Title: Numerical Relativity on Constant Mean Curvature Hypersurfaces

Date: Oct 13, 2011 01:00 PM

URL: http://pirsa.org/11100089

Abstract: Constant mean curvature (uniform K) hypersurfaces extend to future null infinity in asymptotically flat spacetimes. With conformal compactification, the entire hypersurface can be covered by a finite spatial grid, eliminating any need an "outgoing wave" boundary condition or for extrapolation to find gravitational wave amplitudes. I will discuss the asymptotic behavior near future null infinity, how this can be simplified by suitable gauge conditions, and how this determines the physical Bondi energy and momentum of the system. Numerical results for how Bowen-York parameters in the conformally flat initial value problem are related to the physical energy and momentum in systems with single and binary black holes will be presented.

Numerical Relativity on CMC Hypersurfaces

James M. Bardeen
University of Washington

References:

V. Moncrief and O. Rinne, arXiv:0811.4109v2
Bardeen, Sarbach, and Buchman, arXiv:1101.5479
Buchman, Pfeiffer, and Bardeen, arXiv:0907.3163
O. Rinne, arXiv:0910.0139
Bardeen and Buchman, on arXiv soon.
L. Andersson and P. Chrusciel, Comm. Math. Phys. 161:533-568, 1994
Chrusciel, Jezierski, and Leski, arXiv:gr-qc/0307109

Perimeter Institute October 14, 2011

Overview

Hypersurfaces with constant mean curvature $K = K^i{}_i > 0$ (Wald sign convention) in an asymptotically flat spacetime extend to future null infinity $\mathfrak{I}^+$ rather than to spatial infinity. With conformal compactification the entire physical spacetime to the future of a CMC hypersurface can be covered by a finite spatial grid, with no need for approximate "outgoing wave" boundary conditions at a finite radius, and gravitational wave amplitudes can be read off at $\mathfrak{I}^+$ without any extrapolation.

The catch is that the evolution and constraint equations contain terms singular at $\mathfrak{I}^+$ which analytically are consistent with regular solutions, but might prove problematic numerically. The evolution system on CMC hypersurfaces must be a mixed hyperbolic-elliptic system, since at least the equation for the lapse must be an elliptic equation.

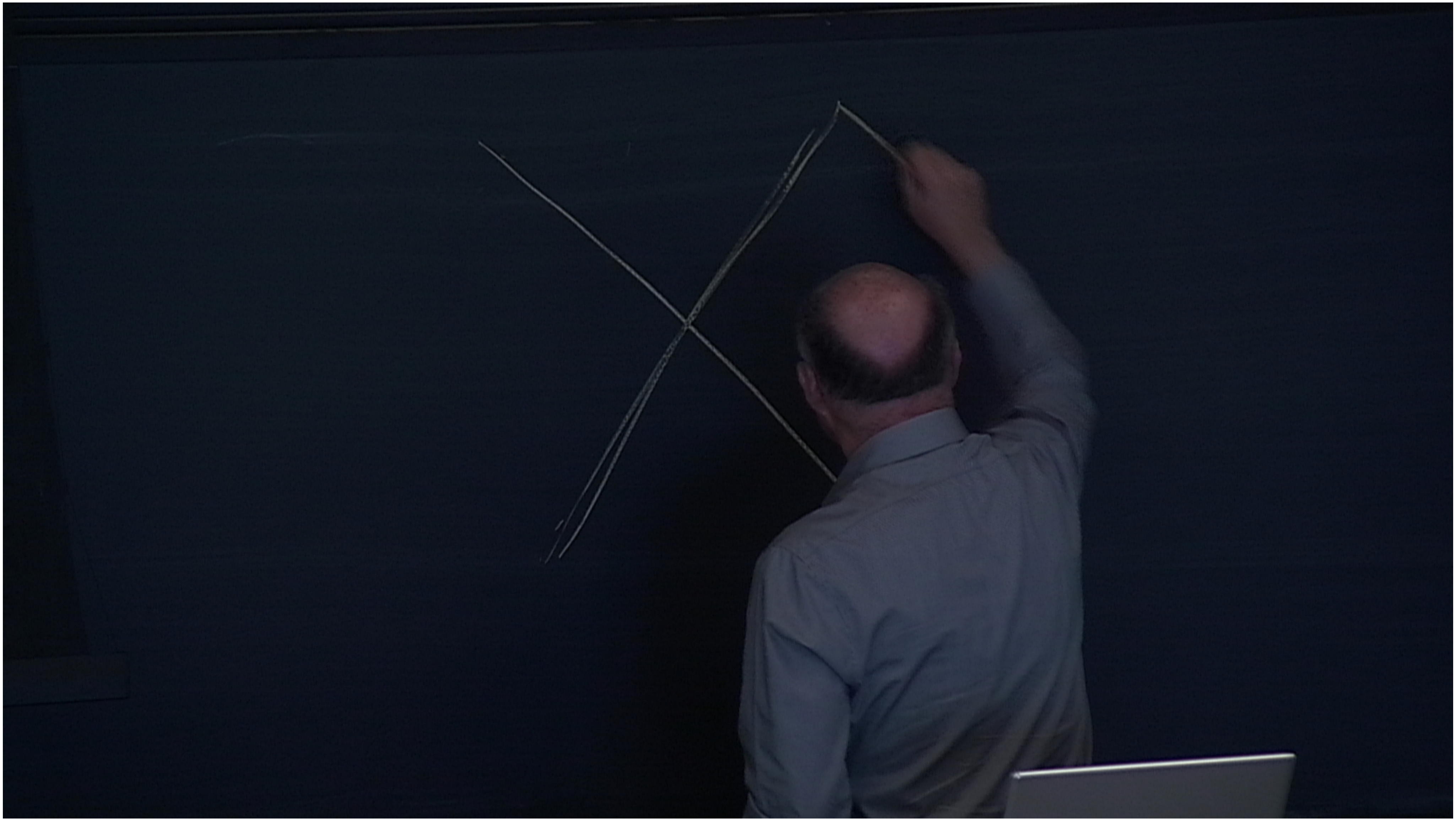
In this talk I will focus on the asymptotic structure of solutions, the calculation of the Bondi-Sachs energy and momentum, and the physical interpretation of numerical solutions of the conformally flat initial value problem.

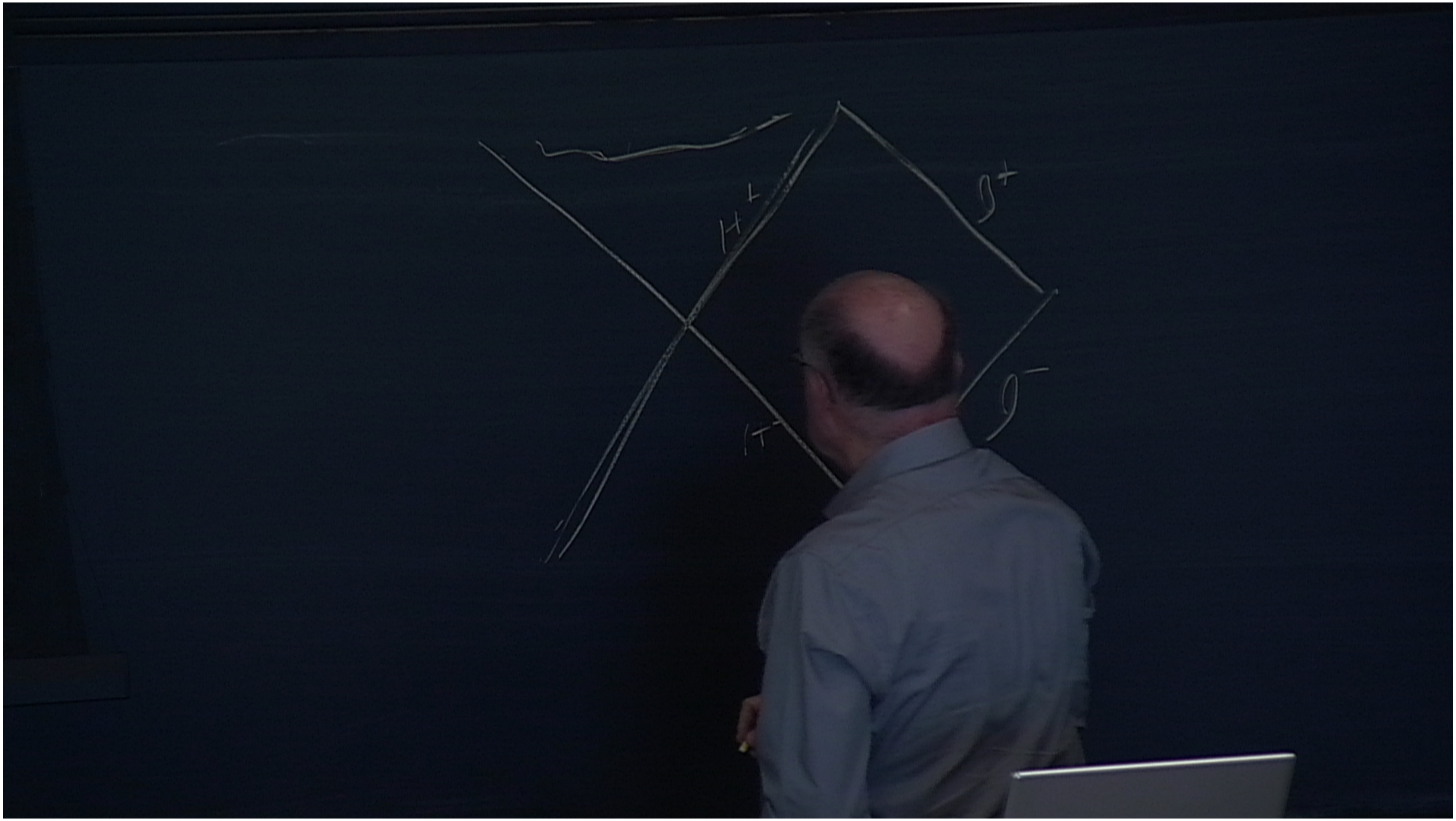
Conformal Compactification

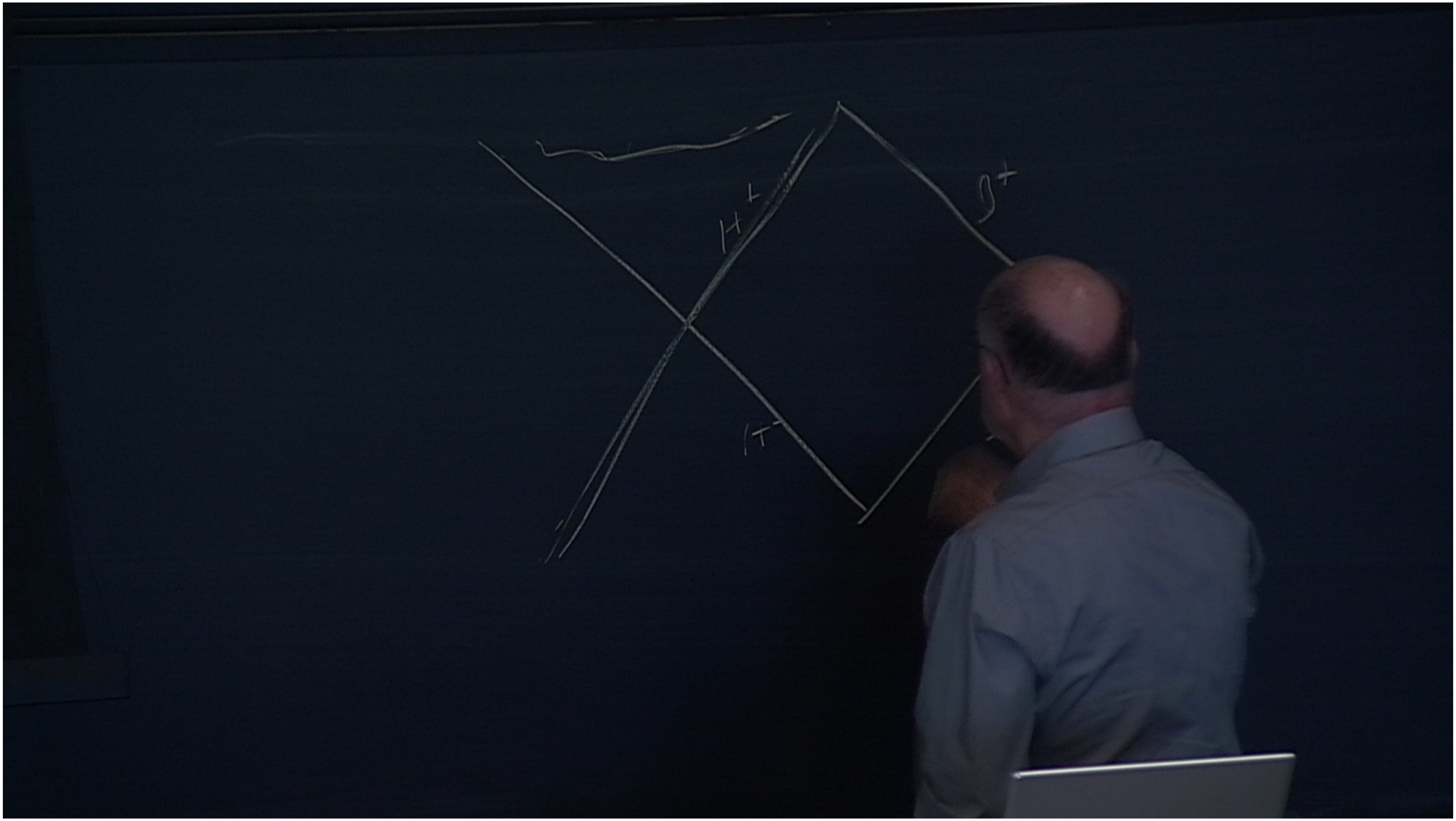
- Physical metric $ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta = \Omega^{-2}\tilde{g}_{\alpha\beta}dx^\alpha dx^\beta$.
- $\Omega = 0$ at future null infinity, chose coordinates so this is a coordinate sphere at $R = \left[\left(x^1\right)^2 + \left(x^2\right)^2 + \left(x^3\right)^2\right]^{1/2} = R_+$.
- Conformal metric $\tilde{g}_{\alpha\beta}$ regular everywhere, make 3+1 split, $d\tilde{s}^2 = -\tilde{\alpha}^2 dt^2 + \tilde{g}_{ij}\left(dx^i + \beta^i dt\right)\left(dx^j + \beta^j dt\right)$.
- CMC hypersurface asymptotically null in physical spacetime, spacelike everywhere in conformal spacetime.
- Define the conformal extrinsic curvature $\tilde{K}_{ij}$ so $\partial_t \tilde{g}_{ij} - \left(L_\beta \tilde{g}\right)_{ij} = 2\tilde{\alpha}\tilde{K}_{ij}$, the traceless part $\hat{\tilde{K}}_{ij} \equiv \Omega^{-1}\hat{K}_{ij}$, the trace $\tilde{K}$ determines the evolution of the conformal gauge, $\partial_t \Omega - \beta^k \partial_k \Omega = \frac{\tilde{\alpha}}{3}\left(\Omega\tilde{K} - K\right)$.
- Vacuum momentum constraint equation $\tilde{\nabla}_j \hat{\tilde{K}}_i^j = 2\left(\Omega^{-1}\partial_j \Omega\right)\hat{\tilde{K}}_i^j$ requires the regularity condition $\partial_j \Omega\, \hat{\tilde{K}}_i^j \doteq 0$ ($\doteq$ signifies equality at $\Im^+$).

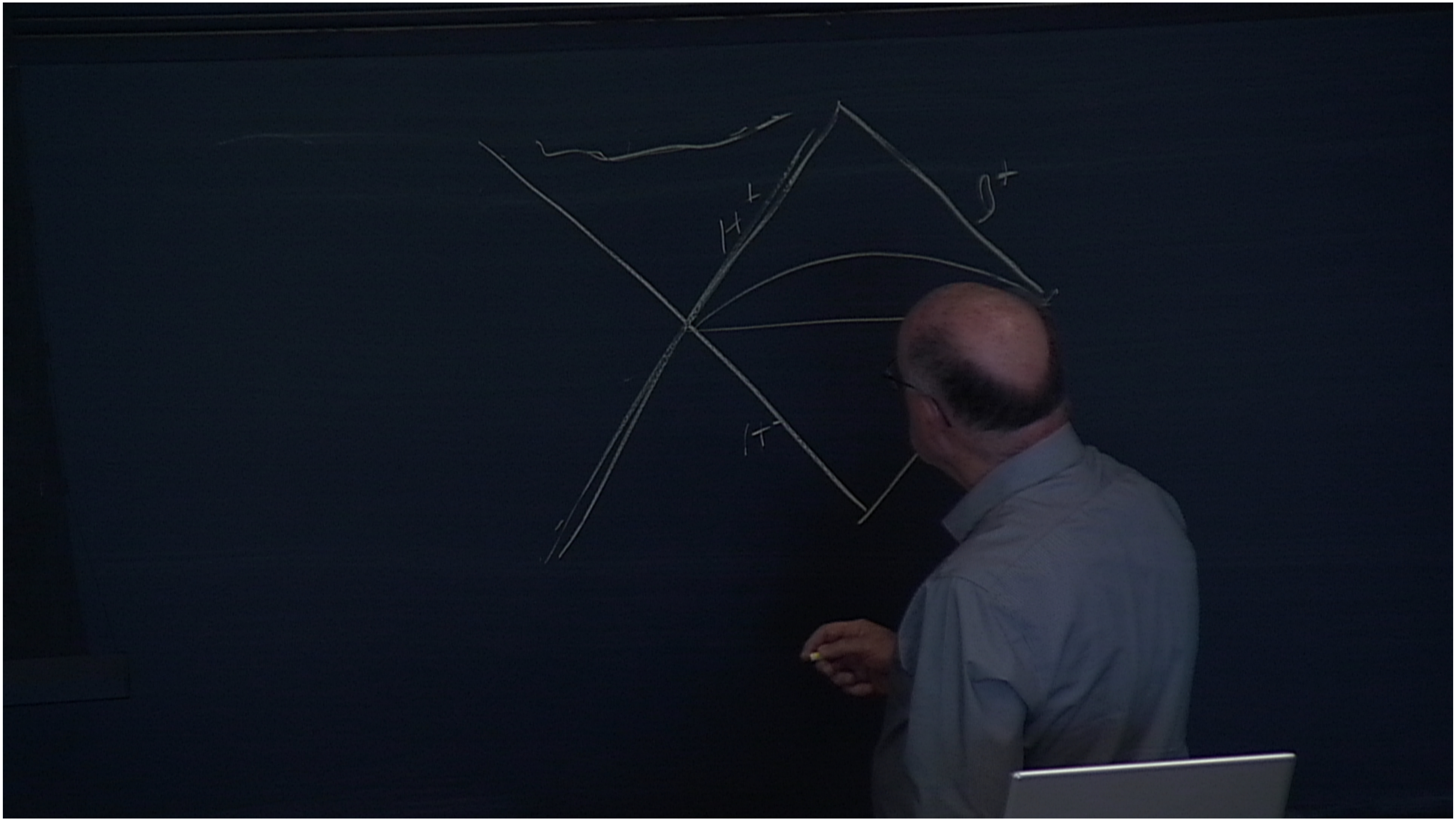
Asymptotic gauge conditions

- Keep the coordinate location of $\Im^+$ fixed during evolution, a boundary condition on β^R.
- Define polar angles $x^A = (\theta, \varphi)$ on $\Im^+$ in the usual way.
- Control the conformal gauge so the 2-surface intersection of the CMC hypersurface with the $\Im^+$ null hypersurface, $\dot{\Im}^+$, has the intrinsic geometry of a 2-sphere.
- If the 2-metric on $\dot{\Im}^+$ has the form $ds^2 = \xi_0^{-2}\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \equiv \xi_0^{-2}\breve{h}_{AB}dx^A dx^B$ initially, it maintains this form with a constant ξ_0 if the angular coordinates are propagated along the null generators of $\Im^+ \left(\Leftrightarrow \beta^A \doteq 0\right)$ and if $\tilde{K}$ satisfies the boundary condition $\tilde{K} \doteq \frac{3}{2}\tilde{\kappa}_0$, where $\tilde{\kappa}_0$ is the trace of the 2D extrinsic curvature of the $\dot{\Im}^+$ 2-surface as embedded in the conformal geometry of the CMC hypersurface.
- The conformal lapse should satisfy the boundary condition $\tilde{\alpha} \doteq \frac{K}{3\xi_0} \equiv \tilde{\alpha}_0$ in order that the CMC time coordinate equal the Minkowski retarded time coordinate on $\Im^+$.

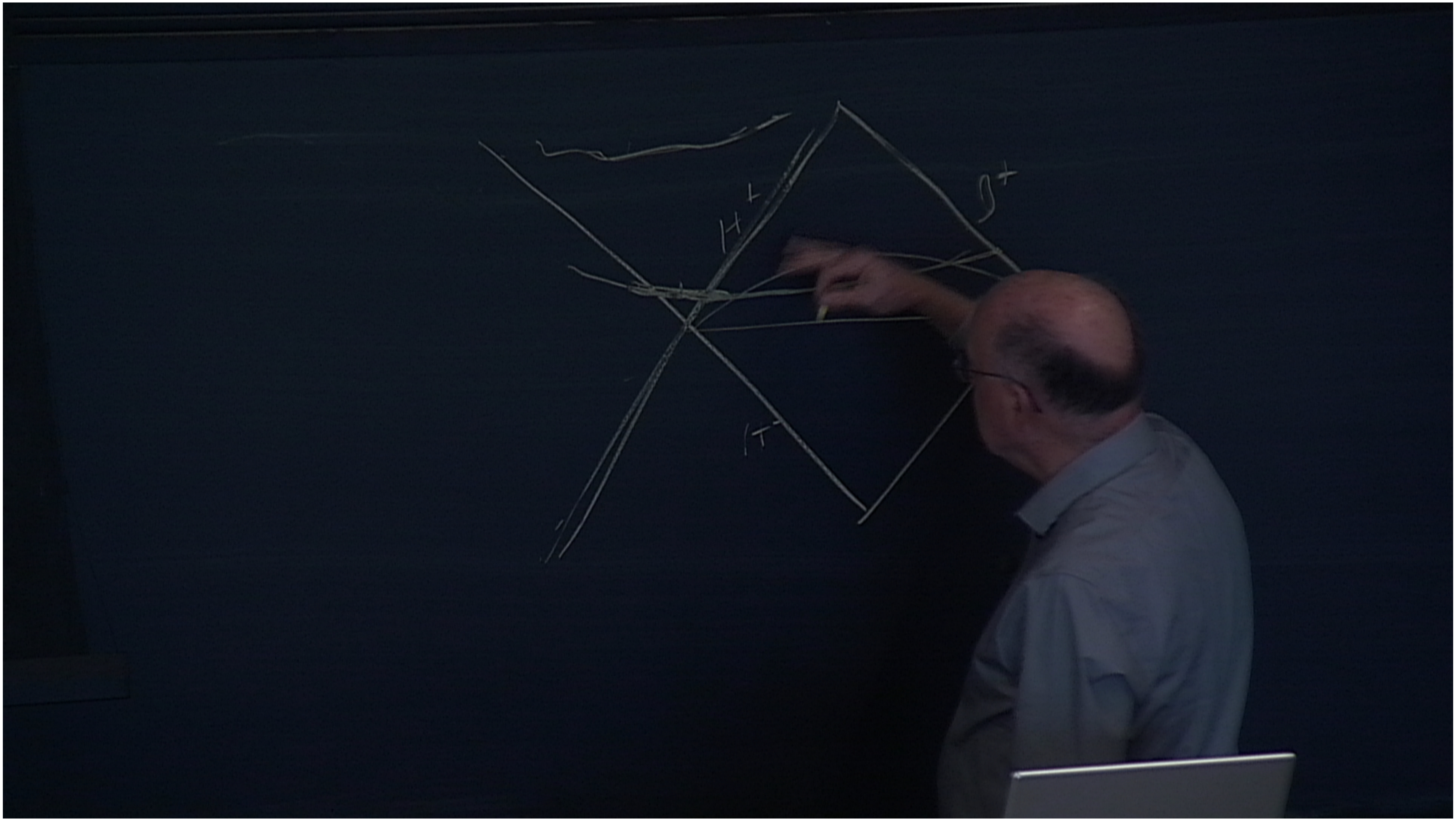


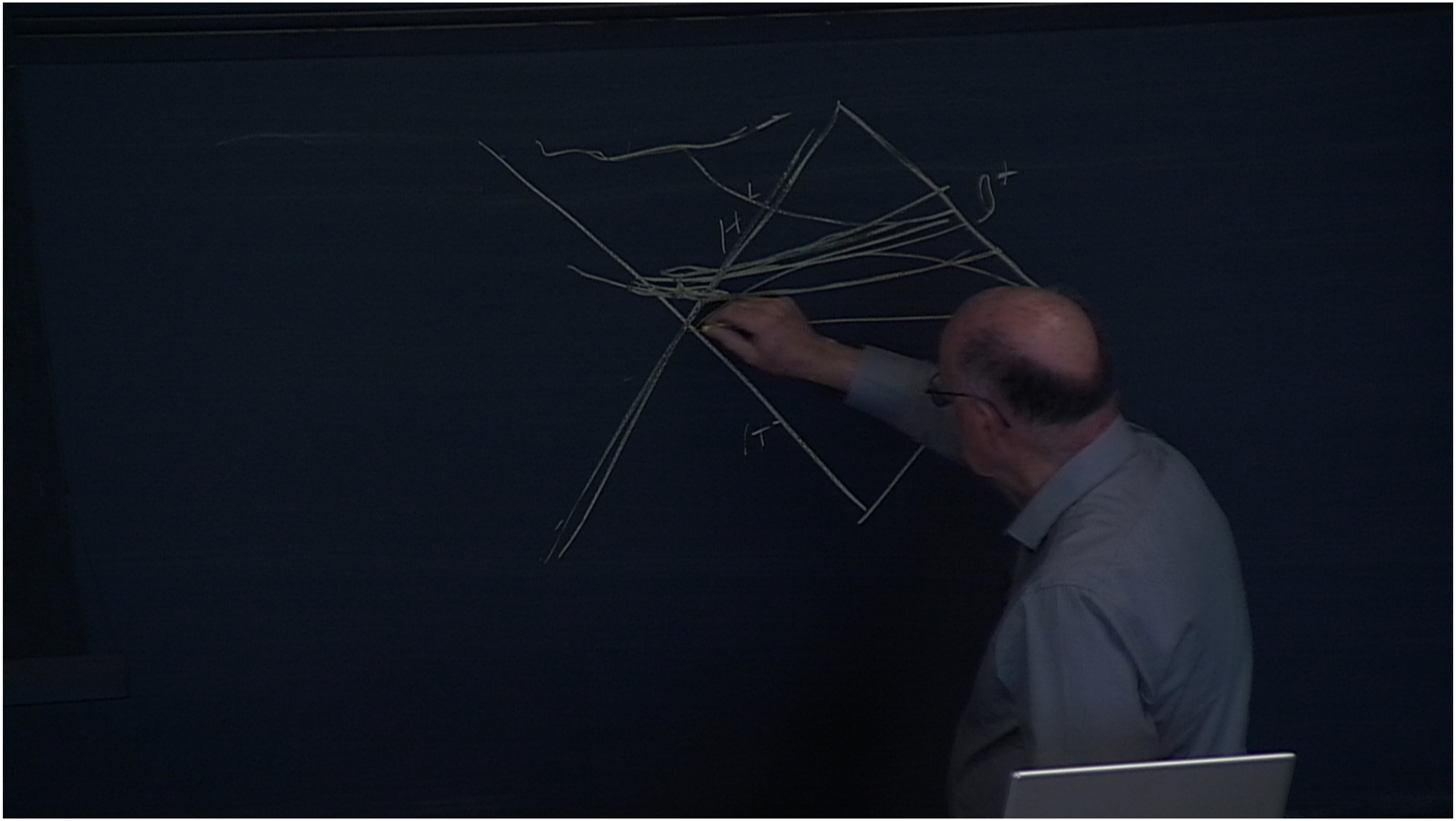

H
H
J+
J−


H^+
H^-
$\mathcal{I}^+$



H^+
$\mathcal{J}^+$
H^-


H^+
$\mathcal{I}^+$
H^-



$\mathcal{I}^+$
$\mathcal{I}^-$

Hamiltonian constraint equation

In vacuum, $\Omega\tilde{\Delta}\Omega = \frac{3}{2}\left[\tilde{\nabla}^k\Omega\tilde{\nabla}_k\Omega - \left(\frac{K}{3}\right)^2\right] + \frac{\Omega^2}{4}\left[\hat{\tilde{K}}^i_j\hat{\tilde{K}}^j_i - \tilde{R}\right]$. The equation is elliptic inside $\dot{\Im}^+$, but degenerate at $\dot{\Im}^+$. Any solution which satisfies the Dirichlet outer boundary condition $\Omega \doteq 0$ automatically has a normal derivative satisfying $\tilde{\nabla}^n\Omega\tilde{\nabla}_n\Omega \doteq \left(\frac{K}{3}\right)^2$. It is not until 4th order in an expansion away from $\dot{\Im}^+$ that the solution has any freedom to deviate from a fixed expansion dictated by the conformal geometry. A unique global solution also requires an inner boundary condition, which we will take to be a minimal surface condition inside the apparent horizon of a black hole in our conformally flat initial data examples.

Lapse equation

Requiring that K remain constant during evolution gives an elliptic equation for the lapse. As an equation for the conformal lapse this is $\Omega\tilde{\Delta}\tilde{\alpha} - 3\tilde{\nabla}^k\Omega\tilde{\nabla}_k\tilde{\alpha} + \left(\tilde{\Delta}\Omega\right)\tilde{\alpha} = \frac{\Omega}{2}\left(3\hat{\tilde{K}}^j_i\hat{\tilde{K}}^i_j - \tilde{R}\right)\tilde{\alpha}$, where I have used the Hamiltonian constraint to eliminate the terms most singular at $\dot{\Im}^+$. The asymptotic behavior of $\tilde{\alpha}$ is also strongly constrained by the degeneracy of the elliptic equation at $\dot{\Im}^+$. The boundary condition at $\dot{\Im}^+$ is the Dirichlet condition $\tilde{\alpha} \doteq \tilde{\alpha}_0$.

Asymptotic solutions in Gaussian normal coordinates

Gaussian normal coordinates based on the $\dot{\mathfrak{I}}^+$ allow a relatively simple way of obtaining solutions of the elliptic equations as power series expansions in the conformal proper distance along the normal spatial geodesics of the conformal geometry. The angular coordinates x^A are propagated inward along these geodesics from their values on $\dot{\mathfrak{I}}^+$. The spatial metric has the physically general form

$$d\tilde{l}^2 = dz^2 + \xi^{-2}\tilde{h}_{AB}dx^A dx^B.$$

Expand $\tilde{h}_{AB} = \xi^{-2}\left[\breve{h}_{AB} - 2\breve{\chi}_{AB}z + \left(\breve{\chi}^C_D\breve{\chi}^D_C\breve{h}_{AB} - \tilde{\psi}_{AB}\right)z^2 + O\left(z^3\right)\right]$, where $\breve{\chi}_{AB}$ and $\breve{\psi}_{AB}$ are arbitrary traceless symmetric tensors on the unit sphere. By construction, $\det\tilde{h}_{AB} = \xi^{-4}\det\breve{h}_{AB} = \xi^{-4}\sin^2\theta$ at all z. We have not allowed any "polyhomogeneous" terms of the form $z^n\left(\log z\right)^m$, unlike Andersson and Chrusciel, because we feel there is no physical justification for doing so in an astrophysical context.

The 2D extrinsic curvature of constant-z 2-surfaces is $\tilde{\kappa}^A_B = \hat{\tilde{\kappa}}^A_B + \frac{1}{2}\delta^A_B\tilde{\kappa}$, with $\hat{\tilde{\kappa}}^A_B = \breve{\chi}^A_B + \tilde{\psi}^A_B z + O\left(z^2\right)$ and $\tilde{\kappa} = 2\partial z\left(\log\xi\right) = \tilde{\kappa}_0 + \tilde{\kappa}_1 z + \tilde{\kappa}_2 z^2 + O\left(z^3\right)$ defines the expansion of $\log\xi$.

The intrinsic scalar curvature of the 2-surfaces is $^2R = 2\xi_0^2\left[1 + \left(\tilde{\kappa}_0 + \frac{1}{2}\breve{\Delta}\tilde{\kappa}_0 - \chi^{AB}{}_{\breve{|}AB}\right)z + O\left(z^2\right)\right]$ and the 3D intrinsic scalar curvature is $\tilde{R} = {}^2\tilde{R} + 2\partial_z\tilde{\kappa} - \frac{3}{2}\tilde{\kappa}^2 - \hat{\tilde{\kappa}}^A_B\hat{\tilde{\kappa}}^B_A$.

Bondi-Sachs Energy and Momentum

The Bondi-Sachs metric

$$ds^2 = -Ve^{2\beta}du^2 - 2e^{2\beta}dudr + r^2\bar{h}_{AB}\left(d\bar{x}^A - U^A du\right)\left(d\bar{x}^B - U^B du\right)$$

has an expansion in powers of $x \equiv r^{-1}$ in neighborhood of $\mathfrak{I}^+$ $(x=0)$, with

$$\bar{h}_{AB} = \breve{h}_{AB} + \bar{\chi}_{AB}x + \frac{1}{4}\bar{\chi}^{CD}\bar{\chi}_{CD}\breve{h}_{AB}x^2 + O\left(x^3\right),$$

$$V = 1 - 2M\left(u,\theta,\varphi\right)x + O\left(x^2\right),$$

$$\beta = -\frac{1}{32}\bar{\chi}^{BC}\bar{\chi}_{BC}x^2 + O\left(x^3\right),$$

$$U^A = -\frac{1}{2}\bar{\chi}^{AB}{}_{|B}x^2 + O\left(x^3\right).$$

The Bondi-Sachs mass aspect is best defined as $M_{\text{Asp}} \equiv M - \frac{1}{4}\bar{\chi}^{AB}{}_{|AB}$, which has the property that $\partial_u M_{\text{Asp}} \leq 0$ at all angles. The Bondi energy and momentum are

$$E_{\text{BS}} = \frac{1}{4\pi}\iint M_{\text{Asp}}\sin\theta d\theta d\varphi,$$

$$P^k_{\text{BS}} = \frac{1}{4\pi}\iint M_{\text{Asp}}N^k\sin\theta d\theta d\varphi, \qquad N^k = \left(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta\right).$$

Replacing M_{Asp} by the metric function M has no effect on the integral values.

Transformation from CMC coordinates to Bondi coordinates

Find the transformations $u(t,z,x^A)$ and $\bar{x}^A(t,z,x^B)$ as power series in z from $\bar{g}^{uu} = \bar{g}^{uA} = 0$, adopting the CMC shift which preserves the Gaussian normal form of the spatial metric, with the result

$$u = t + \frac{z}{2\tilde{\alpha}_0}\left[1 + \frac{1}{4}\tilde{\kappa}_0 z + O(z^2)\right], \qquad \bar{x}^A = x^A + \frac{1}{6}\xi_0^2\left(\breve{\chi}^{AB}{}_{\mid B} - \frac{1}{2}\tilde{\kappa}_0^{\mid A}\right)z^3 + O(z^4).$$

Taking the determinant of the transformation to $\bar{g}^{AB} = x^2\bar{h}^{AB}$ gives

$$x = \Omega\xi\left[1 + \frac{1}{12}\xi_0^2\left(\breve{\chi}^{AB}{}_{\mid AB} - \frac{1}{2}\Delta\tilde{\kappa}_0\right)z^3 + O(z^4)\right].$$

Substituting back, we find $\bar{\chi}^{AB} = -2\left(\frac{3}{K\xi_0}\right)\breve{\chi}^{AB}$, which relates the CMC $\breve{\chi}^{AB}$ to the Bondi news. Finally, evaluating $\bar{g}^{xx} = x^4 Ve^{-2\beta}$ gives for the mass aspect

$$-\frac{3}{K\xi_0^3}\left[4c_3 + \frac{1}{2}d_1 + \frac{1}{8}Q - \frac{1}{6}\xi_0^2\left(\tilde{\kappa}_0 + \frac{3}{4}\Delta\tilde{\kappa}_0\right) + \frac{1}{48}\tilde{\kappa}_0^3 - \frac{5}{24}\tilde{\kappa}_0\left(\tilde{\kappa}_1 + \breve{\chi}_D^C\breve{\chi}_C^D\right) + \frac{1}{2}\tilde{\kappa}_2\right].$$

An alternative derivation is possible from the expression for the mass aspect in terms of the asymptotic Weyl scalar Ψ_2,

$$M_{\text{Asp}} = -\frac{3}{2K\xi_0^3}\left[\partial_z\hat{\tilde{E}}_z^z + \breve{\chi}_D^C\partial_t\breve{\chi}_C^D\right] + \frac{3}{2K\xi_0}\breve{\chi}^{CD}{}_{\mid CD}.$$

The conformally flat IVP

With conformal flatness for the spatial metric, $\xi = R^{-1}$, $\xi_0 = R_+^{-1}$, and $\tilde{\kappa}_0 = 2\xi_0$, $\tilde{\kappa}_1 = 2\xi_0^2$, $\tilde{\kappa}_2 = 2\xi_0^3$, and $\hat{\tilde{\kappa}}^A_B = 0$. The momentum constraint equation is trivial, and admits the same Bowen-York class of solutions as is well known for maximal hypersurfaces,

$$\Omega^{-2}\hat{\tilde{K}}_{ij} \equiv \tilde{A}_{ij} = \frac{C}{R_{\mathbf{D}}^3}\left[3n_i n_j - \delta_{ij}\right] - \frac{3}{2R_{\mathbf{D}}^2}\left[P_i n_j + P_j n_i + P^k n_k\left(n_i n_j - \delta_{ij}\right)\right]$$
$$-\frac{3}{R_{\mathbf{D}}^3}\left[\varepsilon_{ikl}S^k n^l n_j + \varepsilon_{jkl}S^k n^l n_i\right] + \frac{3}{2R_{\mathbf{D}}^4}\left[Q_i n_j + Q_j n_i + Q^k n_k\left(\delta_{ij} - 5n_i n_j\right)\right],$$

with $R_{\mathbf{D}} = |\mathbf{x} - \mathbf{D}|$ and $n^i = \left(x^i - D^i\right)/R_{\mathbf{D}}$. The expansion of the conformal factor simplifies to

$$\Omega = \frac{KR_+}{3}\bar{z}\left[1 - \frac{1}{2}\bar{z} + \bar{c}_3\left(1 + \bar{z}\right)\bar{z}^3 + O\left(\bar{z}^5\right)\right],$$

using a dimensionless expansion parameter $\bar{z} \equiv z/R_+$ and $\bar{c}_3 \equiv R_+^3 c_3$. The expression for the mass aspect reduces to

$$\bar{M}_{\text{Asp}} \equiv \frac{K}{3}M_{\text{Asp}} = -4\bar{c}_3 - \bar{A}_{RR}, \quad \bar{A}_{RR} \equiv \frac{1}{2}\left(\frac{K}{3}\right)^2 R_+^3 N^i \tilde{A}_{ij} N^j.$$

Numerical results

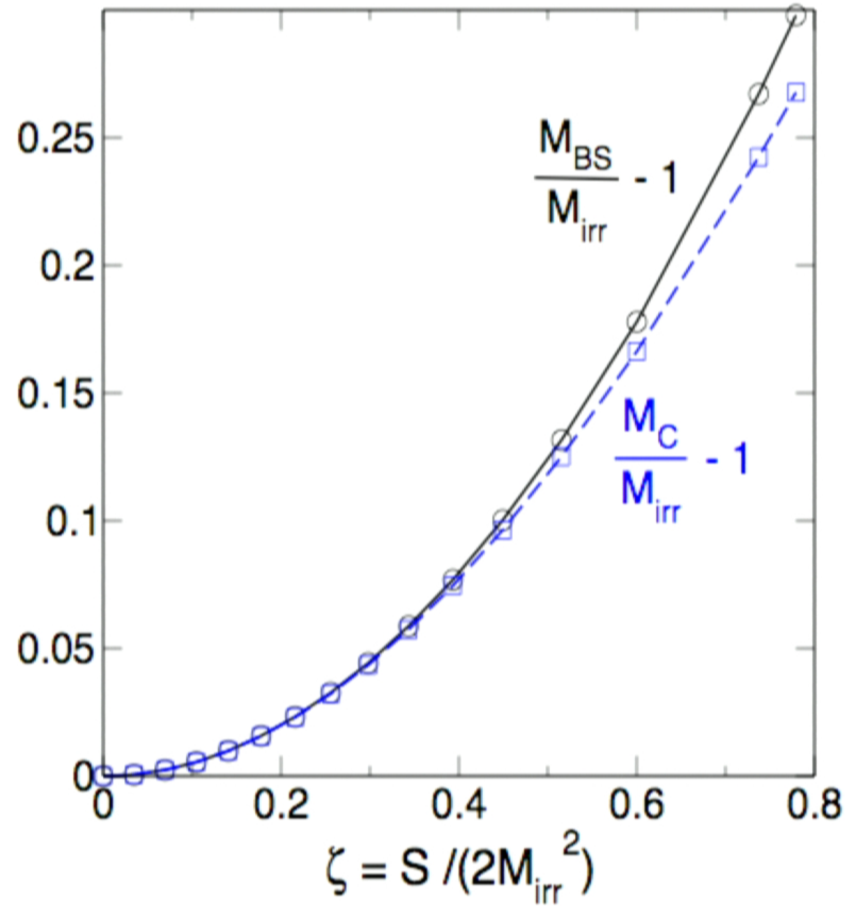
Solve for Ω using the elliptic solver of the Cornell-Caltech SPEC code, and fit to the analytic form of the expansion to extract $\bar{c}_3$. The angular integrals on $\mathfrak{I}^+$ over $\bar{A}_{RR}$ can be done analytically. The latter contributions to the physical energy and momentum are

$$(E_{\text{BS}})_{\text{K}} = -\frac{KC}{3} + \frac{K}{3}\mathbf{D}\cdot\mathbf{P},$$

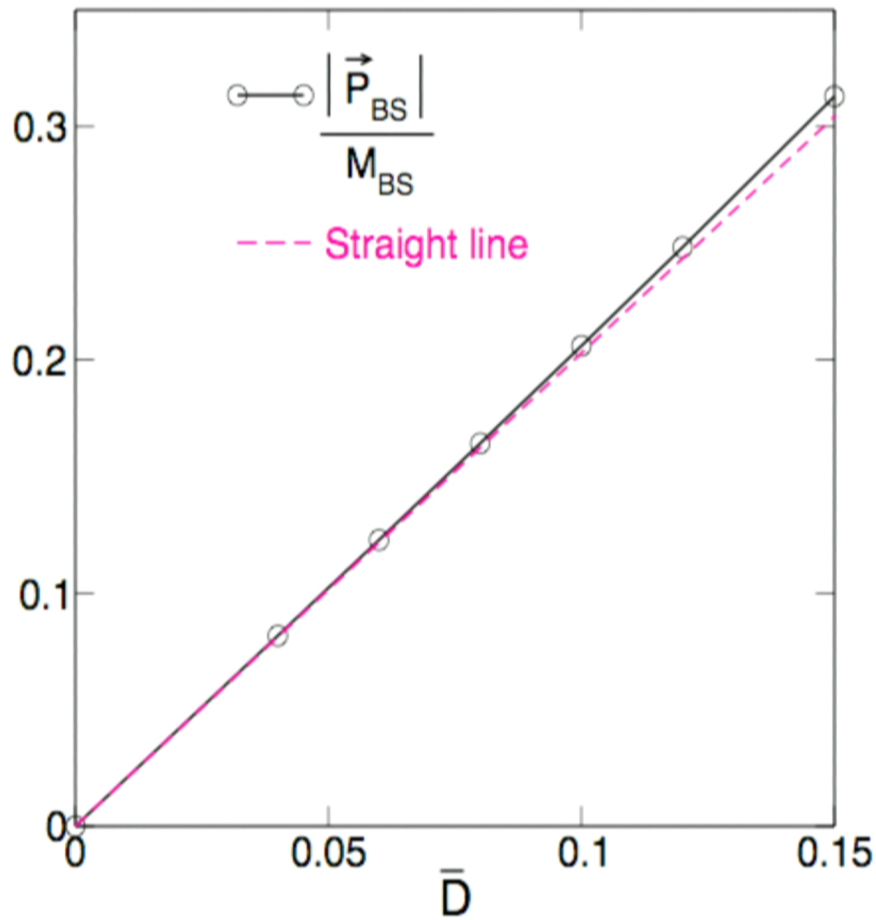
$$(\mathbf{P}_{\text{BS}})_{\text{K}} = \frac{KR_+}{6}\mathbf{P}\left(1-\left|\frac{\mathbf{D}}{R_+}\right|^2\right) - \frac{KC}{3R_+}\mathbf{D} + \frac{K}{3R_+}(\mathbf{D}\cdot\mathbf{P})\mathbf{D} + \frac{K}{6R_+}\mathbf{Q} + \frac{K}{3R_+}\mathbf{D}\times\mathbf{S}.$$

The analytic result for the total angular momentum of the system is $\mathbf{J} = \mathbf{S} + \mathbf{D}\times\mathbf{P}$.

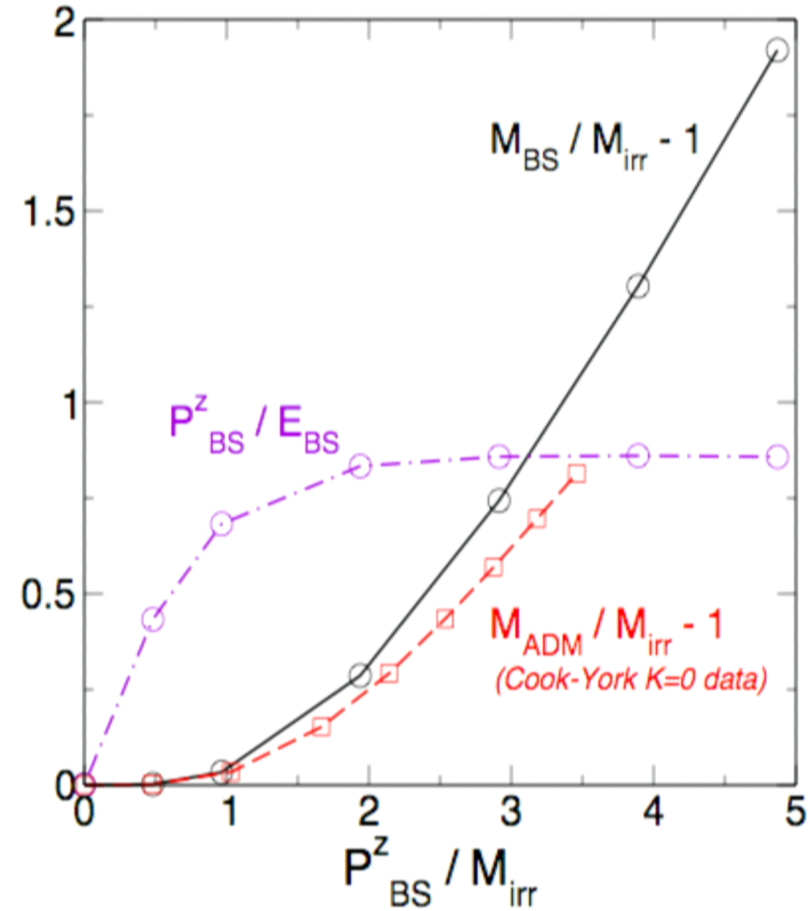
The problem is scale-invariant in two separate ways, in the choice of units for physical quantities and a uniform scale in the conformal factor. Only quantities invariant under both rescalings are physically relevant. Invariant input parameters are $\bar{C} \equiv (K/3)^2 C$, $\bar{\mathbf{P}} \equiv \left(\frac{K^2 R_+}{18}\right)\mathbf{P}$, $\bar{\mathbf{S}} \equiv \mathbf{S}/C$, $\bar{\mathbf{D}} \equiv \mathbf{D}/R_+$, and the coordinate distance from the black hole center at which the minimal surface inner boundary condition is imposed relative to R_+, $\bar{R}_{\text{ms}} \equiv R_{\text{ms}}/R_+$. Our results for single black holes typically have $\bar{C} \approx 0.0011$ and $\bar{R}_{\text{ms}} \approx 0.0013$. A factor of $K/3$ is used make physical masses and momenta dimensionless. For Schwarzschild black holes typically $KM/3 \approx \bar{C}^{1/2}$ as long as $\bar{R}_{\text{ms}}$ is sufficiently small.



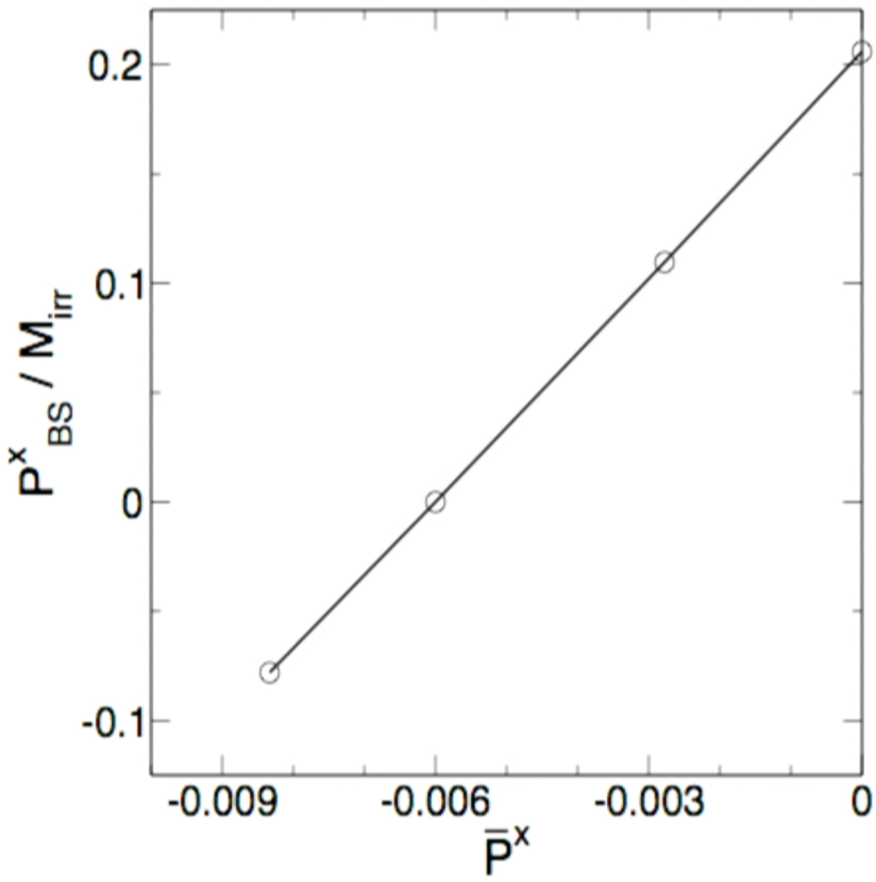
0.25
0.2
0.15
0.1
0.05
0
0
0.2
0.4
0.6
0.8
$\zeta = S/(2M_{irr}^2)$
$\frac{M_{BS}}{M_{irr}} - 1$
$\frac{M_C}{M_{irr}} - 1$



$\frac{|\vec{P}_{BS}|}{M_{BS}}$
Straight line
0.3
0.2
0.1
0
0
0.05
0.1
0.15
$\bar{D}$



$M_{BS} / M_{irr} - 1$
P^z_{BS} / E_{BS}
$M_{ADM} / M_{irr} - 1$
(Cook-York K=0 data)
P^z_{BS} / M_{irr}
0
0.5
1
1.5
2
0
1
2
3
4
5


0.2
0.1
0
-0.1
P^x_{BS} / M_{irr}
-0.009
-0.006
-0.003
0
$\bar{P}^x$

Discussion

- The coefficient Q of the leading polyhomogeneous terms in the conformal factor and extrinsic curvature is equal to $\xi_0^2 \breve{\chi}^{AB}\breve{\gamma}_{AB} + \breve{\chi}^A_B \partial_t \breve{\chi}^B_A$, which is almost always non-zero whenever radiation is present at $\mathfrak{I}^+$. This is a gauge artifact of the CMC hypersurface condition; no terms of this type are present in the Bondi-Sachs metric. Will these terms create convergence problems for spectral methods such as those of the SPEC code? Rinne's axisymmetric code did not seem to have problems, but this was a finite difference code without any attempt at high accuracy.
- Can the elliptic gauge conditions which preserve nice properties of $\mathfrak{I}^+$ be implemented efficiently enough to compete with the purely hyperbolic codes that dominate numerical relativity today?
- Will the analytic cancellation of singular terms in the evolution equations for the extrinsic curvature hold up in the presence of numerical errors? The regularity conditions may need to be enforced during the evolution, not just in the initial data. This was not a problem in the Rinne calculations, but absence of symmetries may make things more difficult.
- The conformal thin sandwich approach to the IVP has significant advantages for producing quasi-equilibrium initial data. Can this be implemented on CMC hypersurfaces in a reasonably simple way?