

Title: MERA and CFT

Date: Oct 25, 2011 10:00 AM

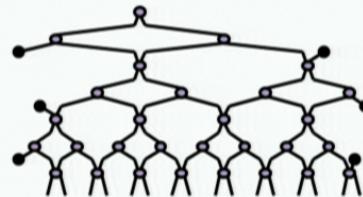
URL: <http://pirsa.org/11100098>

Abstract: The MERA offers a powerful variational approach to quantum field theory. While the continuous MERA may allow us to directly address field theories in the continuum, the MERA on the lattice has already demonstrated its ability to characterize conformal field theories. In this talk I will explain how to extract the conformal data (central charge, primary fields, and their scaling dimensions and OPE) of a CFT from a quantum spin chain at a quantum critical point. I will consider both homogeneous systems (translation invariant) and systems with an impurity (where translation invariance is explicitly broken). Key to the success of the MERA is the exploitation of both scale and translation invariance. I will show how translation invariance can still be exploited even in the presence of an impurity, even if the system is no longer translation invariant. This follows from an intriguing "causality principle" in the RG flow. I will also discuss the relation of these results with Wilson's famous resolution of the Kondo impurity problem.

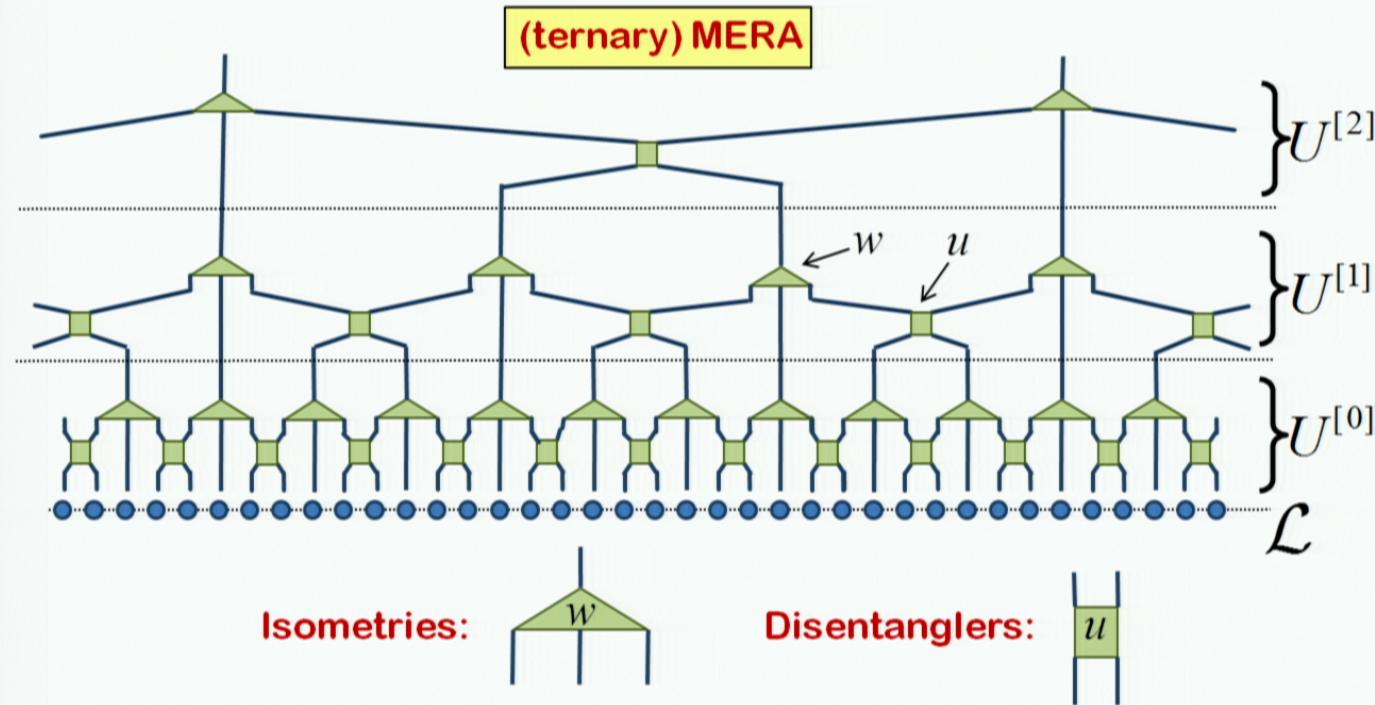
The MERA and CFT's

Outline

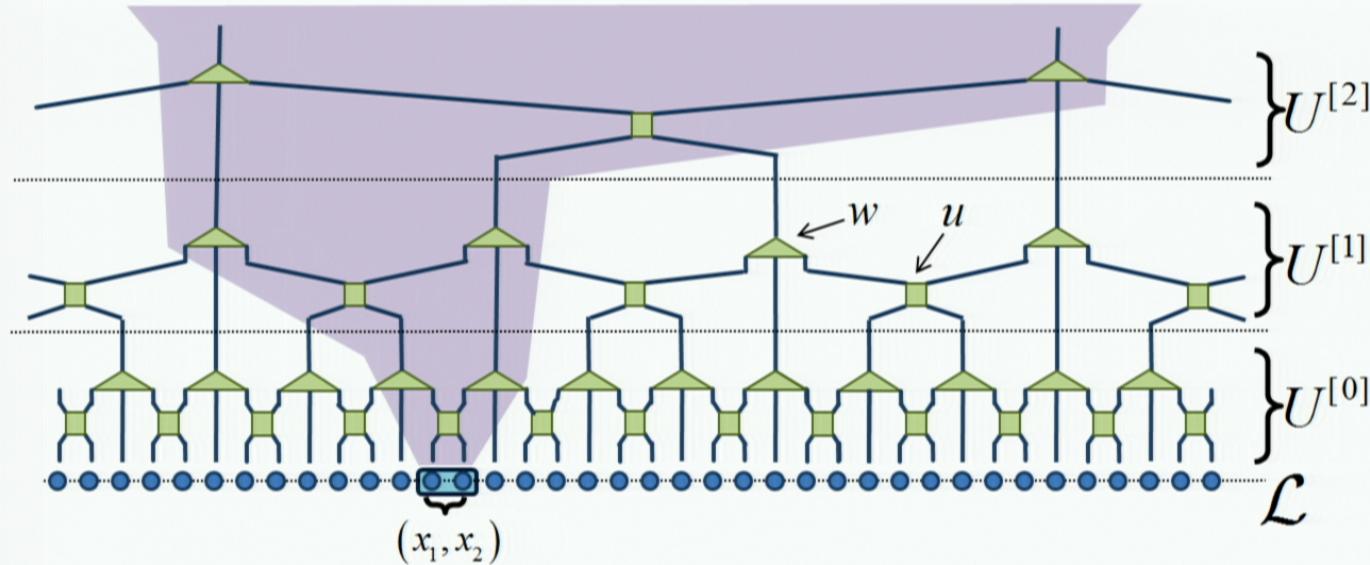
- Introduction to the MERA
 - causal cones
 - coarse-graining local Hamiltonians
- The scale-invariant MERA and CFTs
- CFTs in the presence of an impurity
- A “Principle of minimal influence” in the RG flow
 - properties of impurity MERA
 - implementation
 - extensions
- Summary and Conclusions



Intro to the MERA



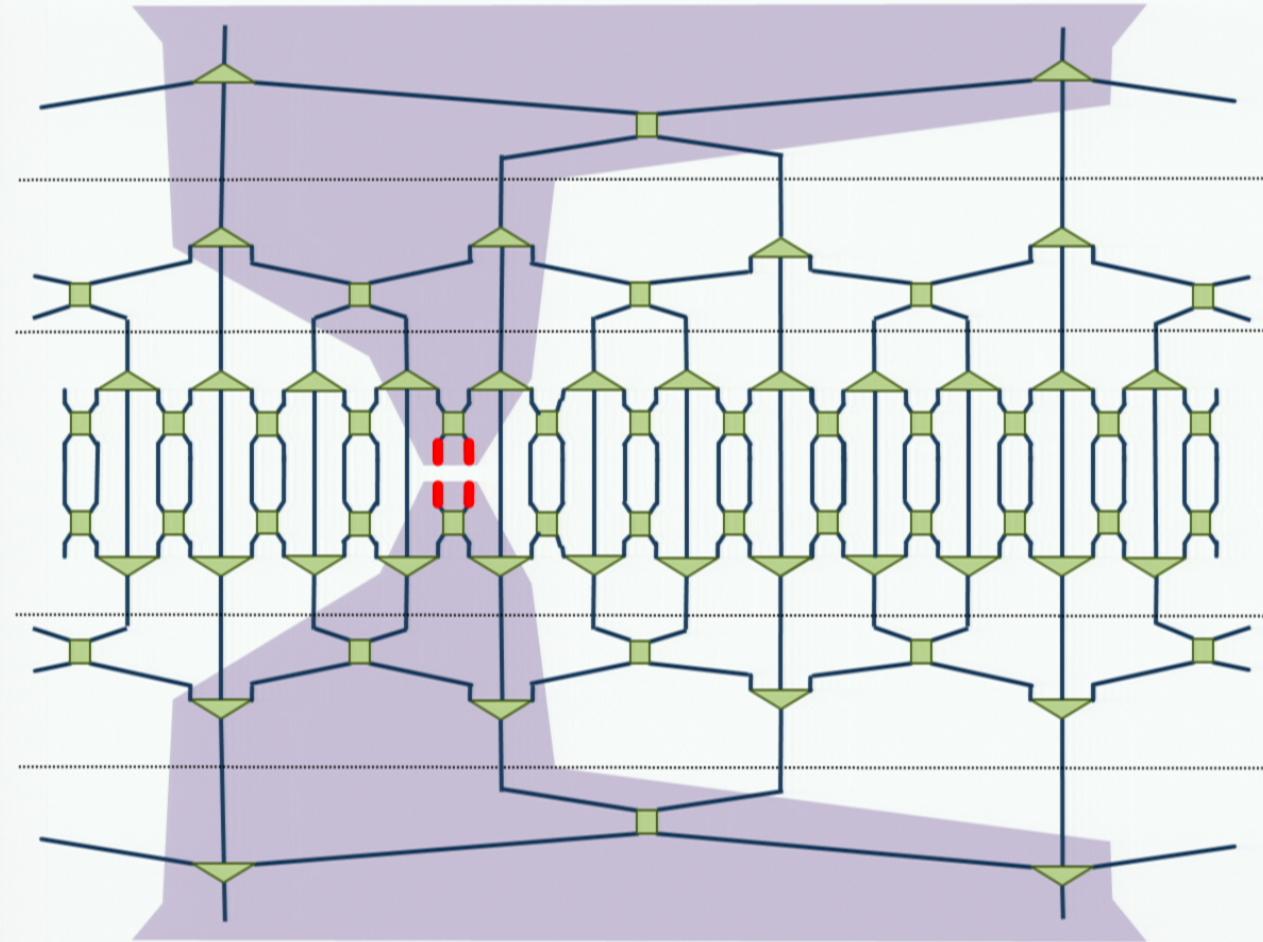
Intro to the MERA: Causal Cones



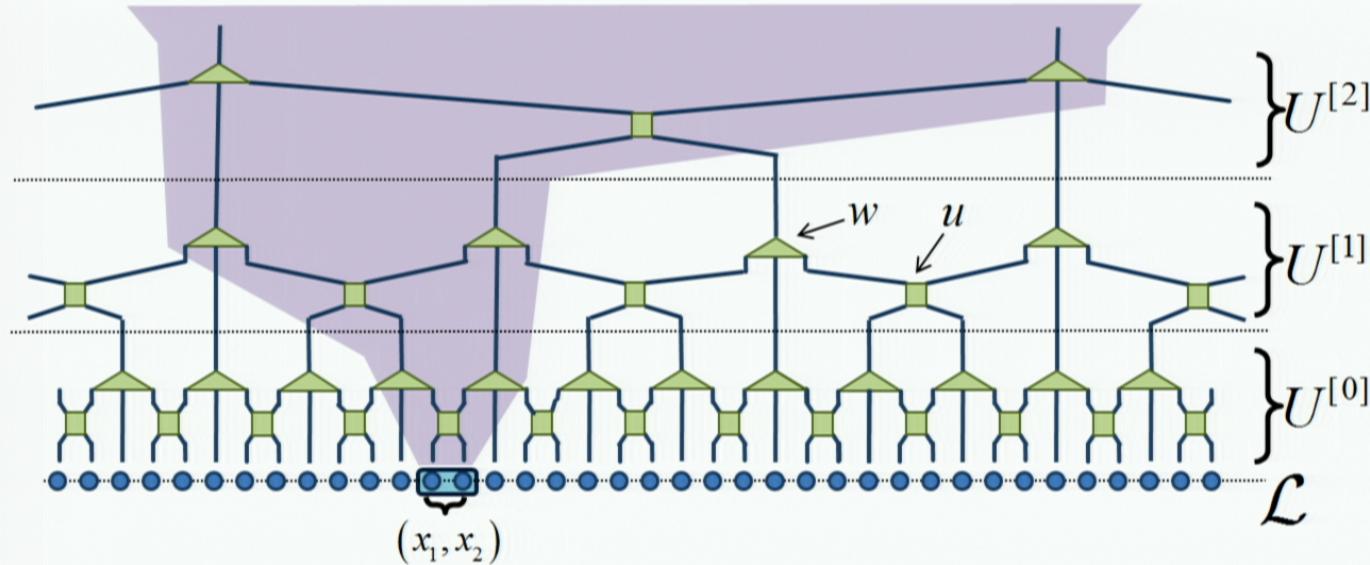
Def: **Causal Cone** of sites (x_1, x_2) = set of tensors that could affect the reduced density matrix $\rho(x_1, x_2)$

$$\rho(x_1, x_2) \equiv \text{tr}_{\overline{(x_1, x_2)}} |\Psi\rangle\langle\Psi|$$

Intro to the MERA: Causal Cones



Intro to the MERA: Causal Cones

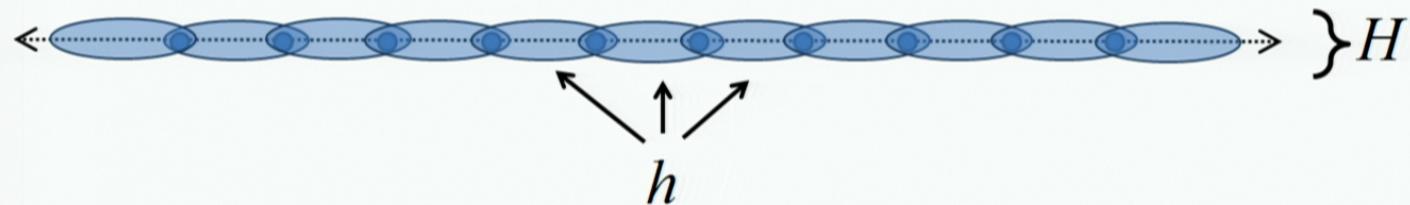


Def: **Causal Cone** of sites (x_1, x_2) = set of tensors that affect the reduced density matrix $\rho(x_1, x_2)$

- In a MERA, the **Causal Cone** of any site has **bounded width** (width = 2 for ternary MERA)
- “Bounded causal cone width” allows for efficient evaluation of the expectation value of local observables

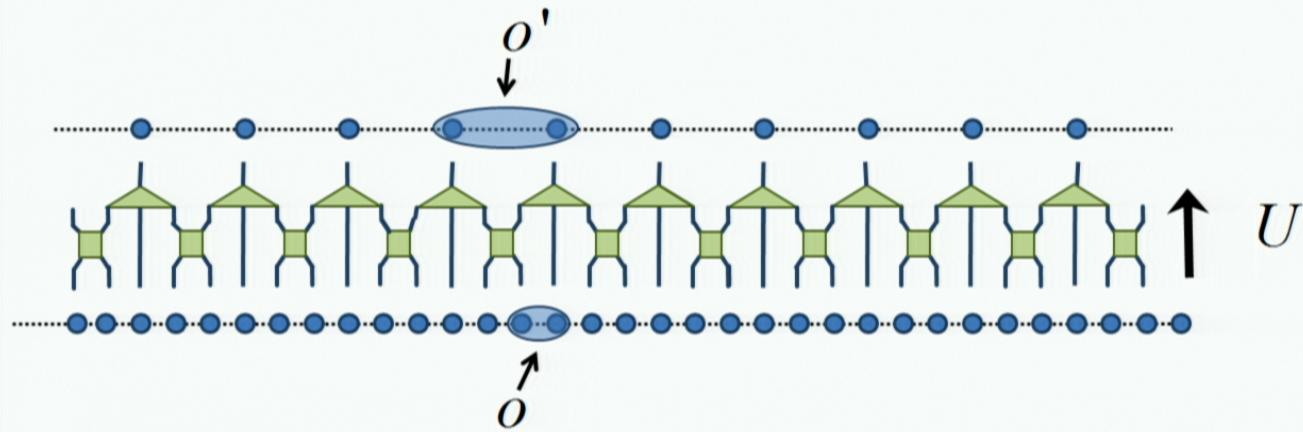
Intro to the MERA: coarse-graining Hamiltonians

Pictorial representation of local Hamiltonians: $H = \sum_{x=-\infty}^{\infty} h(x, x+1)$



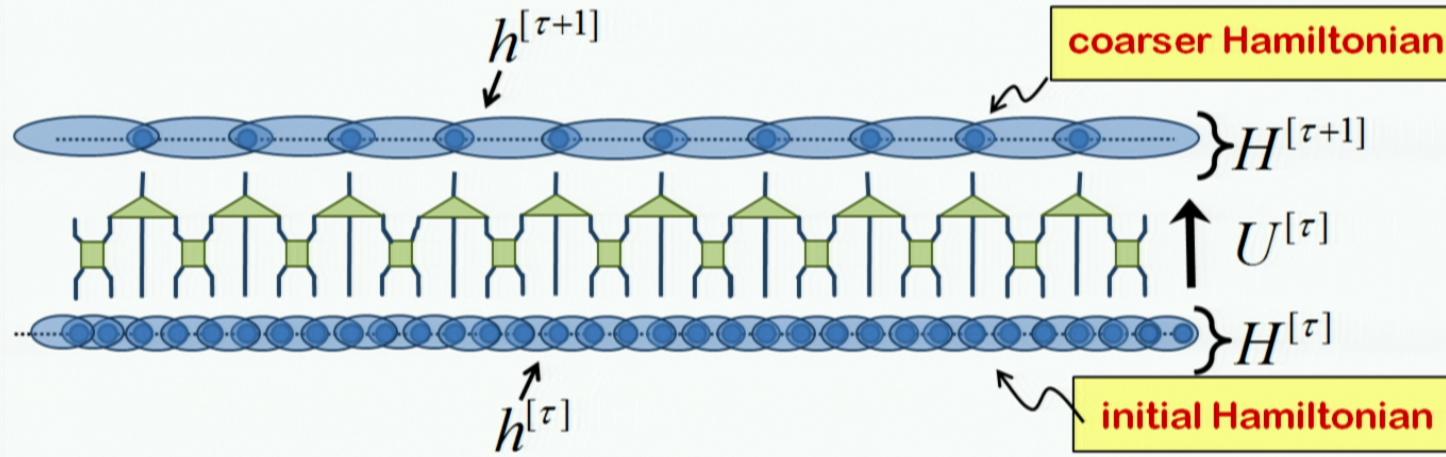
Intro to the MERA: coarse-graining Hamiltonians

Coarse-graining of a local operator: $o' = U o U^\dagger$



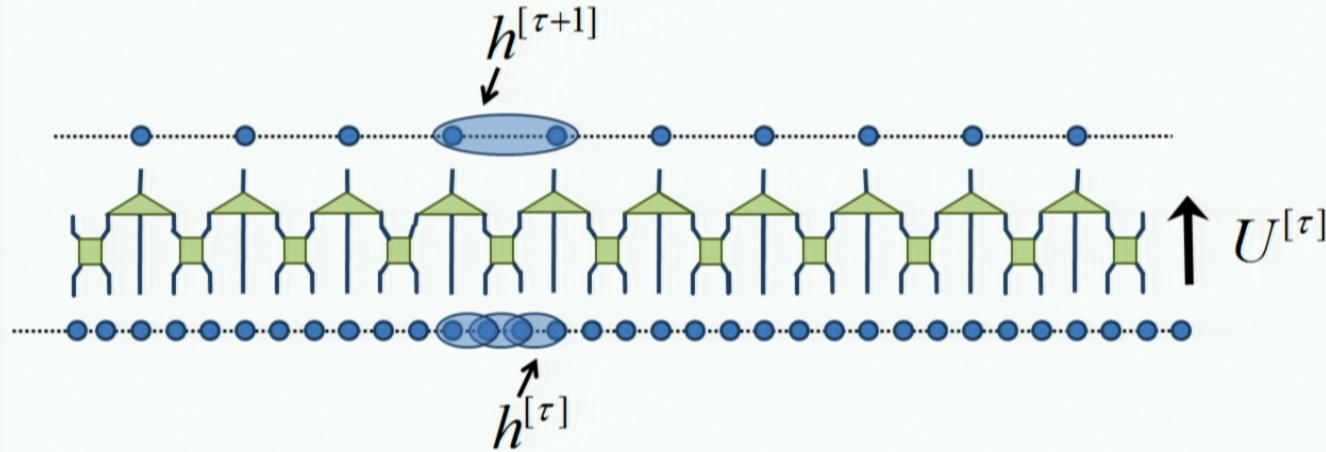
Intro to the MERA: coarse-graining Hamiltonians

Coarse-graining the Hamiltonian: $H^{[\tau+1]} = U^{[\tau]} H^{[\tau]} \left(U^{[\tau]} \right)^\dagger$

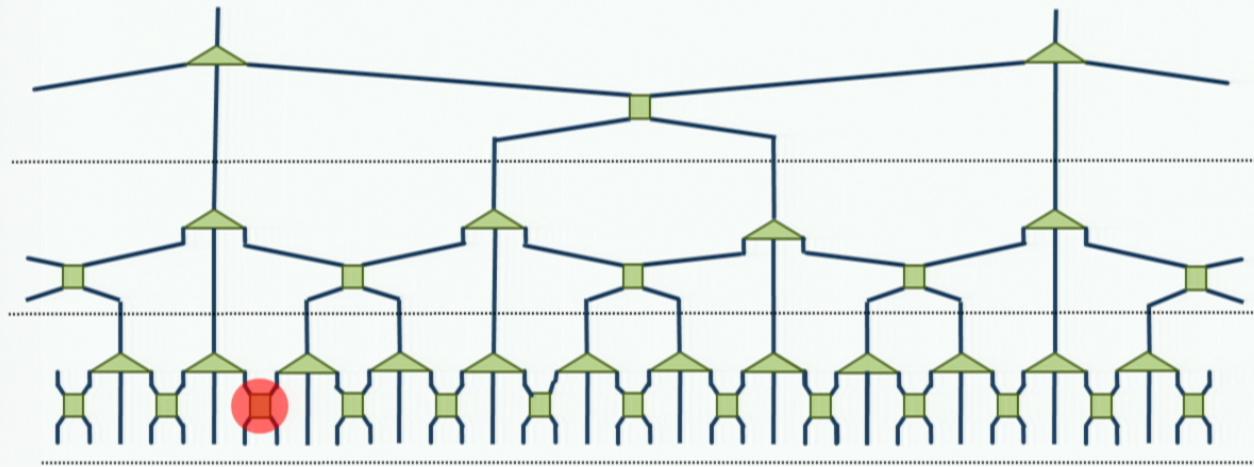


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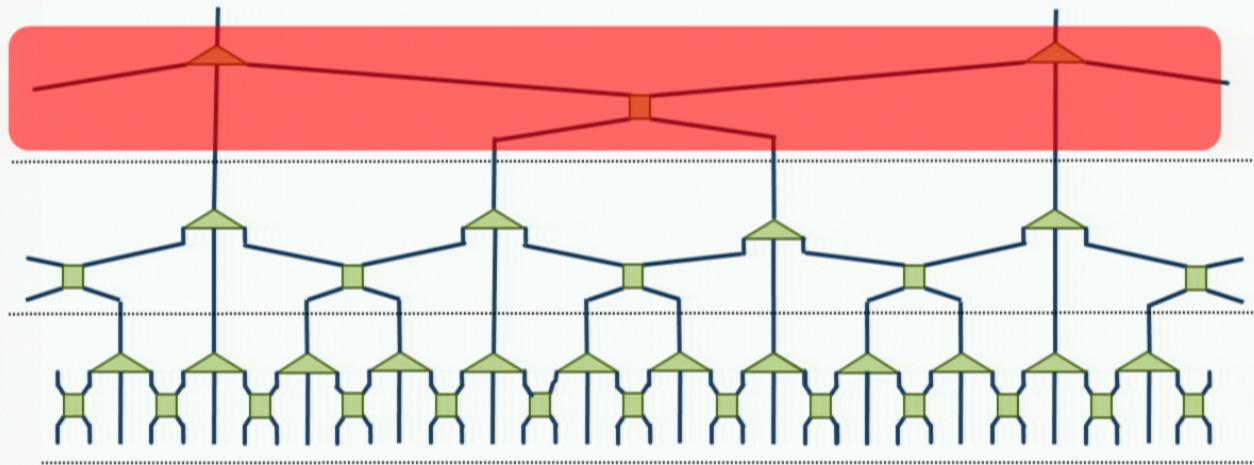


Scale-invariant MERA



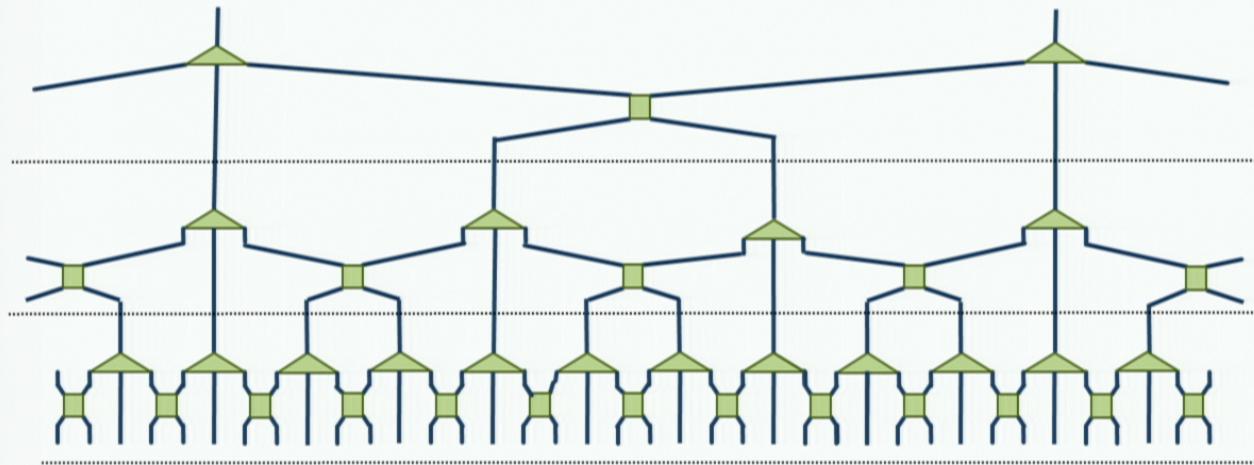
- exploits translation invariance

Scale-invariant MERA



- exploits translation invariance
- enforces and exploits scale invariance

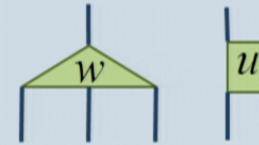
Scale-invariant MERA



- exploits translation invariance
- enforces and exploits scale invariance
- contains $O(1)$ different tensors

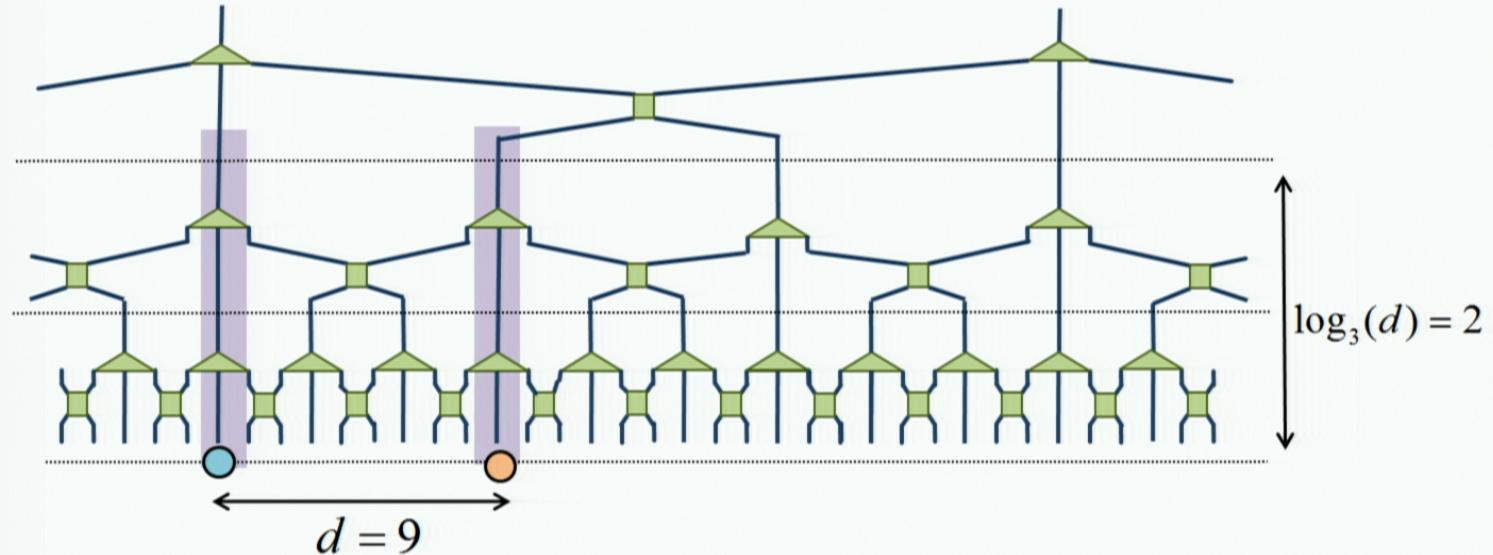
Properties of Scale-Invariant MERA?

characterised by
a single pair of tensors



Scale-invariant MERA: Correlators

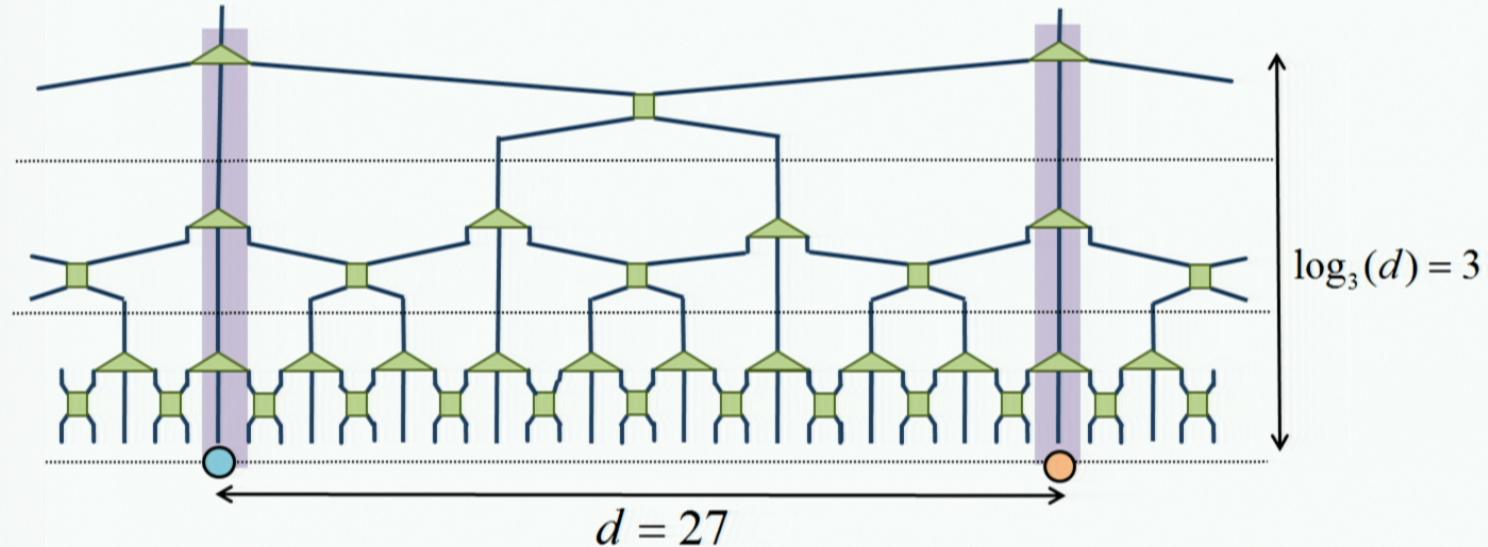
Two-Point Correlator: $\langle \psi | o_1(x) o_2(x+d) | \psi \rangle$



- operators are coarse-grained onto nearest neighbour sites after $\log_3(d)$ layers

Scale-invariant MERA: Correlators

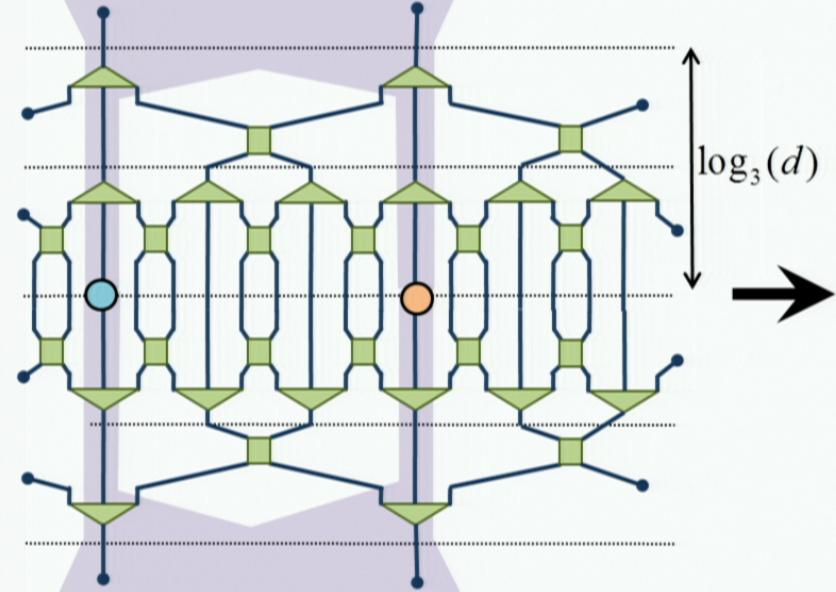
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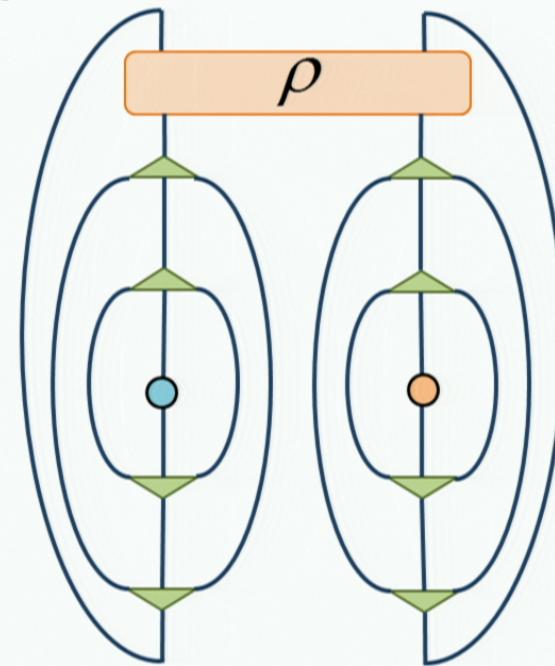
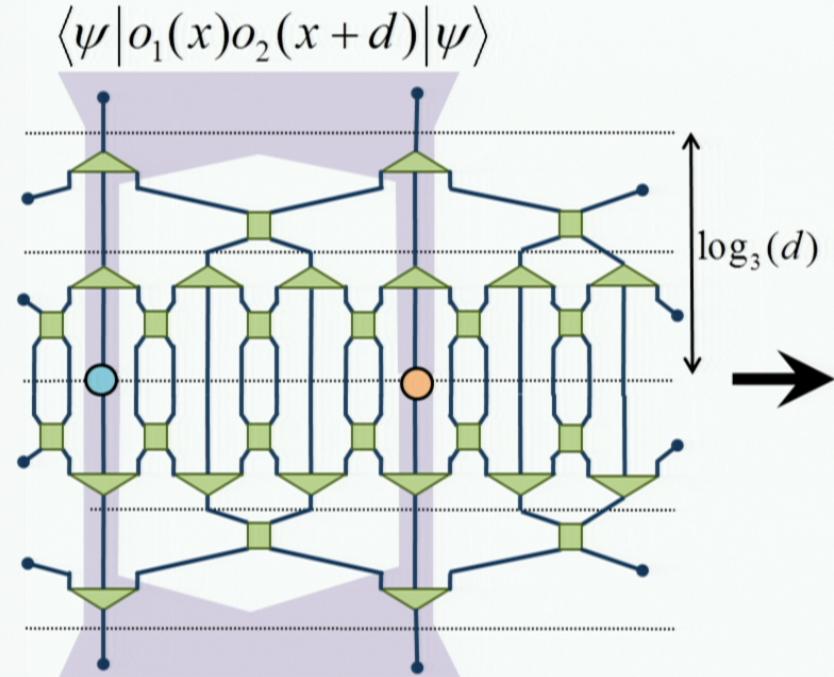
Scale-invariant MERA: Two-point Correlators

$$\langle \psi | o_1(x) o_2(x+d) | \psi \rangle$$



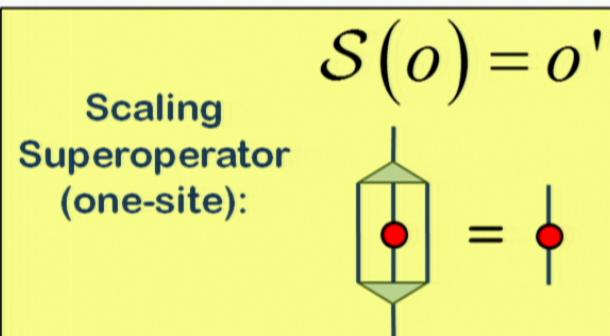
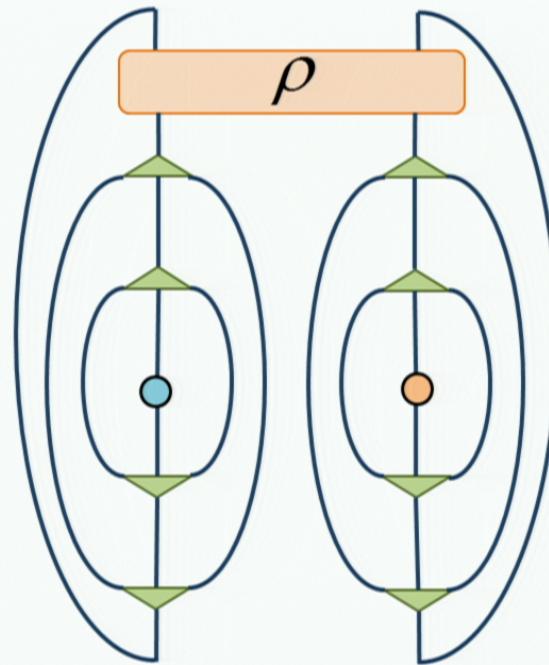
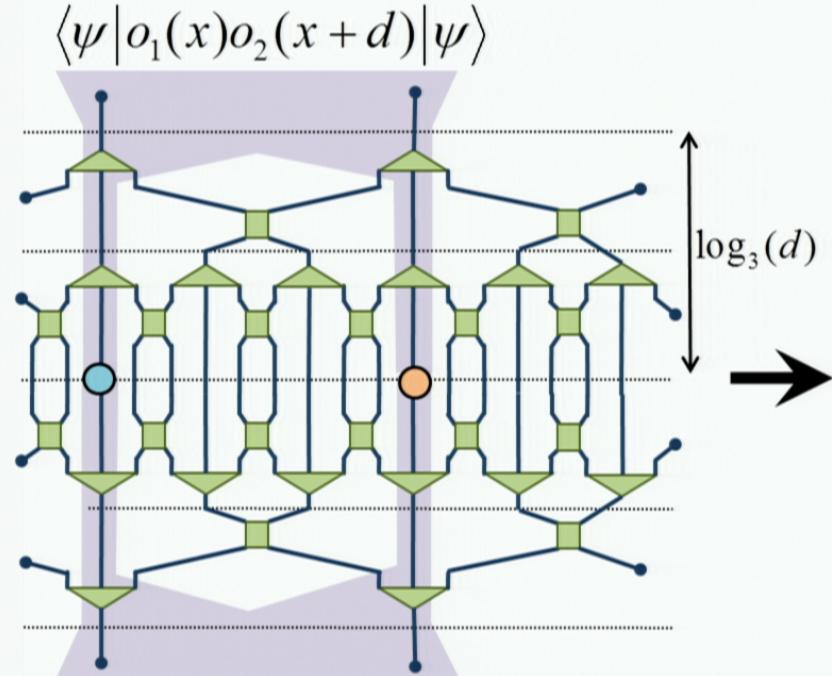
Scale-invariant MERA: Two-point Correlators

$$\langle \psi | o_1(x) o_2(x+d) | \psi \rangle$$



Scale-invariant MERA: Two-point Correlators

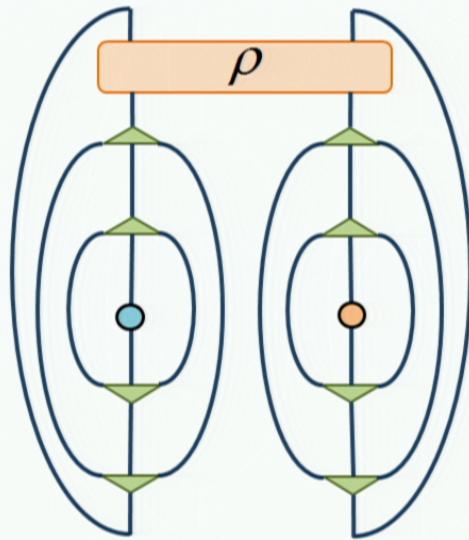
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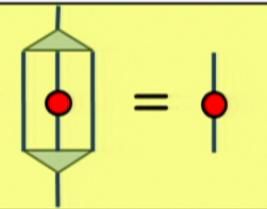
Two-Point Correlator:

$$\langle \psi | o_1(x) o_2(x+d) | \psi \rangle = \text{tr}(\rho \mathcal{S}^{\log_3 d}(o_1) \mathcal{S}^{\log_3 d}(o_2))$$

Scale-invariant MERA: Two-point Correlators



Scaling
Superoperator
(one-site):

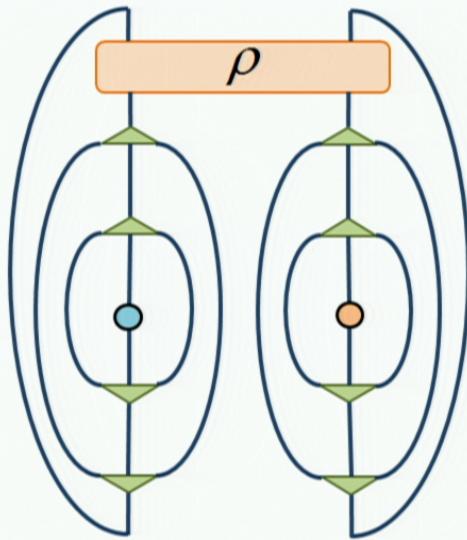


diagonalize: $\mathcal{S}(\phi_\alpha) = \lambda_\alpha \phi_\alpha$

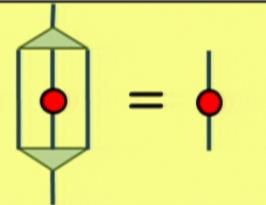
scaling
operators
 ϕ_α

scaling dimensions
 $\Delta_\alpha = -\log_3(\lambda_\alpha)$

Scale-invariant MERA: Two-point Correlators



Scaling
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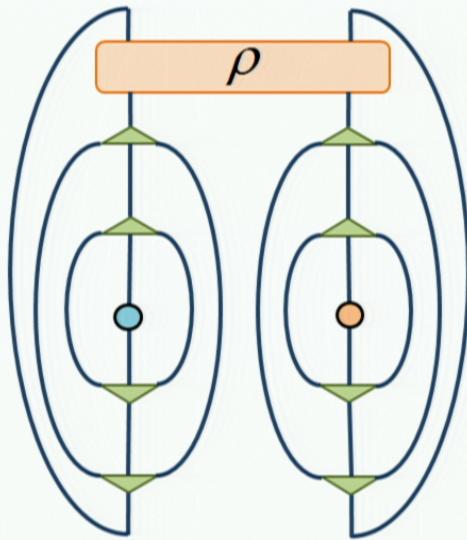


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Scale-invariant MERA: Two-point Correlators



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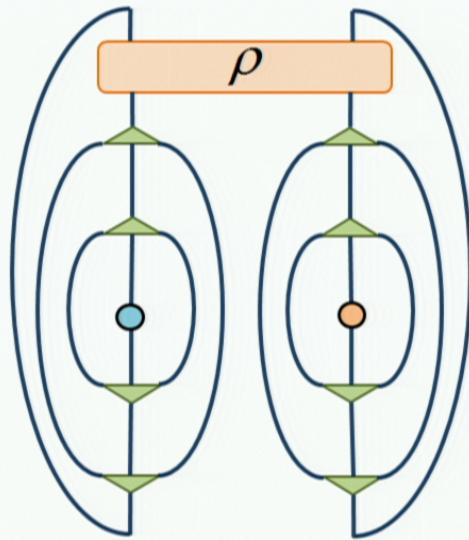
scaling operators
 ϕ_α

scaling dimensions
 $\Delta_\alpha = -\log_3(\lambda_\alpha)$

$$\langle \phi_\alpha(x) \phi_\beta(x+d) \rangle = (\lambda_\alpha \lambda_\beta)^{\log_3(d)} C_{\alpha\beta}$$

where $C_{\alpha\beta} = \text{tr}((\phi_\alpha \otimes \phi_\beta) \rho)$

Scale-invariant MERA: Two-point Correlators



diagonalize: $\mathcal{S}(\phi_\alpha) = \lambda_\alpha \phi_\alpha$

scaling operators
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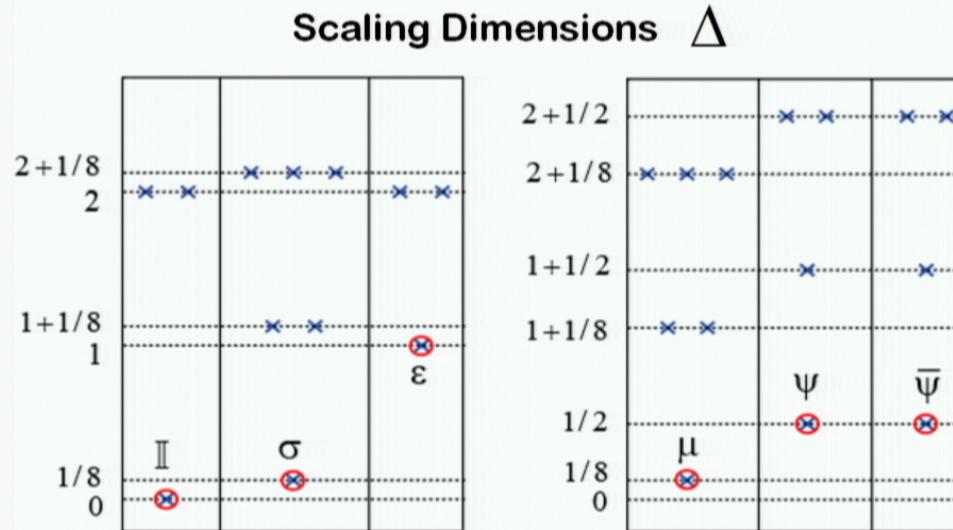
$$\Rightarrow \langle \phi_\alpha(x) \phi_\beta(x+d) \rangle = \frac{C_{\alpha\beta}}{d^{\Delta_\alpha + \Delta_\beta}}$$

- **scale invariant MERA** exhibits polynomial decay of correlations

- **scaling dimensions** describe powers of polynomial decay

Numerical Example:

1D critical Ising model: $H = \sum(-XX + Z)$



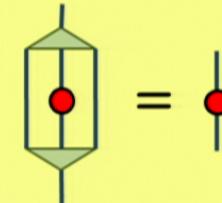
primary fields:

identity:
spin:
energy:
disorder:
fermion:

exact

MERA

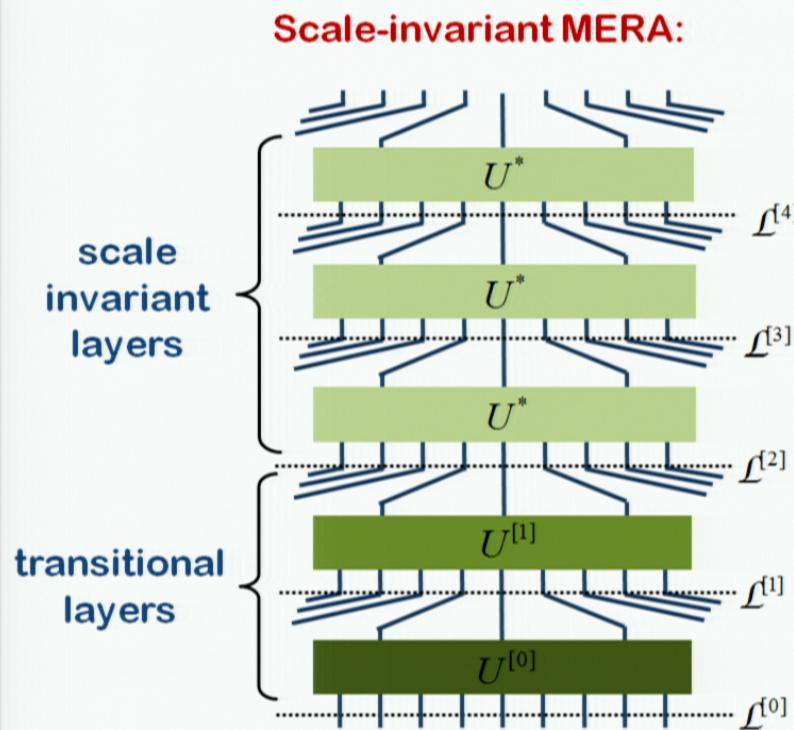
Scaling Superoperator (one-site):



diagonalize:

$$\mathcal{S}(\phi_\alpha) = \lambda_\alpha \phi_\alpha$$

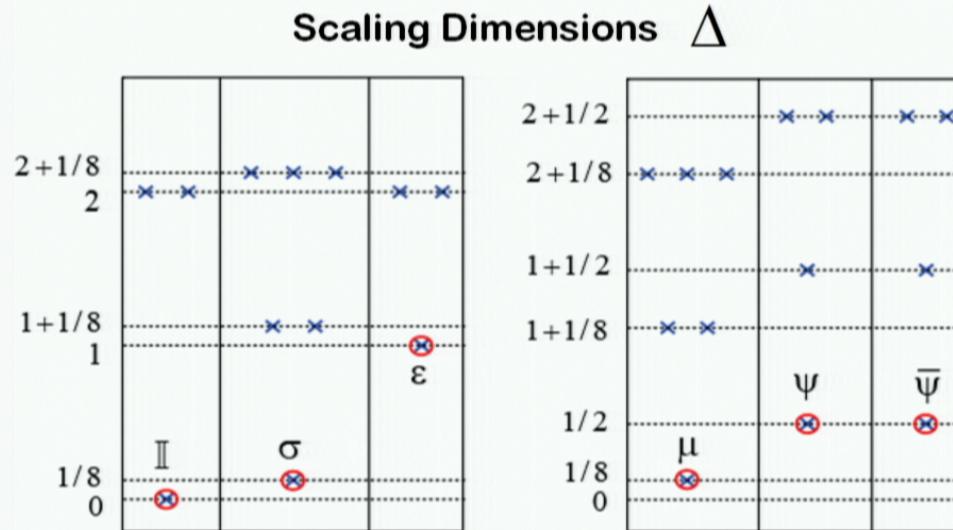
Summary: scale-invariant MERA



- incorporates **scale invariance**
- can be used to compute the **conformal data** of underlying CFT
- transitional layers deal with transitional, short-range effects (=RG irrelevant terms in H)

Numerical Example:

1D critical Ising model: $H = \sum(-XX + Z)$



primary fields:

identity:
spin:
energy:
disorder:
fermion:

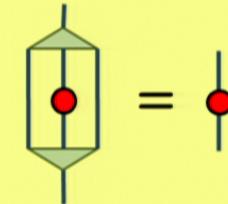
exact

$I = 0$
 $\sigma = 0.125$
 $\epsilon = 1$
 $\mu = 0.125$
 $\psi = 0.5$

MERA

(0)
0.1250003
1.0000200
0.1250002
0.5000004

Scaling Superoperator (one-site):



diagonalize:

$$\mathcal{S}(\phi_\alpha) = \lambda_\alpha \phi_\alpha$$

Conformal field theories with a defect/impurity

CFT without impurity: H_{bulk}

bulk scaling
operators

$$\phi_\alpha$$

bulk scaling
dimensions

$$\Delta_\alpha$$

two-point correlators

$$\langle \phi_\alpha(x) \phi_\alpha(x+d) \rangle = \frac{1}{|d|^{2\Delta_\alpha}}$$

Conformal field theories with a defect/impurity

CFT without impurity: H_{bulk}

bulk scaling
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dimensions

$$\phi_\alpha \qquad \Delta_\alpha$$

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$$\langle \phi_\alpha(x) \phi_\alpha(x+d) \rangle = \frac{1}{|d|^{2\Delta_\alpha}}$$

translation invariant, critical “bulk”

$$H_{\text{imp.}} = H_{\text{bulk}} + h_{\text{imp.}}(0)$$

↓ ↘
impurity (local!)

↑
Impurity Hamiltonian:

- ✗ no translation invariance
- ✓ scale invariance (conformal defect)

Conformal field theories with impurity

$$H_{\text{imp.}} = H_{\text{bulk}} + h_{\text{imp.}}(0)$$

- bulk scaling operators in the **absence** of an impurity:

$$\langle \phi_\alpha \rangle = 0$$

- bulk scaling operators in the **presence** of an impurity:

$$\langle \phi_\alpha(d) \rangle_{\text{imp.}} \sim \frac{1}{|d|^{\Delta_\alpha}}$$

Conformal field theories with impurity

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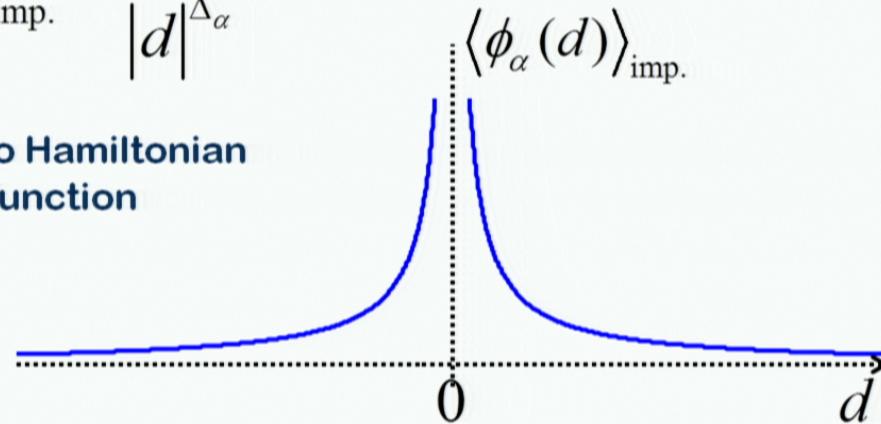
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$$\langle \phi_\alpha \rangle = 0$$

- bulk scaling operators in the **presence** of an impurity:

$$\langle \phi_\alpha(d) \rangle_{\text{imp.}} \sim \frac{1}{|d|^{\Delta_\alpha}}$$

- addition of **local impurity** to Hamiltonian affects ground state wavefunction **everywhere**



Conformal field theories with impurity

$$H_{\text{imp.}} = H_{\text{bulk}} + h_{\text{imp.}}(0)$$

- new **scaling operators** and **scaling dimensions** associated to the impurity
 - ↓ $\tilde{\phi}_\alpha$
 - ↓ $\tilde{\Delta}_\alpha$
- two-point correlator between **impurity scaling operator** and **bulk scaling operator**

$$\left\langle \tilde{\phi}_\alpha(0)\phi_\beta(d) \right\rangle_{\text{imp.}} = \frac{\tilde{C}_{\alpha\beta}}{|d|^{\tilde{\Delta}_\alpha + \Delta_\beta}}$$


Conformal field theories with impurity: Summary

$$H_{\text{imp.}} = H_{\text{bulk}} + h_{\text{imp.}}(0)$$

CFT without impurity: H_{bulk}

bulk scaling
operators
 ϕ_α

bulk scaling
dimensions
 Δ_α

local expectation values

$$\langle \phi_\alpha(x) \rangle = 0$$

two-point correlators

$$\langle \phi_\alpha(x) \phi_\beta(x+d) \rangle = \frac{C_{\alpha\beta}}{|d|^{\Delta_\alpha + \Delta_\beta}}$$

CFT with impurity: $H_{\text{imp.}}$

local expectation values

$$\langle \phi_\alpha(d) \rangle_{\text{imp.}} \sim 1/d^{\Delta_\alpha}$$

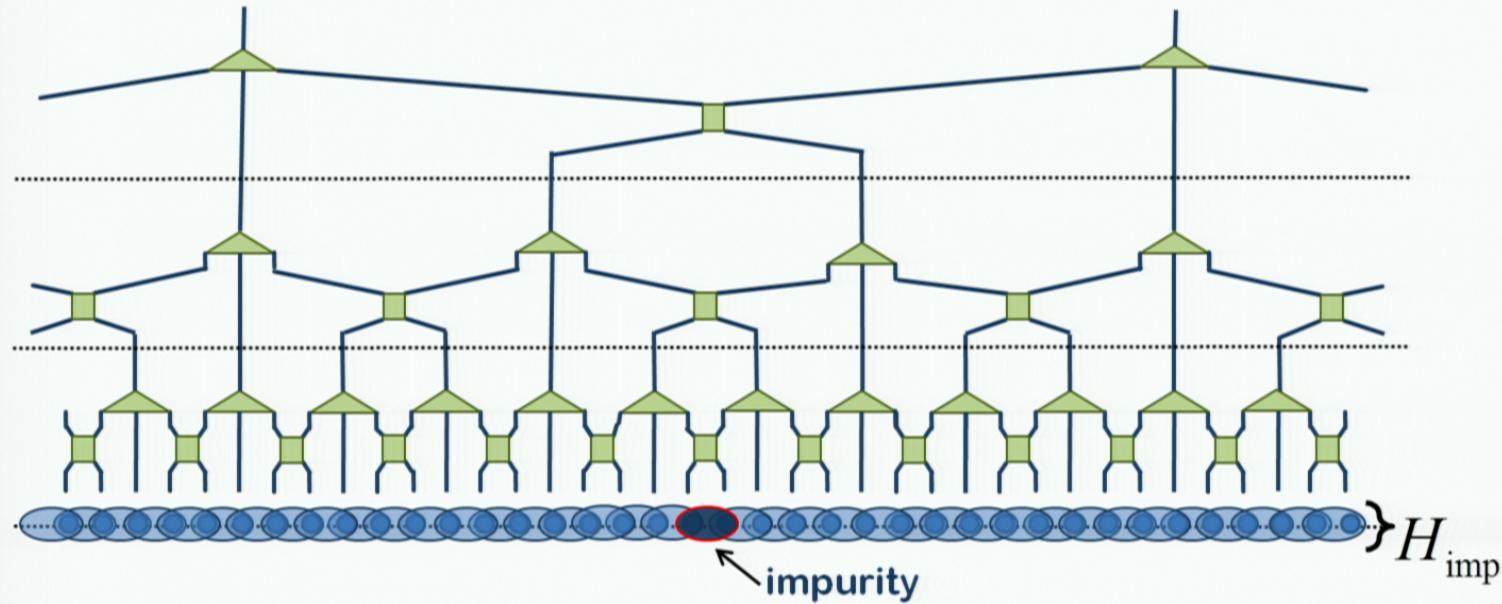
impurity scaling
operators

impurity scaling
dimensions
 $\tilde{\Delta}_\alpha$

impurity-bulk correlator

$$\langle \tilde{\phi}_\alpha(0) \phi_\beta(d) \rangle_{\text{imp.}} = \frac{\tilde{C}_{\alpha\beta}}{|d|^{\tilde{\Delta}_\alpha + \Delta_\beta}}$$

Impurity problems with MERA



Difficulty: impurity system does not have translation invariance.

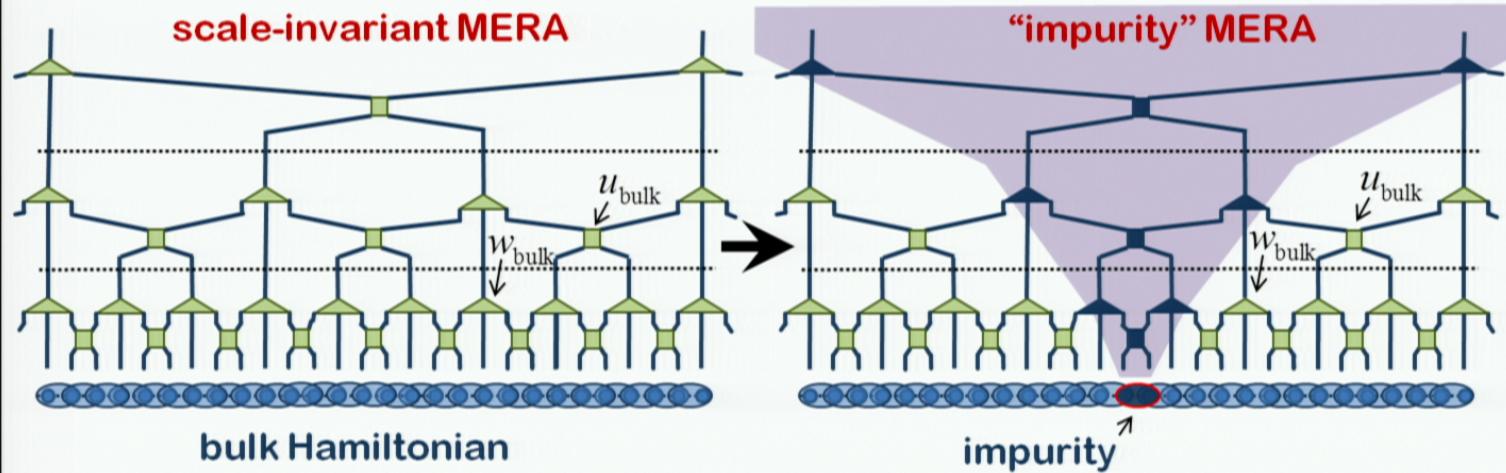
Impurity problems with MERA

local Hamiltonian:
ground state
MERA:

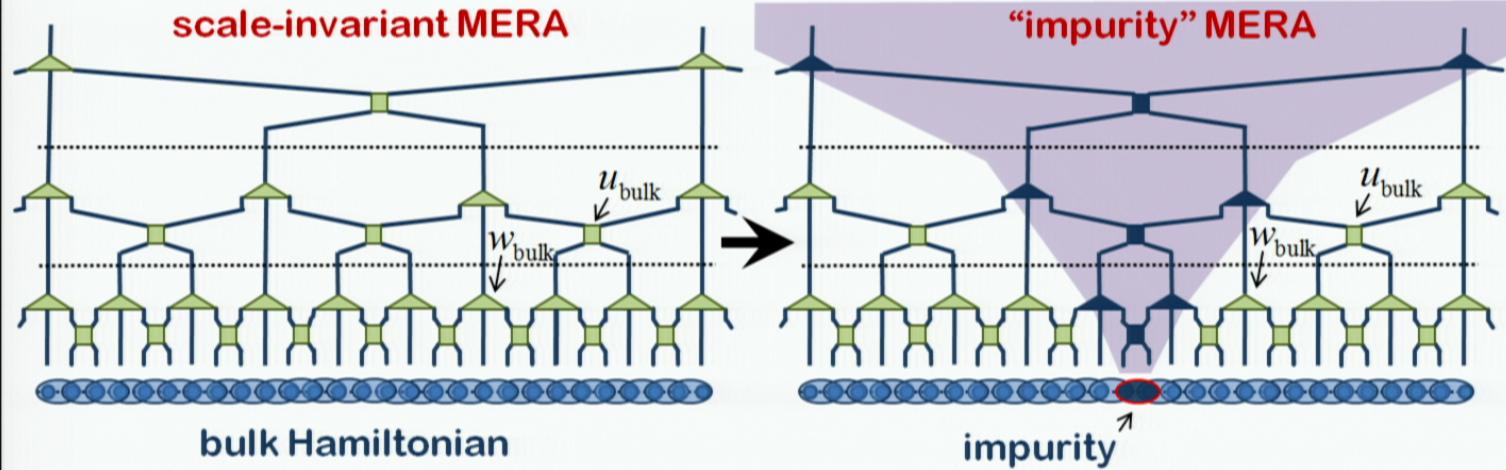
$$\begin{array}{c} H_0 \\ |\psi_0\rangle \end{array} \rightarrow \begin{array}{c} H_1 = H_0 + h(R) \\ |\psi_1\rangle \end{array}$$

Principle of minimal influence:

Given MERA for $|\psi_0\rangle$, we can obtain MERA for $|\psi_1\rangle$ by changing tensors in the causal cone of region R only.



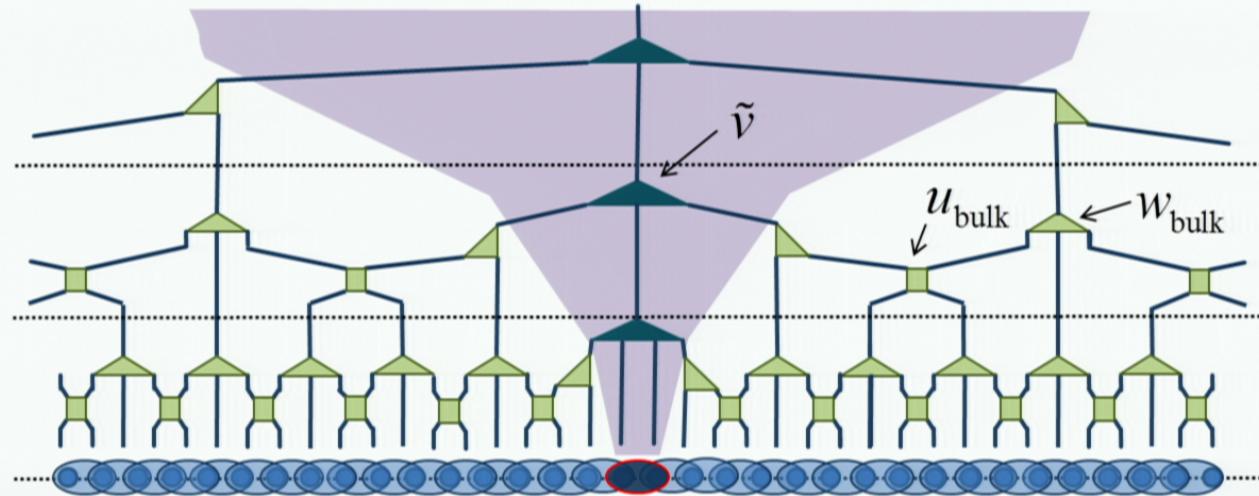
Impurity problems with MERA



Principle of minimal influence

- We have no proof...
- Plausible (changes in causal cone of R already produce changes in all the system)
- strong numerical evidence
- Exceptions: e.g. cases involving spontaneous symmetry breaking

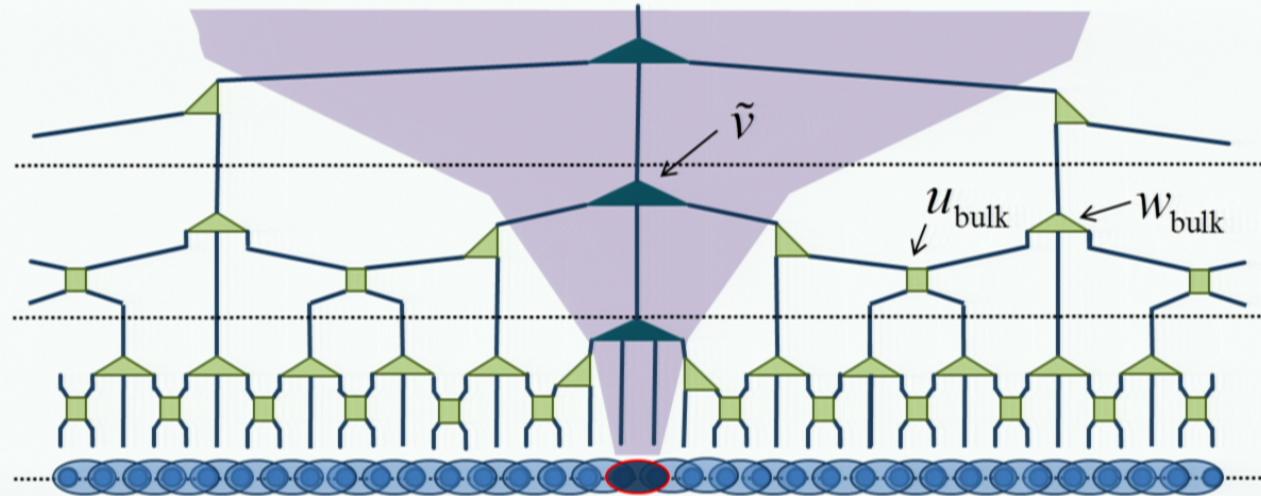
Impurity problems with MERA



"impurity" MERA

- exploit the translation invariance of the bulk
- can also enforce scale invariance
- cost is $O(1)$, same as bulk scale-invariant MERA!

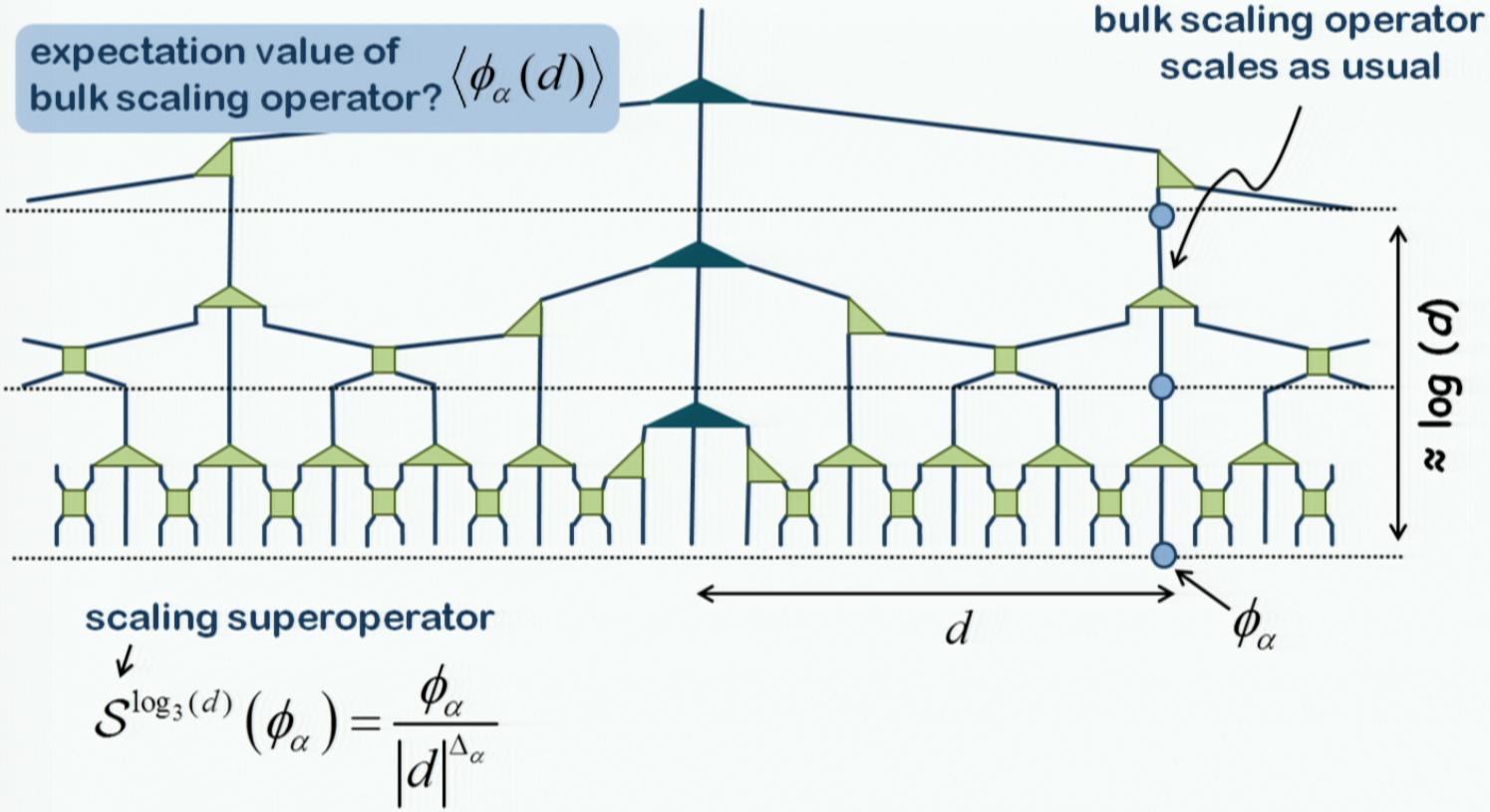
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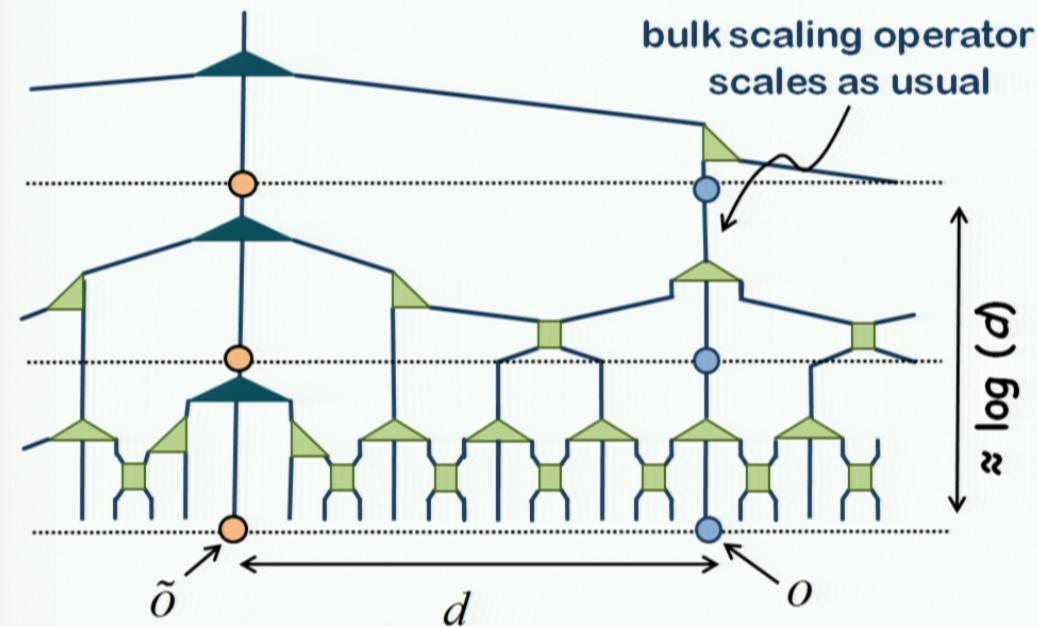
"impurity" MERA

- exploit the translation invariance of the bulk
- can also enforce scale invariance
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Impurity MERA: local observables



Impurity MERA: correlators



Two-Point Correlator: $\langle \tilde{o}(0)o(d) \rangle$

operator at impurity

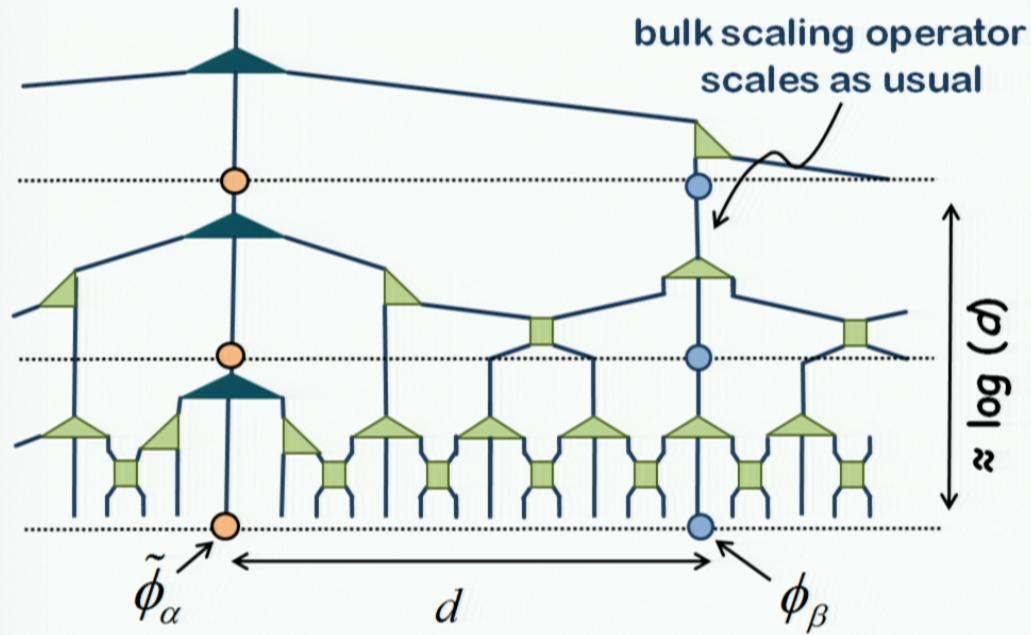
operator in
the bulk

diagonalize: $\tilde{S}(\tilde{\phi}_\alpha) = \tilde{\lambda}_\alpha \tilde{\phi}_\alpha$

impurity scaling operators

impurity scaling dimensions

Impurity MERA: correlators



$$\langle \tilde{\phi}_\alpha(0) \phi_\beta(d) \rangle = \text{tr} \left(\rho \tilde{\mathcal{S}}^{\log_3(d)}(\tilde{\phi}_\alpha) \mathcal{S}^{\log_3(d)}(\tilde{\phi}_\alpha) \right)$$

Impurity Scaling Superoperator

$$\tilde{\mathcal{S}}(\tilde{\phi}_\alpha) = \tilde{\lambda}_\alpha \tilde{\phi}_\alpha$$

$=$

Summary

CFT with impurity:

local expectation values

$$\langle \phi_\alpha(d) \rangle_{\text{imp.}} \sim 1/d^{\Delta_\alpha}$$

impurity scaling
operators

$$\tilde{\phi}_\alpha$$

impurity scaling
dimensions

$$\tilde{\Delta}_\alpha$$

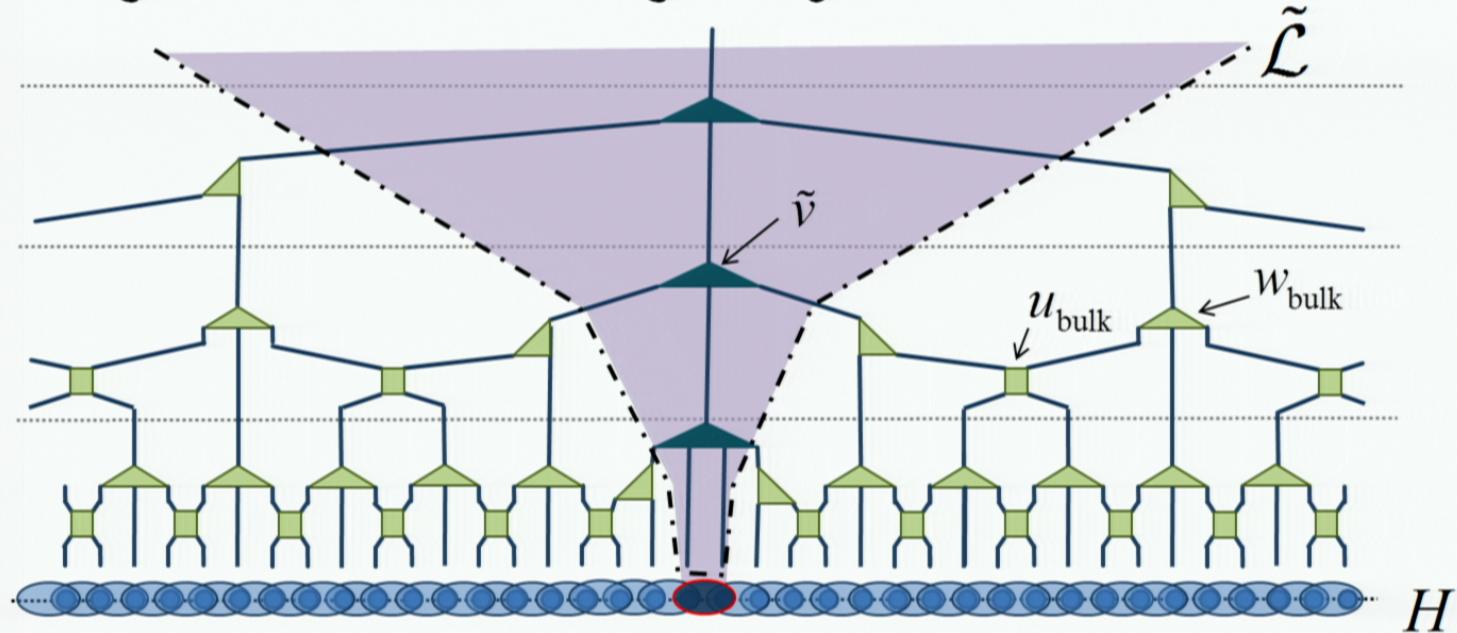
impurity-bulk correlator

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impurity MERA:



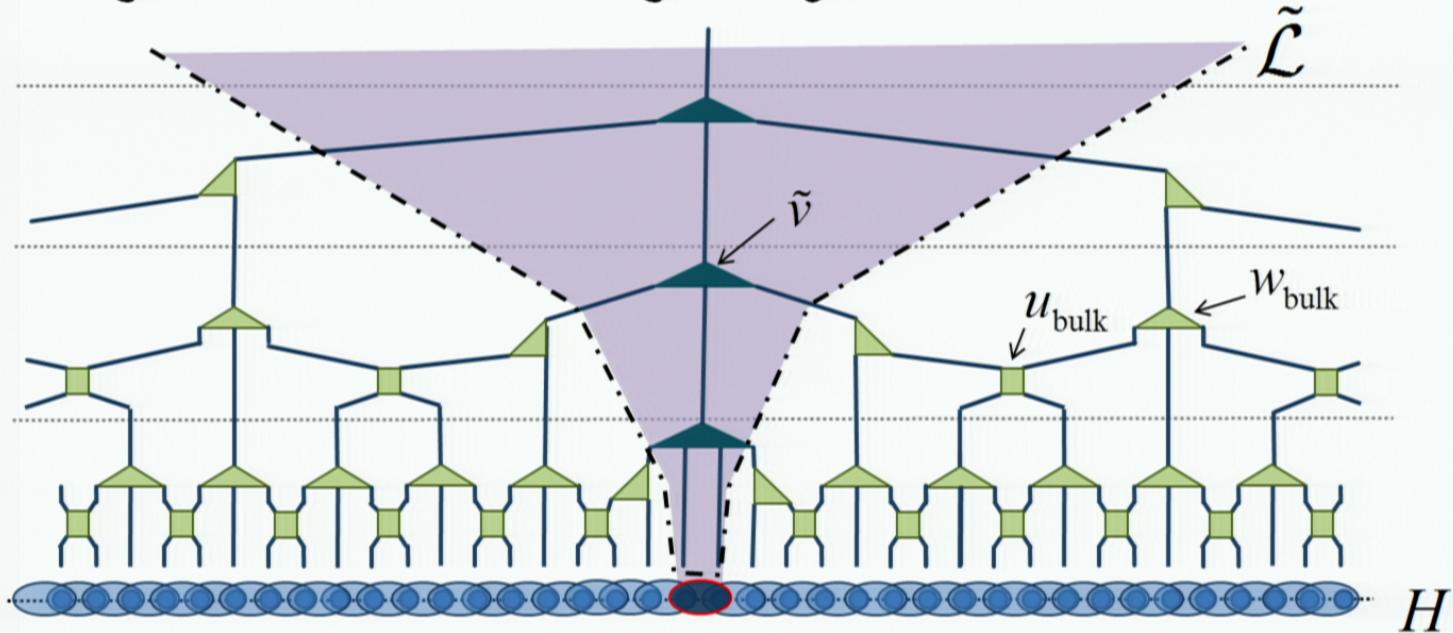
Optimization of Impurity MERA



Steps for optimising an impurity MERA:

- 1) Optimise **bulk tensors** with bulk Hamiltonian (no impurity)

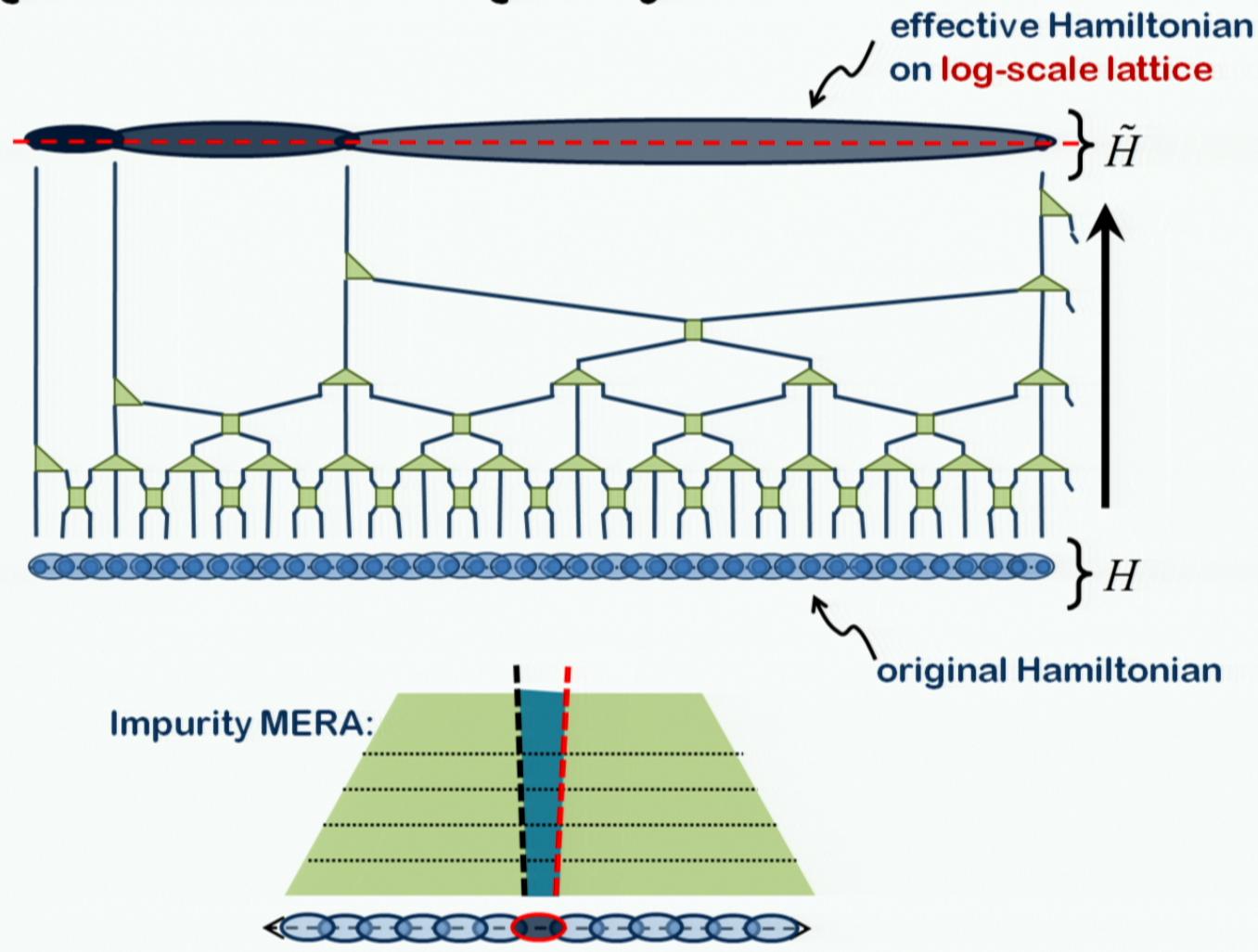
Optimization of Impurity MERA



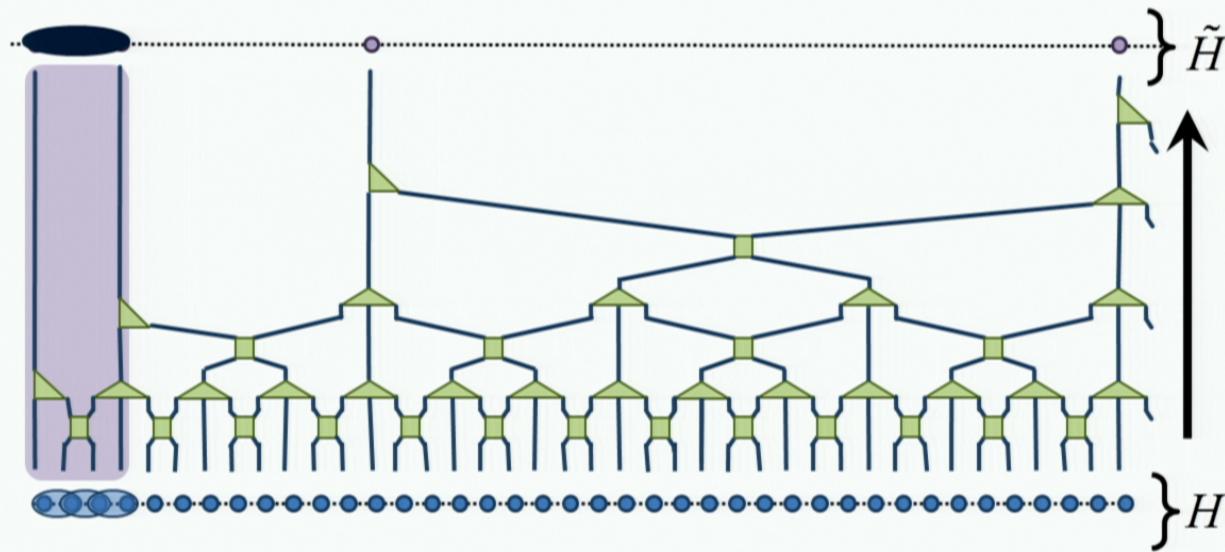
Steps for optimising an impurity MERA:

- 1) Optimise **bulk tensors** with bulk Hamiltonian (no impurity)
- 2) Use bulk tensors to map original impurity problem to an **effective impurity problem** on log-scale lattice $\tilde{\mathcal{L}}$
- 3) Optimise for impurity tensors from **effective Hamiltonian** on the log-scale lattice

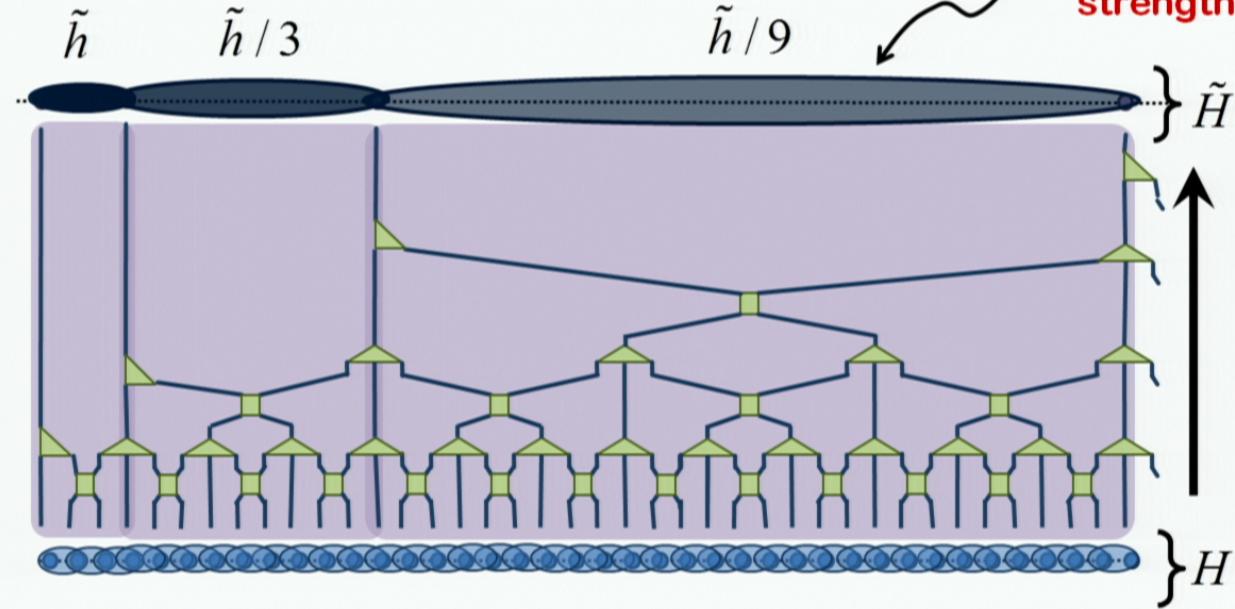
Optimization of Impurity MERA



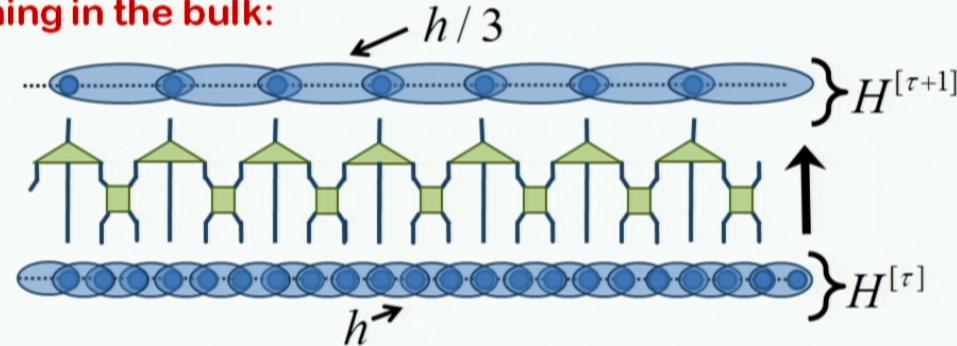
Optimization of Impurity MERA



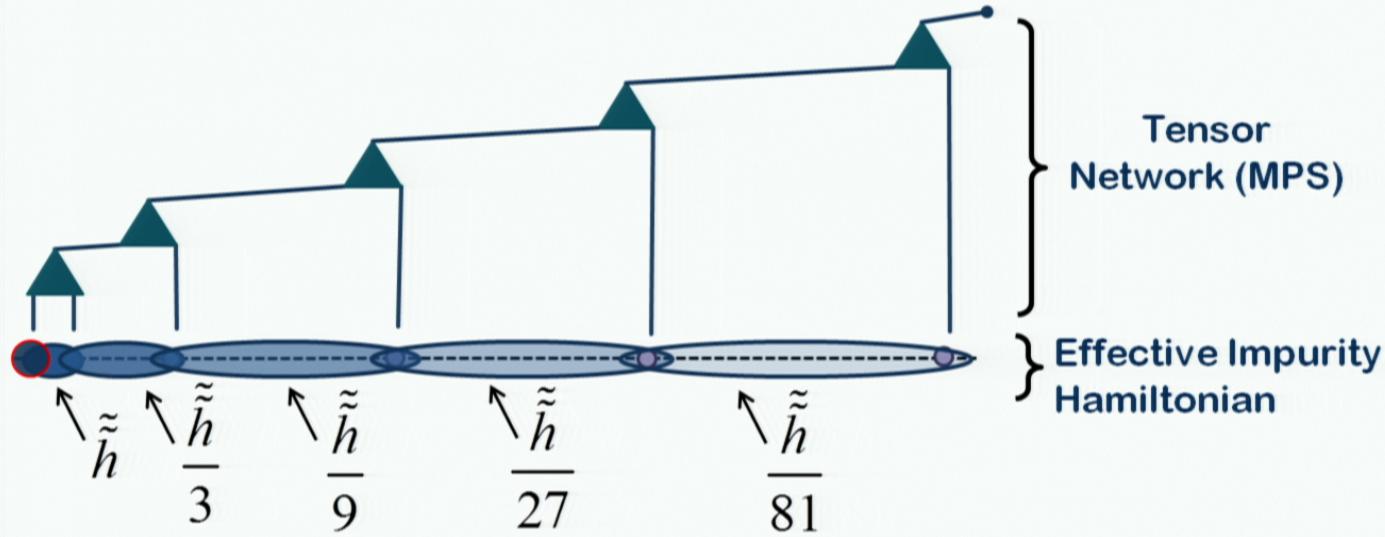
Optimization of Impurity MERA



Coarse-graining in the bulk:



Optimization of Impurity MERA



- same form of effective impurity Hamiltonian as arises in **Wilson's solution** to the **Kondo Impurity** problem!

magnetic impurity in a metal

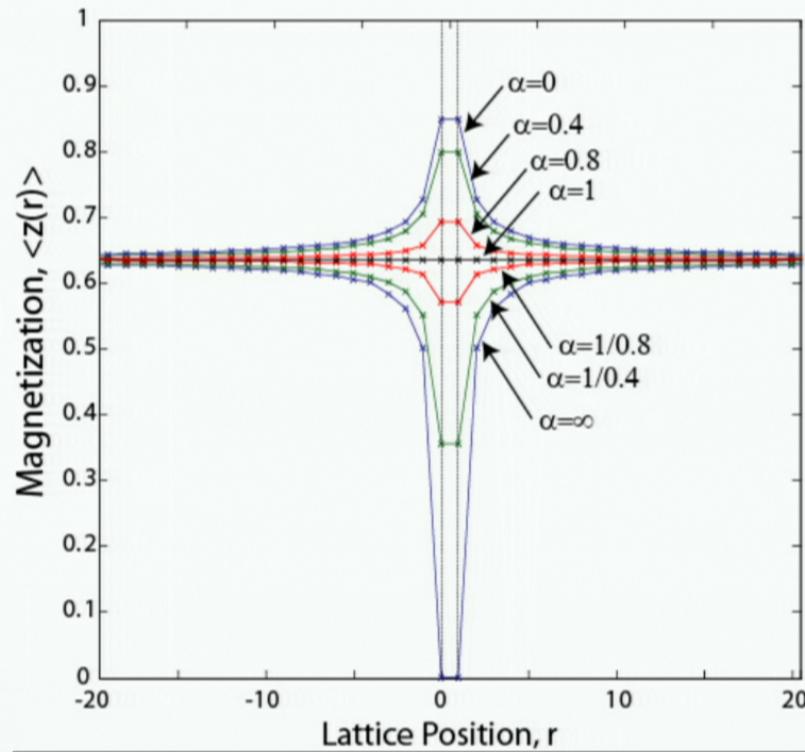
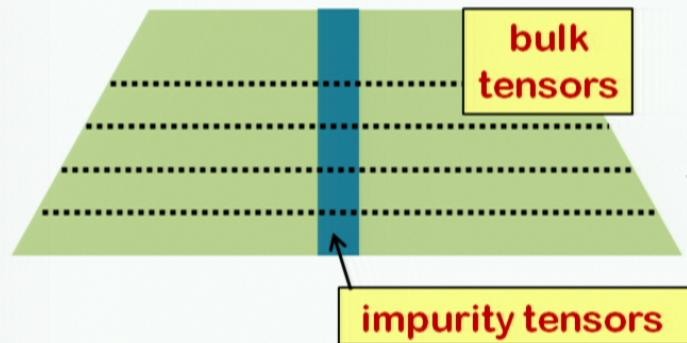
Example: the Ising model with an Impurity

critical 1D Ising chain with conformal impurity:

$$H = H_{\text{bulk}} + h_{\text{Imp.}}$$

$$H_{\text{bulk}} = \sum (-XX + Z)$$

$$h_{\text{Imp.}} = (1 - \alpha)XX$$



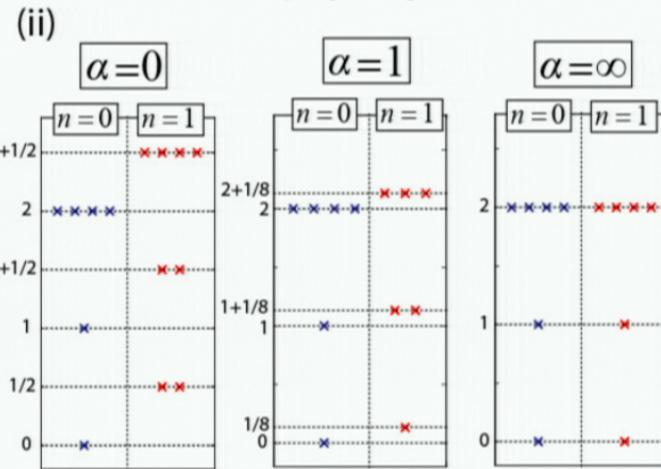
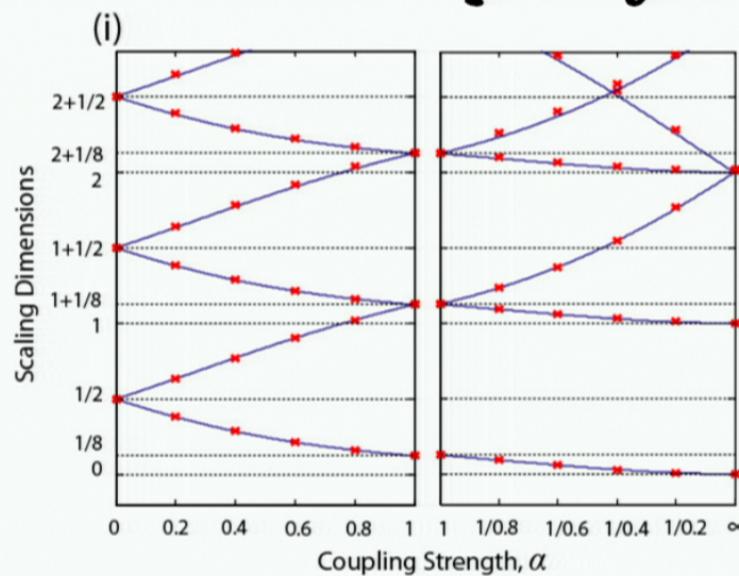
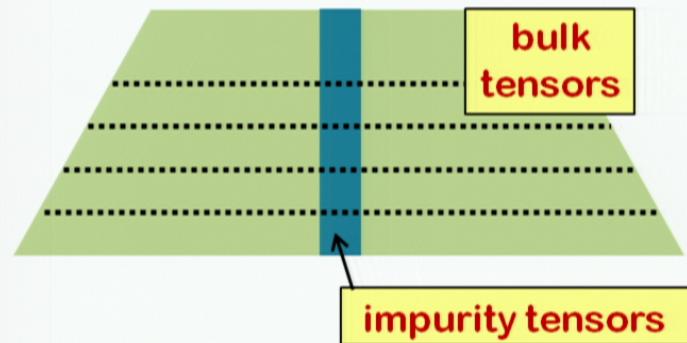
Example: the Ising model with an Impurity

critical 1D Ising chain with conformal impurity:

$$H = H_{\text{bulk}} + h_{\text{Imp.}}$$

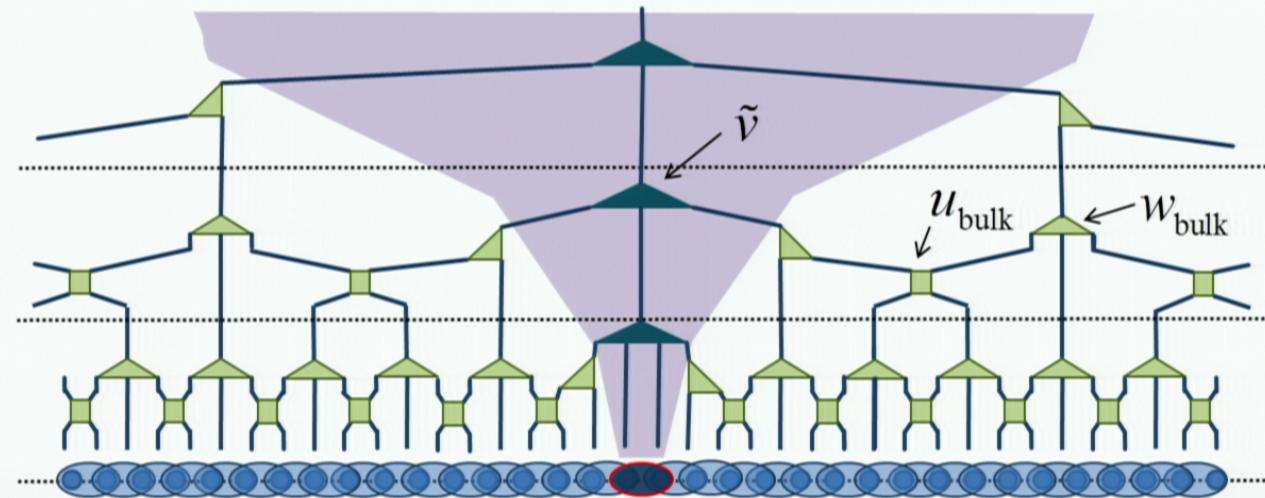
$$H_{\text{bulk}} = \sum (-XX + Z)$$

$$h_{\text{Imp.}} = (1-\alpha)XX$$



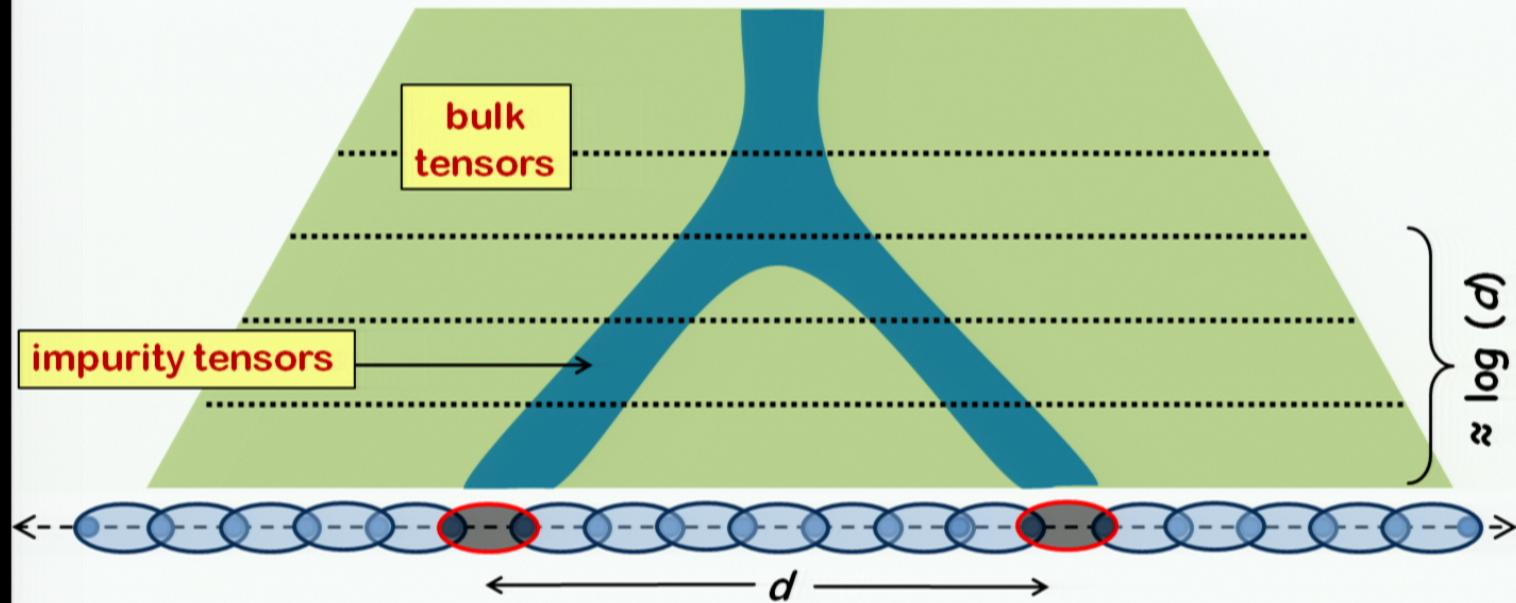
Impurity problems with MERA: Summary

Principle of minimal influence \Rightarrow Impurity MERA



- exploits **translation invariance** (of the bulk) and **scale-invariance**; thus cost $O(1)$ in system size
- natural ansatz for ground state; can reproduce **local observables**, **correlators** and **conformal data** of a CFT to high accuracy
- many similarities to **Wilson's approach**, but does not rely on free-fermion bulk

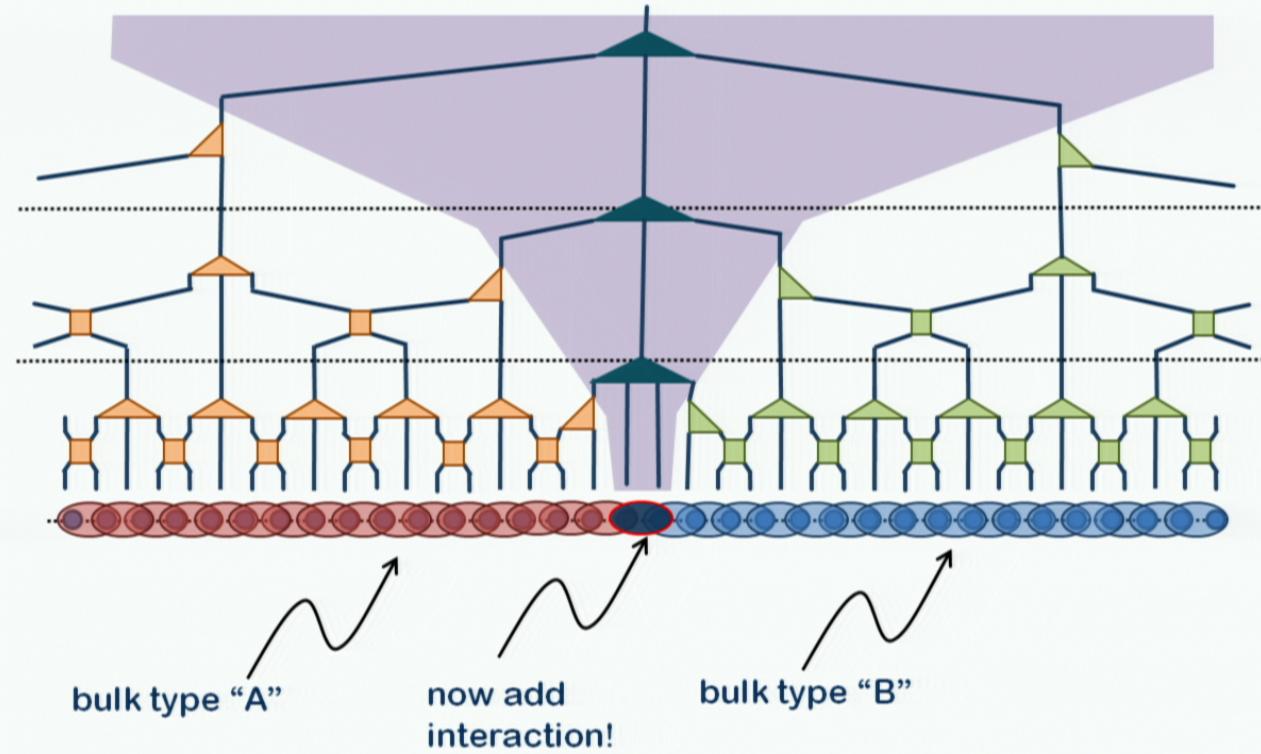
Extensions: Multiple Impurities



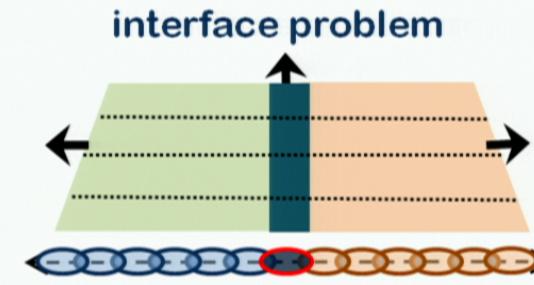
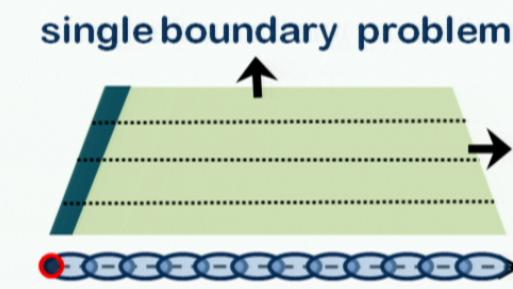
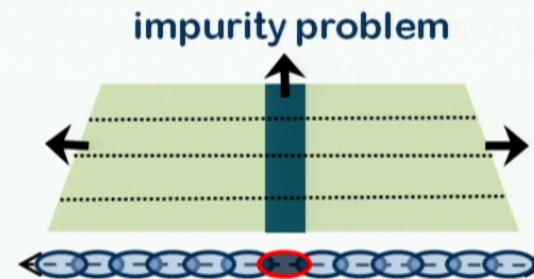
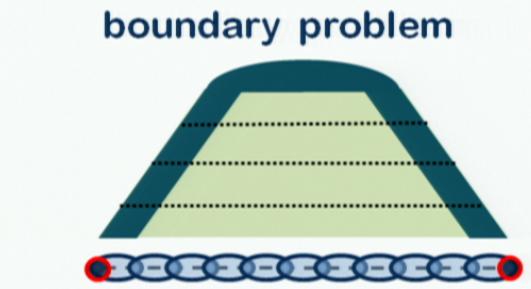
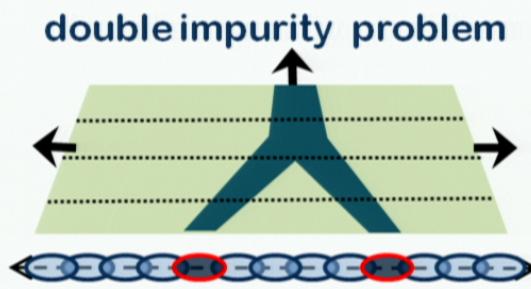
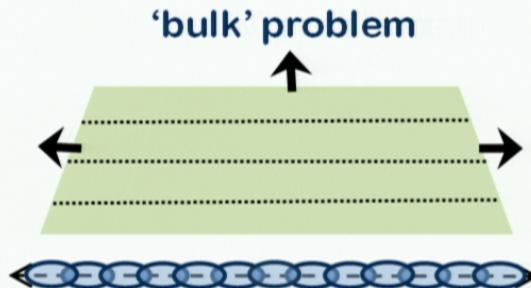
- causal cones of impurities fuse together after $\log(d)$ coarse-graining transforms

Extensions: Interfaces

“boundary” MERA → “interface” MERA



Summary: principle of minimal influence



Real-space Renormalization Group

Old thinking: real-space RG is not so good...

K.Wilson: real-space RG can work if your problem has
separation of energy scales...

S.White: real-space RG can work for general problems if
you have the **density matrix...**

Here: real-space RG can work **without the density matrix** if your
coarse-graining approach has a built in separation of energy scales

density matrix not that important!

