Title: Fluctuations and Viscosity

Date: Oct 13, 2011 04:00 PM

URL: http://www.pirsa.org/11100097

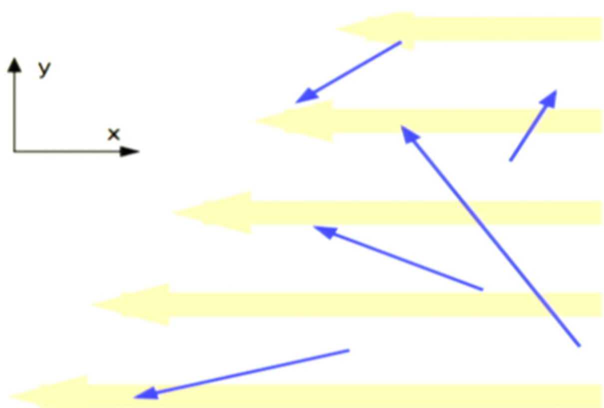
Abstract: Sound waves with long-distance propagation are both a consequence of hydrodynamics, and a danger to hydrodynamics' very existence, as they violate the assumption of local equilibration. In the talk, I will discuss what the thermally excited sound and shear waves do to viscosity. In 2+1 dimensions, the shear viscosity and the diffusion constant cease being independent transport coefficients. In 3+1 dimensions, the fluctuations render the second-order hydrodynamics invalid.

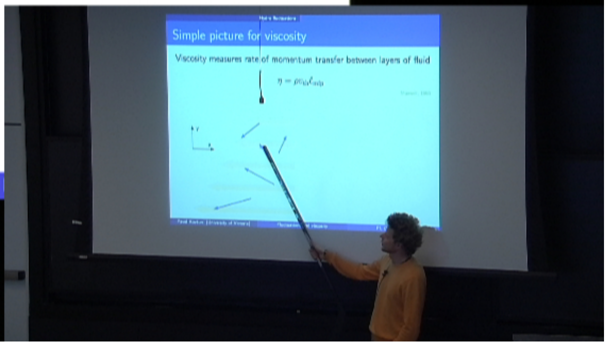
Simple picture for viscosity

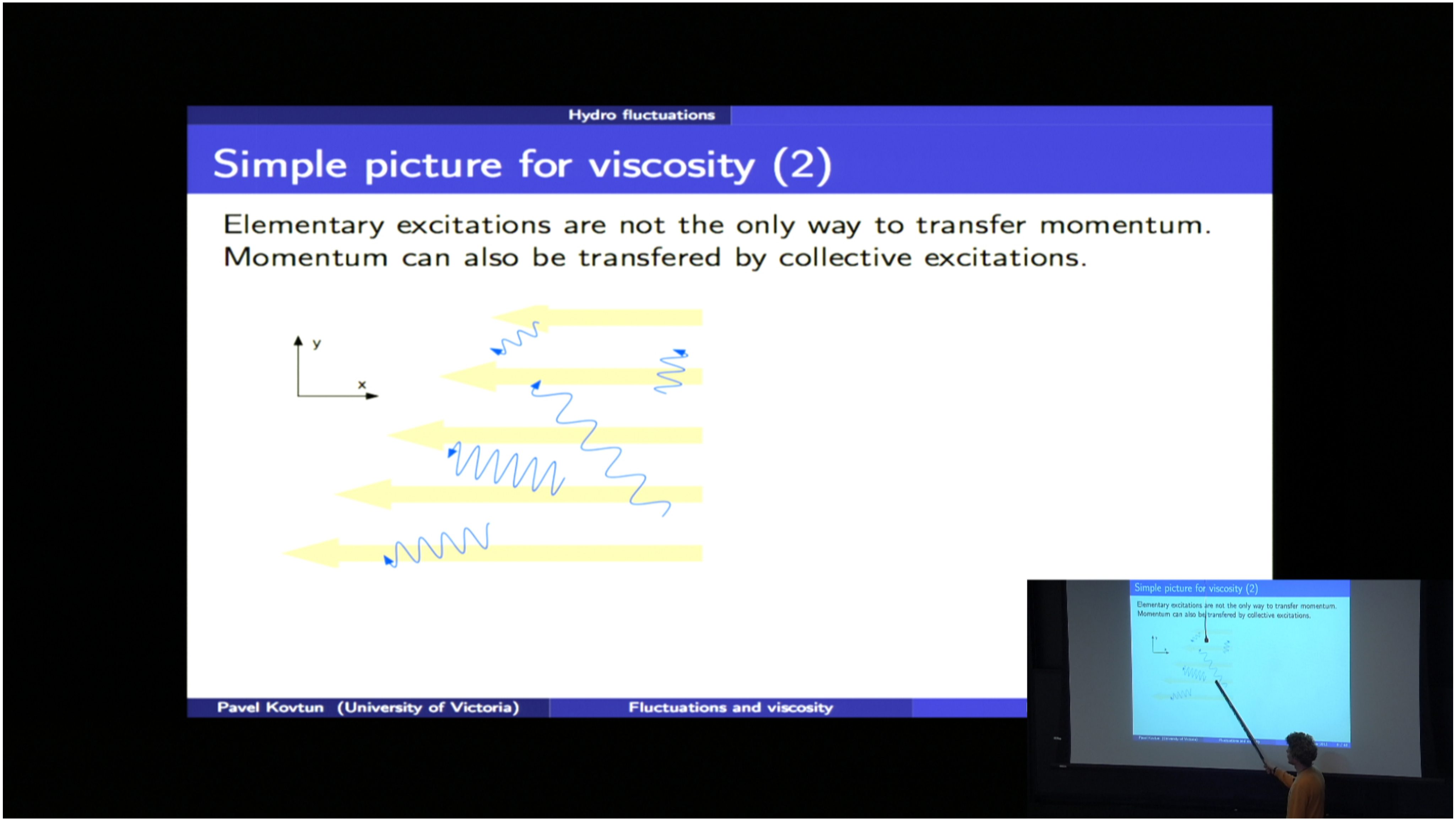
Viscosity measures rate of momentum transfer between layers of fluid

$$\eta = \rho v_{\rm th} \ell_{\rm mfp}$$

Maxwell, 1860






Hydro fluctuations
Simple picture for viscosity (2)
Elementary excitations are not the only way to transfer momentum.
Momentum can also be transfered by collective excitations.
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Fluctuations and viscosity

Simple picture for viscosity (2)

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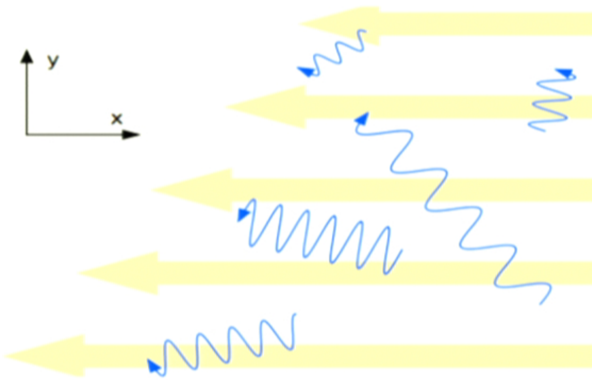

$$\ell_{\rm mfp} \sim \frac{1}{\frac{\eta}{\epsilon+P}\boldsymbol{k}^2}$$

$$\eta_1 \sim \int^{k_{\rm max}} d^3k \, \frac{T}{\frac{\eta_0}{\epsilon+P}\boldsymbol{k}^2} \sim \frac{k_{\rm max} T^2}{\eta_0/s}$$

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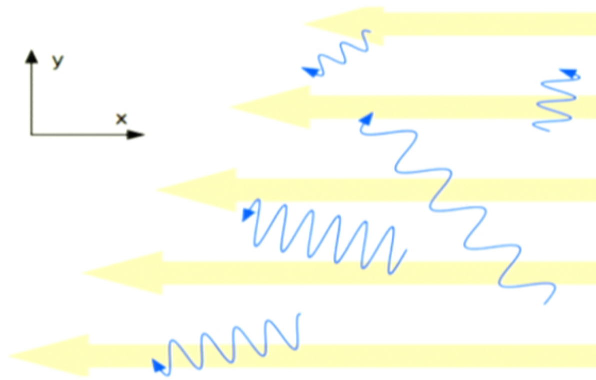

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1. Total viscosity $\eta_{\rm total} = \eta_0 + \eta_1$ is bounded from below
2. This integral IR finite in $d = 3{+}1$, but IR divergent in $d = 2{+}1$

Forster+Nelson+Stephen, 1977

Interaction of hydro modes

- In hydro, there are no arbitrary "coupling constants" like g
- Coefficients of non-linear terms are fixed by symmetry (Galilean or Lorentz) E.g.

$$J^\mu = n u^\mu + \nu^\mu \,, \qquad T^{\mu\nu} = (\epsilon + P) u^\mu u^\nu + P \eta^{\mu\nu} + \tau^{\mu\nu} \,.$$

  All transport coefs η, ζ, κ are present already in linearized hydro
- Interaction of modes will change hydro correlation functions
- Was known since late 1960's – "mode-mode coupling"

Long-time tails

Start with $\boldsymbol{J} = -D\boldsymbol{\nabla} n + n\boldsymbol{v}$, take $\boldsymbol{k} = 0$. Schematically:

$$\begin{aligned}\langle \boldsymbol{J}(t)\boldsymbol{J}(0)\rangle &\supset \int d^d x\, \langle n(t,\boldsymbol{x})\boldsymbol{v}(t,\boldsymbol{x})n(0)\boldsymbol{v}(0)\rangle \\ &= \int d^d x\, \langle n(t,\boldsymbol{x})n(0)\rangle\langle \boldsymbol{v}(t,x)\boldsymbol{v}(0)\rangle \\ &\sim \int d^d k\, e^{-D\boldsymbol{k}^2 t} e^{-\gamma_\eta \boldsymbol{k}^2 t} \\ &\sim \left[\frac{1}{(D+\gamma_\eta)t}\right]^{d/2}\end{aligned}$$

See e.g. Arnold+Yaffe, PRD 1997
(known since late 1960's)

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When FT, the convective contribution to $S(\omega)$ is

$$S(\omega) \sim \omega^{1/2}\,, \qquad d = 3$$
$$S(\omega) \sim \ln(\omega)\,, \qquad d = 2$$

Correction to Kubo formulas

Recall Kubo formula for the diffusion constant:

$$D\chi T = \lim_{\omega \to 0} \frac{1}{2d} S_{ii}(\omega, \boldsymbol{k}=0)$$

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This was derived in linear response. With the non-linear temrs:

$$D^{\text{full}} = \lim_{\omega\to 0} \left(D + \text{const } \omega^{1/2}\right) , \qquad d=3$$
$$D^{\text{full}} = \lim_{\omega\to 0} \left(D + \text{const } \ln(\omega)\right) , \qquad d=2$$

Same applies to shear viscosity:

$$\eta^{\text{full}} = \lim_{\omega\to 0} \left(\eta + \text{const } \omega^{1/2}\right) , \qquad d=3$$
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Comment

- In AdS/CFT, the $\ln(\omega)$ correction is $1/N^{3/2}$ suppressed
- Transport coefficients come out finite in $3+1$ dimensional *classical* gravity
- Long-time tails come from quantum corrections to classical gravity

Kovtun+Yaffe, 2003
Caron-Huot + Saremi, 2009

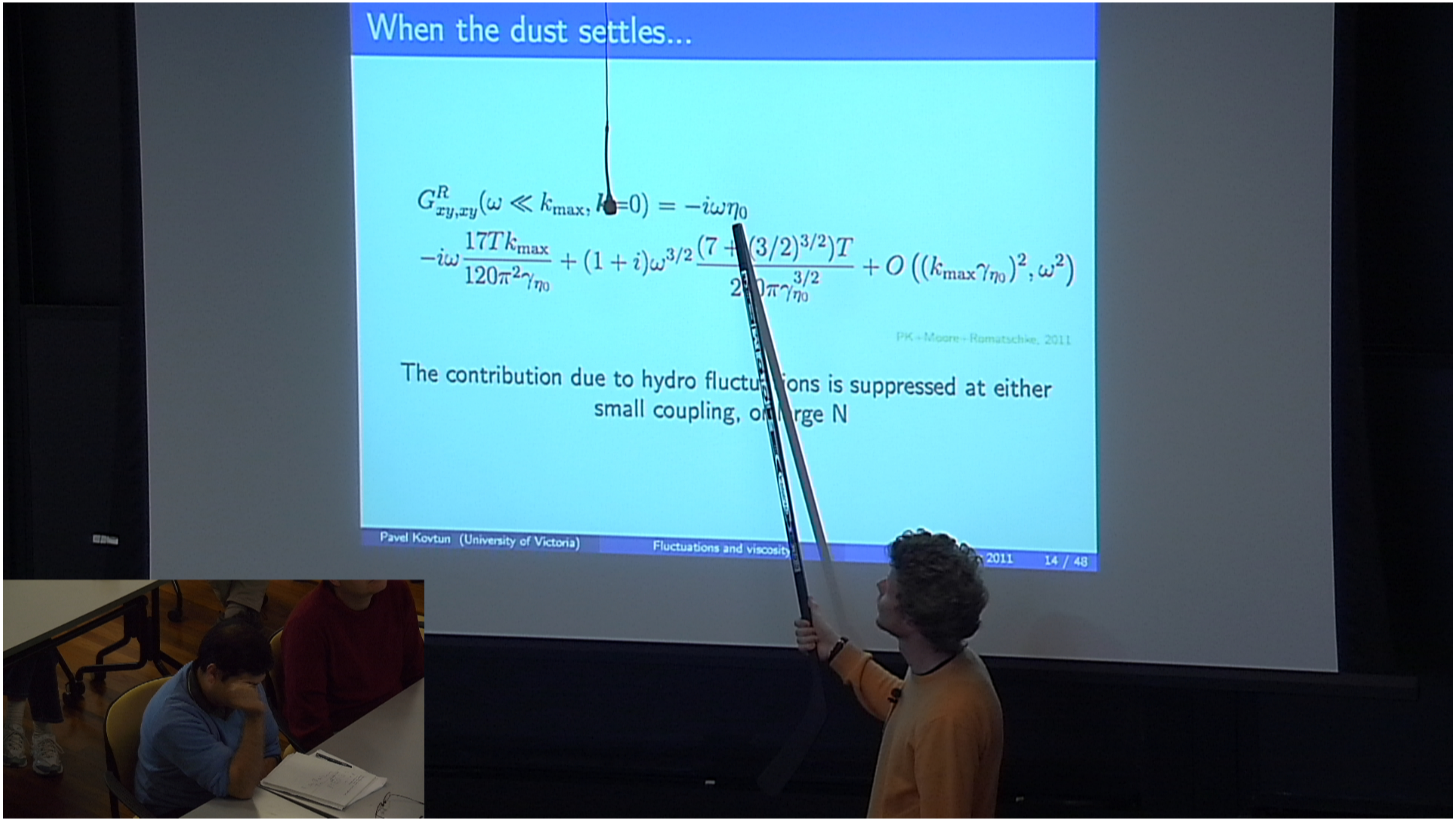
- This is an example where long-time limit does not commute with large-N limit

Can do the same calculation in momentum space

One-loop diagram with sound and/or shear waves in the loop

$$S_{xy,xy}(\omega, \boldsymbol{k}{=}0) = (\epsilon{+}P)^2 \int \frac{d\omega'}{2\pi} \frac{d^3k}{(2\pi)^3}$$
$$\Big(\Delta_{xx}(\omega', \boldsymbol{k})\Delta_{yy}(\omega{-}\omega', -\boldsymbol{k}) + \Delta_{xy}(\omega', \boldsymbol{k})\Delta_{yx}(\omega{-}\omega', -\boldsymbol{k})\Big)$$

where Δ_{ij} = FT of $\langle u_i(x) u_j(0)\rangle$


When the dust settles...
PK+Moore+Romatschke, 2011
The contribution due to hydro fluctu ions is suppressed at either
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Implications for the shear viscosity

- The function $\eta + c/\eta$ has a minimum, hence *viscosity is bounded from below*

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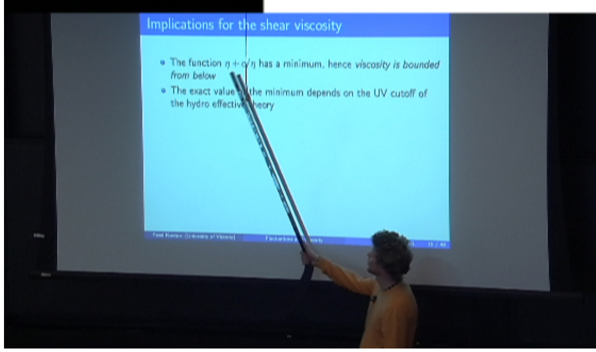

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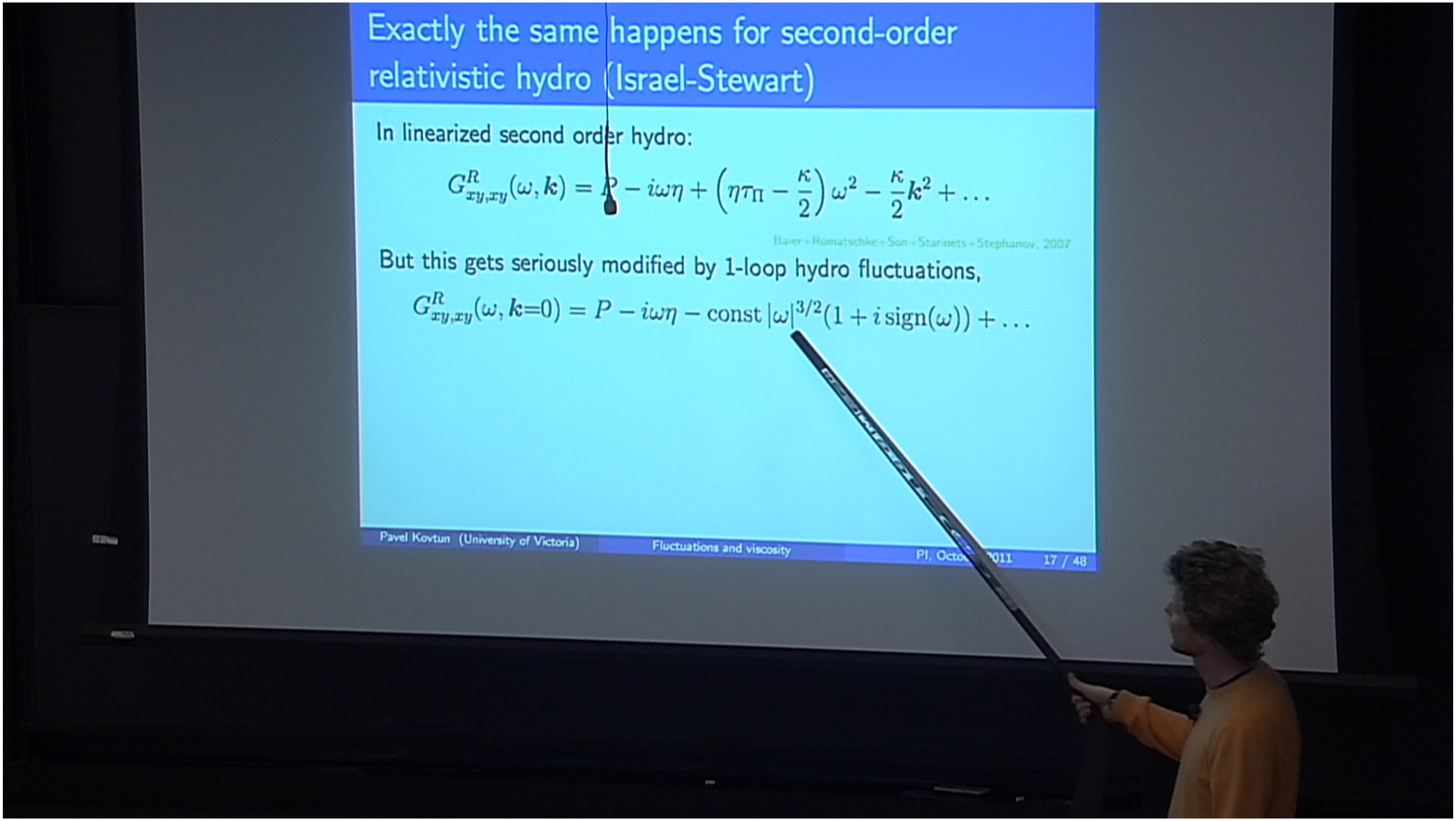
  in QCD at $T \gtrsim T_c$
- Current hydro simulations of QGP are blind to these effects because they simply solve the classical hydro equations and ignore the fluctuations

Exactly the same happens for second-order relativistic hydro (Israel-Stewart)

In linearized second order hydro:

$$G^R_{xy,xy}(\omega, \boldsymbol{k}) = P - i\omega\eta + \left(\eta\tau_\Pi - \frac{\kappa}{2}\right)\omega^2 - \frac{\kappa}{2}\boldsymbol{k}^2 + \dots$$

Baier+Romatschke+Son+Starinets+Stephanov, 2007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But this gets seriously modified by 1-loop hydro fluctuations,
$G^R_{xy,xy}(\omega, \boldsymbol{k}{=}0) = P - i\omega\eta - \text{const}\,|\omega|^{3/2}(1 + i\,\text{sign}(\omega)) + \dots$
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Blindly apply Kubo formula

$$\eta\tau_\Pi - \frac{\kappa}{2} = \lim_{\omega\to 0}\frac{1}{2}\frac{\partial^2}{\partial\omega^2}\text{Re}\,G^R_{xy,xy}(\omega, \boldsymbol{k}{=}0) \to \infty$$

This means τ_Π does not exist

Can we save second-order hydro?

- Can estimate when $\omega^{3/2}$ term becomes comparable to ω^2 term
- 2nd-order hydro breaks down below some ω_* depends on η_0/s
- If $\eta_0/s \sim 0.16$, then $\omega_* \sim T/20$,
  2nd-order hydro OK for heavy-ion collisions
- If $\eta_0/s \sim 0.08$, then $\omega_* \sim 2.5T$,
  2nd order hydro makes no sense for heavy-ion collisions

PK+Moore+Romatschke, 2011

Is there hope for hydrodynamics?

So is 2+1 hydro meaningless?
Is 3+1 hydro meaningless beyond first derivatives?

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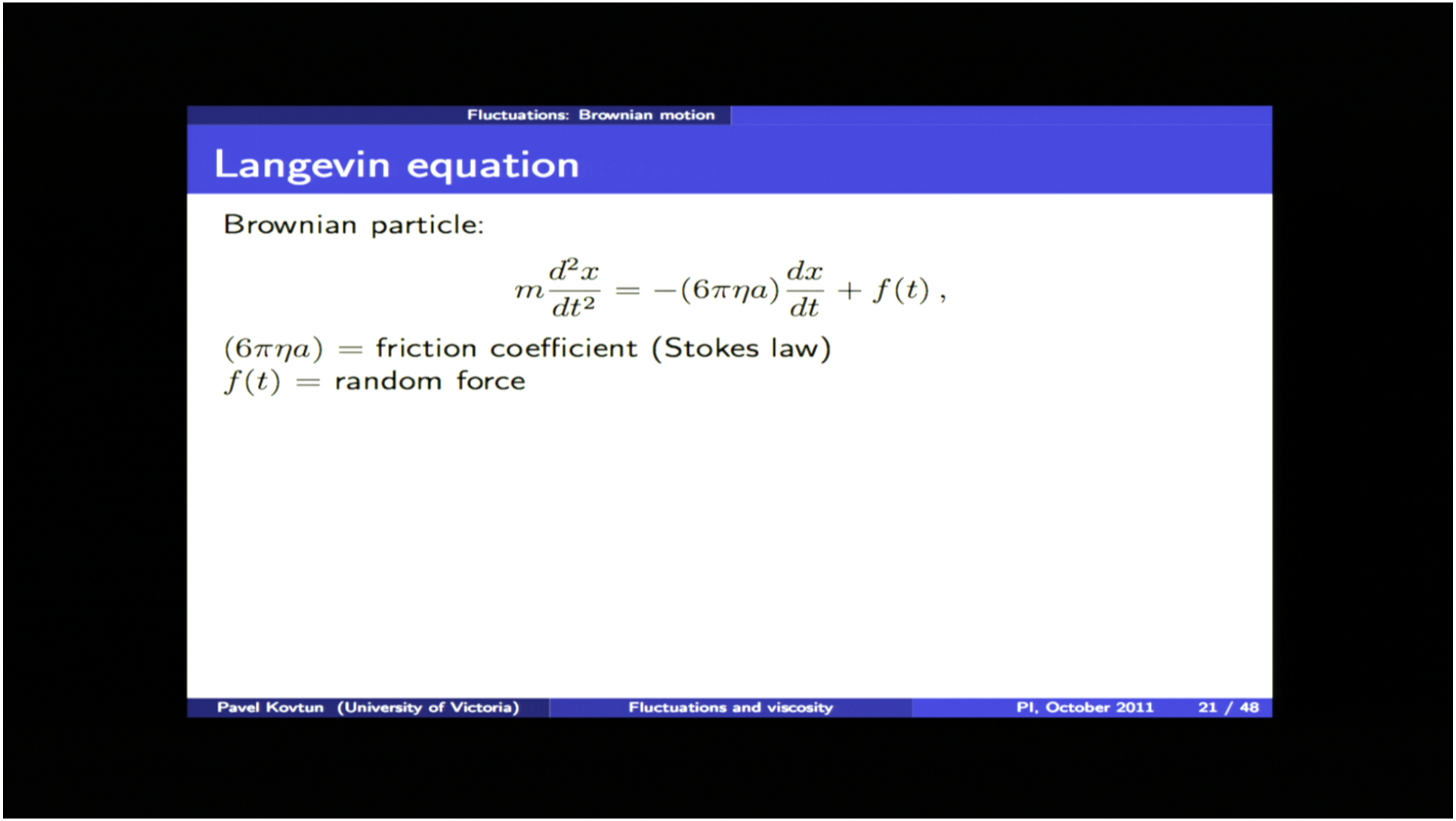
Hydro is not meaningless.
Rather, viscosity, conductivity etc become scale-dependent "running masses" in the low-energy effective hydro theory


Fluctuations: Brownian motion
Langevin equation
Brownian particle:
$$m\frac{d^2x}{dt^2} = -(6\pi\eta a)\frac{dx}{dt} + f(t)\,,$$
$(6\pi\eta a)$ = friction coefficient (Stokes law)
$f(t)$ = random force
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Fluctuations and viscosity
PI, October 2011
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Langevin equation

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Take $q \equiv \frac{dx}{dt}$, $\Rightarrow$ Langevin equation:

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Noise properties:

$$\langle \xi(t) \rangle = 0\,, \qquad \langle \xi(t)\xi(t') \rangle = C\delta(t - t')\,.$$

C determines the strength of the noise

Correlation function of $q(t)$

- Take the Langevin equation $\dot{q}(t) + \gamma q(t) = \xi(t)$
- Solve for $q(t)$ in terms of $\xi(t)$
- Find $\langle q(t)q(t')\rangle$ by averaging over $\xi(t)$
- When $\gamma t, \gamma t' \gg 1$, find

$$\langle q(t)q(t')\rangle = \frac{C}{2\gamma} e^{-\gamma|t-t'|}$$

- Fourier transform:

$$S(\omega) = \frac{C}{\omega^2 + \gamma^2}$$

Noise strength

Recall $\langle \xi(t)\xi(t')\rangle = C\delta(t-t')$

What determines the noise strength C?

- Assume thermal equilibrium
- Demand that the correlation functions satisfy the FDT:

$$\mathrm{Im}\, G_R(\omega) = \frac{\omega}{2T} S(\omega)$$

- To find G_R, introduce source (external force)

$$\delta q(t) = \int dt'\, G_R(t-t')\, \delta f(t')$$

- Langevin equation gives $G_R(\omega) = \frac{i}{\omega + i\gamma}$
- Demand FDT:

$$\boxed{C = 2T}$$

Path integral for Brownian particle

Let us now represent the Brownian motion as Quantum Mechanics
(0+1 dimensional quantum field theory)

Path integral for Brownian particle

Step 1 Write Langevin equation as $EoM \equiv (\dot{q} + \frac{\partial F}{\partial q} - \xi) = 0$

Step 2 Gaussian noise:

$$\langle \ldots \rangle = \int \mathcal{D}\xi \, e^{-W[\xi]} (\ldots) \,, \quad \text{where} \quad W[\xi] = \frac{1}{2C} \int dt' \, \xi(t')^2 \,.$$

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Step 3 Recall $\delta(f(x)) \sim \delta(x - x_0)$, where x_0 solves $f(x_0) = 0$. So

$$\int \mathcal{D}q \, J \, \delta(EoM) \, q(t_1) \, q(t_2) \ldots = \underbrace{q_\xi(t_1) \;\; q_\xi(t_2)}_{\text{satisfy } EoM(q,\xi)=0} \ldots$$

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$$\text{satisfy } EoM(q, \xi) = 0$$

Step 4 Write $\delta(EoM) = \int \mathcal{D}p \, e^{i \int p \, EoM}$, do the integral over $\xi(t)$.

Path integral for Brownian particle (2)

When the dust settles:

$$\langle\, q(t_1) \ldots q(t_n) \,\rangle = \int \mathcal{D}q\, \mathcal{D}p\, J\, e^{iS[q,p]}\, q(t_1) \ldots q(t_n)$$

where

$$S[q,p] = \int dt \left(p\dot{q} + p\frac{\partial F}{\partial q} + \frac{iC}{2} p^2 \right) .$$

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For the simple Langevin equation $F(q) = \frac{1}{2}\gamma q^2$,

$$S(\omega) = \mathrm{FT\ of\ } \langle q(t)q(t')\rangle = \frac{C}{\omega^2+\gamma^2}\,,$$

as expected.

Bottomline:

In the stochastic model

$$\dot{q}(t) + \underbrace{\frac{\partial F(q)}{\partial q}}_{\text{relaxation term}} = \underbrace{\xi(t)}_{\text{noise term}}$$

correlation functions can be derived from field theory with

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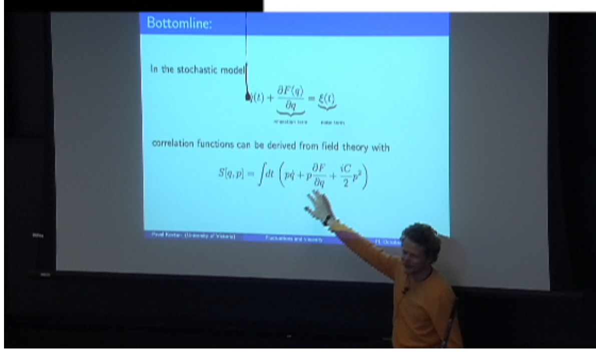

Functional integral for hydro

Correlation functions in linearized hydro:

$$\langle \epsilon(\boldsymbol{x},t)\pi_k(\boldsymbol{x}',t')\ldots\rangle = \int \mathcal{D}\epsilon\, \mathcal{D}\boldsymbol{\pi}\, \mathcal{D}\eta\, \mathcal{D}\boldsymbol{\lambda}\, e^{iS} \epsilon(\boldsymbol{x},t)\pi_k(\boldsymbol{x}',t')\ldots$$

$$S = \int_{t,\boldsymbol{x}} \Big(\eta\big(\tfrac{\partial\epsilon}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{\pi}\big) + \lambda_i\Big(\tfrac{\partial\pi_i}{\partial t} + v_s^2\partial_i\epsilon - M_{ij}\pi_j\Big) - i\varpi T\,\lambda_i M_{ij}\lambda_j\Big)$$

Can integrate out the auxiliary field $\boldsymbol{\lambda}$:

$$S_{\rm eff}[\epsilon,\boldsymbol{\pi}] = \frac{1}{2}\int_{t,\boldsymbol{x},\boldsymbol{x}'} E_i(t,\boldsymbol{x}) D_{ij}(\boldsymbol{x},\boldsymbol{x}') E_j(t,\boldsymbol{x}')$$

where $E_i \equiv (\frac{\partial\pi_i}{\partial t} + v_s^2\partial_i\epsilon - M_{ij}\pi_j)$, and $M_{ij}D_{jk} = -\frac{1}{2\varpi T}\delta(\boldsymbol{x}-\boldsymbol{x}')\delta_{ik}$

Note the action $S_{\rm eff}[\epsilon,\boldsymbol{\pi}]$ is time-reversal invariant, as it should be

This effective action produces the correct hydro response functions

Correlation functions

Once know $S_{\pi_i \pi_j}(\omega, \boldsymbol{k})$, the others follow from energy conservation:

$$\omega S_{\epsilon\pi_i}(\omega, \boldsymbol{k}) = k_l S_{\pi_l \pi_i}(\omega, \boldsymbol{k}) \,,$$
$$\omega S_{\epsilon\epsilon}(\omega, \boldsymbol{k}) = k_l S_{\pi_l \epsilon}(\omega, \boldsymbol{k}) \,.$$

Can read off correlation functions from the effective action $S_{\text{eff}}[\epsilon, \boldsymbol{\pi}]$:

$$S_{\pi_i \pi_j}(\omega, \boldsymbol{k}) = \left(\delta_{ij} - \frac{k_i k_j}{\boldsymbol{k}^2}\right) \frac{2\gamma_\eta \bar{w} T \boldsymbol{k}^2}{\omega^2 + (\gamma_\eta \boldsymbol{k}^2)^2} + \frac{k_i k_j}{\boldsymbol{k}^2} \frac{2\gamma_s \bar{w} T \boldsymbol{k}^2 \omega^2}{(\omega^2 - v_s^2 \boldsymbol{k}^2)^2 + (\gamma_s \boldsymbol{k}^2 \omega)^2}$$

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Stochastic model for linearized hydro

$$\frac{\partial \epsilon}{\partial t} = -\boldsymbol{\nabla}\cdot\boldsymbol{\pi}\,,$$
$$\frac{\partial \pi_i}{\partial t} = -v_s^2 \partial_i \epsilon + M_{ij}\pi_j + \xi_i(\boldsymbol{x}, t)\,.$$

Dissipative terms:

$$M_{ij} \equiv \gamma_\eta(\boldsymbol{\nabla}^2\delta_{ij} - \partial_i\partial_j) + \gamma_s\partial_i\partial_j$$

Noise correlations:

$$\langle \xi_i(\boldsymbol{x}, t)\xi_j(\boldsymbol{x}', t')\rangle = -2\bar{w}T M_{ij}\delta(\boldsymbol{x}-\boldsymbol{x}')\delta(t-t')$$

Note the same M_{ij} must appear both in the hydro equations, and in the noise correlations

Bottomline

- We have an effective action for linearized relativistic hydro
- This effective action is **not** meant to reproduce the classical hydro equations
- Rather, it is to be used to construct the generating functional for the correlation functions of $T^{0\mu}(x)$
  - *i)* at low energies
  - *ii)* in real time
  - *iii)* near thermal equilibrium

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Now that we know how to construct the effective action for linearized hydro, can look at the full non-linear hydrodynamics

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A simple toy model

- Incompressible fluid: impose $\boldsymbol{\nabla}\cdot\boldsymbol{\pi} = 0$ Forster+Nelson+Stephen, 1977
- Momentum conservation:

$$\partial_t \pi_i = -\partial_j T_{ij} + \xi_i \,, \qquad T_{ij} = P\delta_{ij} - \gamma_\eta(\partial_i \pi_j + \partial_j \pi_i) + \frac{\pi_i \pi_j}{\bar{w}}$$

- Current conservation:

$$\partial_t n = -\partial_i J_i + \theta \,, \qquad J_i = -D\partial_i n + \frac{n\pi_i}{\bar{w}}$$

- Stochastic model:

$$\partial_t \pi_i = -\partial_i P + \gamma_\eta \boldsymbol{\nabla}^2 \pi_i - \frac{(\boldsymbol{\pi}\cdot\boldsymbol{\nabla})\pi_i}{\bar{w}} + \xi_i \,,$$

$$\partial_t n = D\boldsymbol{\nabla}^2 n - \frac{(\boldsymbol{\pi}\cdot\boldsymbol{\nabla})n}{\bar{w}} + \theta \,,$$

Effective action for the toy model

$$S_{\text{eff}} = \int dt\, d^d x \left(\mathcal{L}^{(2)} + \mathcal{L}^{(int)} \right)$$

$$\mathcal{L}^{(2)} = -\frac{\sigma}{2}\rho\nabla^2\rho - \frac{\tilde{\sigma}}{2}\lambda_i\nabla^2\lambda_i - i\rho(\partial_t n - D\nabla^2 n) - i\lambda_i(\partial_t\pi_i - \Gamma\nabla^2\pi_i) + \bar{\psi}_i(\partial_t - \Gamma\nabla^2)\psi_i + \bar{\psi}_{\rm n}(\partial_t - D\nabla^2)\psi_{\rm n}\,,$$

$$\mathcal{L}^{(int)} = -\frac{i}{w}\rho\pi_i\partial_i n - \frac{i}{w}\lambda_i\pi_j\partial_j\pi_i + \frac{1}{w}\bar{\psi}_i\partial_k\pi_i\,\psi_k + \frac{1}{w}\bar{\psi}_i\pi_k\partial_k\psi_i + \frac{1}{w}\bar{\psi}_{\rm n}\partial_i n\,\psi_i + \frac{1}{w}\bar{\psi}_{\rm n}\pi_k\partial_k\psi_{\rm n}\,,$$

plus the constraints $\partial_i\pi_i = 0$, $\partial_i\lambda_i = 0$, $\partial_i\bar{\psi}_i = 0$, $\partial_i\psi_i = 0$.
The constants are $\sigma = 2TD\chi$, $\tilde{\sigma} = 2T\Gamma w$, $\Gamma = \eta/w$.

One-loop correlation functions in the toy model

As $\boldsymbol{k}\to 0$:

$$\langle T_{0i}T_{0j}\rangle = \frac{2Tw\Gamma(\omega)\boldsymbol{k}^2}{\omega^2 + \Big(\Gamma(\omega)\boldsymbol{k}^2\Big)^2}\,, \qquad \langle J_0 J_0\rangle = \frac{2T\chi D(\omega)\boldsymbol{k}^2}{\omega^2 + \Big(D(\omega)\boldsymbol{k}^2\Big)^2}\,.$$

This looks like the familiar linear response functions, except D and η now depend on ω.

In $d{=}3$ dimensions:

$$\Gamma(\omega) = \Gamma - \frac{23}{30\pi s}\frac{\sqrt{|\omega|}}{(4\Gamma)^{3/2}}\,, \qquad D(\omega) = D - \frac{1}{3\pi s}\frac{\sqrt{|\omega|}}{[2(\Gamma{+}D)]^{3/2}}\,.$$

Conventional Kubo formulas make sense:

$$D = \frac{1}{2T\chi}\lim_{\omega\to 0}\lim_{\boldsymbol{k}\to 0}\frac{\omega^2}{\boldsymbol{k}^2}G_{nn}(\omega,\boldsymbol{k})$$

One–loop correlation functions in the toy model

As $\boldsymbol{k}\to 0$:

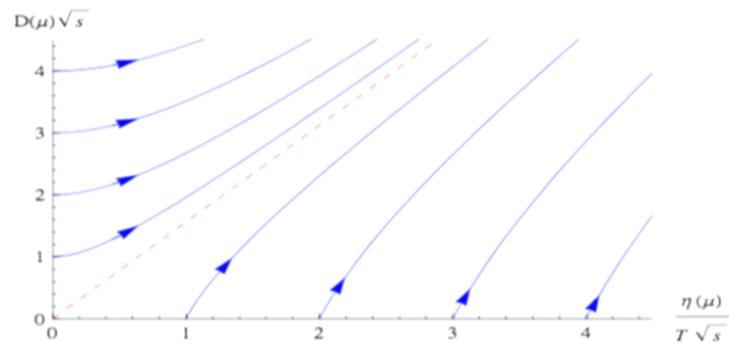
$$\langle T_{0i}T_{0j}\rangle = \frac{2Tw\Gamma(\omega)\boldsymbol{k}^2}{\omega^2+\left(\Gamma(\omega)\boldsymbol{k}^2\right)^2}\,,\qquad \langle J_0 J_0\rangle = \frac{2T\chi D(\omega)\boldsymbol{k}^2}{\omega^2+\left(D(\omega)\boldsymbol{k}^2\right)^2}\,.$$

This looks like the familiar linear response functions, except D and η now depend on ω.

In $d{=}2$ dimensions:

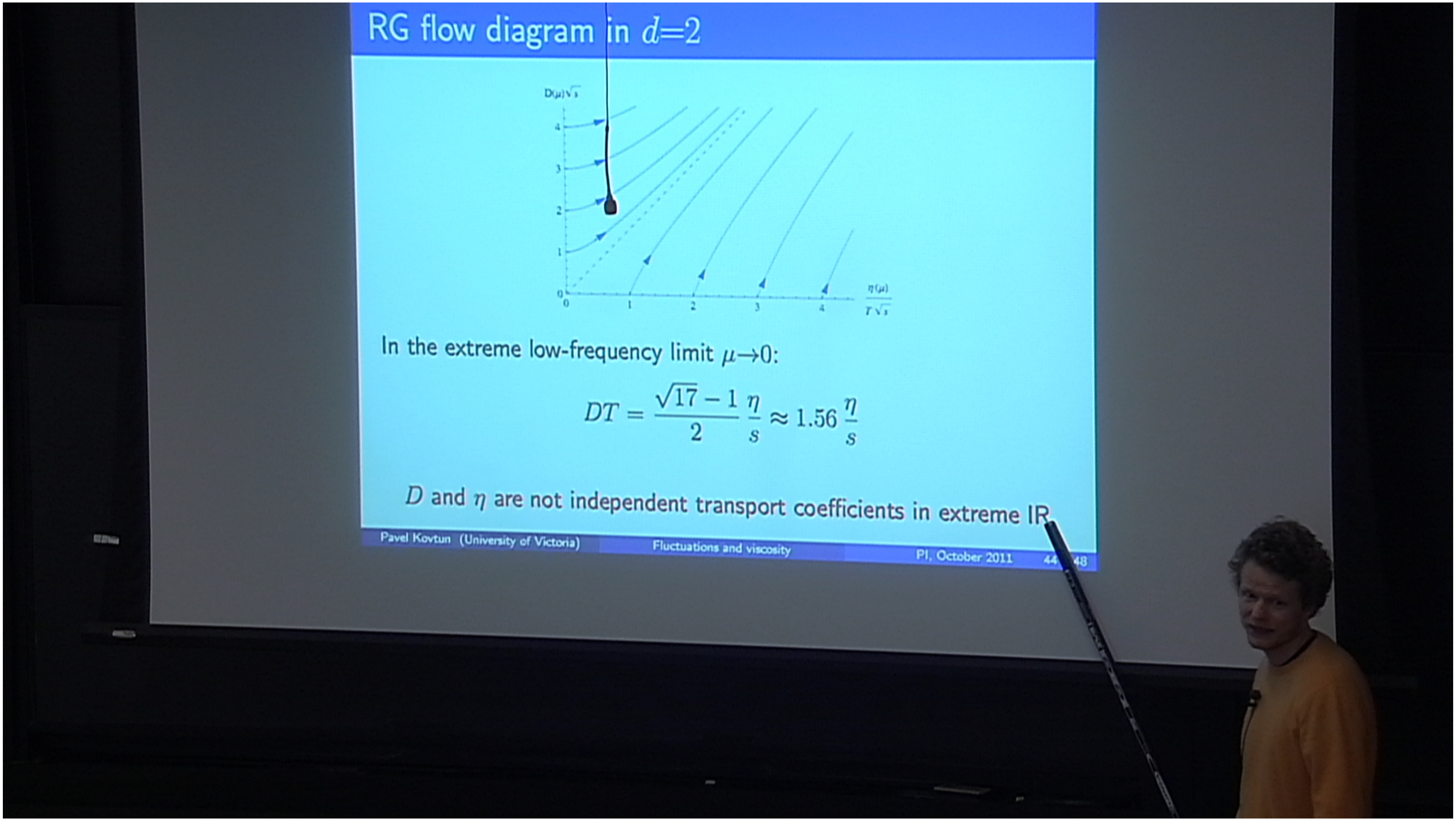
$$\Gamma(\omega) = \Gamma(\mu) + \frac{1}{32\pi s}\frac{1}{\Gamma(\mu)}\ln\frac{\mu}{\omega}\,,\quad D(\omega) = D(\mu) + \frac{1}{8\pi s}\frac{1}{\Gamma(\mu)+D(\mu)}\ln\frac{\mu}{\omega}\,.$$

RG flow diagram in $d=2$



In the extreme low-frequency limit $\mu \to 0$:

$$DT = \frac{\sqrt{17}-1}{2}\,\frac{\eta}{s} \approx 1.56\,\frac{\eta}{s}$$


RG flow diagram in $d=2$
$D(\mu)\sqrt{s}$
$\frac{\eta(\mu)}{T\sqrt{s}}$
In the extreme low-frequency limit $\mu\to 0$:
$$DT = \frac{\sqrt{17}-1}{2}\,\frac{\eta}{s} \approx 1.56\,\frac{\eta}{s}$$
D and η are not independent transport coefficients in extreme IR
Pavel Kovtun (University of Victoria)
Fluctuations and viscosity
PI, October 2011

Hydro fluctuations imply that

- $\frac{\eta}{s}$ is bounded from below in real-world QCD
- Second-order relativistic hydrodynamics strictly speaking does not exist
- However, 2nd order hydro still OK for heavy-ion collisions if η/s is sufficiently large
- Fluctuation effects disappear in the $N \to \infty$ limit

What I would like to understand

- I only showed the effective action for linearized hydro and the toy model. Can one find the covariant action for the full non-linear relativistic hydro? Work in progress with GM and PR!
- Effective action for hydro from AdS/CFT?
- Effective action for relativistic superfluids?
- How do transport coefficients in 2+1 dim flow at non-zero density?
- How do transport coefficients in 2+1 dim flow in external magnetic field?
- Other 2-nd order transport coefficients in relativistic hydro?