

Title: The Projection Operator Method for Quantum Constraints

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Abstract: Classical constraints come in various forms: first and second class, irreducible and reducible, regular and irregular, all of which will be illustrated. They can lead to severe complications when classical constraints are quantized. An additional complication involves whether one should quantize first and reduce second or vice versa, which may conflict with the axiom that canonical quantization requires Cartesian coordinates. Most constraint quantization procedures (e.g., Dirac, BRST, Faddeev) run into difficulties with some of these issues and may lead to erroneous results. The Projection Operator Method involves no gauge fixing, no auxiliary variables of any kind, and can treat simultaneously any and all kinds of constraints. It also admits a phase space path integral formulation with similar features.

The Projection Operator Method for Quantum Constraints

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Outline

- *Classical constraints*
- *Dirac & Faddeev quantization:
the good and the bad*
- *Projection operator method:
how it can lead to better results*
- *Coherent state path integrals:
how they can include the projection
operator*

Constrained Dynamics (1)

Action $I = \int [p_j \dot{q}^j - H(p, q) - \lambda^\alpha \varphi_\alpha(p, q)] dt$

Equations $dq^j / dt = \partial H / \partial p_j + \lambda^\alpha \partial \varphi_\alpha / \partial p_j$, $\underline{\varphi_\alpha = 0}$
 $dp_j / dt = -\partial H / \partial q^j - \lambda^\alpha \partial \varphi_\alpha / \partial q^j$

Poisson brackets

$$\{A, B\} \equiv (\partial A / \partial q^j)(\partial B / \partial p_j) - (\partial A / \partial p_j)(\partial B / \partial q^j)$$

$$dA / dt = \{A, H\} + \lambda^\beta \{A, \varphi_\beta\}$$

$$d\varphi_\alpha / dt = \{\varphi_\alpha, H\} + \lambda^\beta \{\varphi_\alpha, \varphi_\beta\} = 0$$

Constrained Dynamics (2)

First class constraints on $\mathcal{C} = \{\varphi_\alpha = 0\}$

$$0 = \{\varphi_\alpha, \varphi_\beta\} = c_{\alpha\beta}{}^\gamma \varphi_\gamma , \quad 0 = \{\varphi_\alpha, H\} = h_\alpha{}^\beta \varphi_\beta$$

(if latter equation is false, then

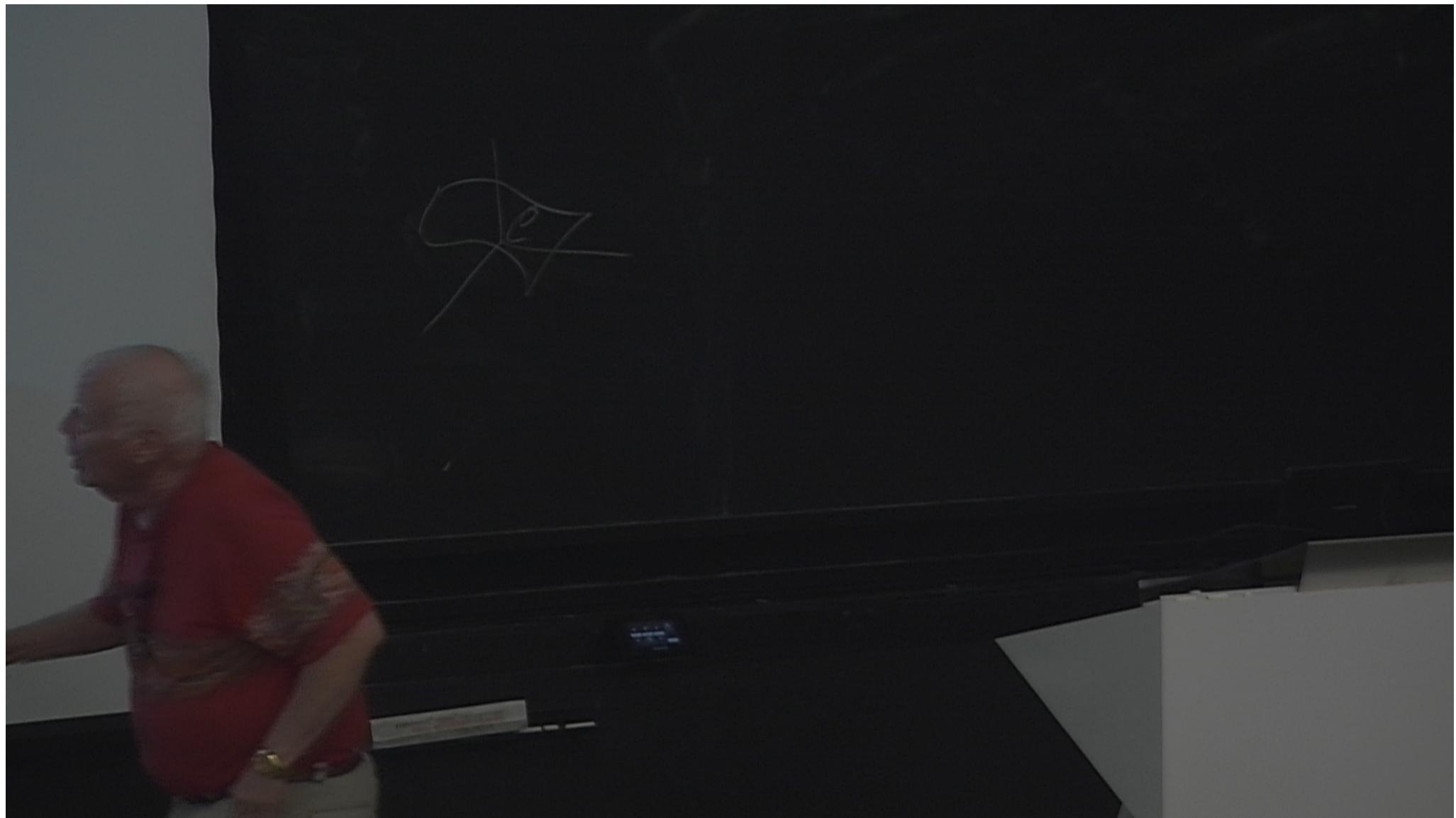
it becomes a new constraint)

Requires 'initial value equation' to put it on \mathcal{C}

Once on \mathcal{C} then dynamics keeps it on \mathcal{C}

No restriction imposed on λ^α variables

Solution requires λ^α variables (''gauge choice'')



Constrained Dynamics (4)

Irreducible constraints : $c^\alpha \varphi_\alpha = 0 \Rightarrow c^\alpha = 0$

Reducible constraints : $c^\alpha \varphi_\alpha = 0 \not\Rightarrow c^\alpha = 0$

Regular constraints :

$\varphi_\alpha(p, q) = 0 ; \quad \partial \varphi_\alpha / \partial p^j \neq 0, \quad \partial \varphi_\alpha / \partial q_k \neq 0$

Irregular constraints : $\varphi_\alpha^{ir}(p, q) = \varphi_\alpha(p, q)^3, \text{ etc.}$

straints: $c^\alpha \varphi_\alpha = 0 \Rightarrow c^\alpha = 0$

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aints:

Constraint Quantization (1)

Dirac procedure: \mathcal{Q} before \mathcal{R}

$$\varphi_\alpha(p, q) \rightarrow \Phi_\alpha(P, Q) [= \Phi_\alpha(P, Q)^+]$$

$$\Phi_\alpha(P, Q) |\psi_{phys}\rangle = 0$$

$$|\psi_{phys}\rangle \in \mathcal{H}_{phys} \subset \mathcal{H}$$

Two possible difficulties

$$\langle \psi_{phys} | \psi_{phys} \rangle = \infty$$

$$[\Phi_\alpha, \Phi_\beta] |\psi_{phys}\rangle = 0 \Rightarrow |\psi_{phys}\rangle = 0$$

Constraint Quantization (2)

Good examples : compact Lie algebras

$$\Phi_a |\psi_{phys}\rangle = 0 , \quad [\Phi_\alpha, \Phi_\beta] |\psi_{phys}\rangle = i\hbar c_{\alpha\beta}{}^\gamma \Phi_\gamma |\psi_{phys}\rangle = 0$$

Special open first class constraints

$$\Lambda_a |\psi_{phys}\rangle = 0 , \quad [\Lambda_\alpha, \Lambda_\beta] |\psi_{phys}\rangle = i\hbar F_{\alpha\beta}{}^\gamma \Lambda_\gamma |\psi_{phys}\rangle = 0$$

Bad examples :

$$Q |\psi_{phys}\rangle = 0 , \quad \langle \psi_{phys} | \psi_{phys} \rangle = \infty$$

Second class example

$$Q |\psi_{phys}\rangle = 0 , \quad P |\psi_{phys}\rangle = 0 ; \quad [Q, P] |\psi_{phys}\rangle = i\hbar |\psi_{phys}\rangle = 0$$

Eliminate second class constraints classically!

Phase Space Path Integral

$$[\mathcal{Q}, P] = i\hbar I \quad ; \quad \mathcal{Q}|q\rangle = q|q\rangle \quad ; \quad P|p\rangle = p|p\rangle$$

$$I = \int |q\rangle\langle q| dq = \int |p\rangle\langle p| dp \quad ; \quad \langle q|p\rangle = e^{iqp/\hbar} / \sqrt{2\pi\hbar}$$

$$\begin{aligned} \langle q'' | e^{-iT\mathfrak{H}/\hbar} | q' \rangle &= \langle q'' | e^{-i\varepsilon\mathfrak{H}} \cdots e^{-i\varepsilon\mathfrak{H}} e^{-i\varepsilon\mathfrak{H}} | q' \rangle \quad ; \quad [T = (N+1)\varepsilon\hbar] \\ &= \int \prod_{n=0}^N \langle q_{n+1} | e^{-i\varepsilon\mathfrak{H}} | q_n \rangle \prod_{n=1}^N dq_n \quad ; \quad (q'' = q_{N+1} \rightarrow q' = q_0) \\ &= \int \prod_{n=0}^N \langle q_{n+1} | p_{n+1/2} \rangle \langle p_{n+1/2} | e^{-i\varepsilon\mathfrak{H}} | q_n \rangle \prod_{n=0}^N dp_{n+1/2} \prod_{n=1}^N dq_n \\ &= \lim_{N \rightarrow \infty} \int \prod_{n=0}^N \langle q_{n+1} | p_{n+1/2} \rangle \langle p_{n+1/2} | [1 - i\varepsilon\mathfrak{H}] | q_n \rangle \prod_{n=0}^N dp_{n+1/2} \prod_{n=1}^N dq_n \\ &= \lim_{N \rightarrow \infty} \int \exp \left[\sum_{n=0}^N i p_{n+1/2} (q_{n+1} - q_n) / \hbar - i\varepsilon H(p_{n+1/2}, q_n) \right] \\ &\quad \times \prod_{n=0}^N dp_{n+1/2} / 2\pi\hbar \prod_{n=1}^N dq_n \quad ; \quad [H(p, q) = \langle p | \mathfrak{H} | q \rangle / \langle p | q \rangle] \\ \langle q'' | e^{-iT\mathfrak{H}/\hbar} | q' \rangle &= \mathcal{D}\mathbb{C} \int e^{(i/\hbar) \int [pq - H(p, q)] dt} Dp Dq \end{aligned}$$

Constraint Quantization (4)

Faddeev procedure : \mathcal{R} before \mathcal{Q}

$$P = \{P^j\} , q = \{q_j\} ; j \in \{1, \dots, J\} , \alpha \in \{1, \dots, A\}$$

$$\begin{aligned} & \int e^{i \int [pq - H(p,q) - \chi^\alpha \varphi_\alpha(p,q)] dt} Dp Dq D\lambda \\ &= \int e^{i \int [pq - H(p,q)] dt} \delta\{\varphi(p,q)\} Dp Dq \end{aligned}$$

Introduce auxiliary conditions :

$$\chi^\alpha(p,q) = 0 ; \Delta_{FP} \equiv \det\{\chi^\alpha, \varphi_\beta\} \neq 0$$

$$\begin{aligned} & \int e^{i \int [pq - H(p,q)] dt} \delta\{\chi(p,q)\} \det\{\chi^\alpha, \varphi_\beta\} \delta\{\varphi(p,q)\} Dp Dq \\ &= \int e^{i \int [p\dot{q} - H(p,\dot{q})] dt} Dp^* Dq^* \end{aligned}$$

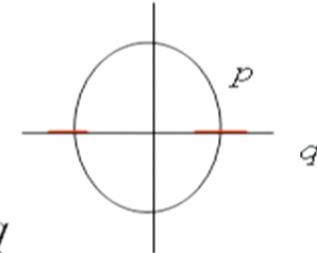
Does $QR = RQ$?

$$A = \int [p\dot{q} - \lambda(p^2 + q^4 - E)]dt$$

What values of E are allowed?

→ $[(P^2 + Q^4) - E] |\psi\rangle = 0 , \quad E \in \{E_n\}$

→ $\int \delta\{P^2 + Q^4 - E\} e^{i\int p\dot{q}dt} DpDq$
 $\int \delta\{P\} \Pi(4q^3) \delta\{P^2 + Q^4 - E\} e^{i\int p\dot{q}dt} DpDq$
 $= \int \Pi(4q^3) \delta\{Q^4 - E\} Dq$
 $= 1 \text{ or } 0 , \quad \text{independent of } E!$



BRST Method

- BRST involves additional variables including anti-commuting Grassmann variables, BRST charge, gauge fixing fermion, etc., all designed to reproduce the Dirac formulation.
- Difficulties when reduced phase space is non-Euclidean and a Gribov problem exists. Nevertheless, useful for gauge invariance in perturbative formulations of first class systems.
- *A different quantization procedure without additional variables that also works for an arbitrary set of constraints is preferred.*

Coherent States

General Heisenberg group coherent states

$$W(p,q) \equiv e^{-iqP/\hbar} e^{ipQ/\hbar} ; [W(p,q)]^+ \equiv e^{-ipQ/\hbar} e^{iqP/\hbar}$$

$$W(p',q') W(p,q) = e^{ip'q/\hbar} W(p' + p, q' + q)$$

$$|p,q\rangle \equiv W(p,q)|\eta\rangle \equiv |p,q;\eta\rangle , \langle \eta|\eta\rangle = 1$$

$$\int |p,q\rangle\langle p,q| d\mu(p,q) = I ; d\mu(p,q) \equiv dp dq / 2\pi\hbar ; \psi(p,q) \equiv \langle p,q|\psi\rangle$$

$$\rightarrow (\psi, \varphi) = \int \psi(p,q)^* \varphi(p,q) d\mu(p,q) = \langle \psi | \varphi \rangle \leftarrow$$

Every $\psi(p,q)$ is a bounded, continuous function

$$\langle p,q|P|\psi\rangle = -i\hbar(\partial/\partial q)\langle p,q|\psi\rangle , \langle p,q|Q|\psi\rangle = [q + i\hbar(\partial/\partial p)]\langle p,q|\psi\rangle$$

$$i\hbar(\partial/\partial t) \psi(p,q) = H(-i\hbar\partial/\partial q, q + i\hbar\partial/\partial p) \psi(p,q)$$

$$\psi(p,q) = \int \eta(x)^* e^{-ipx/\hbar} \psi(x+q) dx$$

$$[2\pi\hbar\eta(0)^*]^{-1} \int \psi(p,q) dp = \psi(q)$$

Reproducing Kernel Hilbert Space

Continuous function of positive type $K(l''; l')$, $l \in \mathcal{L}$

$$\sum_{m,n=1}^{N,N} \bar{a}_m a_n K(l_m; l_n) \geq 0 , \quad N < \infty \iff K(l''; l') = \langle l'' | l' \rangle$$

Elements of a dense set of abstract vectors

$$|\psi\rangle = \sum_{n=1}^N a_n |l_n\rangle , \quad N < \infty ; \quad |\phi\rangle = \sum_{m=1}^M b_m |l_m\rangle , \quad M < \infty$$

Functional representatives

$$\psi(l) \equiv \sum_{n=1}^N a_n \langle l | l_n \rangle = \langle l | \psi \rangle ; \quad \phi(l) \equiv \sum_{m=1}^M b_m \langle l | l_m \rangle = \langle l | \phi \rangle$$

Inner product; complete the space by proper limits

$$(\psi, \phi) \equiv \sum_{m,n=1}^{M,N} \bar{a}_n b_m \langle l_n | l_m \rangle = \langle \psi | \phi \rangle$$

→ All elements determined by $K(l''; l')$ ←

Projection Operator Method (1)

Classical constraints :

$$\phi_\alpha(p, q) = 0 \quad ; \quad \sum_{\alpha=1}^4 \phi_\alpha(p, q)^2 = 0$$

Quantum constraints :

$$\Phi_\alpha(P, Q) \quad ; \quad \sum_{\alpha=1}^4 \Phi_\alpha(P, Q)^2 \equiv \int_0^\infty u \, dE(u)$$

Dirac :

$$\Phi_\alpha(P, Q) |\psi_{phys}\rangle = 0 \quad \Leftrightarrow \quad \sum_{\alpha=1}^4 \Phi_\alpha(P, Q)^2 |\psi_{phys}\rangle = 0$$

Projection operator method :

$$E \equiv E(\sum_\alpha \Phi_\alpha^2 \leq \delta^2(\hbar)) \equiv \int_0^{\delta^2(\hbar)} dE(u) \quad ; \quad \mathcal{H}_{phys} \equiv E\mathcal{H}$$

Projection Operator Method (2)

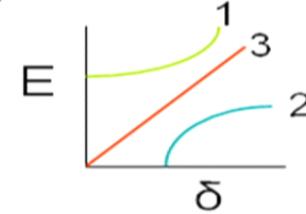
Three basic examples :

1. $\{\Phi_1 = J_1, \Phi_2 = J_2, \Phi_3 = J_3\} ; \quad \sum_{k=1}^3 J_k^2$

$$\mathbf{E} = \mathbf{E}(\sum_{k=1}^3 J_k^2 \leq \hbar^2 / 2) = \mathbf{E}(\sum_{k=1}^3 J_k^2 = 0)$$

2. $\{\Phi_1 = P, \Phi_2 = Q\} ; \quad P^2 + Q^2$

$$\mathbf{E} = \mathbf{E}(P^2 + Q^2 \leq \hbar) = |0\rangle\langle 0|$$



3. $\{\Phi_1 = Q\} ; \quad Q^2$

$$\mathbf{E} = \mathbf{E}(Q^2 \leq \delta^2) = \mathbf{E}(-\delta < Q < \delta)$$

$$\lim_{\delta \rightarrow 0} \langle p', q' | \mathbf{E} | p, q \rangle / \langle \eta | \mathbf{E} | \eta \rangle \equiv \langle\langle p', q' | p, q \rangle\rangle$$

Projection Operator Method (3)

Additional examples :

$$4. \quad \mathbf{E}(Q^2 + Q^2 \leq \delta^2) = \mathbf{E}(Q^2 \leq \delta'^2)$$

$$5. \quad \mathbf{E}(Q^{2\Omega} \leq \delta^2) = \mathbf{E}(Q^2 \leq \delta''^2) ; \quad \Omega > 0$$

$$6. \quad \mathbf{E}(Q^2 + Q^4 \leq \delta^2) = \mathbf{E}(Q^2 \leq \delta'''^2)$$

(all lead to same RK as example 3)

$$7. \quad \mathbf{E}(P^2 + Q^4 \leq c\hbar^{4/3}) = |0'\rangle\langle 0'|$$

$$8. \quad \mathbf{E}(Q^2(1 - Q)^2 \leq \delta^2) ; \quad (\delta^2 \ll 1)$$

$$= \mathbf{E}_0(Q^2 \leq \delta^2) + \mathbf{E}_1((1 - Q)^2 \leq \delta^2)$$

Classical/Quantum Connection

Quantum action : $A_Q = \int \langle \psi(t) | [i\hbar\partial/\partial t - \mathcal{H}] | \psi(t) \rangle dt$

Classical action : $A_C = \int [p(t)\dot{q}(t) - H(p(t), q(t))] dt$

State space : \mathcal{Q} : $|\psi(t)\rangle \in \mathbb{H}$; C : $(p(t), q(t)) \in \mathbb{R}^2$

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Restricted quantum action : $|\psi(t)\rangle \rightarrow |p(t), q(t)\rangle$

$$\begin{aligned} \longrightarrow A_{Q(C)} &= \int \langle p(t), q(t) | [i\hbar\partial/\partial t - \mathcal{H}(P, Q)] | p(t), q(t) \rangle dt \\ &= \int [p(t)\dot{q}(t) - H(p(t), q(t))] dt \end{aligned}$$

Classical theory is a restricted version of quantum theory!

Classical/Quantum Connection

Quantum action : $A_Q = \int \langle \psi(t) | [i\hbar\partial/\partial t - \mathcal{H}] | \psi(t) \rangle dt$

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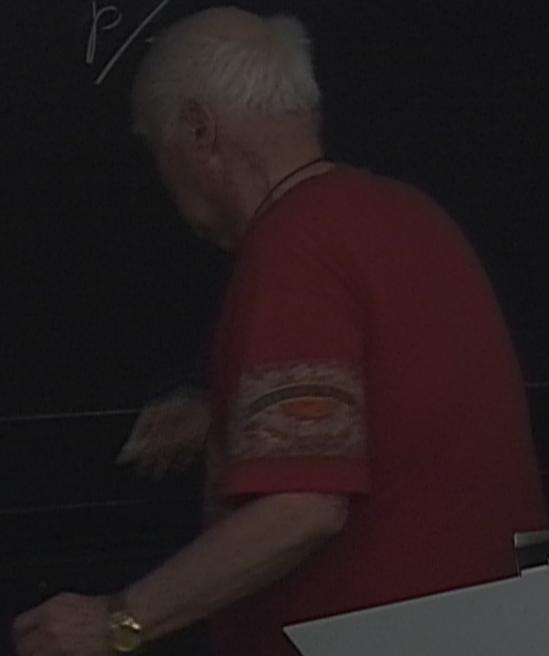
$$\begin{aligned} \longrightarrow A_{Q(C)} &= \int \langle p(t), q(t) | [i\hbar\partial/\partial t - \mathcal{H}(P, Q)] | p(t), q(t) \rangle dt \\ &= \int [p(t)\dot{q}(t) - H(p(t), q(t))] dt \end{aligned}$$

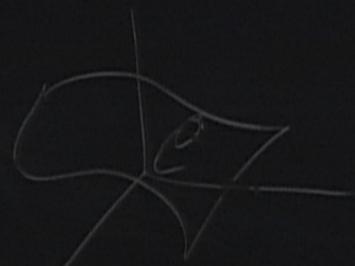
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$$g = \bar{g}^3$$

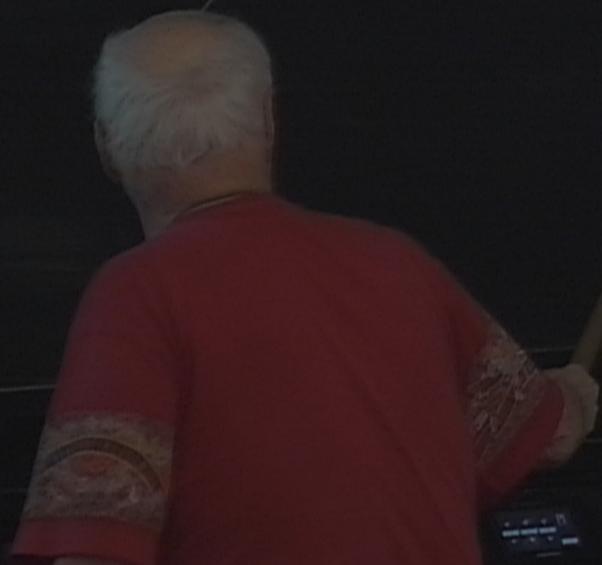
$$p = \bar{p}$$





$$g = \bar{g}^3$$

$$p = \bar{P}/3\bar{q}^2$$



Coherent State Path Integral (2)

$$\begin{aligned}\langle p'', q'' | e^{-iT\mathcal{H}/\hbar} | p', q' \rangle &= \langle p'', q'' | e^{-i\varepsilon\mathcal{H}} e^{-i\varepsilon\mathcal{H}} \dots e^{-i\varepsilon\mathcal{H}} | p', q' \rangle \\&= \int \prod_{n=0}^N \langle p_{n+1}, q_{n+1} | e^{-i\varepsilon\mathcal{H}} | p_n, q_n \rangle \prod_{n=1}^N d\mu(p_n, q_n) \\&= \lim_{N \rightarrow \infty} \int \prod_{n=0}^N \langle p_{n+1}, q_{n+1} | [1 - i\varepsilon\mathcal{H}] | p_n, q_n \rangle \prod_{n=1}^N d\mu(p_n, q_n) \\&= \lim_{N \rightarrow \infty} \int \prod_{n=0}^N \{1 + ii\langle n+1 | (|n+1\rangle - |n\rangle) - i\varepsilon\langle n+1 | \mathcal{H} | n \rangle\} \\&\quad \times \prod_{n=1}^N d\mu(p_n, q_n)\end{aligned}$$

$$\begin{aligned}\langle p'', q'' | e^{-iT\mathcal{H}/\hbar} | p', q' \rangle &= \mathcal{D}\tilde{\zeta} \int e^{(i/\hbar)\int [i\hbar\langle p, q | d/dt | p, q \rangle - \langle p, q | \mathcal{H} | p, q \rangle] dt} Dp Dq \\&= \mathcal{D}\tilde{\zeta} \int e^{(i/\hbar)\int [pq - H(p, q)] dt} Dp Dq\end{aligned}$$

Making the Projection Operator (1)

Constraints form a compact Lie algebra :

$$[\Phi_a, \Phi_b] = i c_{ab}{}^c \Phi_c \quad ; \quad 1 = \int d\mu(g)$$

$$d\mu(g_0 \circ g) = d\mu(g \circ g_0) = d\mu(g^{-1}) = d\mu(g)$$

$$\mathbf{E} = \int e^{ig^a \Phi_a} d\mu(g) = \int U(g) d\mu(g) = \mathbf{E} (\sum_a \Phi_a^2 = 0)$$

$$\mathbf{E} = \mathbf{E}^+ = \mathbf{E}^2 = U(g_0) \mathbf{E}$$

Good, but limited use.

Need a more general construction.

Making the Projection Operator (1)

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Abstract Dynamics

Combination of dynamics (\mathcal{K}) and constraints (\mathbf{E})

Case 1. Hamiltonian is an observable : $[\mathcal{K}, \mathbf{E}] = 0$

Evolution operator : $U_E(t) \equiv \exp(-it\mathcal{K}/\hbar) \mathbf{E}$

Case 2. Hamiltonian is NOT an observable : $[\mathcal{K}, \mathbf{E}] \neq 0$

Evolution operator : $U_{E,N}(t) \equiv \mathbf{E}e^{-i\varepsilon\mathcal{K}}\mathbf{E}e^{-i\varepsilon\mathcal{K}}\dots e^{-i\varepsilon\mathcal{K}}\mathbf{E}$

$U_E(t) \equiv \lim_{N=t/\varepsilon\hbar \rightarrow \infty} \mathbf{E}e^{-i\varepsilon\mathcal{K}}\mathbf{E}e^{-i\varepsilon\mathcal{K}}\dots e^{-i\varepsilon\mathcal{K}}\mathbf{E}$

$U_E(t) = \mathbf{E}e^{-it(\mathbf{E}\mathcal{K}\mathbf{E})/\hbar} \mathbf{E} = \mathbf{E}e^{-it\mathcal{K}^g/\hbar} \mathbf{E}$

If $\mathcal{K} \geq 0$ then $\mathcal{K}^g = \mathbf{E}\mathcal{K}\mathbf{E} \geq 0$; SA extensions exist

Coh. State P. I. & Constraints

$$\begin{aligned} & \langle p'', q'' | \mathbf{E} e^{-iT(\mathbf{E}\mathfrak{K}\mathbf{E})/\hbar} \mathbf{E} | p', q' \rangle \\ &= \lim_{N \rightarrow \infty} \langle p'', q'' | (\mathbf{E} e^{-i\varepsilon\mathfrak{K}} \mathbf{E})(\mathbf{E} e^{-i\varepsilon\mathfrak{K}} \cdots \mathbf{E})(\mathbf{E} e^{-i\varepsilon\mathfrak{K}} \mathbf{E}) | p', q' \rangle \\ &= \lim_{N \rightarrow \infty} \int \prod_{n=0}^N \langle p_{n+1}, q_{n+1} | \mathbf{E} e^{-i\varepsilon\mathfrak{K}} \mathbf{E} | p_n, q_n \rangle \prod_{n=1}^N d\mu(p_n, q_n) \\ &= Y'' Y' \lim_{N \rightarrow \infty} \int \prod_{n=0}^N \langle \langle p_{n+1}, q_{n+1} | e^{-i\varepsilon\mathfrak{K}} | p_n, q_n \rangle \rangle \prod_{n=1}^N d\mu_Y(p_n, q_n) \\ &= Y'' Y' \int e^{(i/\hbar) \int [i\hbar \langle \langle p, q | (d/dt) | p, q \rangle \rangle - \langle \langle p, q | \mathfrak{K} | p, q \rangle \rangle] dt} D\mu_Y(p, q) \end{aligned}$$

Can rewrite the former equation in terms of $|p, q\rangle$

$$Y(p, q) \equiv \langle p, q | \mathbf{E} | p, q \rangle^{1/2} ; \quad d\mu_Y(p, q) \equiv \langle p, q | \mathbf{E} | p, q \rangle dp dq / 2\pi\hbar$$
$$|p, q\rangle \equiv \mathbf{E}|p, q\rangle / \langle p, q | \mathbf{E} | p, q \rangle^{1/2}$$

An Example

$$A = \int [p\dot{q} + r\dot{s} - H(p, r, q, s) - \lambda_1 r - \lambda_2 s] dt$$

$$\mathbf{E} = \mathbf{E}(R^2 + S^2 \leq \hbar) = I_1 \otimes |0_2\rangle\langle 0_2|$$

$$|p, r, q, s\rangle\rangle = |p, q\rangle \otimes |0_2\rangle\langle 0_2| r, s\rangle / |\langle 0_2 | r, s\rangle|$$

$$i\hbar \langle\langle p, r, q, s | (d/dt) | p, r, q, s \rangle\rangle = p\dot{q} + \dot{\alpha}(r, s)$$

$$\langle\langle p, r, q, s | \mathcal{H} | p, r, q, s \rangle\rangle = H(p, 0, q, 0)$$

$$d\mu_Y(p, r, q, s) = |\langle 0_2 | r, s\rangle|^2 dp dq dr ds / (2\pi\hbar)^2$$

An Example (2)

$$\begin{aligned} & \langle p'', r'', q'', s'' | \mathbf{E} e^{-i(\mathbf{E}^\dagger \mathbf{E})T/\hbar} \mathbf{E} | p', r', q', s' \rangle \\ &= M \int e^{(i/\hbar) \int [i\hbar \langle \langle p, r, q, s | (d/dt) | p, r, q, s \rangle \rangle - \langle \langle p, r, q, s | \mathcal{H} | p, r, q, s \rangle \rangle] dt} D\mu_Y(p, r, q, s) \\ &= Y'' Y' \int e^{(i/\hbar) \int [pq + \alpha(r, s) - H(p, 0, q, 0)] dt} \prod_t |\langle \mathbf{0}_2 | r, s \rangle|^2 dp dq dr ds / (2\pi\hbar)^2 \\ &= Z''^* Z' \int e^{(i/\hbar) \int [pq - H(p, 0, q, 0)] dt} \prod_t dp dq / (2\pi\hbar) \\ & Z''^* = \langle r'', s'' | \mathbf{0}_2 \rangle ; \quad Z' = \langle \mathbf{0}_2 | r', s' \rangle ; \quad Y = |Z| \end{aligned}$$

Same result as if we had eliminated r and s classically

Folding property of the propagator still holds

A Second Example

For what values of E can the following system be quantized?

$$A = \int [p\dot{q} - \lambda(p^2 + q^4 - E)]dt$$

$$(P^2 + Q^4 - E)|n\rangle = 0 ; \quad \mathbf{E} = \mathbf{E}(\{P^2 + Q^4 - E\}^2 \leq \delta^2) = |n\rangle\langle n|$$

$$\begin{aligned}\langle p'', q'' | \mathbf{E} | p', q' \rangle &= M \int e^{i \int [p\dot{q} - \lambda(p^2 + q^4 - E)] dt} Dp Dq DR(\lambda) \\ &= \int \prod_{n=0}^N \langle p_{n+1}, q_{n+1} | p_n, q_n \rangle \prod_{n=1}^N d\mu(p_n, q_n) \langle p_0, q_0 | \mathbf{E} | p', q' \rangle d\mu(p_0, q_0) \\ &= \langle p'', q'' | \mathbf{E} | p', q' \rangle = \langle p'', q'' | n \rangle \langle n | p', q' \rangle\end{aligned}$$

Result : A reproducing kernel for a 1-d Hilbert space

Another Coh. St. Path Integral

$$I - i\varepsilon \mathcal{K} = \int [1 - i\varepsilon h(p, q)] |p, q\rangle\langle p, q| d\mu(p, q)$$

$$e^{-i\varepsilon \mathcal{K}} = \int e^{-i\varepsilon h(p, q)} |p, q\rangle\langle p, q| d\mu(p, q) + O(\varepsilon^2)$$

$$\begin{aligned} & \langle p'', q'' | e^{-iT\mathcal{H}} | p', q' \rangle \\ &= \lim_{N \rightarrow \infty} \int \prod_{n=0}^N \langle p_{n+1}, q_{n+1} | p_n, q_n \rangle e^{-i\varepsilon h(p_n, q_n)} \prod_{n=1}^N d\mu(p_n, q_n) \\ &= M \int e^{i \int [pq - h(p, q)] dt} \prod_t dp dq \end{aligned}$$

$$\langle p_{n+1}, q_{n+1} | p_n, q_n \rangle = e^{i(p_{n+1} + p_n)(q_{n+1} - q_n)/2 - [(p_{n+1} - p_n)^2 + (q_{n+1} - q_n)^2]/4}$$

Yet Another Coh. St. Path Int.

Formal phase space path integral

$$M \int e^{i \int [pq - h(p,q)] dt} DpDq$$

Wiener measure regularization

$$\begin{aligned} & \lim_{\nu \rightarrow \infty} N \int e^{i \int [pq - h(p,q)] dt} e^{-(1/2\nu) \int (p^2 + q^2) dt} DpDq \\ &= \lim_{\nu \rightarrow \infty} 2\pi e^{\nu T/2} \int e^{i \int [pdq - h(p,q)dt]} d\mu_W(p, q) \\ &= \langle p'', q'' | e^{-i \mathcal{H}T} | p', q' \rangle \end{aligned}$$

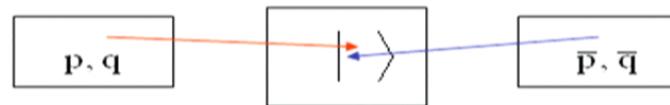
Automatically leads to coherent state representation

Yet Another Coh.St.Path Int. (2)

Canonical coordinate trasformation

$$pdq = \bar{p}d\bar{q} + d\bar{G}(\bar{p}, \bar{q}) ; h(p, q) = \bar{h}(\bar{p}, \bar{q})$$

$$|p, q\rangle = |\bar{p}, \bar{q}\rangle$$



Wiener measure regularization (still 2-d flat space!)

$$\begin{aligned} \langle p'', q'' | e^{-i\Omega T} | p', q' \rangle &= \lim_{\nu \rightarrow \infty} 2\pi e^{\nu T/2} \int e^{i \int [p dq - h(p, q) dt]} d\mu_W(p, q) \\ &= \lim_{\nu \rightarrow \infty} 2\pi e^{\nu T/2} \int e^{i \int [\bar{p} d\bar{q} + d\bar{G}(\bar{p}, \bar{q}) - \bar{h}(\bar{p}, \bar{q}) dt]} d\bar{\mu}_W(\bar{p}, \bar{q}) \\ &= \langle \bar{p}'', \bar{q}'' | e^{-i\Omega T} | \bar{p}', \bar{q}' \rangle \end{aligned}$$

Covariant under canonical coordinate transformations

Yet Another Coh.St.Path Int. (3)

Constraints plus Weiner measure regularization

$$\mathbf{E} = \mathbf{E}(\sum \Phi_\alpha^2 \leq \delta(\hbar)^2) = \int \mathbf{T} e^{-i \int \lambda(t) \Phi_\alpha dt} DR(\lambda)$$

Wiener measure regularization

$$\begin{aligned} & \langle p'', q'' | \mathbf{E} e^{-i(\mathbf{E}^\dagger \mathbf{E})^T} \mathbf{E} | p', q' \rangle \\ &= \lim_{\nu \rightarrow \infty} M \int e^{i \int [pq - h(p,q) - \lambda \phi_a(p,q)] dt} e^{-(1/2\nu) \int (\dot{p}^2 + \dot{q}^2) dt} Dp Dq DR(\lambda) \\ &= \lim_{\nu \rightarrow \infty} 2\pi e^{\nu T/2} \int e^{i \int [p dq - h(p,q) dt - \lambda \phi_a(p,q) dt]} d\mu_W(p, q) DR(\lambda) \end{aligned}$$

Summary

- Classical constraints have various forms that lead to a reduction of phase space
- Canonical quantization requires Cartesian coordinates; may conflict with reduction
- Projection operator method quantizes first and works for all kinds of constraints
- Universal construction of projection operator
- Coherent state path integral representations accommodate the projection operator