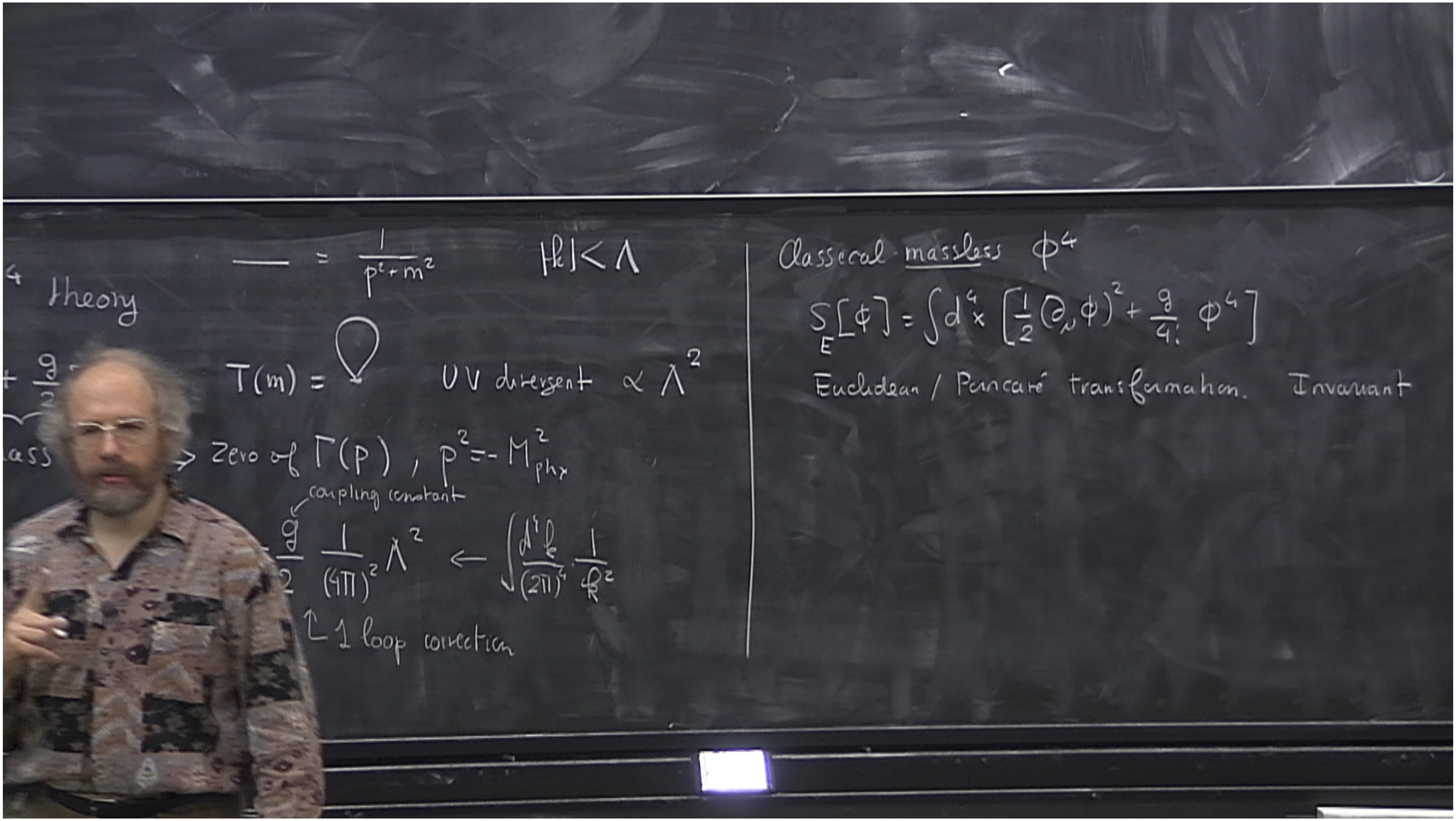
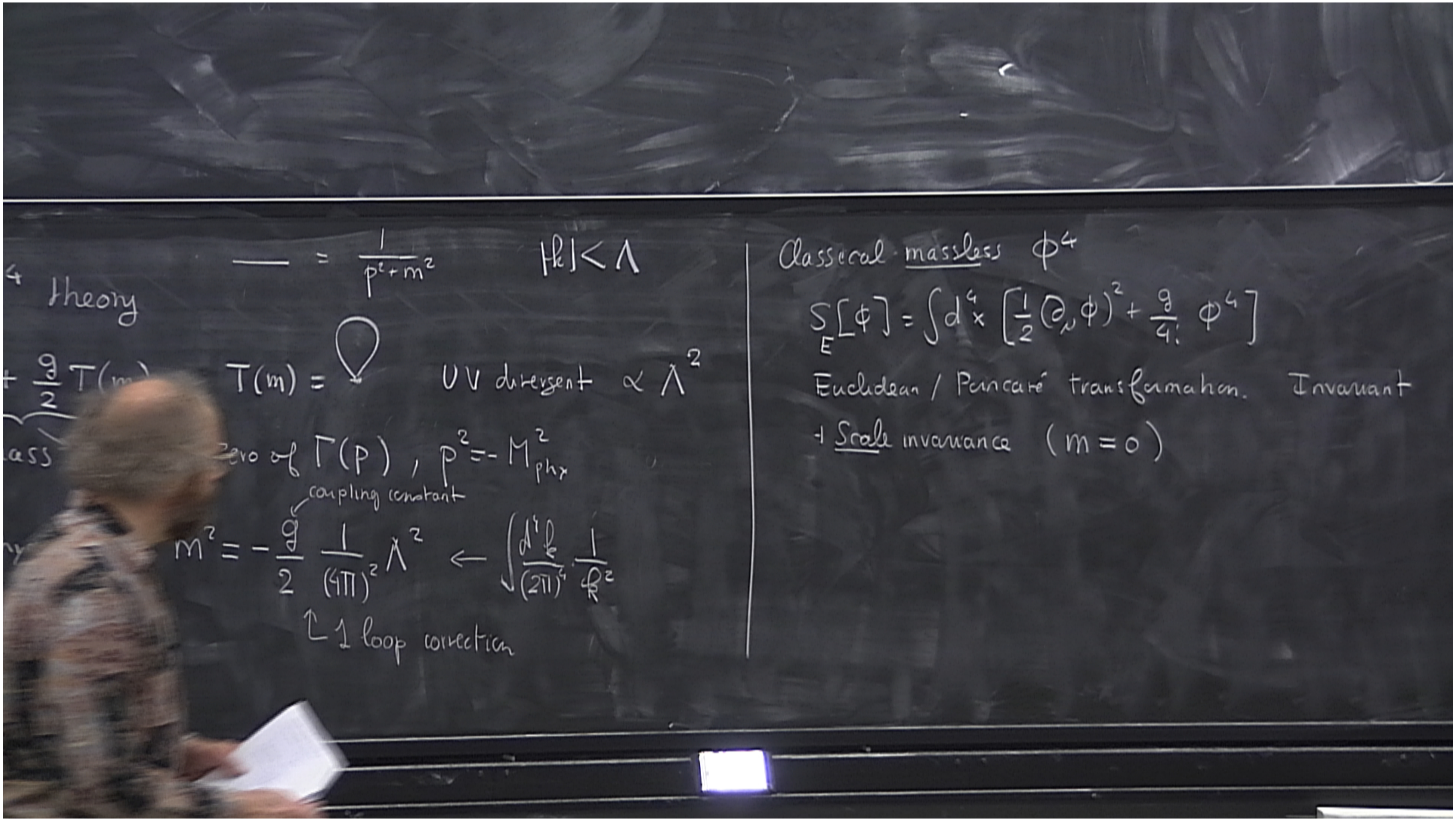
Title: Quantum Field Theory II - Lecture 7

Date: Nov 08, 2011 09:00 AM

URL: http://pirsa.org/11110011

Abstract:


theory
$= \frac{1}{p^2+m^2}$
$|k| < \Lambda$
$T(m) =$
UV divergent $\propto \Lambda^2$
Zero of $\Gamma(p)$, $p^2 = -M^2_{phys}$
coupling constant
$\frac{g}{2}\frac{1}{(4\pi)^2}\Lambda^2 \leftarrow \int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2}$
1 loop correction
Classical massless ϕ^4
$S_E[\phi] = \int d^4x \left[\frac{1}{2}(\partial_\mu\phi)^2 + \frac{g}{4!}\phi^4\right]$
Euclidean / Poincaré transformation. Invariant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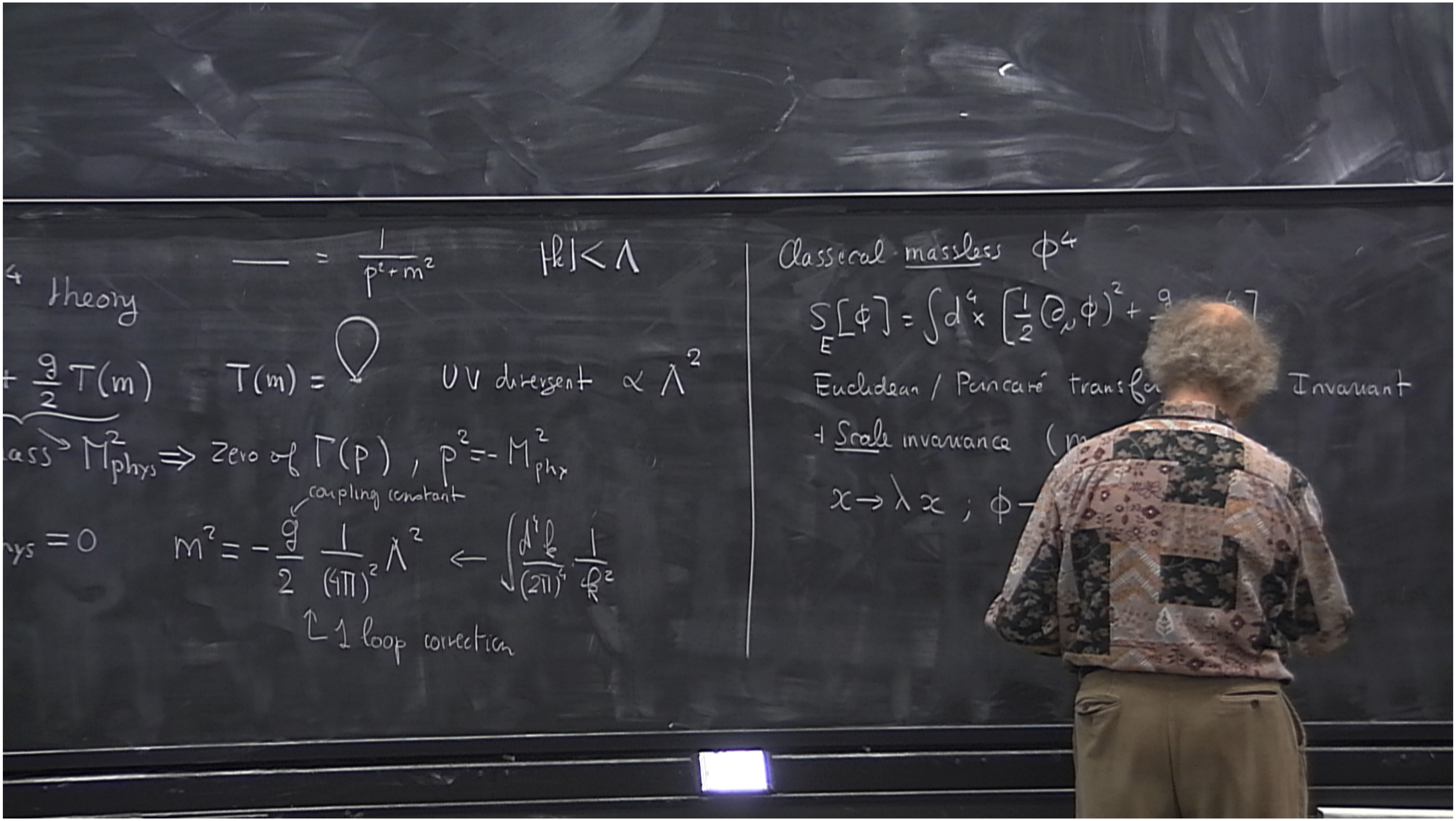
Euclidean / Poincaré transformation. Invariant
+ Scale invariance $(m=0)$


4 theory
$= \frac{1}{p^2+m^2}$
$|k| < \Lambda$
$+ \frac{g}{2} T(m)$
$T(m) =$
UV divergent $\propto \Lambda^2$
ass $\rightarrow M^2_{phys} \Rightarrow$ Zero of $\Gamma(p)$, $p^2 = -M^2_{phys}$
coupling constant
ys $= 0$
$m^2 = -\frac{g}{2} \frac{1}{(4\pi)^2} \Lambda^2 \leftarrow \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2}$
1 loop correction
Classical massless ϕ^4
$S_E[\phi] = \int d^4x \left[\frac{1}{2} (\partial_\mu \phi)^2 + \right.$
Euclidean / Poincaré transf
Invariant
+ Scale invariance (m
$x \rightarrow \lambda x$; ϕ

$|k| < \Lambda$

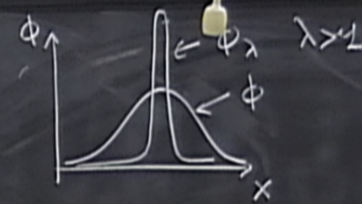
... V divergent $\propto \Lambda^2$

... $^2 = -M^2_{phys}$

$\leftarrow \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2}$

...ection

Classical massless ϕ^4



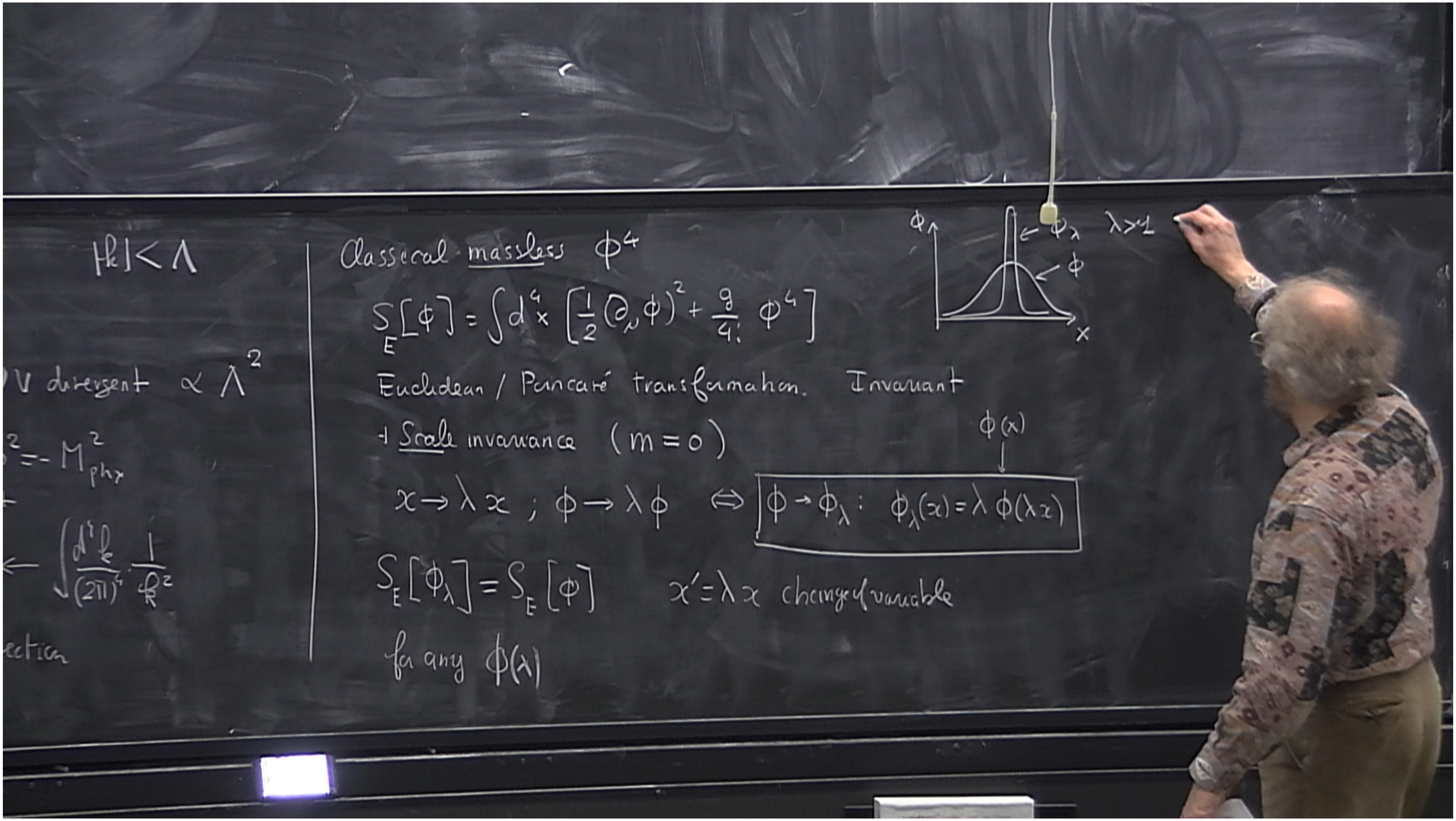
$$S_E[\phi] = \int d^4 x \left[\frac{1}{2} (\partial_\mu \phi)^2 + \frac{g}{4!} \phi^4 \right]$$

Euclidean / Poincaré transformation. Invariant

+ Scale invariance $(m = 0)$

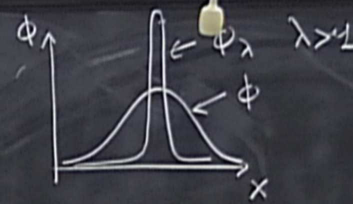
$\phi(x)$

$$x \rightarrow \lambda x \; ; \; \phi \rightarrow \lambda \phi \quad \Leftrightarrow \quad \boxed{\phi \rightarrow \phi_\lambda : \quad \phi_\lambda(x) = \lambda \phi(\lambda x)}$$


|k| < Λ
V divergent ∝ Λ²
= - M²phys
ection
Classical massless φ⁴
S_E[φ] = ∫d⁴x [1/2 (∂_μ φ)² + g/4! φ⁴]
Euclidean / Poincaré transformation. Invariant
+ Scale invariance (m = 0)
x → λx ; φ → λφ ⟺ φ → φ_λ : φ_λ(x) = λ φ(λx)
φ(x)
S_E[φ_λ] = S_E[φ]
x′ = λx change of variable
for any φ(x)
φ_λ
φ
x
λ>1

Classical massless ϕ^4

$$S_E[\phi] = \int d^4x \left[\frac{1}{2}(\partial_\mu \phi)^2 + \frac{g}{4!}\phi^4\right]$$



Euclidean / Poincaré transformation. Invariant

+ Scale invariance $(m = 0)$

$\phi(x)$

$$x \to \lambda x \;;\; \phi \to \lambda \phi \quad \Leftrightarrow \quad \boxed{\phi \to \phi_\lambda : \quad \phi_\lambda(x) = \lambda\,\phi(\lambda x)}$$

$$S_E[\phi_\lambda] = S_E[\phi] \qquad x' = \lambda x \text{ change of variable}$$

for any $\phi(x)$

$$S[\phi_\lambda]$$

$$= \int d^4x \, \frac{1}{2}\left[\frac{\partial}{\partial x}\lambda\phi(\lambda x)\right]^2 + \frac{g}{4!}\lambda^4\phi^4(\lambda x)$$

$$= \int d^4x \, \frac{1}{2}\lambda^2\lambda^2\left[(\partial\phi)(\lambda x)\right]^2 + \frac{g}{4!}\lambda^4\phi^4(\lambda x)$$

$\lambda x = x'$

$$= \int d^4x' \, \frac{1}{2}\left[\partial\phi(x')\right]^2 + \frac{g}{4!}\phi(x')^4$$

$\frac{\partial}{\partial x}\phi$

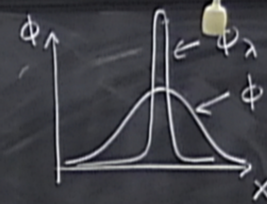
$\lambda x =$

$|k| < \Lambda$

...sent $\propto \Lambda^2$

Classical massless ϕ^4

$$S_E[\phi] = \int d^4x \left[\frac{1}{2} (\partial_\mu \phi)^2 + \frac{g}{4!} \phi^4 \right]$$



Euclidean / Poincaré transformation. Invariant

+ Scale invariance $(m = 0)$

$$x \to \lambda x \ ; \ \phi \to \lambda \phi \iff \boxed{\phi \to \phi_\lambda : \ \phi_\lambda(x) = \lambda \phi(\lambda x)}$$

$\phi(x)$

$$S_E[\phi_\lambda] = S_E[\phi]$$

$x' = \lambda x$ change of variable

for any $\phi(x)$ Symmetry of the theory

$S[\phi_\lambda]$

$$= \int d^4x \, \frac{1}{2} \left[\frac{\partial}{\partial x} \lambda \phi(\lambda x) \right]^2 + \frac{g}{4!} \lambda^4 \phi^4(\lambda x)$$

$$= \int d^4x \, \frac{1}{2} \lambda^2 \lambda^2 \left[(\partial \phi)(\lambda x) \right]^2 + \frac{g}{4!} \lambda^4 \phi^4(\lambda x)$$

$\lambda x = x'$

$$= \int d^4x' \, \frac{1}{2} \left[\partial \phi(x') \right]^2 + g/4! \, \phi(x')^4$$

$\uparrow$ 1 loop correction

$S_E[\phi_\lambda] = S_E$ ...

for any $\phi(x)$

4 point irreducible function $\quad m = 0$

$$\Gamma^{(4)}_{\mathrm{Irr}}(p_1,p_2,p_3,p_4) = g \times - \frac{g^2}{2}\left[(1,2,3,4) + \text{"} + \text{"} \right] = g - \frac{g^2}{2}\left[B(p_1+p_2) + B(p_1+p_3) + B(p_1+p_4) \right]$$

depends on the details of the regularization scheme ch...

$p_1 + p_2 + p_3 + p_4 = 0$

$$B(p) = \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2} \cdot \frac{1}{(k-p)^2} = \frac{1}{(4\pi)^2} \log\left(\frac{\Lambda^2}{p^2}\right) + C_{st}$$

only 2 momentum scales $|p|$ & Λ

UV divergent

"universal" UV divergence

terms of order 1

C_{st} can be set to 0 in the calculation

What is the physical coupling constant? Scattering Process $\leftarrow |S_{\mathrm{matrix}}|^2 \leftarrow$ Irr 4pt

momenta $p = (E, \vec{p})$

$\uparrow$ 1 loop correction

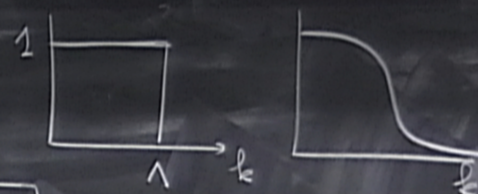
$S_E[\phi_\lambda] = S_E[\phi]$ $\quad x' = \lambda x$ change of variable

for any $\phi(x)$ Symmetry of the theory $\boxed{\dim[\phi] = 1, \ldots}$

$m = 0$

$\ldots + \, '' \, + \, '' \Big] = g - \frac{g^2}{2}\Big[B(p_1+p_2) + B(p_1+p_3) + B(p_1+p_4)\Big]$

depends on the details of the regularization scheme choosen



$B(p) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} \cdot \frac{1}{(k-p)^2} = \frac{1}{(4\pi)^2} \log\left(\frac{\Lambda^2}{p^2}\right) + C_{st}$

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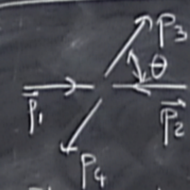


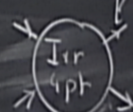
Elastic collision in the rest frame

(E, θ)

momenta $p = (E, \vec{p}_a)$ $\quad a = 1, \ldots 4$

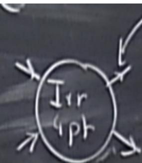
? Scattering Process

$\leftarrow |S_{matrix}|^2 \longleftarrow$



Scattering Process

$\leftarrow |S_{matrix}|^2 \longleftarrow$



$p = (E, p_a)$ $\quad a = 1, \ldots 4$ $\quad (E, \theta)$

energy or momentum scale

...lant g_R : $\quad p^2 = (p_0^2 + \vec{p}^{\,2}) = (-E^2 + \vec{p}^{\,2})$

Euclidean $\qquad$ Minkowski

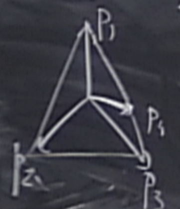
some fixed value of momenta such that

$p_a \approx \mu$ some energy / momentum scale

⚠ $p_1 \cdots p_4$ are not on mass-shell

$p_1^2 = 0$

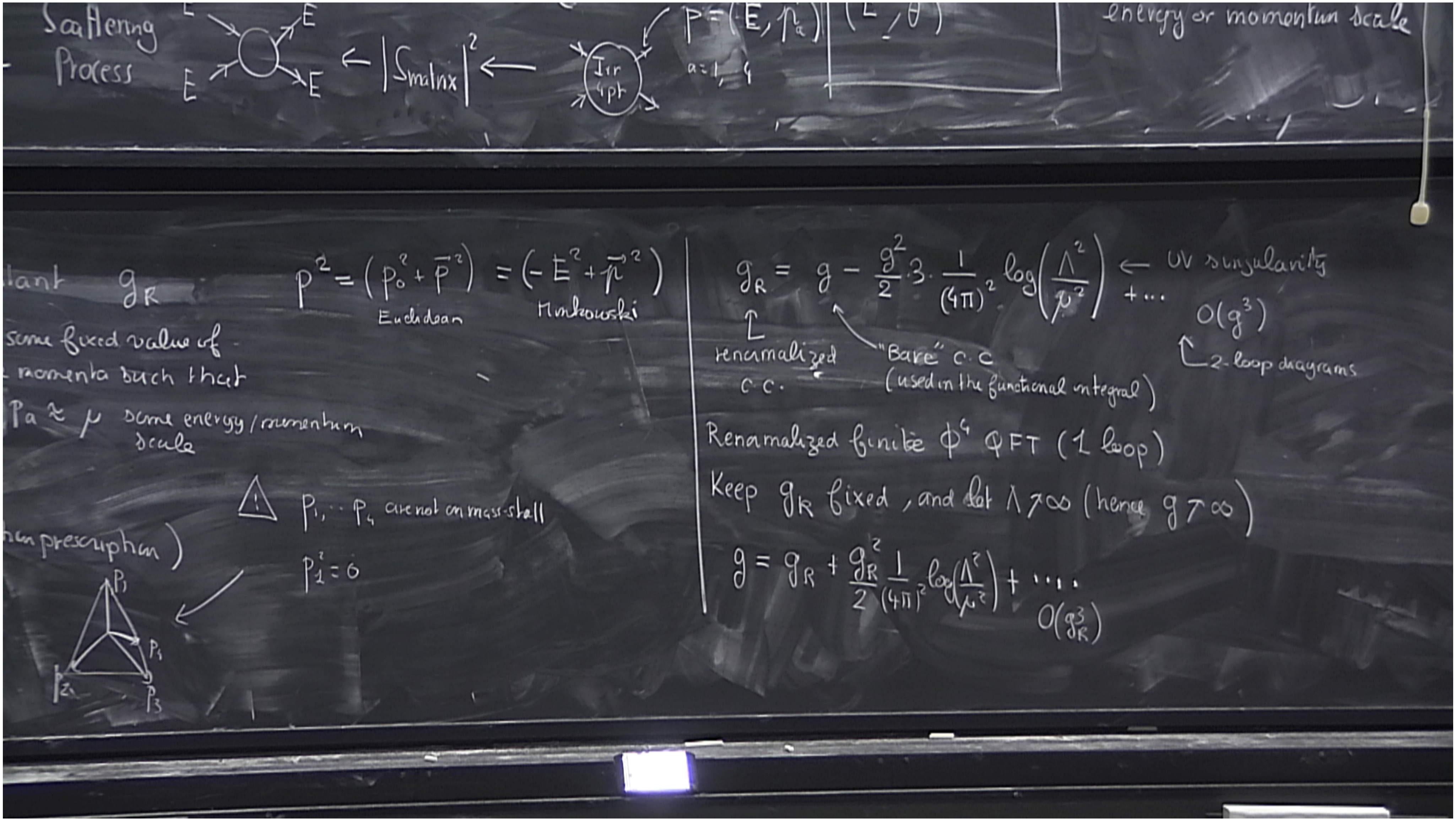
...ion prescription)



$$g_R = g - \frac{g^2}{2} \cdot 3 \cdot \frac{1}{(4\pi)^2} \log\left(\frac{\Lambda^2}{\mu^2}\right) \leftarrow \text{UV singularity}$$

renormalized c.c.

"Bare" c.c (used in the functional integral)



Scattering Process
|Smatrix|^2
Irr 4pt
p = (E, p_a)
a = 1, 4
energy or momentum scale
g_R
p^2 = (p_0^2 + p^2) = (-E^2 + p^2)
Euclidean
Minkowski
some fixed value of
momenta such that
p_a ≈ μ some energy/momentum scale
p_1 ⋯ p_4 are not on mass-shell
p_1^2 = 0
g_R = g − g^2/2 · 3 · 1/(4π)^2 log(Λ^2/μ^2) + ...
UV singularity
O(g^3)
2-loop diagrams
renormalized c.c.
"Bare" c.c (used in the functional integral)
Renormalized finite φ^4 QFT (1 loop)
Keep g_R fixed, and let Λ → ∞ (hence g → ∞)
g = g_R + g_R^2/2 · 1/(4π)^2 log(Λ^2/μ^2) + ...
O(g_R^3)

2-loop

4 point function of the renormalized theory

$$\Gamma_R(p_1, \dots p_4) = g_R - \frac{g_R^2}{2}\frac{1}{(4\pi)^2}\left[\log\left(\frac{\mu^2}{(p_1+p_2)^2}\right) + \log\left(\frac{\mu^2}{(p_1+p_2)^2}\right) + \log\left(\frac{\mu^2}{(p_1+p_4)^2}\right)\right] + O(g_R^3) = \text{Ren}$$

4 momenta

Is finite for any p_a

$$\Gamma_p(p_1, p_2) = \text{Ren}$$

4 point function of the renormalized theory

2-loop

$$\Gamma_R(p_1, \dots p_4) = g_R - \frac{g_R^2}{2}\frac{1}{(4\pi)^2}\left[\log\left(\frac{\mu^2}{(p_1+p_2)^2}\right) + \log\left(\frac{\mu^2}{(p_1+p_3)^2}\right) + \log\left(\frac{\mu^2}{(p_1+p_4)^2}\right)\right] + O(g_R^3) = \text{Ren}$$

4 momenta

Is finite for any p's

2 loops

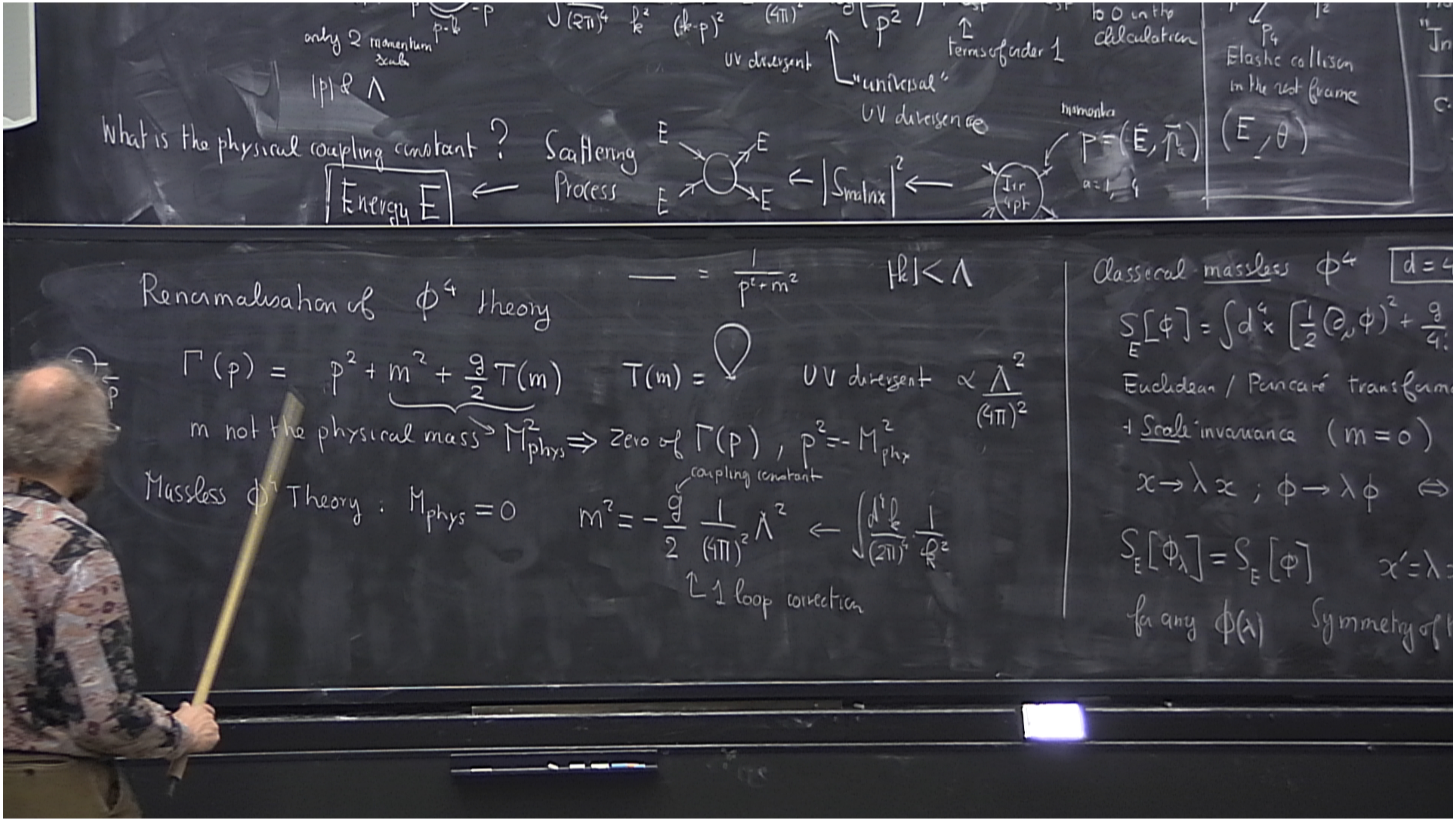
$$\Gamma_p(p_1, p_2) = -\text{Ren}- = p^2 + O(g_R^2)$$ also UV finite

all the N-point function are still UV finite

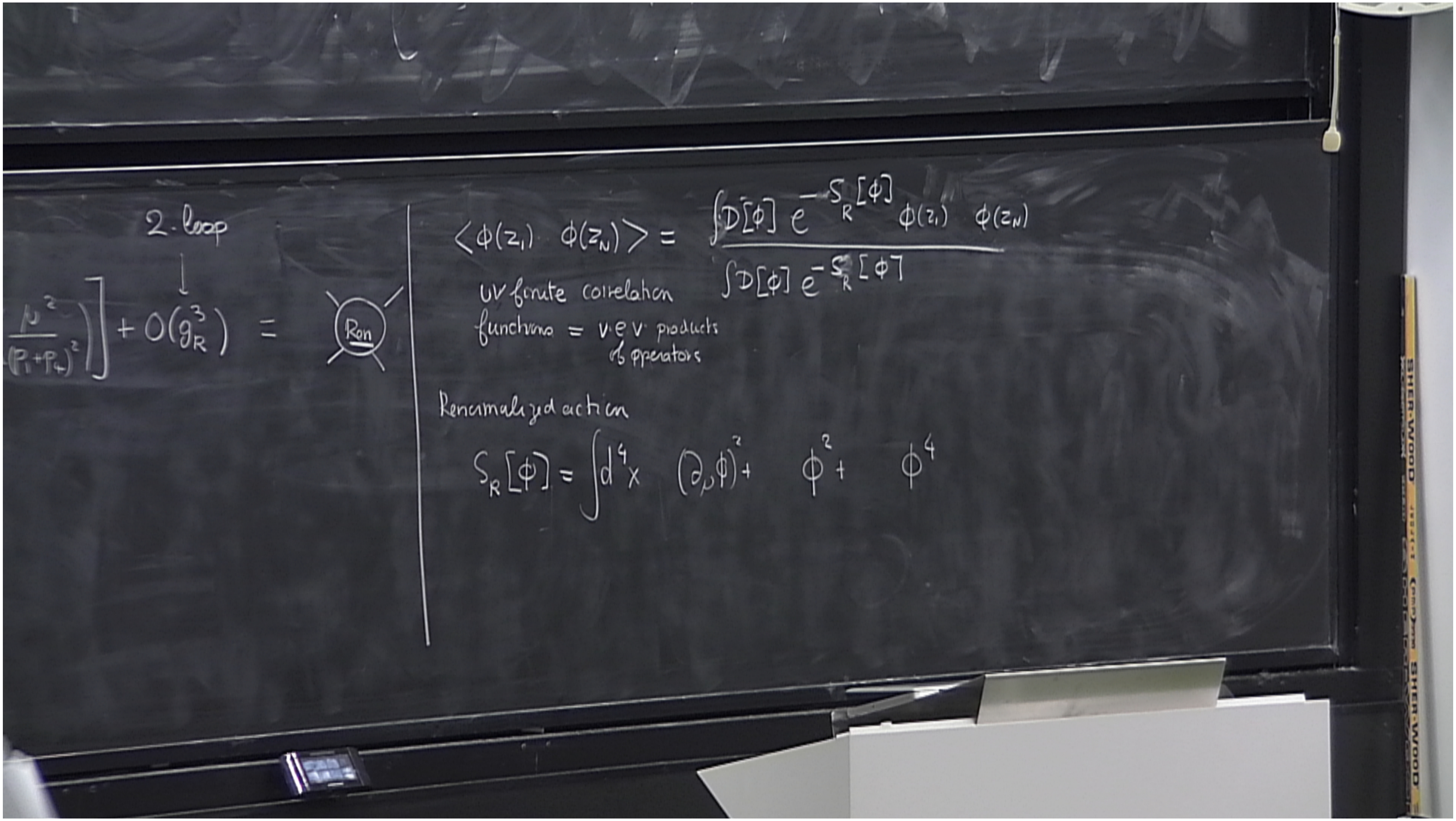
2-loop diagrams

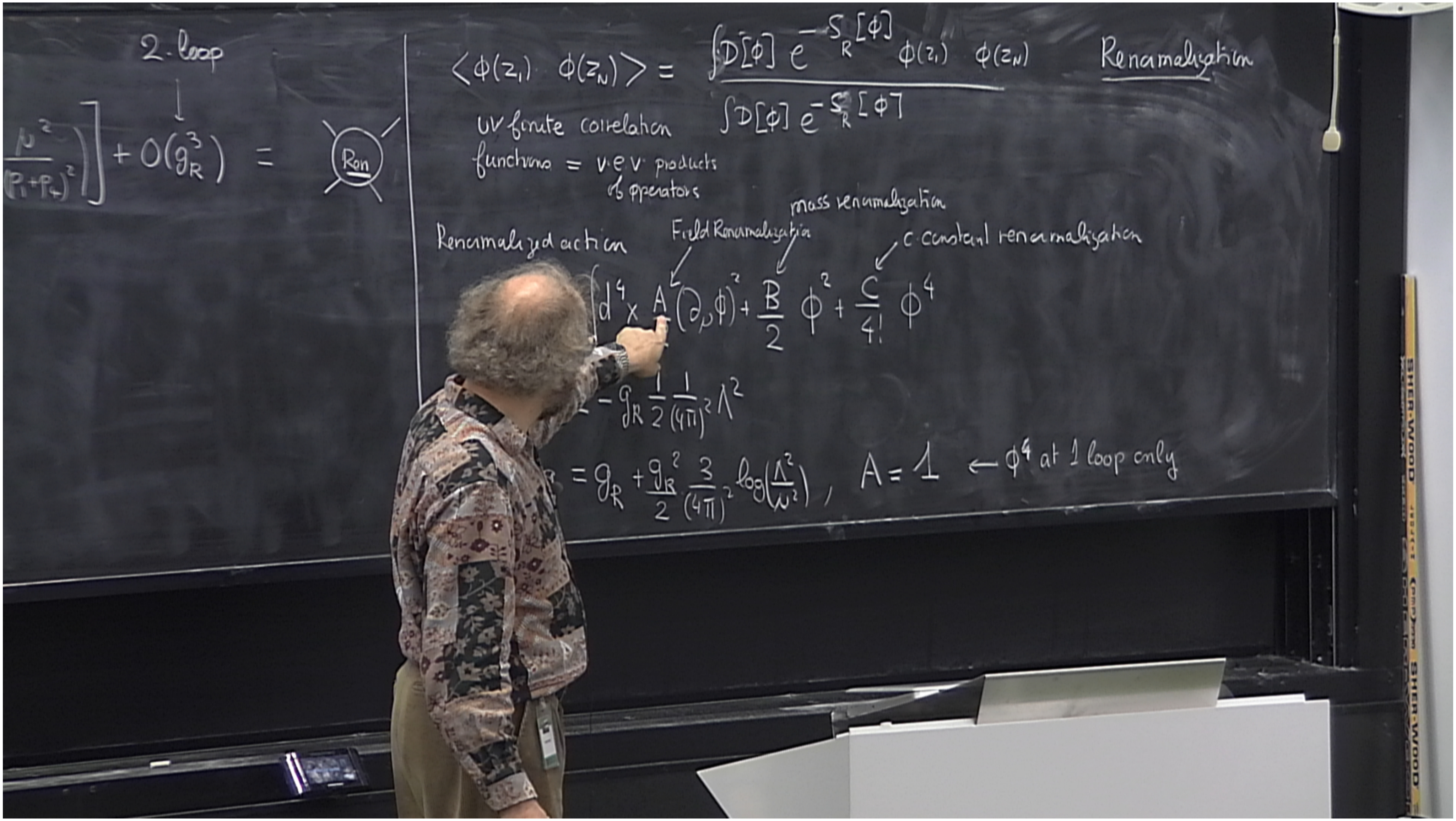
$$\text{Irr.} = g^3 \int^\infty \frac{d^4k}{|k|^6} = g_R^3 + O(g_R^4)$$

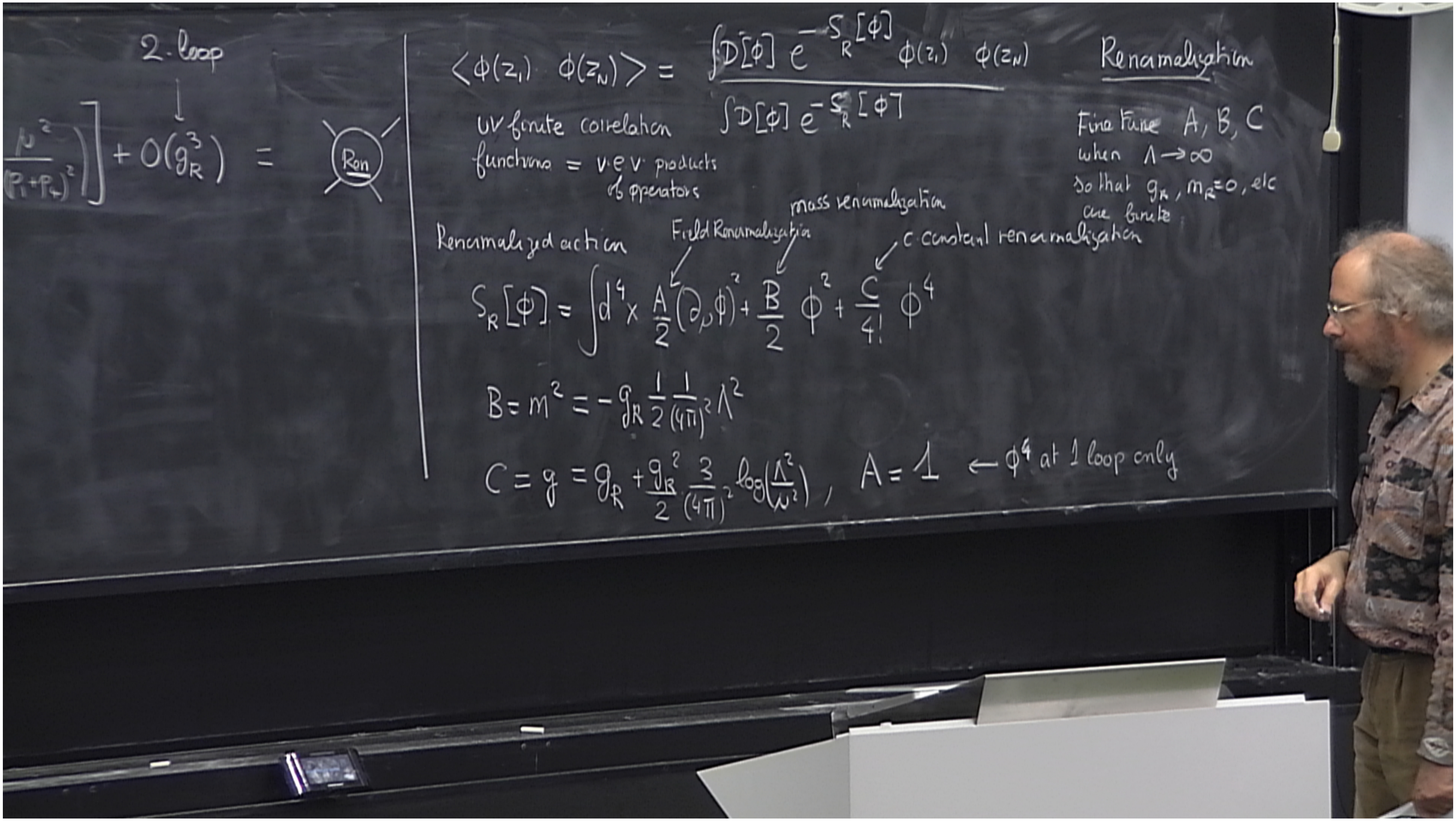
UV finite



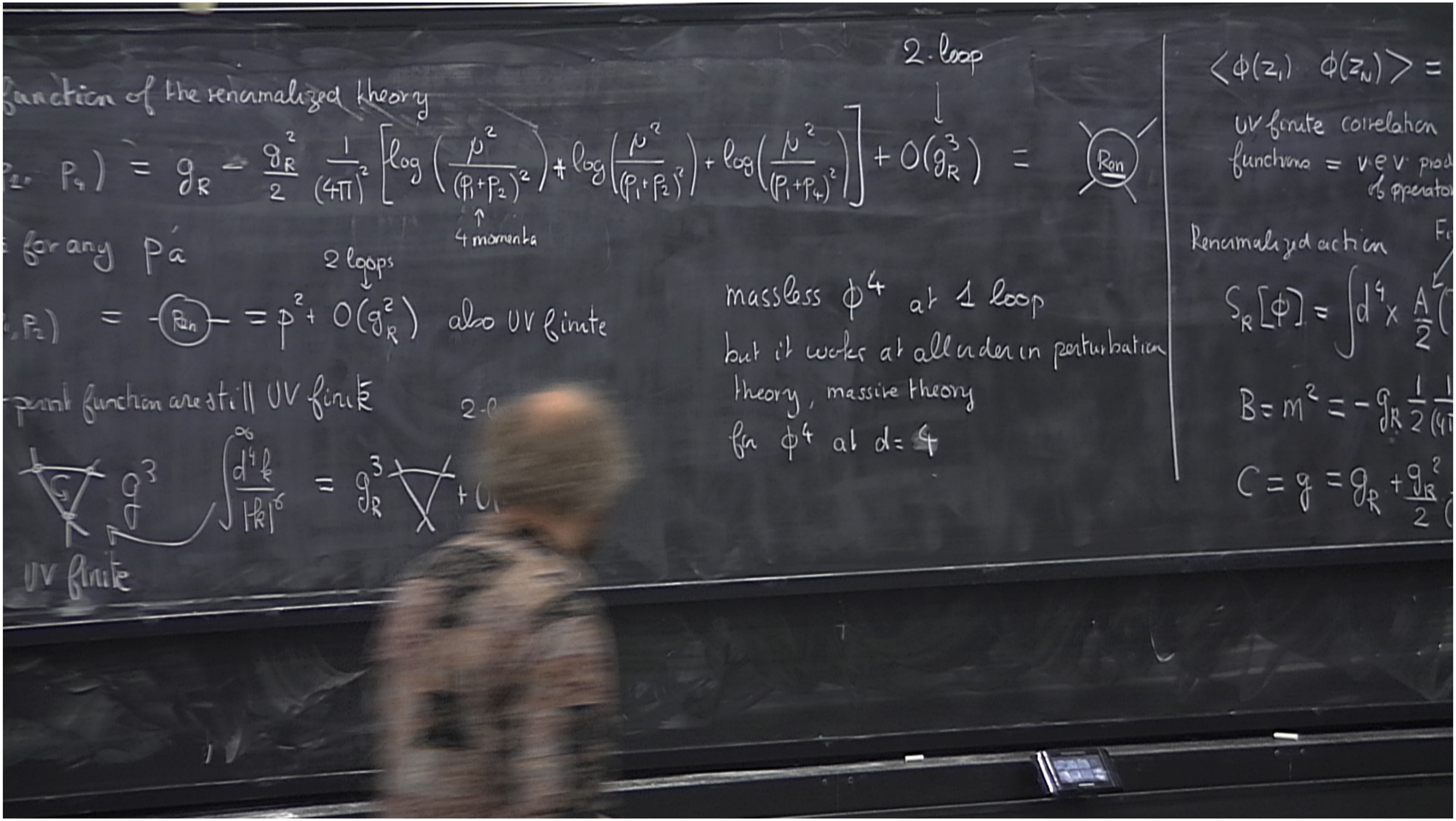
only 2 momentum scales
|p| & Λ
UV divergent
"universal" UV divergence
terms of order 1
to 0 in the calculation
Elastic collision in the rest frame
momenta
p = (E, p)
(E, θ)
What is the physical coupling constant?
Energy E
Scattering Process
|S matrix|²
Renormalisation of φ⁴ theory
|k| < Λ
Γ(p) = p² + m² + g/2 T(m)
T(m) =
UV divergent ∝ Λ²/(4π)²
m not the physical mass → M²phys ⇒ zero of Γ(p), p² = − M²phys
coupling constant
Massless φ⁴ Theory : Mphys = 0
m² = − g/2 1/(4π)² Λ² ← ∫ d⁴k/(2π)⁴ 1/k²
1 loop correction
Classical massless φ⁴
d = 4
Euclidean / Poincaré transform
+ Scale invariance (m = 0)
x → λx ; φ → λφ
S_E[φ_λ] = S_E[φ]
for any φ(λ)
Symmetry of

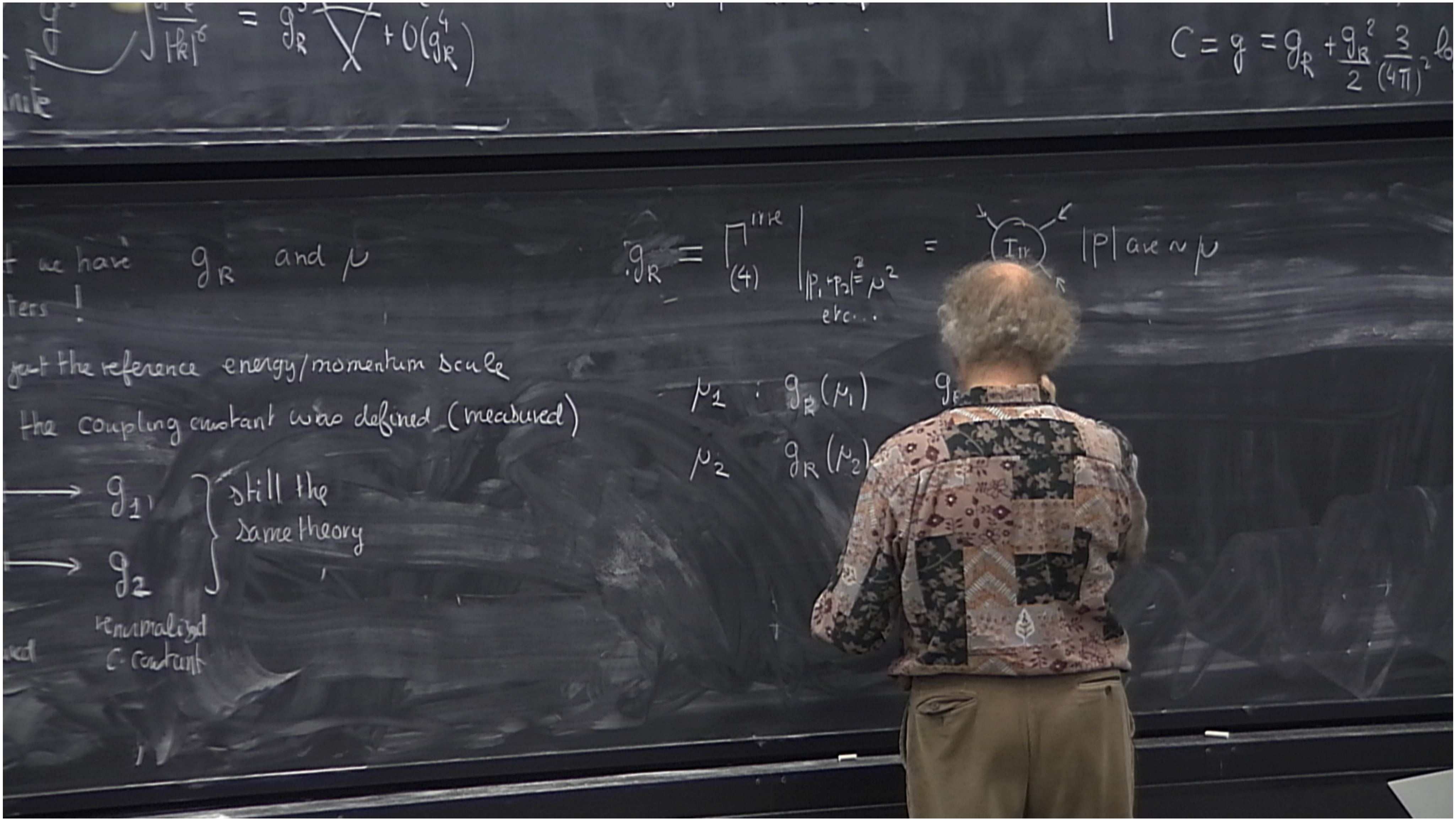

2-loop
$\frac{\mu^2}{(p_1+p_4)^2} \Big) \Big] + O(g_R^3) =$ Ron
$\langle \phi(z_1) \cdots \phi(z_N) \rangle = \frac{\int D[\phi]\, e^{-S_R[\phi]}\, \phi(z_1) \cdots \phi(z_N)}{\int D[\phi]\, e^{-S_R[\phi]}}$
UV finite correlation functions = v.e.v. products of operators
Renormalized action
$S_R[\phi] = \int d^4x \ (\partial_\mu \phi)^2 + \ \phi^2 + \ \phi^4$


2-loop
Renormalisation
UV finite correlation functions = v.e.v. products of operators
Renormalized action
Field Renormalisation
mass renormalisation
c. constant renormalisation
A = 1 ← φ⁴ at 1 loop only



2-loop
$\frac{\mu^2}{(p_1+p_4)^2})\Big] + O(g_R^3) =$
Ren
$\langle \phi(z_1) \cdots \phi(z_N) \rangle = \dfrac{\int D[\phi]\, e^{-S_R[\phi]}\, \phi(z_1) \cdots \phi(z_N)}{\int D[\phi]\, e^{-S_R[\phi]}}$
UV finite correlation functions = v.e.v. products of operators
Renormalisation
Fine tune A, B, C when $\Lambda \to \infty$ so that g_R, $m_R = 0$, etc are finite
mass renormalization
Renormalized action
Field Renormalization
c constant renormalisation
$S_R[\phi] = \int d^4x\, \frac{A}{2} (\partial_\mu \phi)^2 + \frac{B}{2} \phi^2 + \frac{C}{4!} \phi^4$
$B = m^2 = -g_R \frac{1}{2} \frac{1}{(4\pi)^2} \Lambda^2$
$C = g = g_R + \frac{g_R^2}{2} \frac{3}{(4\pi)^2} \log\left(\frac{\Lambda^2}{\mu^2}\right)$, $A = 1$ $\leftarrow \phi^4$ at 1 loop only

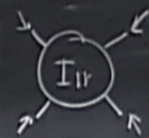

2-loop
⟨φ(z_1) φ(z_N)⟩ =
function of the renormalized theory
UV finite correlation functions = v.e.v. prod of operators
Ren
4 momenta
for any p_a
2 loops
Renormalized action
massless φ^4 at 1 loop
but it works at all order in perturbation theory, massive theory
for φ^4 at d = 4
also UV finite
point function are still UV finite
UV finite


$= g_R^3 + O(g_R^4)$
$C = g = g_R + \frac{g_R^2}{2} \frac{3}{(4\pi)^2}$
we have g_R and μ
ters !
$g_R = \Gamma^{irr}_{(4)} \Big|_{|p_1+p_2|^2=\mu^2}$
etc...
$|p|$ are $\sim \mu$
the reference energy/momentum scale
the coupling constant was defined (measured)
$\mu_1 : g_R(\mu_1)$
$\mu_2 \quad g_R(\mu_2)$
g_1
g_2
still the same theory
renormalized
C-constant

$+ O(g_R^4)$

$C = g = g_R + \frac{g_R^2}{2}\frac{3}{(4\pi)^2}\log\left(\frac{\Lambda^2}{\mu^2}\right)$, $A = 1$ ← ϕ^4 at

μ

$g_R = \Gamma^{irr}_{(4)}\Big|_{|p_1+p_2|^2=\mu^2}$ etc... $=$ 

$|p|$ are $\sim \mu$

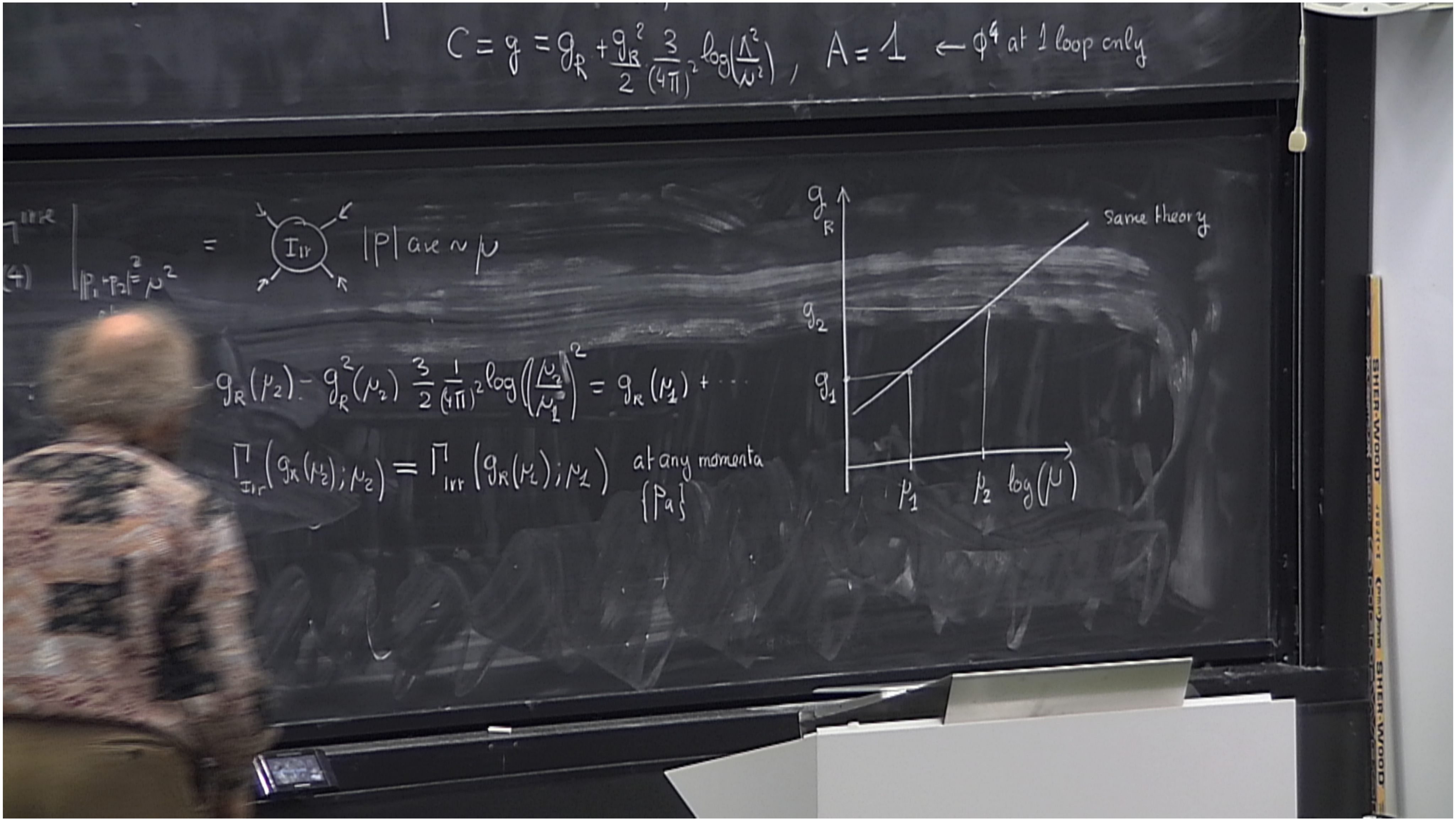
omentum scale

o defined (measured)

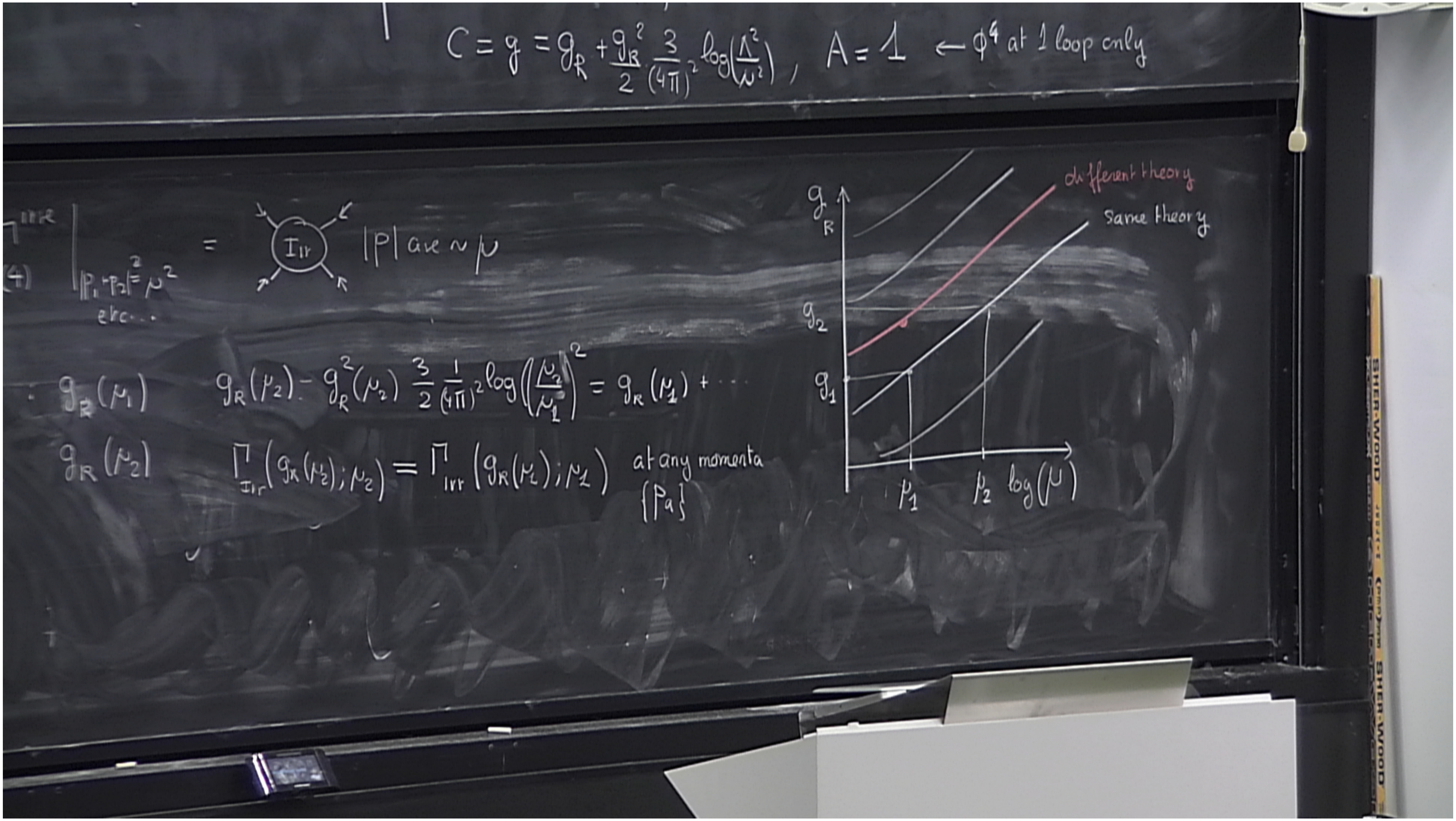
μ_1 : $g_R(\mu_1)$ $\qquad g_R(\mu_2) - g_R^2(\mu_2)\,\frac{3}{2}\frac{1}{(4\pi)^2}\log\left(\frac{\mu_2}{\mu_1}\right)^2 = g_R(\mu_1) + \cdots$

μ_2 $\quad g_R(\mu_2)$ $\qquad \Gamma_{Irr}(g_R(\mu_2);\mu_2) = \Gamma_{irr}(g_R(\mu_1);\mu_1)$ at any momenta $\{p_a\}$

ory



C = g = g_R + g_R^2/2 3/(4π)^2 log(Λ^2/μ^2) , A = 1 ← φ^4 at 1 loop only
|p| are ~ μ
same theory
g_R(μ_2) - g_R^2(μ_2) 3/2 1/(4π)^2 log((μ_2/μ_1)^2) = g_R(μ_1) + ...
Γ_Irr(g_R(μ_2); μ_2) = Γ_Irr(g_R(μ_1); μ_1) at any momenta {p_a}
g_R
g_2
g_1
μ_1
μ_2
log(μ)



$C = g = g_R + \frac{g_R^2}{2} \frac{3}{(4\pi)^2} \log\left(\frac{\Lambda^2}{\mu^2}\right)$, $A = 1$ ← ϕ^4 at 1 loop only
$\Gamma^{irr}_{(4)} \Big|_{|p_1+p_2|^2=\mu^2}$ etc...
$= I_{irr}$ $|p|$ ave $\sim \mu$
$g_R(\mu_1)$
$g_R(\mu_2)$
$g_R(\mu_2) - g_R^2(\mu_2) \frac{3}{2} \frac{1}{(4\pi)^2} \log\left(\frac{\mu_2}{\mu_1}\right)^2 = g_R(\mu_1) + \cdots$
$\Gamma_{irr}(g_R(\mu_2);\mu_2) = \Gamma_{irr}(g_R(\mu_1);\mu_1)$ at any momenta $\{p_a\}$
g_R
g_2
g_1
μ_1
μ_2
$\log(\mu)$
different theory
same theory

In a QFT (renormalization): coupling constant(s) depend on the energy scale!

"running coupling constants" $\leftarrow$ encoded into a differential equation.

Renormalization Group Flow equation

Fixed energy scale μ_0, g_R^0 Fixed

In a QFT (renormalization) : coupling constant(s) depend on the energy scale !

"running coupling constants" ← encoded into a differential equation.

Renormalization Group Flow equation

Fixed energy scale μ_0, g_R^0 Fixed

$\mu \neq \mu_0$. $g_R(\mu) = ?$

$\mu = \mu_0 + d\mu$

$g_R(\mu) = g_R^0 + d g_R$

$$\boxed{\mu \frac{d g_R(\mu)}{d\mu} = \frac{3}{(4\pi)^2} g_R^2(\mu) =}$$

...lization) : coupling constant(s) depend on the energy scale !

...constants" ← encoded into a differential equation.

Renormalization Group Flow equation.

μ_0, g_R^0 Fixed $\qquad \mu = \mu_0 + d\mu$

$\mu \neq \mu_0$. $g_R(\mu) = ?$ $\qquad g_R(\mu) = g_R^0 + d g_R$

$$\boxed{\mu \frac{d g_R(\mu)}{d\mu} = \frac{3}{(4\pi)^2} g_R^2(\mu) = \beta(g_R(\mu))}$$

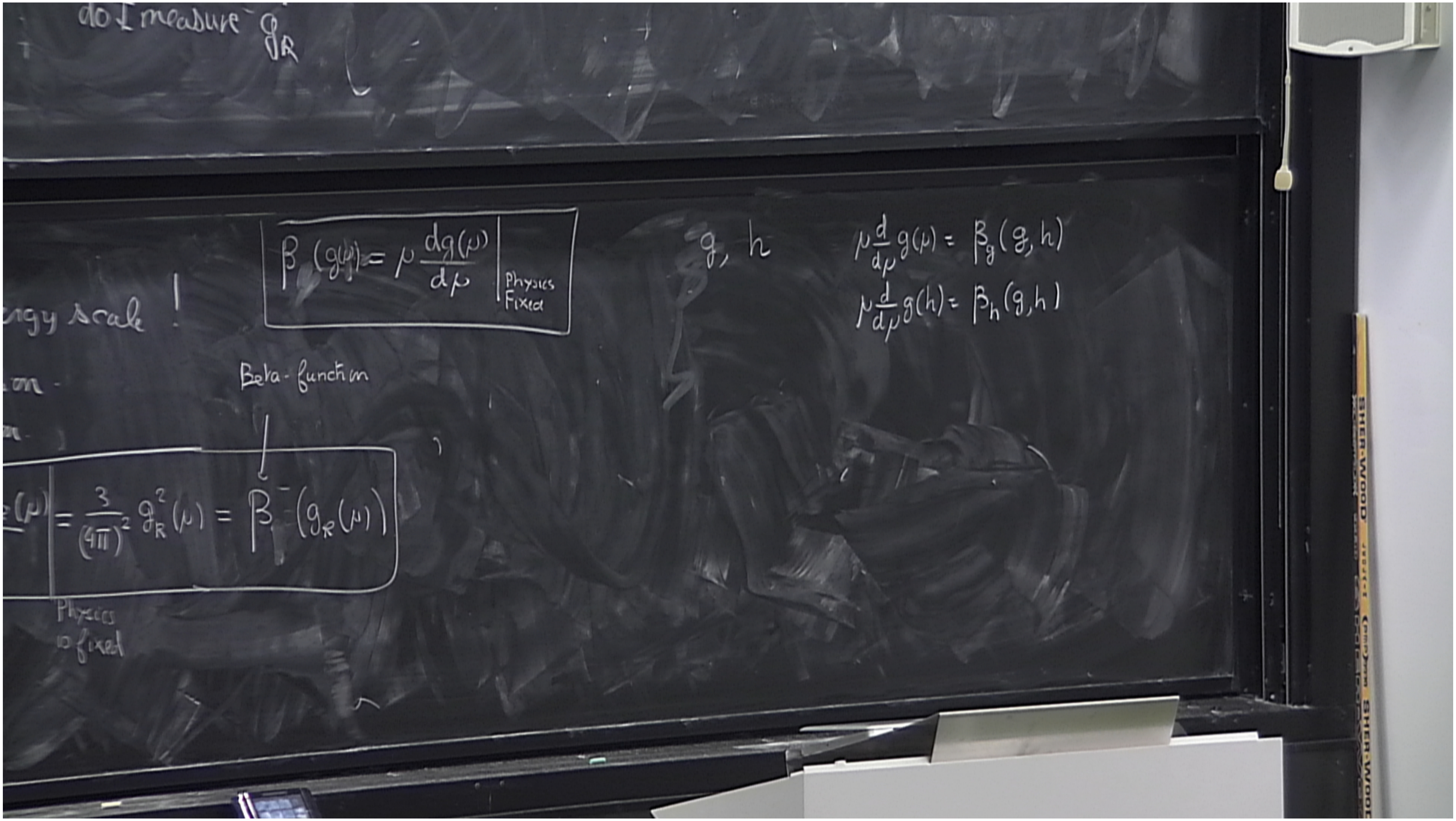
Physics is fixed

Beta-function

$$\boxed{\beta(g(\mu)) = \mu \frac{dg(\mu)}{d\mu}}$$

Physics Fixed

g, h


do I measure g_R
$\boxed{\beta(g(\mu)) = \mu \frac{dg(\mu)}{d\mu}\Big|_{\text{Physics Fixed}}}$
g, h
$\mu \frac{d}{d\mu} g(\mu) = \beta_g(g, h)$
$\mu \frac{d}{d\mu} g(h) = \beta_h(g, h)$
ergy scale !
Beta-function
$(\mu)\Big| = \frac{3}{(4\pi)^2} g_R^2(\mu) = \beta(g_R(\mu))$
Physics is fixed

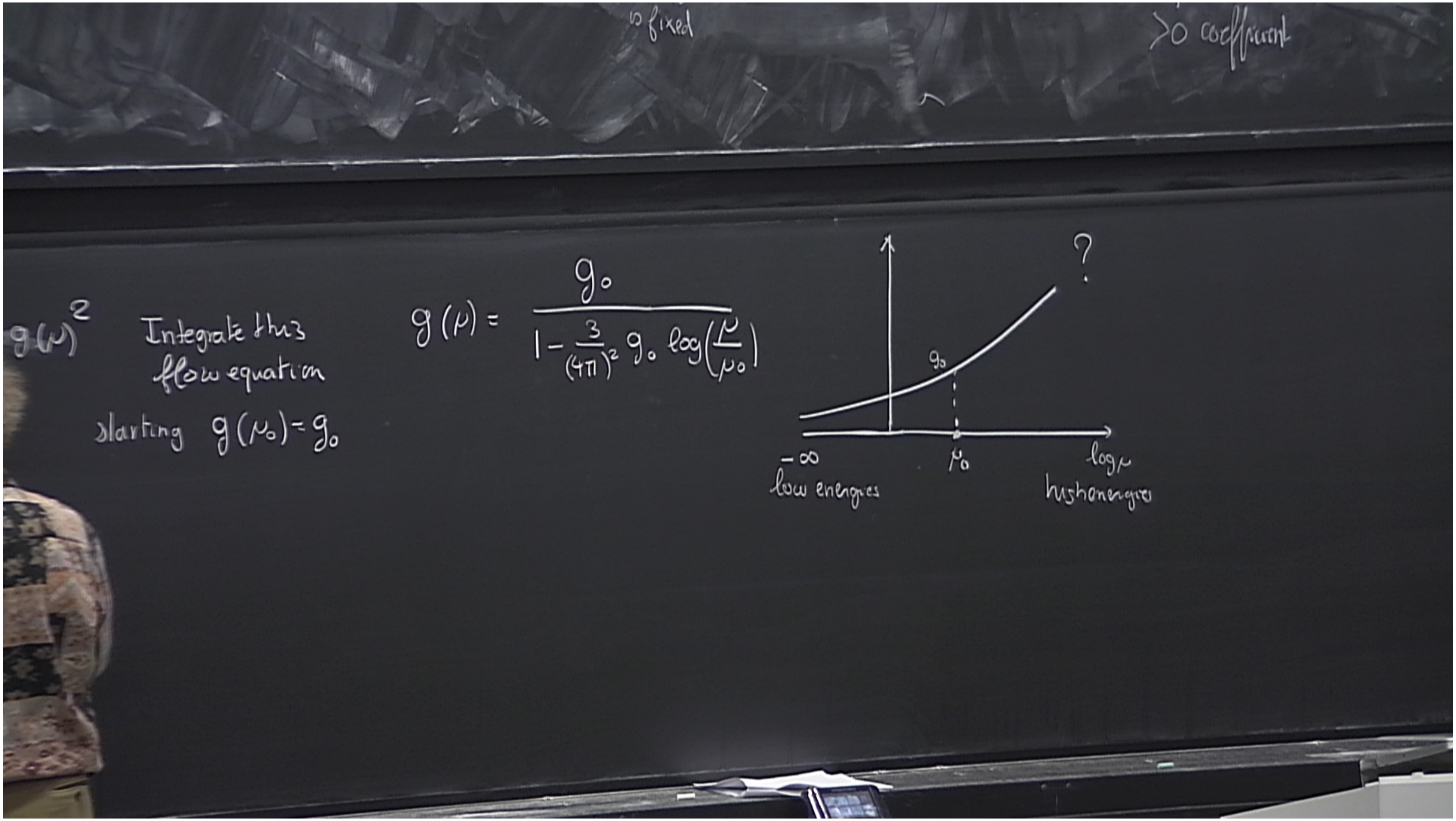
fixed

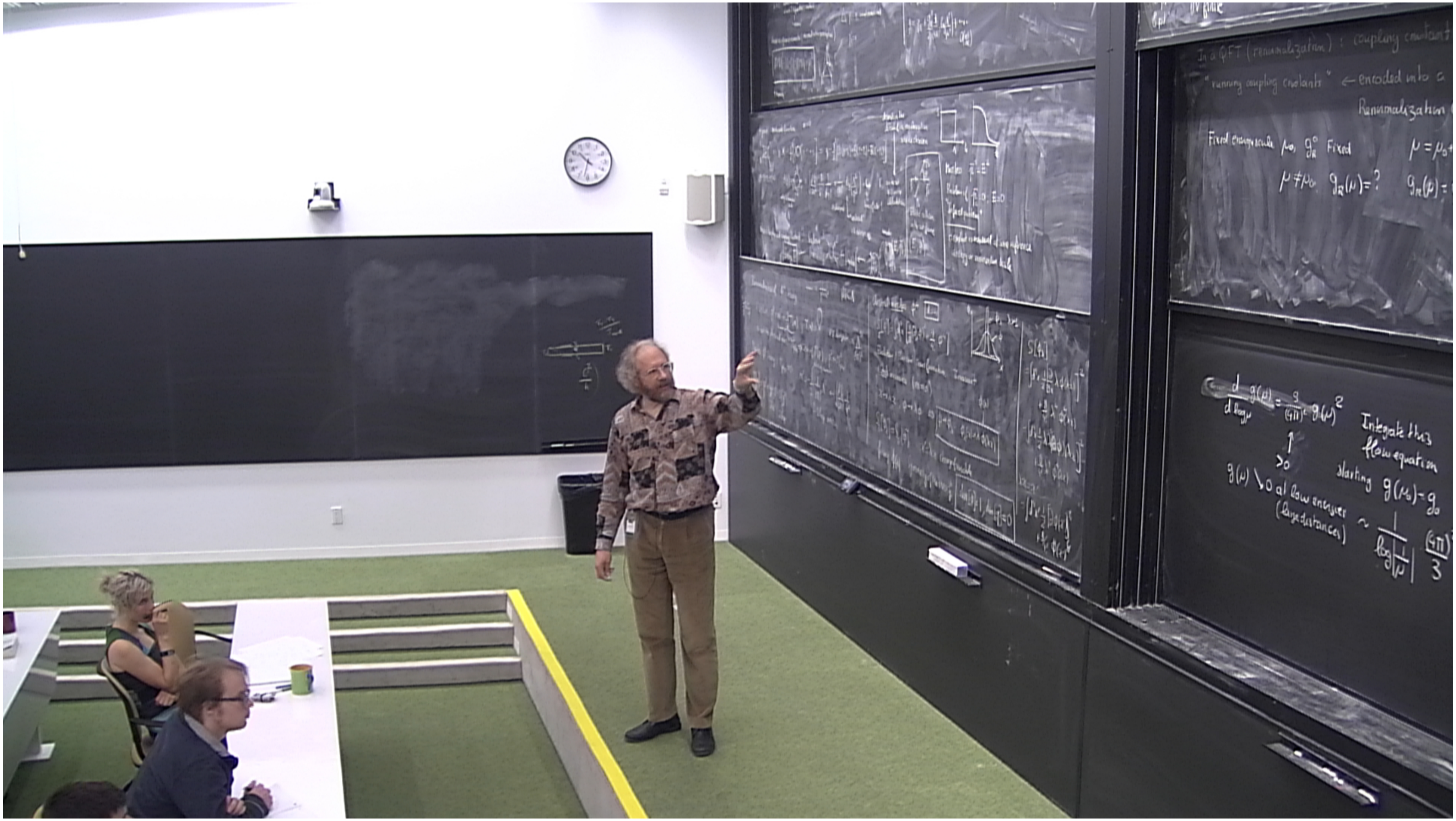
$$\frac{d}{d\log\mu} g(\mu) = \frac{3}{(4\pi)^2} g(\mu)^2$$

Integrate this flow equation

starting $g(\mu_0) = g_0$

$$g(\mu) = \frac{g_0}{1 - \frac{3}{(4\pi)^2} g_0 \log\left(\frac{\mu}{\mu_0}\right)}$$


fixed
>0 coefficient
$g(\mu)^2$
Integrate this flow equation
starting $g(\mu_0)=g_0$
$g(\mu) = \frac{g_0}{1 - \frac{3}{(4\pi)^2} g_0 \log\left(\frac{\mu}{\mu_0}\right)}$
?
g_0
$-\infty$
low energies
μ_0
$\log\mu$
high energies


In a QFT (renormalization):
"running coupling constants" ← encoded into a
Fixed energy scale μ_0, g_R^0 Fixed
$\mu \neq \mu_0$ $g_R(\mu) = ?$
$\frac{d}{d\log\mu} g(\mu) = \frac{3}{(4\pi)^2} g(\mu)^2$
Integrate this flow equation
>0
starting $g(\mu_0) = g_0$
$g(\mu) \searrow 0$ at low energies (large distances)
$\sim \frac{1}{\log|\frac{1}{\mu}|} \frac{(4\pi)^2}{3}$

is fixed

>0 coefficient

Callan-Symanzik + Gell-Mann-Low

$\frac{3}{(4\pi)^2}\, g(\mu)^2$

Integrate this flow equation

$$g(\mu) = \frac{g_0}{1 - \frac{3}{(4\pi)^2}\, g_0 \log\left(\frac{\mu}{\mu_0}\right)}$$

starting $g(\mu_0) = g_0$

low energies (large distances) $\sim \dfrac{1}{\log\left|\frac{1}{\mu}\right|} \; \dfrac{(4\pi)^2}{3} \leftarrow$ coefficient of the β-function