Title: Stability of Frustration-Free Hamiltonians

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Abstract: We generalize the result of Bravyi et al. on the stability of the spectral gap for frustration-free, commuting Hamiltonians, by removing the assumption of commutativity and weakening the assumptions needed for stability.

Counterexamples to stability: Opening the gap.

Splitting the groundstate degeneracy.

Example

1. Consider the $N \times N$ **Ising Hamiltonian** H_N and a **perturbation** Δ_N:

$$H_N = -\sum_{p\sim q} \sigma_p^z \otimes \sigma_q^z, \quad \Delta_N = \delta_N \sum_p \sigma_p^z, \quad \delta_N \sim 1/N^2.$$

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4. **Good classical memory, bad quantum memory:** Encoded state $|+\rangle = |000\ldots 0\rangle + |111\ldots 1\rangle$ flips to $|-\rangle = |000\ldots 0\rangle - |111\ldots 1\rangle$, since $e^{itH_N'}|+\rangle \sim |000\ldots 0\rangle + e^{it}|111\ldots 1\rangle$.

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Localized excitations.

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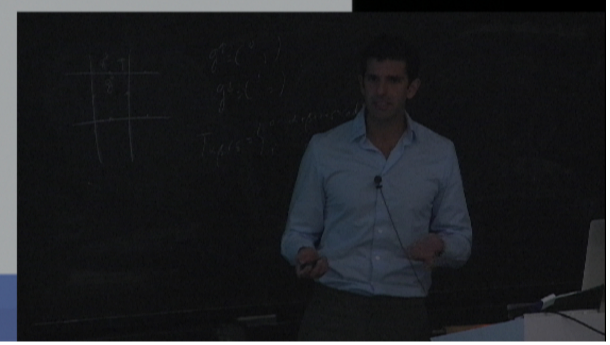

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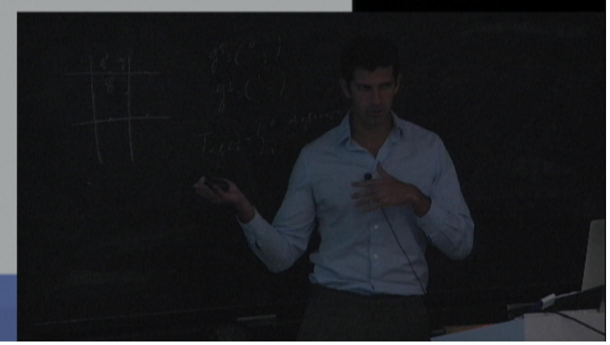
4. $H'_N = H_N + \delta_N \sum_p \sigma_p^z$ has **degenerate groundstate** space spanned by $|000 \cdots 0\rangle$ and $|111 \cdots 1\rangle$, for **vanishing** $\delta_N \sim 1/N^2$.

Distinguishability implies instability!

Hamiltonians are **unstable** because **local order parameters** can act as perturbations to **split the groundstate subspace**, or **close the gap** between **groundstates** and **local, low-energy excitations**.



Frustration-free Hamiltonians.

Definition

1. We say $\mathbf{H_0} = \sum_{\mathbf{u}\in\boldsymbol{\Lambda}} \mathbf{Q_u}$ is **frustration-free**, if the groundstate subspace P_0 satisfies for all $u \in \Lambda \subset \mathbb{Z}^d$:

$$\mathbf{Q_u}\mathbf{P_0} = \lambda_{\mathbf{u}}\mathbf{P_0},$$

where λ_u is the **smallest eigenvalue** of Q_u. Substitute Q_u with $Q_u - \lambda_u \mathbb{1} \geq 0$. This generates a **global energy shift**, which is **irrelevant for spectral gaps**.

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2. NOT all COMMUTING Hamiltonians are FRUSTRATION-FREE!
Take 3 **qubits on the vertices** $\{u, v, w\}$ **of a triangle**, with Ising Hamiltonian $\mathbf{H}_\triangle = \sigma^\mathbf{z}_\mathbf{u} \otimes \sigma^\mathbf{z}_\mathbf{v} + \sigma^\mathbf{z}_\mathbf{v} \otimes \sigma^\mathbf{z}_\mathbf{w} + \sigma^\mathbf{z}_\mathbf{u} \otimes \sigma^\mathbf{z}_\mathbf{w}$.
Since $\sigma^z_u \otimes \sigma^z_w = (\sigma^z_u \otimes \sigma^z_v) \cdot (\sigma^z_v \otimes \sigma^z_w)$, it is **impossible to have common groundstate** for all three terms.

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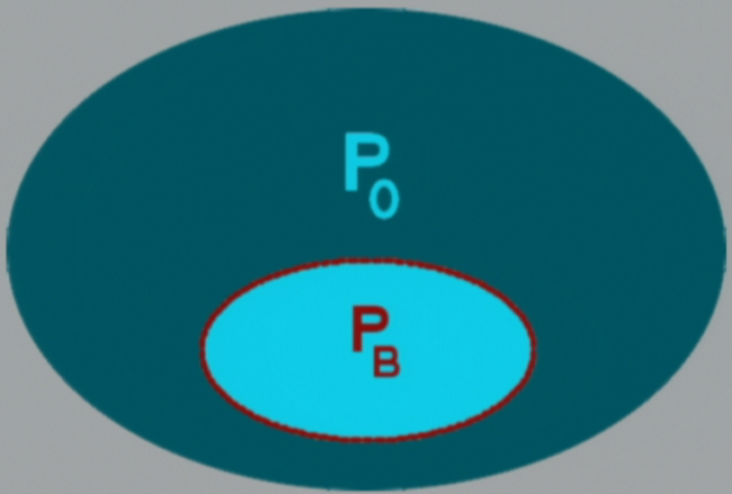
Local Groundstates



Every frustration-free Hamiltonian H_0 on Λ is the extension of another frustration-free Hamiltonian H_B on $B \subset \Lambda$. This implies that the **local groundstate projector** P_B **contains** P_0; that is, $\mathbf{P_B P_0 = P_0}$.

Proof:

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- And so, $\gamma_B(\mathbb{1} - P_B) \leq H_B \implies P_0(\mathbb{1} - P_B)P_0 \leq 0$.

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- But, $A^\dagger A = 0 \implies A = 0$, so $(\mathbb{1} - P_B)P_0 = 0$.
- Done. But, only works for frustration-free Hamiltonians!

Topological Quantum Order

Macroscopic indistinguishability of groundstates.

TQO: $\mathbf{P_0}$ satisfies **Topological Quantum Order**, if for all O_A:

$$P_0 O_A P_0 = c(O_A) P_0, \quad c(O_A) = \mathrm{Tr}(O_A\, P_0)/\, \mathrm{Tr}\, P_0, \quad (1)$$

where $A = b_u(r)$, $\mathbf{r} \leq \mathbf{L}^* \sim \mathbf{L}^\alpha$, $\alpha \in (0, 1]$.

Topological Quantum Order

You win some, you lose some.

1. **The Ising Hamiltonian does not satisfy the TQO condition.**
2. $\mathbf{P_0 O_A P_0 = c(O_A) P_0} \implies \langle \psi_0 | \mathbf{O_A} | \psi_0 \rangle = \langle \phi_0 | \mathbf{O_A} | \phi_0 \rangle = c(O_A)$, for any groundstates $|\psi_0\rangle, |\phi_0\rangle$. But, $\langle 000 \cdots 0 | \sigma_p^z | 000 \cdots 0 \rangle = 1$ and $\langle 111 \cdots 1 | \sigma_p^z | 111 \cdots 1 \rangle = -1$.

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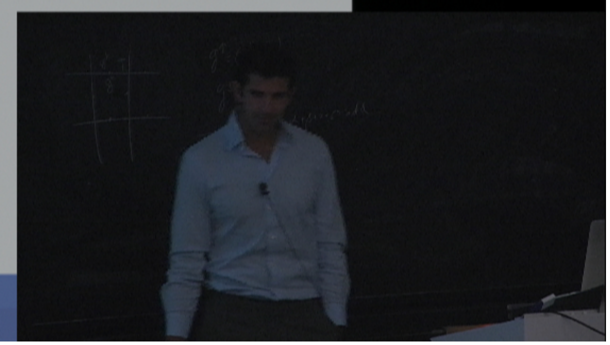
3. It is no coincidence that $\sum_p \sigma_p^z$ is used to split the groundstates.
4. **Kitaev's Toric Code**, a four-fold degenerate groundstate subspace, **satisfies the TQO condition** with $\alpha = 1$, so $L^* \sim L$.



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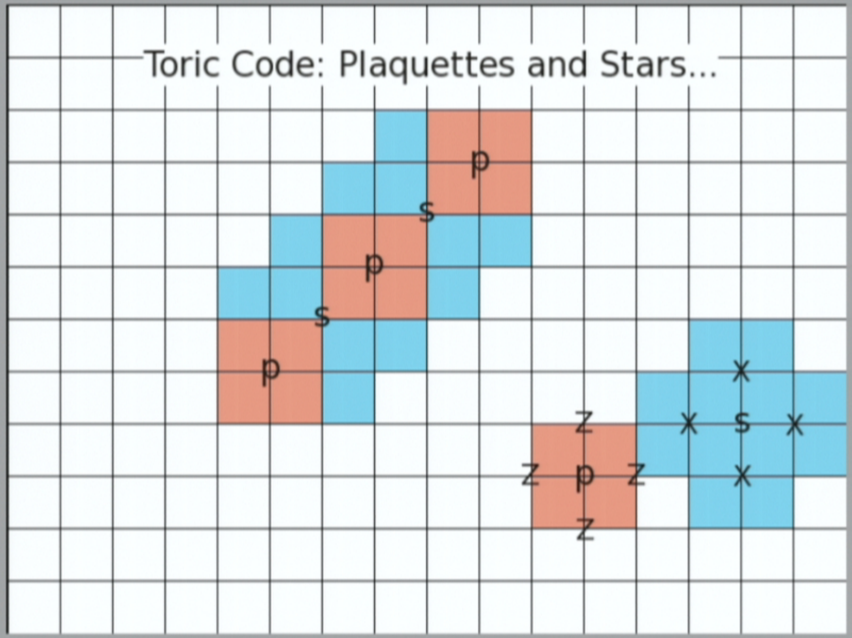
5. Of course, so does every Hamiltonian with a **unique groundstate**.

But the second counterexample had a unique groundstate!

The toric code...



Plaquettes have $B_p = \prod_{j \in \mathrm{edges(p)}} \sigma_j^z$ and **stars** have $A_s = \prod_{j \in \mathrm{star(s)}} \sigma_j^x$.
Note that $[\mathbf{A_s}, \mathbf{B_p}] = \mathbf{0}$ for all s and p.

Kitaev's Toric Code

The toric code model

Example

The standard toric code model is defined by the Hamiltonian:

$$H_{tc} = -\sum_p B_p - \sum_s A_s,$$

where qubits live on the edges of a lattice on a **torus**.

- Lowest-energy subspace P_0 (toric code) has $B_p = 1$, $A_s = 1$ for all p and s. That is, for any ground state $|\Psi_0\rangle$ we have:

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- Since $\prod_s A_s = \prod_p B_p = \mathbb{1}$, there are 4 such ground states on the torus, distinguished only through **macroscopic string operators**.

Breaking the Toric Code

The "Ising toric code with a defect" model.

Example

$$H_0 = -\sum_{p\sim p'} B_p \otimes B_{p'} - B_0 - \sum_s A_s.$$

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- **Is this gapped Hamiltonian stable?**
- **Perturb** H_0 by adding the vanishing **B-field**: $\delta_\Lambda \sum_p B_p$, with $\delta_\Lambda \sim 1/|\Lambda|$ and Λ the lattice (torus) on which H_0 is defined.

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- The subspace $A_s = 1, B_p = -1$, becomes the **new groundstate**.

TQO is not enough for stability. Now what?

Stability needs...

Local Groundstate Indistinguishability.

Local-TQO: $\mathbf{H}_0$ satisfies **Local-TQO**, if there exists a **rapidly-decaying function** $\Delta_0(\ell)$, such that:

$$\|P_{A(\ell)} O_A P_{A(\ell)} - c(O_A) P_{A(\ell)}\| \leq \|O_A\| \, \Delta_0(\ell). \quad (3)$$

Here, $A = b_u(r)$, $r \leq L^*$ and $A(\ell) := b_u(r+\ell)$.

Local-TQO implies that groundstates on $A(\ell)$ are identical when viewed on the bulk A, up to error $2\Delta_0(\ell)$.

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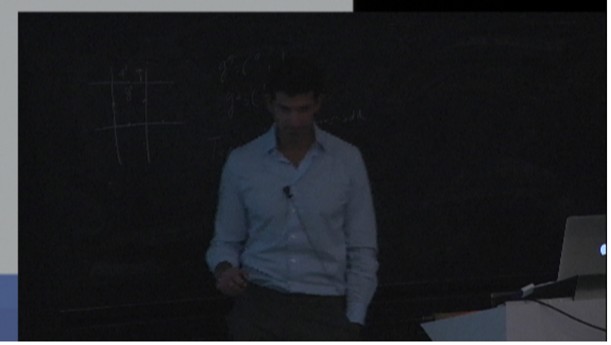


Proof:

By definition of the trace norm and $\rho_A = \mathrm{Tr}_{A(\ell)\setminus A} |\psi_{A(\ell)}\rangle\langle\psi_{A(\ell)}|$:

$$\|\rho_A^1 - \rho_A^2\|_1 = \sup_{\|O_A\|=1} \left| \left\langle \psi^1_{A(\ell)} \right| O_A \left| \psi^1_{A(\ell)} \right\rangle - \left\langle \psi^2_{A(\ell)} \right| O_A \left| \psi^2_{A(\ell)} \right\rangle \right|.$$

Local-TQO implies $\left| \langle \psi_{A(\ell)} | O_A | \psi_{A(\ell)} \rangle - c(O_A) \right| \leq \Delta_0(\ell)$. Use the triangle inequality:

$$\|\rho_A^1 - \rho_A^2\|_1 \leq 2\Delta_0(\ell).$$

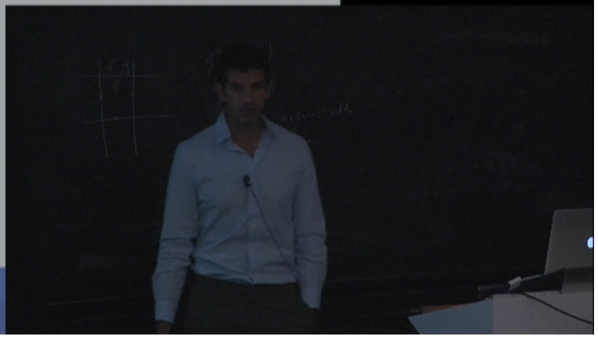
Local-TQO implies Entanglement bound.

Corollary

Any groundstate $|\Psi_0\rangle$ *of* H_0 **satisfying Local-TQO**, *also satisfies a* **bound for the entanglement entropy** *of* $\rho_A := \mathrm{Tr}_{A^c} |\Psi_0\rangle\langle\Psi_0|$, *with* $A = b_u(r)$, $r \leq L^*$:

$$\mathbf{S}(\rho_\mathbf{A}) \leq (\mathbf{c_d} \ln \mathbf{D})\,(\mathbf{1}+\mathbf{r})^{\mathbf{d-1}} \cdot \ell_\mathbf{0}, \tag{4}$$

where c_d *and* D *are constants and* $\ell_\mathbf{0} = \min\{\ell : \mathbf{\Delta_0}(\ell) \leq \ell/(\mathbf{1}+\mathbf{r})\}$.



Local Gaps.

Definition

Local-Gap: We define $\mathbf{H_0}$ to be **locally gapped** w.r.t. a function $\gamma(r)$, if $\mathbf{H_B} \geq \gamma(\mathbf{r})(\mathbf{1} - \mathbf{P_B})$, where $B = b_u(r)$. If $\gamma(r)$ decays at most **polynomially**, we say that H_0 satisfies the **Local-Gap** condition.

Open Problem: Is the **Local-Gap** condition always satisfied if H_0 is a sum of local projections with frustration-free ground-state and a spectral gap?

Open Problem: Is **Local-TQO** important for **Local-Gap** in this setting?

A brief History of Stability...

1. (Euclid, 314 B.C., Kato, '66) Let H_0 have spectral gap $\gamma > 0$ and **unique** groundstate. Then, $H_0 + V$ retains a gap if $\|V\| < \gamma/2$.

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2. (Datta, et al. '95, Yarotzky, '00) Let H_0 be sum of **classical terms**, with gap γ and **unique, frustration-free** groundstate. Then, for $V = \sum_u V_u$, with exponentially decaying V_u, $\exists J_0 : \|V_u\| \leq J_0 \implies$ **stable gap**. (common product eigenbasis)

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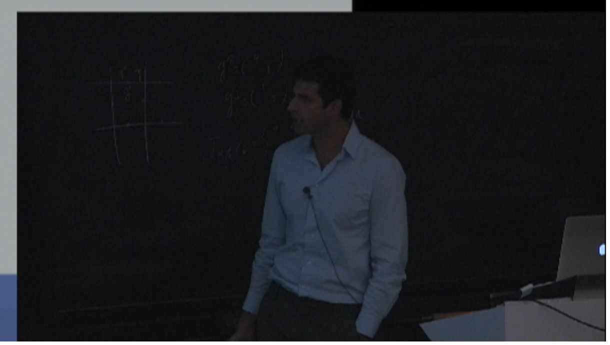
3. (Bravyi, Hastings, M., '10) H_0 is sum of **commuting projections**, with spectral gap γ and **frustration-free** groundstate subspace, satisfying a form of **Local Topological Order**. Then, for V a sum of **rapidly decaying terms** V_u, there exists a J_0 such that for $\|V_u\| \leq J_0 \implies$ **stable gap**. (common eigenbasis)



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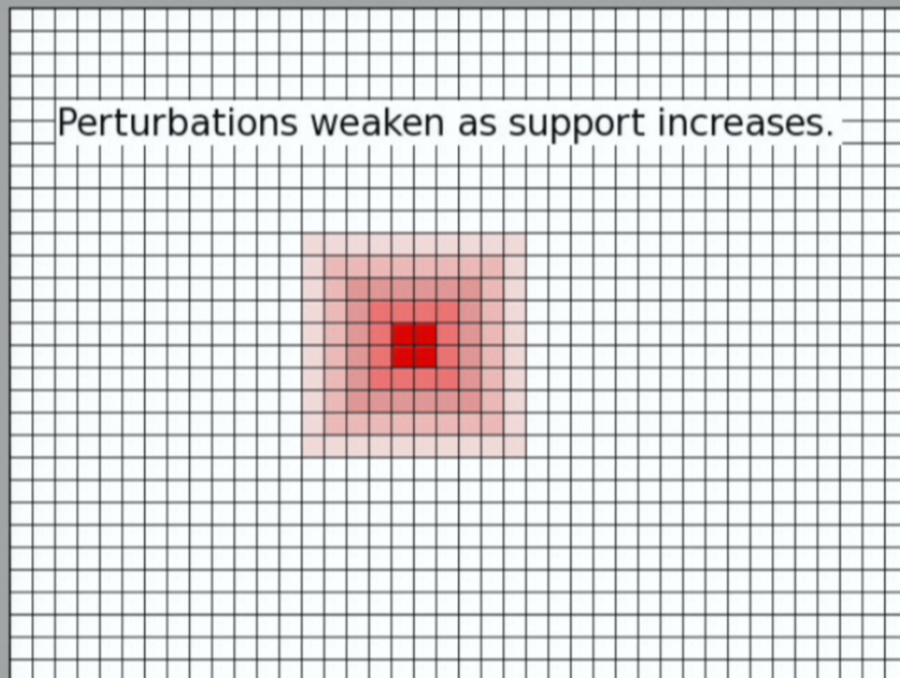
4. (M., Pytel, '11) Let H_0 have gap γ and **frustration-free** groundstate subspace, satisfying **Local-TQO** and **Local-Gap**. Then, stability holds for all perturbations V, as above. (common groundstate)

Decaying perturbations...



For each site $u \in \Lambda$, we allow perturbations supported on $b_u(r)$. As the radius of the support increases, the norm of the perturbation decreases rapidly.

The Perturbations: Local decomposition and strength.

Definition
We say that V **has strength** J **and rapid decay** f, if we can write

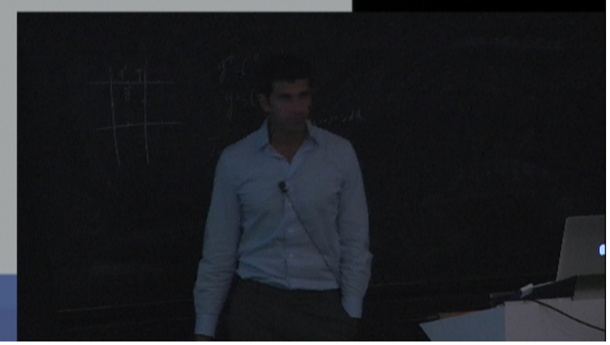
$$V = \sum_{u \in \Lambda} V_u, \quad V_u := \sum_{r \geq 0} V_u(r),$$

such that $V_u(r)$ has support on $b_u(r)$ and $\|V_u(r)\| \leq J f(r)$, $r \geq 0$.

The main result.

Theorem

- *Let H_0 be a* **frustration-free** *Hamiltonian satisfying* **Local-TQO** *and* **Local-Gap** *with decay given by $\Delta_0(r)$ and $\gamma(r)$, respectively.*



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$$(1 - c_0 J)\gamma - c_1\, J\, L^d \left(\sqrt{\Delta_0(L^*)} + w(L^*) \right),$$

where $c_0 = \sum_{r=1}^{L} r^d \cdot [w(r)/\gamma(r)]$ *and* $w(r)$ *has rapid decay related to the decay rate of* $f(r)$.

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- **Groundstate splitting** *is bounded by $J \, L^d \left(\sqrt{\Delta_0(L^*)} + w(L^*) \right)$. Since $L^* \sim L^\alpha$, this implies* **exponentially small splitting** *for rapidly decaying Δ_0 and w.*

Here is one of the tools used in the proof. Hot stuff.

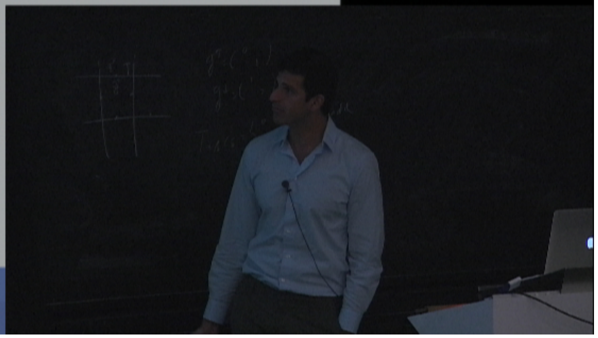
Spectral Flow - Hastings' Quasi-Adiabatic Evolution

Definition

Define the unitary operator U_s (**Spectral Flow**), by:

$$\partial_s U_s \equiv i\mathcal{D}_s U_s, \quad U_0 = \mathbb{1}, \tag{5}$$

where $\mathcal{D}_s$ **simulates the generator of the adiabatic evolution** for a family of gapped Hamiltonians H_s. (Next slide.)



Generators of quasi-adiabatic evolution.

Definition

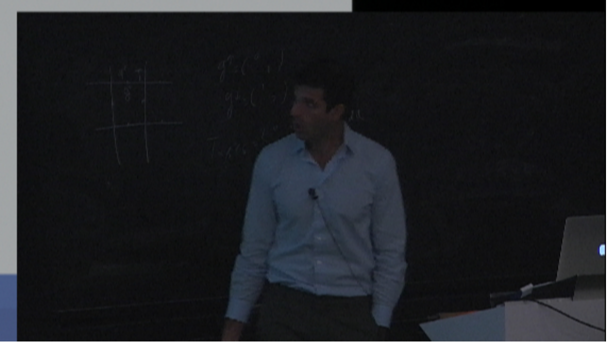
For $H_s = H_0 + sV$, **define the generator** $\mathcal{D}_s$ by:

$$\mathcal{D}_s \equiv \int_{-\infty}^{\infty} \mathrm{d}t \, s_\gamma(t) \int_0^t \mathrm{d}u \, e^{iuH_s}(V)e^{-iuH_s}, \qquad (6)$$

where the function $s_\gamma(t)$ (called a **filter function**) is chosen to satisfy the following properties:

1. The Fourier transform of $s_\gamma(t)$, denoted by $\hat{s}_\gamma(\omega)$, obeys:

$$|\omega| \geq \gamma/2 \quad \rightarrow \quad \hat{s}_\gamma(\omega) = 0 \quad \text{(compact support)}. \qquad (7)$$



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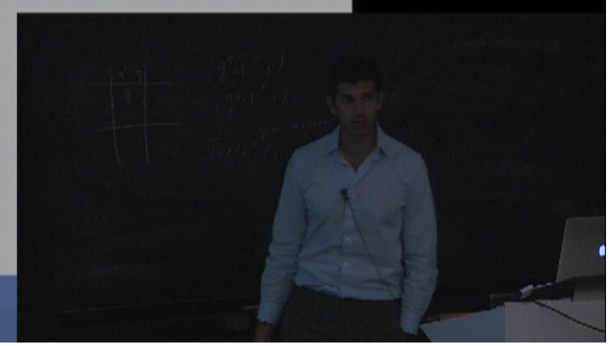
4. Note: This magical function $s_\gamma(t)$ exists and can be quite the ice-breaker on a first date.

Spectral Lemma + Local Decomposition

Lemma

- *Let H_s be a* **differentiable family of Hamiltonians**.



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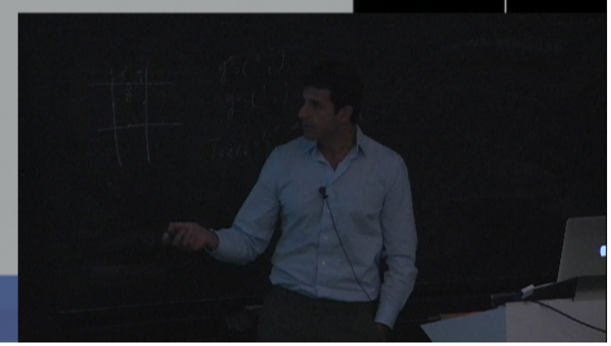

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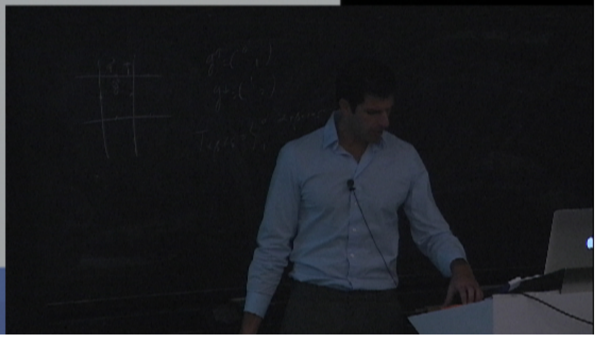
Overview of the proof...

- Assume that s^* **is the largest number in** $[0, 1]$ such that $H_s = H_0 + sV$ **has gap at least** $\gamma/2$ **for all** $s \in [0, s^*]$.(assume gap)



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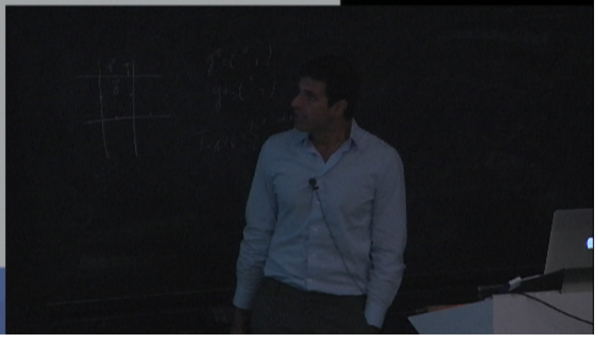

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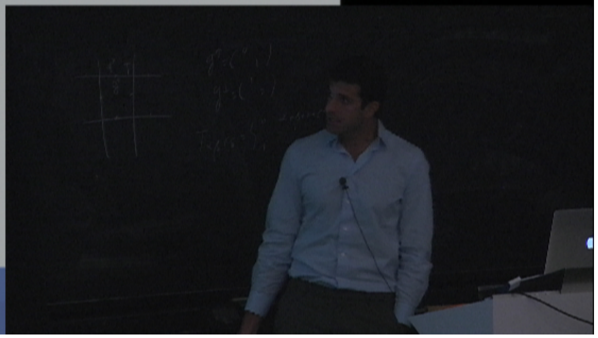

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- **Decompose** $V'_u = W_u + \Delta_u + E_u \mathbb{1}$, where:
  - $\Delta_u = (V'_u - E_u)P_0$,
  - $W_u = (V'_u - E_u)(1 - P_0)$ and
  - E_u is a constant energy. (energy-shift)

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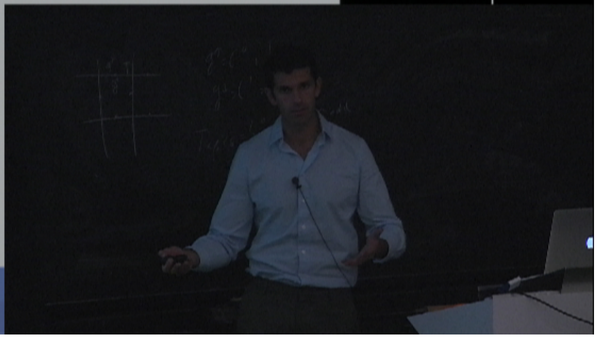

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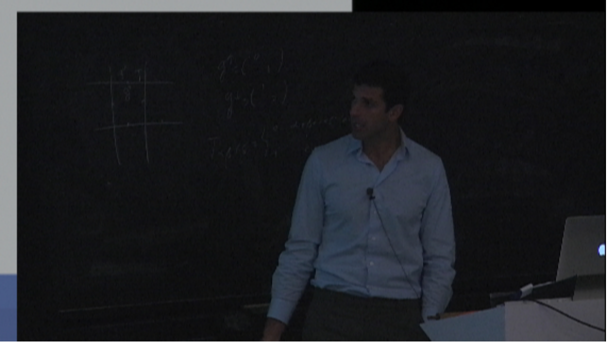

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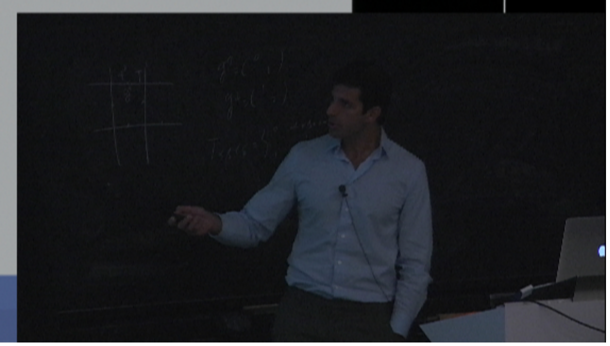

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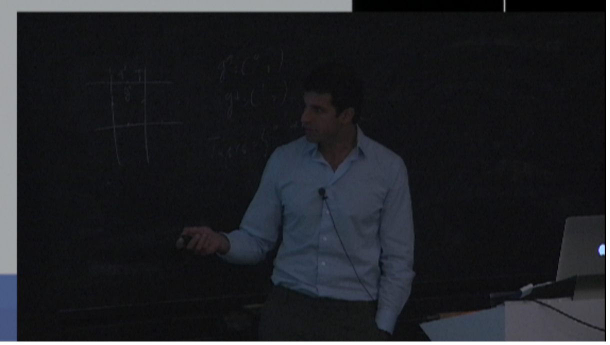

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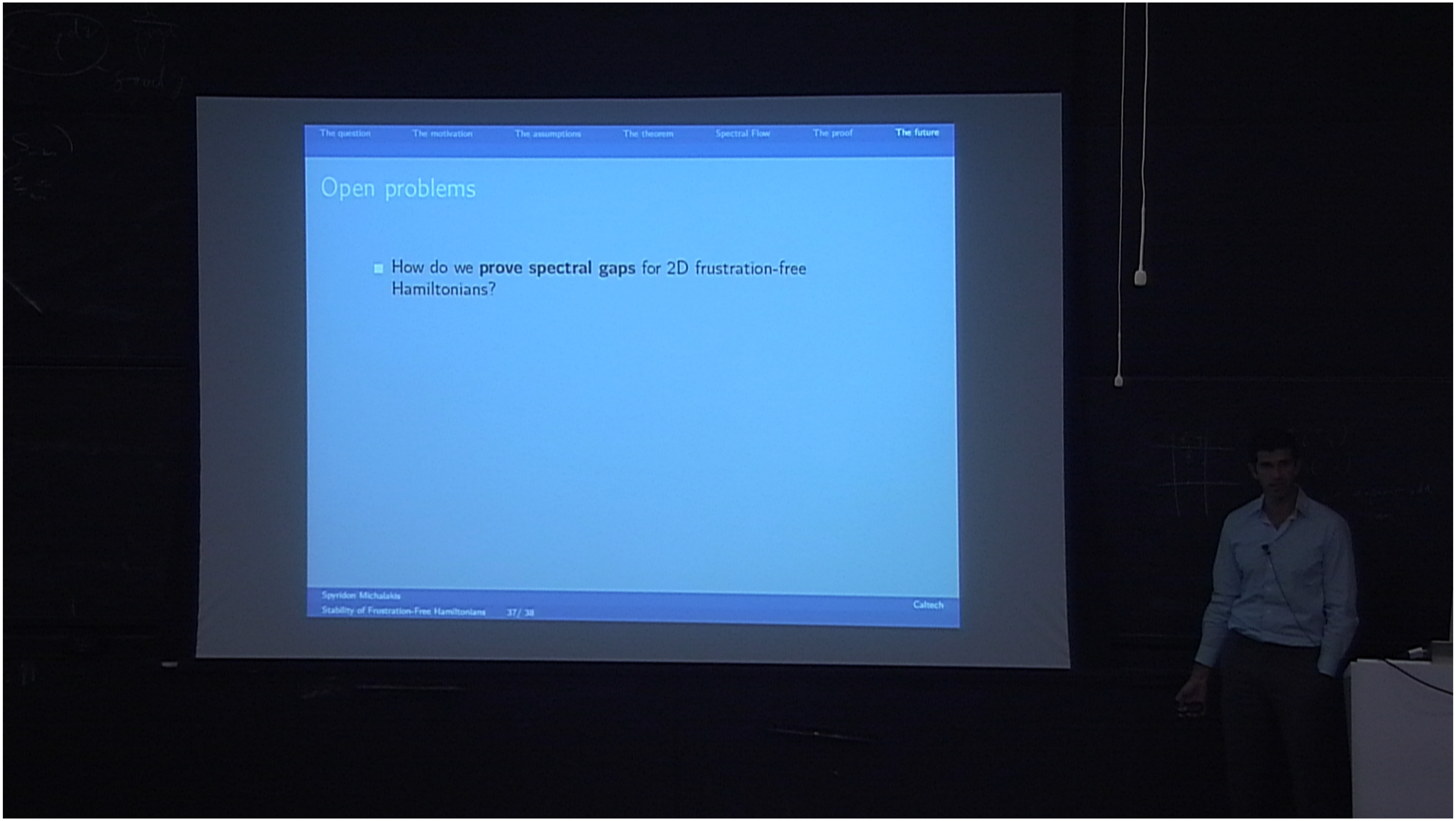
- **Contradiction!** $H_0 + s^*V$ has gap at most $\gamma/2$, by assumption! So, $s^* = 1$, for $J \leq J_0$.


The question
The motivation
The assumptions
The theorem
Spectral Flow
The proof
The future
Open problems
How do we **prove spectral gaps** for 2D frustration-free Hamiltonians?
Spyridon Michalakis
Stability of Frustration-Free Hamiltonians
37/ 38
Caltech

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Questions?

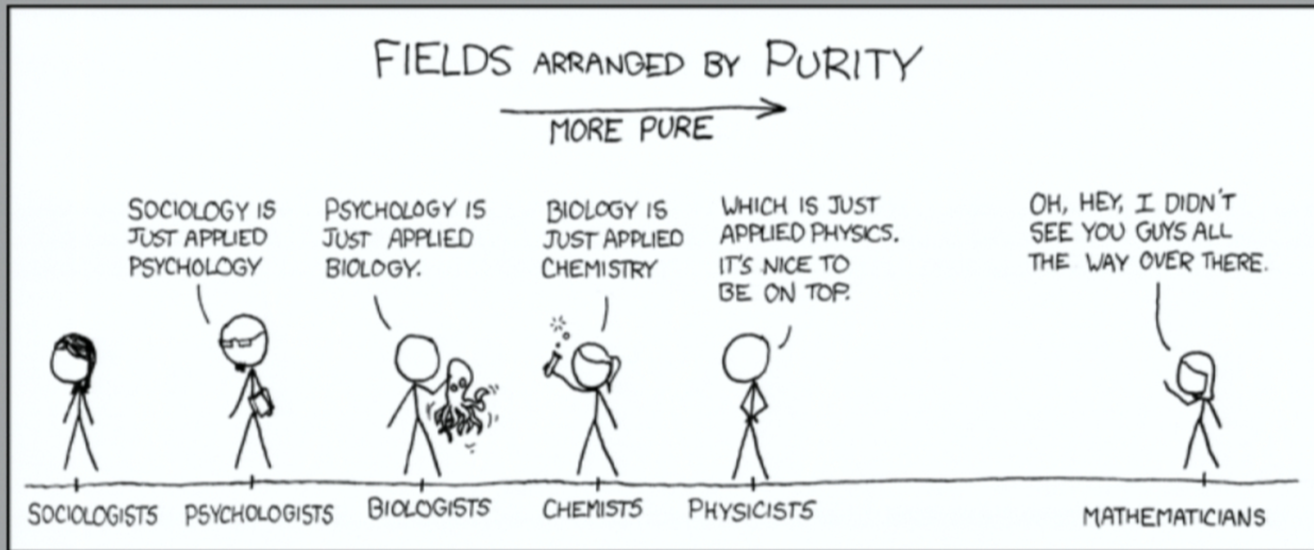

Figure: There was no space for string theory on the right.

Thank you!

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