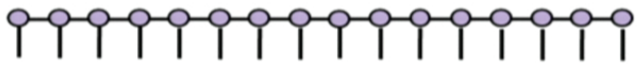
Title: Pedagogical Introduction to Tensor Networks: MPS, PEPS and MERA

Date: Oct 24, 2011 09:00 AM

URL: http://pirsa.org/11100075

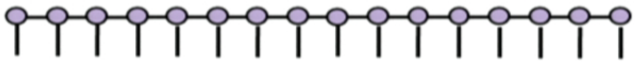
Abstract: This introductory talk aims to answer a few basic questions (What is a tensor network? Under which circumstance is a tensor network useful?) and describe the tensor network states that will be discussed during the workshop (matrix product state [MPS], projected entangled pair states [PEPS], and the multi-scale entanglement renormalization ansatz [MERA]). I will then briefly describe the recent developments that motivated this workshop on

Scientific Program



Monday **MPS/PEPS**

Scientific Program



Monday **MPS/PEPS**

morning

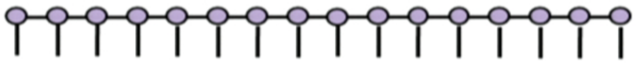
Introduction

lattice

applications:

- fermions (Corboz)
- phases of matter (Gu)

Scientific Program



Monday **MPS/PEPS**

morning

Introduction

lattice

applications:

- fermions (Corboz)
- phases of matter (Gu)

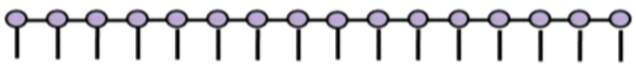
afternoon

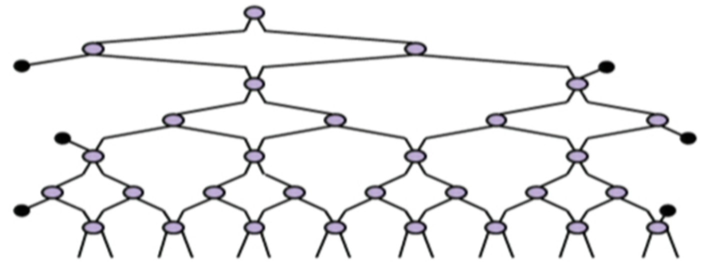
continuum

- non-relativistic (Verstraete)
- relativistic (Haegeman)

Discussion

Scientific Program





Monday **MPS/PEPS**

Tuesday **MERA**

morning

Introduction

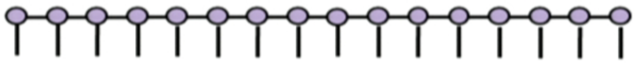
lattice
applications:
• fermions (Corboz)
• phases of matter (Gu)

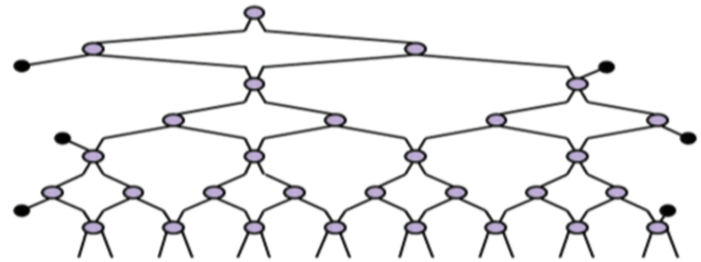
afternoon

continuum
• non-relativistic (Verstraete)
• relativistic (Haegeman)

Discussion

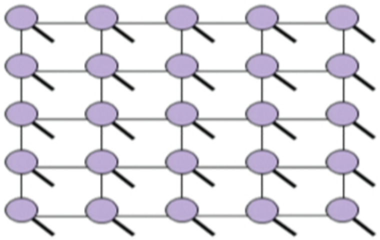
Scientific Program



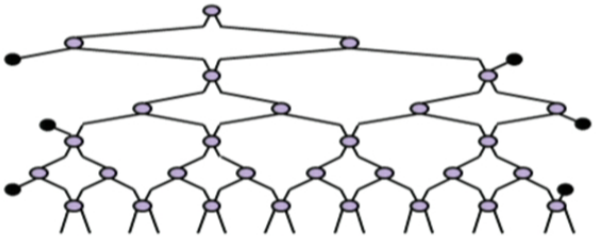


	Monday **MPS/PEPS**	Tuesday **MERA**
morning	Introduction	Introduction
	lattice applications: • fermions (Corboz) • phases of matter (Gu)	**lattice** application: • criticality (Evenbly) • holographic duals (Lee)
afternoon	**continuum** • non-relativistic (Verstraete) • relativistic (Haegeman)	**continuum** • non-relativistic (Osborne) • relativistic (Haegeman)
	Discussion	

Outline



- Diagrammatic notation
- What is a tensor network?
- Examples of tensor networks:
  MPS, PEPS, MERA
- What makes a tensor network useful? (MPS)
- Correlations and entanglement entropy in MPS

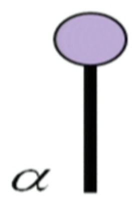


Graphical representation of matrices/tensors

$$A = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} \qquad B = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{pmatrix}$$

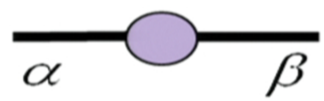
- vector

$$a_\alpha$$



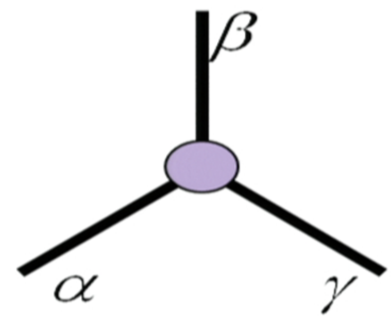
- matrix

$$b_{\alpha\beta}$$



- rank 3 tensor

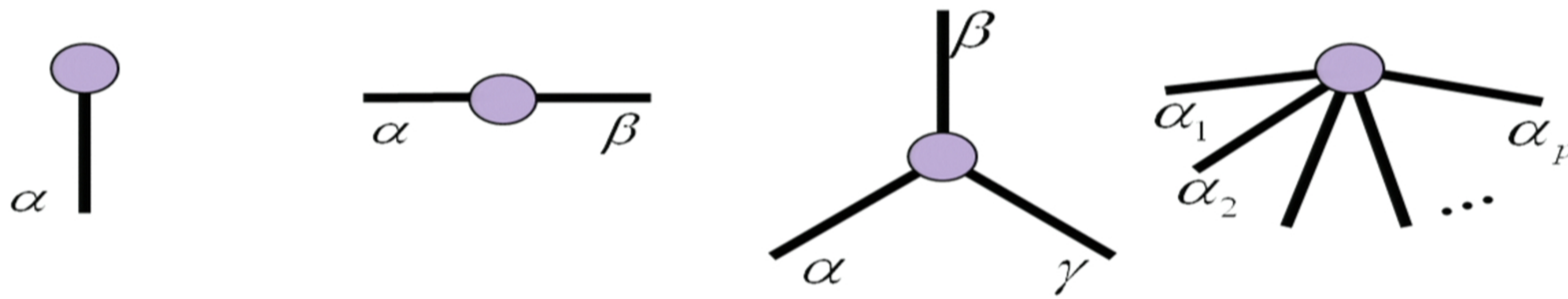
$$c_{\alpha\beta\gamma}$$



Graphical representation of matrices/tensors

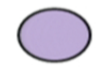
$$A = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} \qquad B = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{pmatrix}$$

- vector a_α
- matrix $b_{\alpha\beta}$
- rank 3 tensor $c_{\alpha\beta\gamma}$
- rank p tensor $t_{\alpha_1\alpha_2\cdots\alpha_p}$



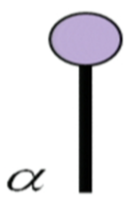
Graphical representation of matrices/tensors

$$A = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} \qquad B = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{pmatrix}$$

 ?

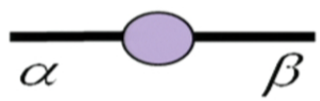
- vector

$$a_\alpha$$



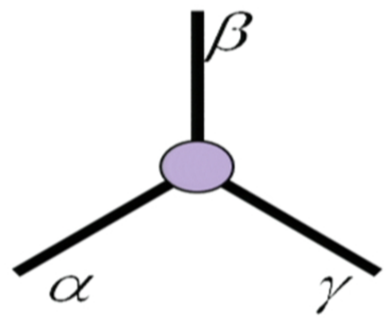
- matrix

$$b_{\alpha\beta}$$



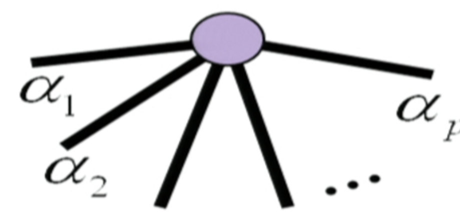
- rank 3 tensor

$$c_{\alpha\beta\gamma}$$



- rank p tensor

$$t_{\alpha_1\alpha_2\cdots\alpha_p}$$

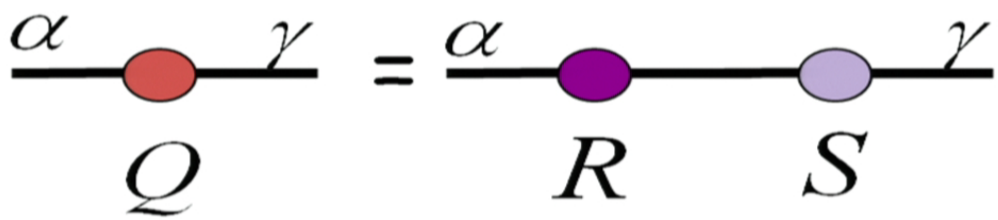


Graphical representation of matrices/tensors

- product of tensors (matrices)

$$Q = RS$$

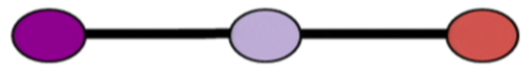
$$q_{\alpha\gamma} = \sum_{\beta} r_{\alpha\beta} s_{\beta\gamma}$$



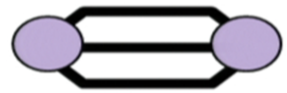
Graphical representation of matrices/tensors

- other examples:

$$x^\dagger A y$$



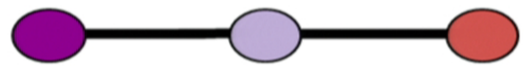
$$\sum_{\alpha\beta\gamma} c_{\alpha\beta\gamma} d_{\alpha\beta\gamma}$$



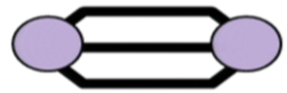
Graphical representation of matrices/tensors

- other examples:

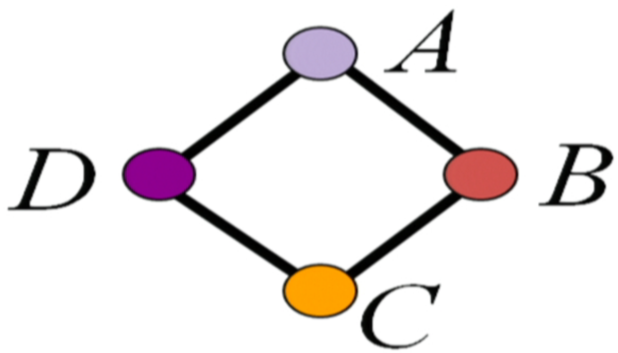
$$x^\dagger A y$$

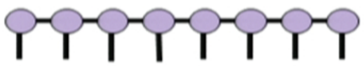


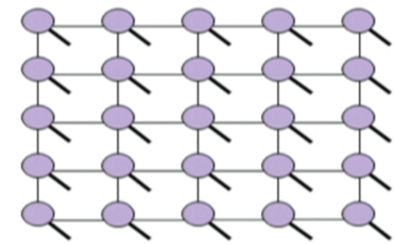
$$\sum_{\alpha\beta\gamma} c_{\alpha\beta\gamma} d_{\alpha\beta\gamma}$$



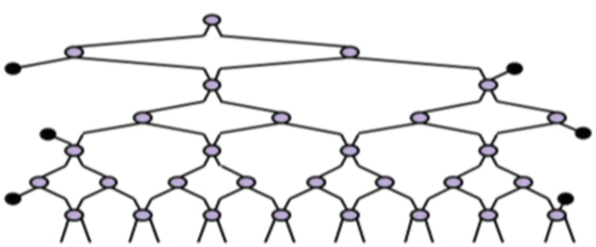
?



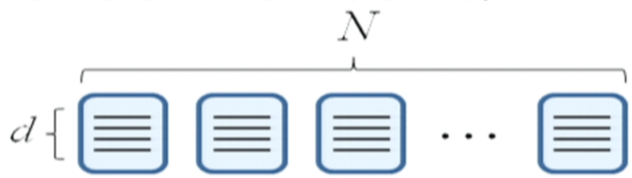




- Diagrammatic notation
- What is a tensor network?
- Examples of tensor networks:
  MPS, PEPS, MERA
- What makes a tensor network useful? (MPS)
- Correlations and entanglement entropy in MPS

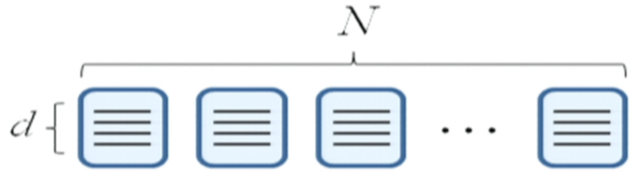


What is a tensor network?



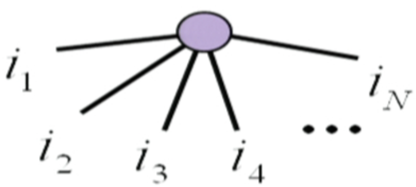
$$\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots \otimes \mathcal{H}_N$$

What is a tensor network?



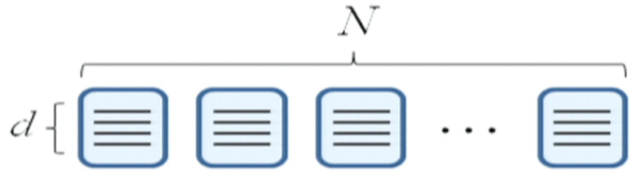
$$\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots \otimes \mathcal{H}_N$$

$$|\Psi\rangle = \sum_{i_1=1}^{d} \sum_{i_2=1}^{d} \cdots \sum_{i_N=1}^{d} c_{i_1 i_2 \ldots i_N} |i_1\rangle |i_2\rangle \ldots |i_N\rangle$$



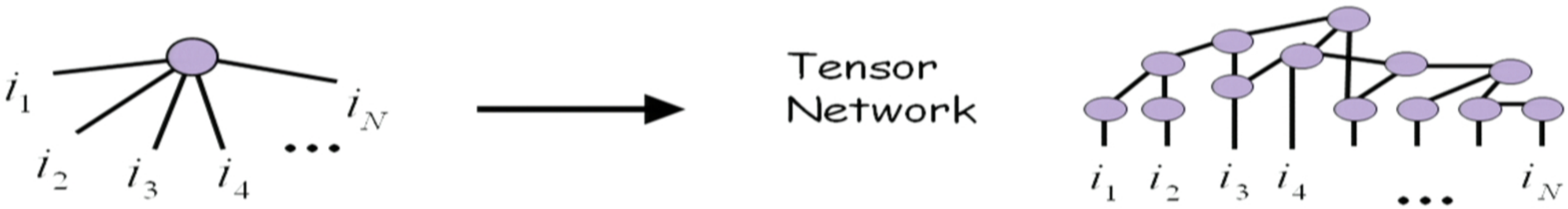
d^N coefficients

What is a tensor network?



$$\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots \otimes \mathcal{H}_N$$

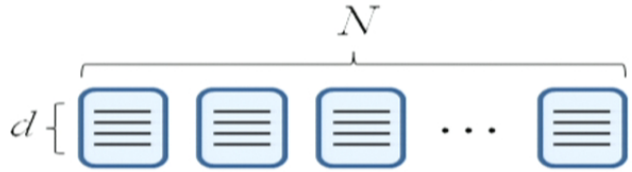
$$|\Psi\rangle = \sum_{i_1=1}^{d} \sum_{i_2=1}^{d} \cdots \sum_{i_N=1}^{d} c_{i_1 i_2 \ldots i_N} |i_1\rangle |i_2\rangle \ldots |i_N\rangle$$



d^N coefficients

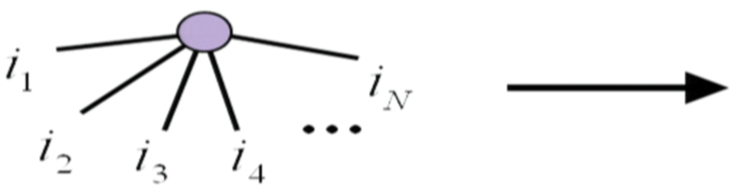
$O(N)$ coefficients

What is a tensor network?

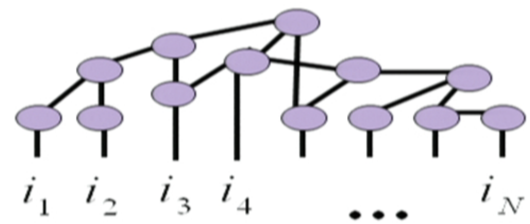


$\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots \otimes \mathcal{H}_N$

$$|\Psi\rangle = \sum_{i_1=1}^{d} \sum_{i_2=1}^{d} \cdots \sum_{i_N=1}^{d} c_{i_1 i_2 \ldots i_N} |i_1\rangle |i_2\rangle \ldots |i_N\rangle$$

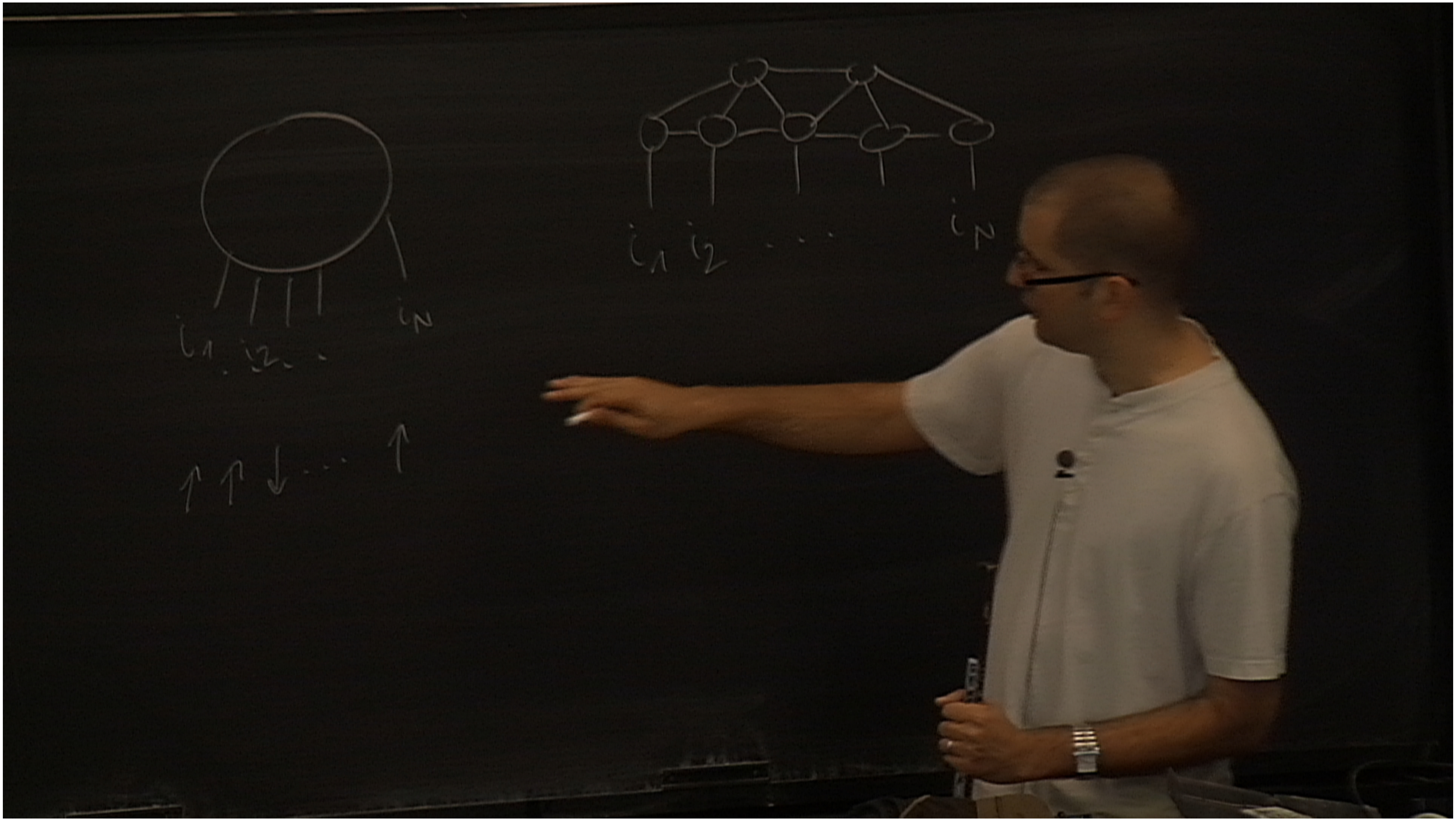


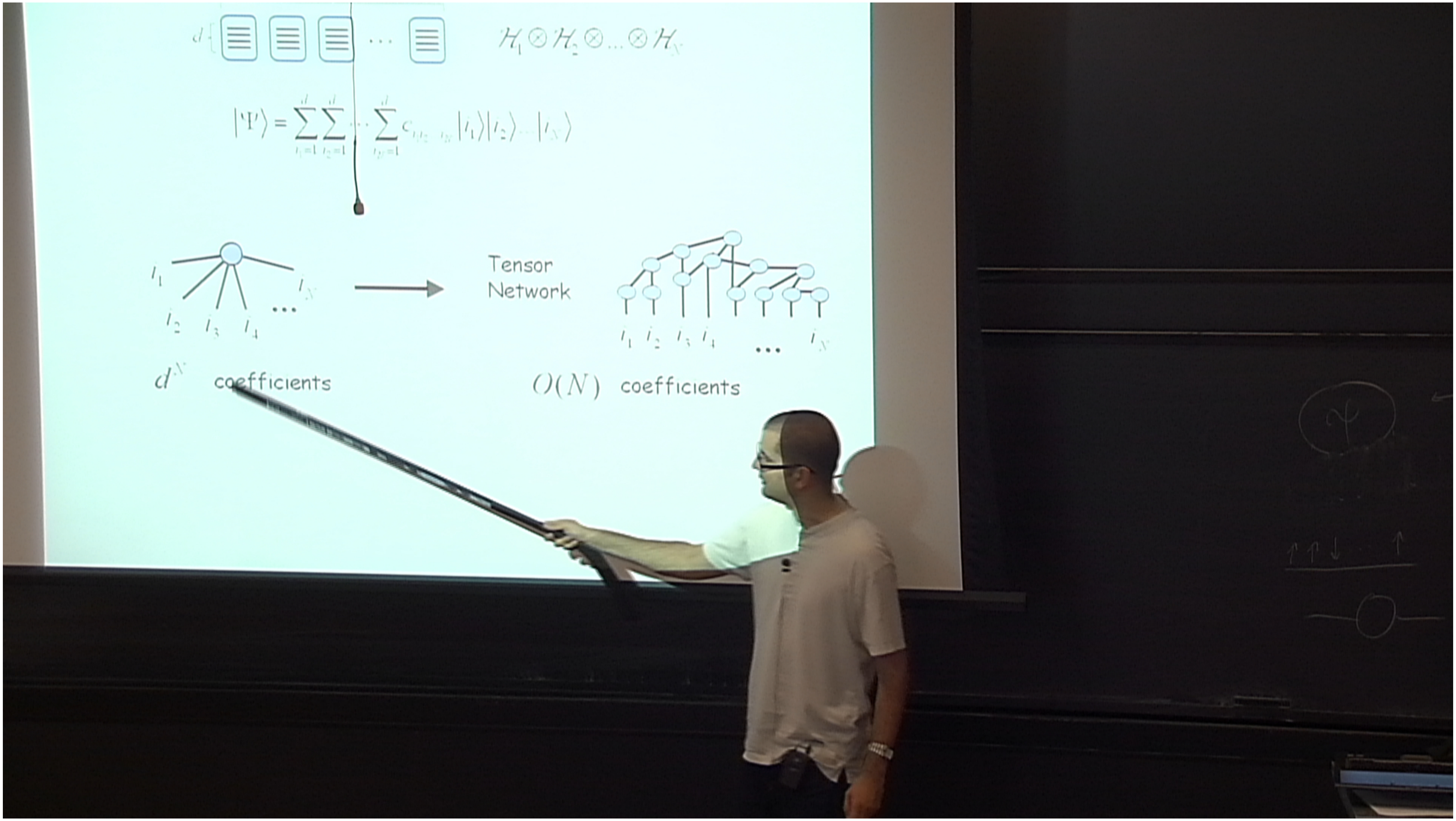
Tensor Network



d^N coefficients

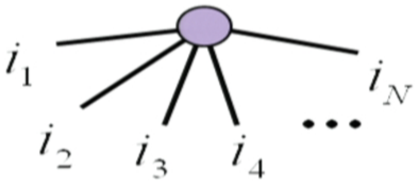
$O(N)$ coefficients



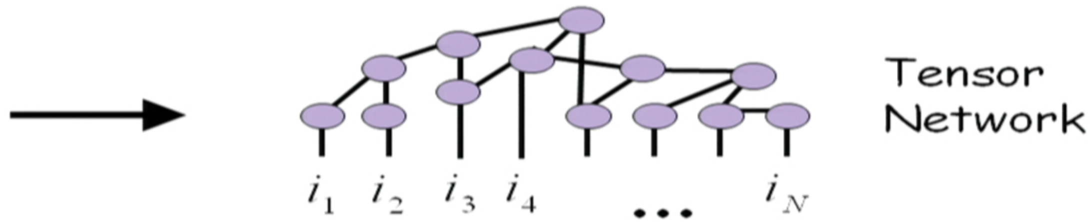

Tensor
Network
coefficients
coefficients

Tensor Networks

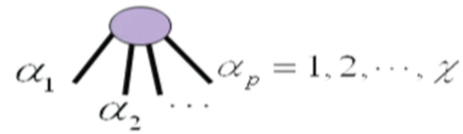
$$|\Psi\rangle = \sum_{i_1=1}^{d}\sum_{i_2=1}^{d}\cdots\sum_{i_N=1}^{d} c_{i_1 i_2 \ldots i_N} |i_1\rangle|i_2\rangle\ldots|i_N\rangle$$

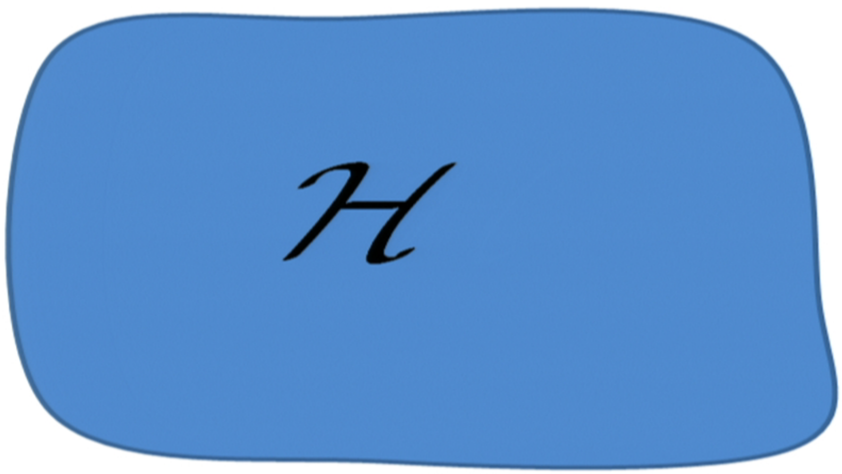


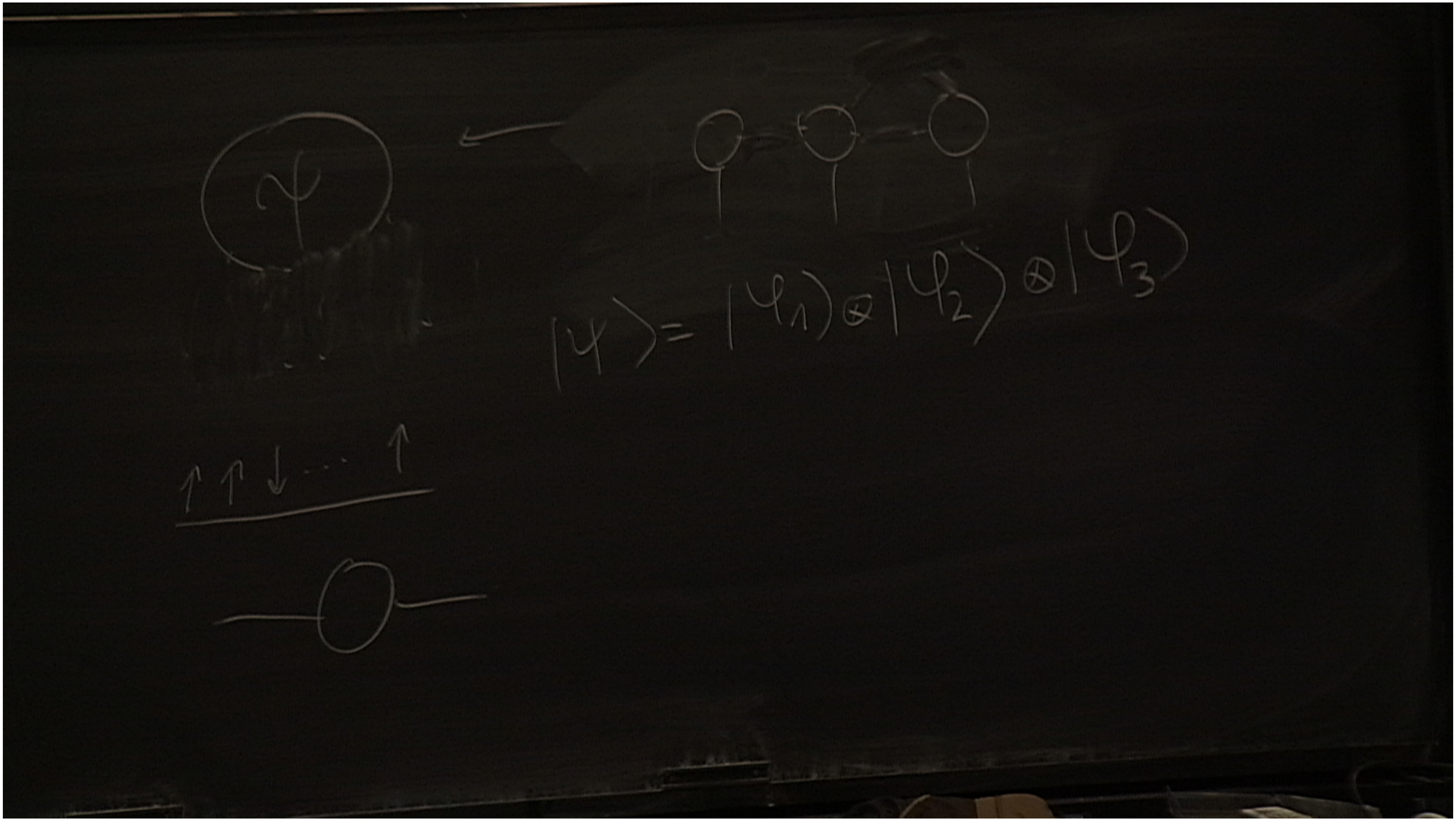
d^N coefficients



$O(N\chi^p)$ coefficients

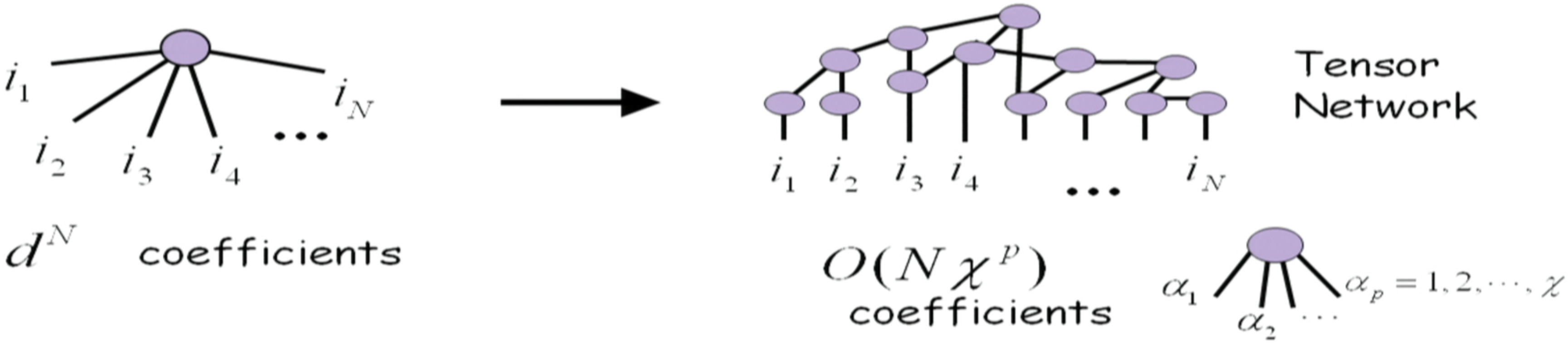


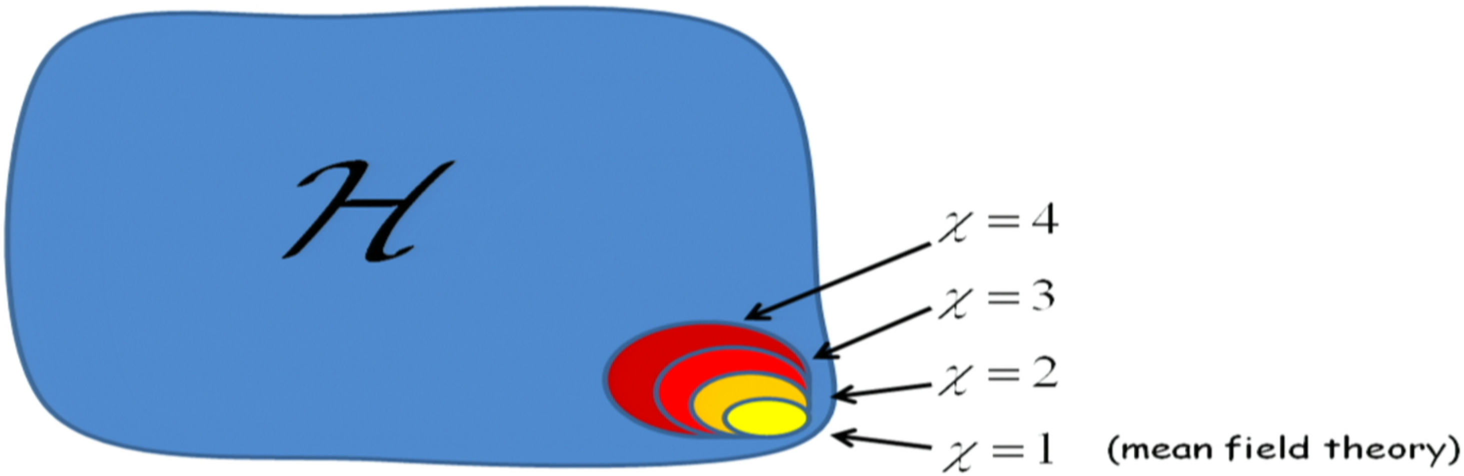



|ψ⟩ = |φ1⟩ ⊗ |φ2⟩ ⊗ |φ3⟩
↑ ↑ ↓ . . . ↑

Tensor Networks

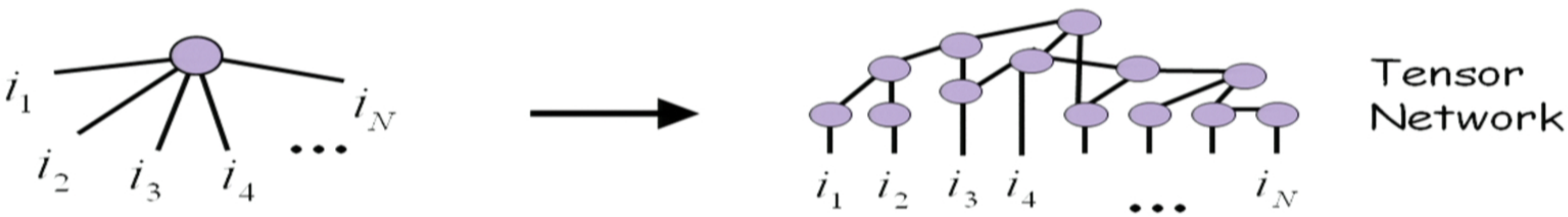
$$|\Psi\rangle = \sum_{i_1=1}^{d}\sum_{i_2=1}^{d}\cdots\sum_{i_N=1}^{d} c_{i_1 i_2 \dots i_N} |i_1\rangle |i_2\rangle \dots |i_N\rangle$$





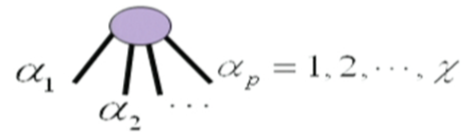
Tensor Networks

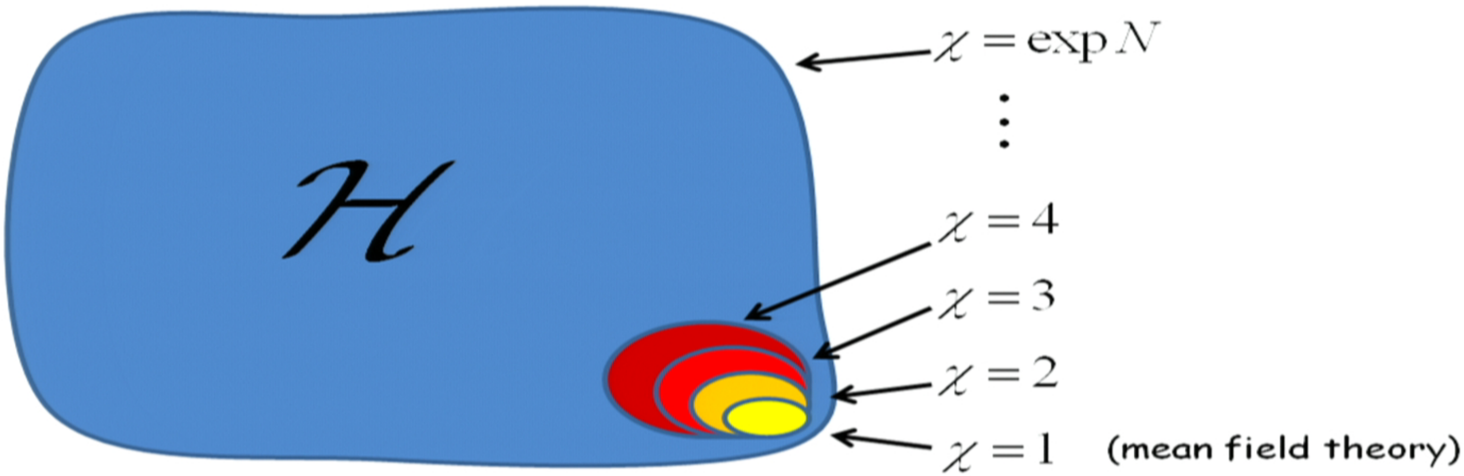
$$|\Psi\rangle = \sum_{i_1=1}^{d}\sum_{i_2=1}^{d}\cdots\sum_{i_N=1}^{d} c_{i_1 i_2 \ldots i_N} |i_1\rangle|i_2\rangle\ldots|i_N\rangle$$



d^N coefficients

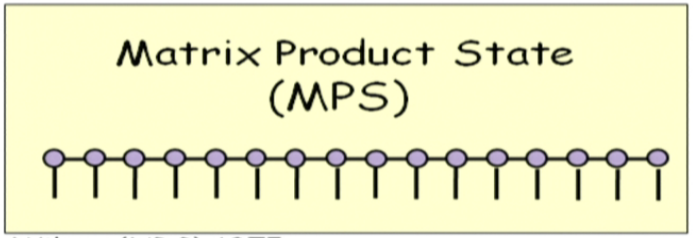
$O(N\chi^p)$ coefficients





Examples of tensor network states:

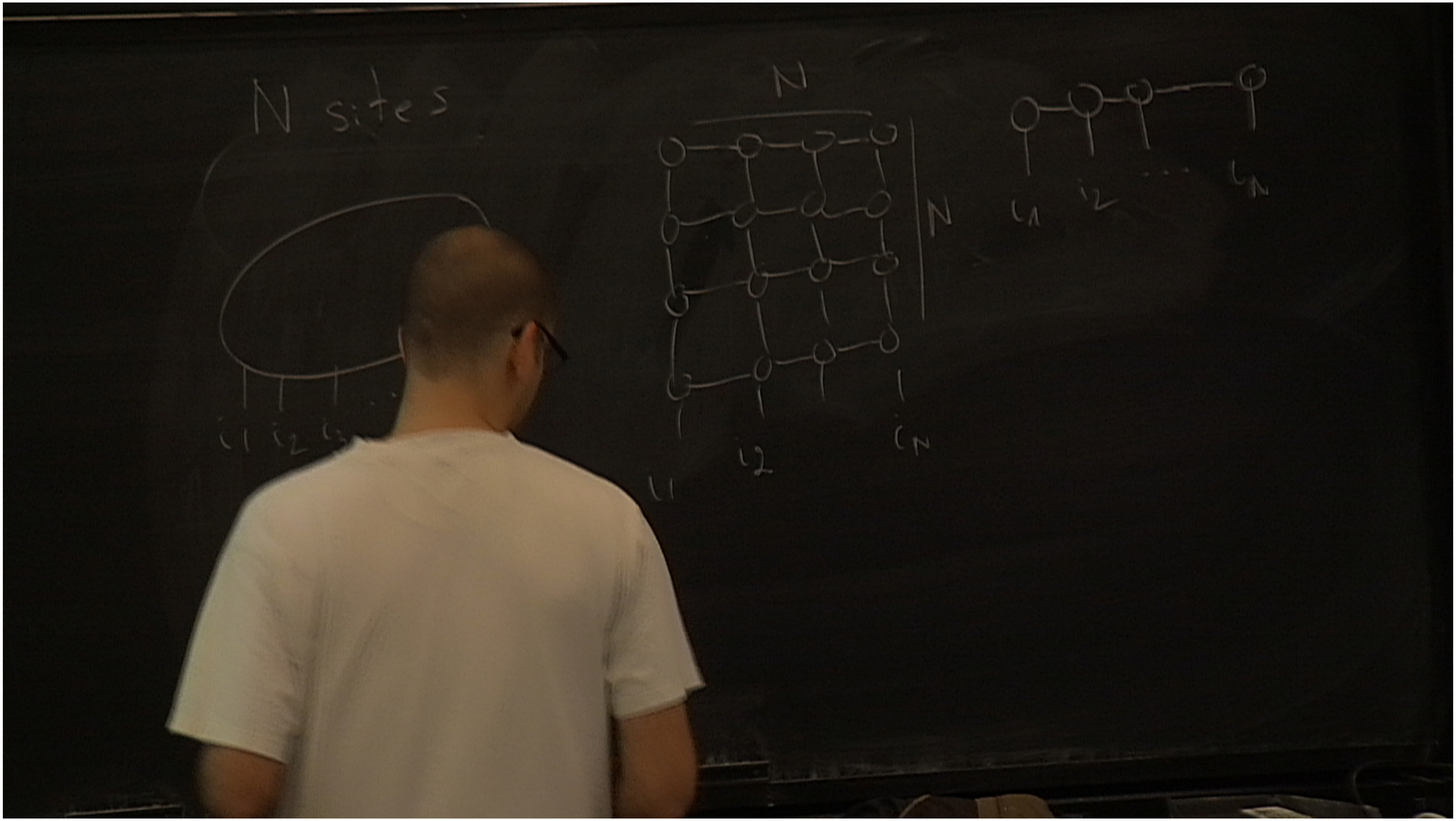
D=1 dimensions

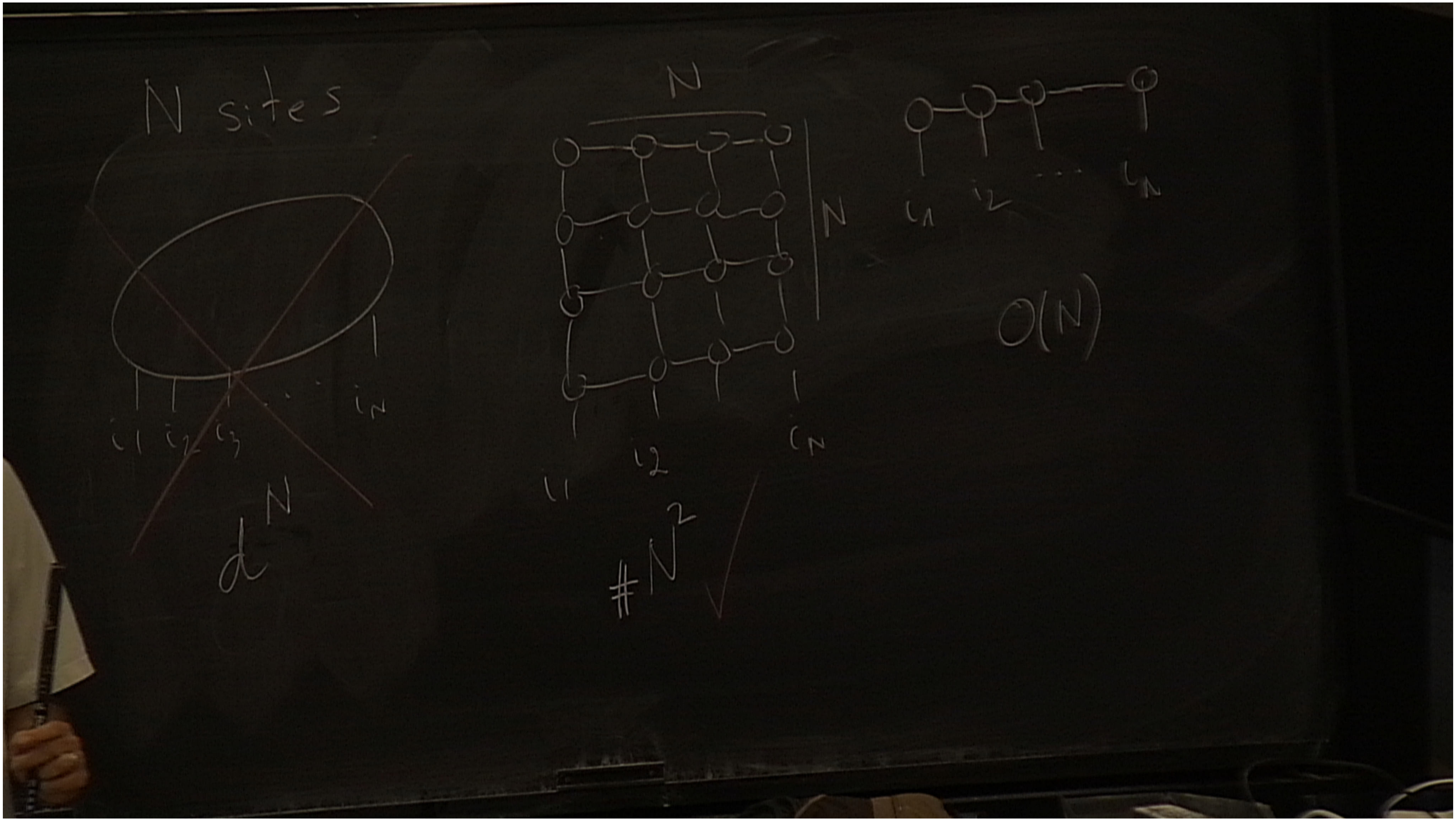


Wilson (NRG) 1975
Fannes, Nachtergaele, Werner 1992
White 1992 (DMRG)
Oestlund, Rommer 1995
time evolution in 1D - 2003
...

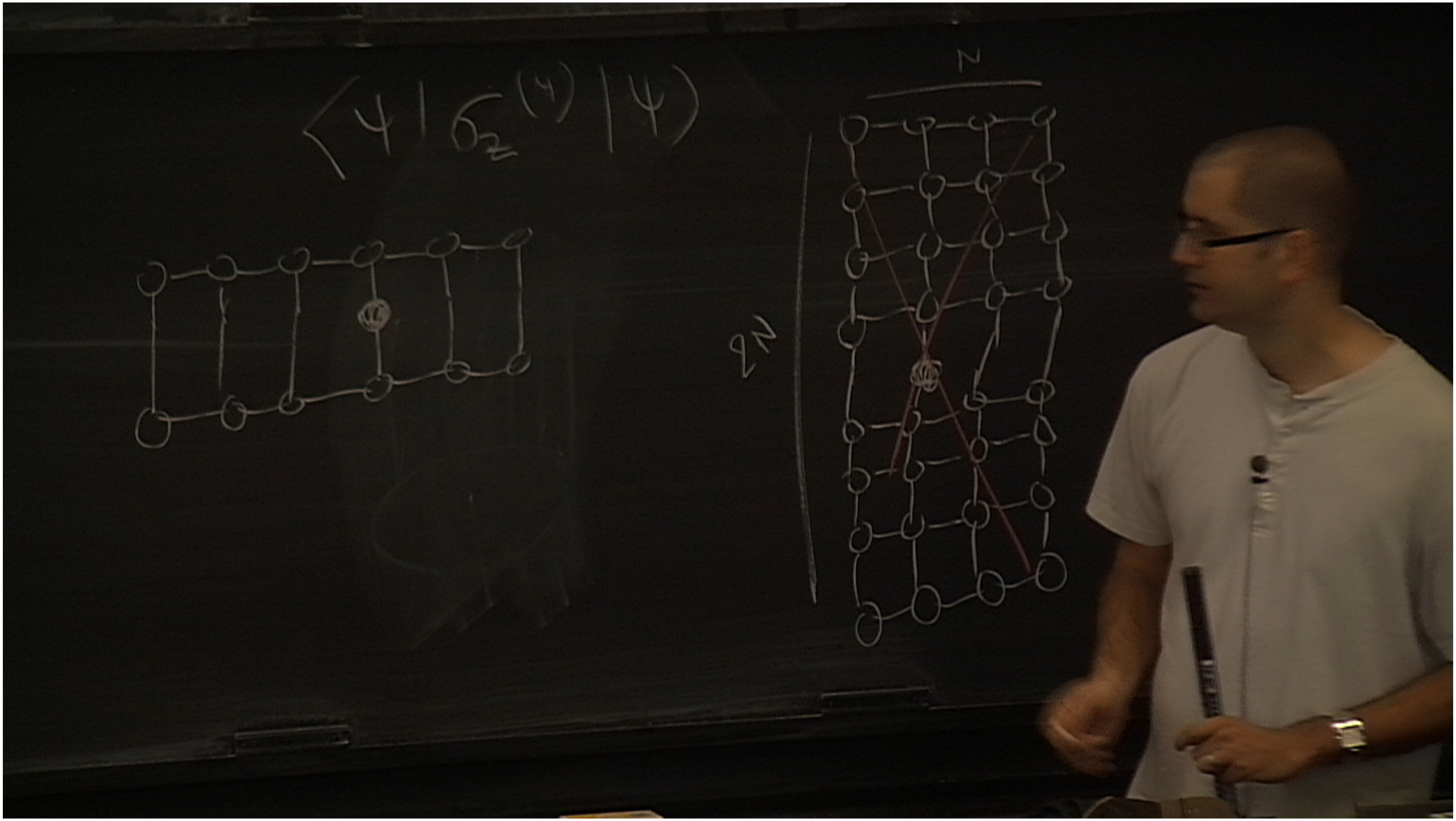
What makes a tensor network useful? (MPS)

- Efficiency (computational cost)

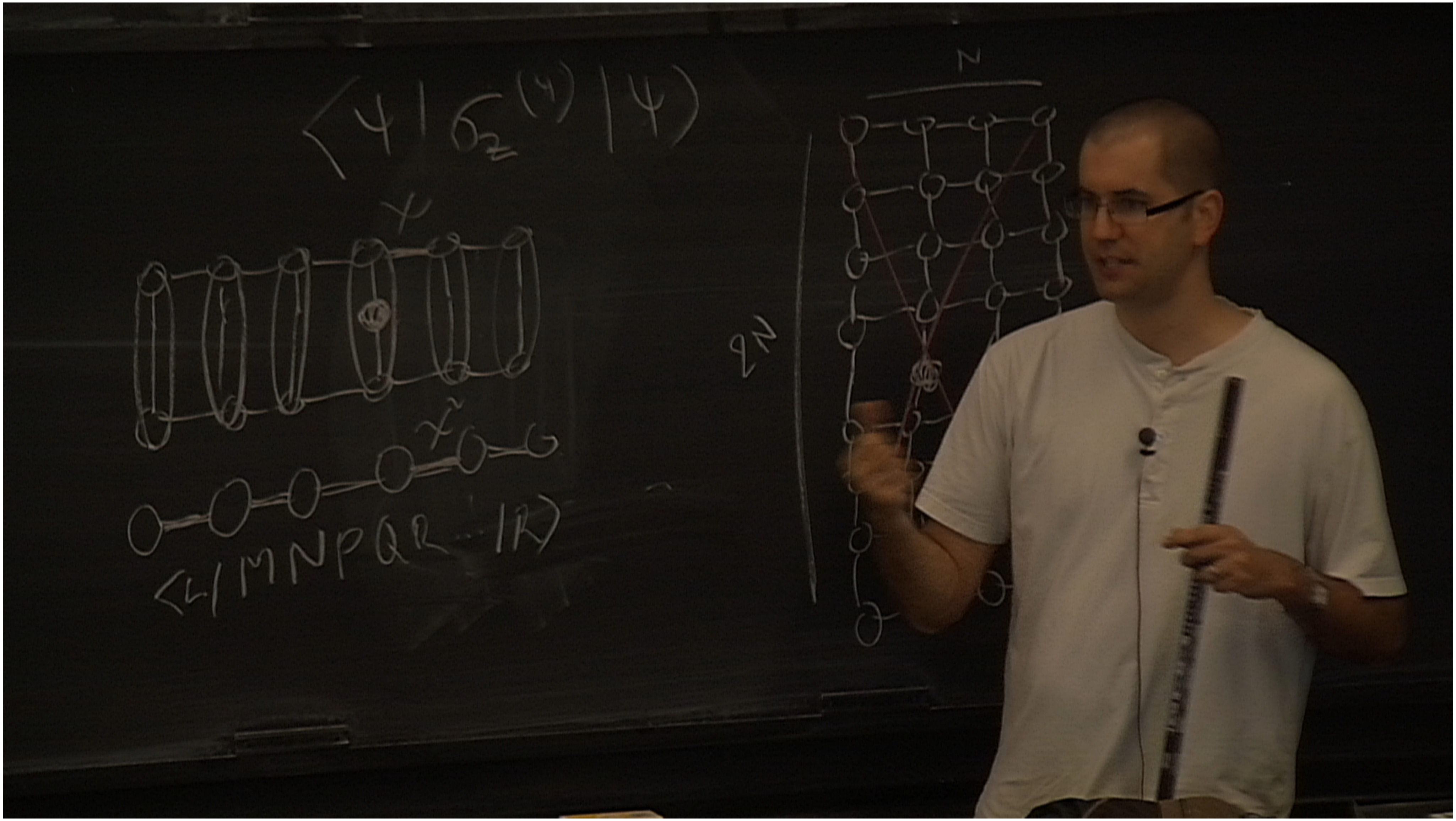

N sites
N
N

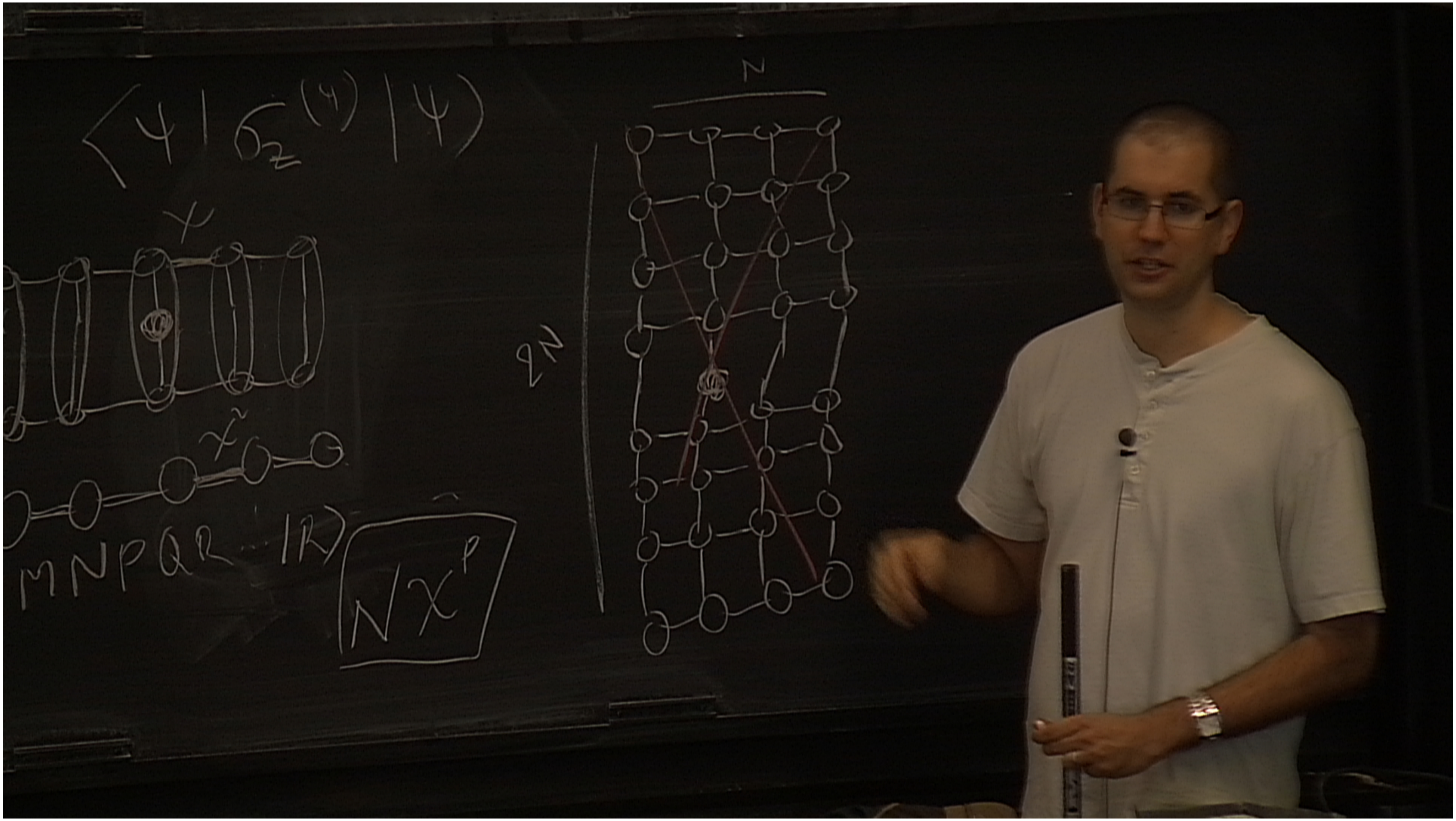

N sites
i_1 i_2 i_3 . . . i_N
d^N
N
N
i_1 i_2 i_N
$\# N^2$ ✓
i_1 i_2 . . . i_N
O(N)

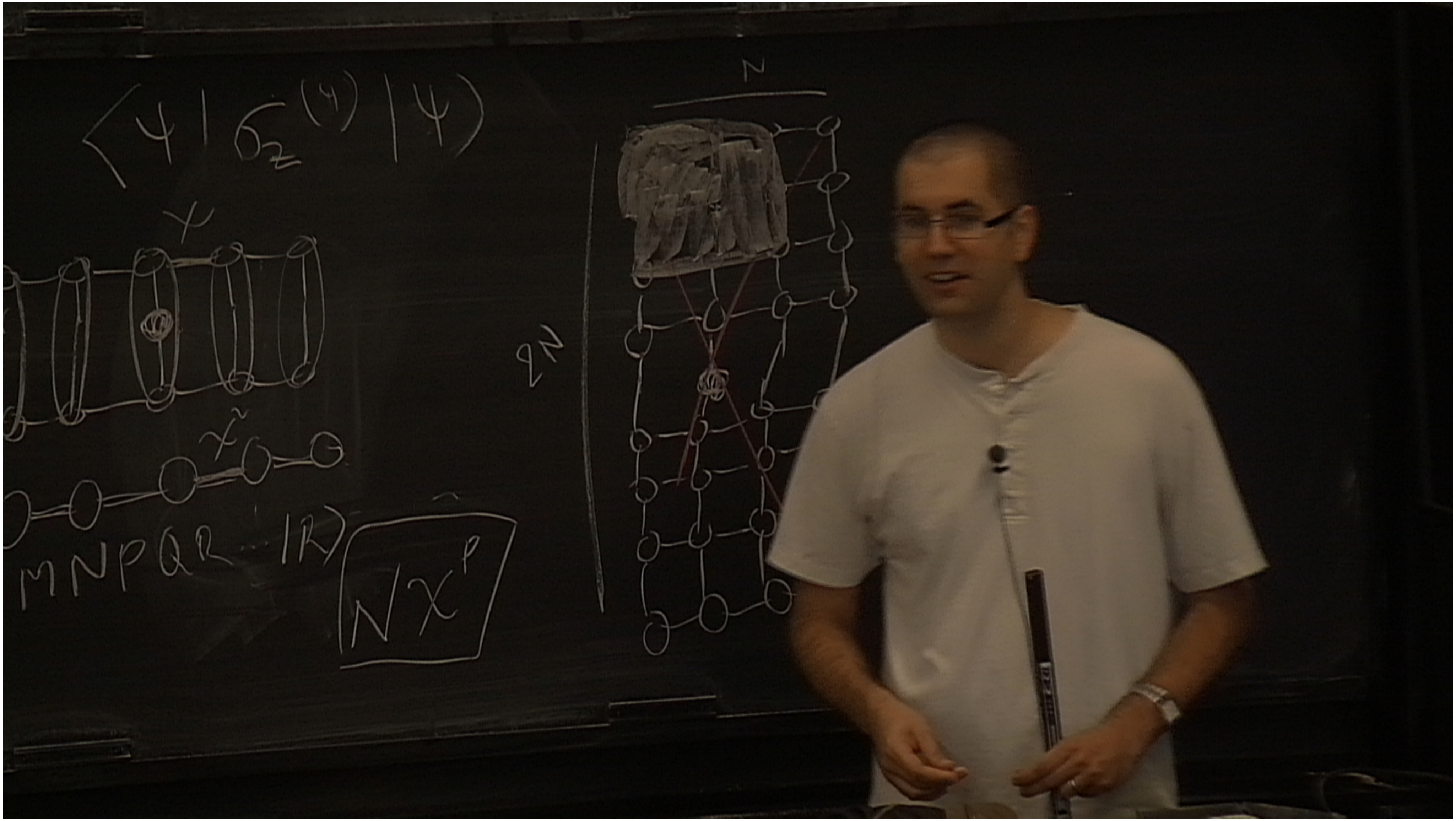

⟨ψ| σ_z^(4) |ψ⟩
ψ
ψ



⟨ψ| σ_z^(4) |ψ⟩
N
2N


N
2N
<L|MNPQR…|R>




N
2N

What makes a tensor network useful? (MPS)

- Efficiency (computational cost)

  1) Efficient representation of a many-body wavefunction

  2) Computation of expectation values

- Accurate approximation of many-body states (e.g. ground states)

What makes a tensor network useful? (MPS)

- Efficiency (computational cost)

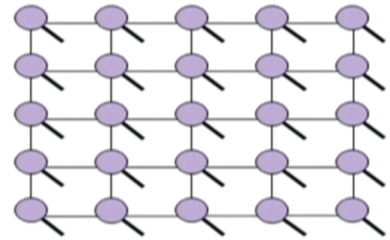
  1) Efficient representation of a many-body wavefunction

  2) Computation of expectation values

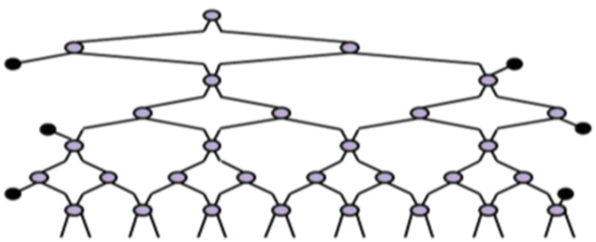
- Accurate approximation of many-body states (e.g. ground states)

  1) Entanglement entropy?

  2) Correlations?



- Diagrammatic notation
- What is a tensor networks
- Examples of tensor networks:
  MPS, PEPS, MERA
- What makes a tensor network useful? (MPS)
- Correlations and entanglement entropy in MPS



Correlations and entanglement entropy in MPS

Correlations and entanglement entropy in MPS

- Gapped systems in D=1 dimensions

local
Hamiltonian

$$H$$

with
energy gap

$$E_1 - E_0 > 0$$

Correlations and entanglement entropy in MPS

- Gapped systems in D=1 dimensions

local
Hamiltonian

H

with
energy gap

$E_1 - E_0 > 0$

⇒

ground
state

$|\Psi\rangle$

Correlations and entanglement entropy in MPS

- Gapped systems in D=1 dimensions

local Hamiltonian

H

with energy gap

$E_1 - E_0 > 0$

→

ground state

$\left|\Psi\right\rangle$

1) Exponential decay of correlations

$$c(x_1, x_2) \equiv \left\langle\Psi\right| A(x_1) B(x_2) \left|\Psi\right\rangle$$

Correlations and entanglement entropy in MPS

- Gapped systems in D=1 dimensions

local Hamiltonian

$$H$$

with energy gap

$$E_1 - E_0 > 0$$

ground state

$$|\Psi\rangle$$

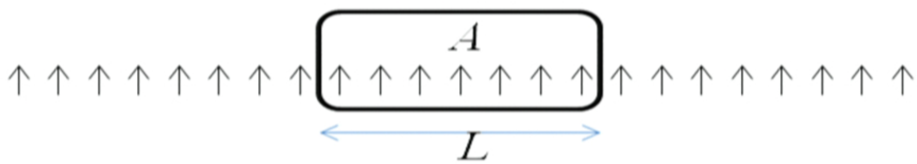
1) Exponential decay of correlations

$$c(x_1, x_2) \equiv \langle \Psi | A(x_1) B(x_2) | \Psi \rangle$$

$$c(x_1, x_2) \approx \exp(-|x_1 - x_2| / \xi)$$

2) Saturation of entanglement entropy

$$\rho_A \equiv \mathrm{tr}_{\bar{A}} |\Psi\rangle\langle\Psi|$$



Correlations and entanglement entropy in MPS

- Gapped systems in D=1 dimensions

local Hamiltonian

H

with energy gap

$E_1 - E_0 > 0$

ground state

$|\Psi\rangle$

1) Exponential decay of correlations

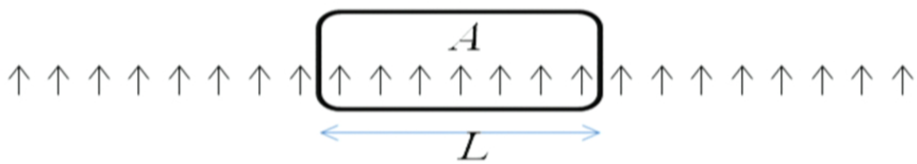
$$c(x_1, x_2) \equiv \langle\Psi| A(x_1) B(x_2) |\Psi\rangle$$

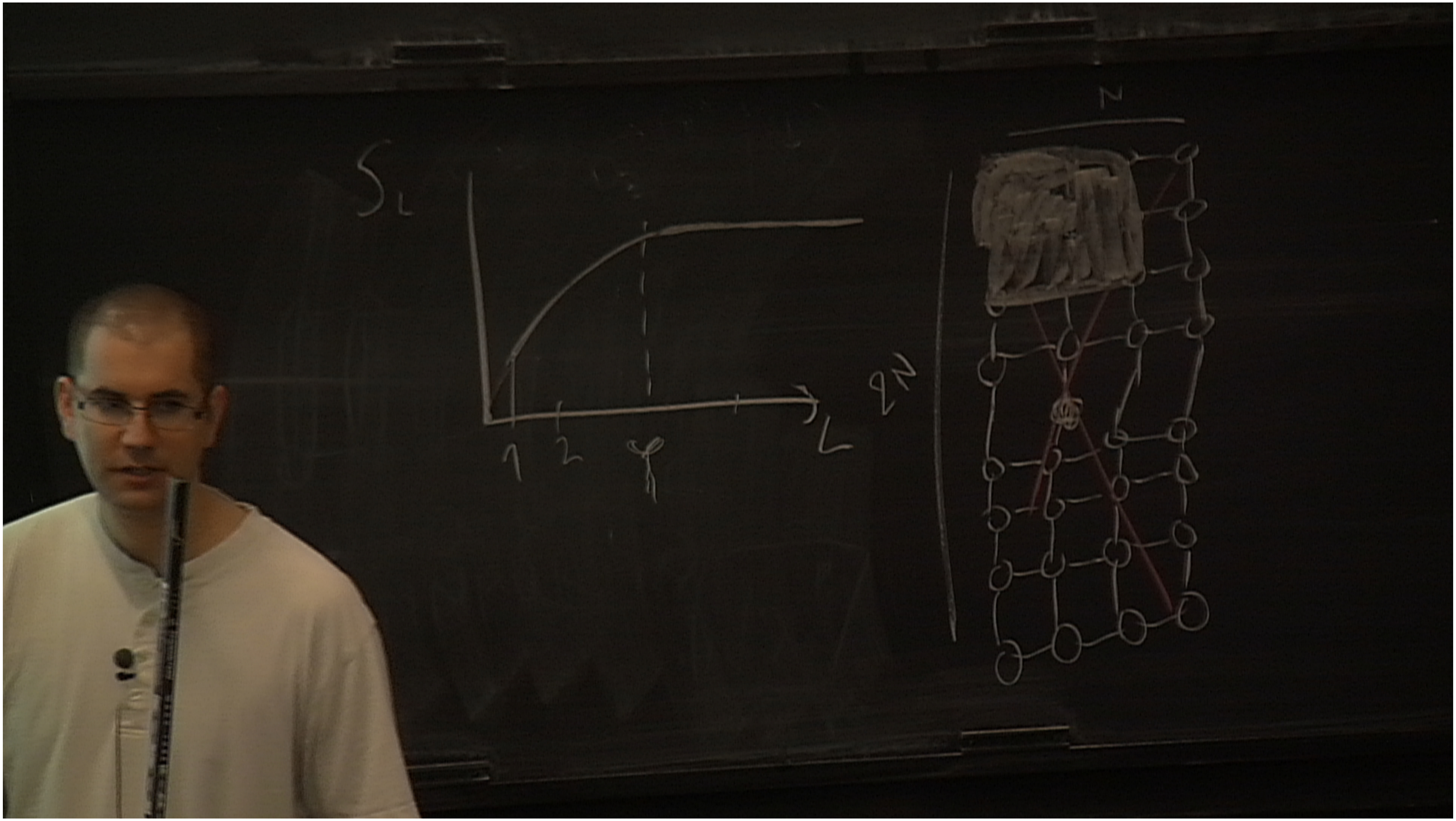
$$c(x_1, x_2) \approx \exp(-|x_1 - x_2| / \xi)$$

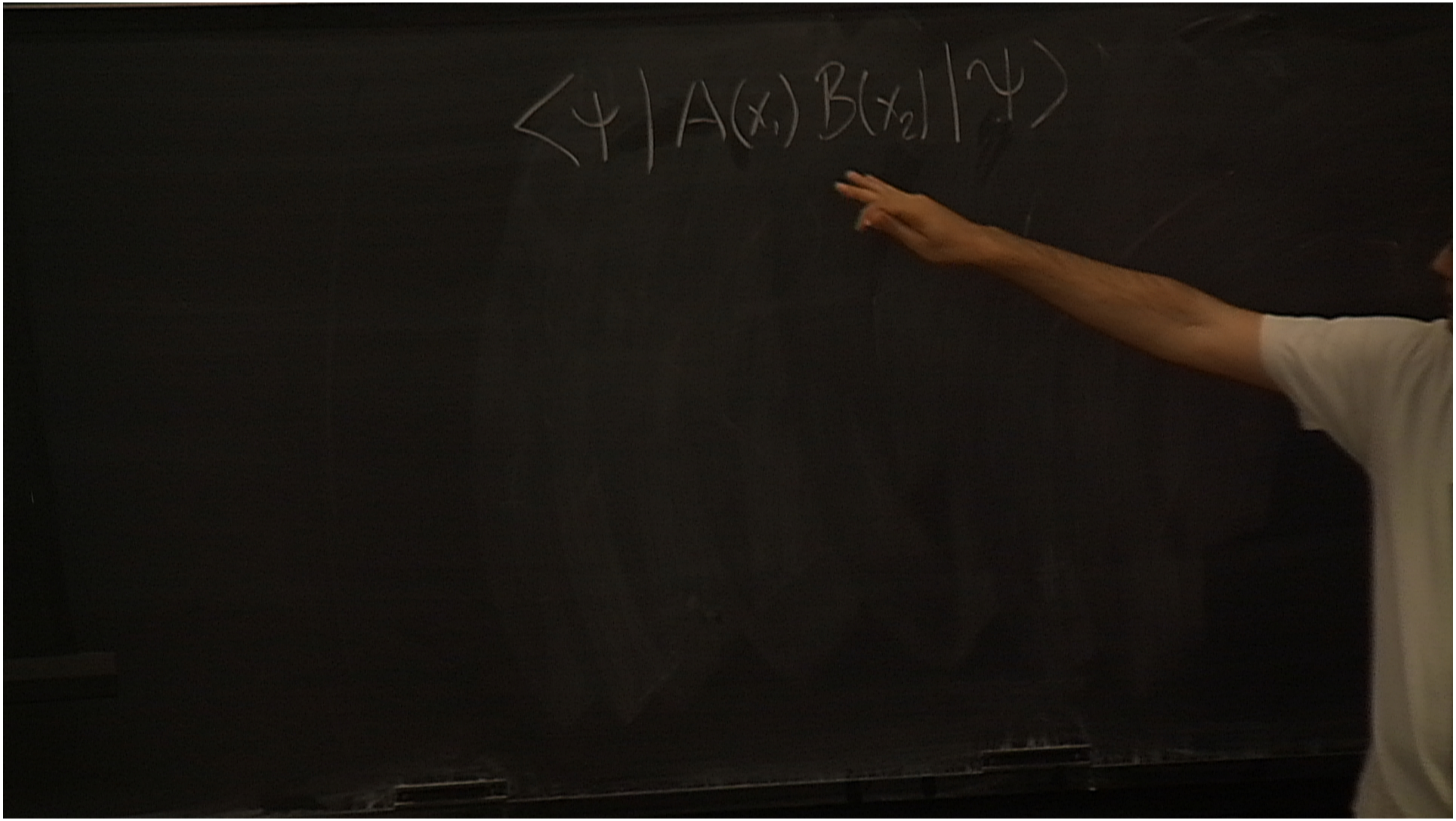
2) Saturation of entanglement entropy

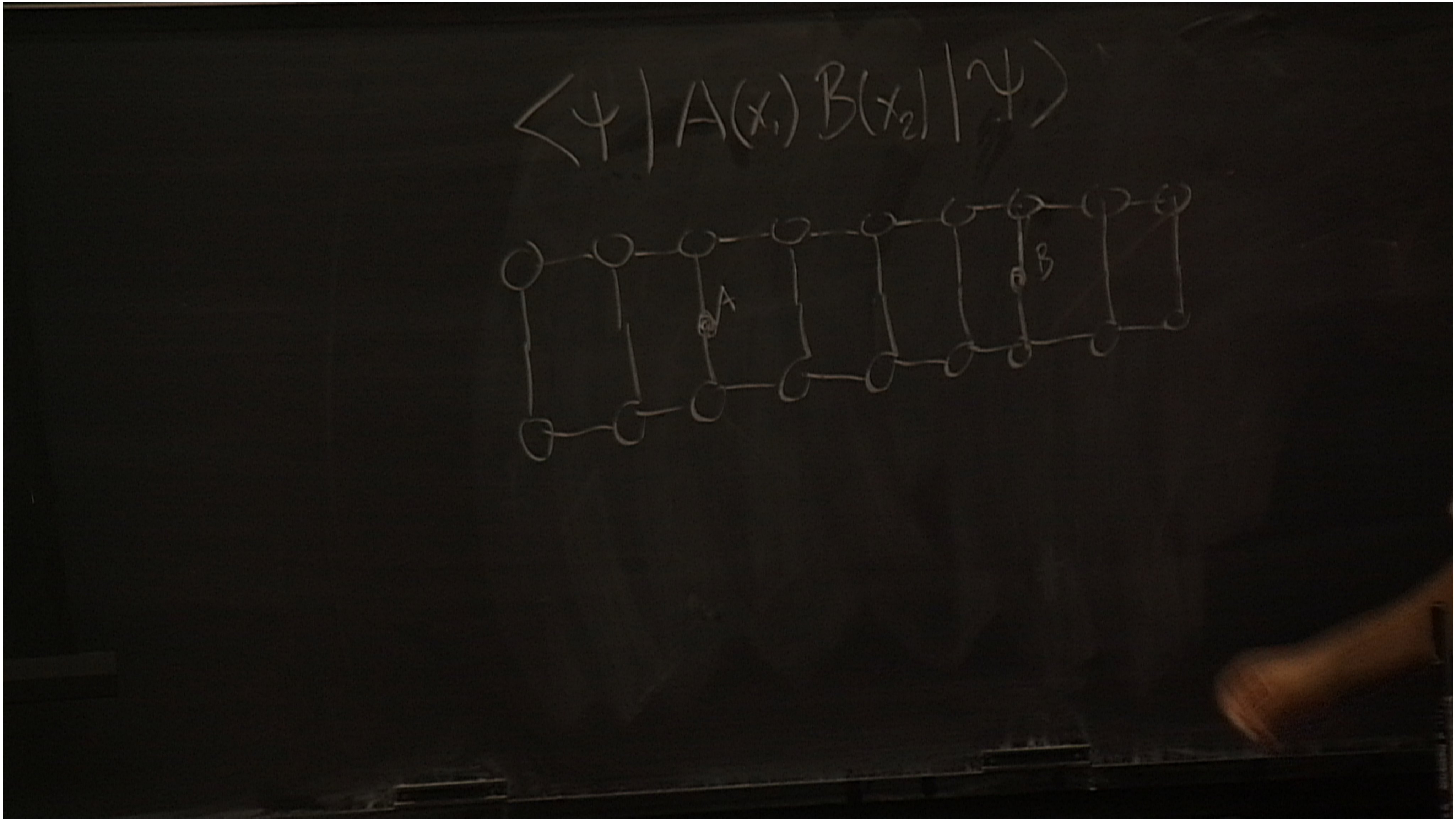
$$\rho_A \equiv \mathrm{tr}_{\bar{A}} |\Psi\rangle\langle\Psi|$$

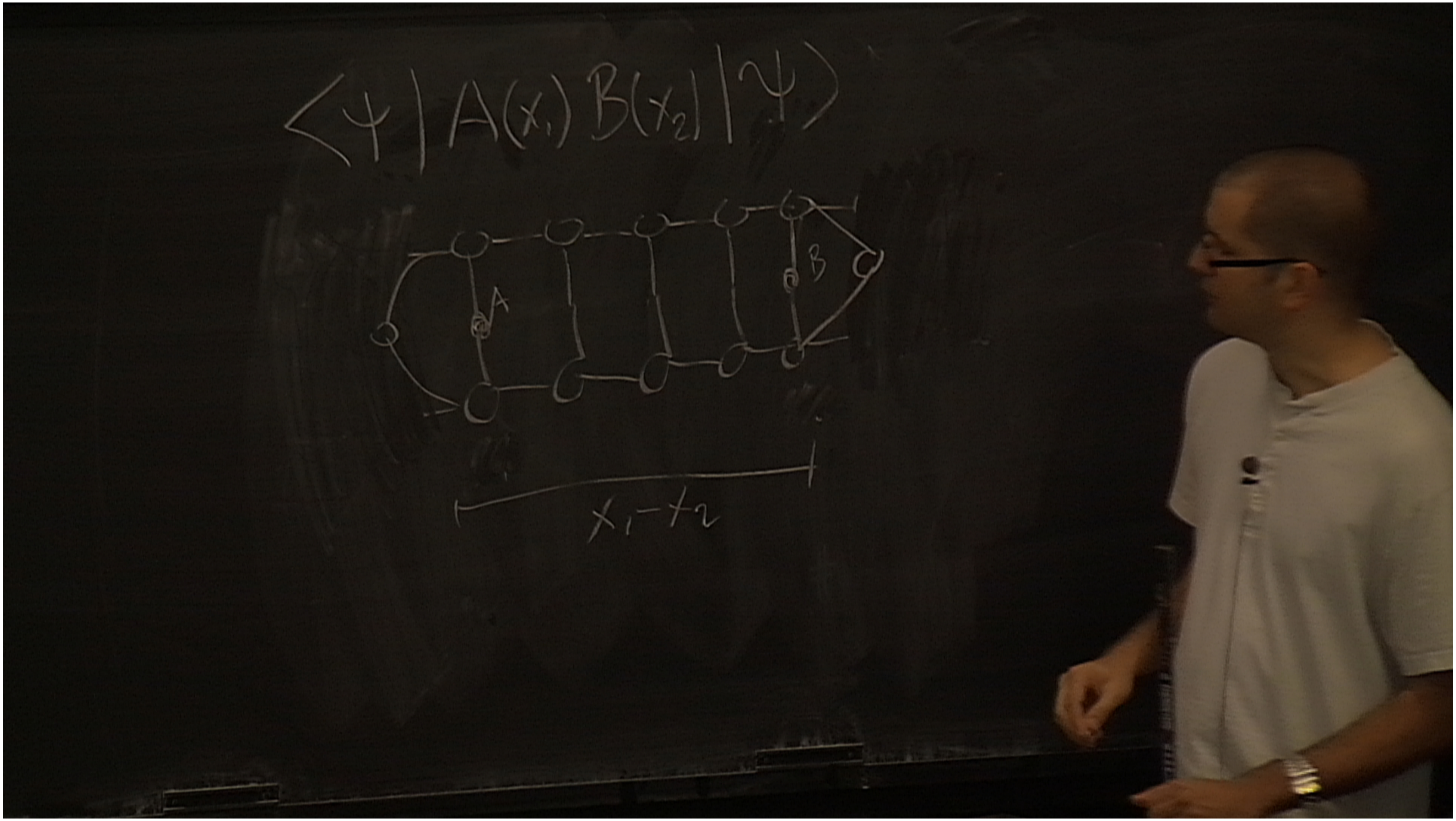
$$S(\rho_A) \equiv -\mathrm{tr}(\rho_A \log \rho_A)$$



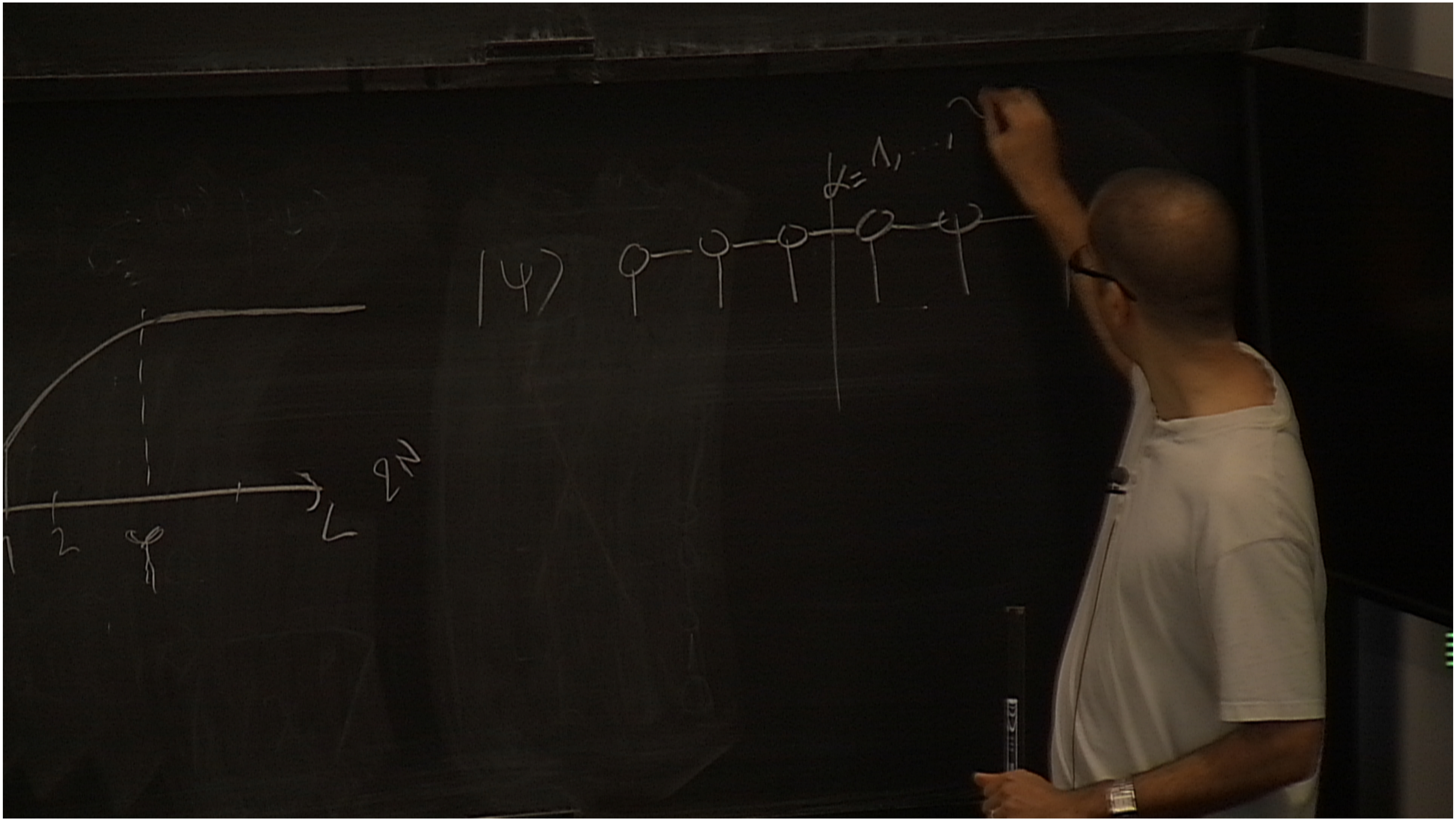

S
L
1
2
L
2N
N

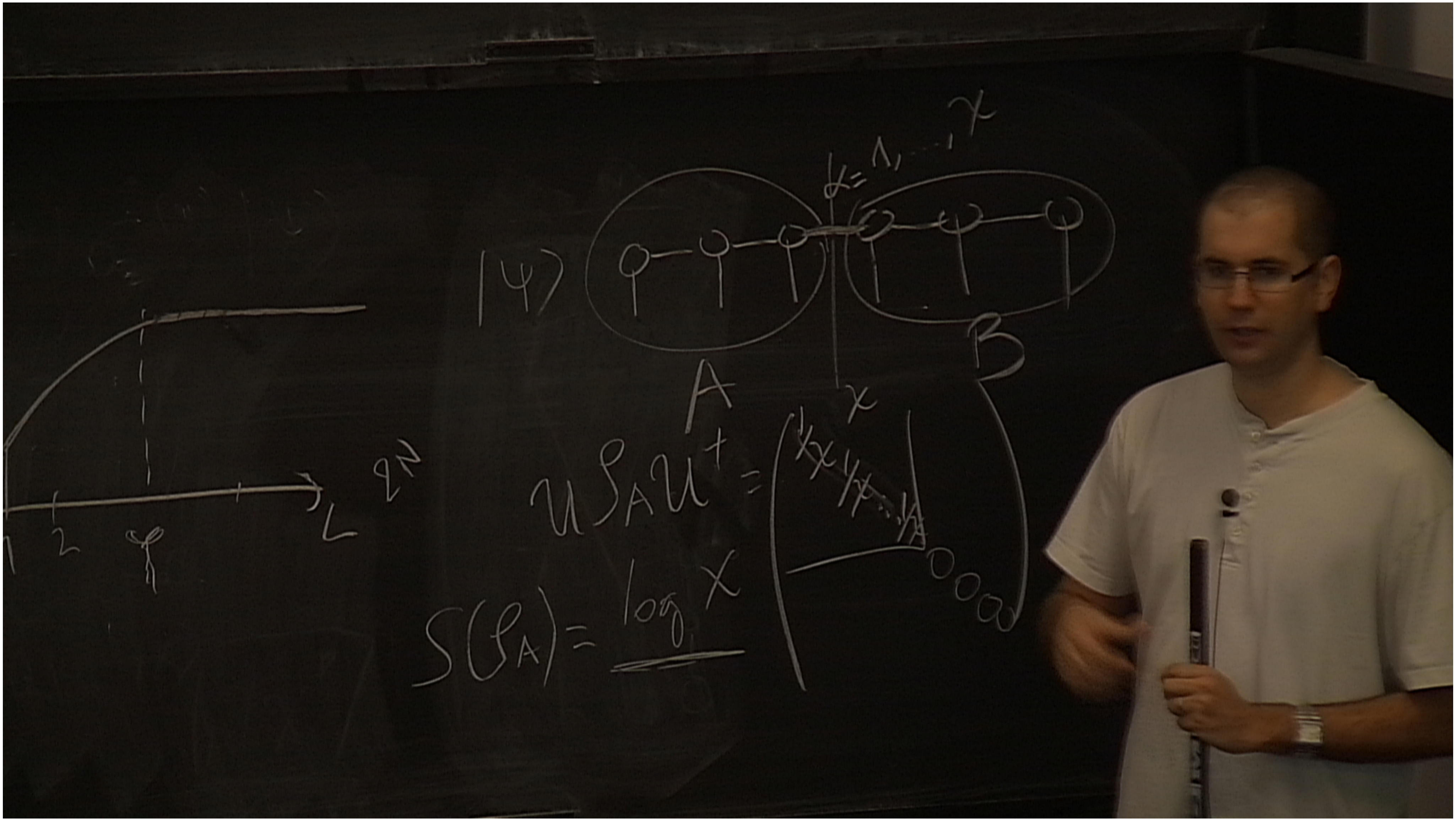

$\langle \psi | A(x_1) B(x_2) | \psi \rangle$


⟨ψ| A(x₁) B(x₂) |ψ⟩
A
B

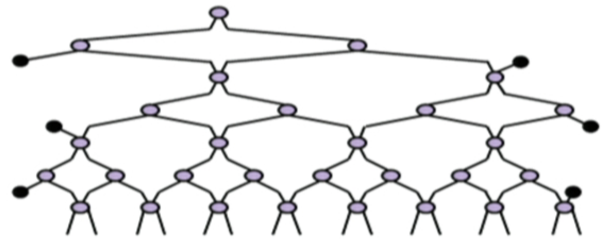


$\langle \Psi | A(x_1) B(x_2) | \Psi \rangle$
A
B
$x_1 - x_2$



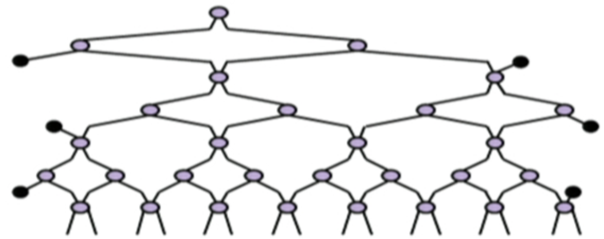

$|\psi\rangle$
A
B
$\alpha = 1, \ldots, \chi$
χ
$U \rho_A U^\dagger =$
$S(\rho_A) = \log \chi$
L
2^N

Summary



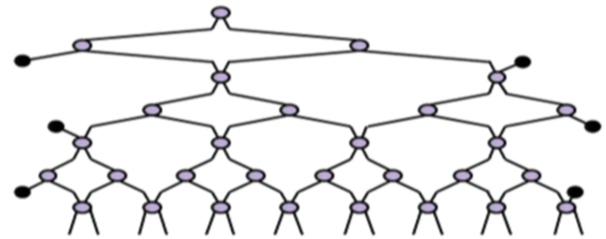
- Diagrammatic notation
- What is a tensor network?
- Examples of tensor networks:
  MPS, PEPS, MERA
- What makes a tensor network useful? (MPS)

Summary



- Diagrammatic notation
- What is a tensor network?
- Examples of tensor networks:
  MPS, PEPS, MERA
- What makes a tensor network useful? (MPS)

Summary



- Diagrammatic notation
- What is a tensor network?
- Examples of tensor networks:
  MPS, PEPS, MERA
- What makes a tensor network useful? (MPS)

Recent development:

Tensor networks

lattice systems → quantum field theories