Title: Tensor Networks and TQFTs

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Abstract: In this talk, I will present a first principle construction of a holographic dual for gauged matrix models that include gauge theories. The dual theory is shown to be a closed string field theory coupled with an emergent two-form gauge field defined in one higher dimensional space. The bulk space with an extra dimension emerges as a well defined classical background only when the two-form gauge field is in the deconfinement phase. Based on this, it is shown that critical phases that admit holographic descriptions form a novel universality class with a non-trivial quantum order.

Tensor-Networks(TNs) & Topological Quantum Field Theories(TQFTs)

Zheng-cheng Gu (KITP)

Collaborators:

Part I(Duality):
Prof. M. Leivn(U. of Marryland)
Part II(TQFTs):
Prof. Xiao-gang Wen (MIT)
Prof. Zhenghan Wang (Station-q)
Dr. Xie Chen(MIT)
Dr. Zhen-Xin Liu(Tsinghua U.)

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Outline

- Intrinsic Topological(IT) order and TQFTs
- Symmetry Protected Topological(SPT) order

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- TNs description of Intrinsic Topological order
- TQFTs of SPT order

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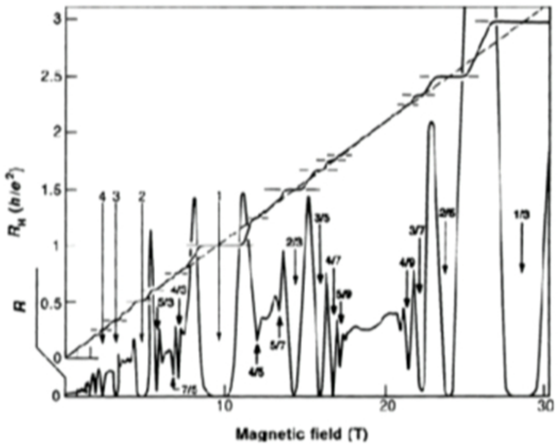
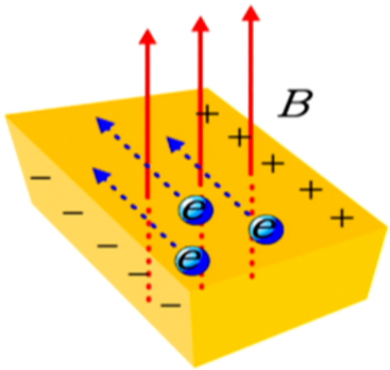
- TNs description of Intrinsic Topological order
- TQFTs of SPT order
- Fermionic TQFTs from Grassmann TNs

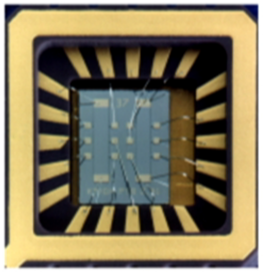
Topological Phenomena in strongly correlated systems

Fractional Quantum Hall Effect

D.C.Tsui, *etal* 1982







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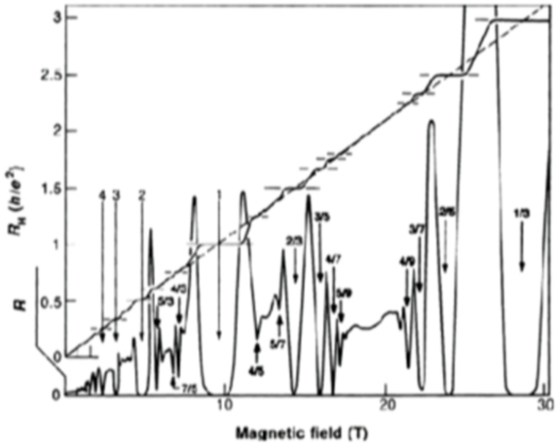
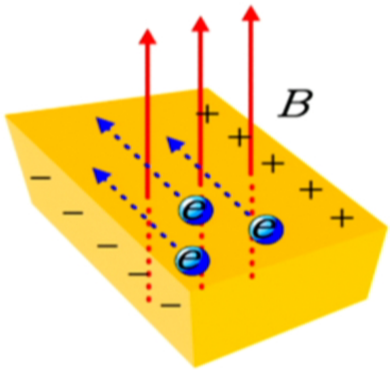
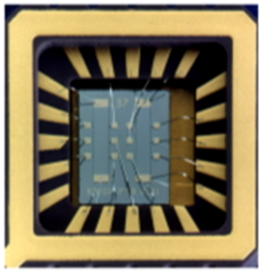
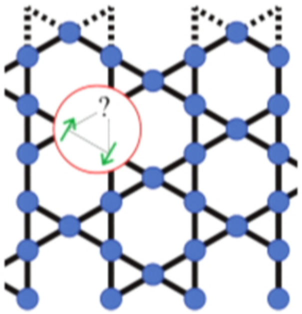


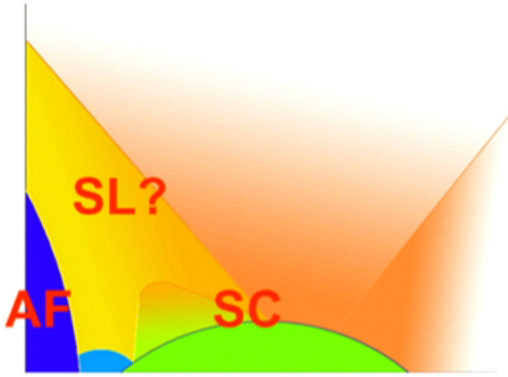




Gapped spin liquid

- Frustrated magnets
- High-Tc cuprates





S. Yan, etal, Science, 2011

New phases of matter: (Intrinsic) Topological order X.-G. Wen, *etal* 1989

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- Ground state degeneracies depend on the topology of the space.
- Ground state degeneracies are robust against any local perturbations.
- Excitations carry fractional statistics.
- Protected chiral edge states(chiral topological order, e.g. FQHE).

Topological quantum field theories (TQFTs)

- TQFTs are special classes of quantum field theories, whose actions are independent on space-time metric.
- TQFTs are stable fixed point actions. They are very powerful tools to help us understand various topological phenomena.

TQFTs for FQHE

$$\mathcal{L}_{\text{eff}} = \frac{2m+1}{4\pi}\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda$$

S-C. Zhang, *etal* 1989
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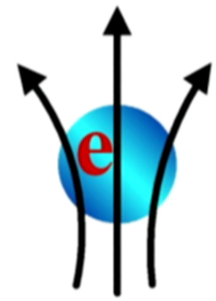

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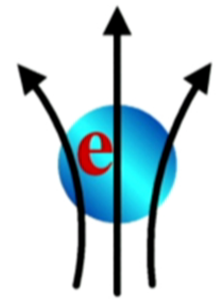


Z_2 spin liquid and its TQFT

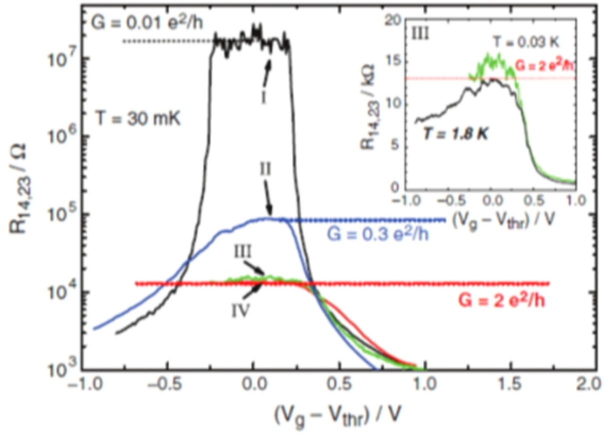
$|\Psi_{RVB}\rangle =$

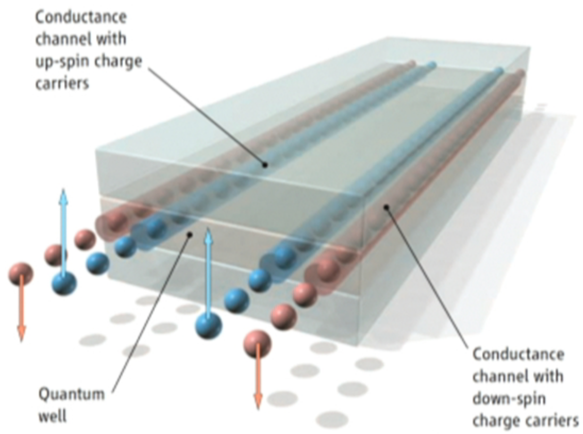
$+\ldots$

$$L = \frac{1}{4\pi}K_{IJ}a_{I\mu}\partial_\nu a_{J\lambda}\epsilon^{\mu\nu\lambda}, \qquad I, J = 1, 2 \qquad K = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$

Symmetry Protected Topological Phenomena

Topological insulator in 2D/3D





C L Kane, *etal*, 2005
S C Zhang's group 2006
W Molenkamp's group 2007
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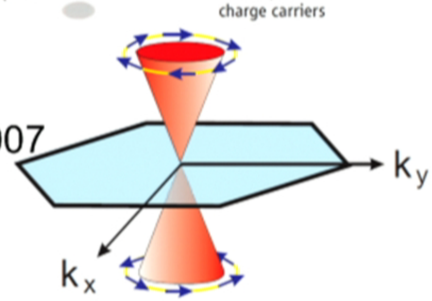

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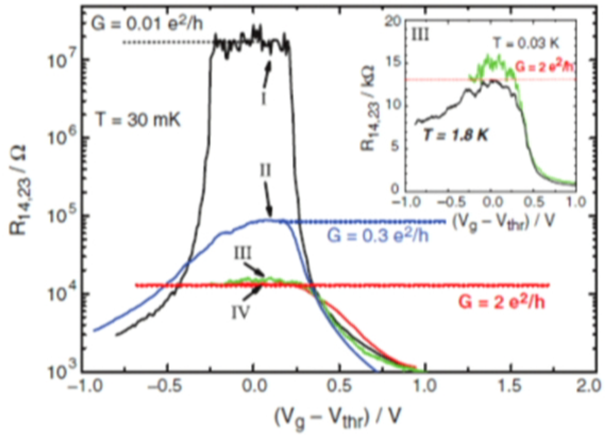

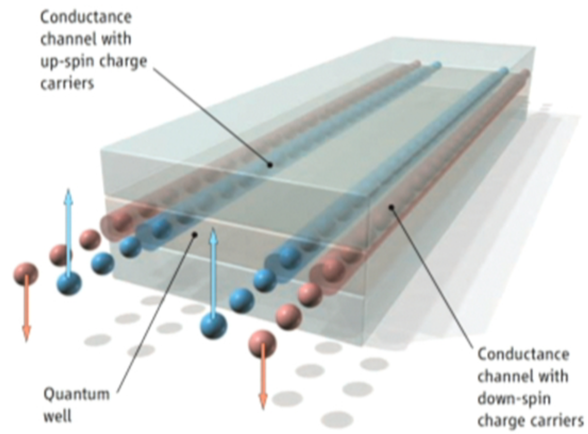

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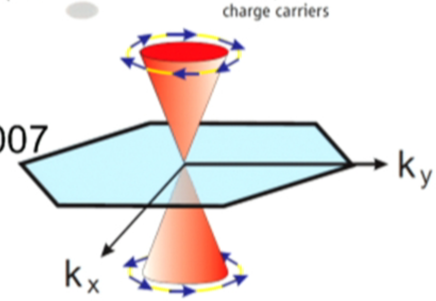

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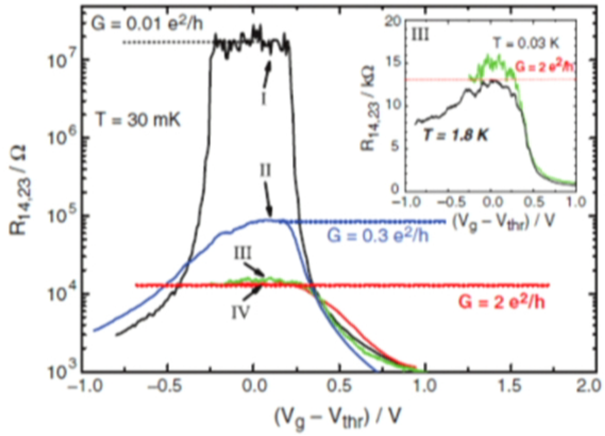

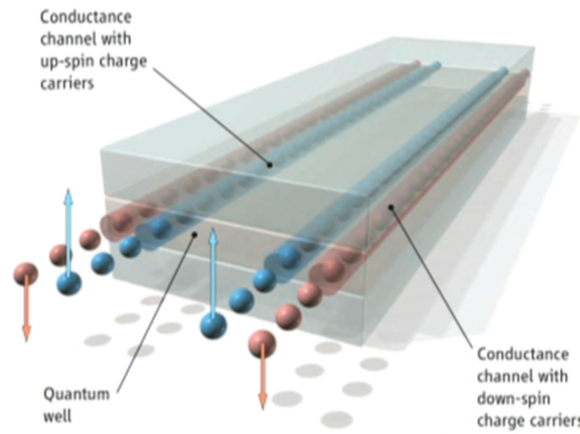

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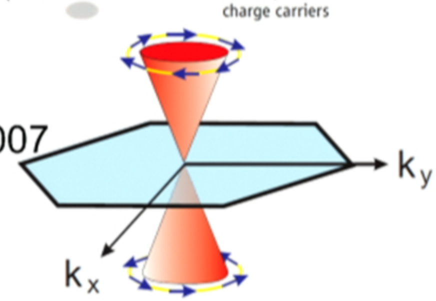
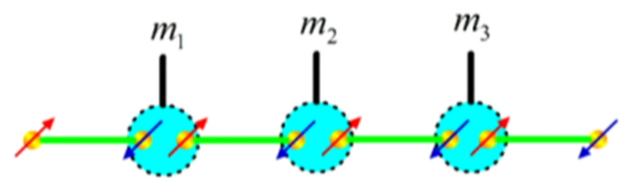


Haldane phase of spin one chain in 1D

$$H = \sum_i P_2(\mathbf{S}_i + \mathbf{S}_{i+1})$$
$$= \sum_i [\tfrac{1}{2}\mathbf{S}_i \cdot \mathbf{S}_{i+1} + \tfrac{1}{6}(\mathbf{S}_i \cdot \mathbf{S}_{i+1})^2 + \tfrac{1}{3}].$$



$m_i = 0, \pm 1$ $\qquad \frac{1}{2}\otimes\frac{1}{2} = \cancel{0}\oplus 1$

FDM Haldane, 1983, Ian Affleck, 1987
Z Gu, *etal*, 2009, F Pollmann, *etal*, 2010

New phases of matter: Symmetry Protected Topological Order

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New phases of matter: Symmetry Protected Topological Order

- Can have the same symmetry as trivial disorder systems.
- No long range correlations.
- Unique Ground state. Excitations do not carry fractional statistics.
- Indistinguishable with trivial disorder systems if symmetry is broken in bulk
- Protected gapless edge states if symmetry is not (spontaneously or explicitly)broken on the edge

How to understand the symmetry protected nature of SPT orders?

Duality!

IT

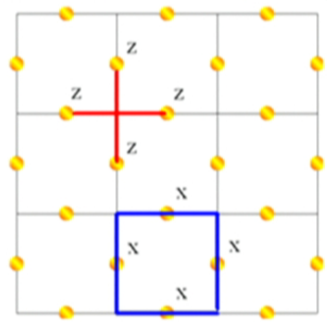
SPT

An exact solvable model of (intrinsic) topological order

toric code model:
(Z_2 spin liquid)

$$H = U\sum_v \left(1 - \prod_{l\in v}\sigma_l^z\right) - t\sum_p \prod_{l\in p}\sigma_l^x$$

Kitaev 2003, M. Levin and X.G. Wen 2005



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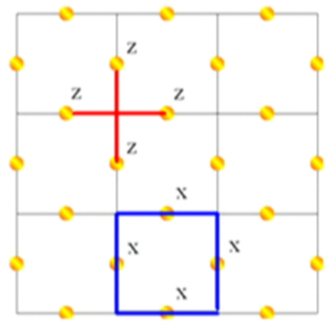

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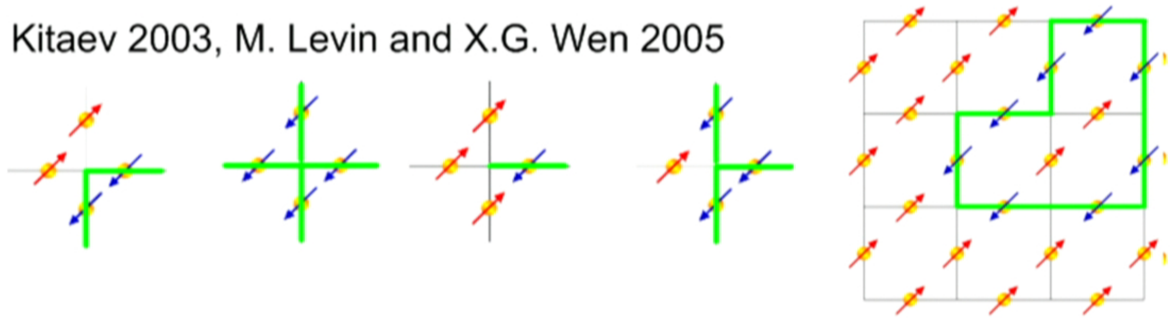

$|\uparrow\rangle \to$ no string; $|\downarrow\rangle \to$ one string

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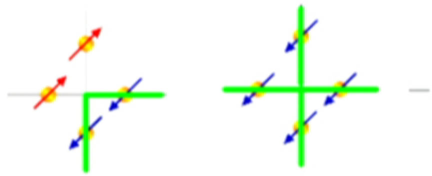

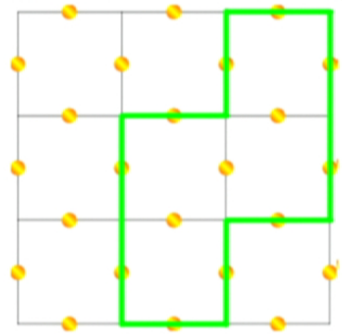

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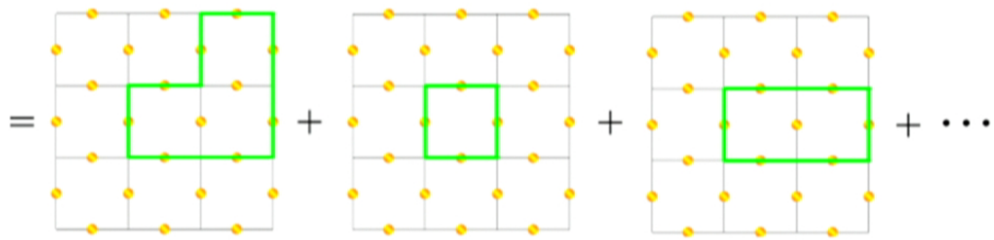
$$|\Psi_{Z_2}\rangle = \sum |X_{\text{close}}\rangle$$



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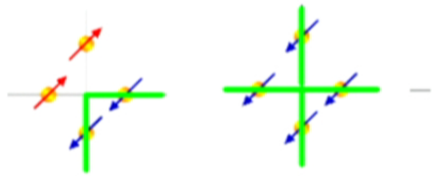

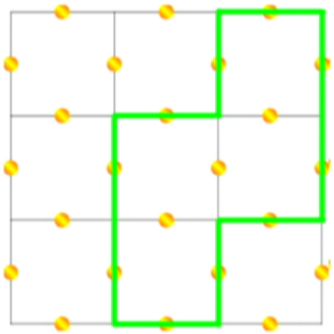

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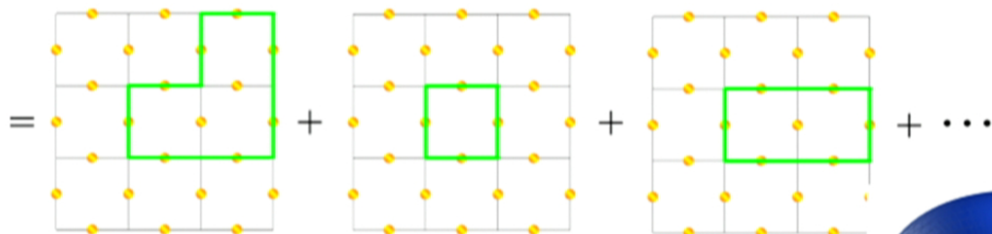
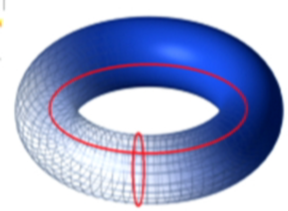


- Four fold ground state degeneracies
- Quasi particles carry fractional statistics



The twisted toric code: the double semion model M. Levin and X.G. Wen 2005

$$H_{\text{dsemion}} = -U \sum_v \prod_{i \in v} \sigma_i^z - \sum_p \left(\prod_{i \in p} \sigma_i^x \prod_{\text{legs of} p} i^{\frac{1+\sigma_l^z}{2}} \right)$$

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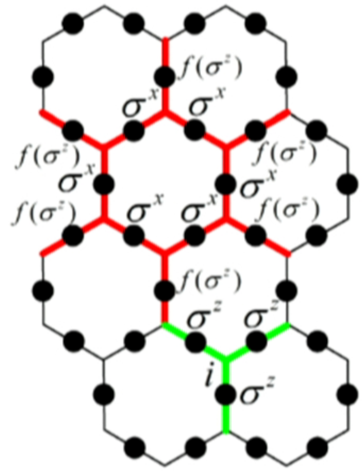
$$f(x) = \frac{1+x}{2}$$



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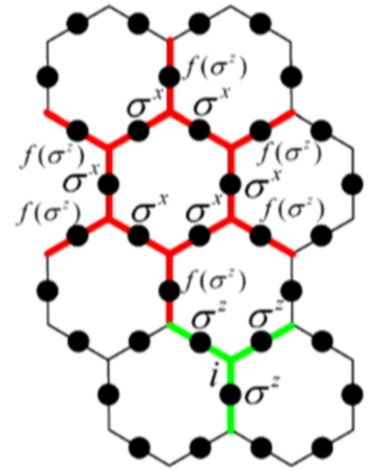

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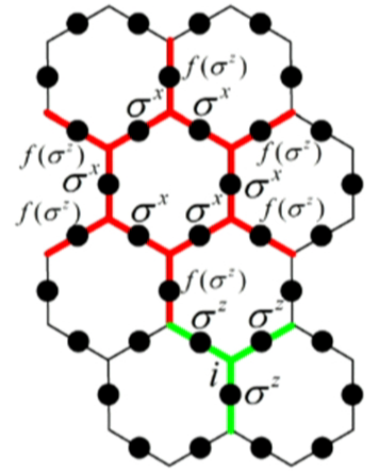

Double semion model describes different intrinsic topological order

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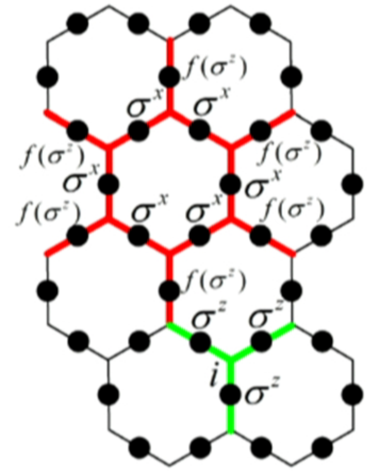

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Quasi-particle in toric code model:

1, e, m, f=em

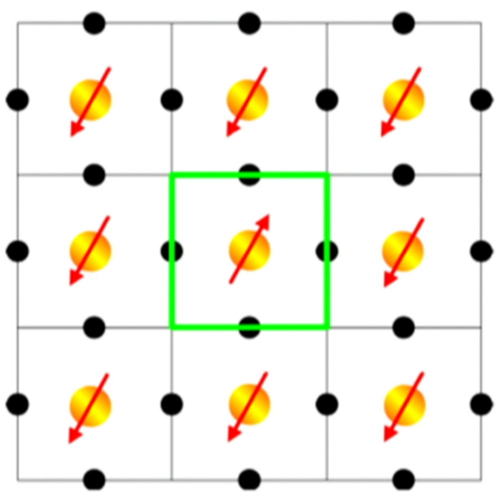
Quasi-particle in double semion model:

1, s, <u>s</u>, b=s<u>s</u>

Duality between Z2 gauge model and Ising model

$$H_{Z_2} = -U\sum_v \prod_{i\in v}\sigma_i^z - \sum_p \prod_{i\in p}\sigma_i^x \quad \Longleftrightarrow \quad H_{\text{Ising}} = -\sum_p \sigma_p^x$$

$$H = U\sum_v \left(1 - \prod_{l\in v}\sigma_l^z\right) - t\sum_p \prod_{l\in p}\sigma_l^x - h\sum_l \sigma_z$$



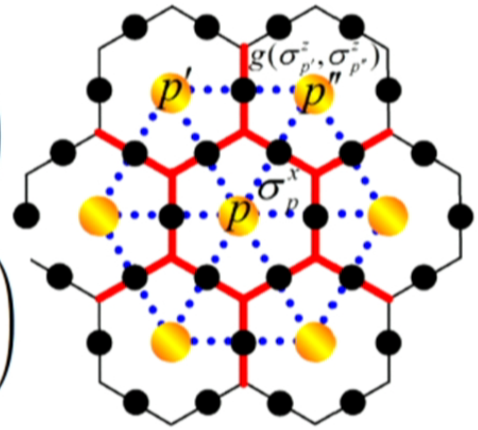
$$H = -t\sum_p \sigma_p^x - h\sum_{\langle pq\rangle}\sigma_p^z\sigma_q^z$$

String-net condensation = domain wall condensation

Dual theory of double semion model

$$H_{\text{dsemion}} = -U\sum_v \prod_{i\in v} \sigma_i^z - \sum_p \left(\prod_{i\in p} \sigma_i^x \prod_{\text{legs of} p} i^{\frac{1+\sigma_l^z}{2}} \right)$$

$$H_{\text{twistIsing}} = -\sum_p \widetilde{\sigma}_p^x = -\sum_p \left(\sigma_p^x \prod_{\text{sites}\in \text{p}} i^{\frac{1+\sigma_{p'}^z \sigma_{p''}^z}{2}} \right)$$



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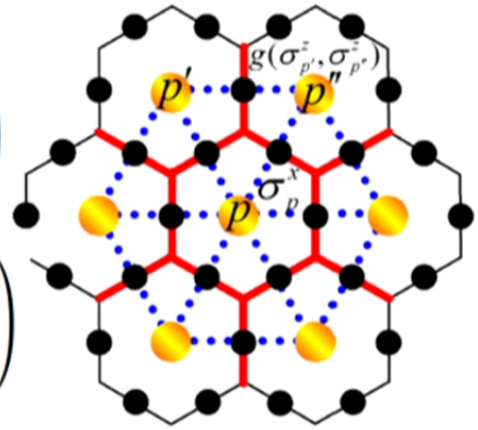
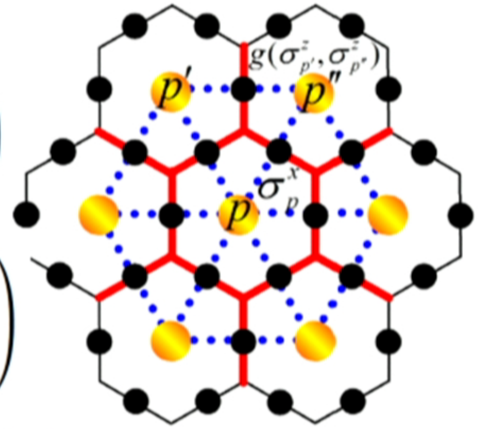


The dual theory of double semion model is a SPT ordered phase!

$$H = -\alpha\sum_p \sigma_p^x - (1-\alpha)\sum_p \widetilde{\sigma}_p^x \Leftrightarrow$$

$$H' = -U\sum_v \prod_{i\in v}\sigma_i^z - \alpha\sum_p\prod_{i\in p}\sigma_i^x - (1-\alpha)\sum_p \left(\prod_{i\in p}\sigma_i^x \prod_{\text{legs of}p} i^{\frac{1+\sigma_l^z}{2}}\right)$$

Dual theory of double semion model



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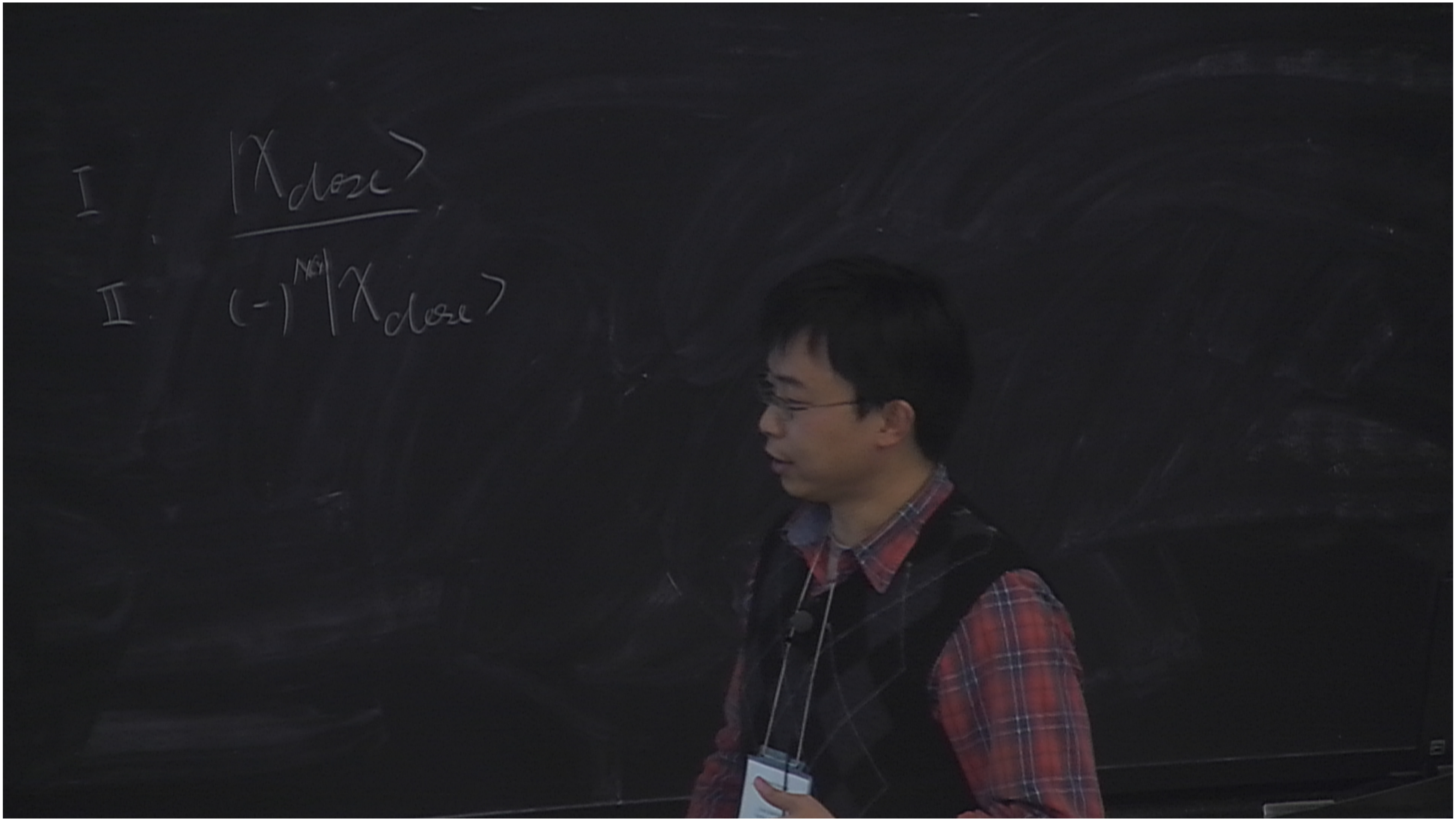
⇔

$$H' = -U \sum_v \prod_{i \in v} \sigma_i^z - \alpha \sum_p \prod_{i \in p} \sigma_i^x - (1-\alpha) \sum_p \left(\prod_{i \in p} \sigma_i^x \prod_{\text{legs of} p} i^{\frac{1+\sigma_l^z}{2}} \right)$$

SPT order

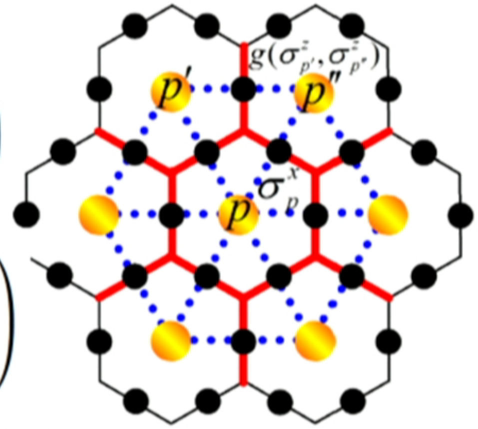
(intrinsic) topological order


I |χ close⟩
II (-) |χ close⟩

Dual theory of double semion model

$$H_{\text{dsemion}} = -U\sum_v \prod_{i\in v}\sigma_i^z - \sum_p \left(\prod_{i\in p}\sigma_i^x \prod_{\text{legs of} p} i^{\frac{1+\sigma_l^z}{2}}\right)$$

$$H_{\text{twistIsing}} = -\sum_p \widetilde{\sigma}_p^x = -\sum_p \left(\sigma_p^x \prod_{\text{sites}\in\text{p}} i^{\frac{1+\sigma_{p'}^z \sigma_{p''}^z}{2}}\right)$$



The dual theory of double semion model is a SPT ordered phase!

$$H = -\alpha\sum_p \sigma_p^x - (1-\alpha)\sum_p \widetilde{\sigma}_p^x$$

⟺

$$H' = -U\sum_v\prod_{i\in v}\sigma_i^z - \alpha\sum_p\prod_{i\in p}\sigma_i^x - (1-\alpha)\sum_p\left(\prod_{i\in p}\sigma_i^x\prod_{\text{legs of}p} i^{\frac{1+\sigma_l^z}{2}}\right)$$

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- Indeed, the above two models are self dual and the transition point is at 0.5

Edge theory of Z_2 (bosonic) SPT order

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$$H_{\text{edge}} = \sum_p \sigma_p^z \sigma_{p+1}^z$$

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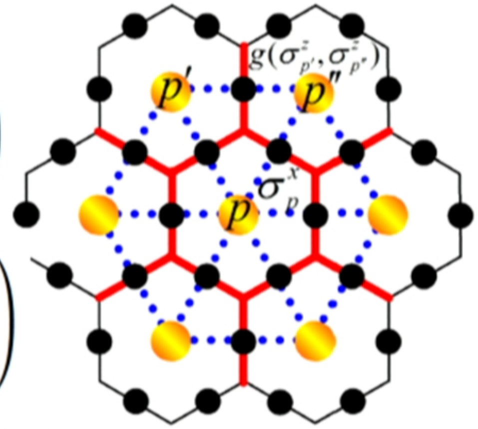

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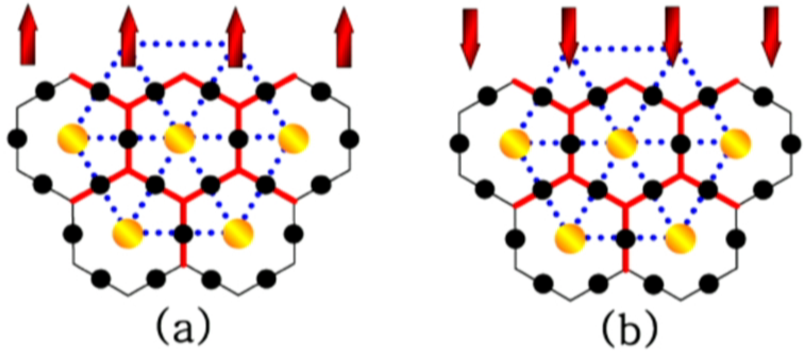
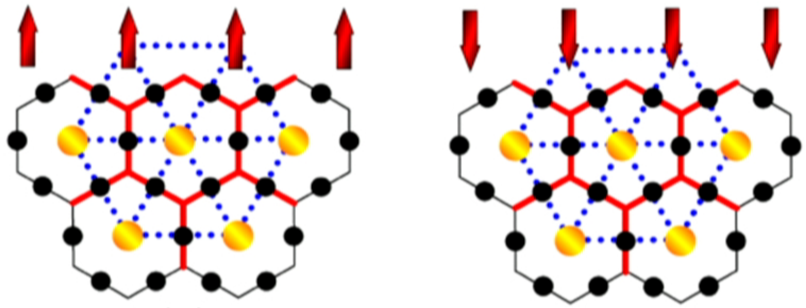


Edge theory of Z2 (bosonic) SPT order



(a) (b)

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$$\sigma^x_p(\downarrow) = i\gamma_{2p-1}\gamma_{2p+2} \qquad E_k = \pm\sqrt{2(1+2\cos 2k)} = \pm 2\sin k; \quad k > 0$$

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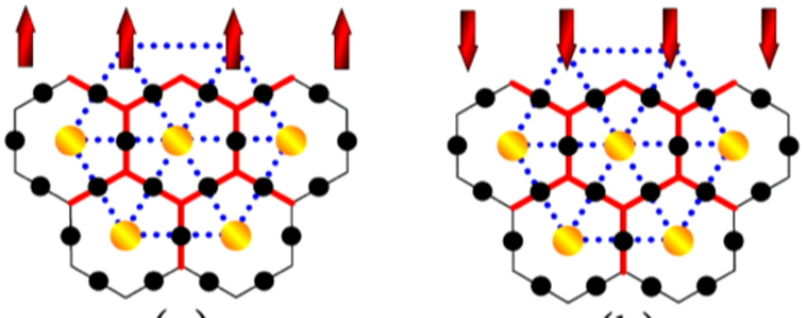

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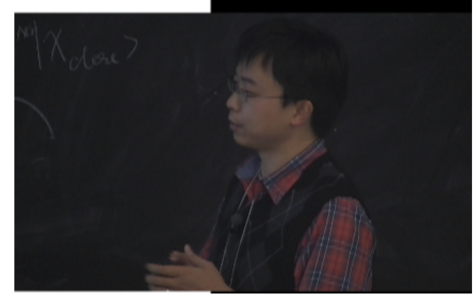

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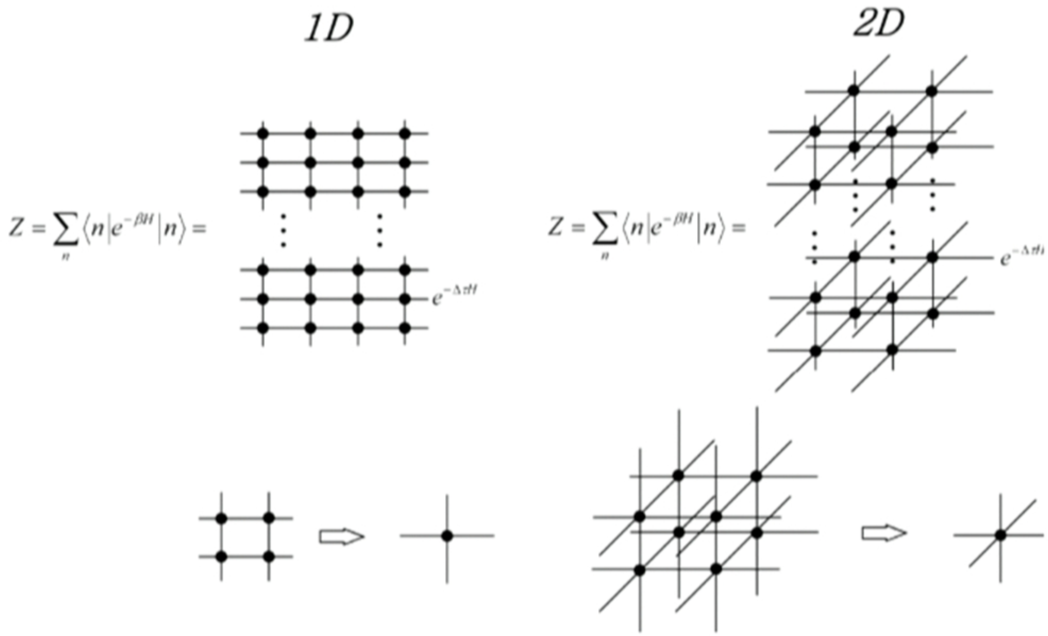
Tensor-Networks(TNs)

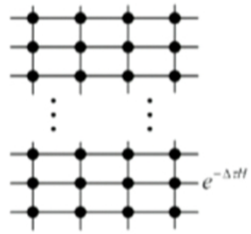
TNs representations for partition functions & their renormalization



TNs representations for partition functions & their renormalization

1D | *2D*

$Z = \sum_n \langle n | e^{-\beta H} | n \rangle =$



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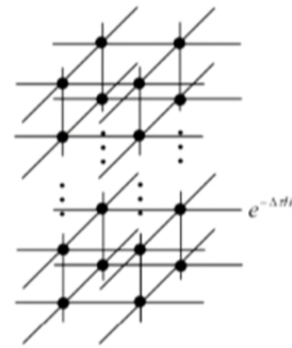

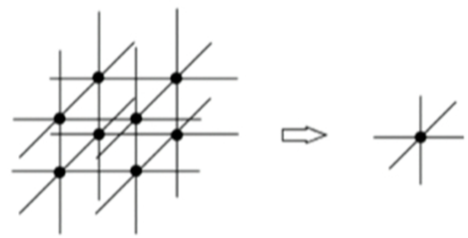

Fixed point TQFTs = fixed point tensors

$$\mathcal{L}^{(0)} \to \mathcal{L}^{(1)} \to \mathcal{L}^{(2)} \to \cdots \to \mathcal{L}^{(*)}$$

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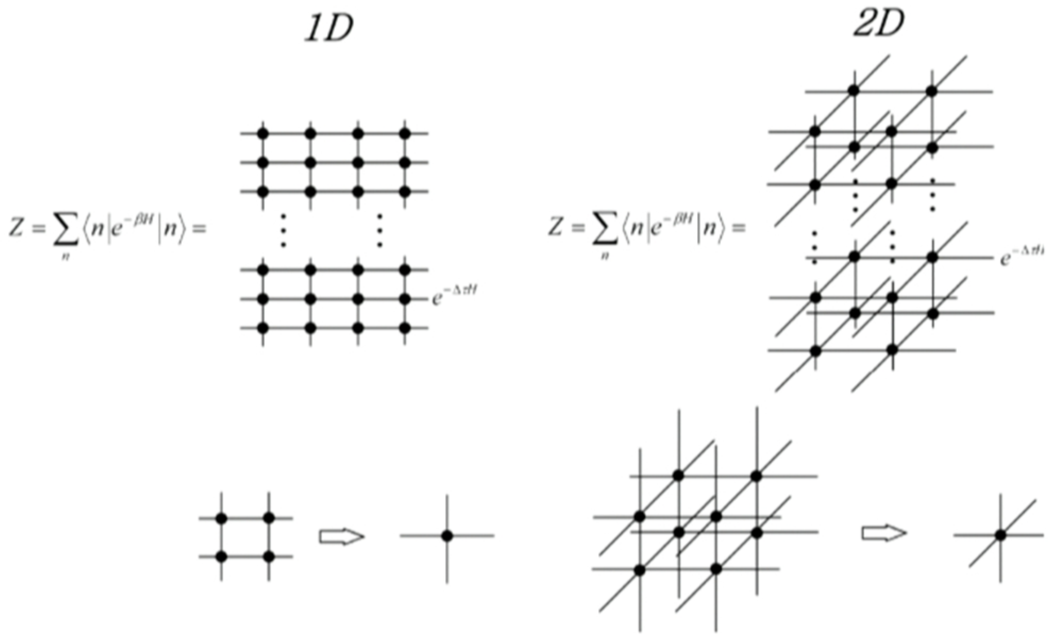

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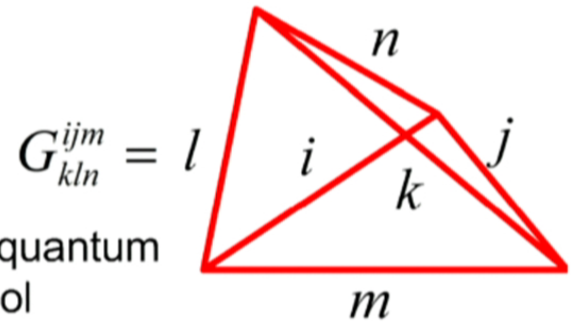
Topological ordered phases are described by isolate fixed ponit tensors.

TNs representation for TQFTs of non-chrial (intrinsic) topological order

Partition function of string-net models: Turaev-Viro states sum invariants

$$Z = \frac{1}{D^{N_v}} \sum_{ijklmn...} \prod_{\text{link}} d_i \prod_{\text{tetrahedron}} G^{ijm}_{kln}$$



- d is the quantum dimension, D is the total quantum dimension and G is the (symmetric) 6j symbol

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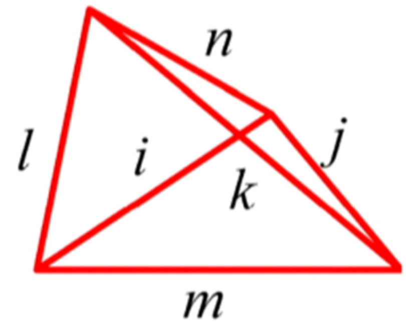
$G^{ijm}_{kln} =$ 

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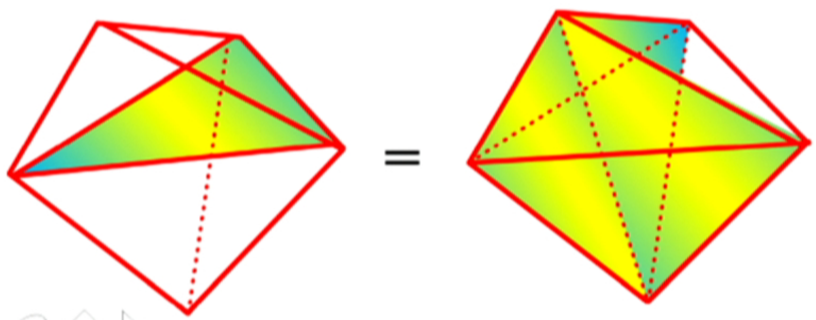

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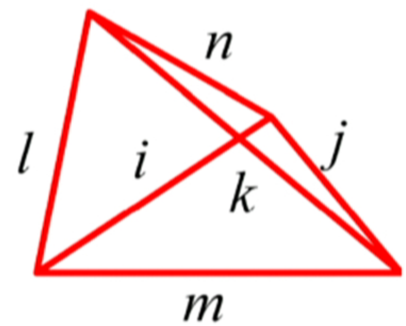

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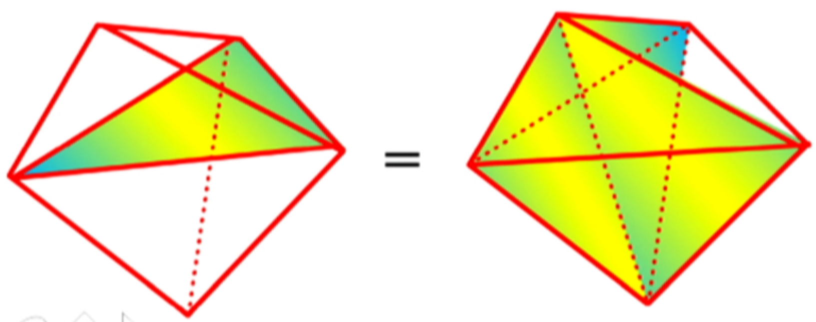

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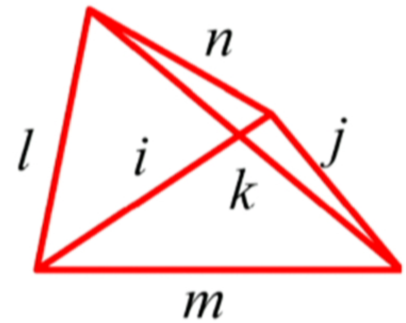

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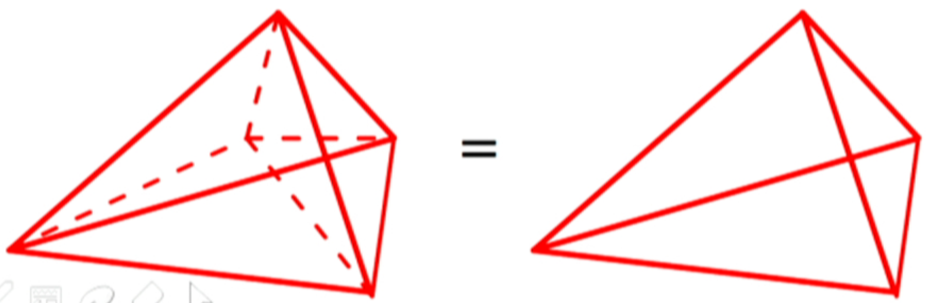

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TQFTs for Z_2 bosonic SPT order

$$\tilde{Z} = \frac{1}{D^{N_v}} \sum_{g_0 g_1 g_2 g_3 \cdots} \prod_{\text{link}} d_i \prod_{\text{tetrahedron}} \tilde{G}(g_0, g_1, g_2, g_3),$$

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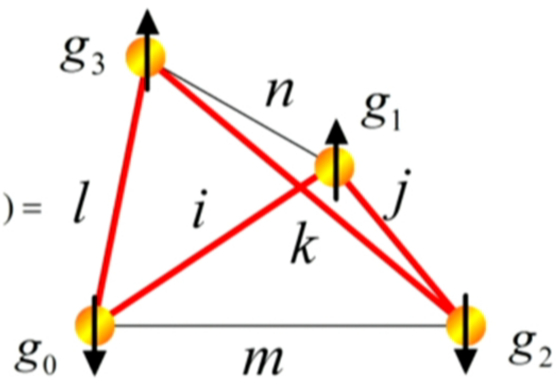
Topological term of (discrete) non-linear sigma model!

Pentagon equation

⇕

Third group co-cycle condition

$$G^{ijm}_{kln} = \widetilde{G}(g_0, g_1, g_2, g_3) =$$



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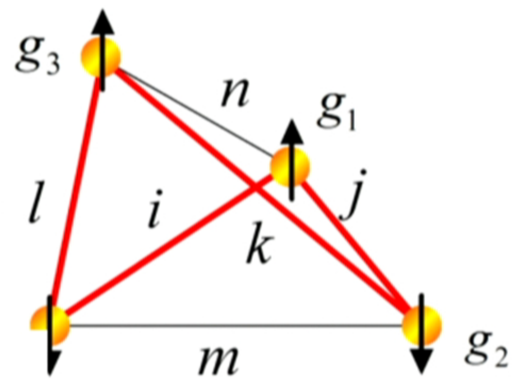

$$\tilde{G}(g_0, g_2, g_3, g_4)\tilde{G}(g_0, g_1, g_2, g_4) = \tilde{G}(g_1, g_2, g_3, g_4)\tilde{G}(g_0, g_1, g_3, g_4)\tilde{G}(g_0, g_1, g_2, g_3)$$

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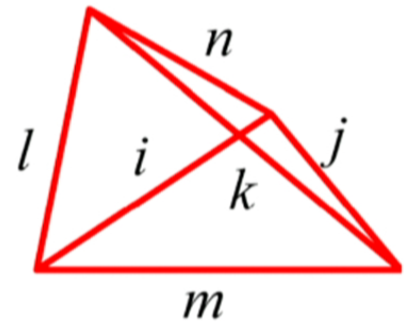

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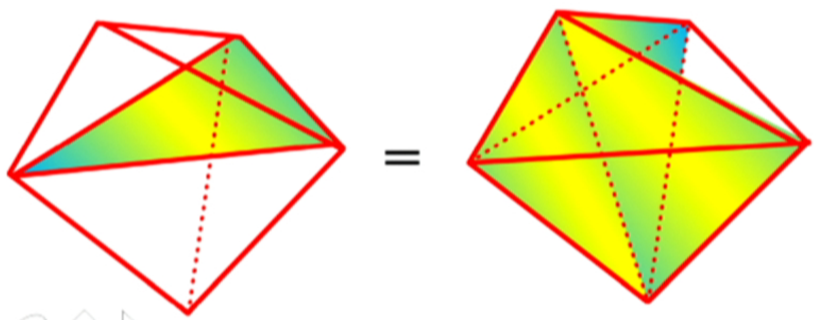

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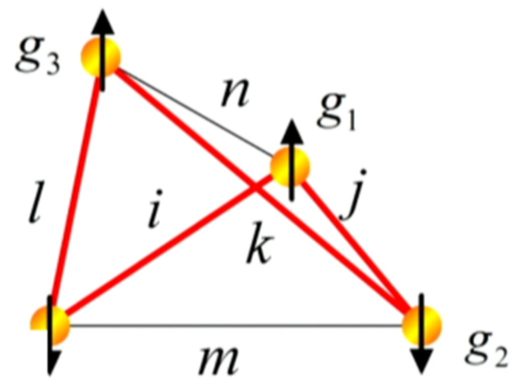

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Z_2 SPT order is classified by $H^3(Z_2, U(1))$

TQFTs of (bosonic) SPT order in any dimensions with any symmetry

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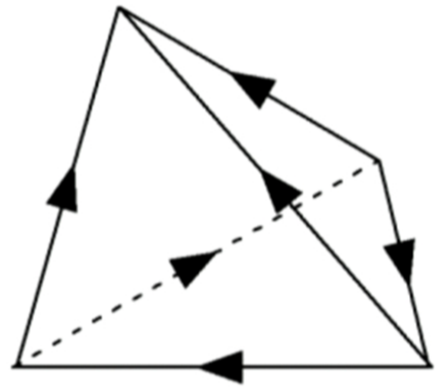

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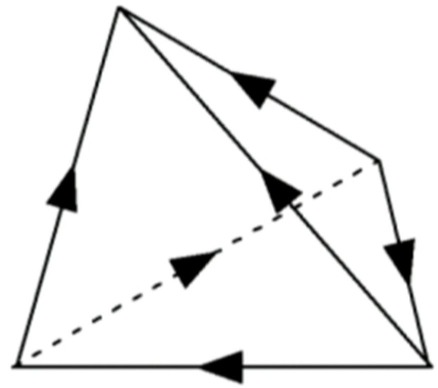

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Unique ground state which connects to a product state through local unitary transformation

$$\Psi_M(\{g_i\}_M) = \frac{\sum_{g_i \in \text{internal}}}{|G|^{N_v^{\text{internal}}}} \prod_{\{ij...k\}} \nu_{d+1}^{s_{ij...k}}(g_i, g_j, ..., g_k)$$

SPT order in any dimension with any group is classified by $\mathcal{H}^{d+1}[G, U_T(1)]$

Symmetry	$d=0$	$d=1$	$d=2$	$d=3$
$U(1) \rtimes Z_2^T$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$
$U(1) \rtimes Z_2^T \times \text{trans}$	$\mathbb{Z}$	$\mathbb{Z} \times \mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_2^3$	$\mathbb{Z} \times \mathbb{Z}_2^8$
$U(1) \times Z_2^T$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_1$	$\mathbb{Z}_2^3$
$U(1) \times Z_2^T \times \text{trans}$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^9$
Z_2^T	$\mathbb{Z}_1$	$\mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_2$
$Z_2^T \times \text{trans}$	$\mathbb{Z}_1$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$
$U(1)$	$\mathbb{Z}$	$\mathbb{Z}_1$	$\mathbb{Z}$	$\mathbb{Z}_1$
$U(1) \times \text{trans}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^4$
Z_n	$\mathbb{Z}_n$	$\mathbb{Z}_1$	$\mathbb{Z}_n$	$\mathbb{Z}_1$
$Z_n \times \text{trans}$	$\mathbb{Z}_n$	$\mathbb{Z}_n$	$\mathbb{Z}_n^2$	$\mathbb{Z}_n^4$
$Z_2^T \times D_2 = D_{2h}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^6$	$\mathbb{Z}_2^9$

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$U(1) \times Z_2^T$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_1$	$\mathbb{Z}_2^3$
$U(1) \times Z_2^T \times \text{trans}$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^9$
Z_2^T	$\mathbb{Z}_1$	$\mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_2$
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$U(1) \times \text{trans}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^4$
Z_n	$\mathbb{Z}_n$	$\mathbb{Z}_1$	$\mathbb{Z}_n$	$\mathbb{Z}_1$
$Z_n \times \text{trans}$	$\mathbb{Z}_n$	$\mathbb{Z}_n$	$\mathbb{Z}_n^2$	$\mathbb{Z}_n^4$
$Z_2^T \times D_2 = D_{2h}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^6$	$\mathbb{Z}_2^9$

Continuum limit analogy

S=1 non-linear sigma model in 1+1D

$$S = \int dxdt \frac{1}{2g}(\partial \boldsymbol{n}(x,t))^2 + \mathrm{i}\theta W, \quad \theta = 2\pi$$

$$W = (4\pi)^{-1} \int dtdx\ \boldsymbol{n}(t,x) \cdot [\partial_t \boldsymbol{n}(t,x) \times \partial_x \boldsymbol{n}(t,x)]$$

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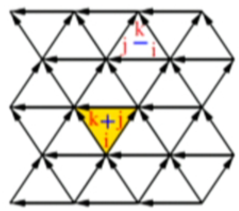
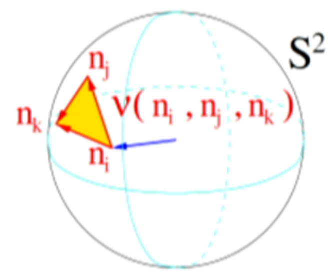
Discrete version of theta term

$$e^{-S} = \prod \nu^{s(i,j,k)}(\boldsymbol{n}_i, \boldsymbol{n}_j, \boldsymbol{n}_k),$$

$$\nu^{s(i,j,k)}(\boldsymbol{n}_i, \boldsymbol{n}_j, \boldsymbol{n}_k) = e^{-\int_\triangle \mathrm{d}x\,\mathrm{d}t}$$

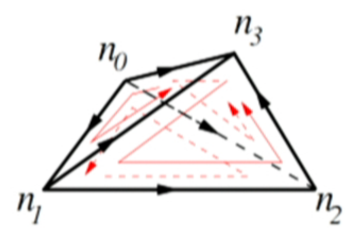


space−time



$$\nu(g\boldsymbol{n}_i, g\boldsymbol{n}_j, g\boldsymbol{n}_k) = \nu(\boldsymbol{n}_i, \boldsymbol{n}_j, \boldsymbol{n}_k), \quad g \in SO(3)$$

$$\frac{\nu(\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3)\nu(\boldsymbol{n}_0, \boldsymbol{n}_1, \boldsymbol{n}_3)}{\nu(\boldsymbol{n}_0, \boldsymbol{n}_2, \boldsymbol{n}_3)\nu(\boldsymbol{n}_0, \boldsymbol{n}_1, \boldsymbol{n}_2)} = 1.$$



Fermionic TQFTs constructed from Grassmann TNs

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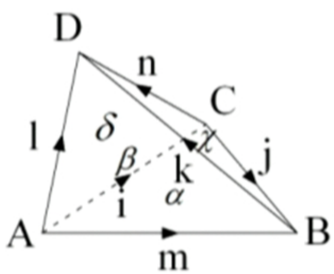
Grassmann TNs construction for fermonic TQFTs

$$Z_{top} = \frac{1}{D^{N_v}} \sum_{\alpha\beta\gamma\delta\cdots} \sum_{ijklmn\cdots} \int \prod_{\text{face}} (\mathrm{d}\theta_\alpha)^{P(\alpha)} (\mathrm{d}\theta_{\alpha'})^{P(\alpha')} \prod_{\text{link}} d_i$$
$$\times \prod_{\text{tetrahedron}} G^{ijm,\alpha\beta}_{kln,\gamma\delta} (\theta_\alpha)^{P(\alpha)} (\theta_\beta)^{P(\beta)} (\theta_\gamma)^{P(\gamma)} (\theta_\delta)^{P(\delta)}$$



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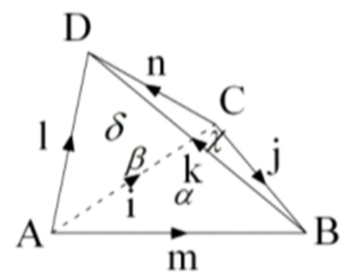

A local fermionic TQFT requires fermion parity conservation

$$P(\alpha) + P(\beta) + P(\gamma) + P(\delta) = 0(mod2)$$

Conclusion

- Intrinsic topological order and symmetry protected topological order can be related through duality map.

- We propose the TQFTs for SPT orders based on TNs representation. They are discrete analogy of topological terms of non-linear sigma model, whose boundary are described by generalized WZW terms.

- The Grassmann generalization of Turaev-Viro states sum invariants allow us to study fermionic TQFTs.

- The duality map can be generalized to any dimension. SPT orders in 3D will correspond to new intrinsic topological orders.

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- TQFTs derived from TNs can be regarded as "background" of physical space-time. Space-time may emerge as collective modes of TQFTs.

- QG in 2+1D is a TQFT and can be regarded as a (double) quantum hall system.

- New TQFTs based on TNs construction could be potential candidates for quantum gravity(QG).

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