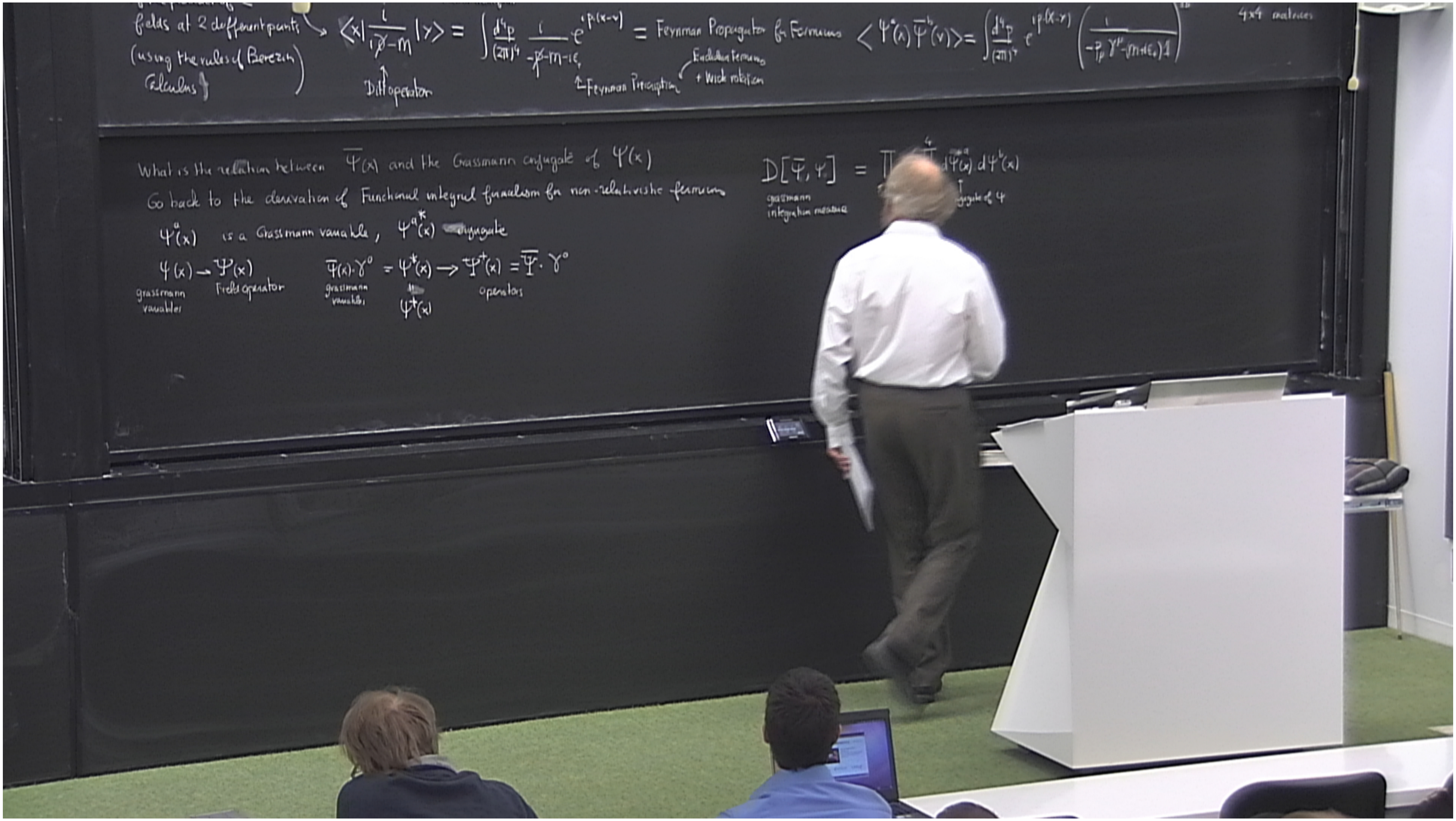


fields at 2 different points
(using the rules of Berezin Calculus)
Diff operator
= Feynman Propagator for Fermions
Feynman Prescription
Euclidean Fermions + Wick rotation
4x4 matrices
What is the relation between $\overline{\Psi}(x)$ and the Grassmann conjugate of $\Psi(x)$
Go back to the derivation of Functional integral for non-relativistic fermions
$\psi^a(x)$ is a Grassmann variable, $\psi^{a*}(x)$ conjugate
$\psi(x) \to \Psi(x)$
Field operator
$\psi^*(x) \to \Psi^+(x)$

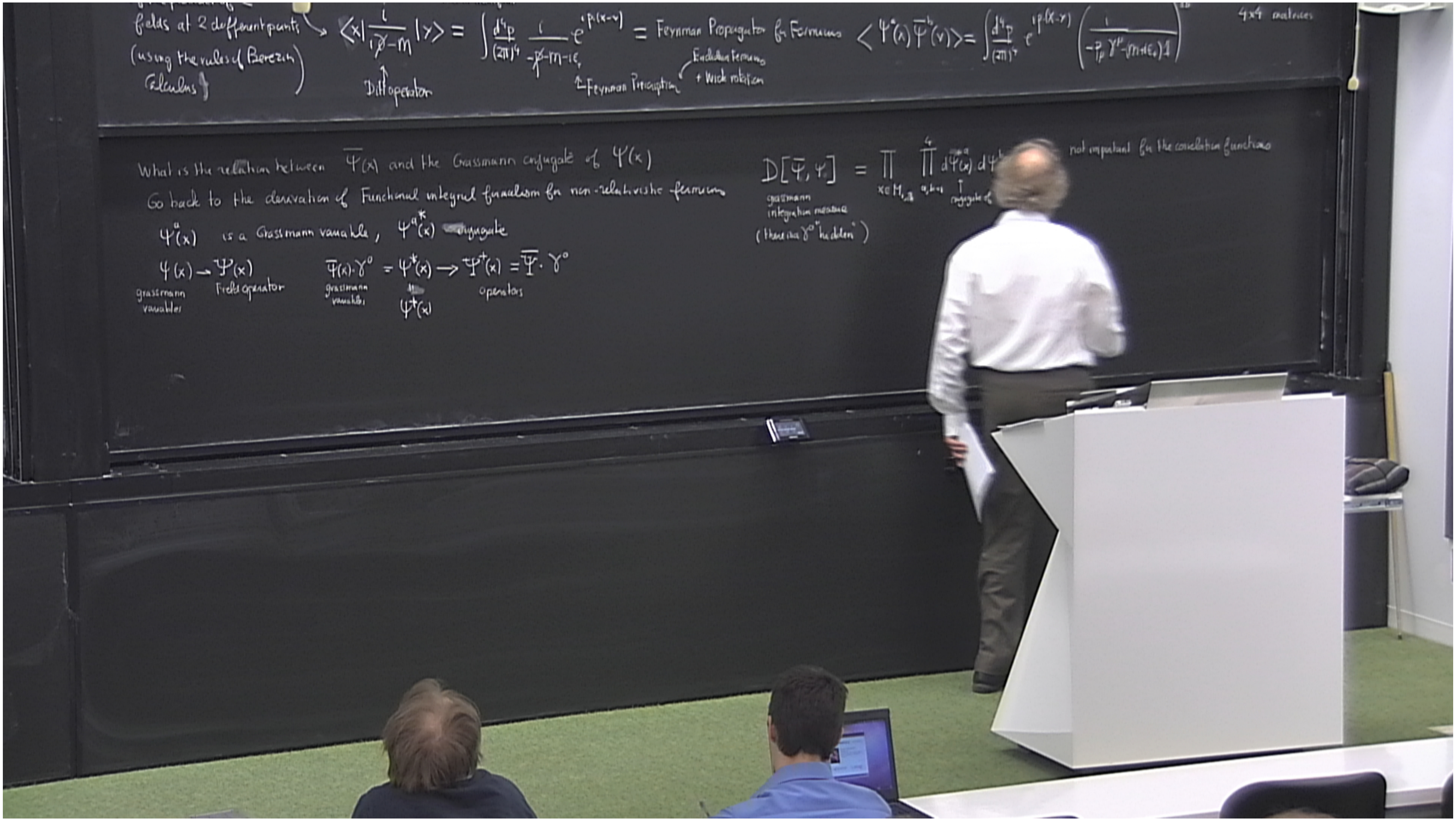

fields at 2 different points (using the rules of Berezin Calculus)
Diff operator
Feynman Propagator for Fermions
Feynman Prescription
Euclidean Fermions + Wick rotation
4x4 matrices
What is the relation between $\bar\Psi(x)$ and the Grassmann conjugate of $\Psi(x)$
to the derivation of Functional integral formalism for non-relativistic fermions
is a Grassmann variable, $\Psi^{a*}(x)$ conjugate
$\bar\Psi(x)\cdot\gamma^0 = \Psi^*(x) \rightarrow \Psi^\dagger(x) = \bar\Psi\cdot\gamma^0$
operators
fields operator

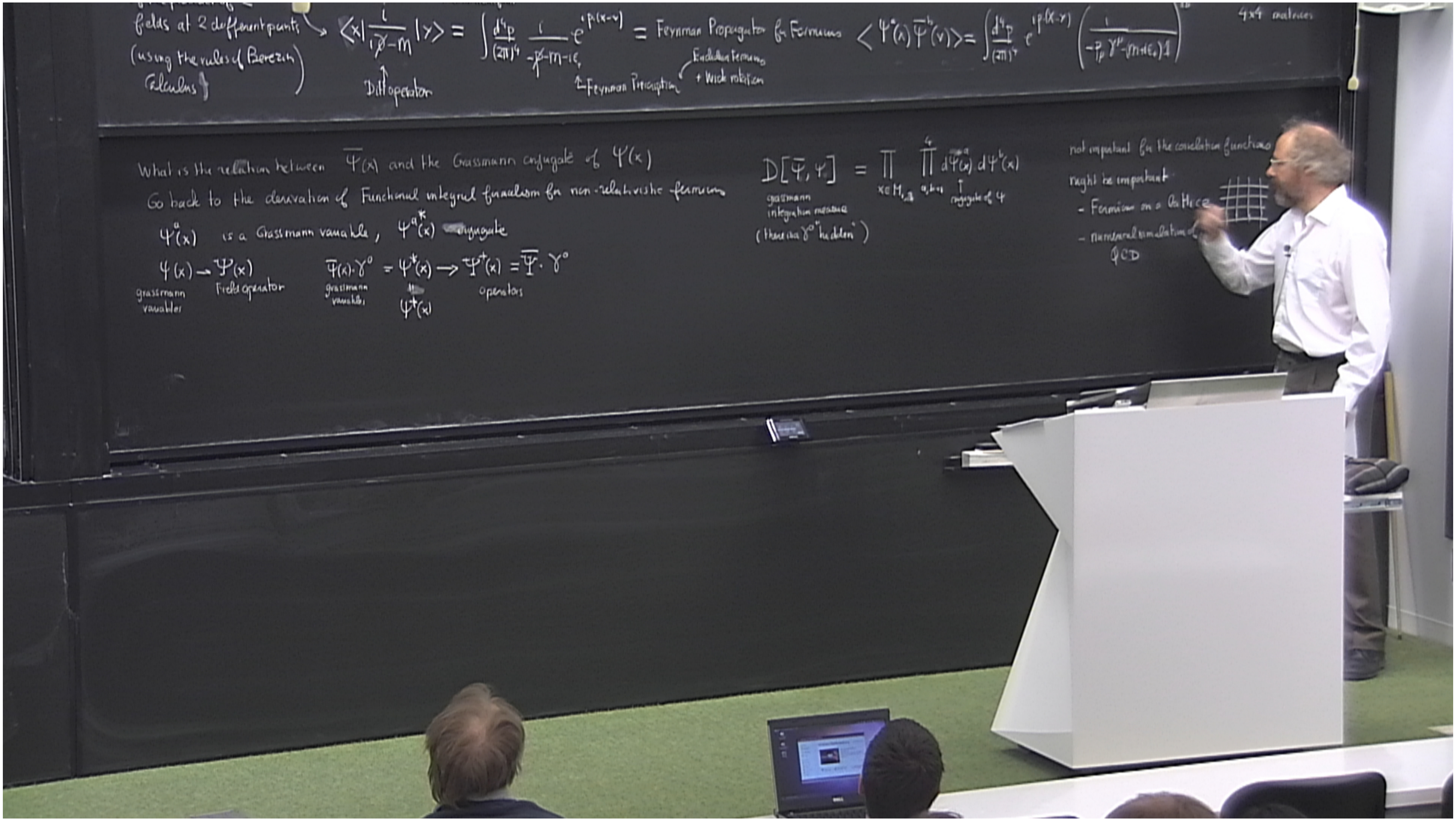

fields at 2 different points
(using the rules of Berezin Calculus)
Diff operator
Feynman Prescription
Euclidean Fermions + Wick rotation
= Feynman Propagator for Fermions
4x4 matrices
What is the relation between $\bar{\Psi}(x)$ and the Grassmann conjugate of $\Psi(x)$
Go back to the derivation of Functional integral formalism for non-relativistic fermions
is a Grassmann variable
conjugate
grassmann variables
Fields operator
grassmann variables
operators
grassmann integration measure

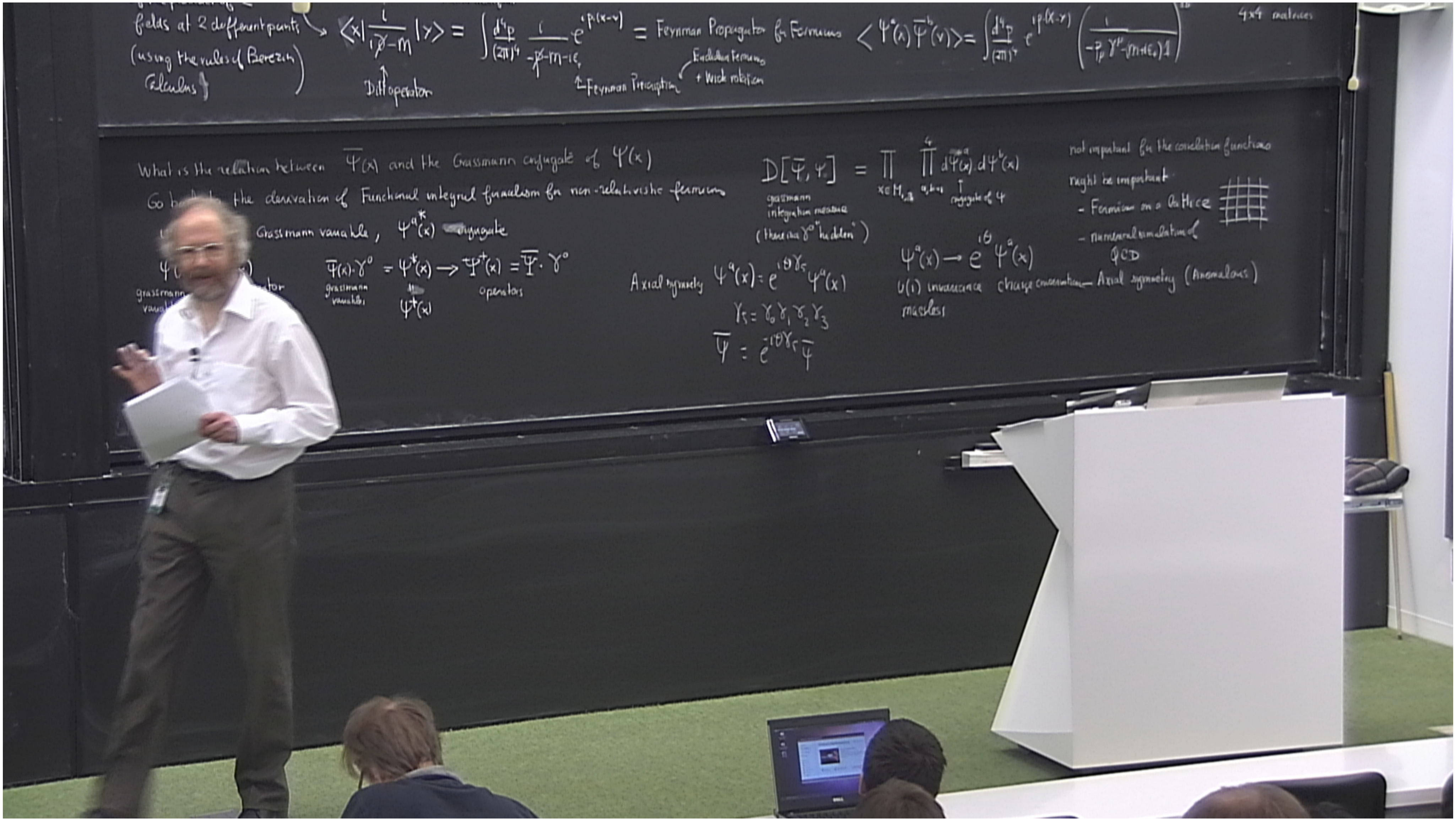

fields at 2 different points
(using the rules of Berezin Calculus)
Diff operator
Feynman Propagator for Fermions
Feynman Prescription
Euclidean Fermions + Wick rotation
4x4 matrices
What is the relation between $\bar{\Psi}(x)$ and the Grassmann conjugate of $\Psi(x)$
Go back to the derivation of F... integral formalism for non-relativistic fermions
$\Psi^a(x)$ is a Grassmann variable
$\Psi(x) \to \Psi(x)$
grassmann variables
Field operator
$\Psi^{+}(x) = \bar{\Psi} \cdot \gamma^0$
operators
grassmann integration measure

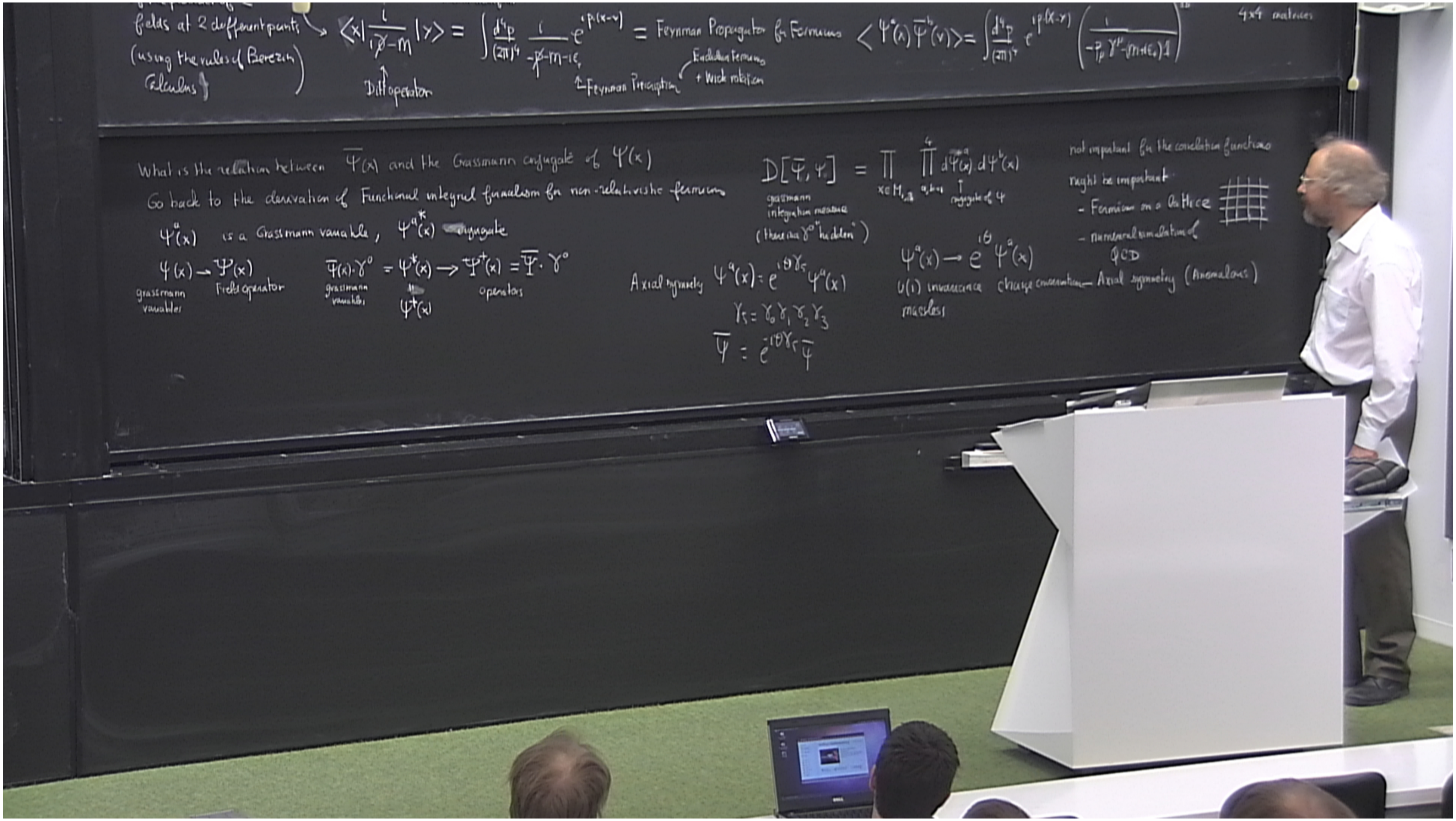


What is the relation between $\overline{\Psi}(x)$ and the Grassmann conjugate of $\Psi(x)$
Go back to the derivation of Functional integral formalism for non-relativistic fermions
$\psi^a(x)$ is a Grassmann variable, $\psi^{a*}(x)$ conjugate
$\psi(x) \to \Psi(x)$
grassmann variables
Fields operator
$\overline{\Psi}(x)\gamma^0 = \psi^*(x) \to \Psi^\dagger(x) = \overline{\Psi}\cdot\gamma^0$
operators
Feynman Propagator for Fermions
Diff operator
Feynman Prescription
+ Wick rotation
4x4 matrices
grassmann
integration measure


fields at 2 different points
(using the rules of Berezin Calculus)
Diff operator
Feynman Propagator for Fermions
Euclidean Fermions
+ Wick rotation
Feynman Prescription
4x4 matrices
What is the relation between Ψ̄(x) and the Grassmann conjugate of Ψ(x)
Go back to the derivation of Functional integral formalism for non-relativistic fermions
is a Grassmann variable,
conjugate
grassmann variables
Fields operator
operators
grassmann integration measure
not important for the correlation functions


fields at 2 different points
(using the rules of Berezin Calculus)
Diff operator
= Feynman Propagator for Fermions
Feynman Prescription
Euclidean Fermions + Wick rotation
4x4 matrices
What is the relation between Ψ̄(x) and the Grassmann conjugate of Ψ(x)
Go back to the derivation of Functional integral formalism for non-relativistic fermions
is a Grassmann variable
conjugate
grassmann variables
Field operator
operators
grassmann integration measure
(there is a γ⁰ "hidden")
not important for the correlation functions
might be important
- Fermions on a Lattice
- numerical simulation of QCD





fields at 2 different points
= Feynman Propagator for Fermions
Diff operator
Feynman Prescription
Euclidean Fermions + Wick rotation
4x4 matrices
What is the relation between $\bar{\Psi}(x)$ and the Grassmann conjugate of $\Psi(x)$
Go back to the derivation of Functional integral formalism for non-relativistic fermions
$\psi^a(x)$ is a Grassmann variable, $\psi^{a*}(x)$ conjugate
grassmann variables
Fields operator
operators
Axial symmetry
$\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$
grassmann integration measure
not important for the correlation functions
might be important
- Fermions on a lattice
- numerical simulation of QCD
U(1) invariance
charge conservation
Axial symmetry (Anomalous)
massless