Title: Statistical Mechanics - Lecture 8

Date: Oct 13, 2011 10:30 AM

URL: http://pirsa.org/11100035

Abstract:

Brownian motion:

Robert Brown (1773-1858) saw particles of pollen "dance around" in fluid under microscope. This motion was caused by many tiny particles hitting the grains of pollen.

The many moving tiny particles are of course **molecules of the liquid**. They were too small to see under a microscope when Brownian motion was discovered, but it was obvious they were there. You can see the molecules of liquid hitting the bigger particle in the animation on the left. (The size of the molecules has been dramatically *increased* in order to make them visible).

http://www.worsleyschool.net/science/files/brownian/motion.html

Albert Einstein (1905) explained this dancing by many, many collisions with molecules in fluid

$$dp/dt = \ldots\ldots + \eta(t) - p/\tau$$

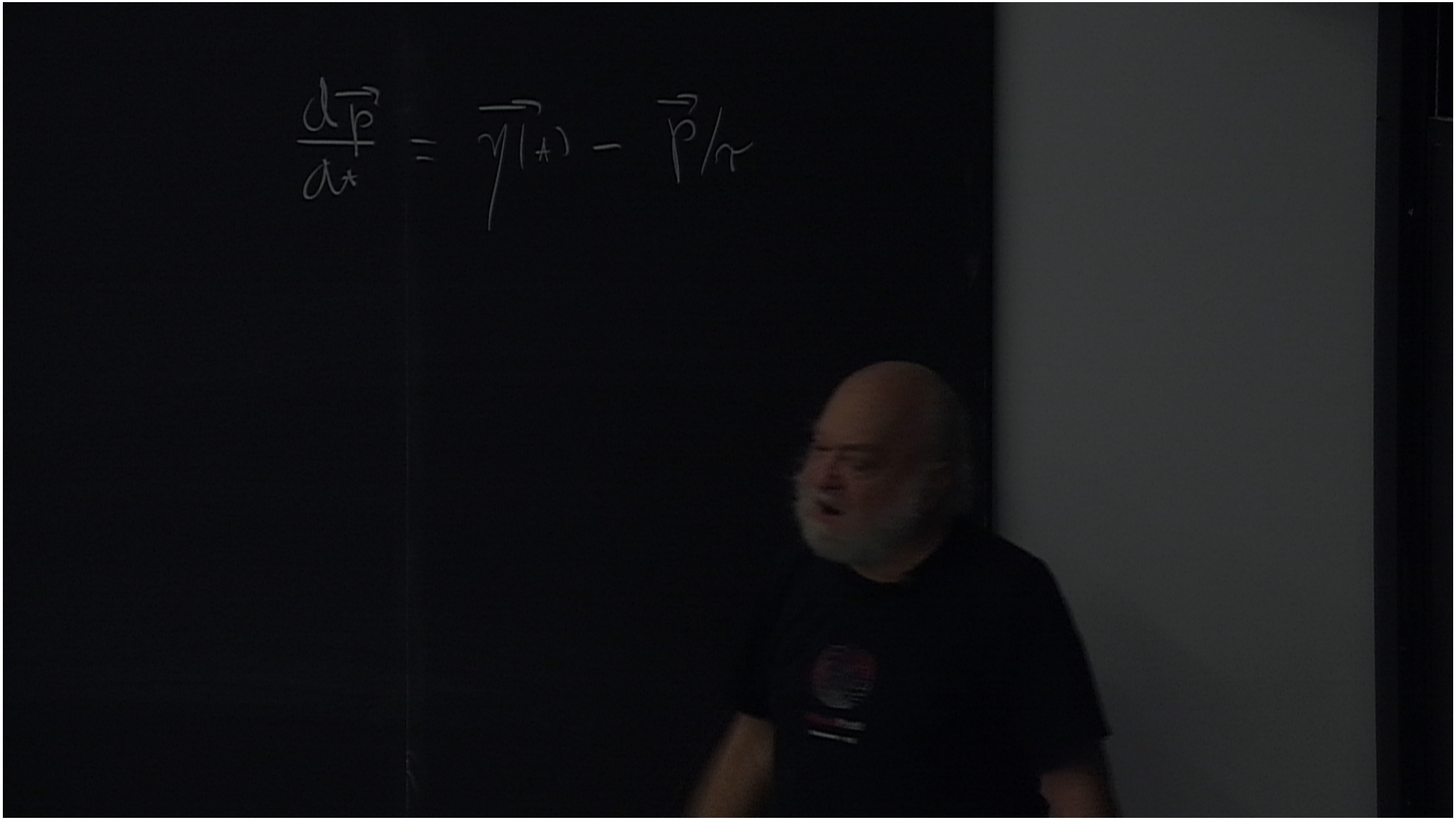
$p=(p_x, p_y, p_z)$ $\eta=(\eta_x, \eta_y, \eta_z)$ v.1

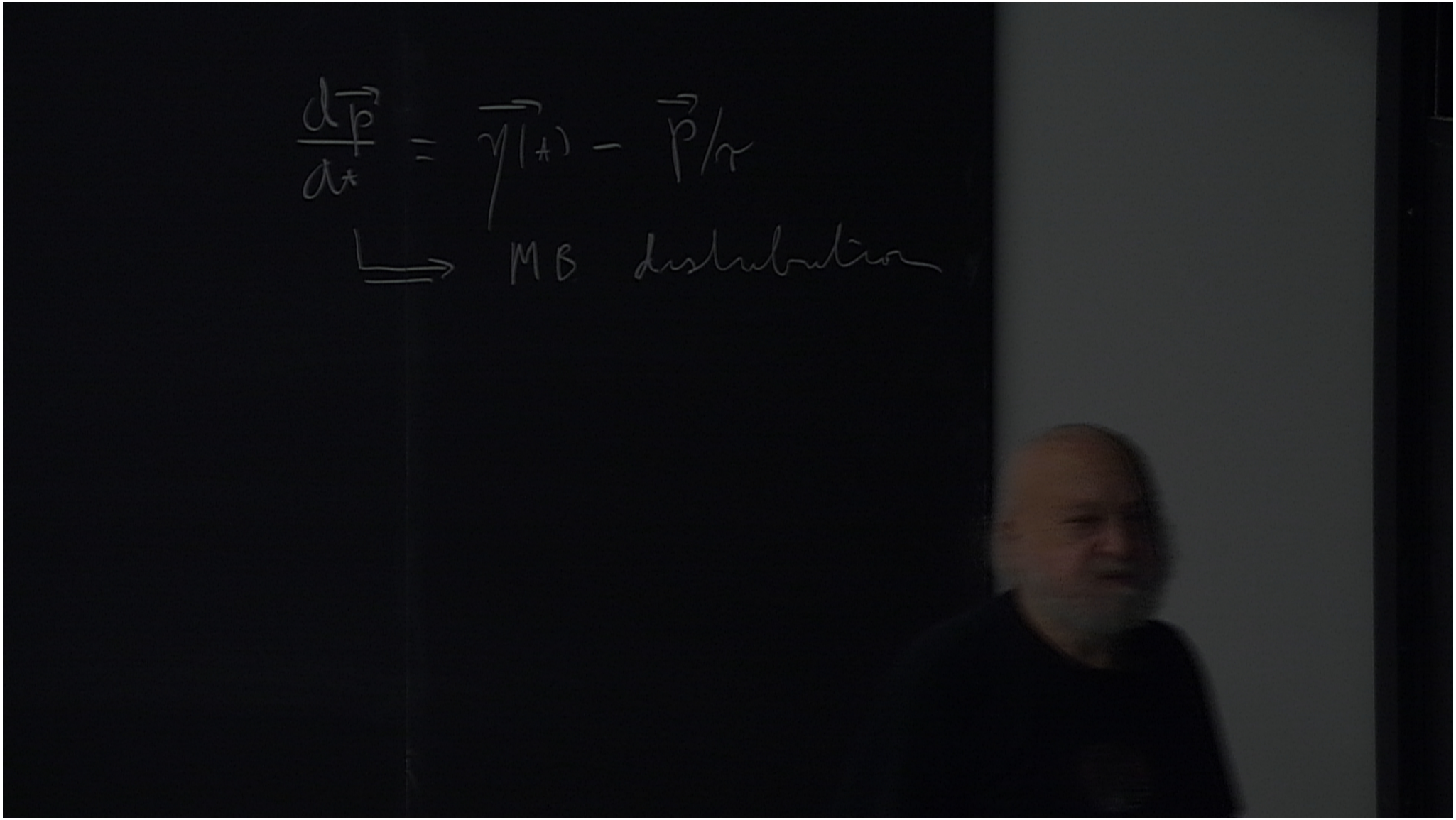
$\eta(t)$ is a Gaussian random variable resulting from random kicks produced by collisions. Since the kicks have random directions $\langle\eta(t)\rangle=0$. Different collisions are assumed to be statistically independent

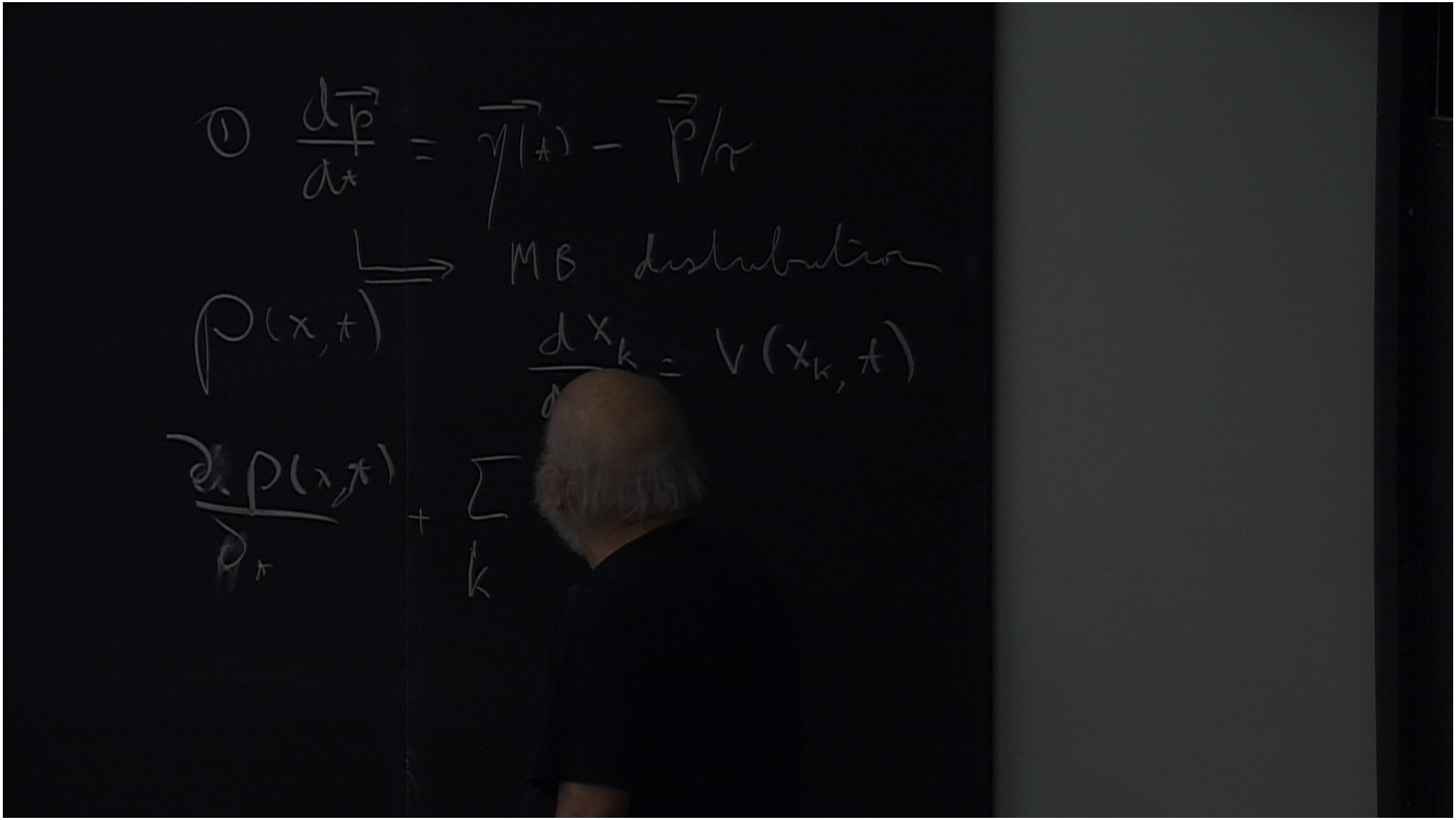
$$\langle\eta_i(t)\,\eta_k(s)\rangle = \Gamma\delta(t-s)\delta_{j,k}$$ v.2

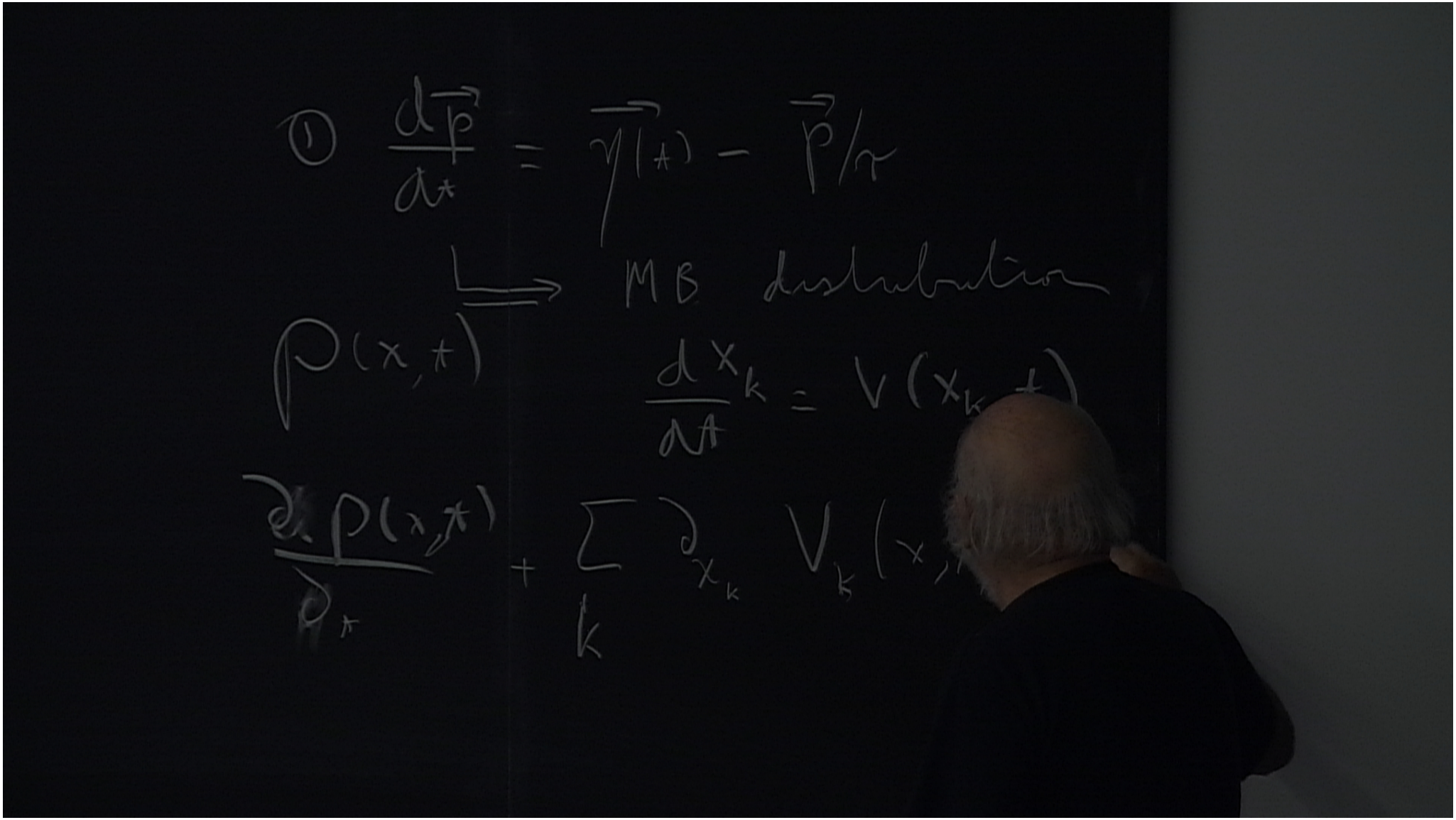
The relaxation time, τ, describes friction slowing down as the particles moves through the medium. In contrast Γ describes the extra momentum picked up via the collisions. Both represent the same physical effect, little particles hitting our big one. However, they operate in a somewhat different fashion. The individual kicks point in every which direction and only in the long run produce any concerted change in momentum. On the other hand the term in τ is a friction tending to continually push our particle toward smaller speeds relative to the medium.

1


$\frac{d\vec{p}}{dt} = \vec{\gamma}(t) - \vec{p}/\tau$


$\frac{d\vec{p}}{dt} = \vec{\eta}(t) - \vec{p}/\tau$
$\Longrightarrow$ MB distribution


① $\frac{d\vec{p}}{dt} = \vec{\eta}(t) - \vec{p}/\tau$
⟹ MB distribution
$\rho(x,t)$
$\frac{dx_k}{dt} = V(x_k,t)$
$\frac{\partial \rho(x,t)}{\partial t} + \sum_k$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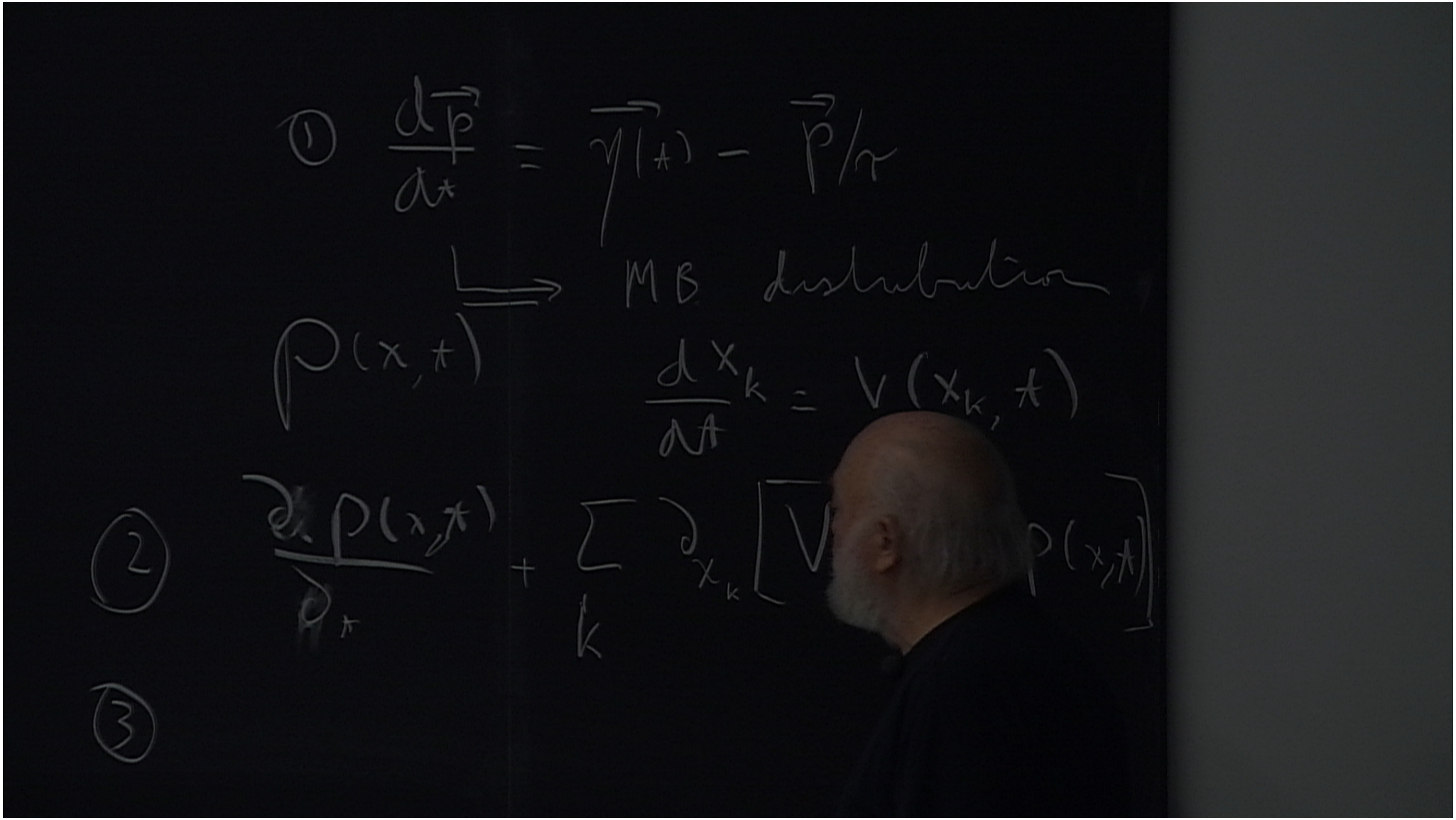
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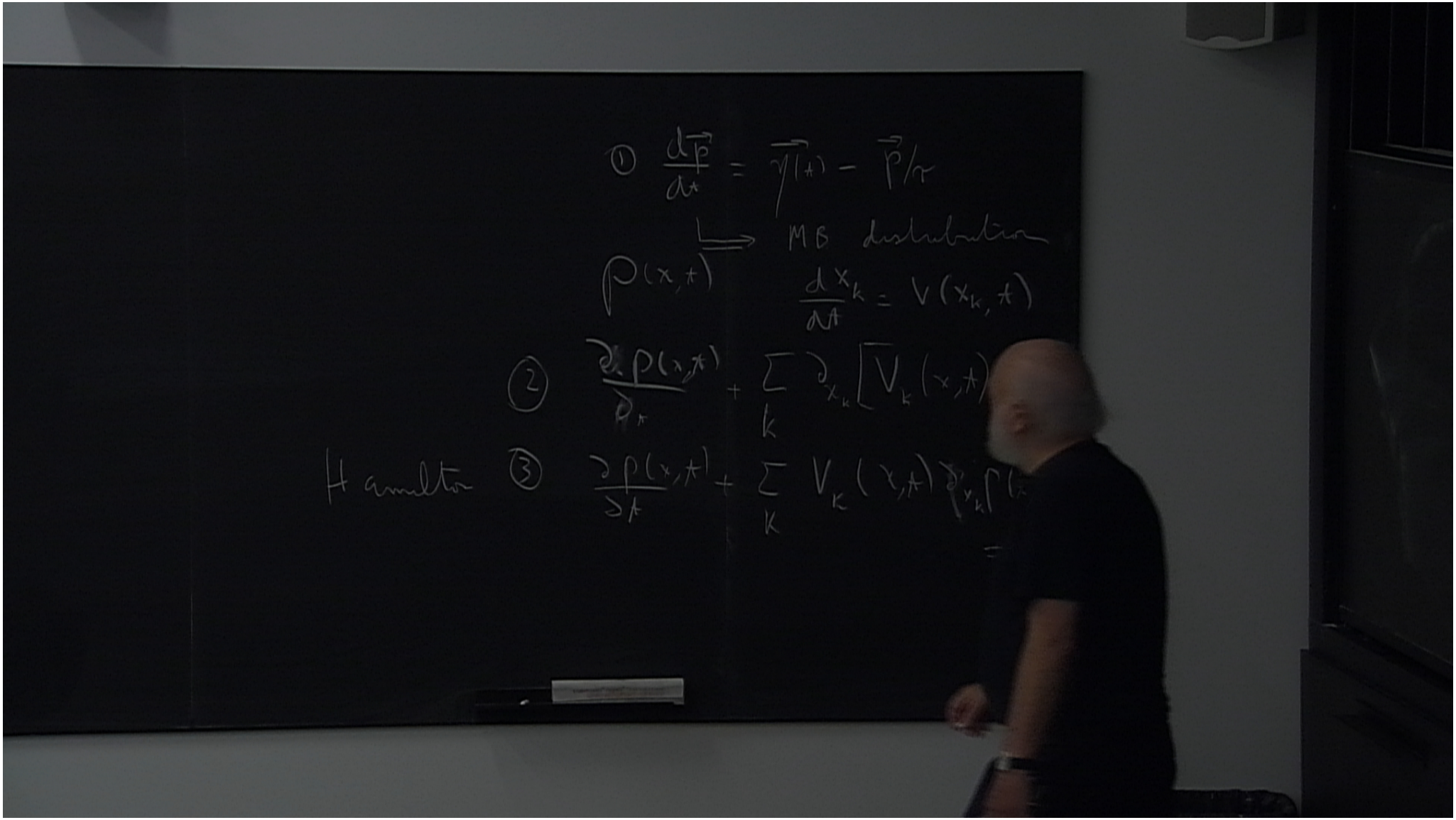
③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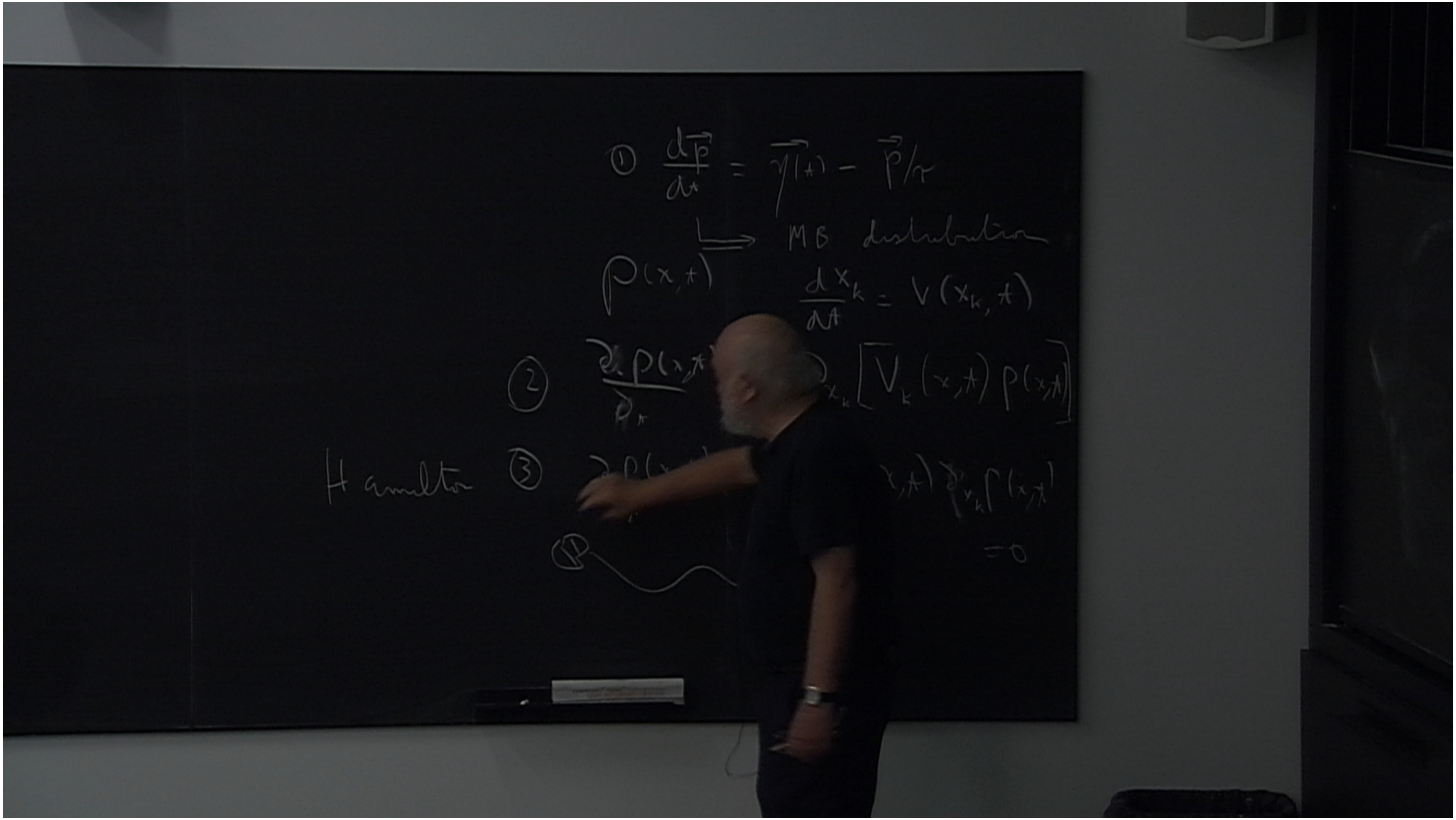
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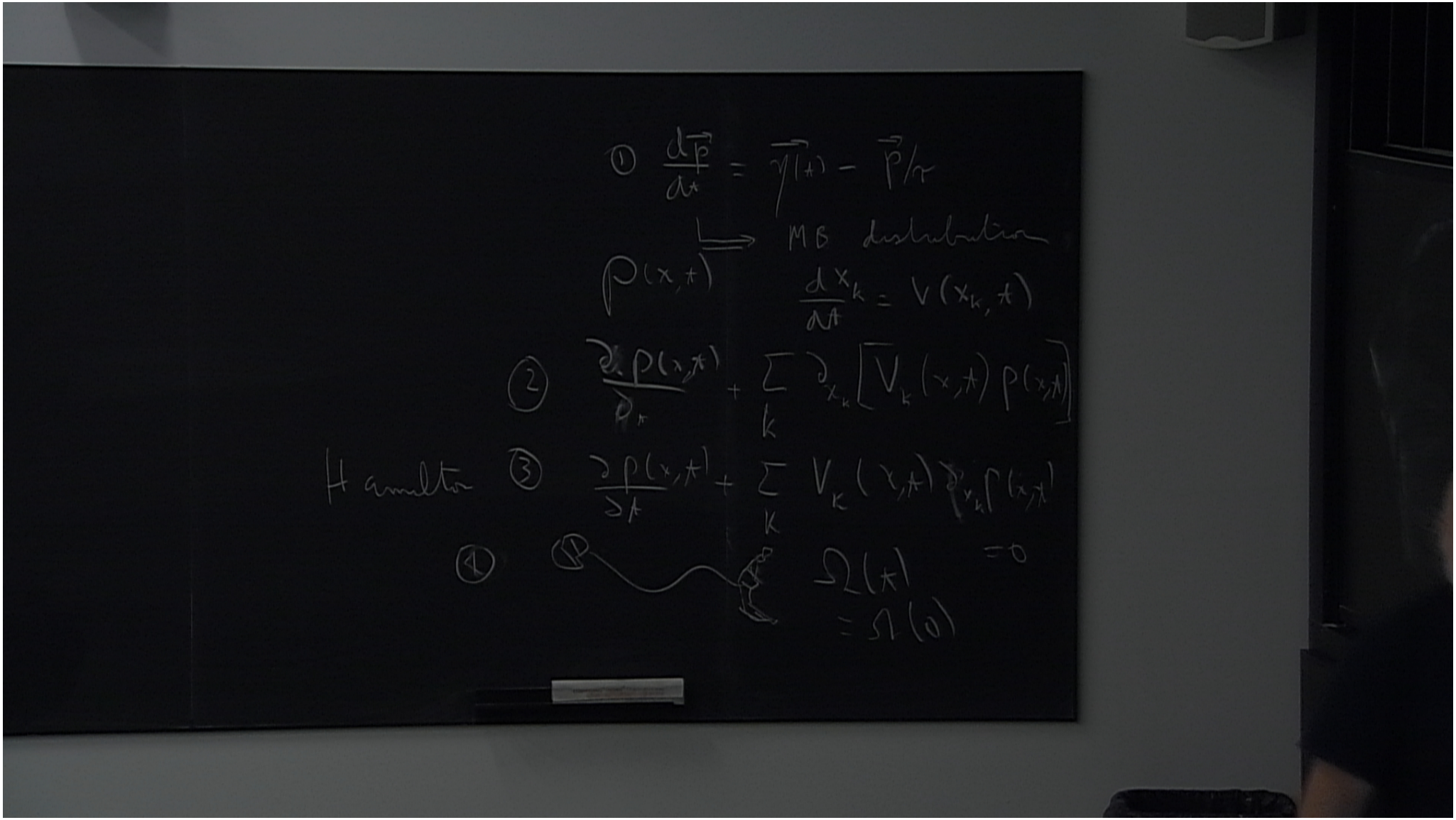
$=0$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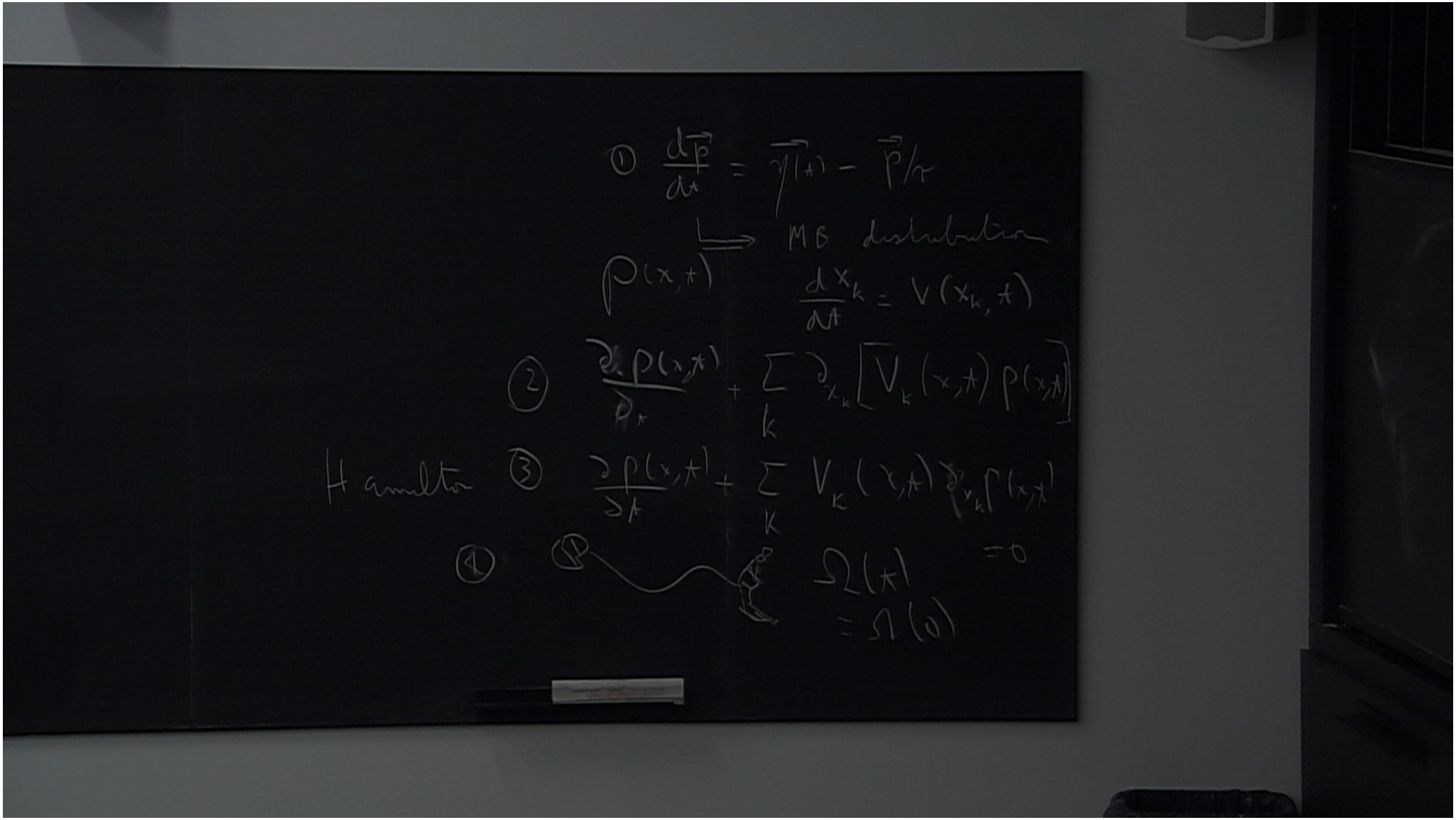
② $\frac{\partial p(x,t)}{\partial t} + \sum_k \partial_{x_k}\left[V_k(x,t)\, p(x,t)\right]$
Hamilton
③ $\frac{\partial p(x,t)}{\partial t} + \sum_k V_k(x,t)\, \partial_{x_k} p(x,t)$
④ $\Omega(t) = 0$
$= \Omega(0)$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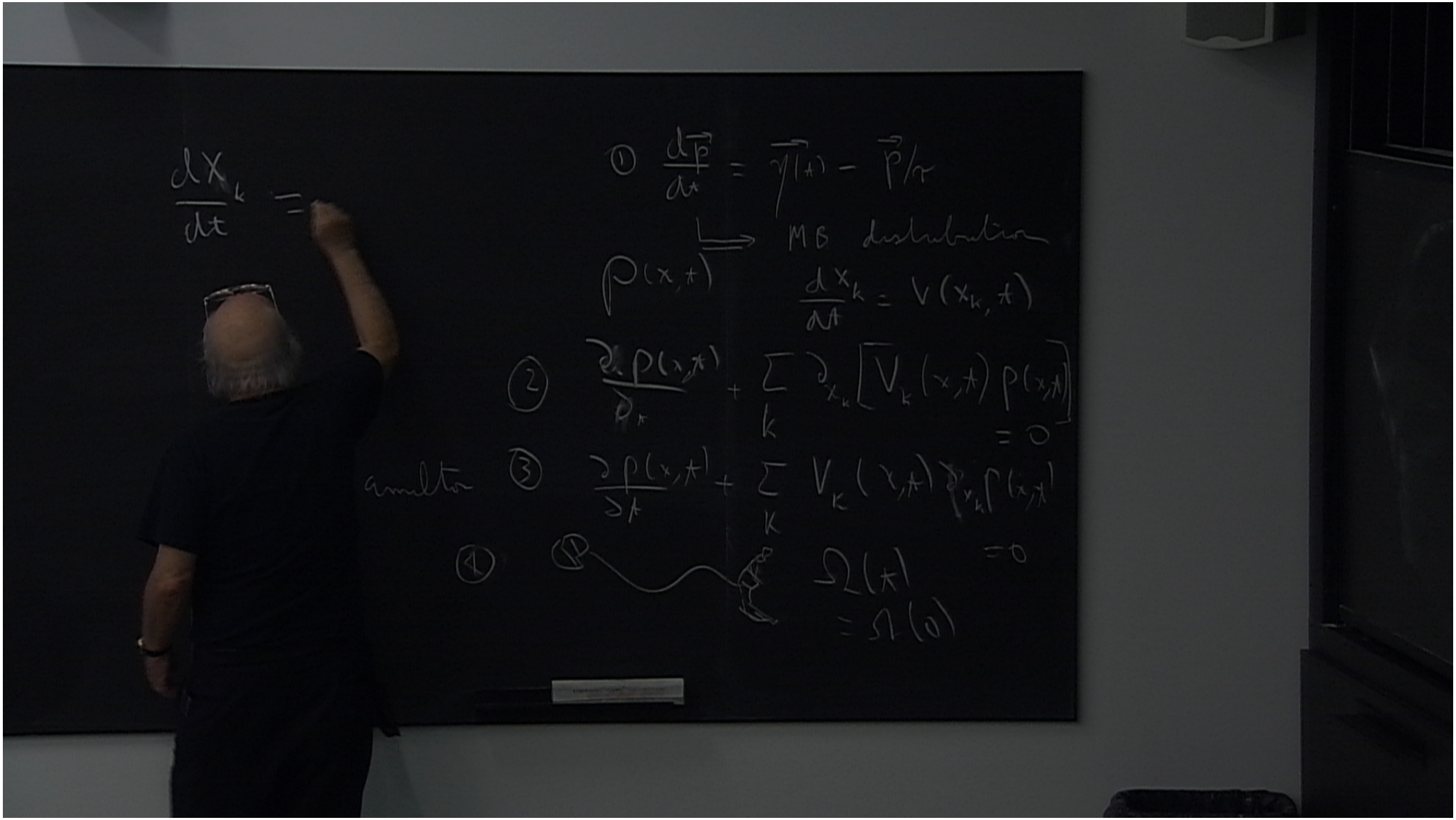
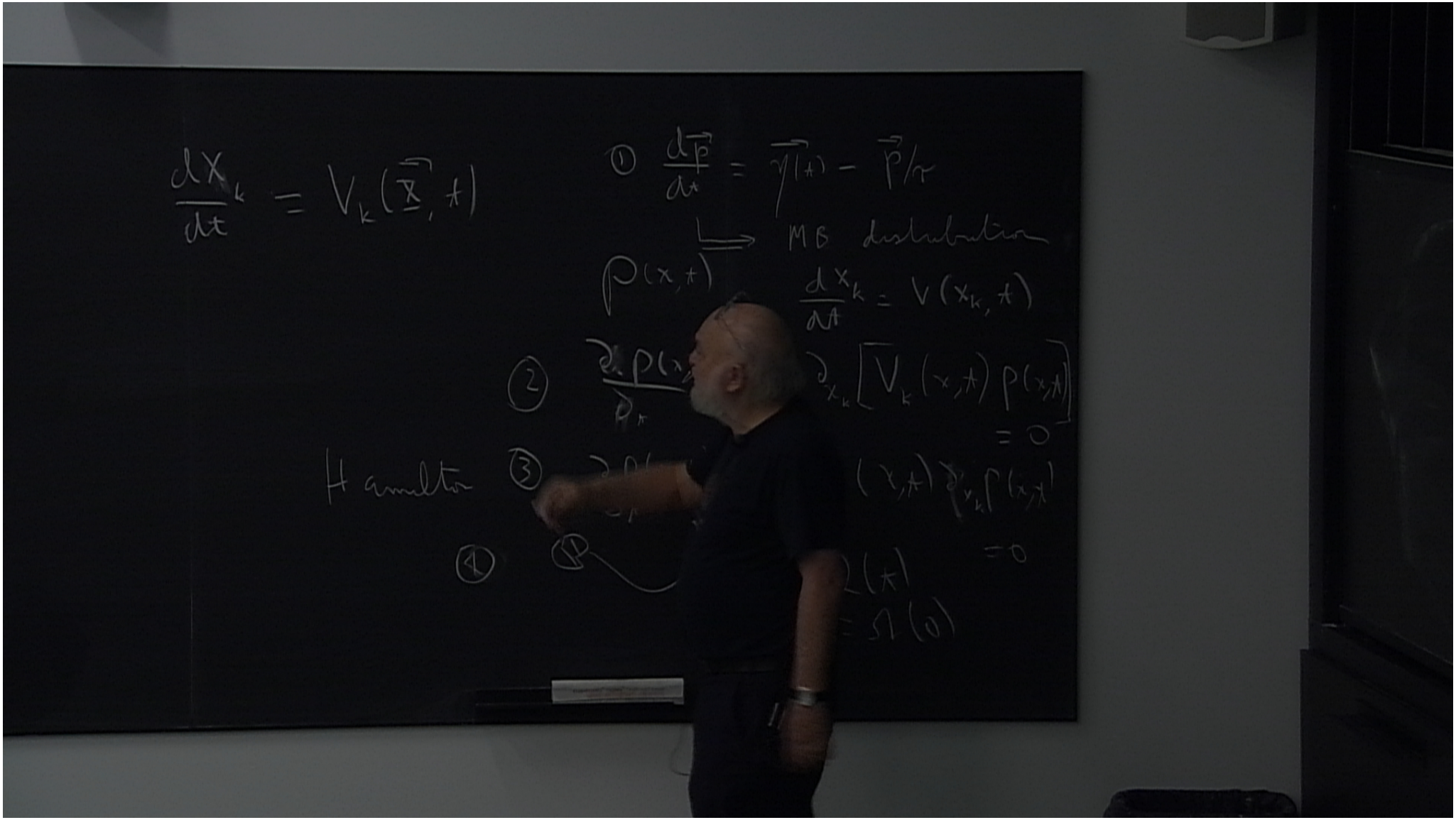

dX_k/dt =
① dp/dt = γ(t) − p/τ
MB distribution
ρ(x,t)
dx_k/dt = V(x_k, t)
② ∂ρ(x,t)/∂t + Σ_k ∂_{x_k}[V_k(x,t) ρ(x,t)] = 0
③ ∂ρ(x,t)/∂t + Σ_k V_k(x,t) ∂_{x_k} ρ(x,t) = 0
Ω(t) = Ω(0)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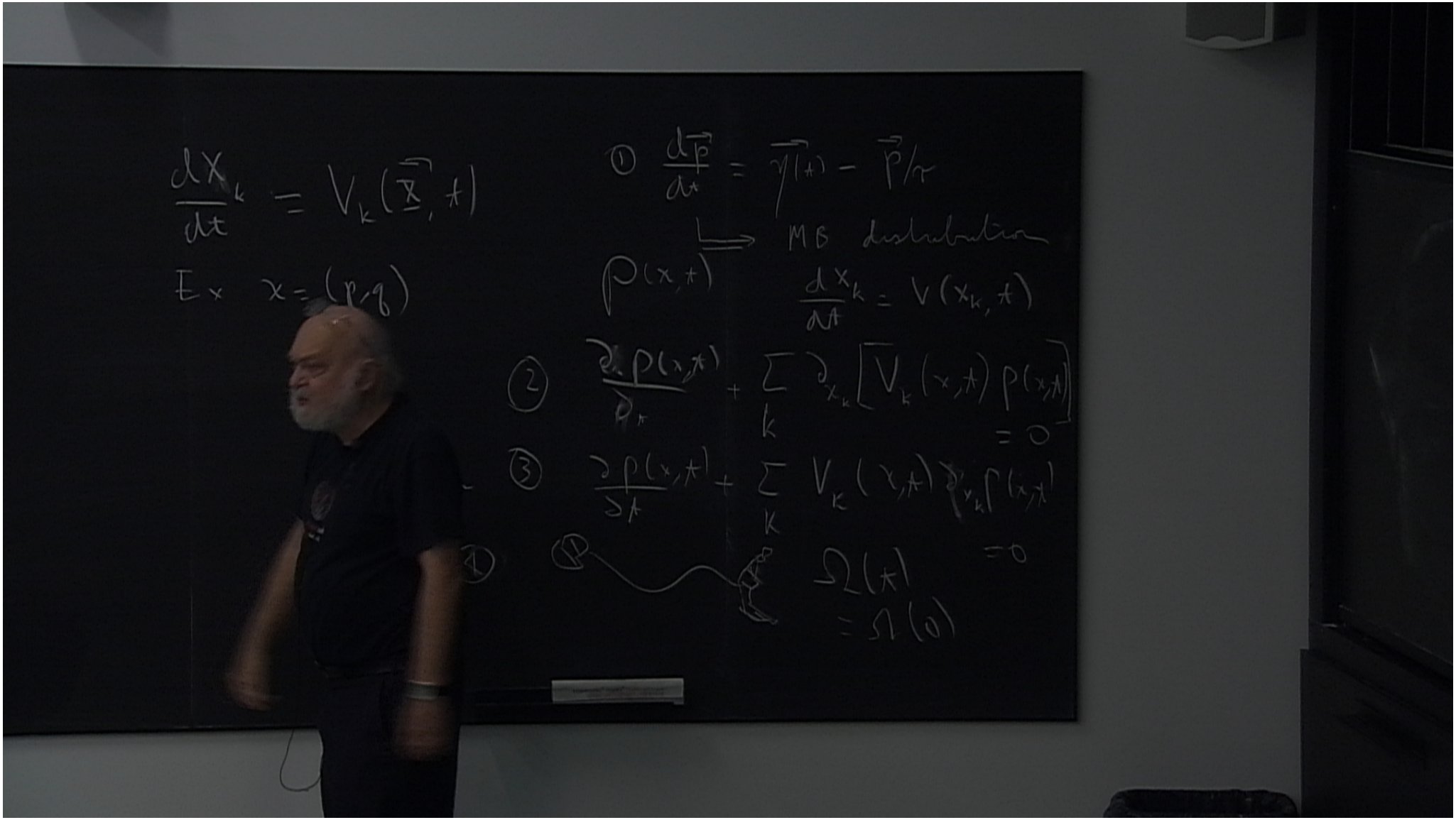
$\frac{dX_k}{dt} = V_k(\vec{X}, t)$
Ex $x = (p, q)$
① $\frac{d\vec{p}}{dt} = \vec{\gamma}(t) - \vec{p}/\tau$
↳ MB distribution
$\rho(x,t)$
$\frac{dx_k}{dt} = V(x_k, t)$
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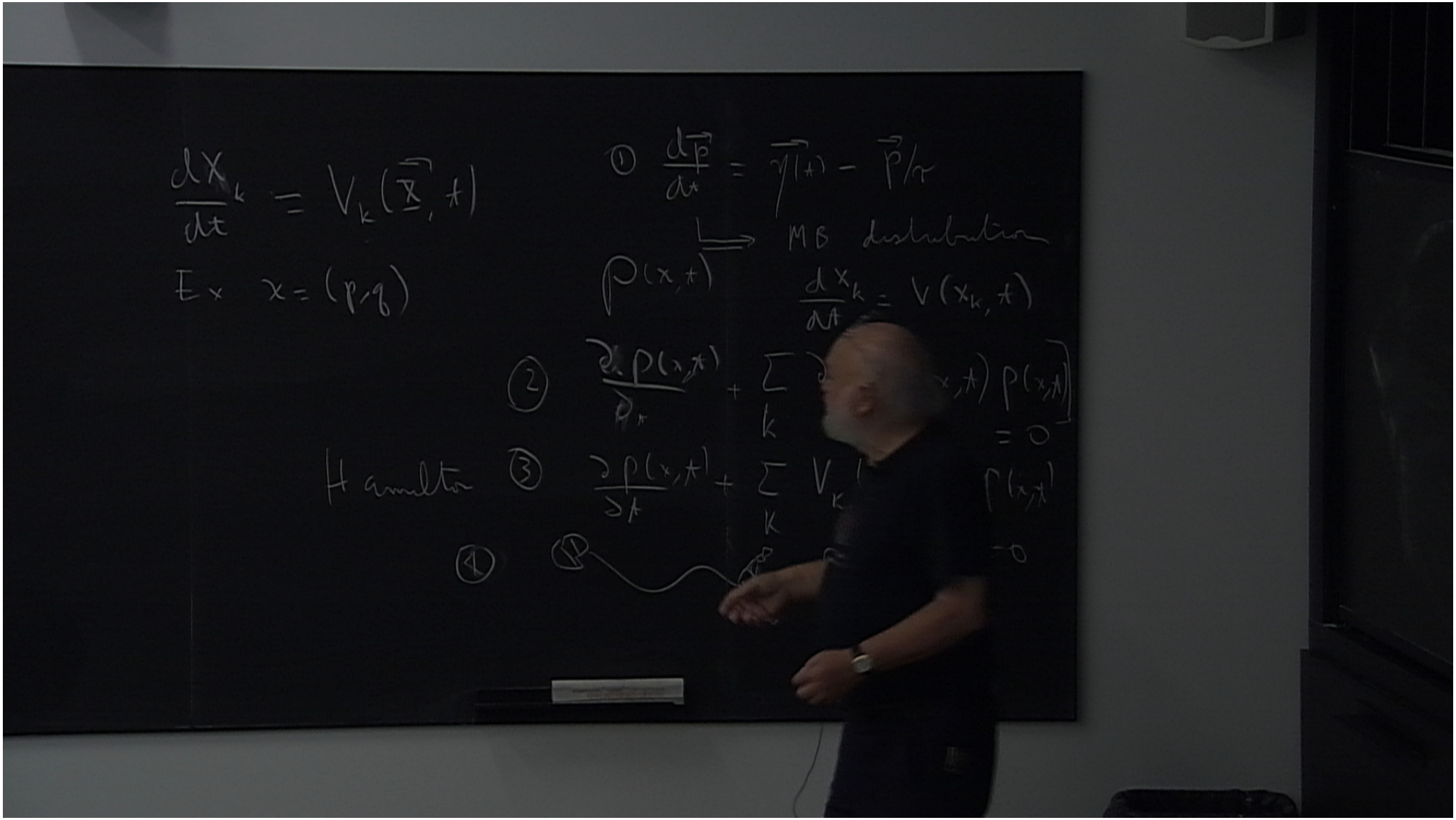
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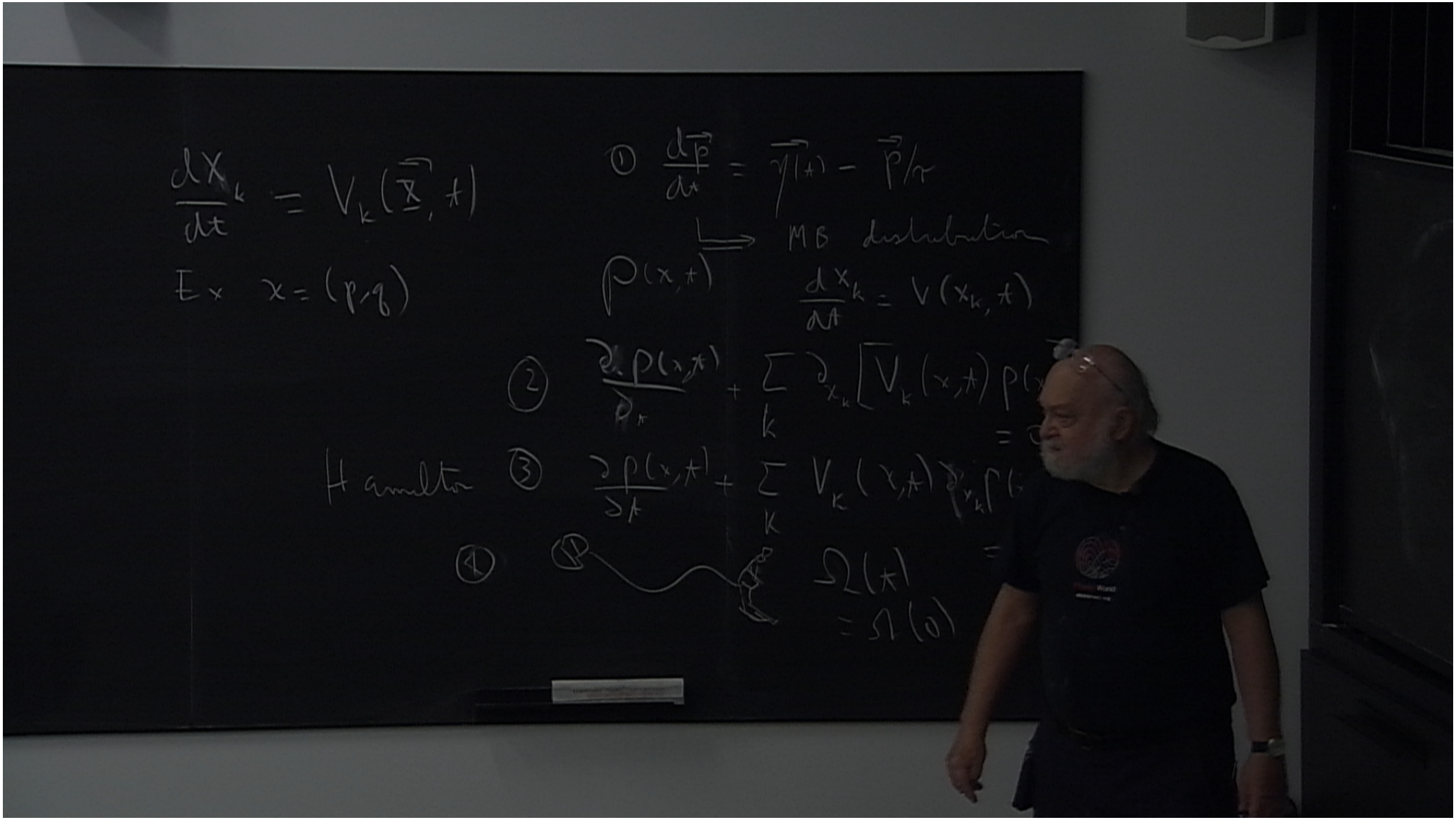
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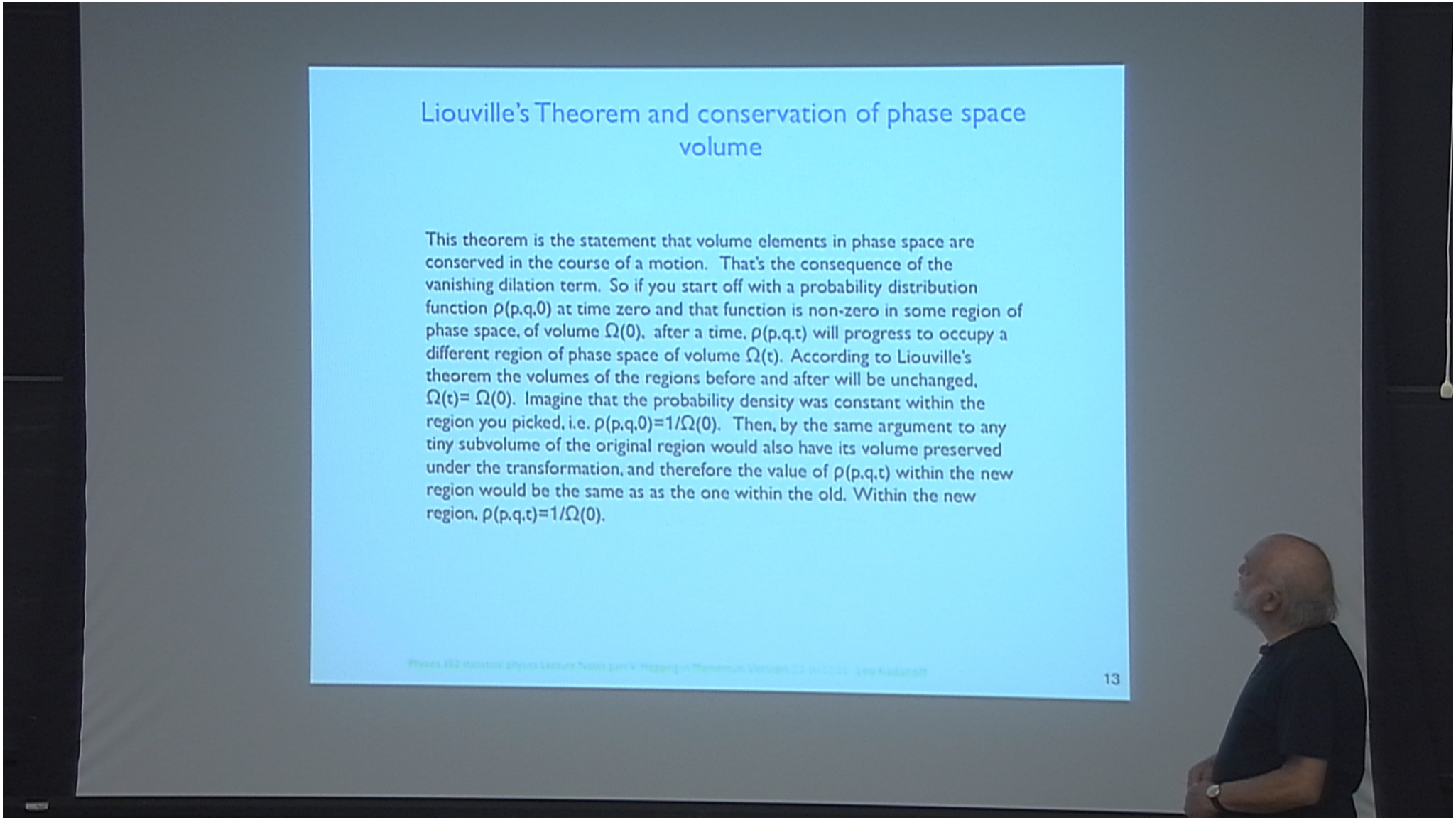
Liouville's Theorem and conservation of phase space volume
This theorem is the statement that volume elements in phase space are conserved in the course of a motion. That's the consequence of the vanishing dilation term. So if you start off with a probability distribution function ρ(p,q,0) at time zero and that function is non-zero in some region of phase space, of volume Ω(0), after a time, ρ(p,q,t) will progress to occupy a different region of phase space of volume Ω(t). According to Liouville's theorem the volumes of the regions before and after will be unchanged, Ω(t)= Ω(0). Imagine that the probability density was constant within the region you picked, i.e. ρ(p,q,0)=1/Ω(0). Then, by the same argument to any tiny subvolume of the original region would also have its volume preserved under the transformation, and therefore the value of ρ(p,q,t) within the new region would be the same as as the one within the old. Within the new region, ρ(p,q,t)=1/Ω(0).
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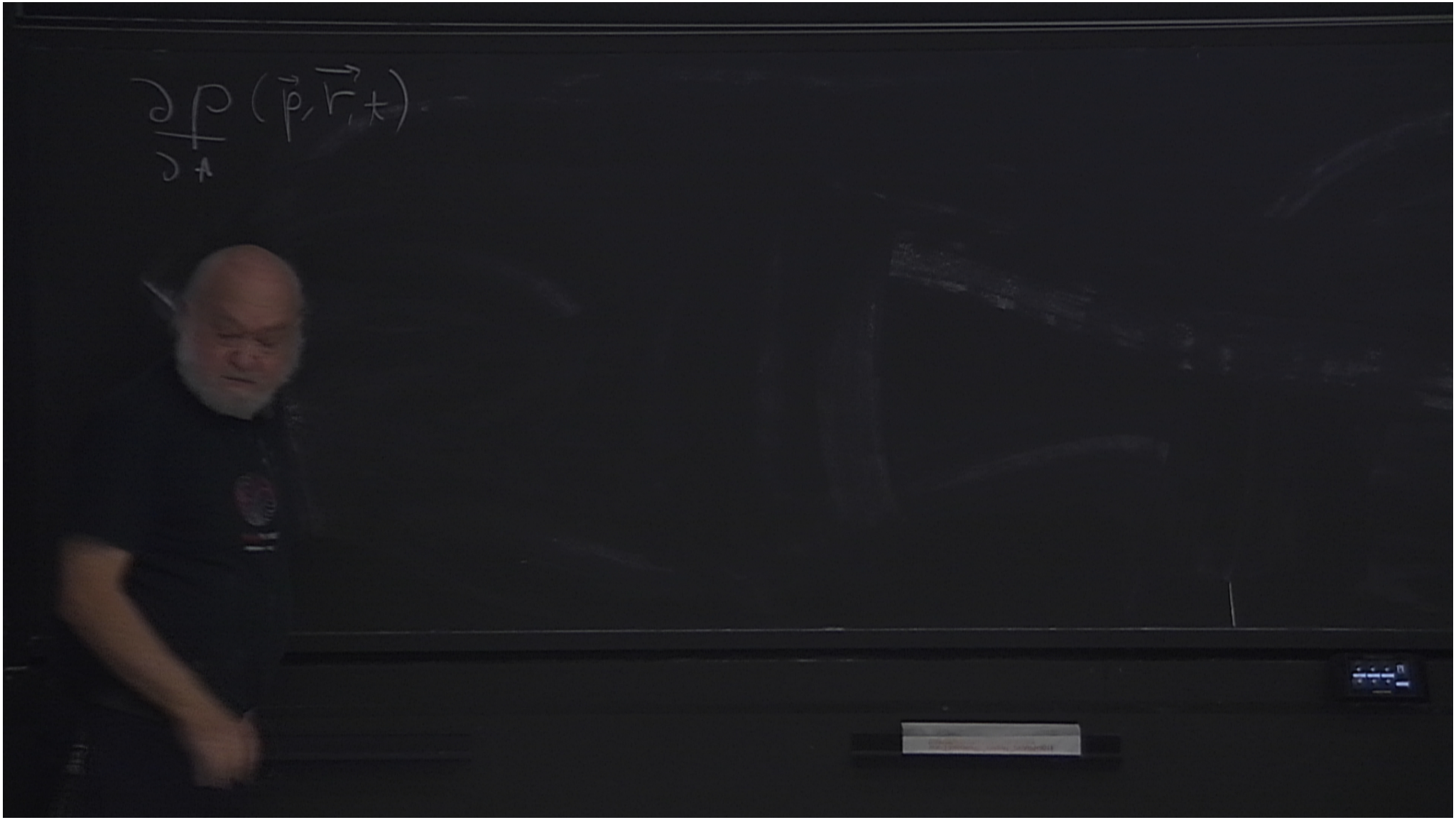
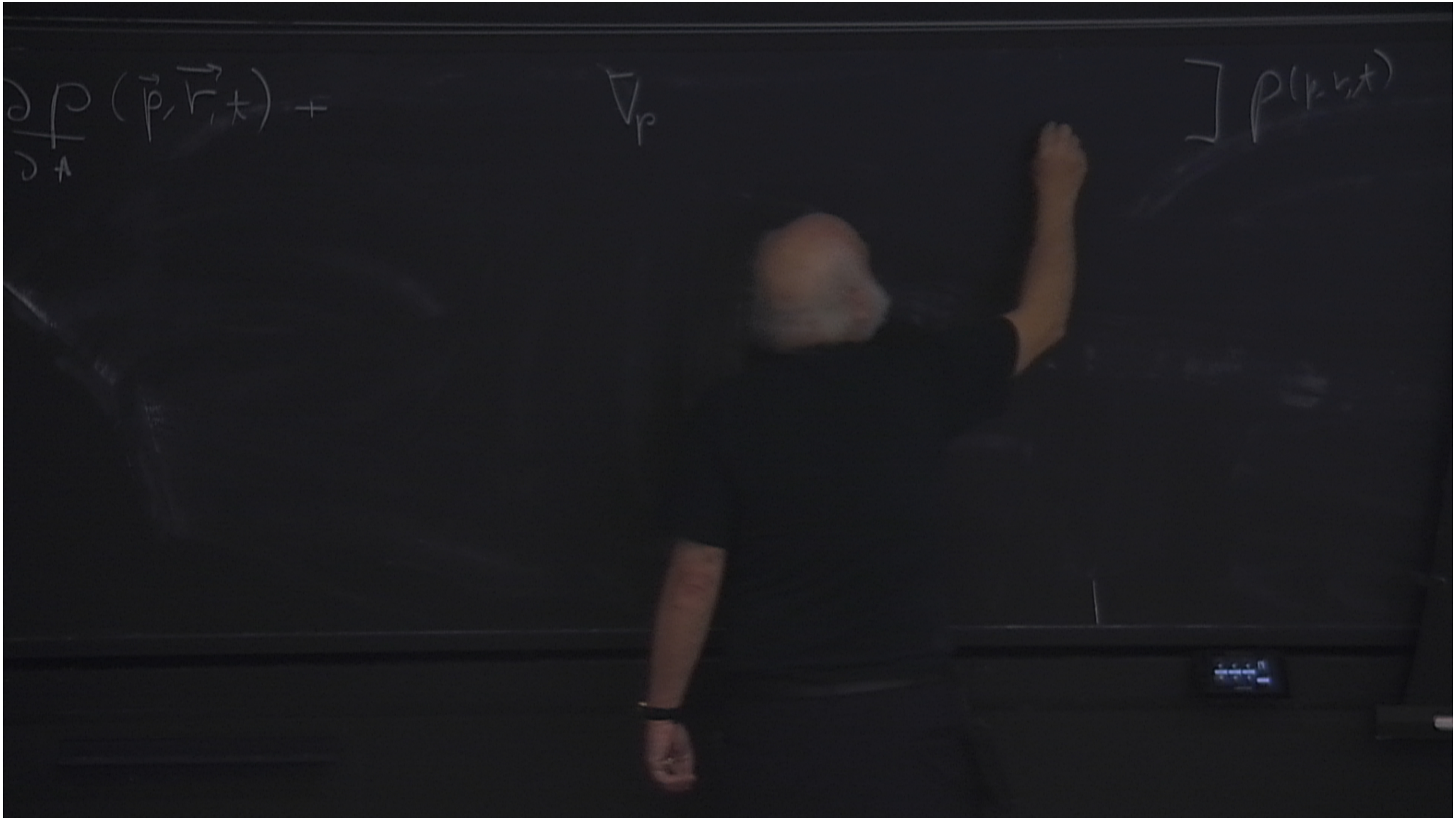
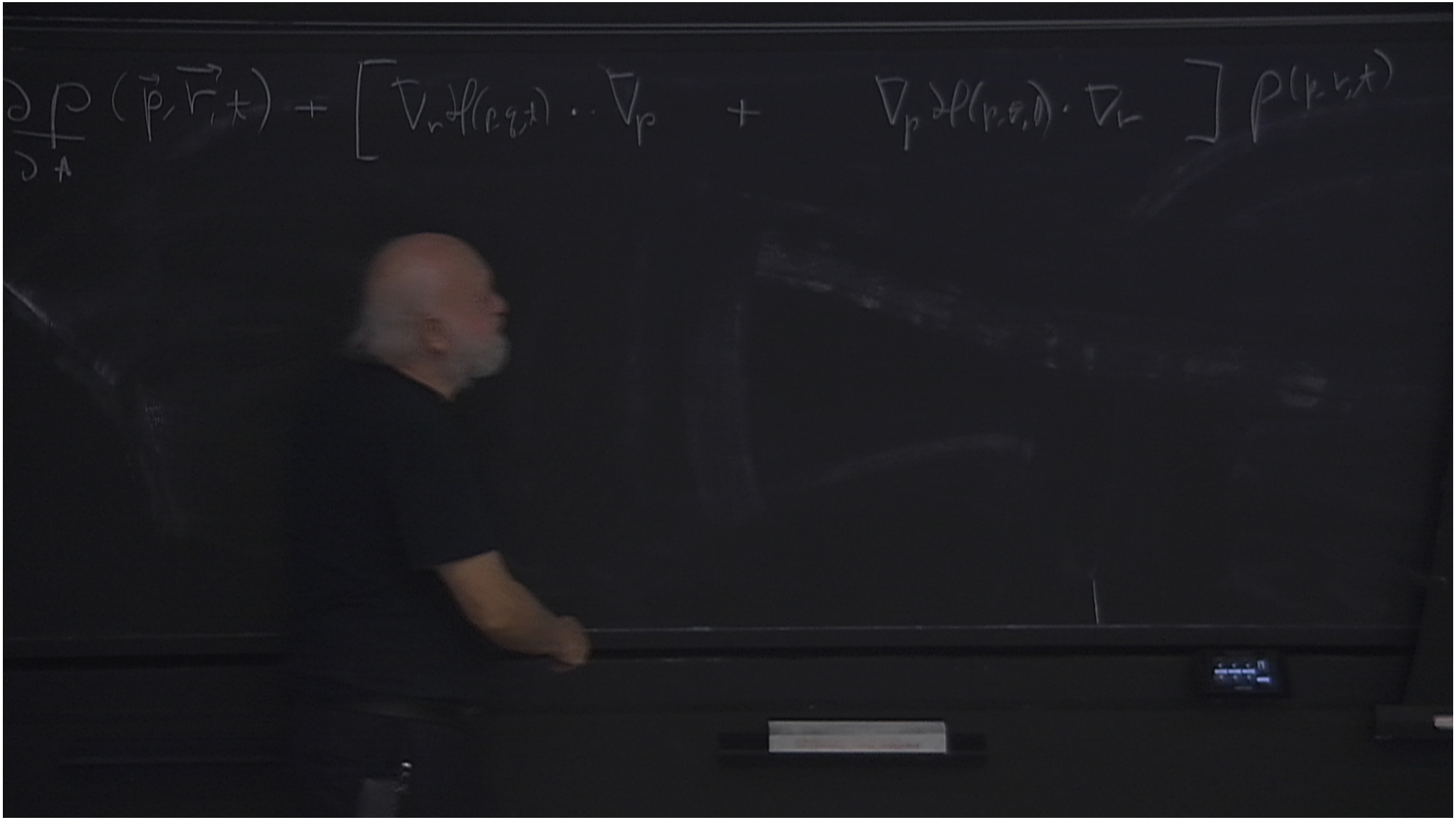
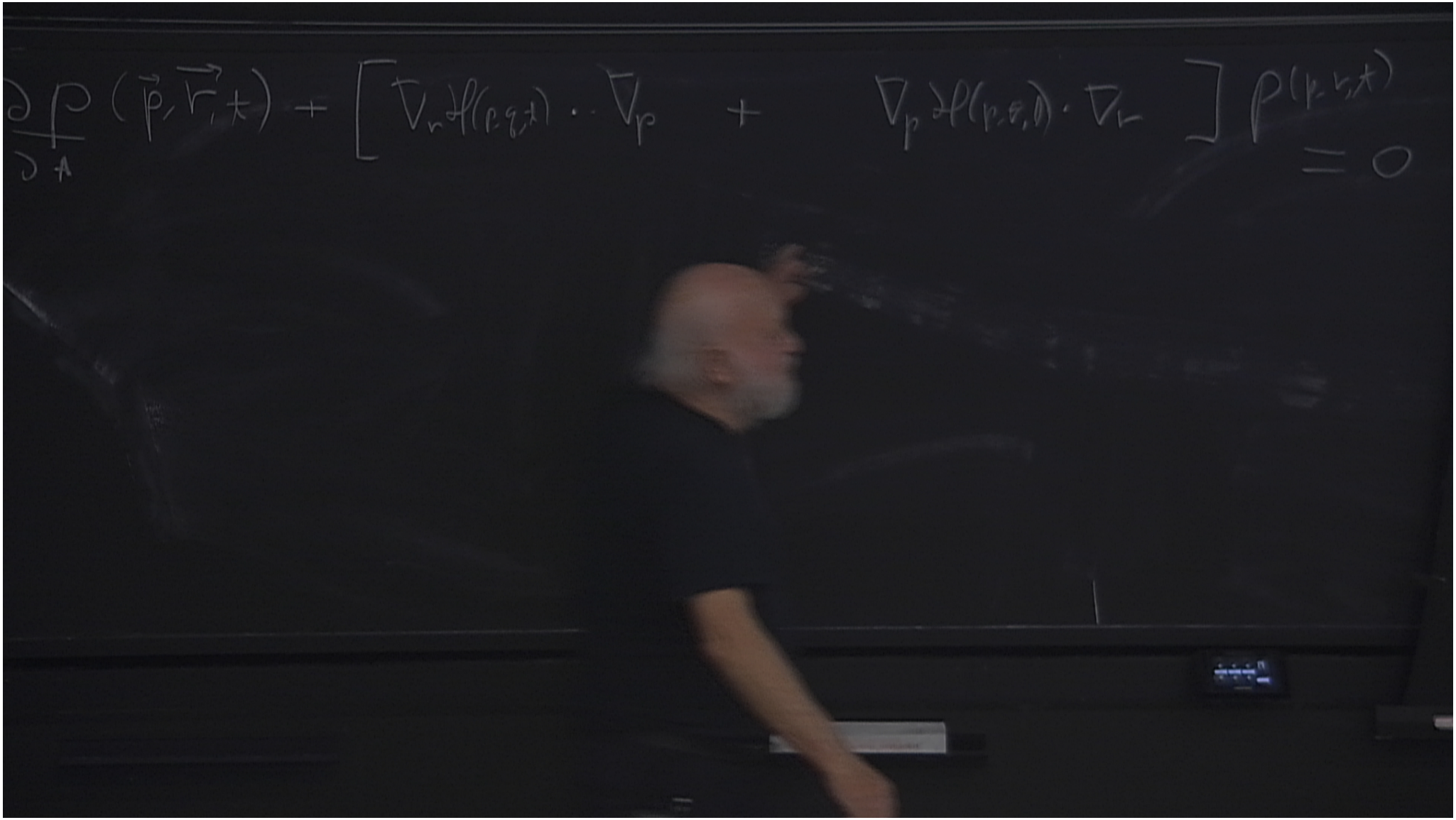
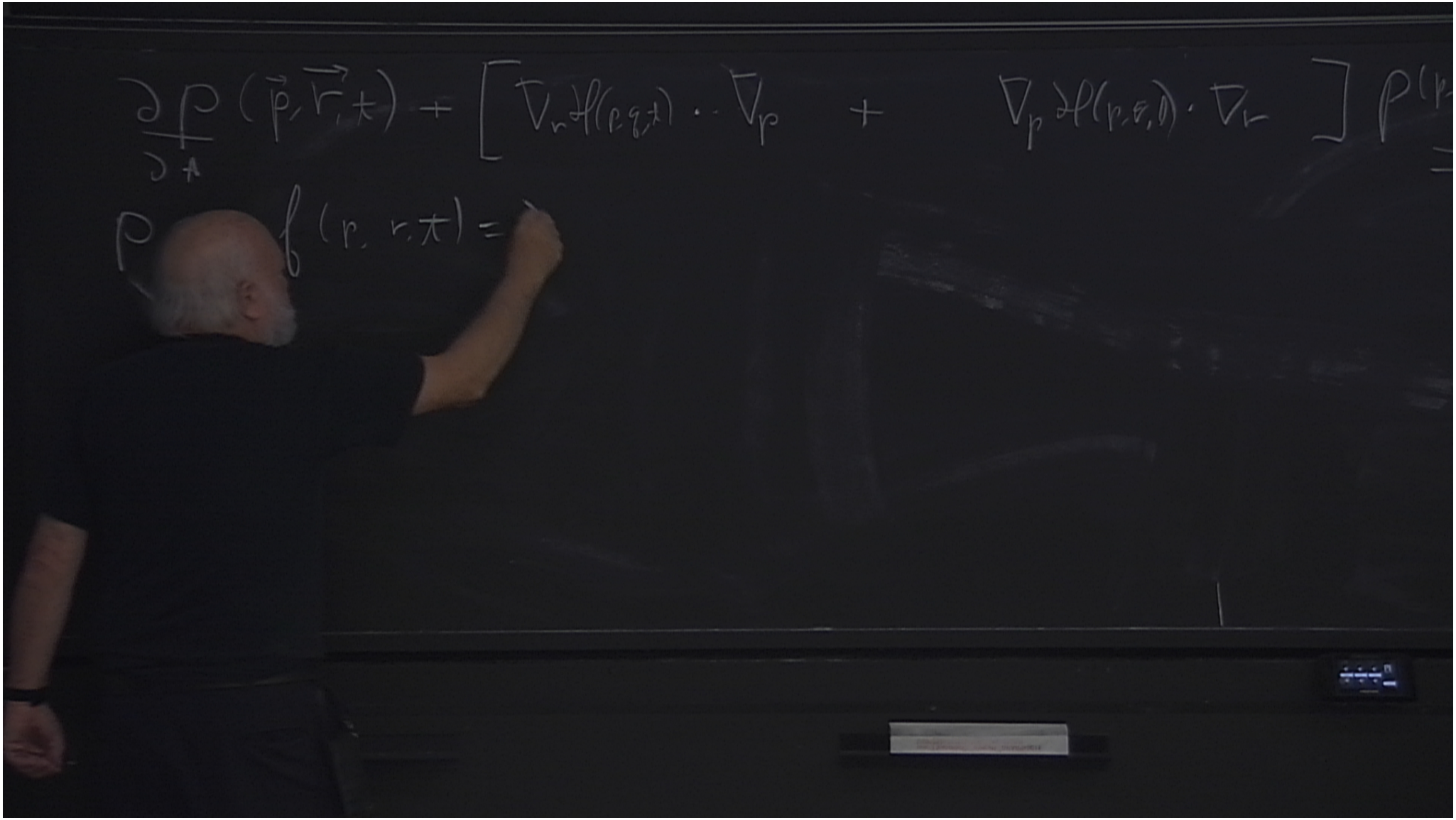
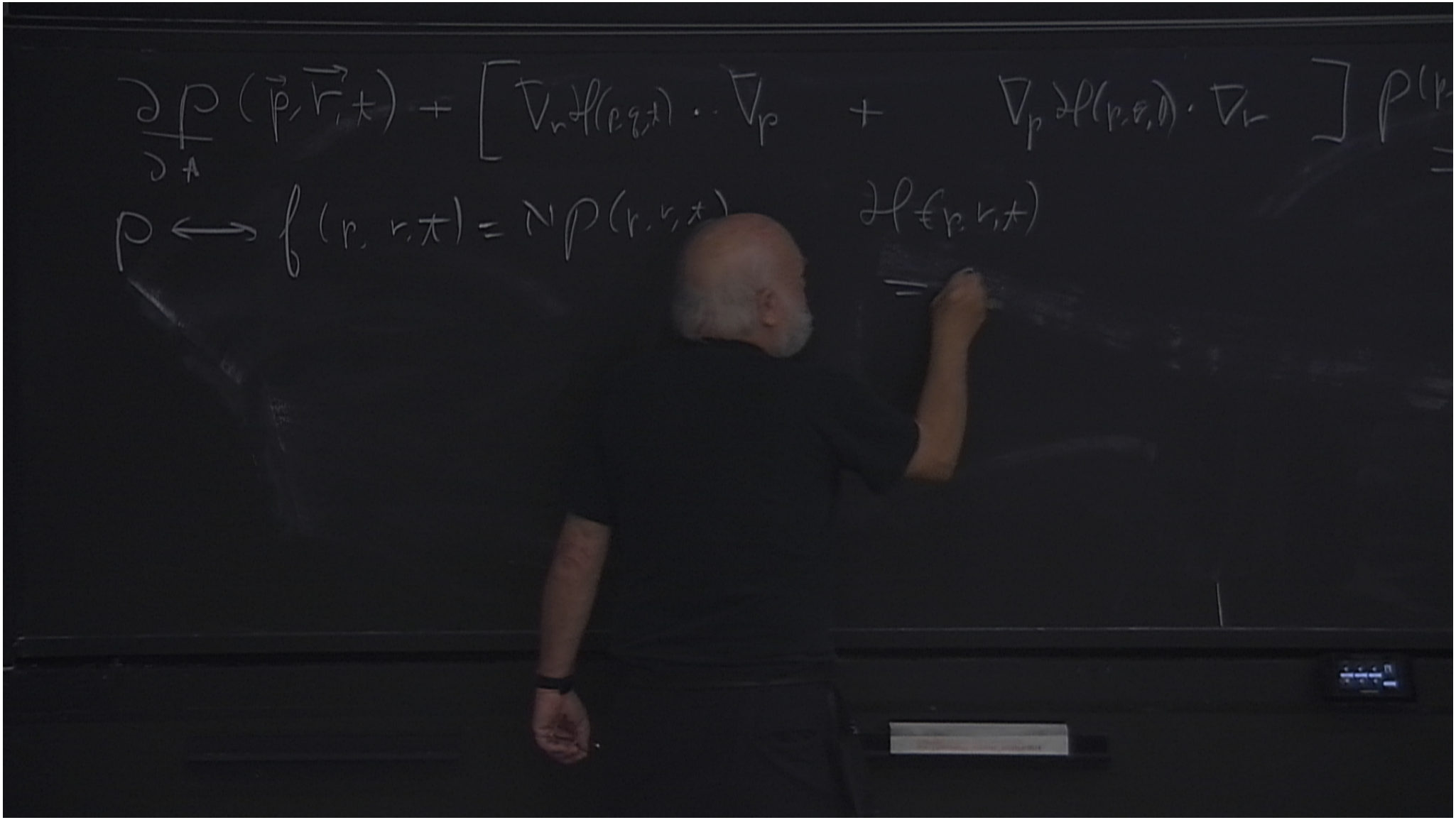

$\frac{\partial \rho}{\partial t}(\vec{p}, \vec{r}, t)$

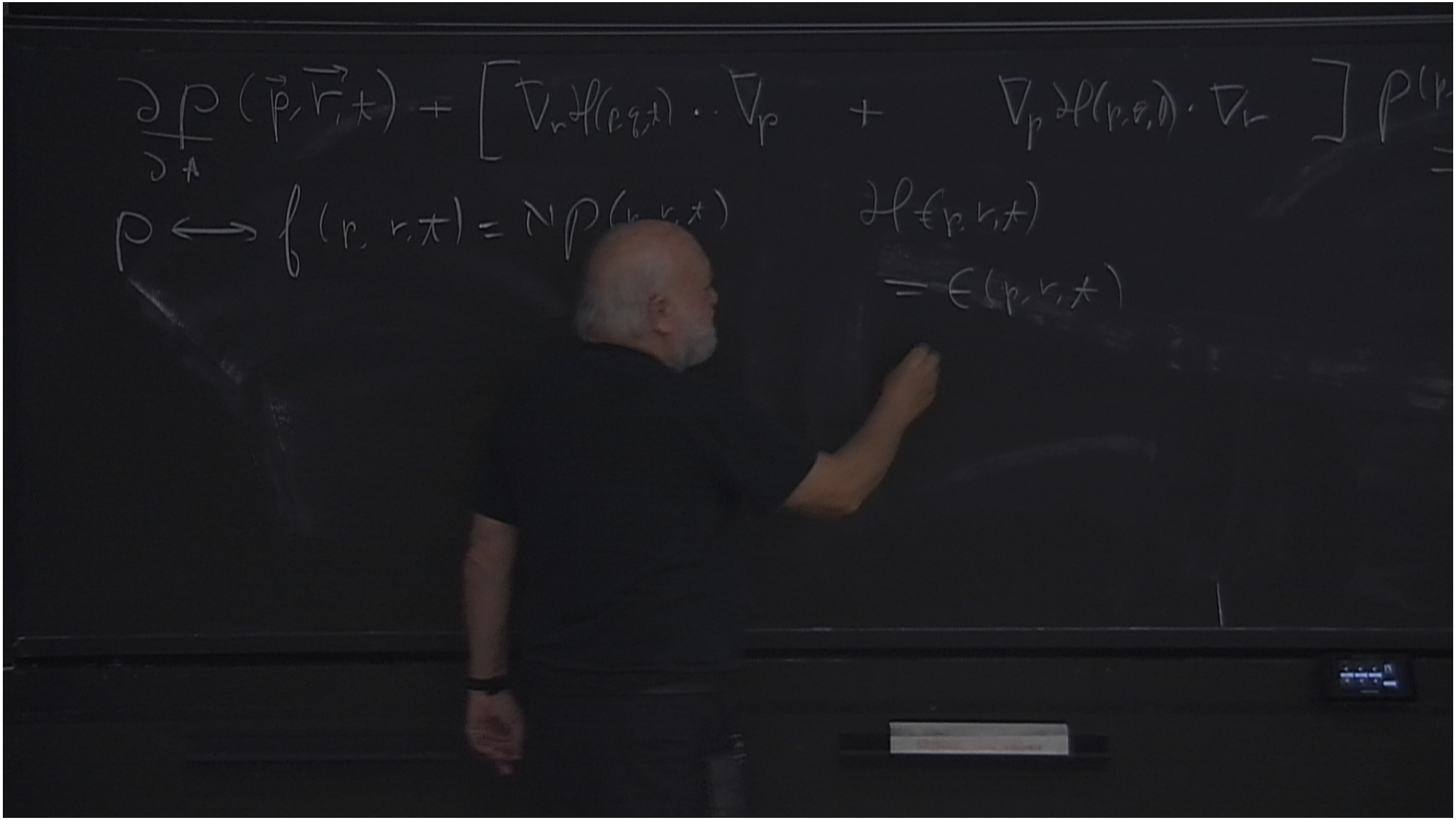

$\frac{\partial f}{\partial t}(\vec{p},\vec{r},t) +$
∇_p
$] \rho(p,r,t)$

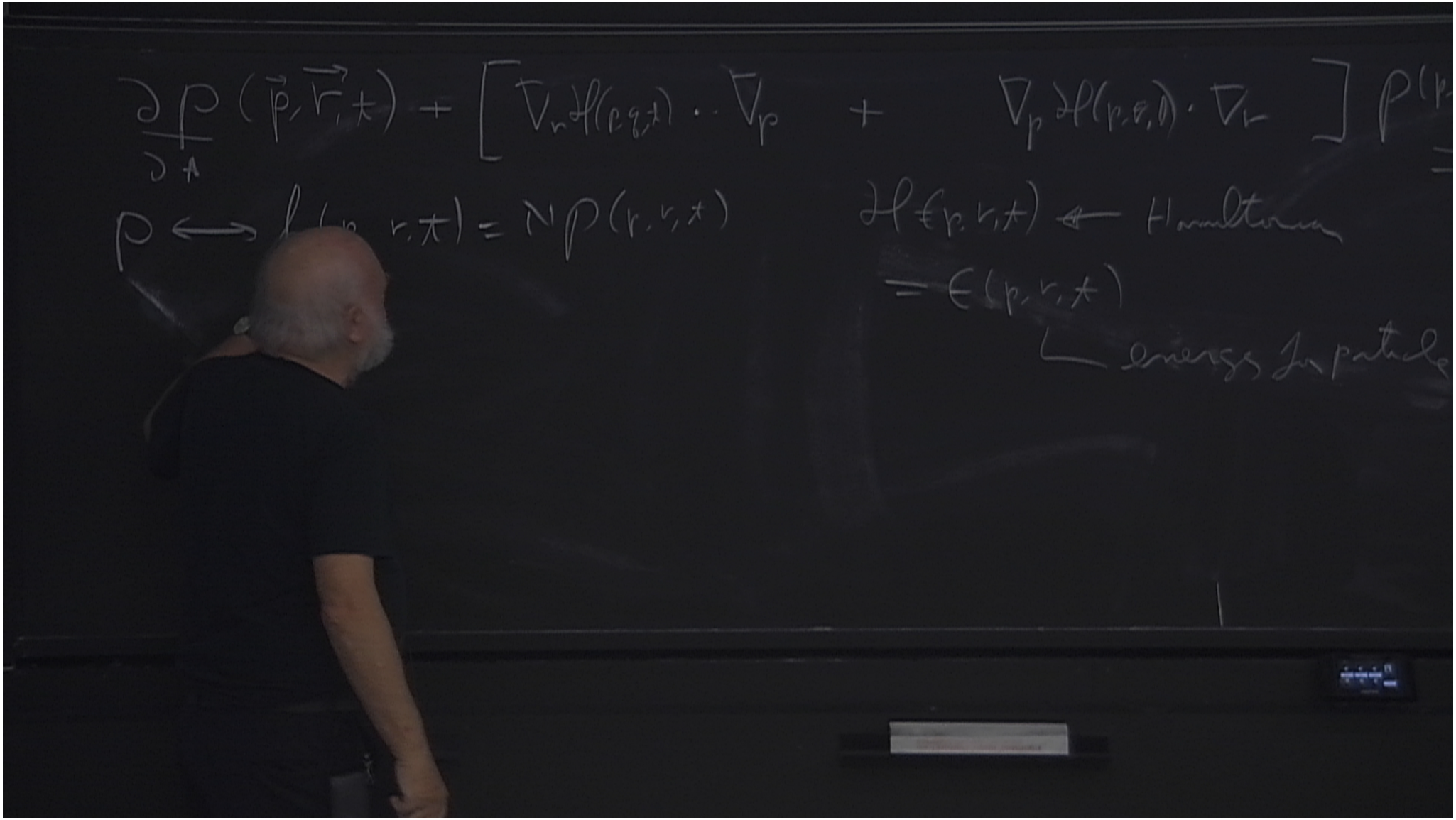



$\frac{\partial f}{\partial t}(\vec{p},\vec{r},t) + \left[\nabla_r \mathcal{H}(r,p,t)\cdot\nabla_p + \nabla_p \mathcal{H}(r,p,t)\cdot\nabla_r\right] f(r,p,t) = 0$






$\frac{\partial f}{\partial A}(\vec{p},\vec{r},t) + \left[\nabla_r \mathcal{H}(r,q,t)\cdot\nabla_p + \nabla_p \mathcal{H}(r,p,t)\cdot\nabla_r\right] f(p$
$\rho \longleftrightarrow f(r,p,t) = N\rho(r,p,t)$
$\mathcal{H}\epsilon(r,p,t)$
$= \epsilon(r,p,t)$


Hamiltonian
energy of particle

Liouville's Theorem and conservation of functions of the energy

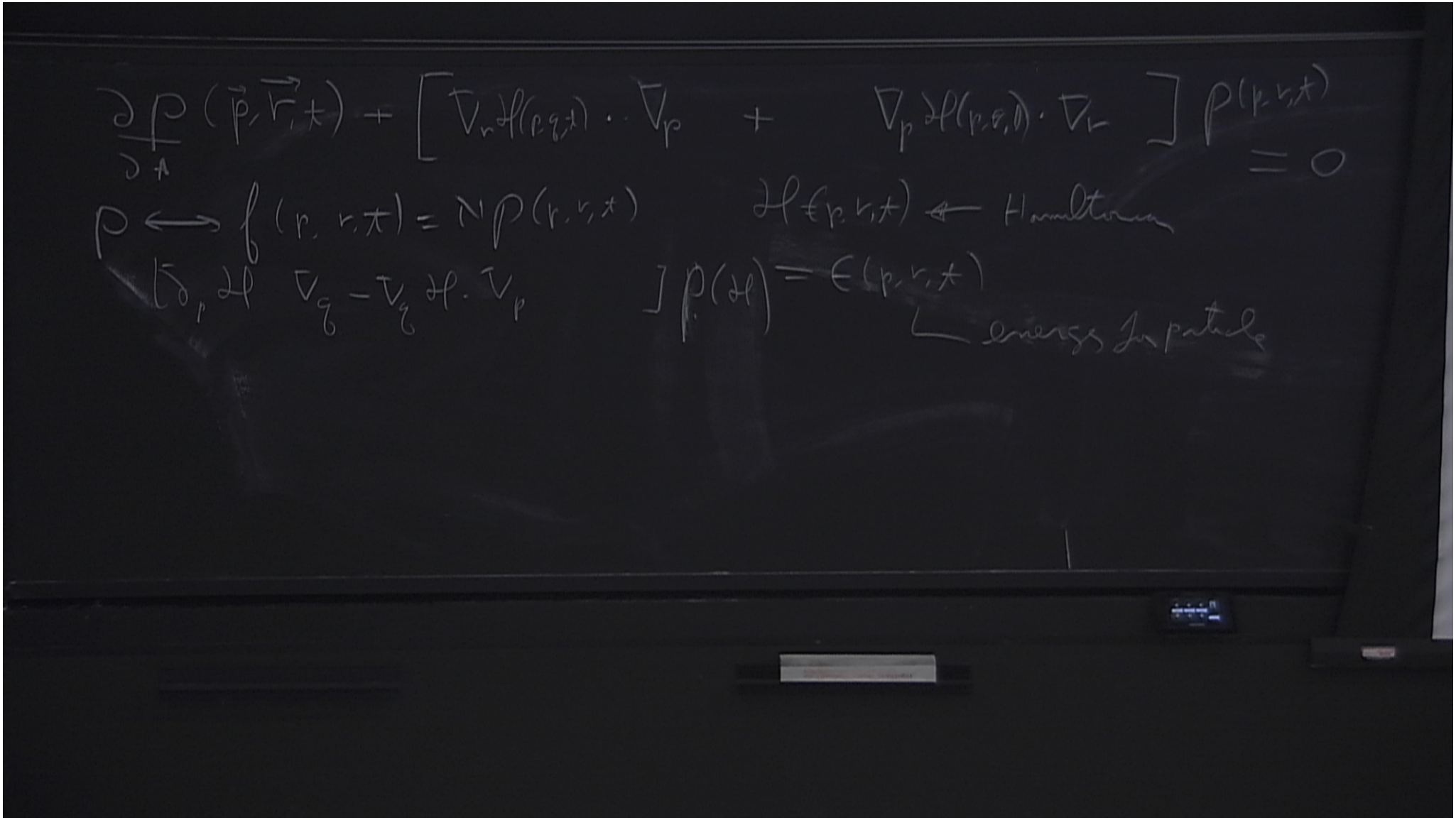
$$\partial_t \rho(p,q,t) + \sum_\alpha [(\partial_{p_\alpha} H)\partial_{q_\alpha} - (\partial_{q_\alpha} H)\partial_{p_\alpha}]\rho(p,q,t) = 0$$

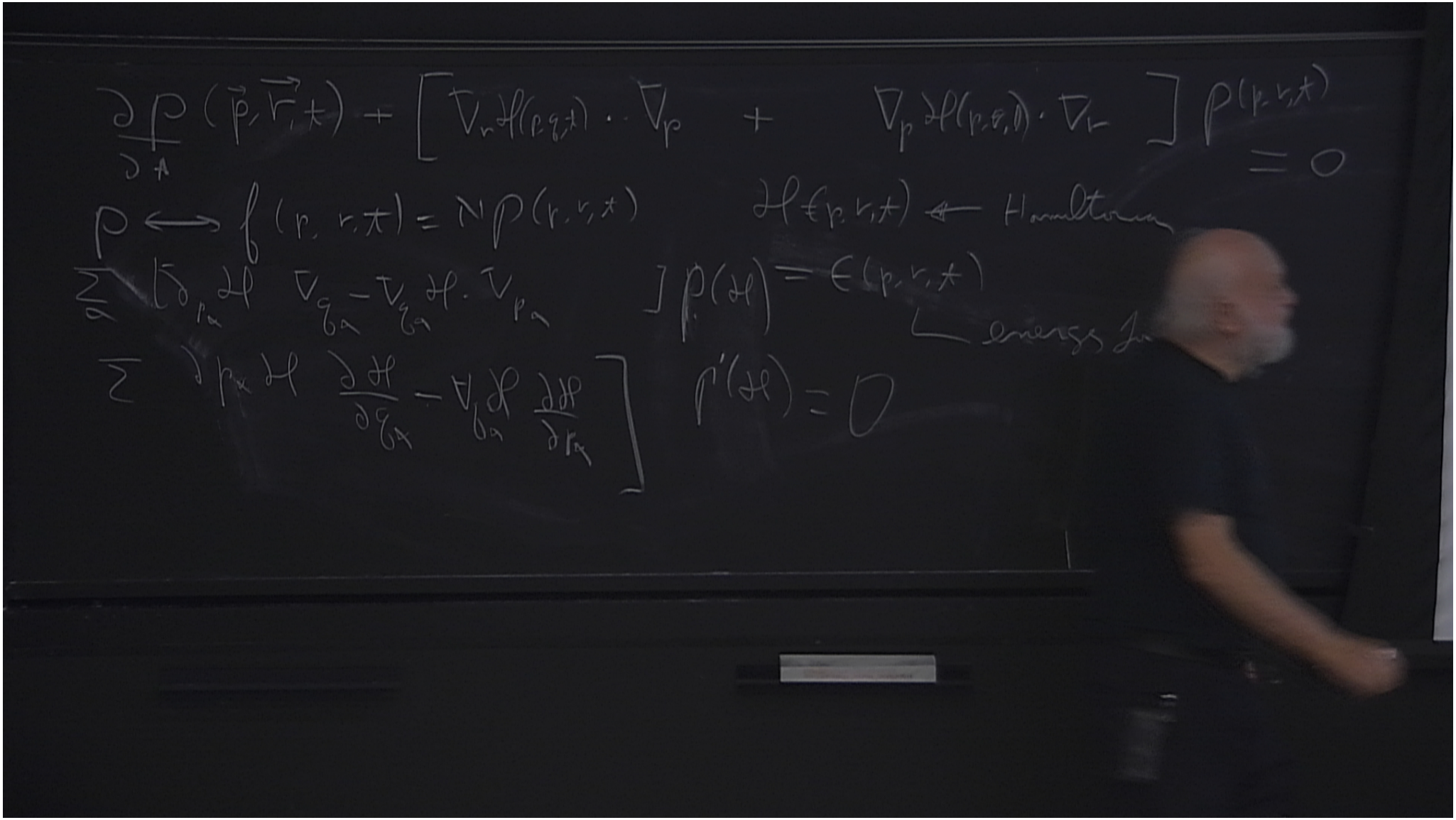
If $\rho(p,q,0)$ is any function of the Hamiltonian, e.g. $\rho(p,q,0) = Z^{-1}\exp[-\beta H(p,q)]$ then this same functional form will hold for all times, assuming that the Hamiltonian has no explicit time dependence. $\rho(p,q,0)=f(H(p,q))$ implies that $\rho(p,q,t)=f(H(p,q))$ for any f.

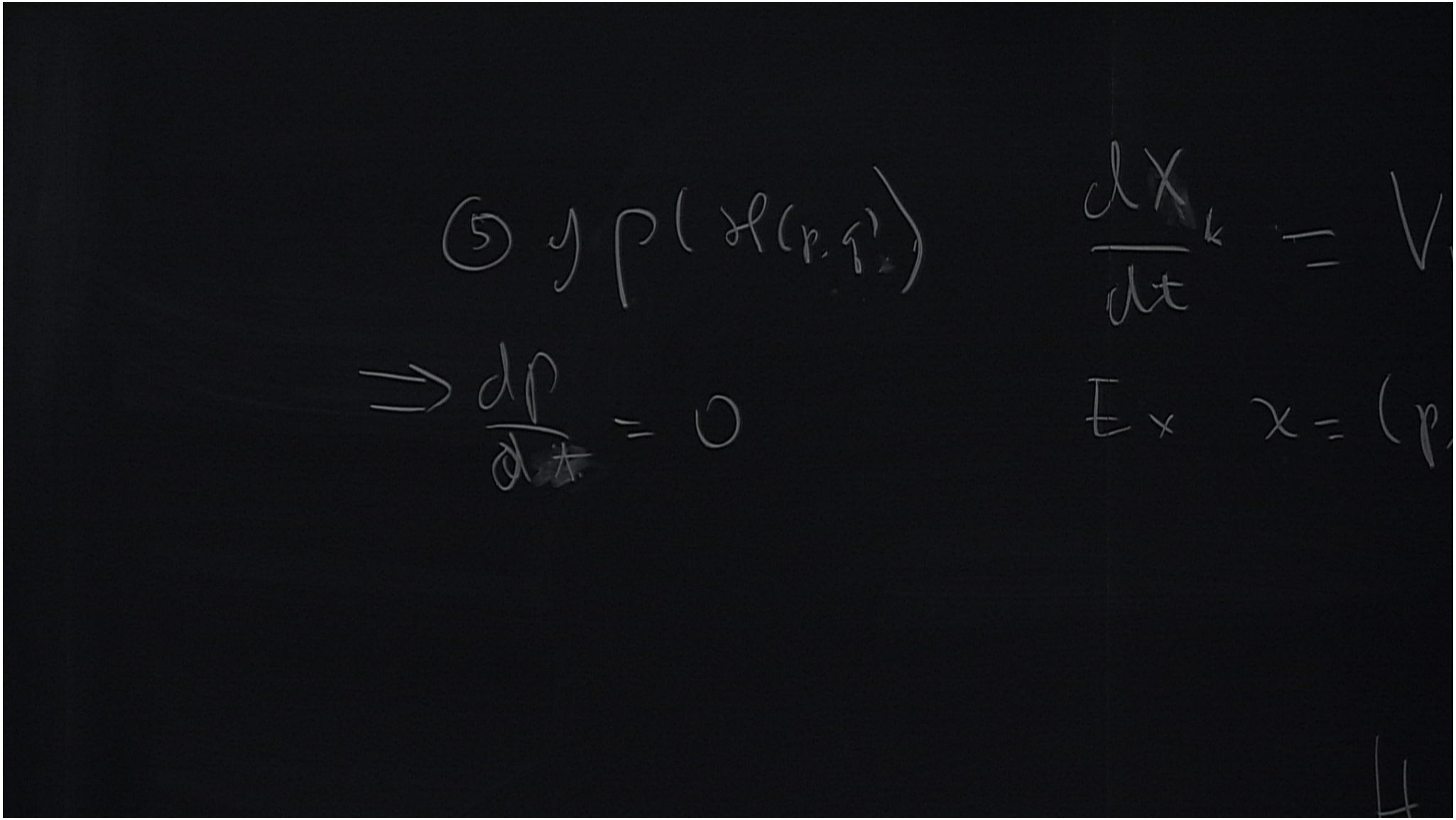
Further, if ρ is any function of a time-independent H and of any other conserved functions of p and q, with no explicit time-dependence, then ρ will be a solution of our equation. Thus, not only is the Maxwell- Boltzmann distribution function a solution describing the equilibrium time-dependence of a Hamiltonian system, there are many other solutions as well.

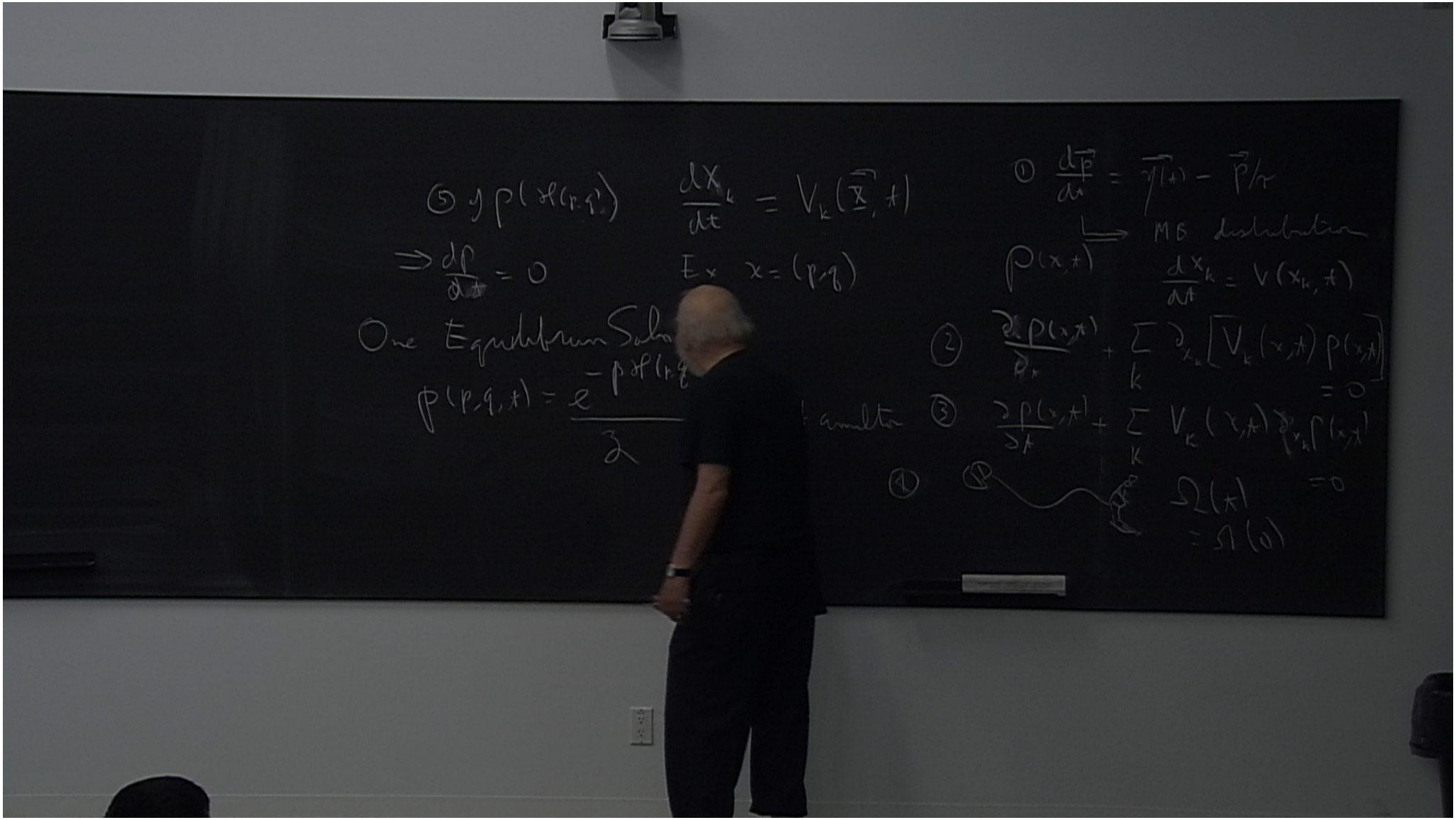
Classical mechanics is not enough to specify a unique equilibrium probability density in a classical system. Something else is needed in addition.

Give some examples of functions of H with and without explicit time-dependence.


$\frac{\partial \rho}{\partial t}(\vec{p},\vec{r},t) + \left[\nabla_r \mathcal{H}(r,p,t)\cdot\nabla_p + \nabla_p \mathcal{H}(r,p,t)\cdot\nabla_r\right]\rho(r,p,t) = 0$
$\rho \longleftrightarrow f(r,p,t) = N\rho(r,p,t)$
$\mathcal{H}(r,p,t)$ ← Hamiltonian
$[\delta_{ij}\mathcal{H}\ \nabla_q - \nabla_q \mathcal{H}\cdot\nabla_p]\ f(\mathcal{H}) = \epsilon(r,p,t)$
└ energy / particle


Hamiltonian
energy


⑤ $\psi\rho(\psi(p,q))$
$\frac{dX}{dt}_k = V$
$\Rightarrow \frac{d\rho}{dt} = 0$
$E_x \quad x = (p$


$\frac{dX_k}{dt} = V_k(\vec{X}, t)$
$\frac{dp}{dt} = 0$
Ex $x = (p, q)$
$\frac{d\vec{p}}{dt} = \vec{\gamma}(t) - \vec{p}/\tau$
MB distribution
$\frac{dx_k}{dt} = V(x_k, t)$
$\frac{\partial \rho(x,t)}{\partial t} + \sum_k \partial_{x_k}[V_k(x,t)\rho(x,t)] = 0$
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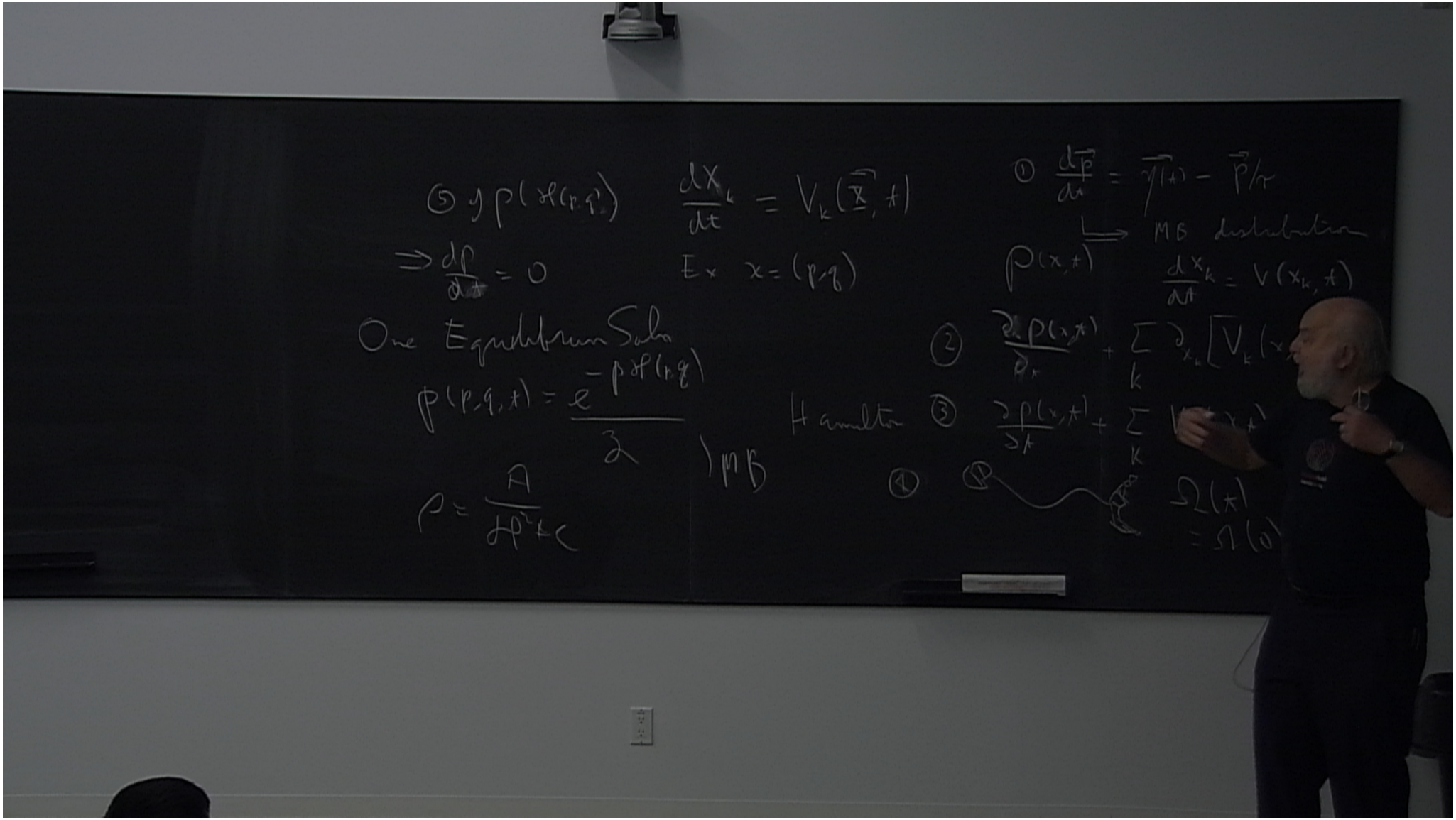
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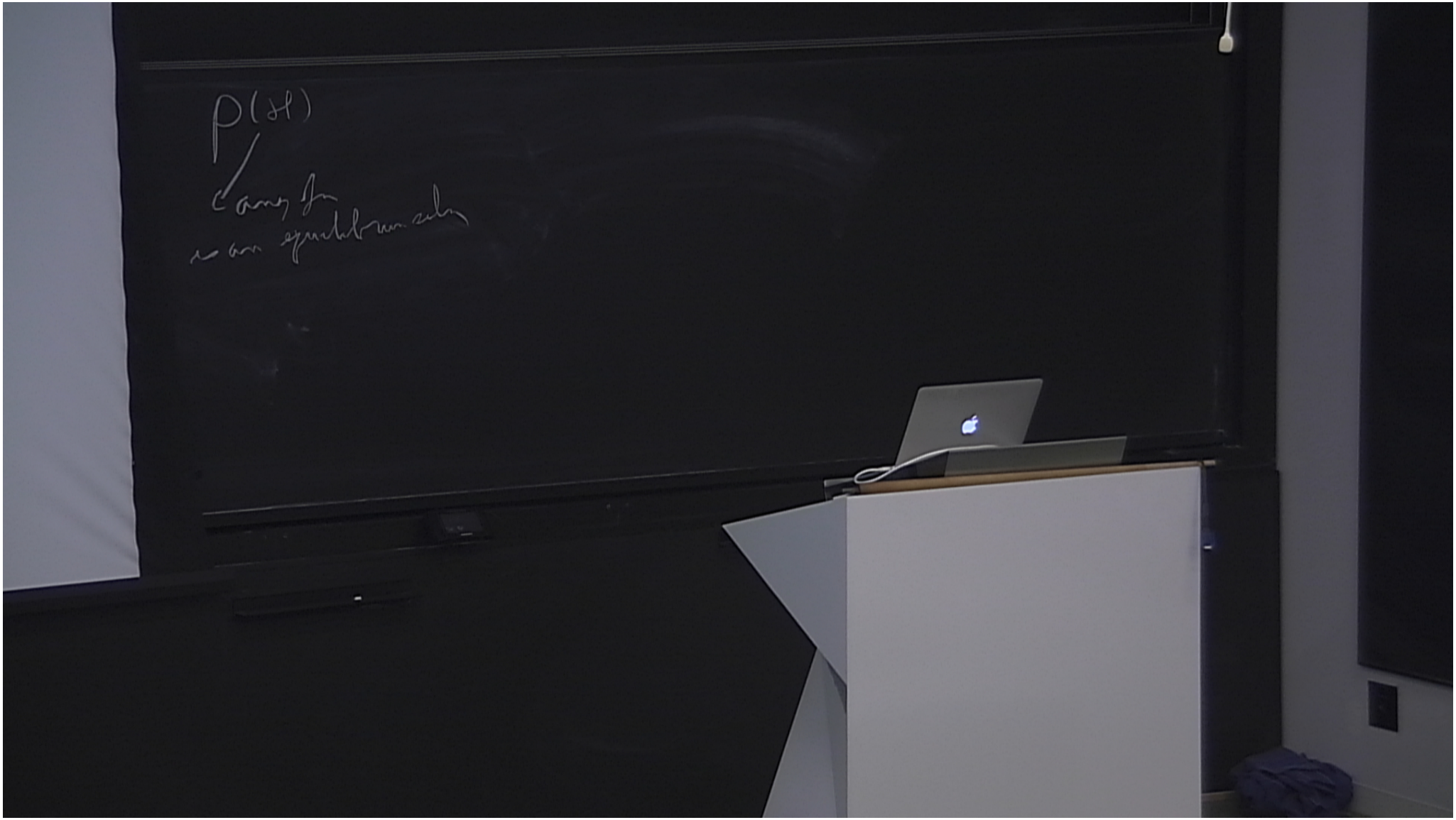
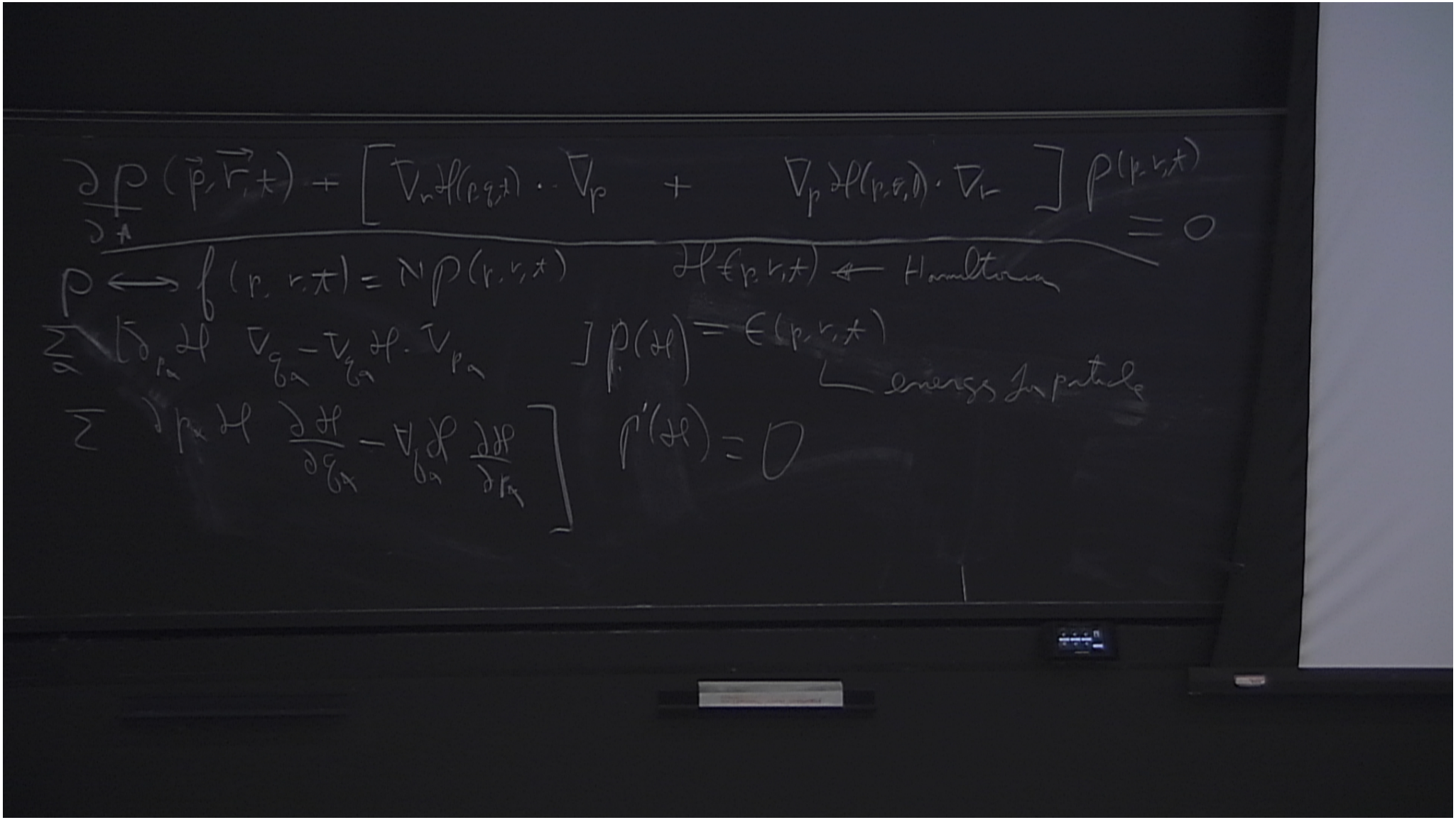
MB distribution
$\frac{dx_k}{dt} = V(x_k, t)$
$\Rightarrow \frac{df}{dt} = 0$
Ex $x = (p, q)$
One Equilibrium Solu
$\rho(p, q, t) = \frac{e^{-\beta \mathcal{H}(p,q)}}{Z}$
MB
Hamilton
$\rho = \frac{A}{2\beta^2 k}$
$\Omega(t) = \Omega(0)$


ρ(x)



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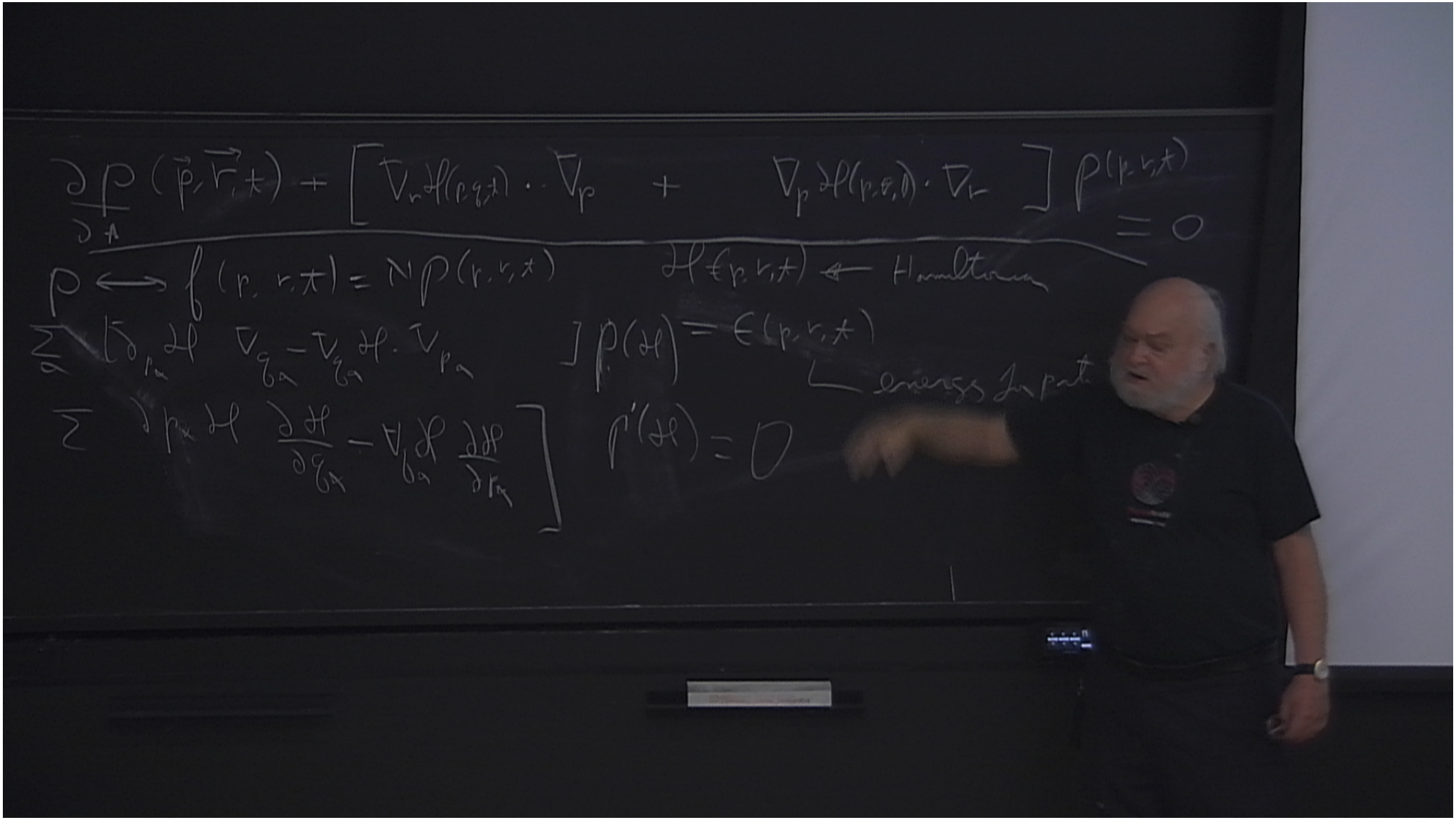
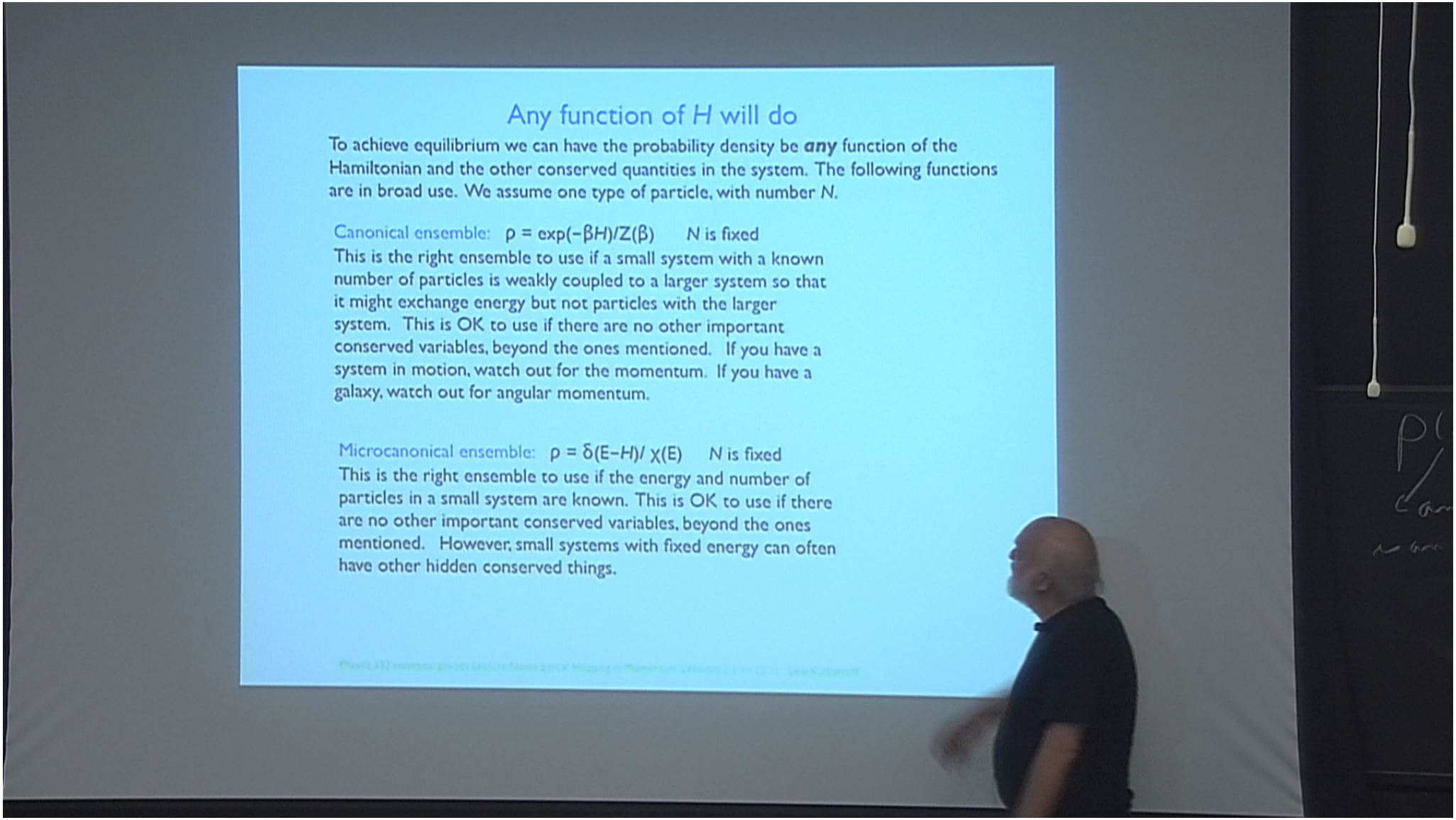

Hamiltonian
energy


Any function of H will do
To achieve equilibrium we can have the probability density be any function of the Hamiltonian and the other conserved quantities in the system. The following functions are in broad use. We assume one type of particle, with number N.
Canonical ensemble: ρ = exp(−βH)/Z(β) N is fixed
This is the right ensemble to use if a small system with a known number of particles is weakly coupled to a larger system so that it might exchange energy but not particles with the larger system. This is OK to use if there are no other important conserved variables, beyond the ones mentioned. If you have a system in motion, watch out for the momentum. If you have a galaxy, watch out for angular momentum.
Microcanonical ensemble: ρ = δ(E−H)/ χ(E) N is fixed
This is the right ensemble to use if the energy and number of particles in a small system are known. This is OK to use if there are no other important conserved variables, beyond the ones mentioned. However, small systems with fixed energy can often have other hidden conserved things.

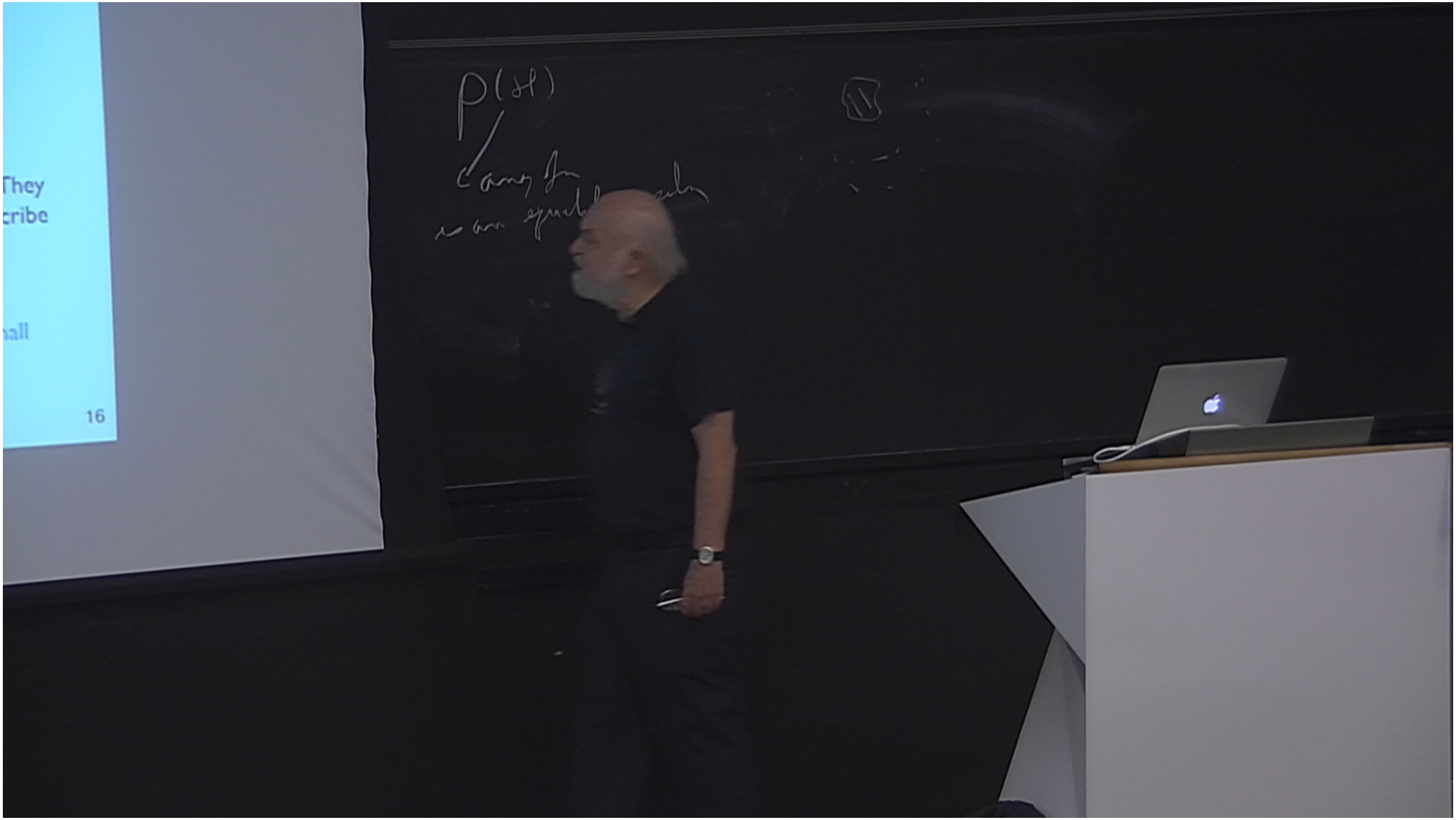
Any function of *H* will do, continued

Grand Canonical ensemble: $\rho = \exp[-\beta(H-\mu N)]/\Xi(\beta,\mu)$
This is the right ensemble to use if a small system is weakly coupled to a larger system so that it might exchange energy and particles with the larger system. This is OK to use if there are no other important conserved variables, beyond the ones mentioned. However watch out for many different kinds of particles, each type has a conserved N. Classical mechanics is provided with extra factors of 1/N! in ρ for each different kind of particle. Particle statistics automatically does this in quantum theory.

For large systems, and for most purposes, all these ensembles are equivalent to one another. Why?

However, the canonical and grand canonical ensembles are particularly convenient. They have the property that if the Hamiltonian is a sum of independent pieces, ρ, will describe independent probabilities for these pieces. This result simplifies both thought and calculations.

We still need a demonstration that the canonical ensemble is a good result for a small system.


They
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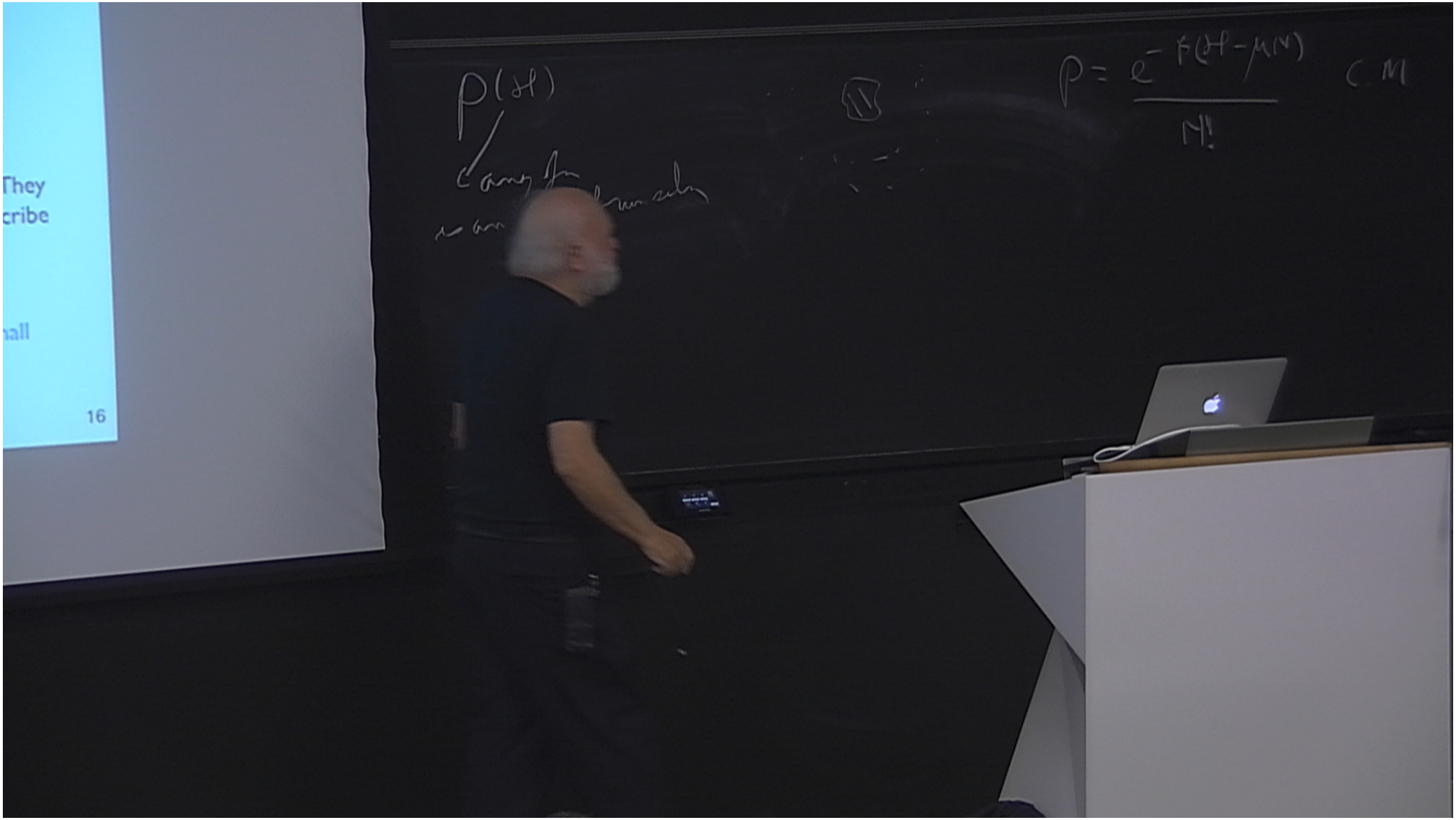
hall
16


They
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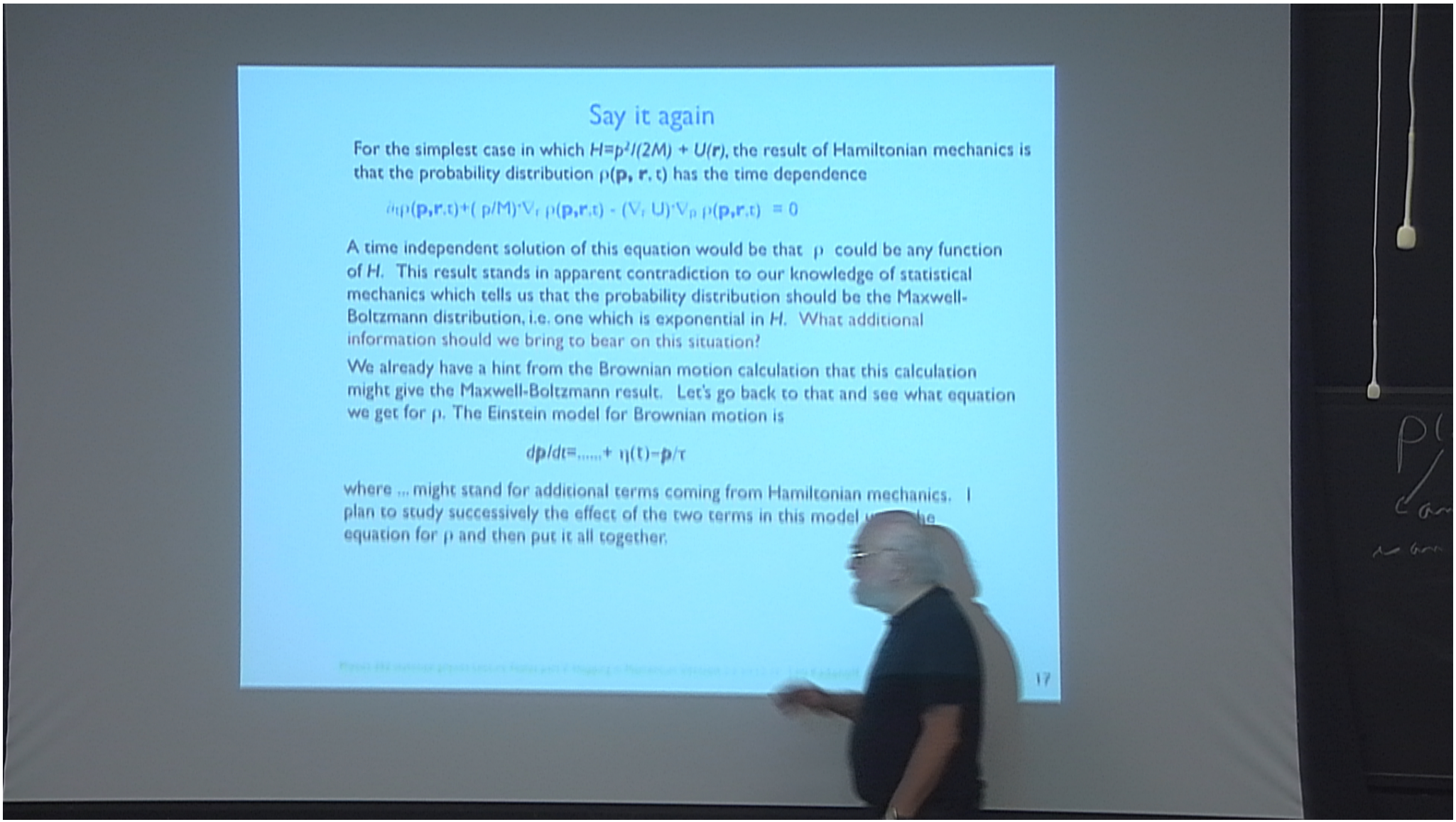
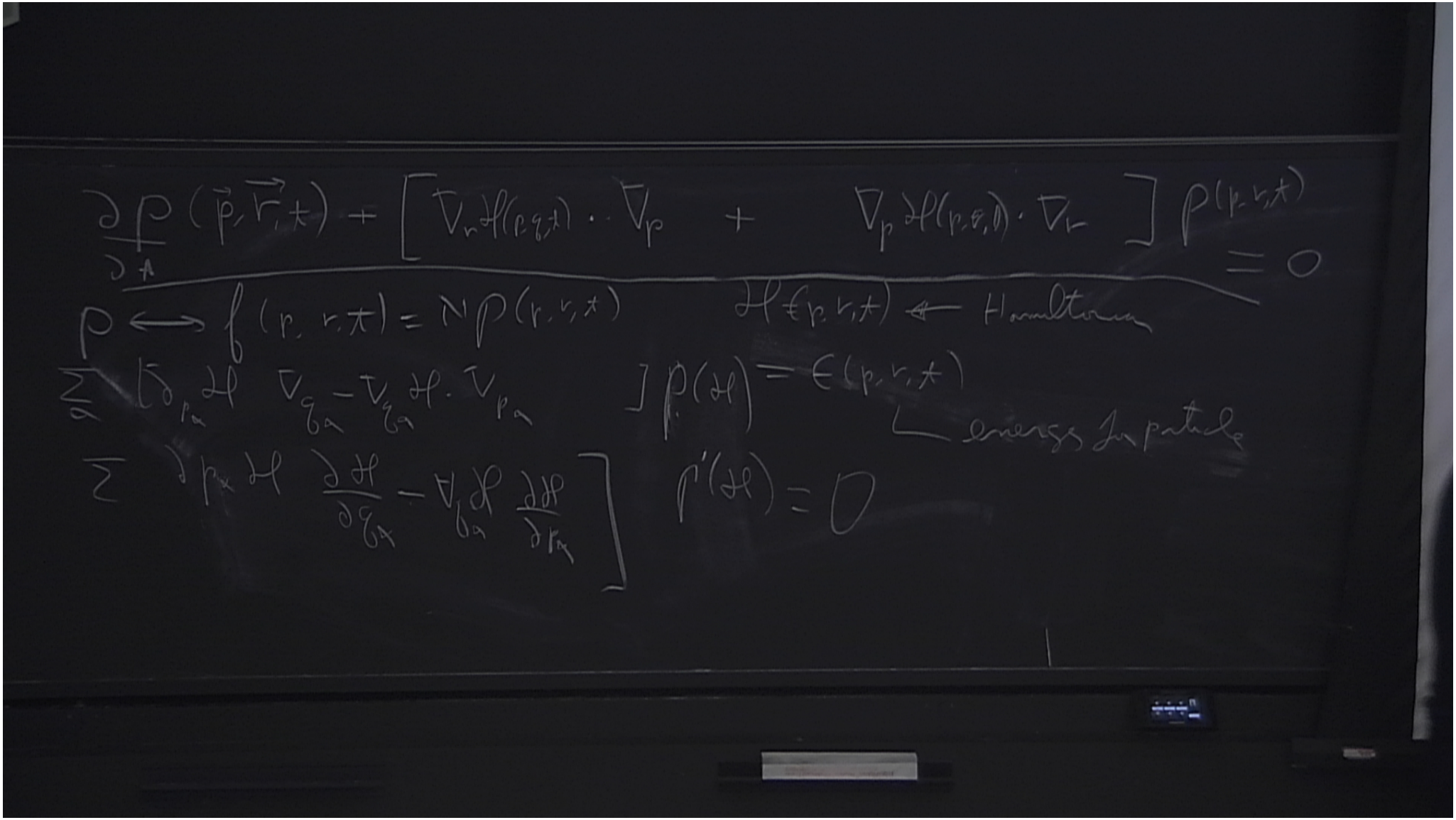
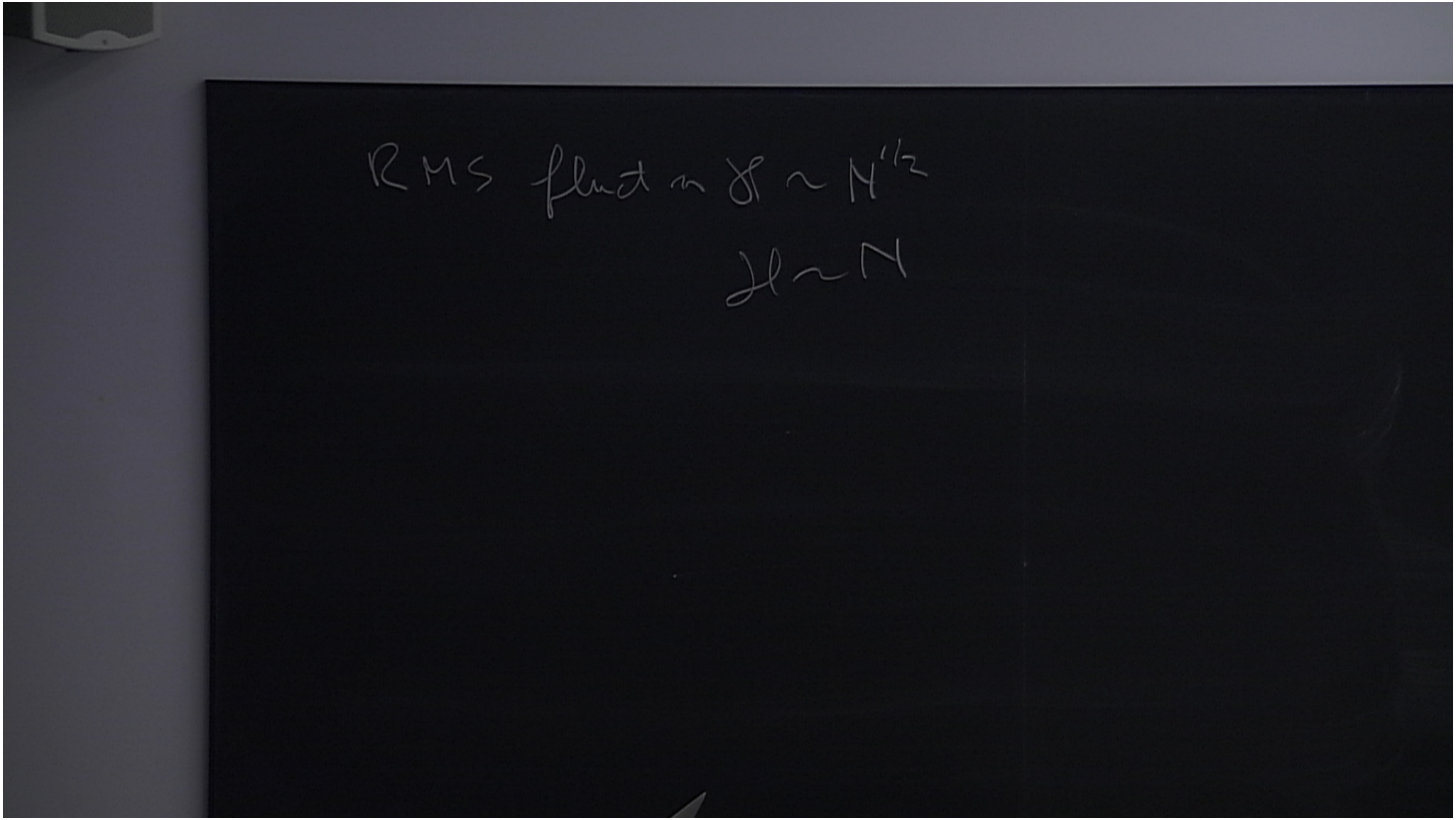
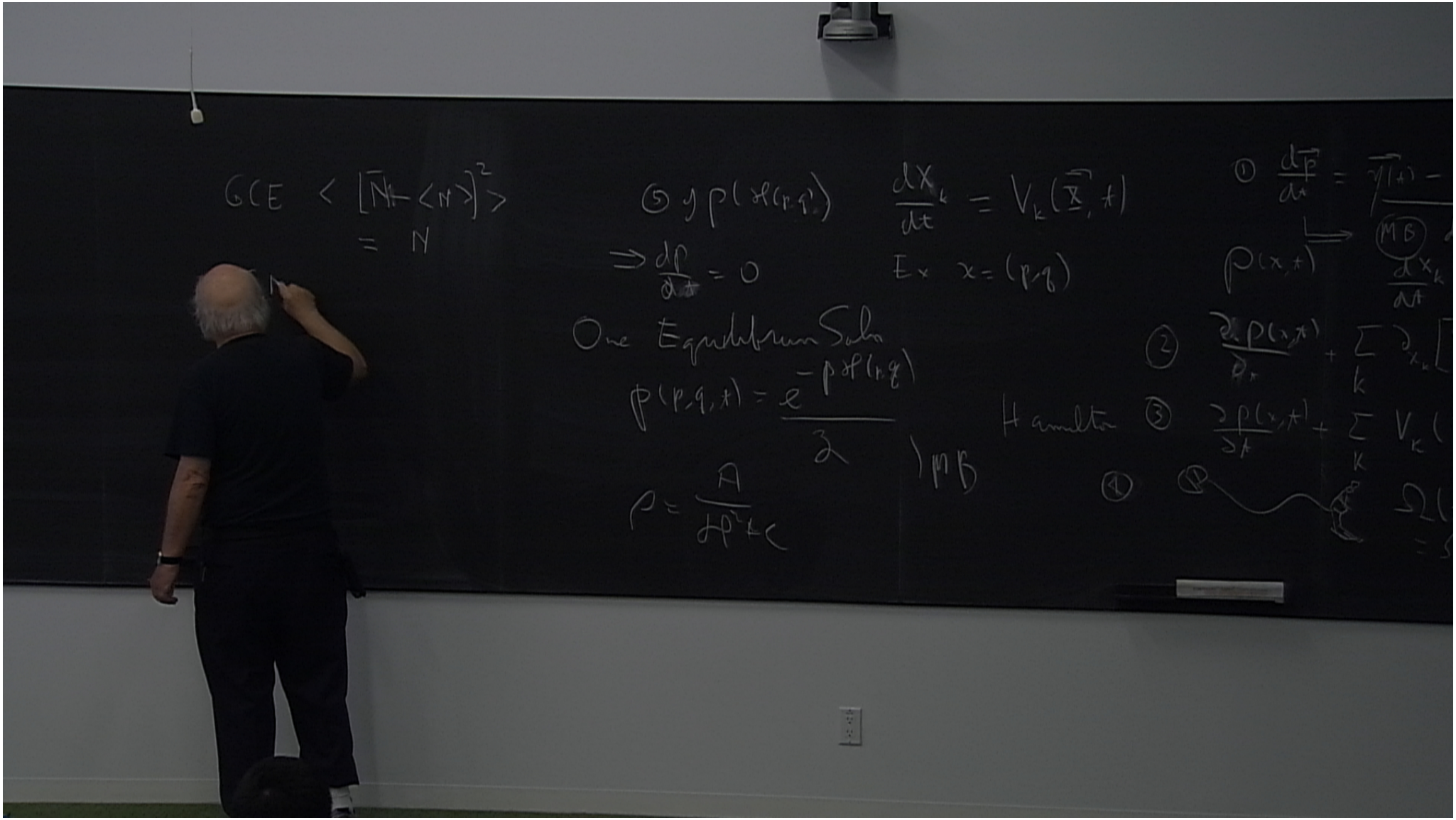
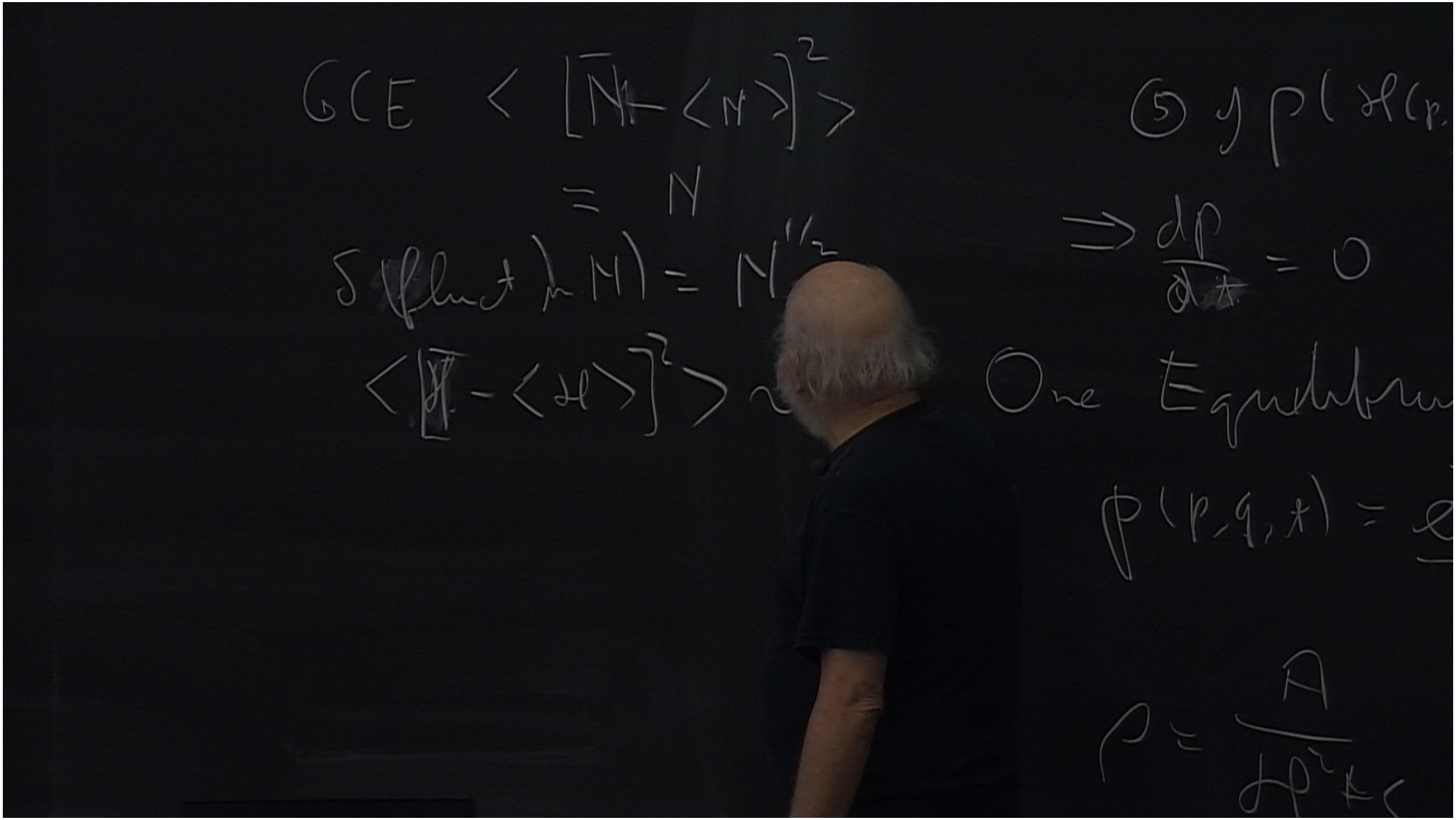
hall
16
P(H)
P = e^{-β(H-μN)} / N!

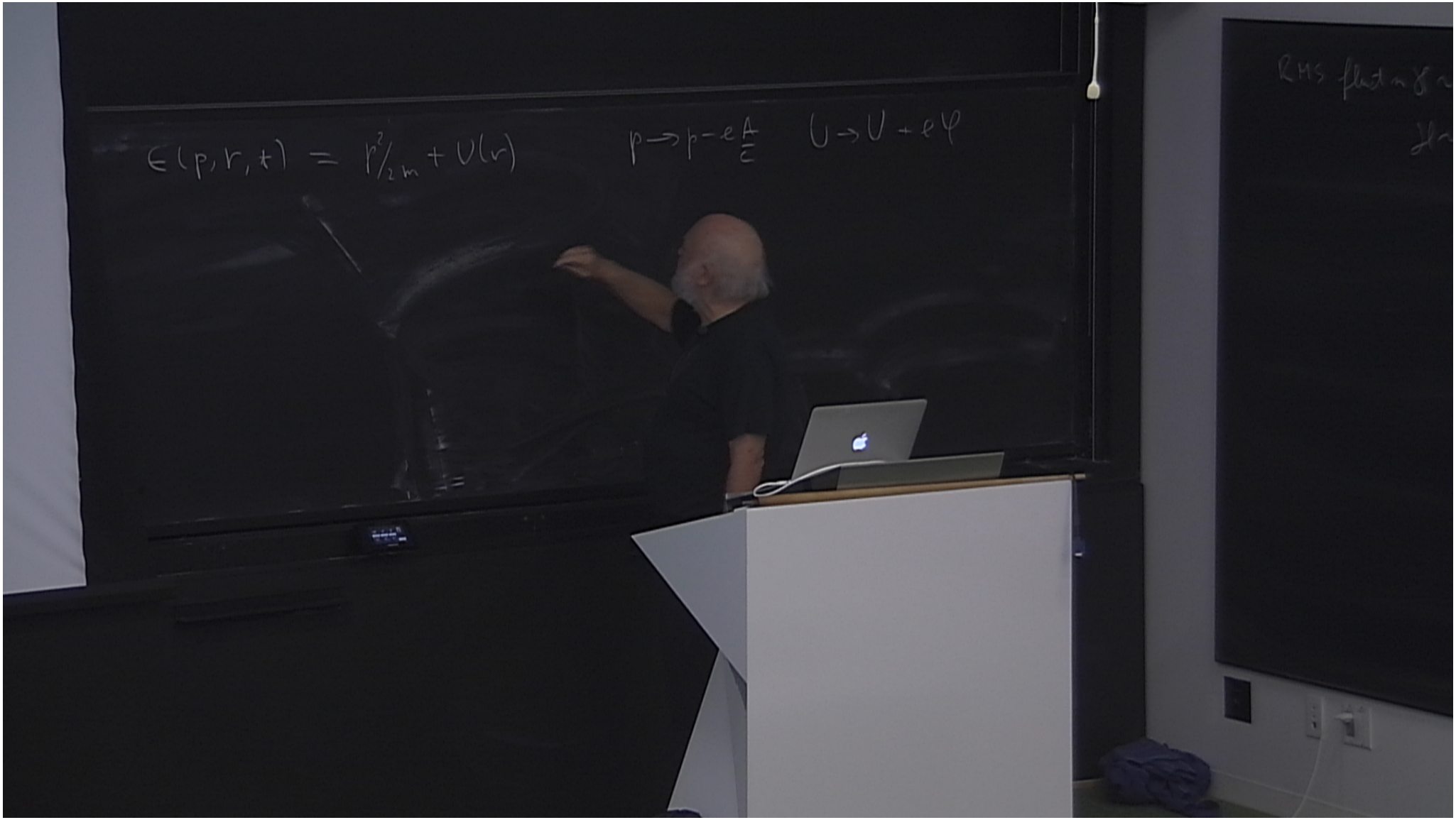

Say it again
For the simplest case in which $H=p^2/(2M) + U(r)$, the result of Hamiltonian mechanics is that the probability distribution $\rho(\mathbf{p}, \mathbf{r}, t)$ has the time dependence
$\partial_t \rho(\mathbf{p},\mathbf{r},t)+(p/M)\cdot\nabla_r \rho(\mathbf{p},\mathbf{r},t) - (\nabla_r U)\cdot\nabla_p \rho(\mathbf{p},\mathbf{r},t) = 0$
A time independent solution of this equation would be that ρ could be any function of H. This result stands in apparent contradiction to our knowledge of statistical mechanics which tells us that the probability distribution should be the Maxwell-Boltzmann distribution, i.e. one which is exponential in H. What additional information should we bring to bear on this situation?
We already have a hint from the Brownian motion calculation that this calculation might give the Maxwell-Boltzmann result. Let's go back to that and see what equation we get for ρ. The Einstein model for Brownian motion is
$d\mathbf{p}/dt=.....+ \eta(t)-\mathbf{p}/\tau$
where ... might stand for additional terms coming from Hamiltonian mechanics. I plan to study successively the effect of the two terms in this model
equation for ρ and then put it all together.
17

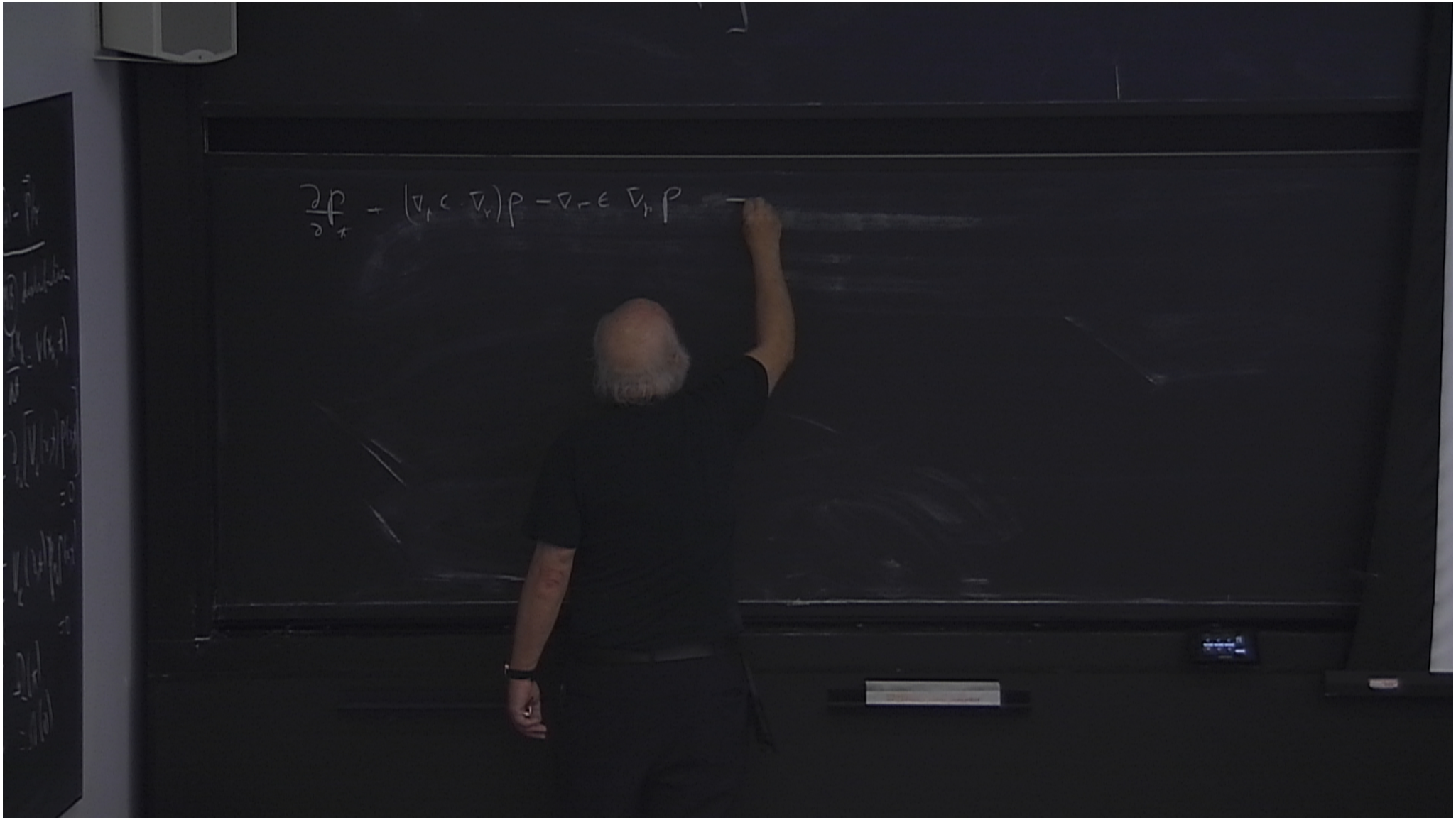

$\frac{\partial \rho}{\partial t}(\vec{p},\vec{r},t) + \left[\nabla_r \mathcal{H}(p,q,t)\cdot\nabla_p + \nabla_p \mathcal{H}(p,r,t)\cdot\nabla_r\right] \rho(p,r,t) = 0$
$\rho \longleftrightarrow f(p,r,t) = N\rho(p,r,t)$
$\mathcal{H}(p,r,t)$ ← Hamiltonian
$\sum_a \left[\nabla_{p_a}\mathcal{H}\,\nabla_{q_a} - \nabla_{q_a}\mathcal{H}\cdot\nabla_{p_a}\right] f(\mathcal{H}) = \epsilon(p,r,t)$
└ energy of particle
$\sum \left[\nabla_{p_a}\mathcal{H}\,\frac{\partial \mathcal{H}}{\partial q_a} - \nabla_{q_a}\mathcal{H}\,\frac{\partial \mathcal{H}}{\partial p_a}\right] f'(\mathcal{H}) = 0$

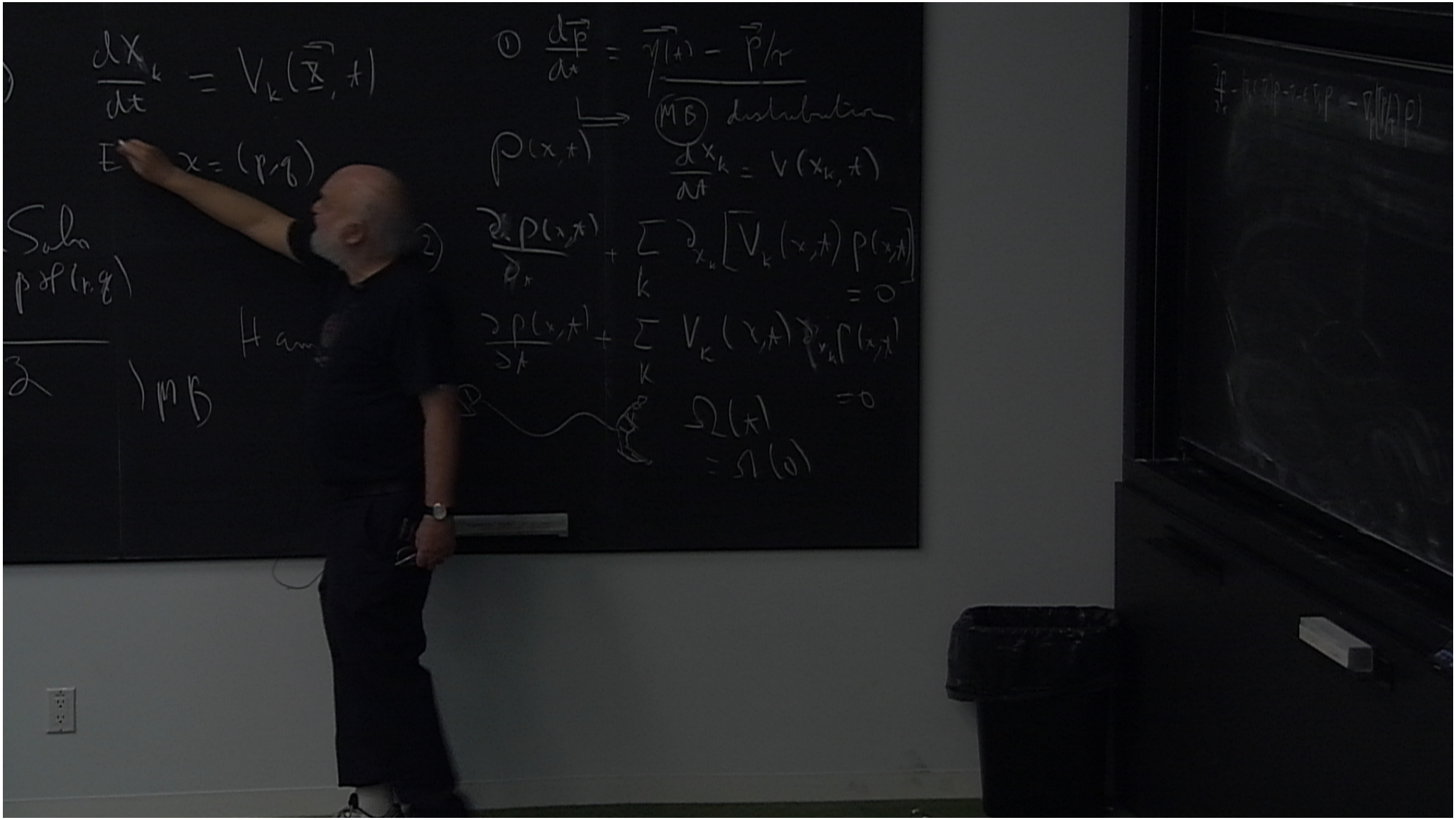

RMS fluct in $\gamma \sim N^{1/2}$
$\gamma \sim N$

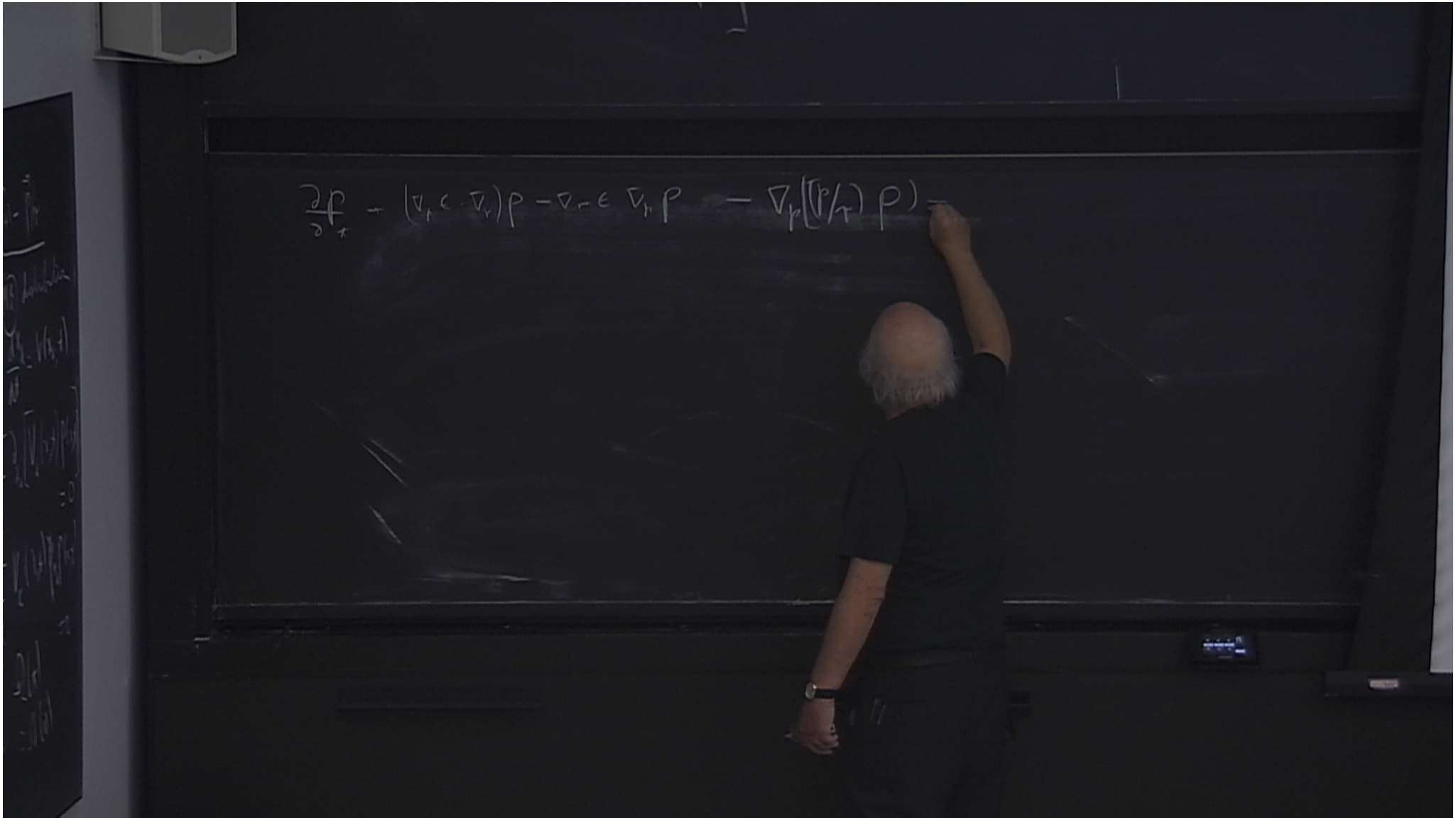

GCE
= N
One Equilibrium Solu
Hamilto

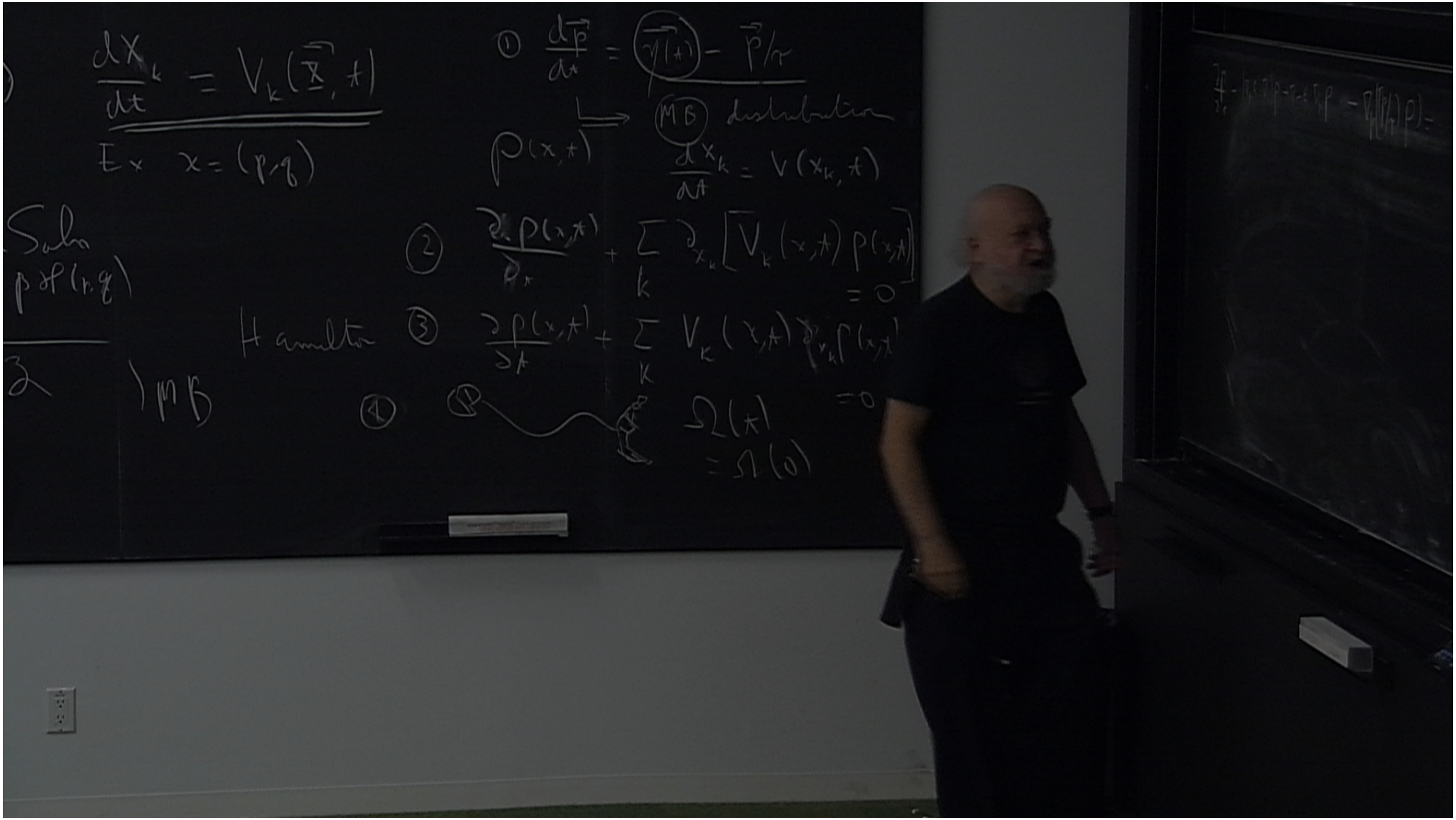

GCE
= N
One


$\epsilon(p,r,t) = p^2/2m + U(r)$
$p \to p - e\frac{A}{c}$
$U \to U + e\varphi$
RMS




$\frac{dX_k}{dt} = V_k(\vec{X}, t)$
$\frac{d\vec{p}}{dt} = \frac{\vec{v}(t) - \vec{p}/\tau}{}$
MB distribution
$\rho(x,t)$
$\frac{dx_k}{dt} = V(x_k, t)$
$\frac{\partial \rho(x,t)}{\partial t} + \sum_k \partial_{x_k}\left[V_k(x,t)\,\rho(x,t)\right] = 0$
$\frac{\partial \rho(x,t)}{\partial t} + \sum_k V_k(x,t)\,\partial_{x_k}\rho(x,t) = 0$
$\Omega(t) = \Omega(0)$


$\frac{\partial f}{\partial t}$ + $(\nabla_r \cdot \nabla_r) f$ − $\nabla_r \cdot \epsilon \nabla_p f$ − $\nabla_p \left[\left(F/m \right) f \right]$ =


$\frac{dx_k}{dt} = V_k(\vec{x}, t)$
$\frac{d\vec{p}}{dt} = \vec{F}(t) - \vec{p}/\tau$
MB distribution
$\rho(x,t)$
$\frac{dx_k}{dt} = V(x_k, t)$
$\frac{\partial \rho(x,t)}{\partial t} + \sum_k \partial_{x_k}[V_k(x,t)\rho(x,t)] = 0$
$\frac{\partial \rho(x,t)}{\partial t} + \sum_k V_k(x,t)\partial_{x_k}\rho(x,t) = 0$
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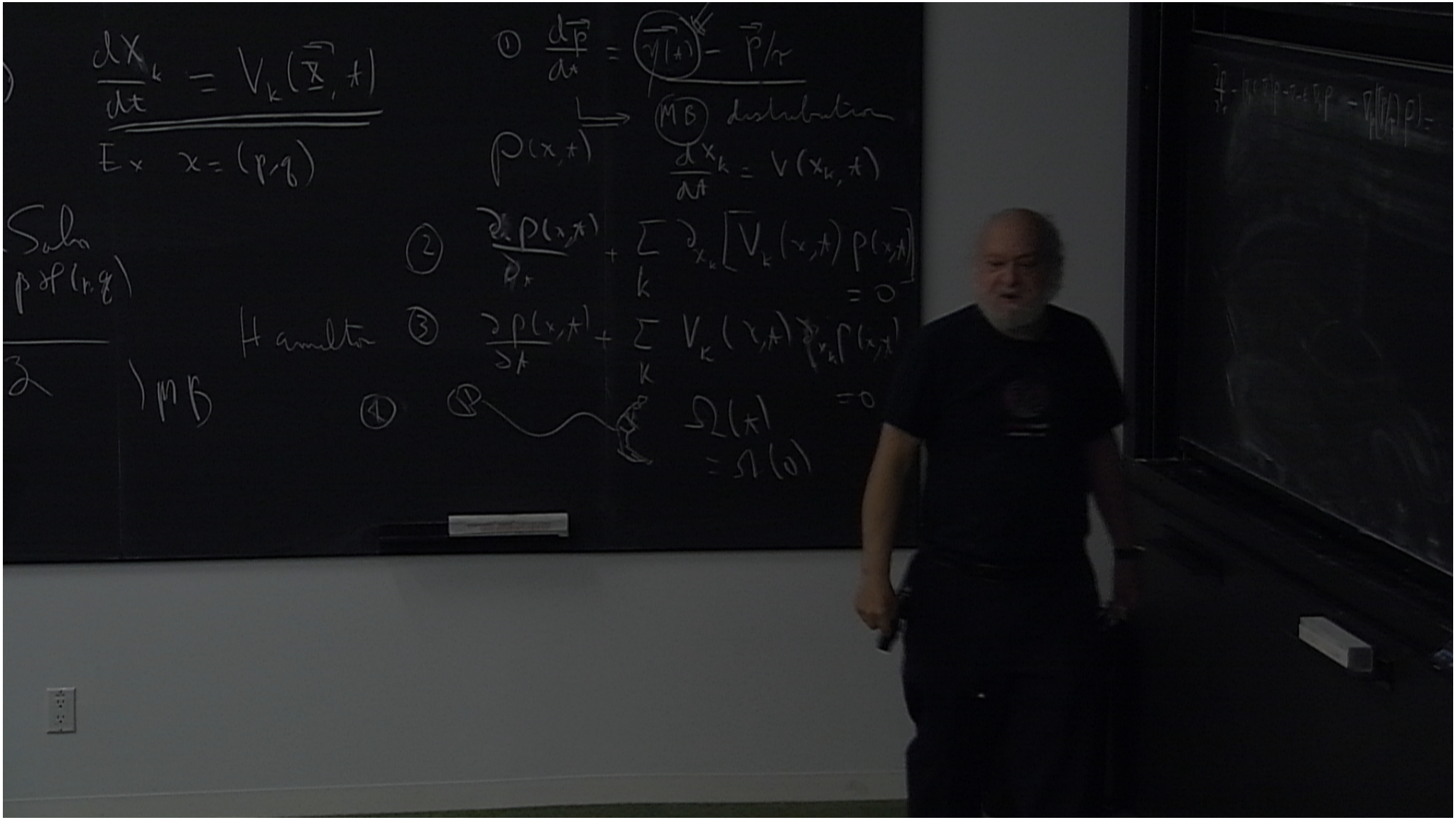
$\frac{dx_k}{dt} = V_k(\vec{x}, t)$
$\frac{d\vec{p}}{dt} = \vec{\gamma}(t) - \vec{p}/\tau$
MB distribution
$\rho(x,t)$
$\frac{dx_k}{dt} = V(x_k, t)$
$\frac{\partial \rho(x,t)}{\partial t} + \sum_k \partial_{x_k}[V_k(x,t)\rho(x,t)] = 0$
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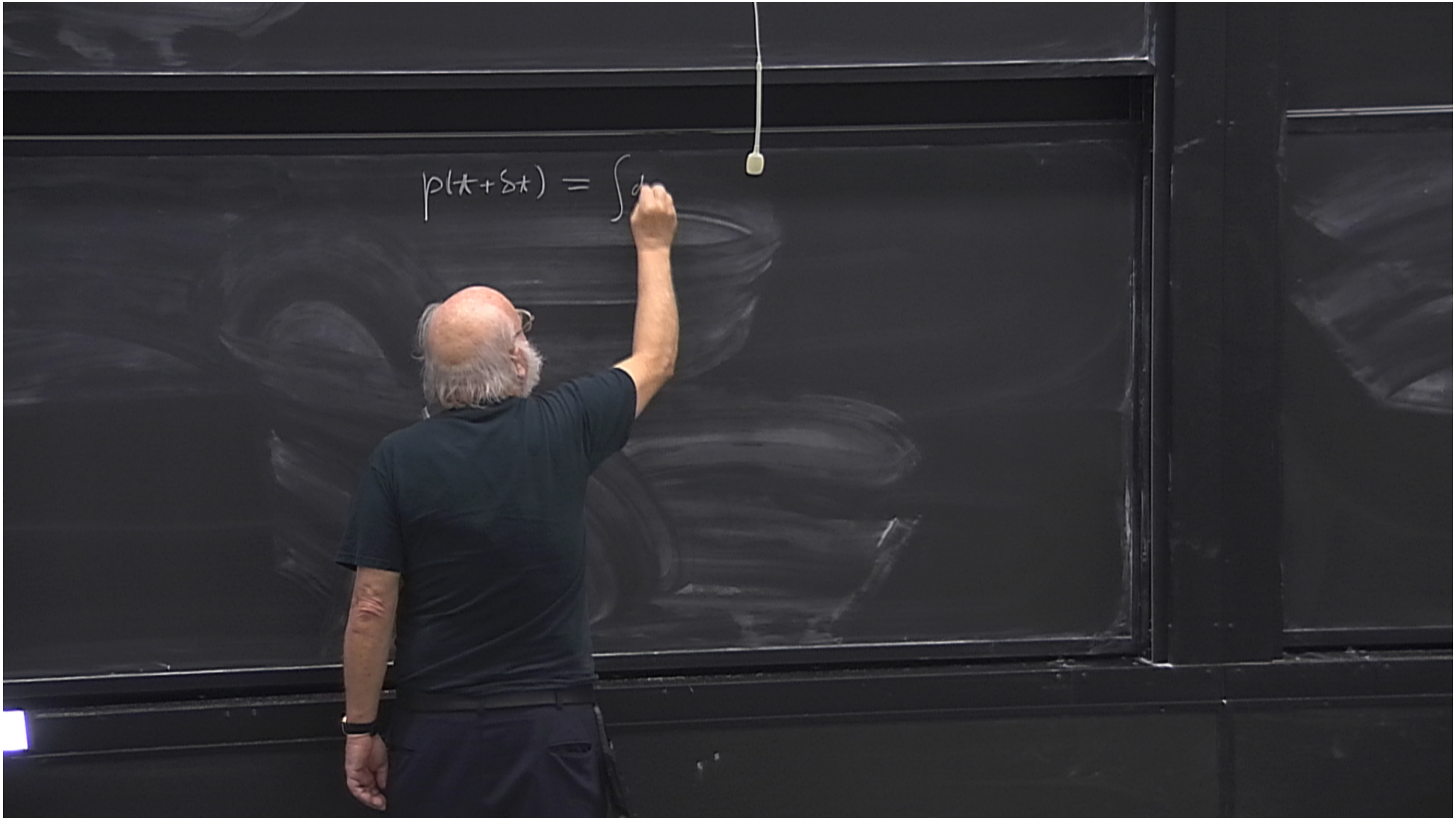
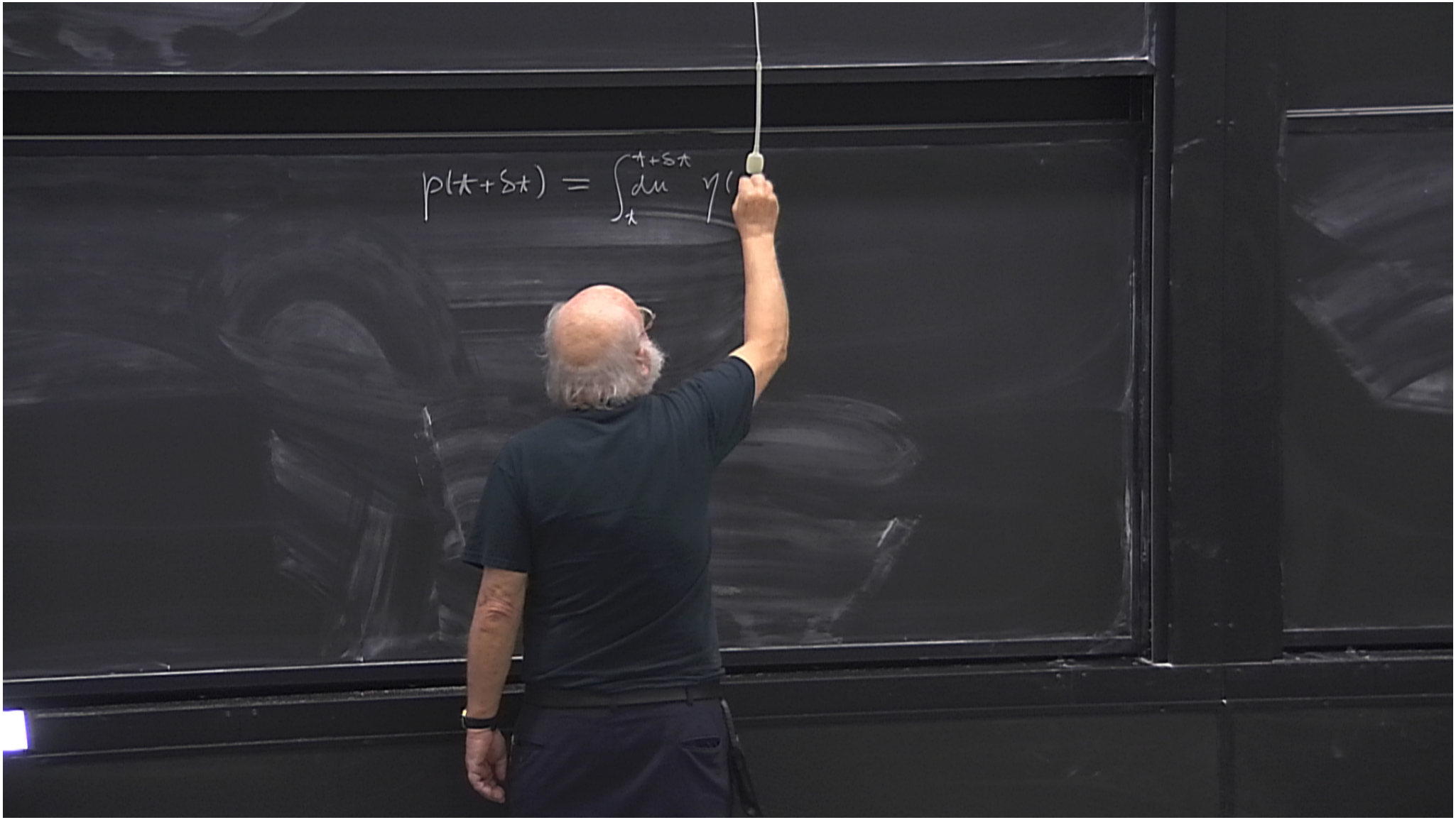
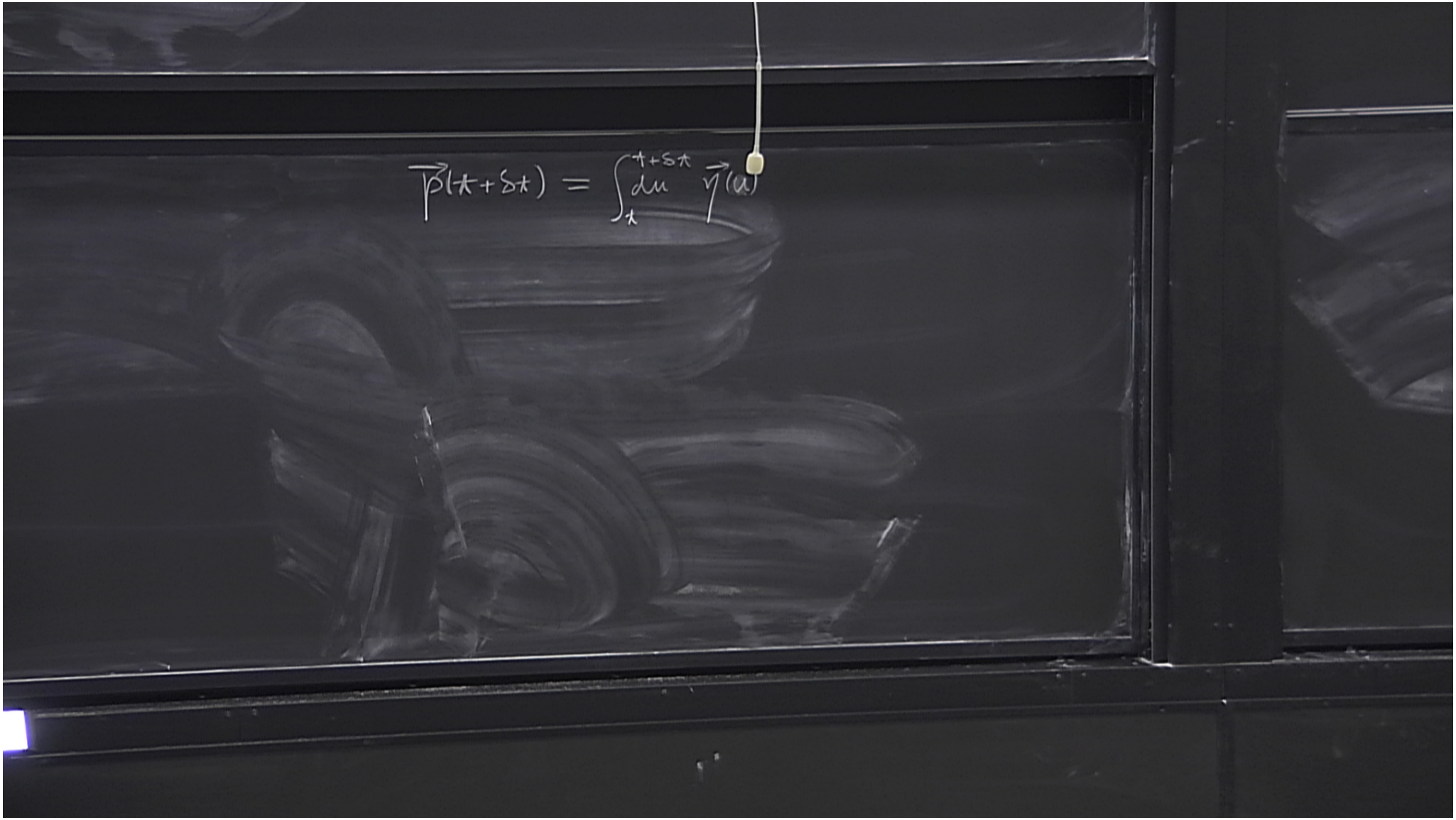
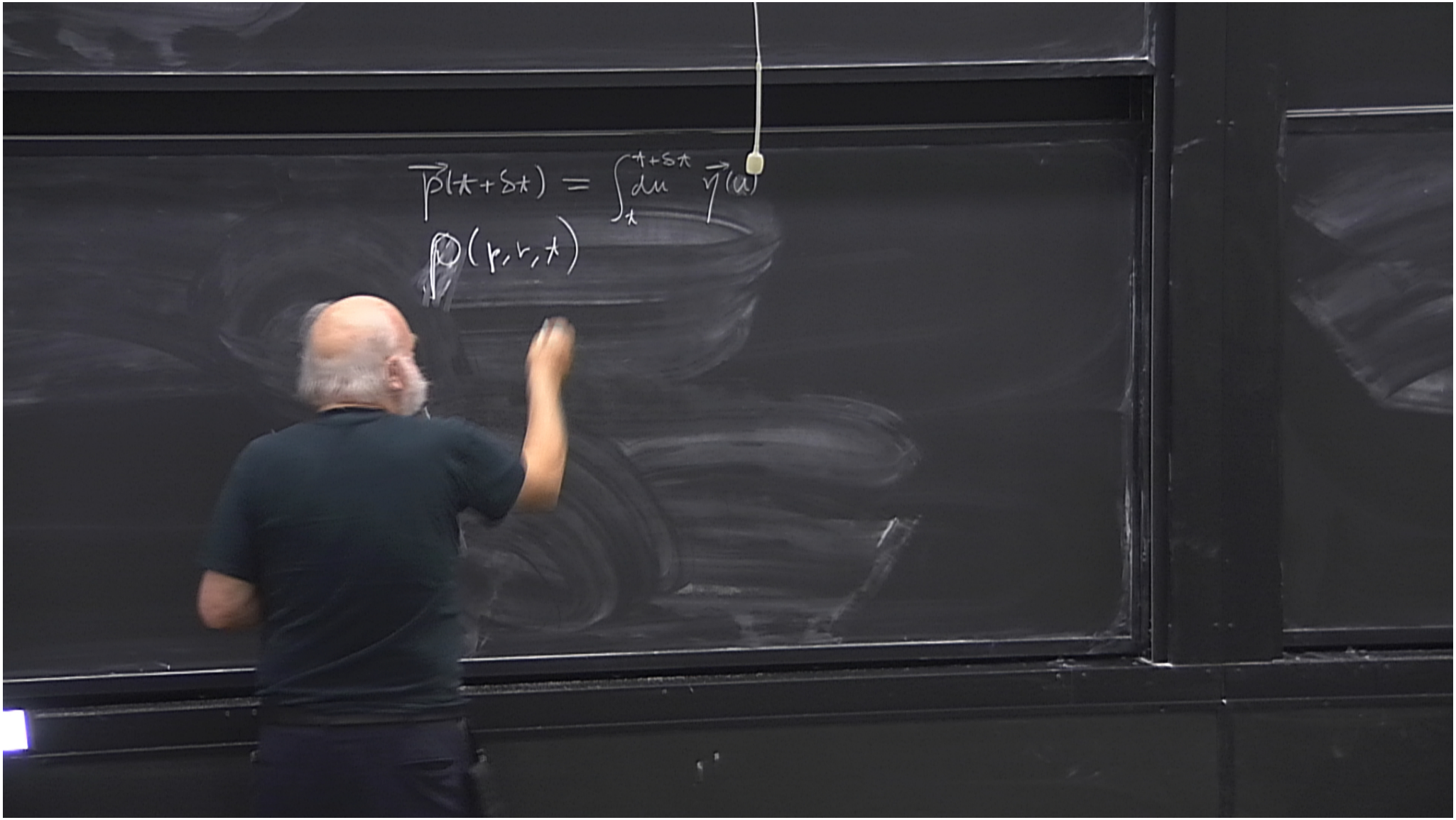
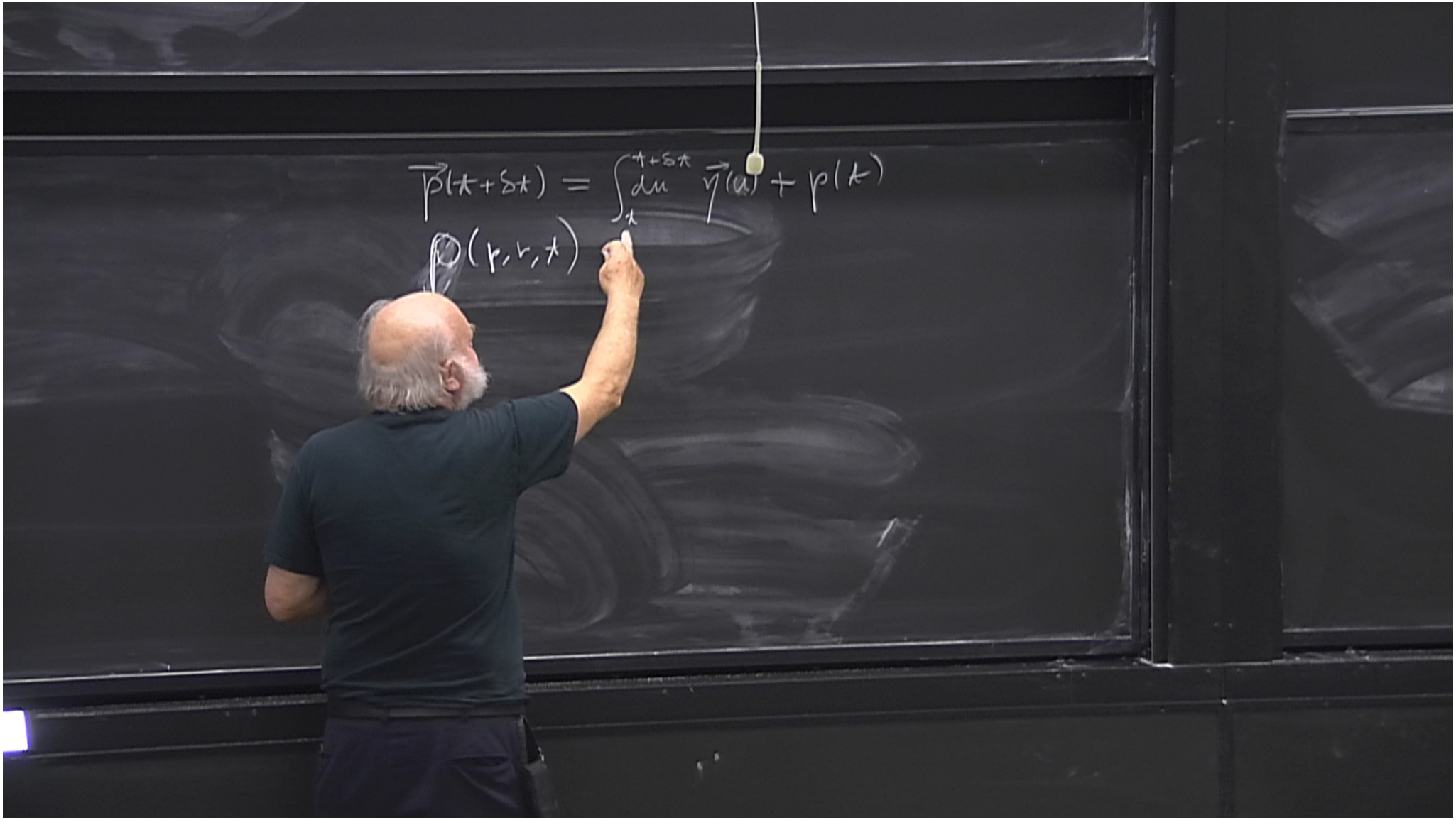
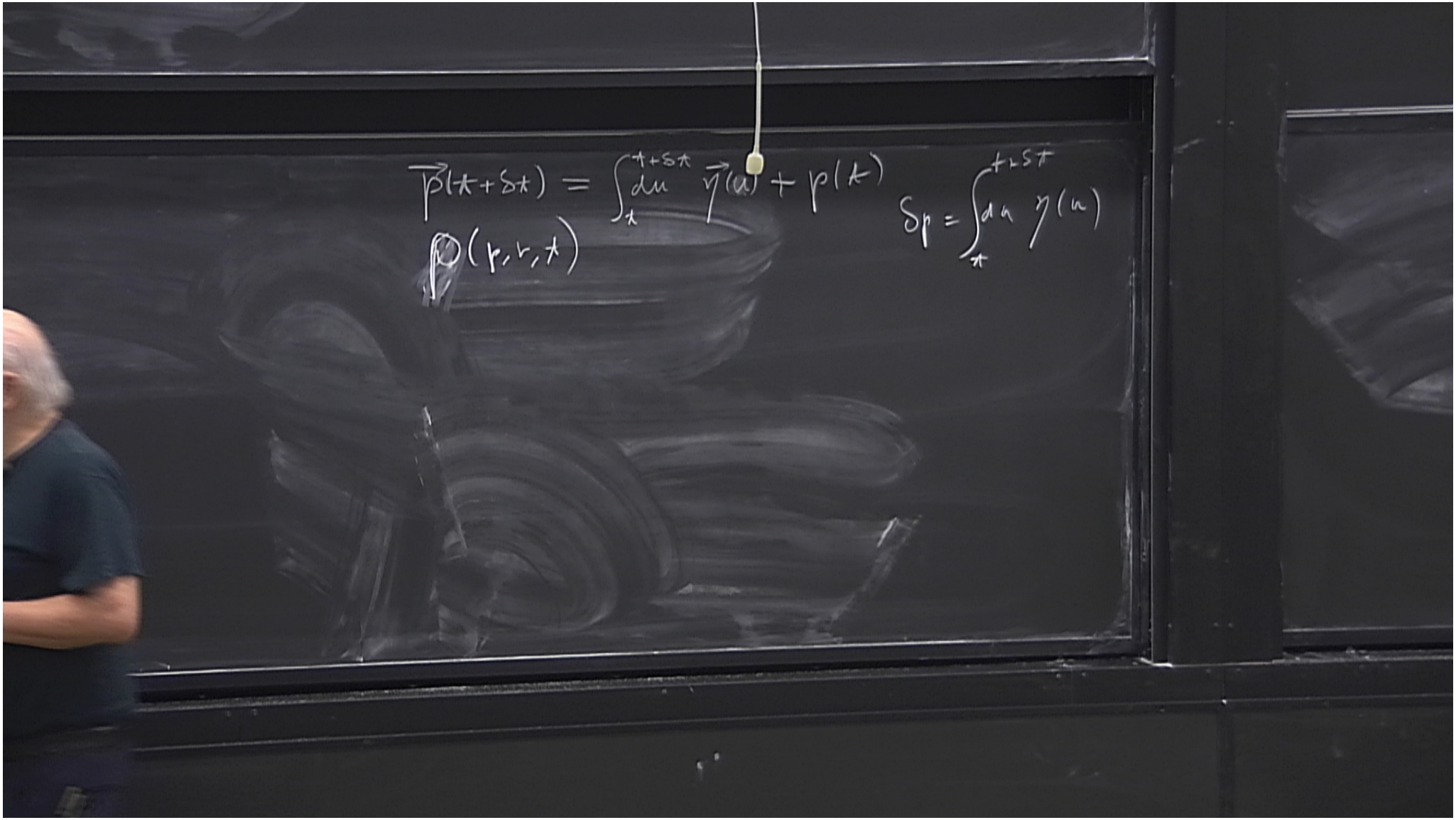

p(t+δt) = ∫

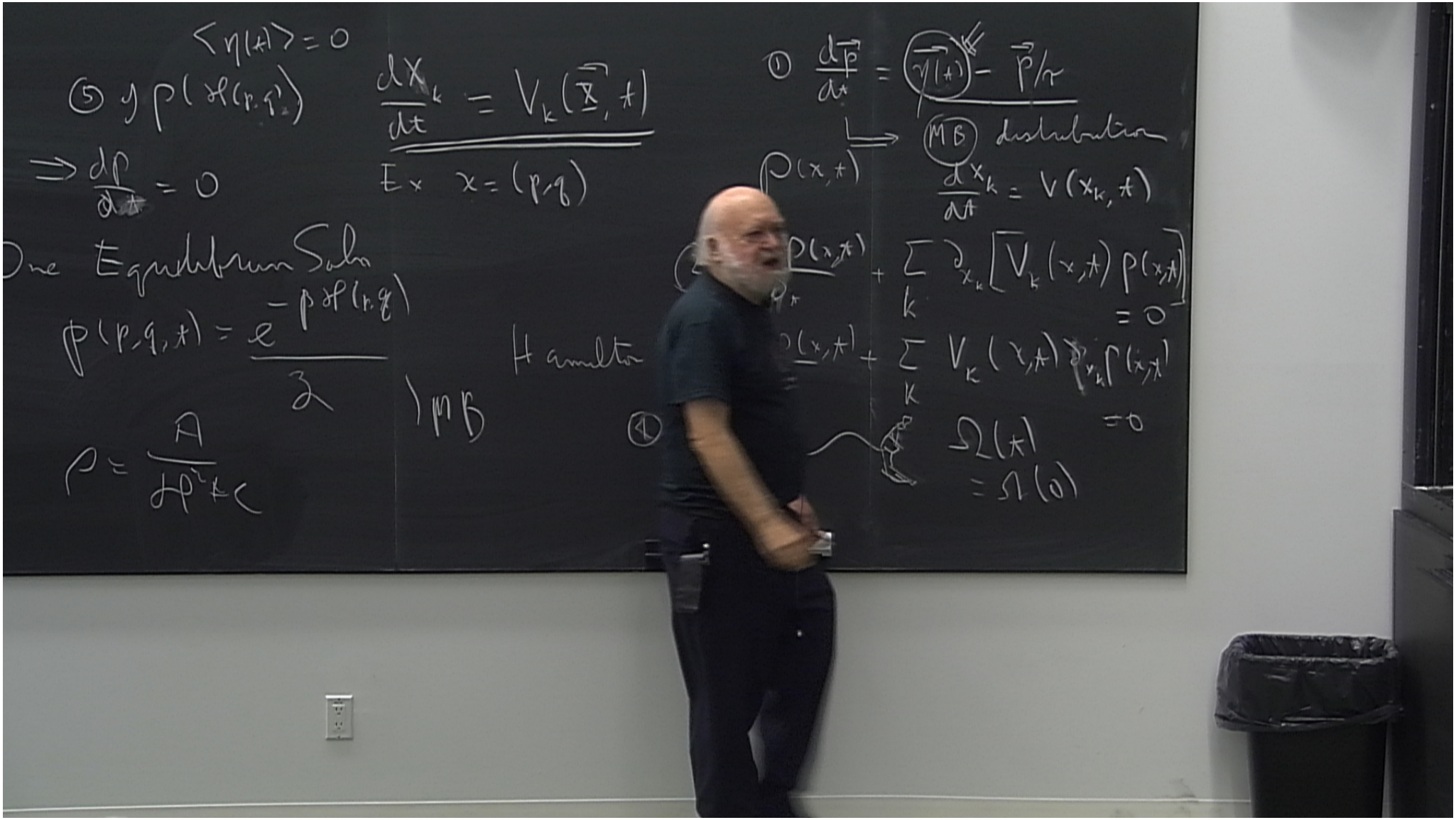

p(t+δt) = ∫ₜ^{t+δt} du


$\vec{p}(t+\delta t) = \int_t^{t+\delta t} du\ \vec{v}(u)$


$\vec{p}(t+\delta t) = \int_t^{t+\delta t} du\ \vec{v}(u)$
$\rho(r,\theta,t)$


$\vec{p}(t+\delta t) = \int_t^{t+\delta t} du\ \vec{F}(u) + p(t)$
$\rho(r, \theta, t)$


$\vec{p}(t+\delta t) = \int_t^{t+\delta t} du\ \vec{\gamma}(u) + p(t)$
$\rho(r,r,t)$
$\delta p = \int_t^{t+\delta t} du\ \gamma(u)$


$\langle \eta(t) \rangle = 0$
$\frac{dx_k}{dt} = V_k(\vec{x}, t)$
Ex $x = (p, q)$
$\Rightarrow \frac{d\rho}{dt} = 0$
One Equilibrium Solu
$\rho(p,q,t) = \frac{e^{-\beta \mathcal{H}(p,q)}}{Z}$
Hamilto
$\rho = \frac{A}{\mathcal{H}^2 + \epsilon}$
$\frac{d\vec{p}}{dt} = \vec{\eta}(t) - \vec{p}/\tau$
MB distribution
$\frac{dx_k}{dt} = V(x_k, t)$
$\rho(x,t)$
$\sum_k \partial_{x_k}[V_k(x,t)\rho(x,t)] = 0$
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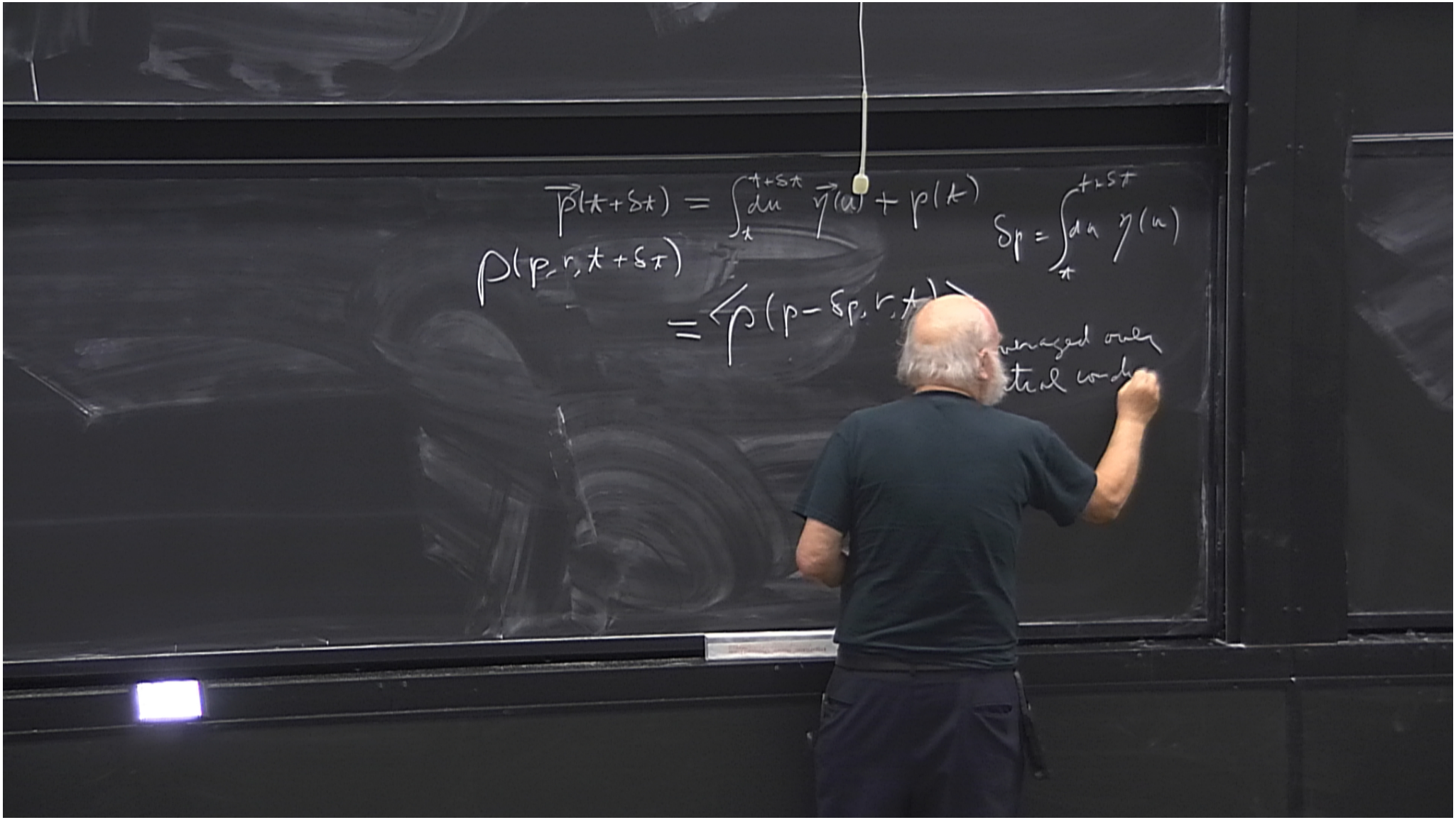
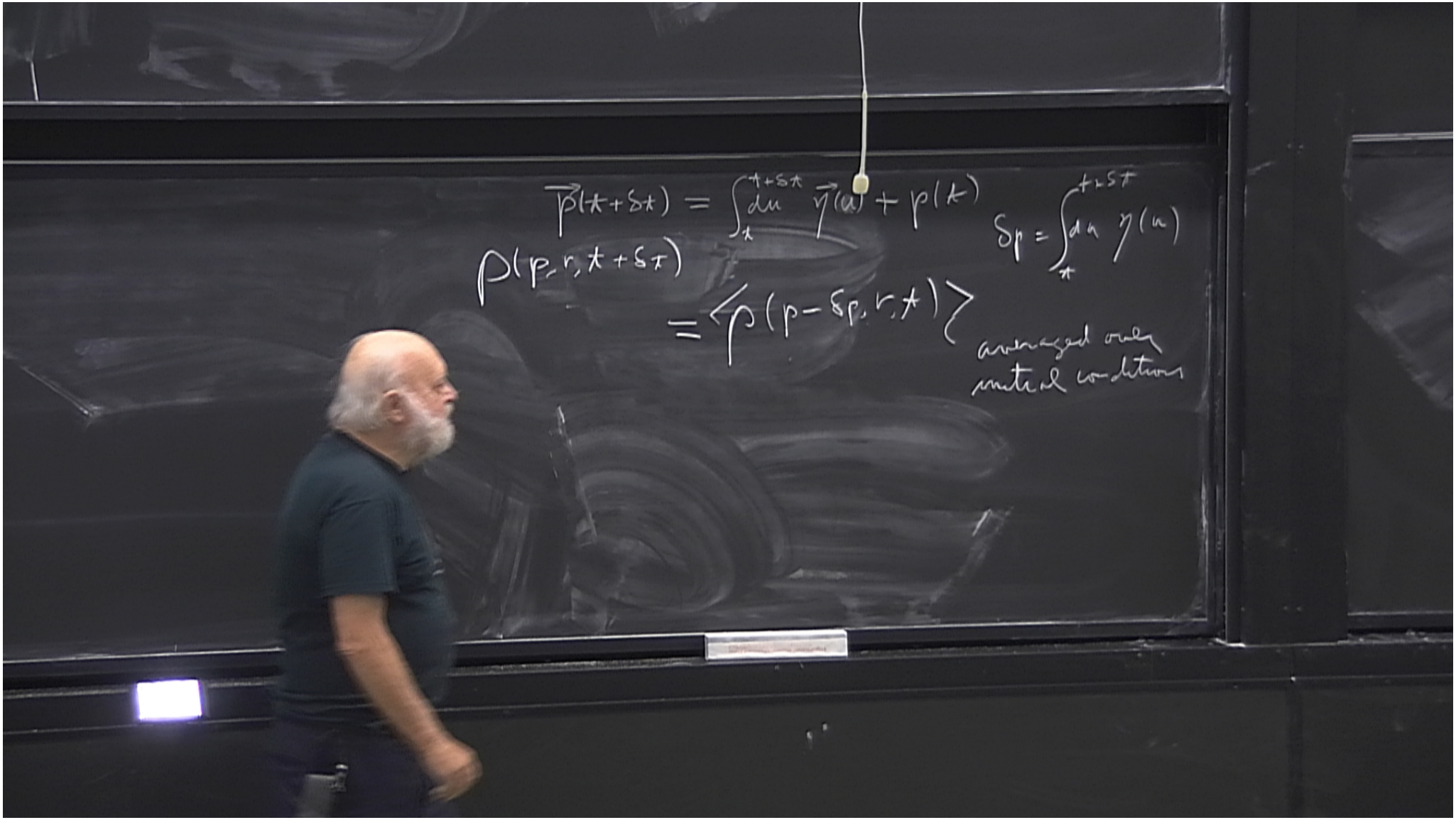

$\vec{p}(t+\delta t) = \int_t^{t+\delta t} du\ \vec{\eta}(u) + p(t)$
$\delta p = \int_t^{t+\delta t} du\ \eta(u)$
$\rho(p, r, t+\delta t)$
$= \langle \rho(p - \delta p, r, t) \rangle$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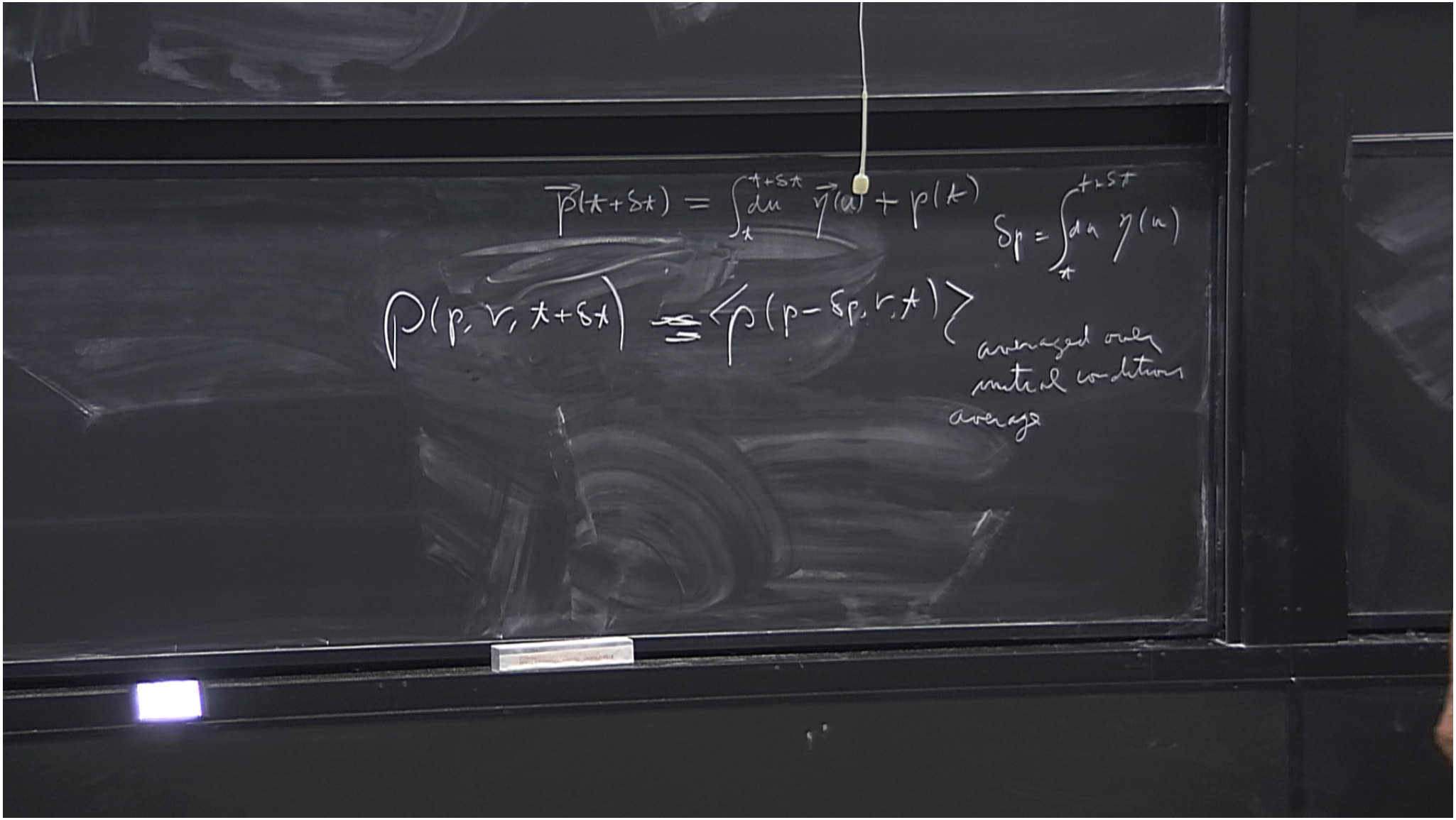
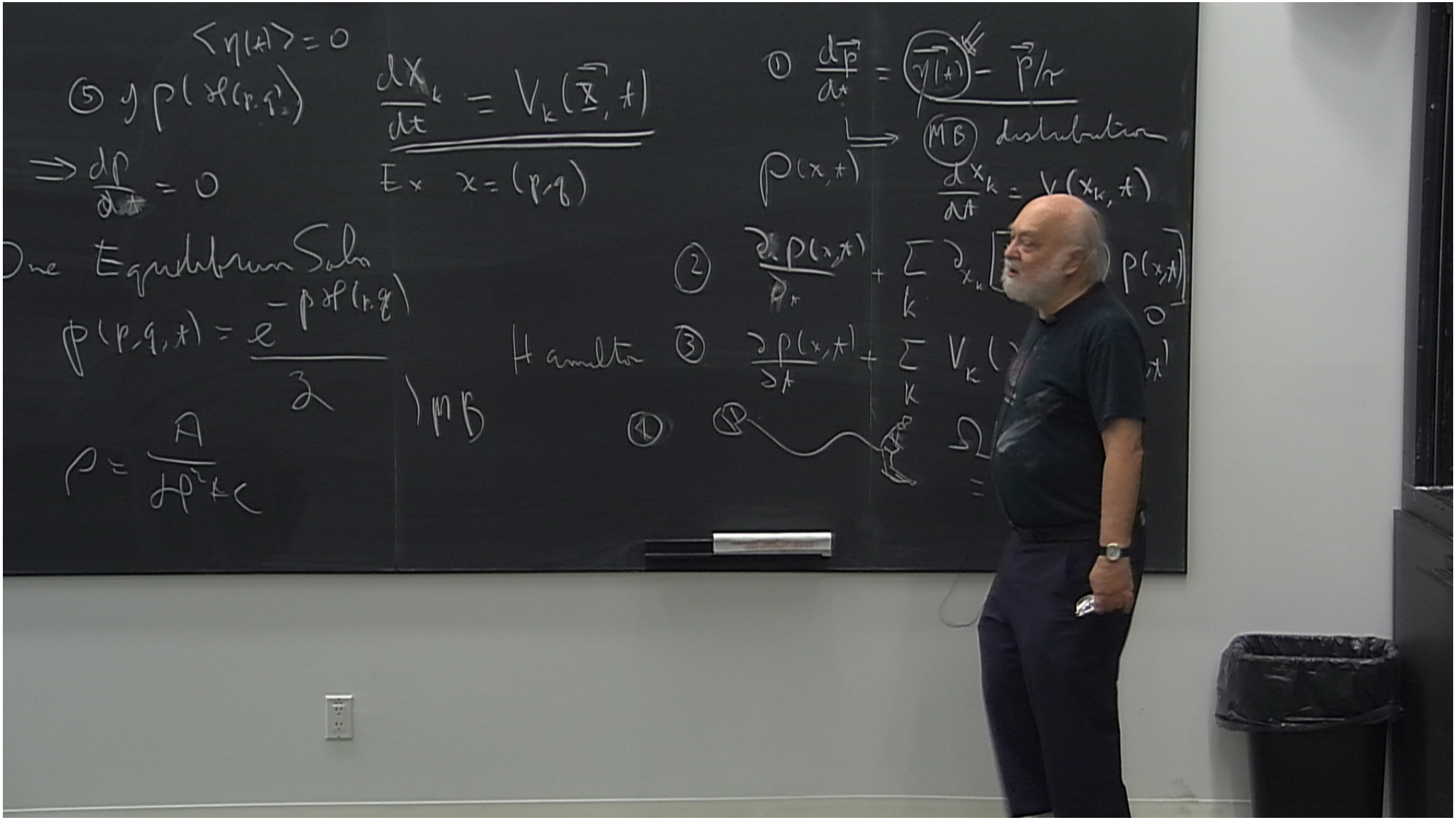
$\delta p = \int_t^{t+\delta\tau} du\ \eta(u)$
$\rho(p, r, t+\delta\tau)$
$= \langle \rho(p-\delta p, r, t) \rangle$
averaged over initial conditions


$\vec{p}(t+\delta t) = \int_t^{t+\delta t} du\ \vec{\eta}(u) + p(t)$
$\delta p = \int_t^{t+\delta t} du\ \eta(u)$
$\rho(p, r, t+\delta t) = \langle \rho(p-\delta p, r, t) \rangle$
averaged over initial conditions
average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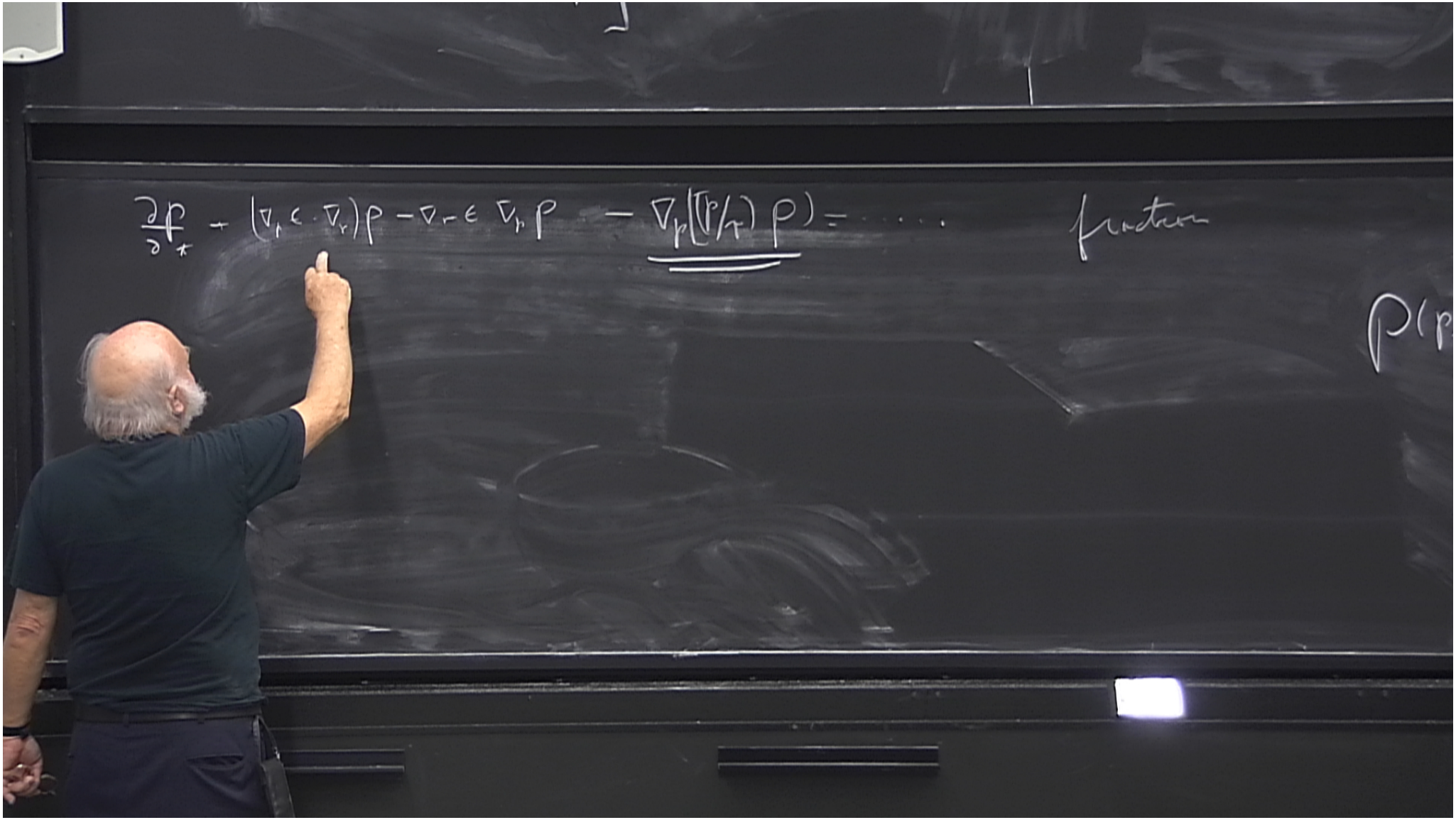
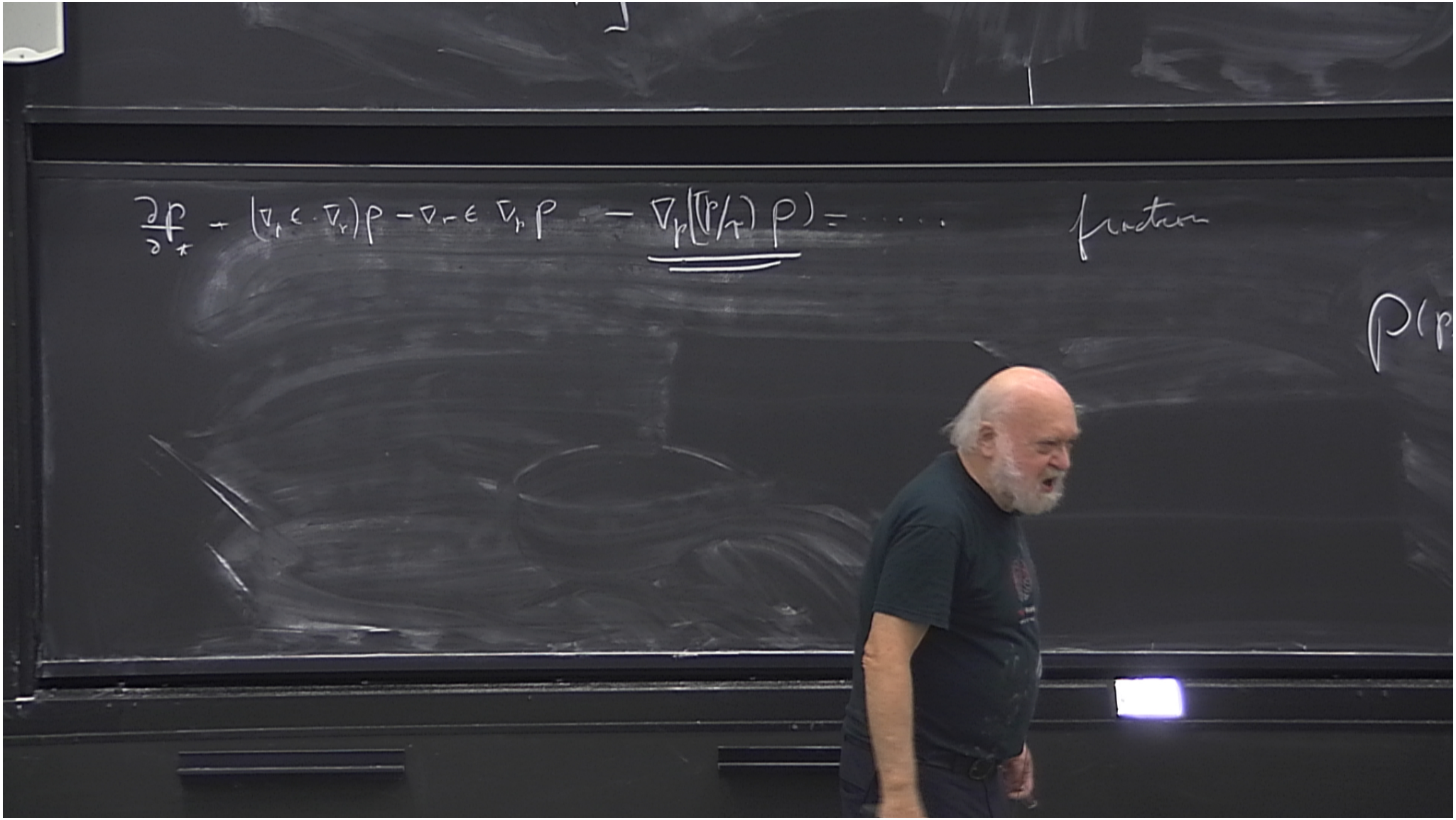
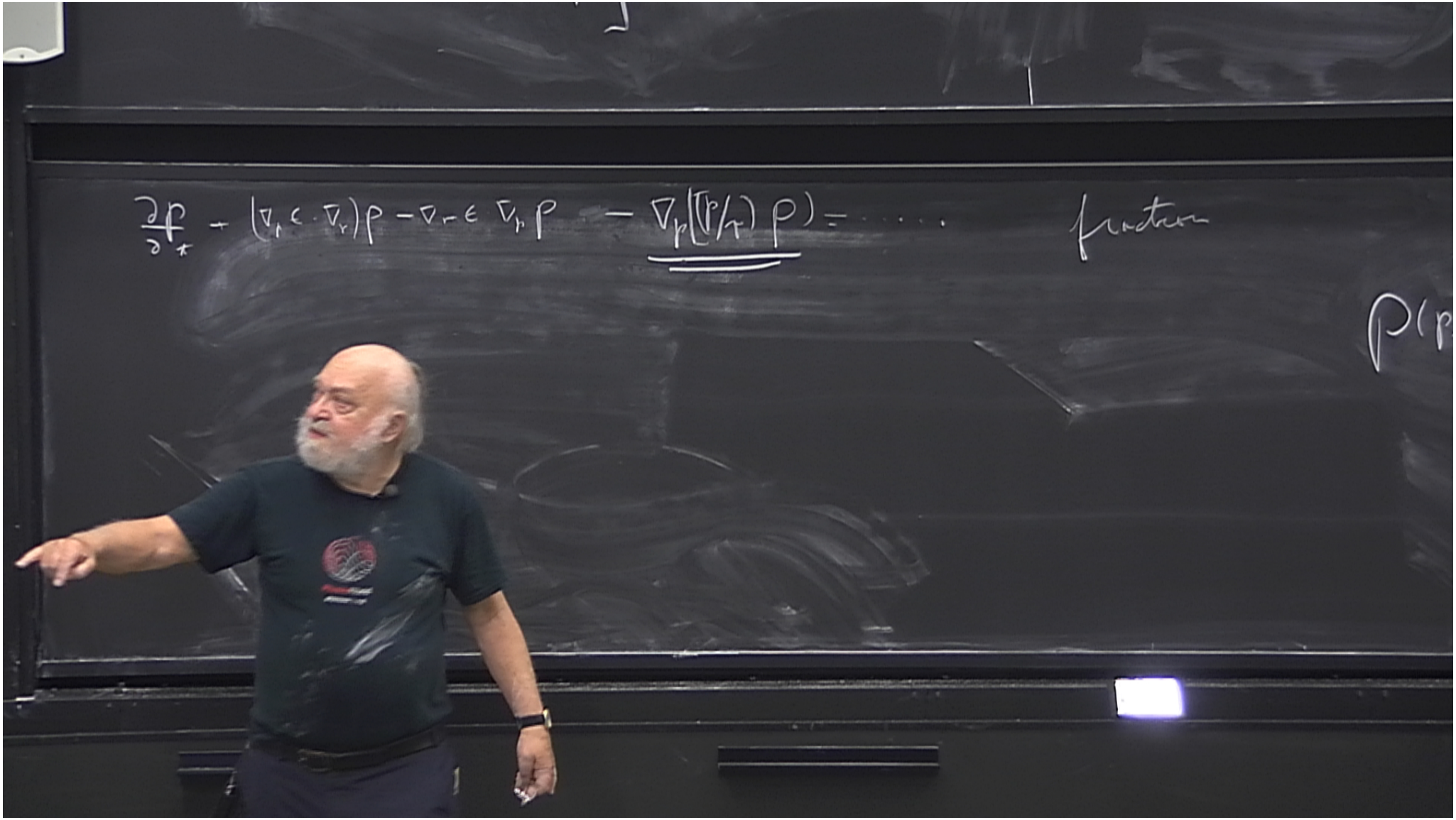
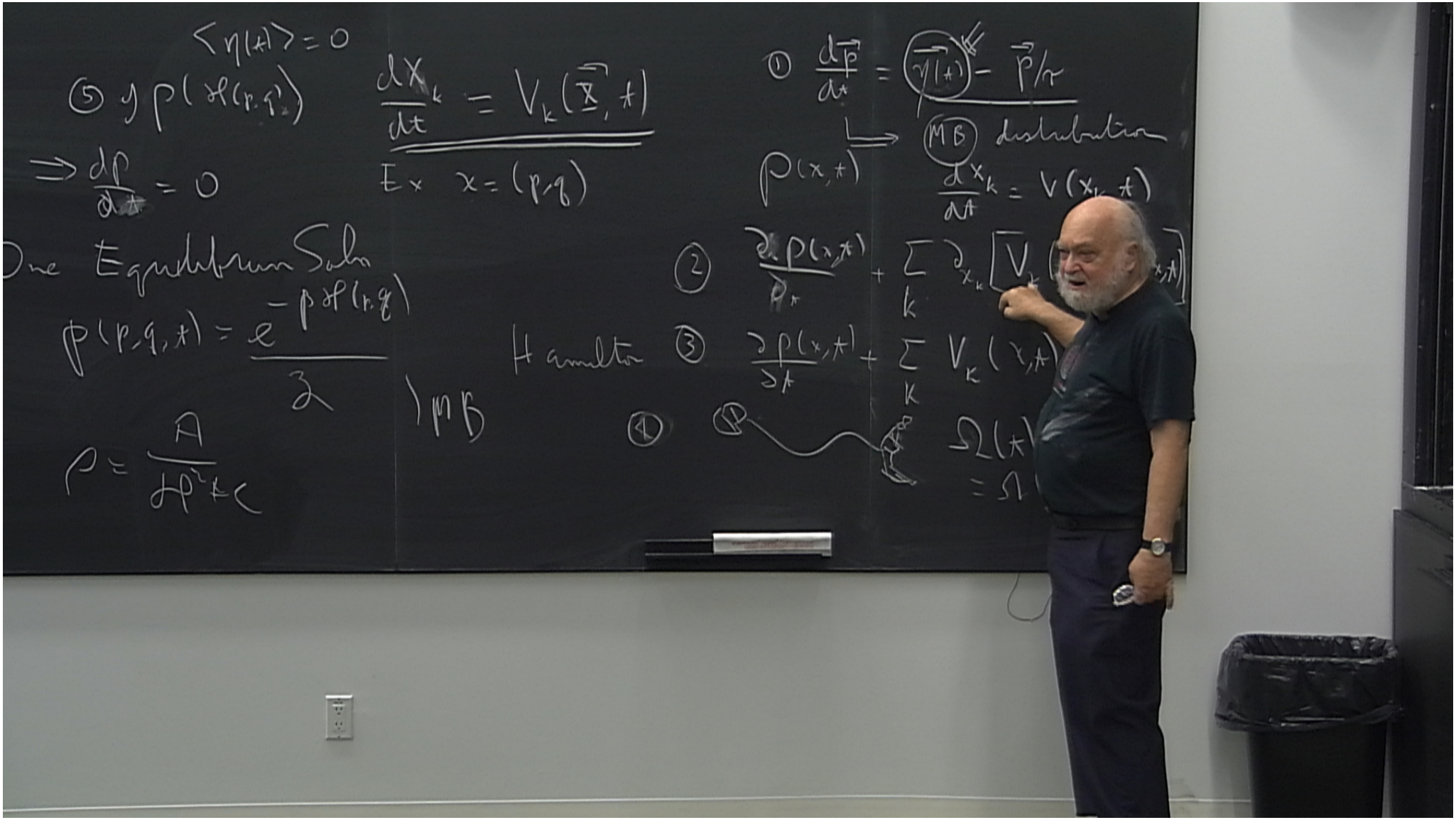
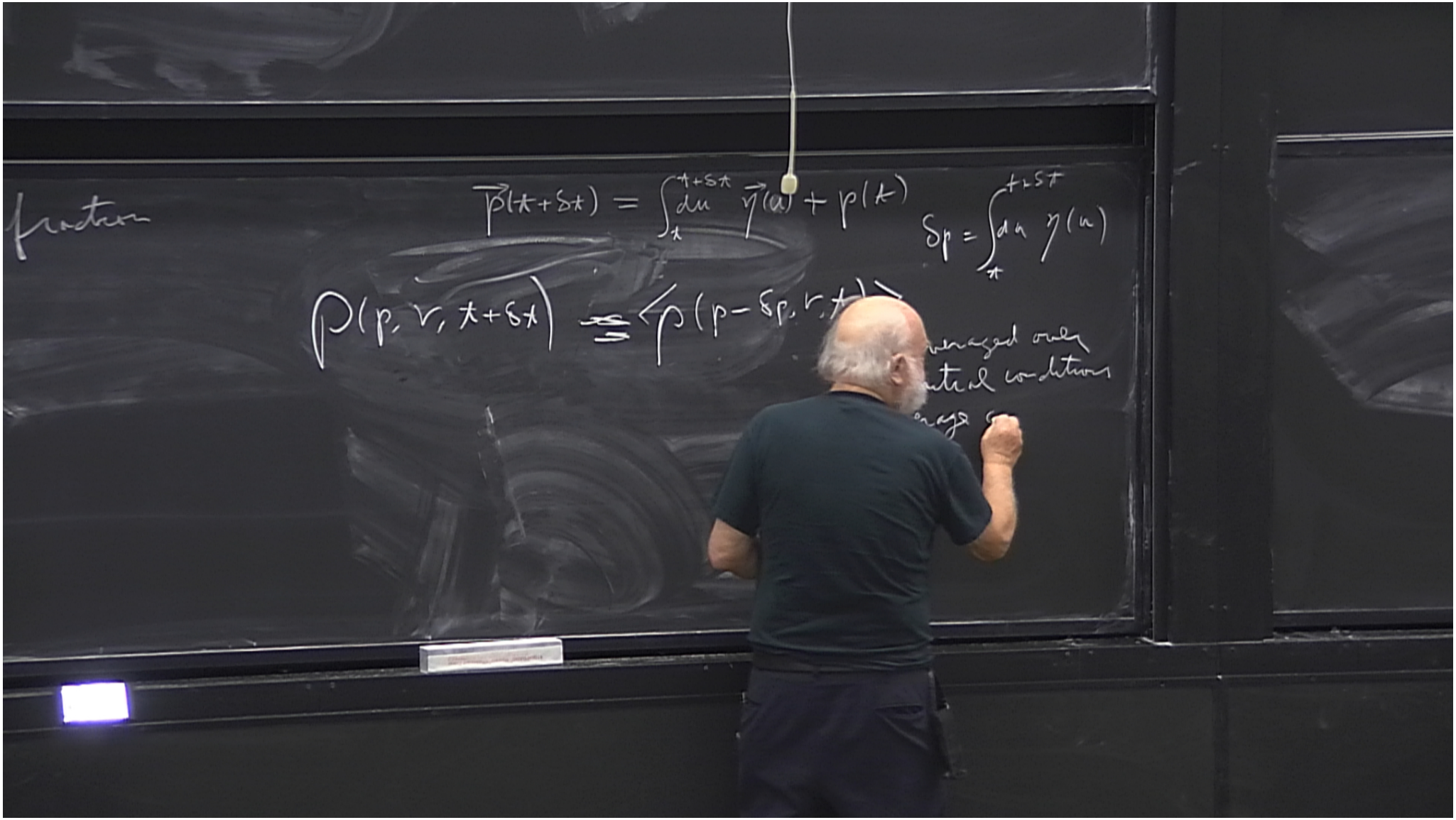
$\rho = \frac{A}{\mathcal{H}^2 + C}$

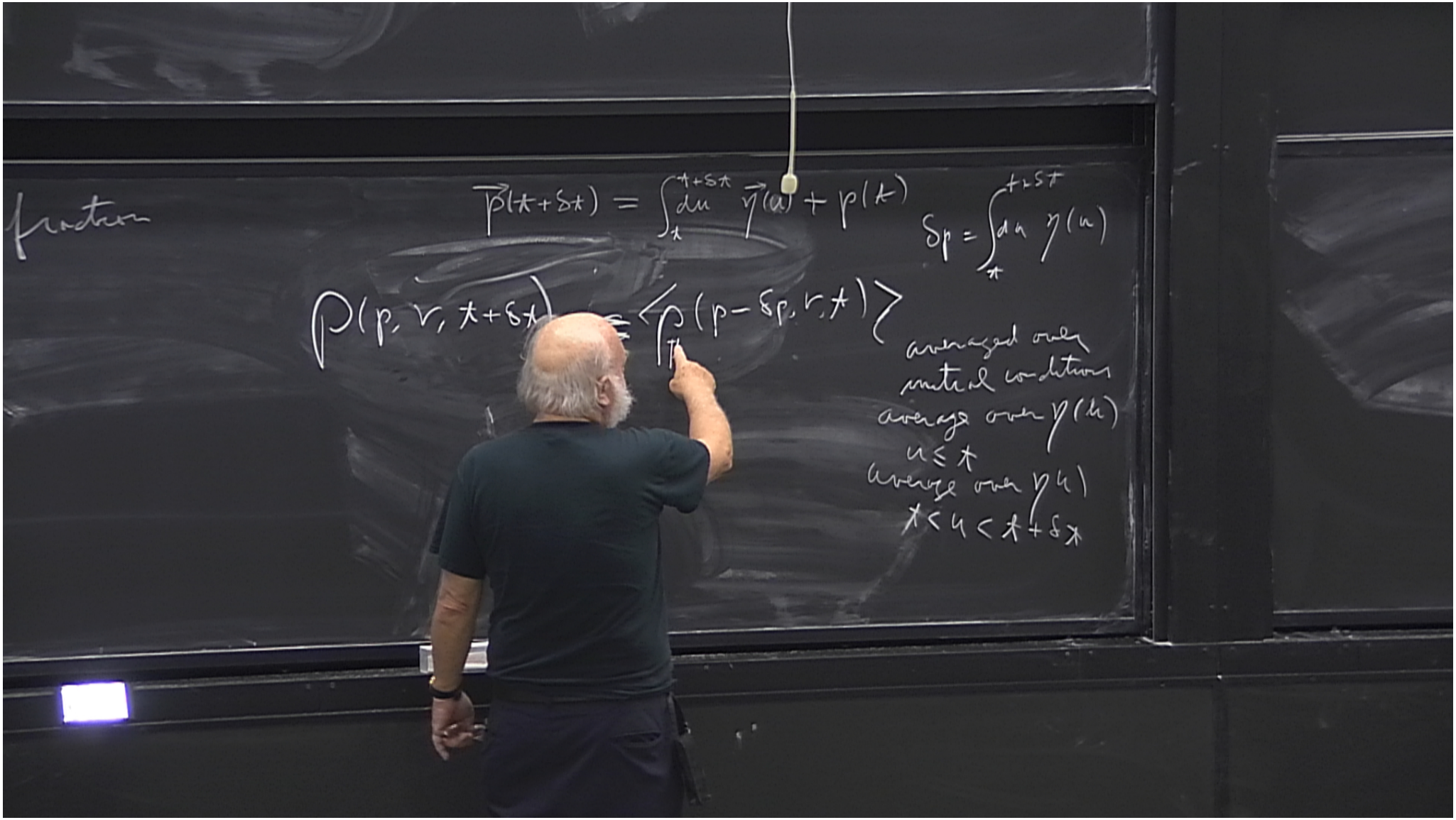

$\frac{\partial f}{\partial t} + (\nabla_r \epsilon \cdot \nabla_r) f - \nabla_r \epsilon \nabla_p f - \nabla_p [(\nabla/\tau) f] = \ldots$
function
$\rho(p$

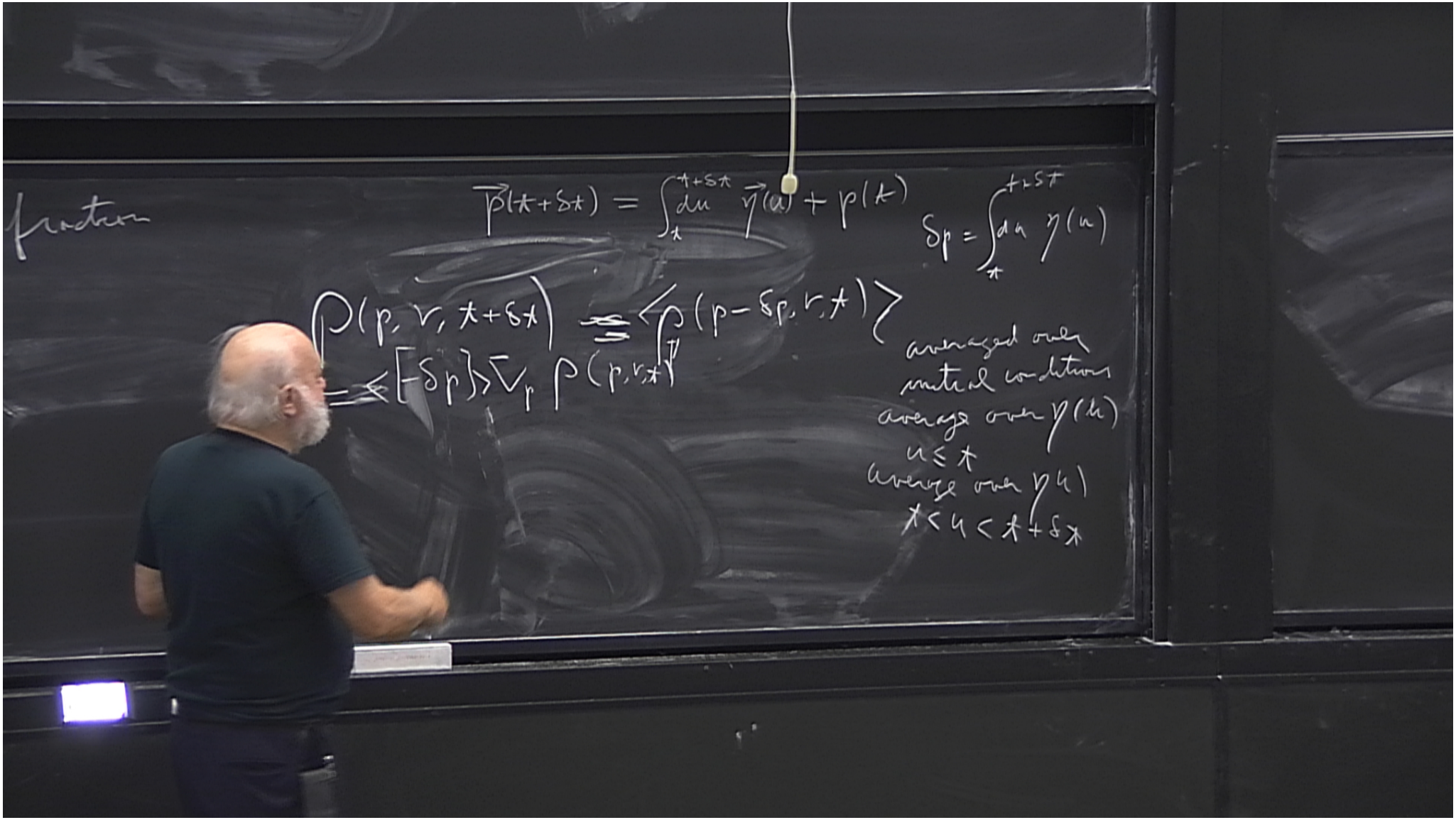

$\frac{\partial f}{\partial t} + (\nabla_r \epsilon \cdot \nabla_r) f - \nabla_r \epsilon \nabla_p f - \nabla_p \left[(\nabla f / \tau) \rho \right] = \ldots$
function
$\rho(p$


$\frac{\partial f}{\partial t} + (v_1 \cdot \nabla_r) f - \nabla_r \cdot \nabla_p f - \nabla_p \left[(F/m) f \right] = \ldots$
function
$f(p$


$\langle \eta(t) \rangle = 0$
$\frac{dx_k}{dt} = V_k(\vec{x}, t)$
Ex $x = (p, q)$
$\rho(x, t)$
MB distribution
$\frac{dx_k}{dt} = V(x_k, t)$
Hamilton
Imp


fraction
$\vec{p}(t+\delta t) = \int_t^{t+\delta t} du\ \vec{\eta}(u) + p(t)$
$\delta p = \int_t^{t+\delta t} du\ \eta(u)$
$\rho(p, r, t+\delta t) = \langle \rho(p-\delta p, r, t) \rangle$


fraction
$\vec{p}(t+\delta t) = \int_t^{t+\delta t} du\ \vec{\eta}(u) + p(t)$
$\delta p = \int_t^{t+\delta t} du\ \eta(u)$
$\rho(p, r, t+\delta t) = \langle \rho(p-\delta p, r, t) \rangle$
averaged over
initial conditions
average over $\eta(u)$
$u \le t$
average over $\eta(u)$
$t < u < t+\delta t$


fraction
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average over $\eta(u)$
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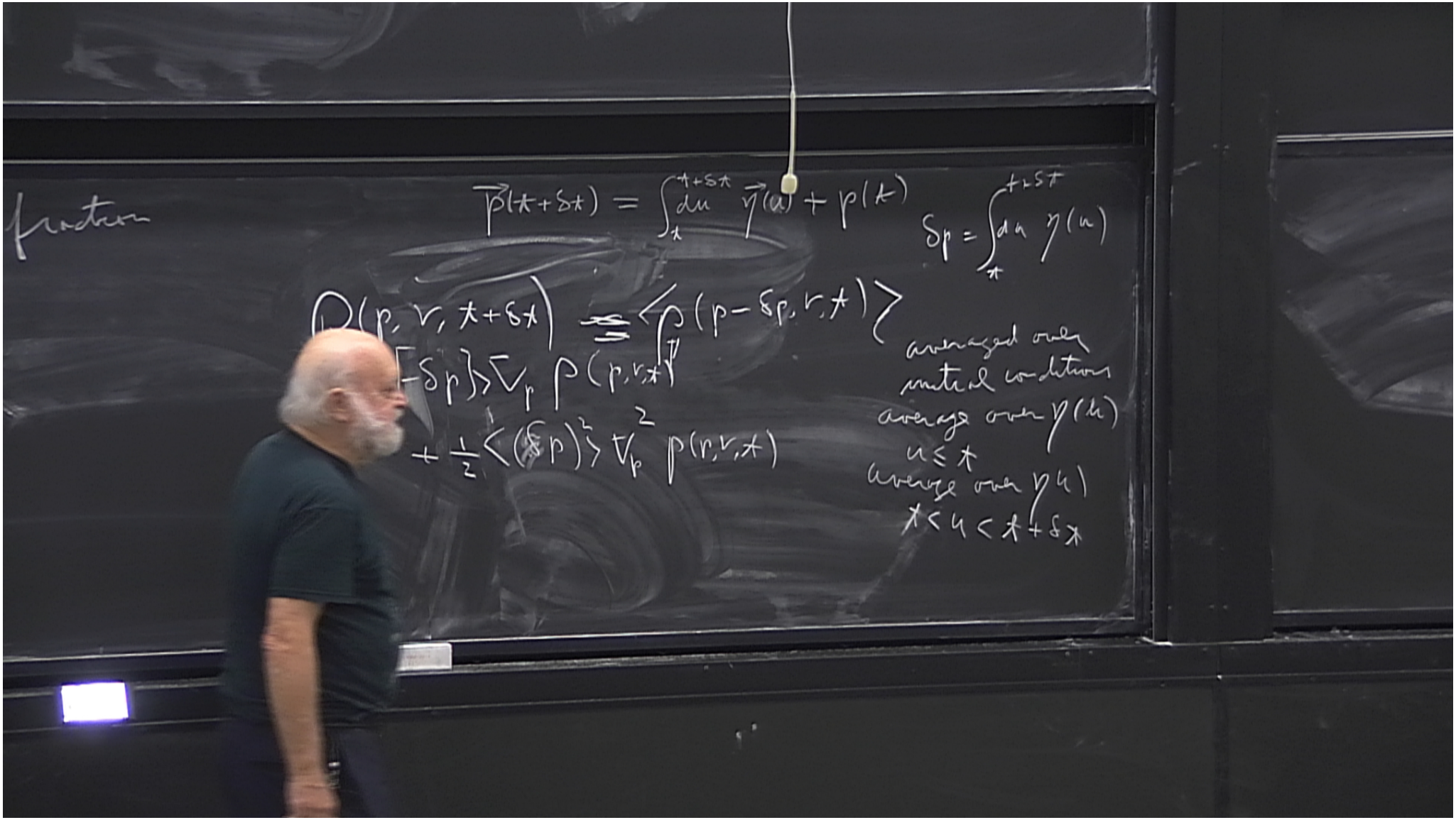
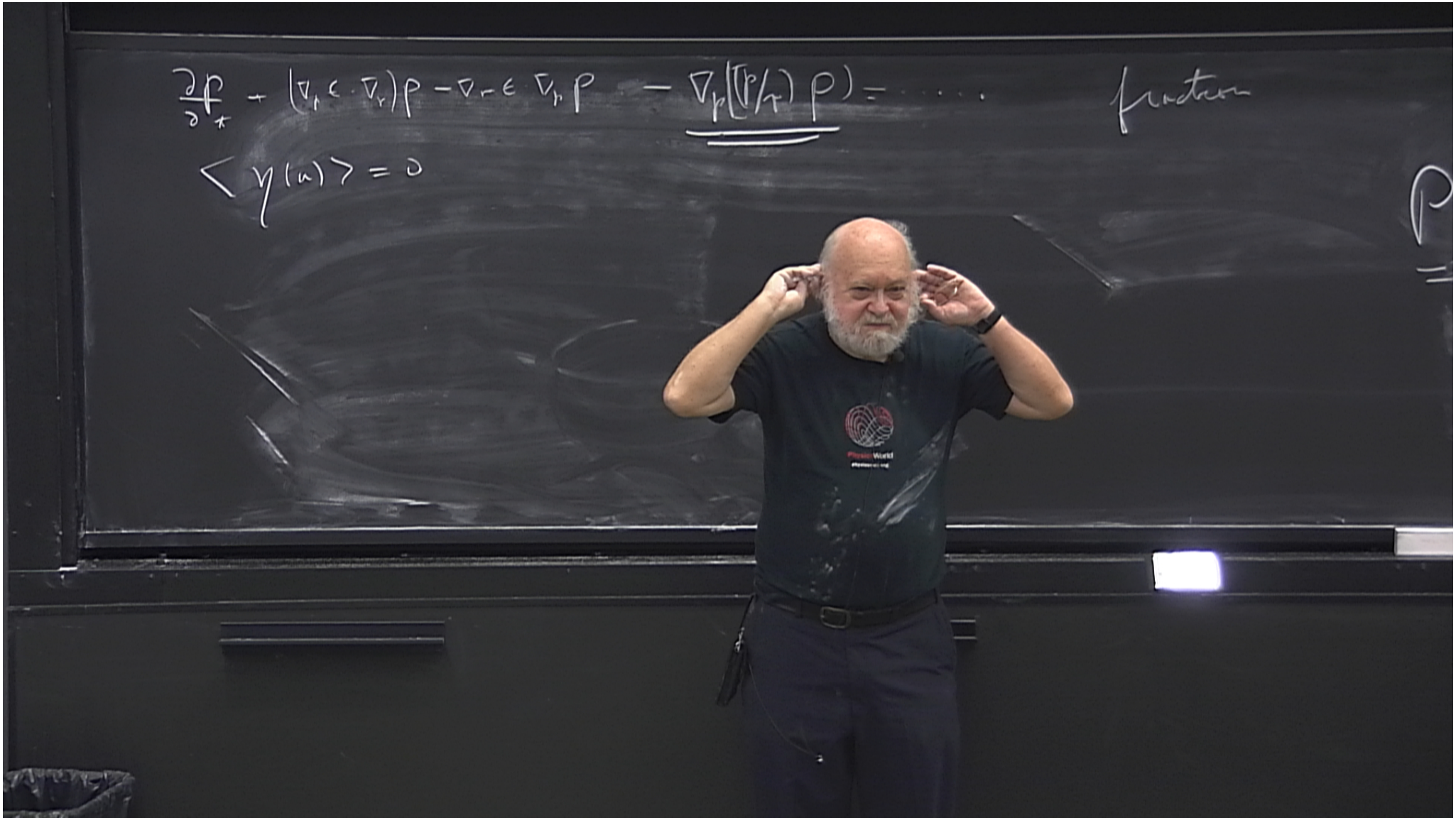
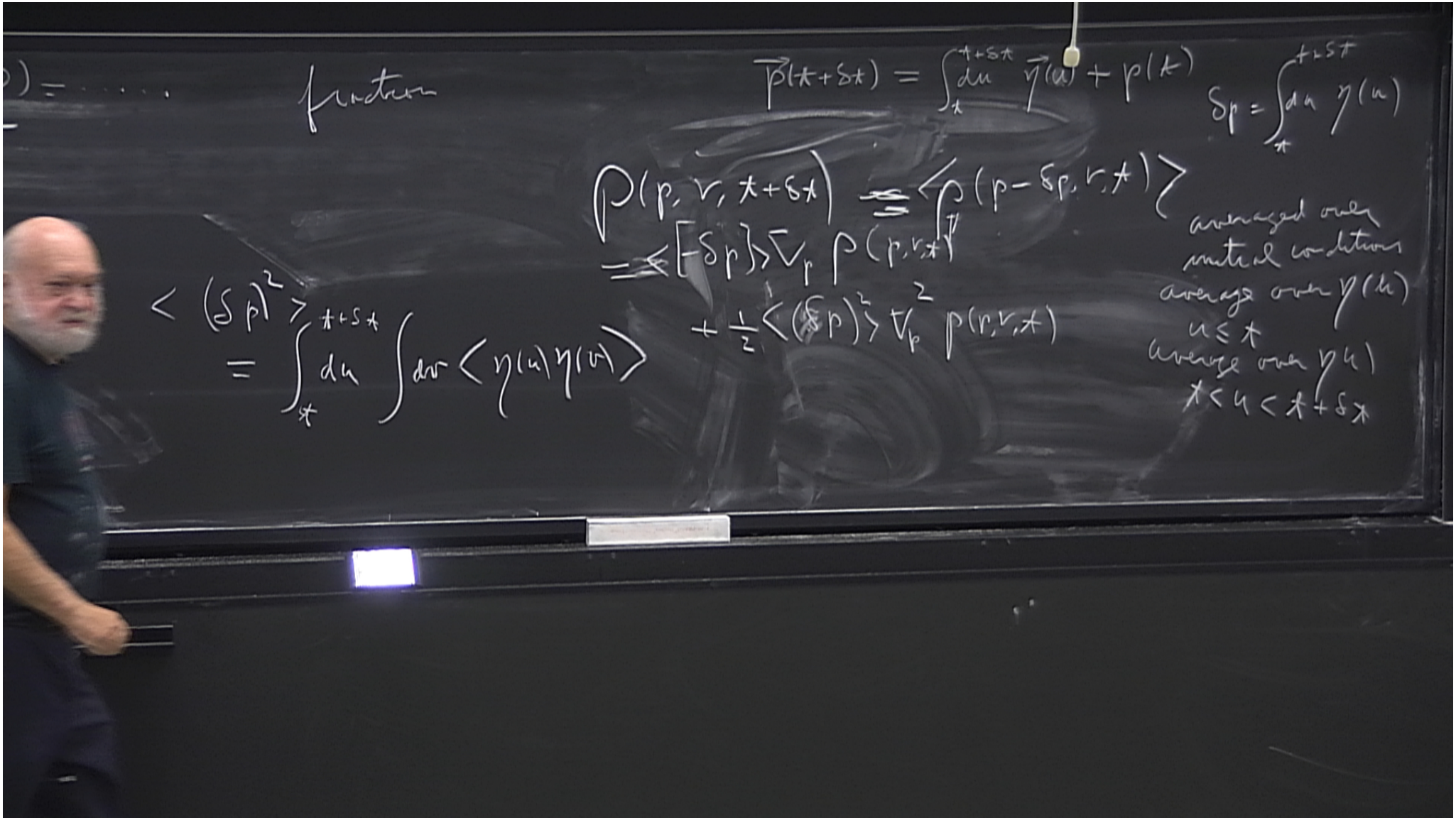
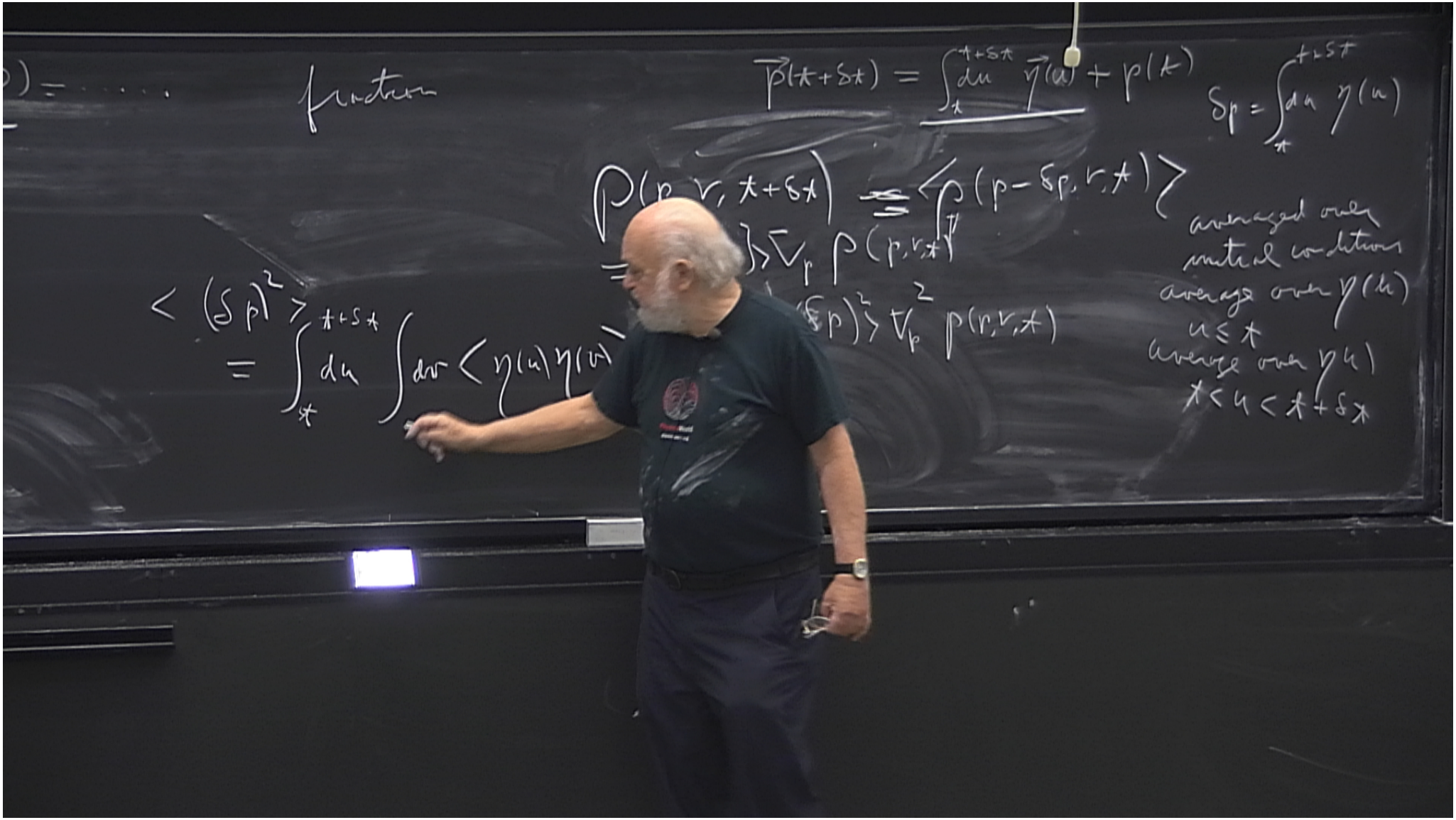
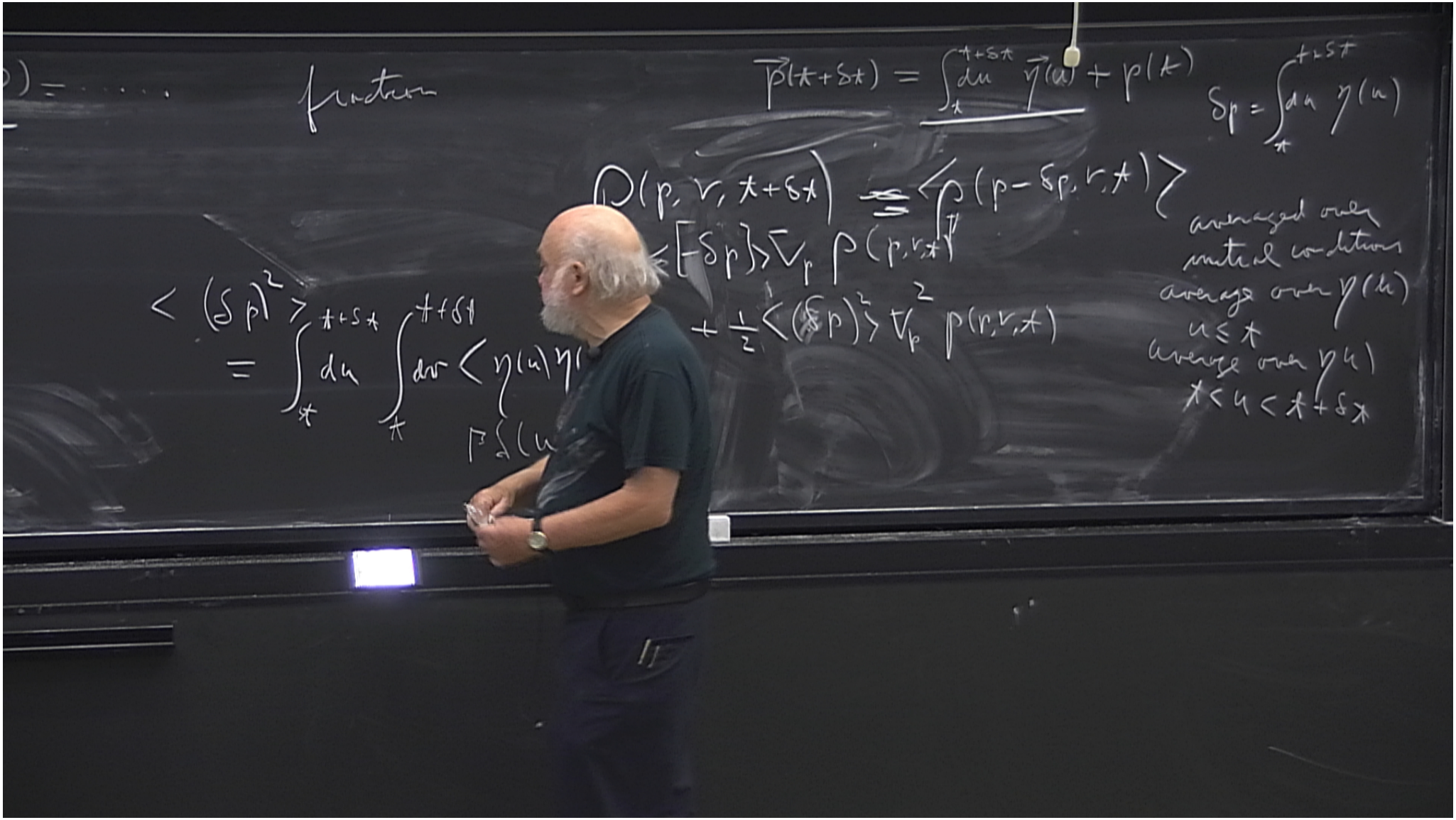
$t < u < t + \delta t$

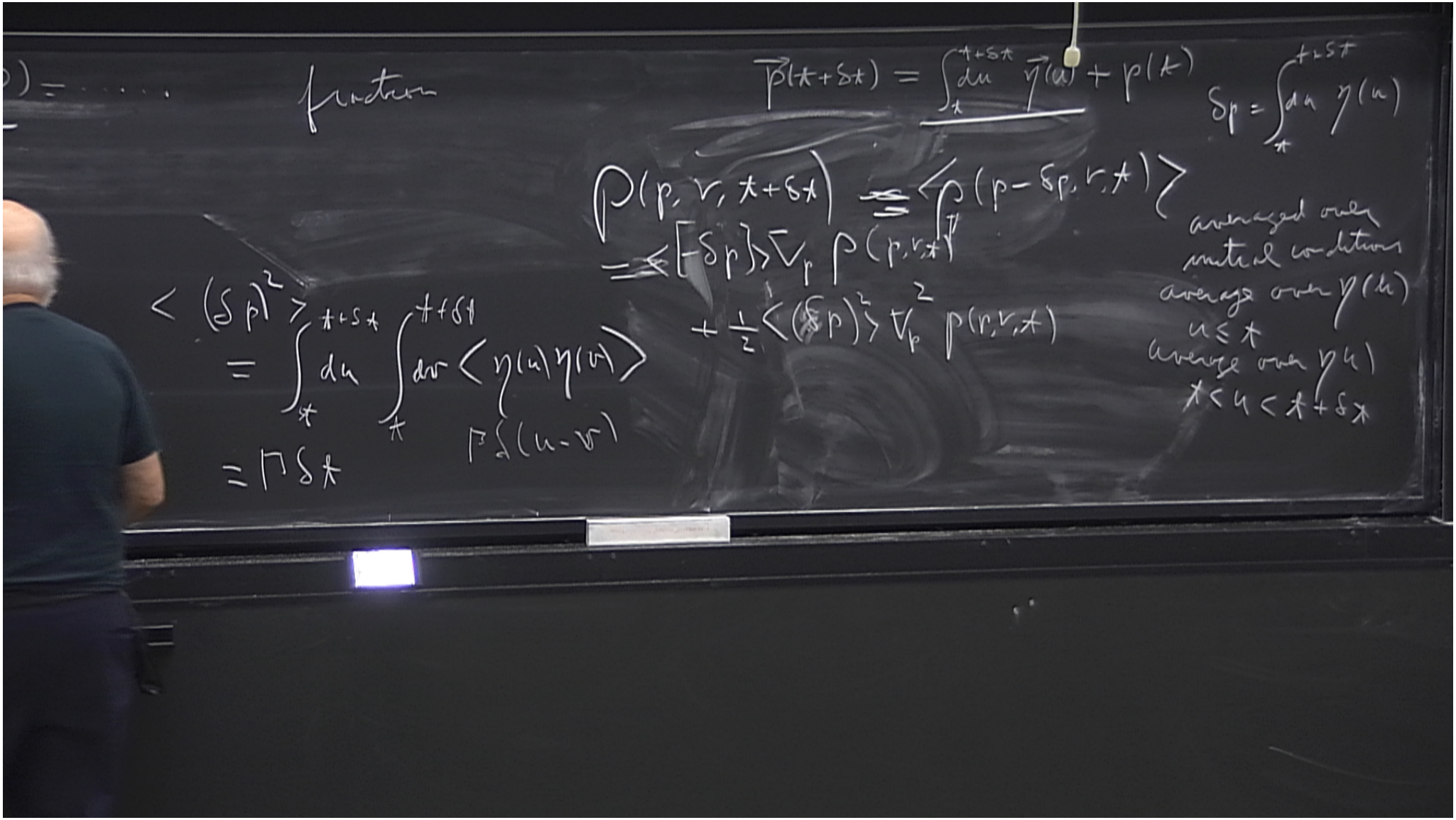

fraction
averaged over
initial conditions
average over η(u)
t ≤ t
average over η(u)
t < u < t + δt

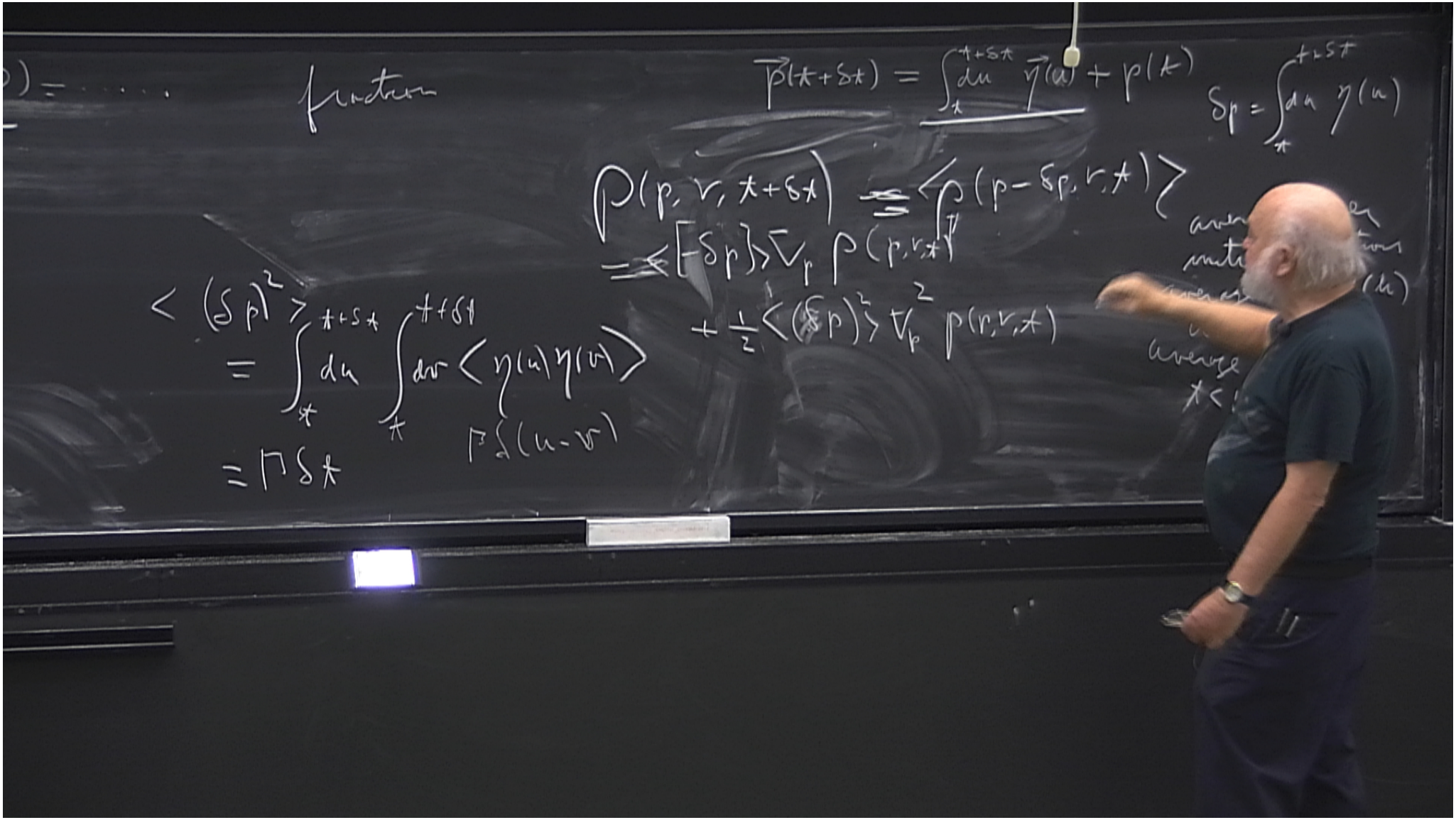

$\frac{\partial f}{\partial t} + (\nabla_1 \epsilon \cdot \nabla_1) f - \nabla_1 \epsilon \nabla_p f - \nabla_p [(\nabla/\tau) f] = \ldots$
function
$\langle \eta(\omega) \rangle = 0$

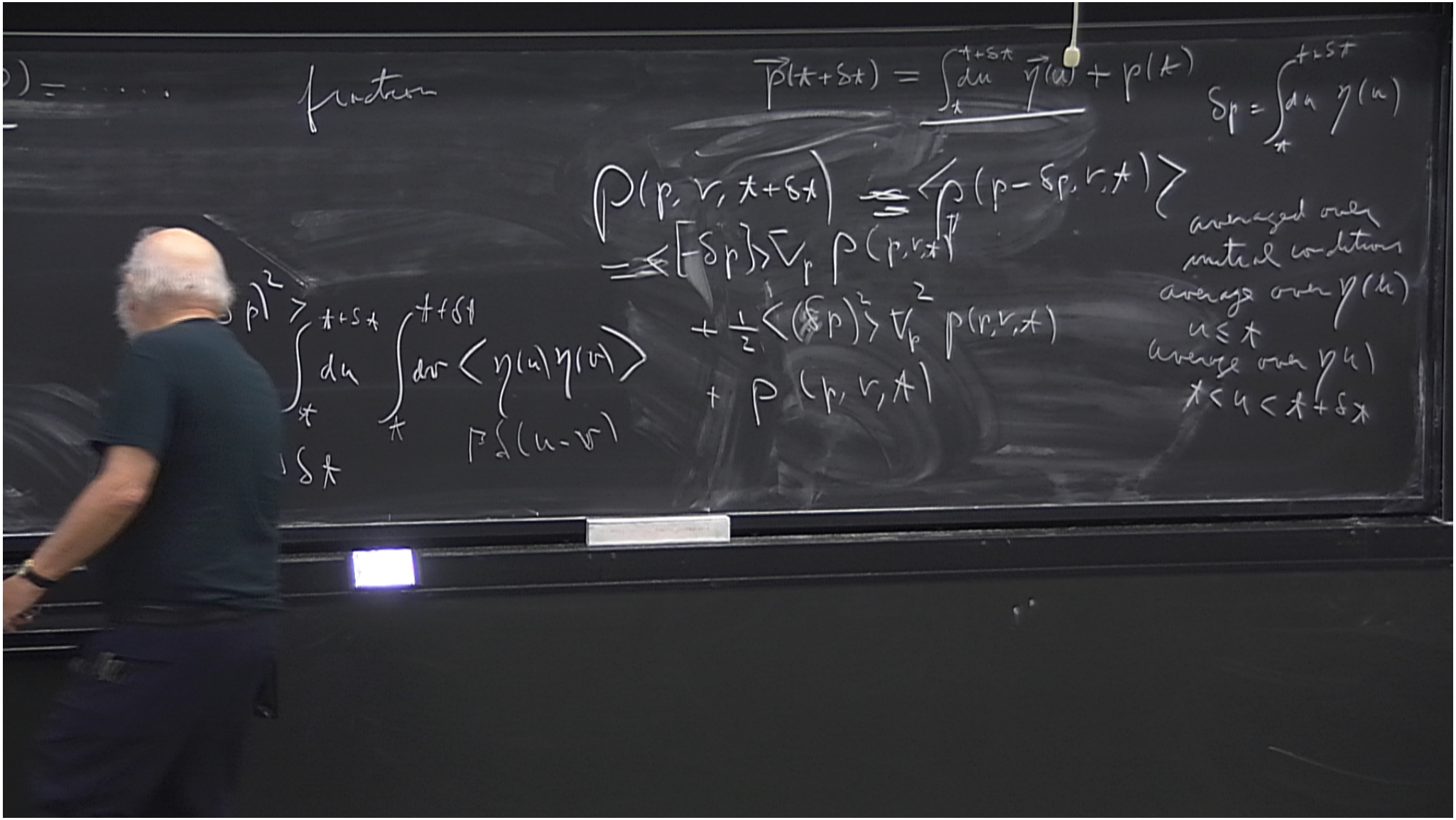

function
$\vec{p}(t+\delta t) = \int_t^{t+\delta t} du\ \vec{\eta}(u) + p(t)$
$\delta p = \int_t^{t+\delta t} du\ \eta(u)$
$\rho(p, r, t+\delta t) = \langle \rho(p-\delta p, r, t) \rangle$
$= \langle [-\delta p] \rangle \nabla_p \rho(p, r, t)$
$+ \frac{1}{2} \langle (\delta p)^2 \rangle \nabla_p^2 \rho(p, r, t)$
averaged over initial conditions
average over $\eta(u)$
$u \le t$
average over $\eta(u)$
$t < u < t + \delta t$
$\langle (\delta p)^2 \rangle = \int_t^{t+\delta t} du \int dw \langle \eta(u) \eta(w) \rangle$

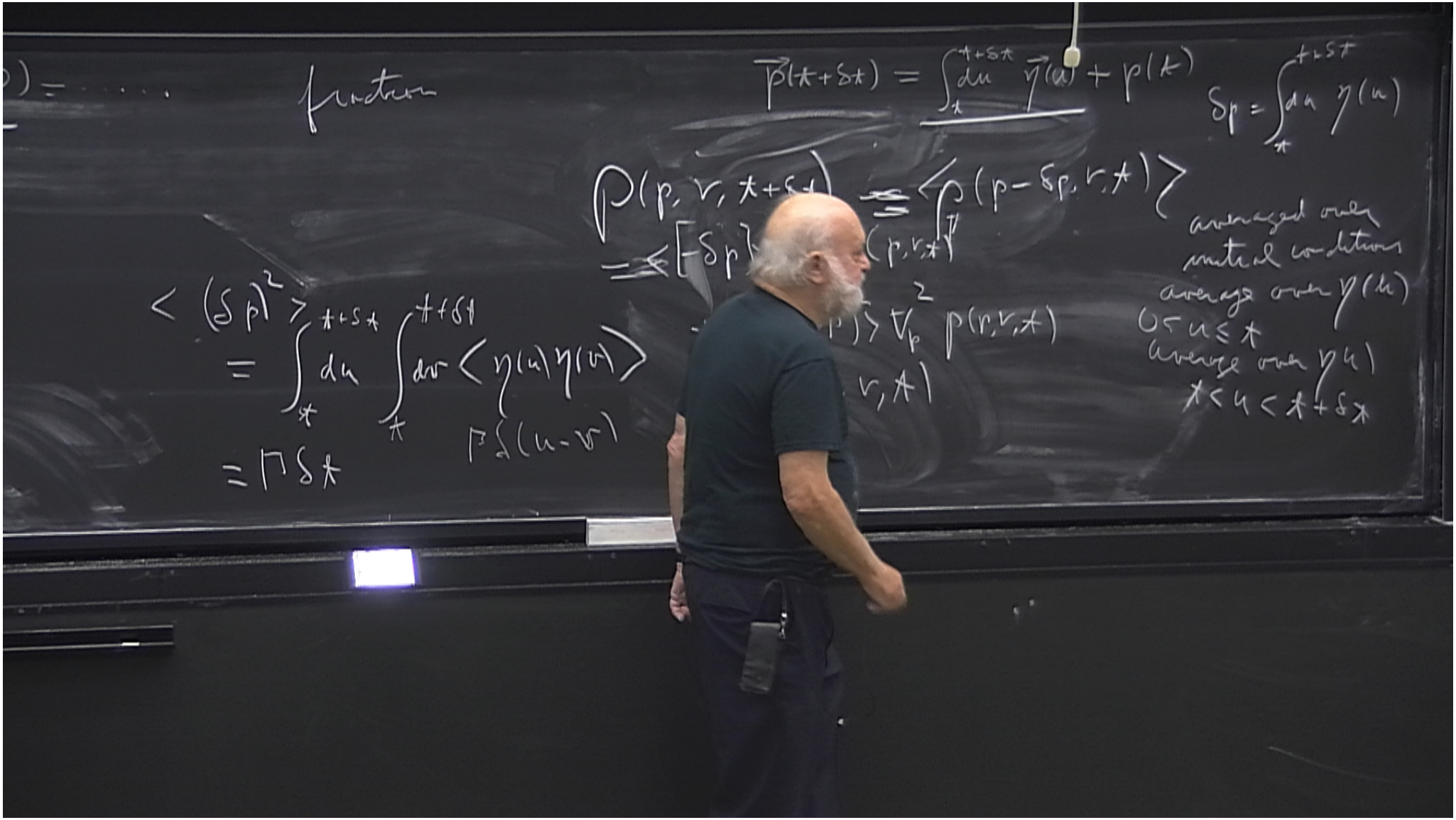

fluctuations
averaged over
initial conditions
average over η(u)
t ≤ u
average over η(u)
t < u < t + δt

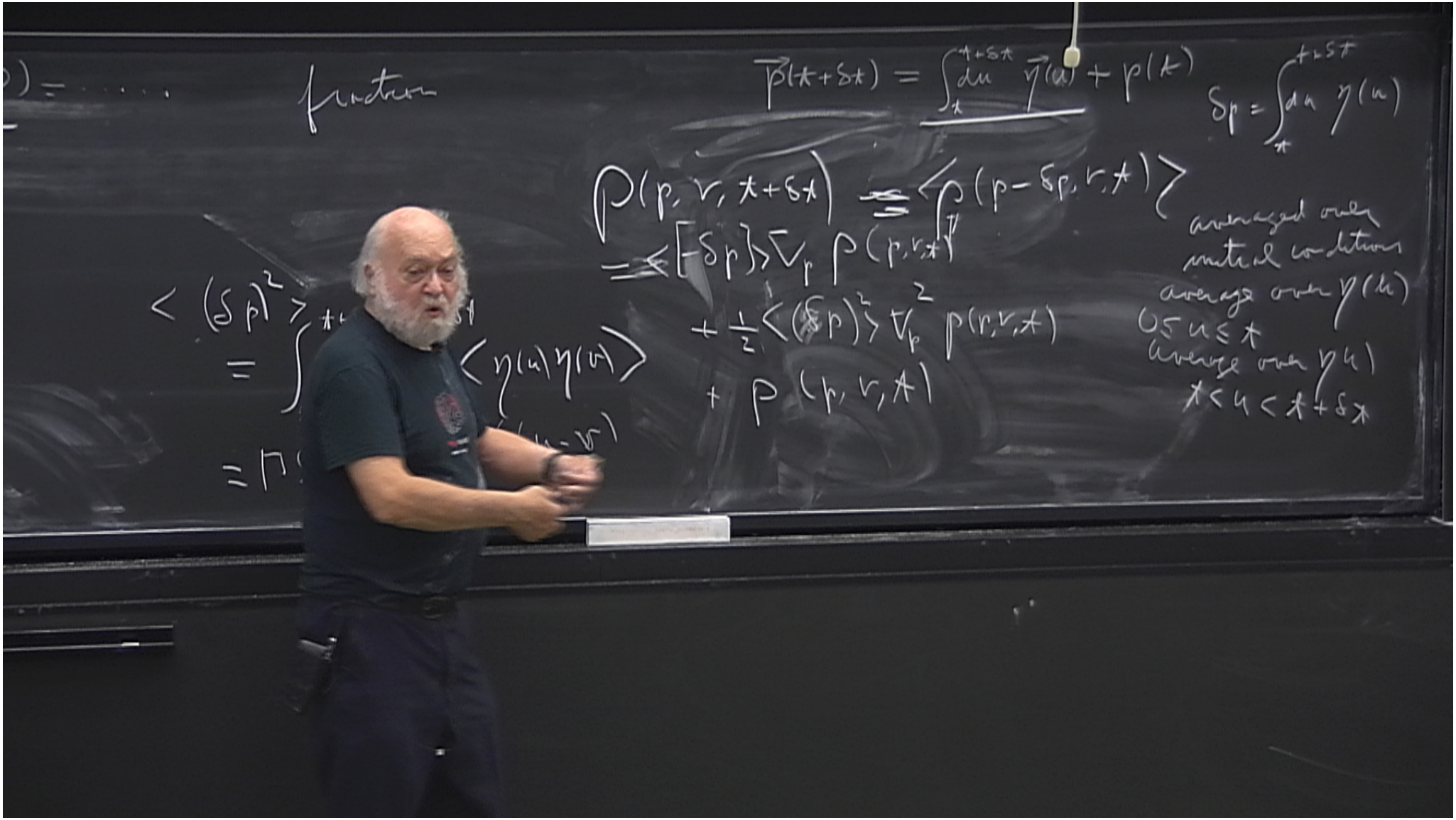

function
averaged over
initial conditions
average over η(u)
t ≤ u
average over η(u)
t < u < t + δt

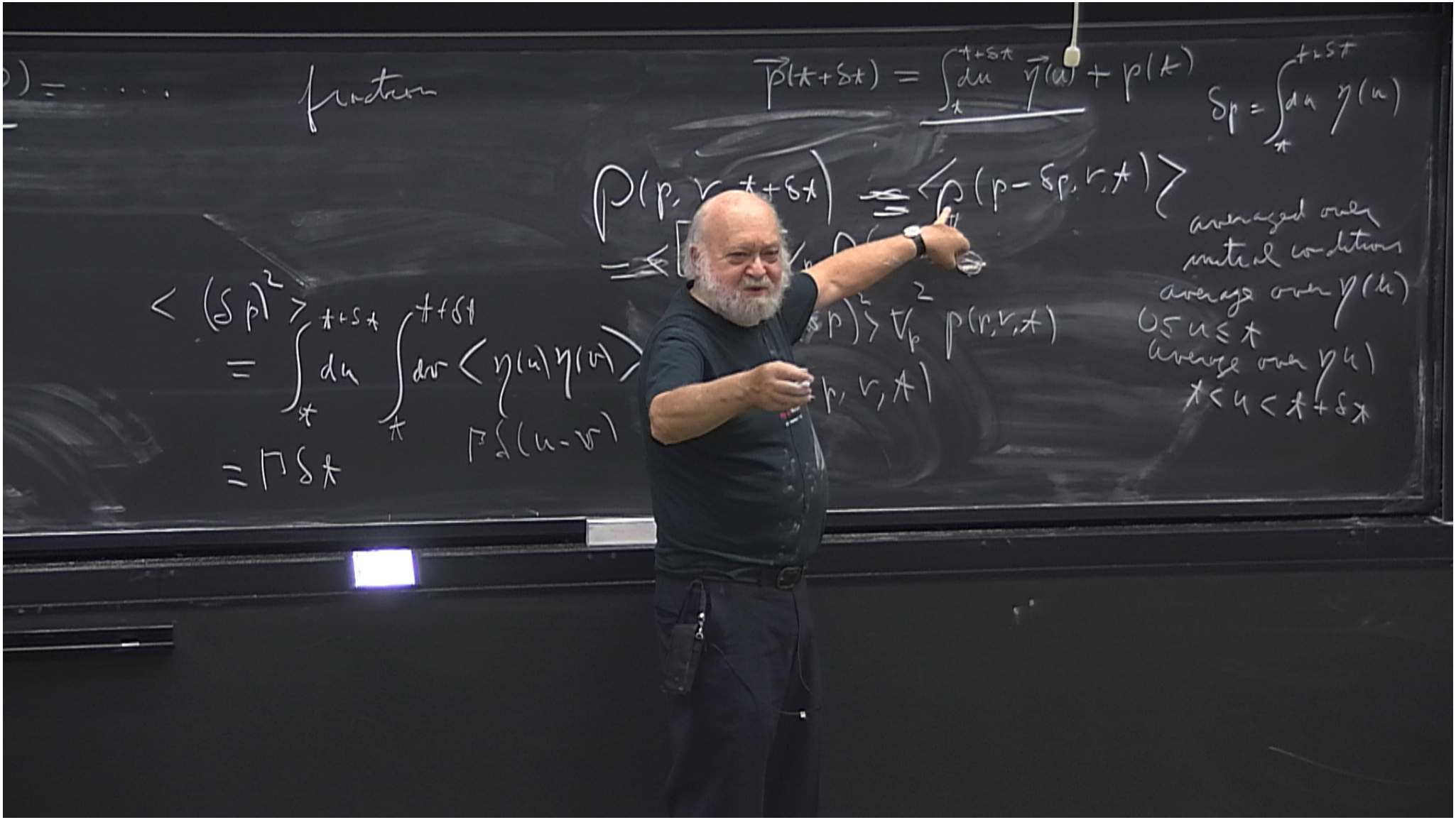

function
$\vec{p}(t+\delta t) = \int_t^{t+\delta t} du\ \vec{\eta}(u) + p(t)$
$\delta p = \int_t^{t+\delta t} du\ \eta(u)$
$\rho(p, r, t+\delta t) = \langle \rho(p-\delta p, r, t) \rangle$
$= \langle [-\delta p] \rangle \nabla_p \rho(p, r, t)$
$+ \frac{1}{2} \langle (\delta p)^2 \rangle \nabla_p^2 \rho(r, r, t)$
averaged over initial conditions
average over $\eta(u)$
$u \le t$
average over $\eta(u)$
$t < u < t + \delta t$
$\langle (\delta p)^2 \rangle = \int_t^{t+\delta t} du \int_t^{t+\delta t} dv \langle \eta(u) \eta(v) \rangle$
$= \Gamma \delta t$
$\Gamma \delta(u-v)$

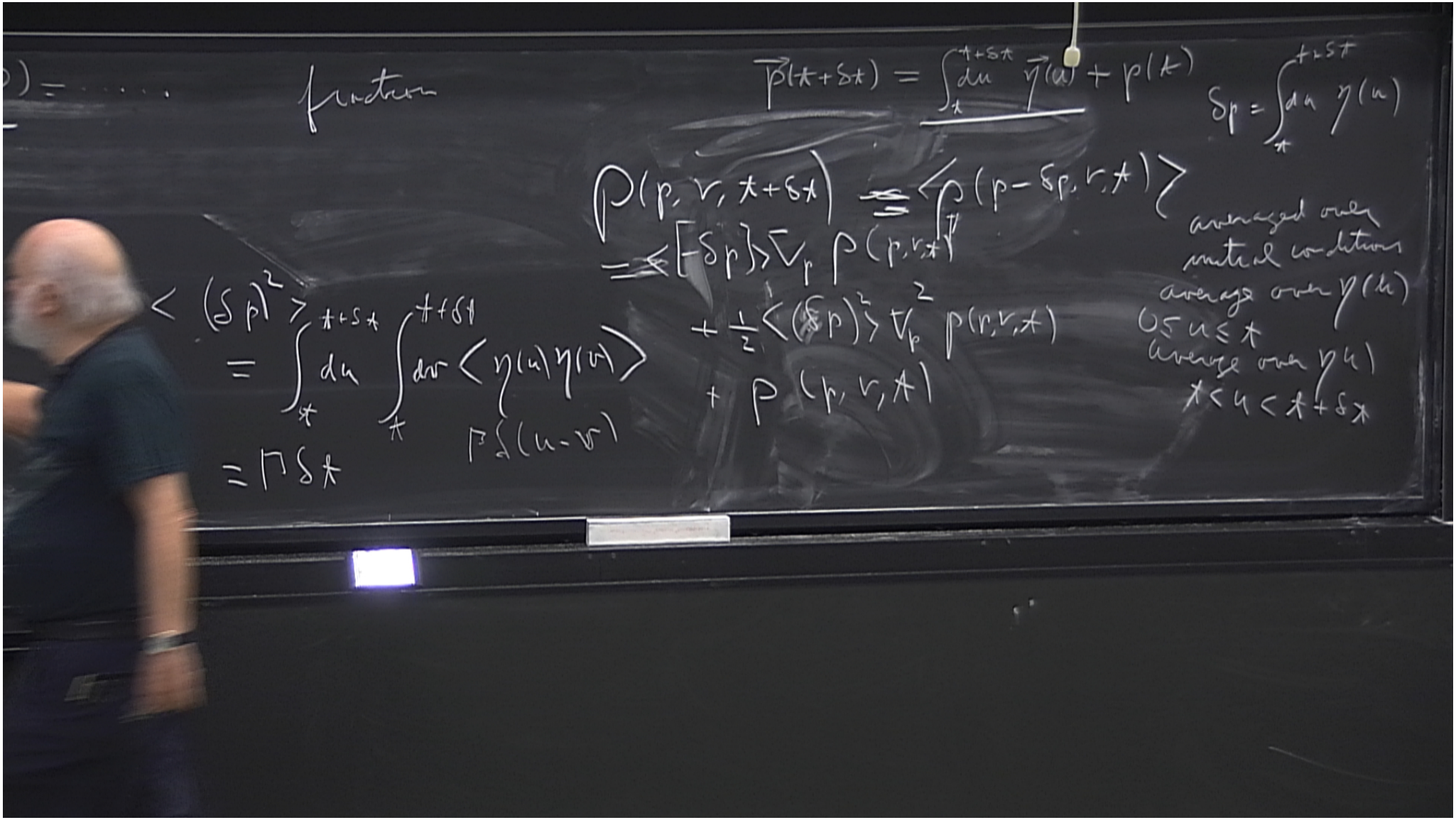

function
$\vec{p}(t+\delta t) = \int_t^{t+\delta t} du\ \vec{\eta}(u) + p(t)$
$\delta p = \int_t^{t+\delta t} du\ \eta(u)$
$\rho(p, v, t+\delta t) = \langle \rho(p - \delta p, v, t) \rangle$
$= \langle [-\delta p] \rangle \nabla_p \rho(p, v, t)$
$+ \frac{1}{2} \langle (\delta p)^2 \rangle \nabla_p^2 \rho(p, v, t)$
$\langle (\delta p)^2 \rangle = \int_t^{t+\delta t} du \int_t^{t+\delta t} dw \langle \eta(u) \eta(w) \rangle$
$= \Gamma \delta t$

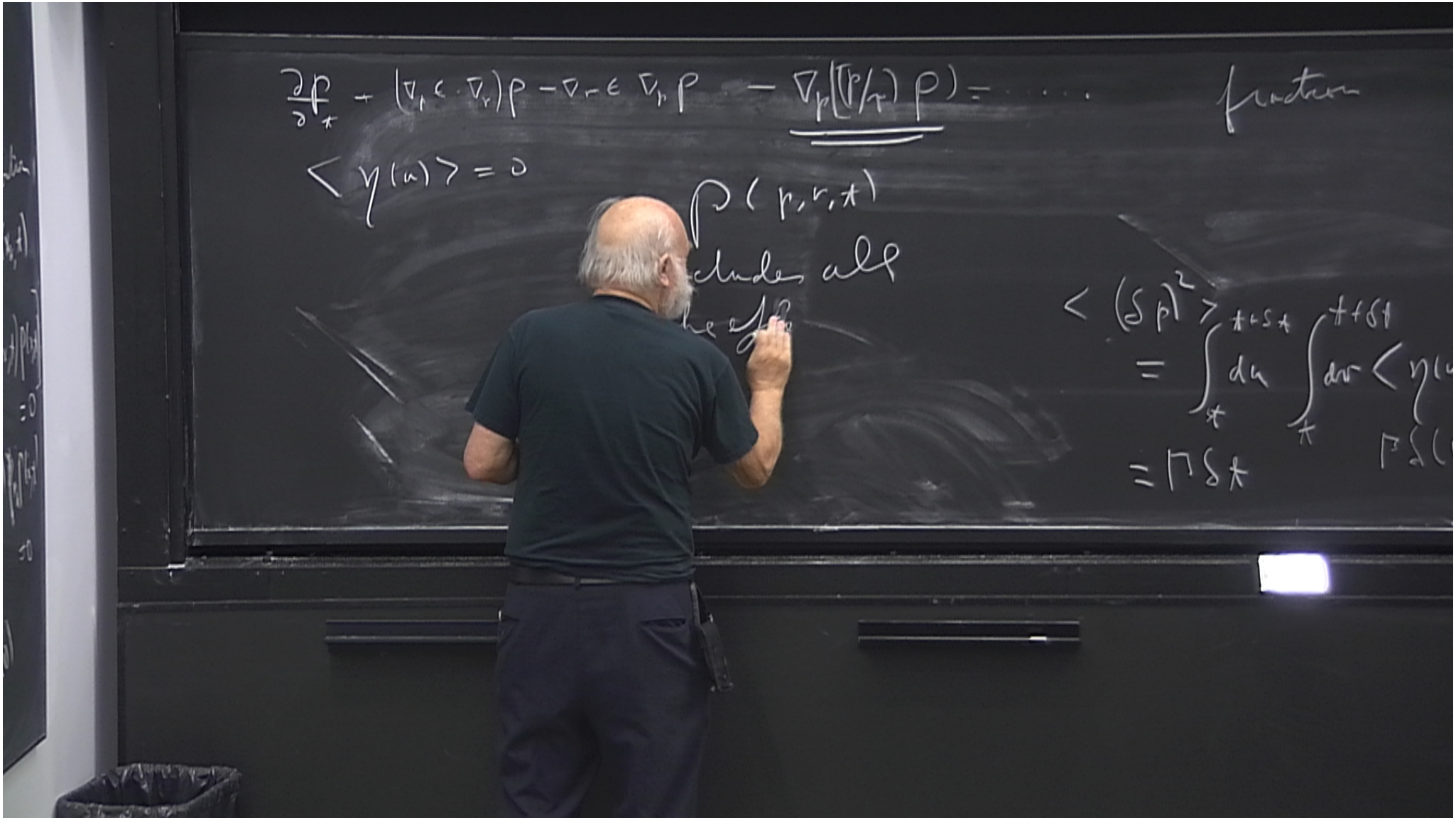

function
averaged over
initial conditions
average over $\eta(u)$
$u \le t$
average over $\eta(u)$
$t < u < t + \delta t$

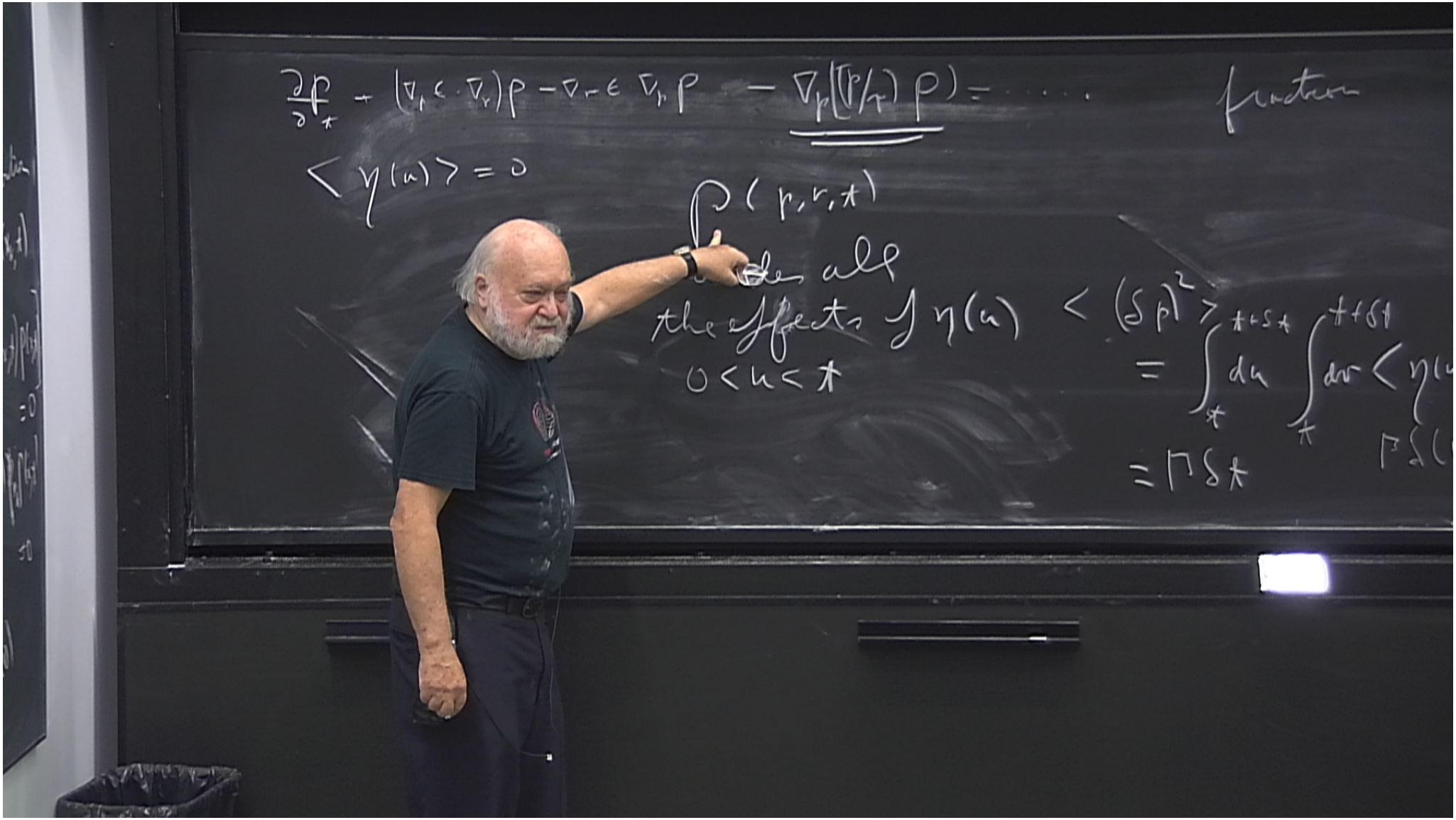

function
averaged over
initial conditions
average over η(u)
0 < u ≤ t
average over η(u)
t < u < t + δt

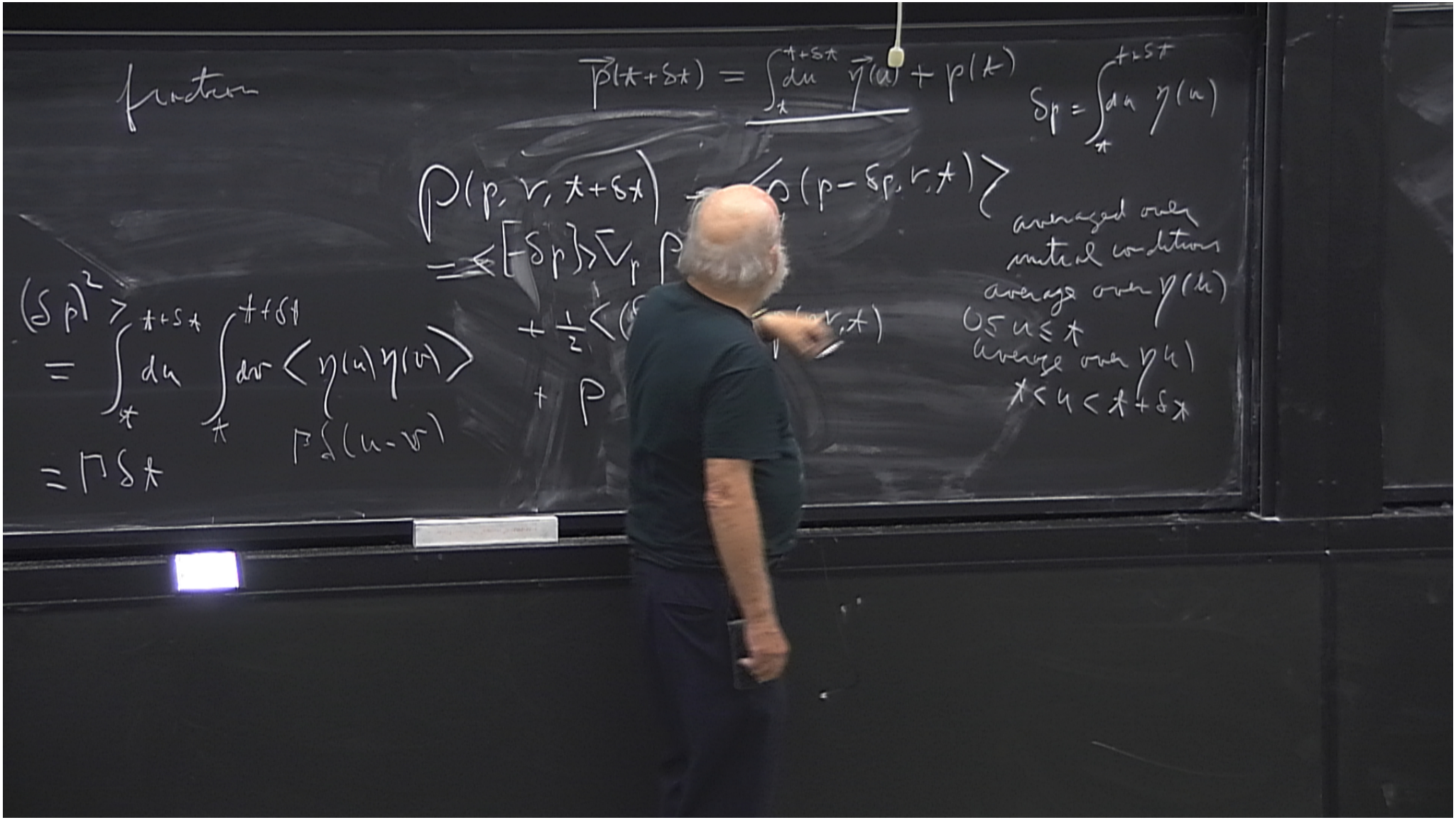

fluctuation
averaged over
initial conditions
average over η(u)
0 ≤ u ≤ t
average over η(u)
t < u < t + δt

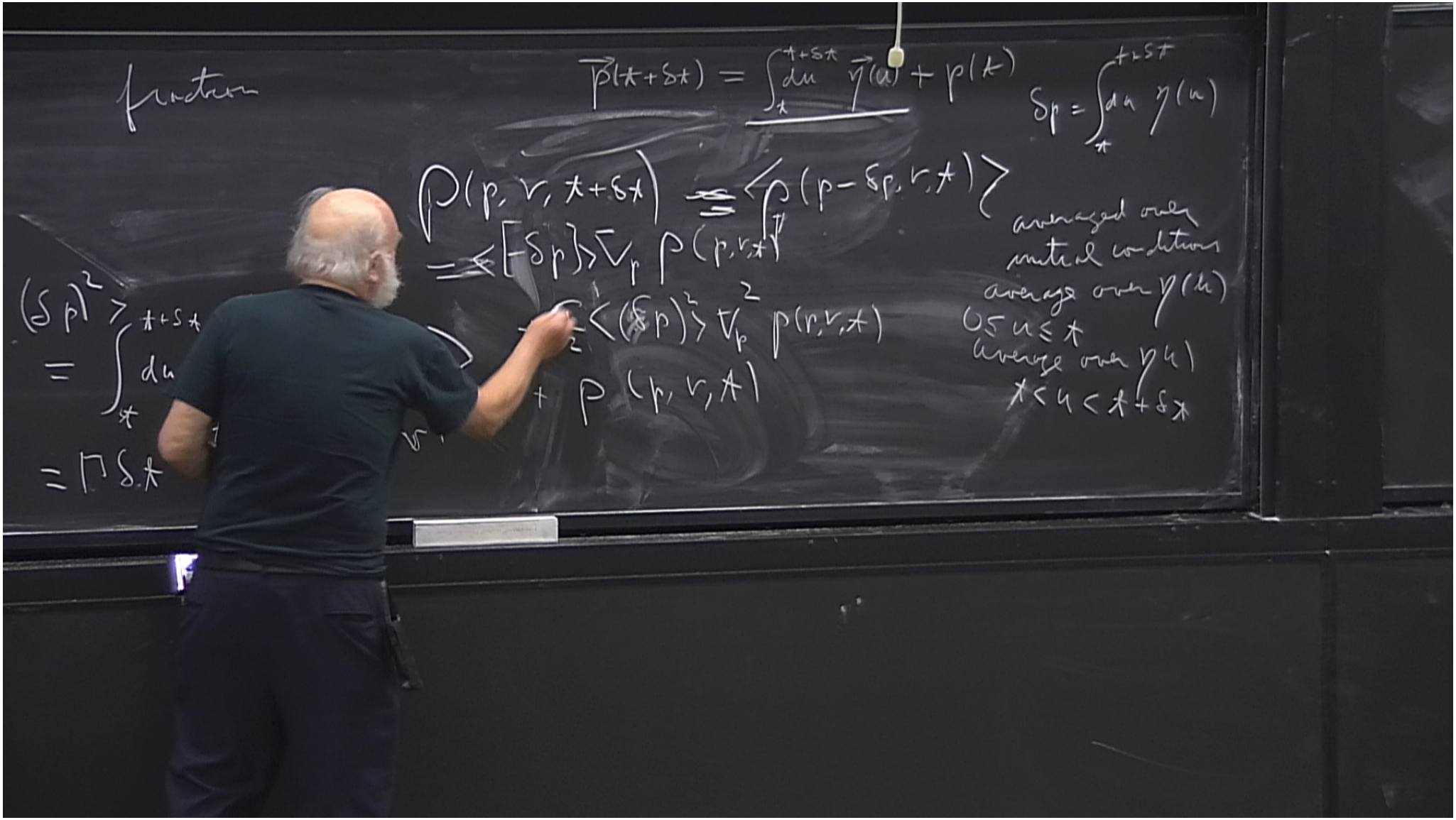

function
averaged over
initial conditions
average over η(u)
0 ≤ u ≤ t
average over η(u)
t < u < t + δt

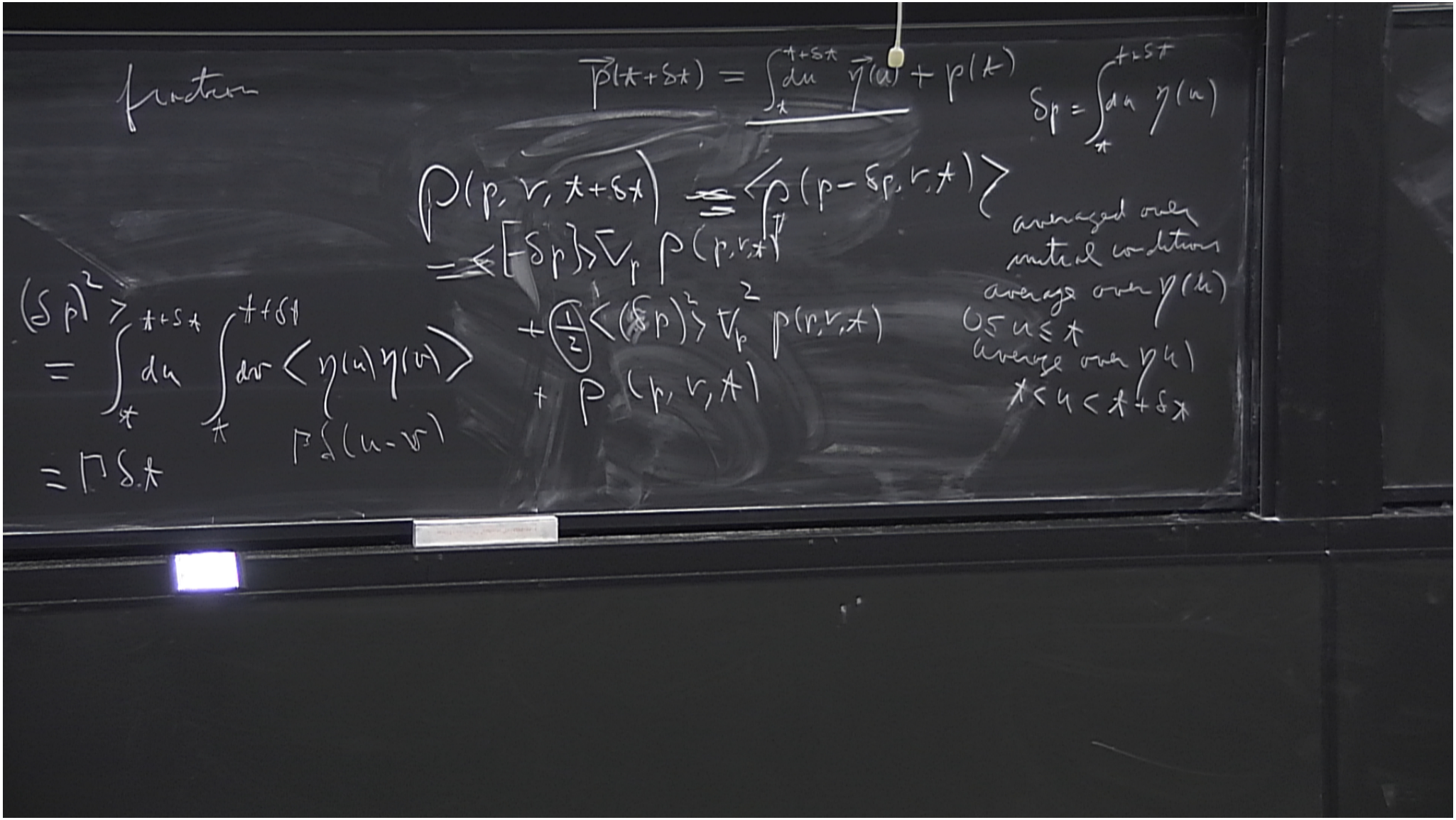

function
$\vec{p}(t+\delta t) = \int_t^{t+\delta t} du\ \vec{\eta}(u) + p(t)$
$\delta p = \int_t^{t+\delta t} du\ \eta(u)$
$\rho(p, r, t+\delta t) = \langle \rho(p-\delta p, r, t) \rangle$
$= \langle [1-\delta p] \rangle \nabla_p \rho(p, r, t)$
$+ \frac{1}{2} \langle (\delta p)^2 \rangle \nabla_p^2 \rho(p, r, t)$
$+ \rho(p, r, t)$
averaged over initial conditions
average over $\eta(u)$
$0 \le u \le t$
average over $\eta(u)$
$t < u < t + \delta t$
$\langle (\delta p)^2 \rangle = \int_t^{t+\delta t} du \int_t^{t+\delta t} dv \langle \eta(u) \eta(v) \rangle$
$= \Gamma \delta t$
$\Gamma \delta(u-v)$

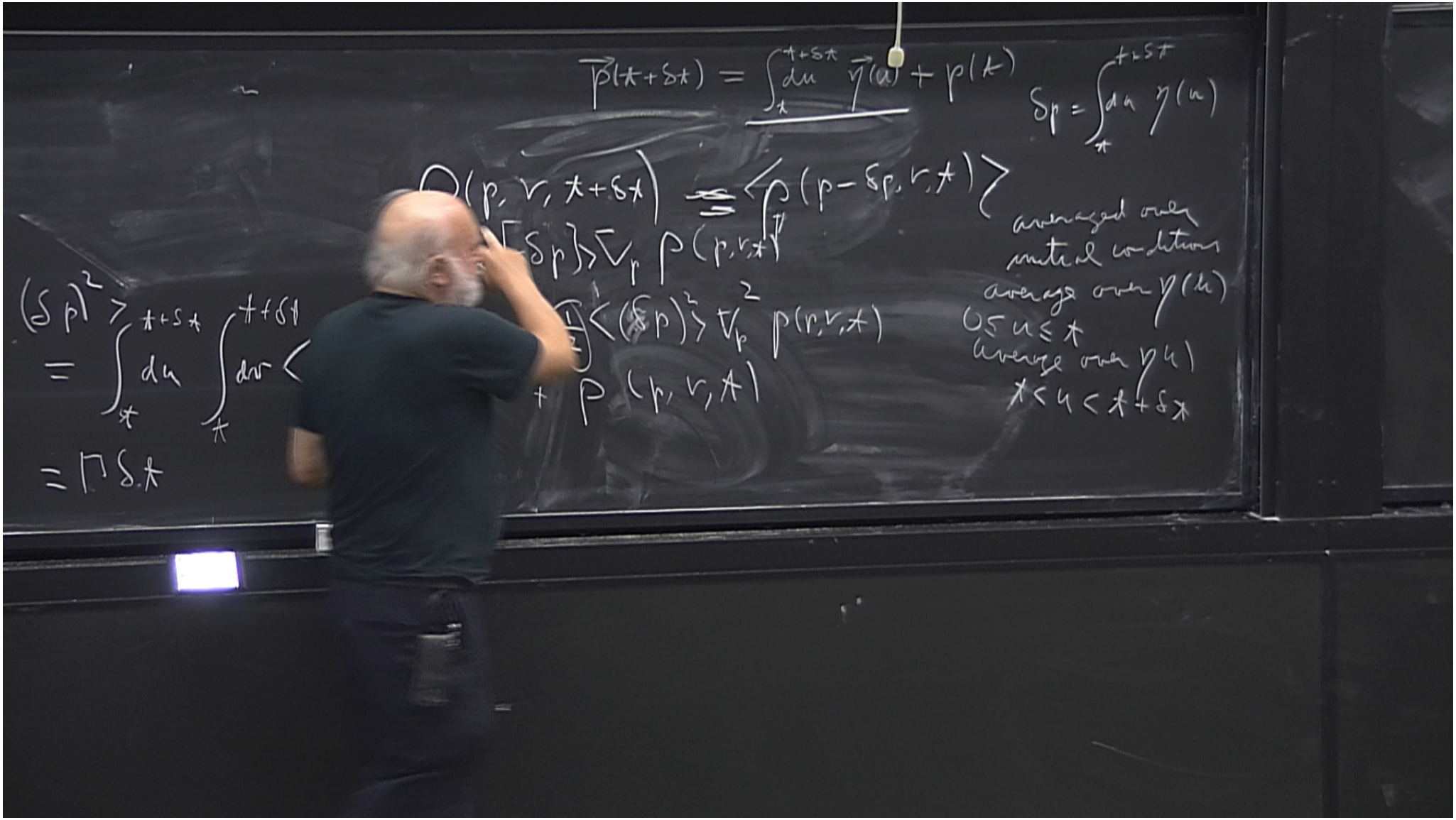

$\frac{\partial f}{\partial t} + (v_1 \cdot \nabla_r) f - \nabla_r \epsilon \nabla_v f - \nabla_v [(F/m) f] = \ldots$
function
$\langle \eta(u) \rangle = 0$
$\rho(r, r, t)$
$\langle (\delta \rho)^2 \rangle = \int_t^{t+\delta t} du \int_t^{t+\delta t} dw \langle \eta($
$= \Gamma \delta t$

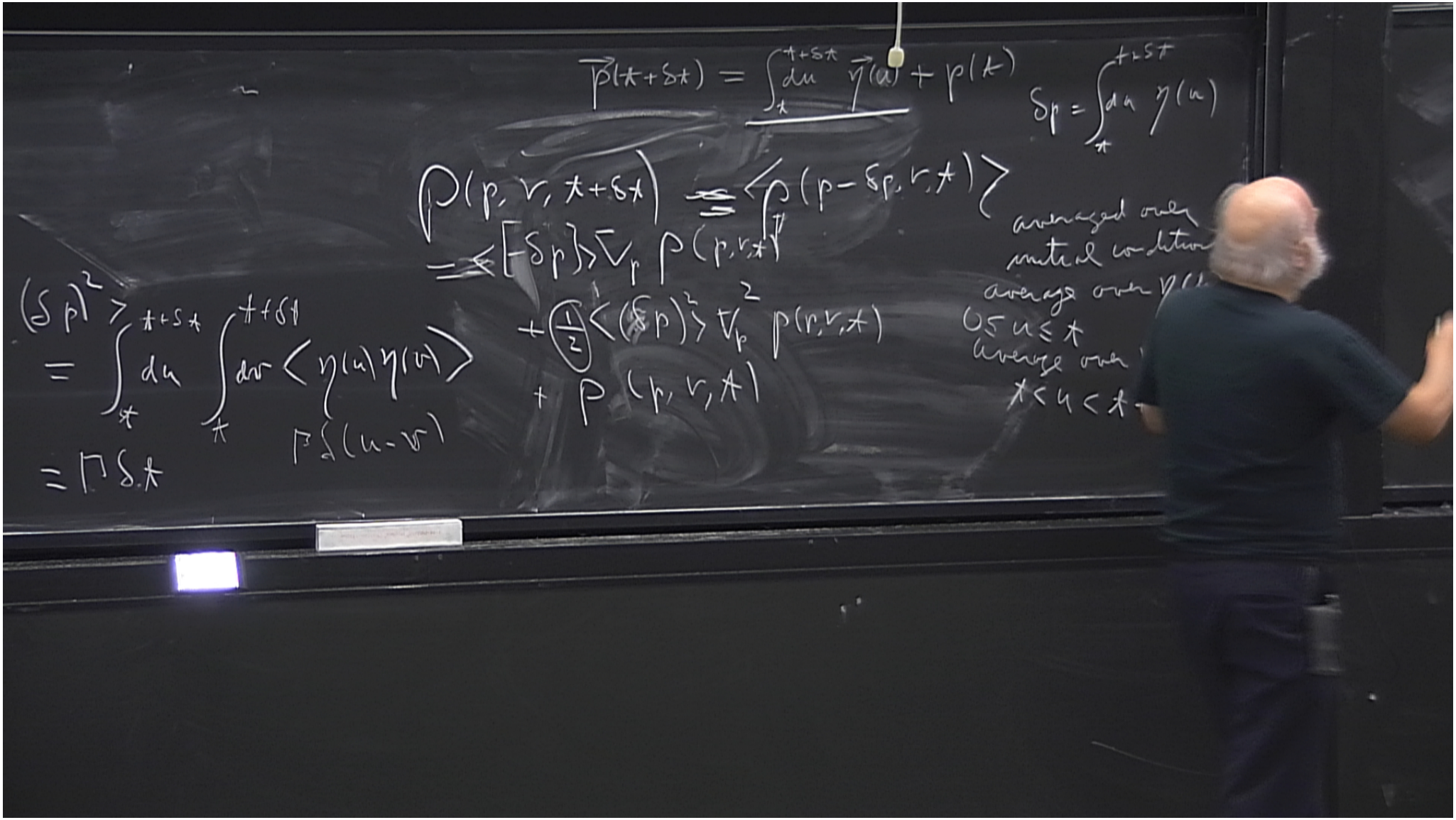

$\frac{\partial f}{\partial t}_* + (\nabla_1 \cdot \nabla_2) f - \nabla_2 \cdot \epsilon \nabla_1 f - \nabla_p([\Gamma/\tau] p) = \ldots$
function
$\langle \eta(u) \rangle = 0$
$\int (r, r, t)$
$\eta(u)$
$\langle (\delta p)^2 \rangle$
$= \int_t^{t+\delta t} du \int_t^{t+\delta t} dv \langle \eta($
$= \Gamma \delta t$

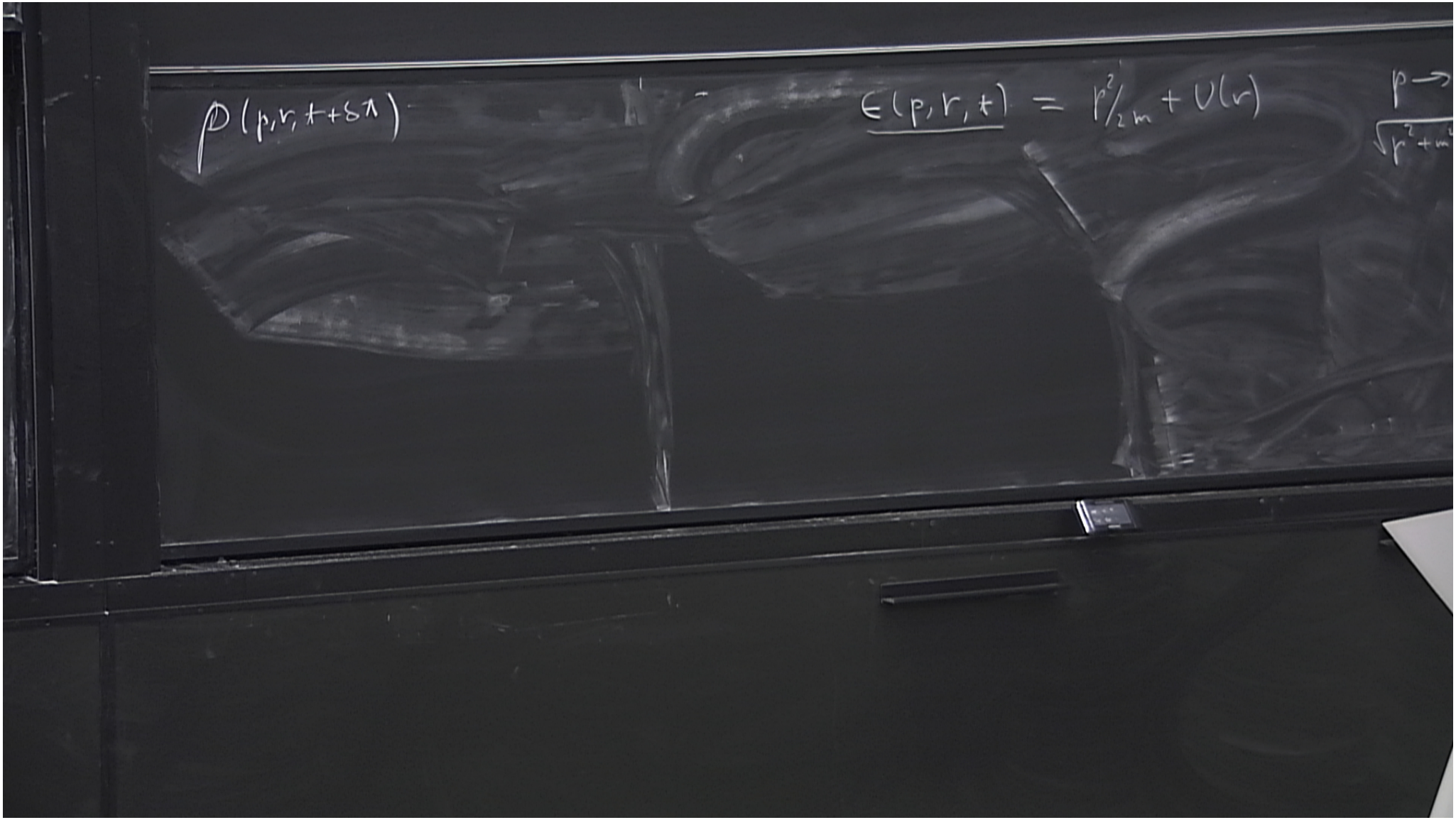

fluctuations
$\vec{p}(t+\delta t) = \int_t^{t+\delta t} du\ \vec{\eta}(u) + p(t)$
$\delta p = \int_t^{t+\delta t} du\ \eta(u)$
$\rho(p, r, t+\delta t) = \langle \rho(p-\delta p, r, t) \rangle$
averaged over initial conditions
average over $\eta(u)$
$0 \le u \le t$
average over $\eta(u)$
$t < u < t+\delta t$
$(\delta p)^2 = \int_t^{t+\delta t} du \int_t^{t+\delta t} dv \langle \eta(u)\eta(v) \rangle$
$= \Gamma \delta t$

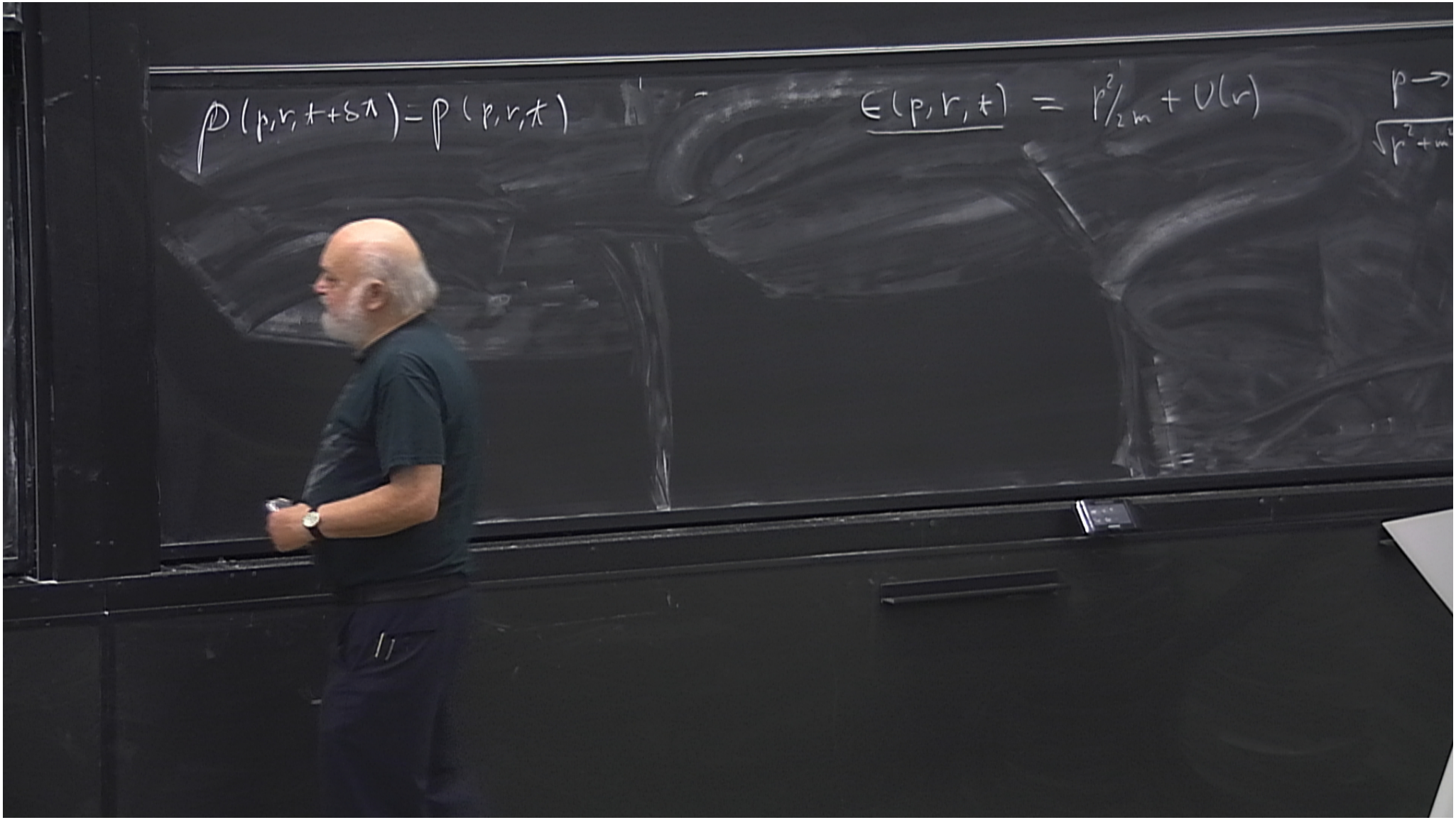

fluctuation
$\vec{p}(t+\delta t) = \int_t^{t+\delta t} du\ \vec{\eta}(u) + p(t)$
$\delta p = \int_t^{t+\delta t} du\ \eta(u)$
$\rho(p, r, t+\delta t) = \langle \rho(p-\delta p, r, t) \rangle$
averaged over initial conditions
average over $\eta(u)$
$0 \le u \le t$
average over $\eta(u)$
$t < u < t+\delta t$
$(\delta p)^2 = \int_t^{t+\delta t} du$

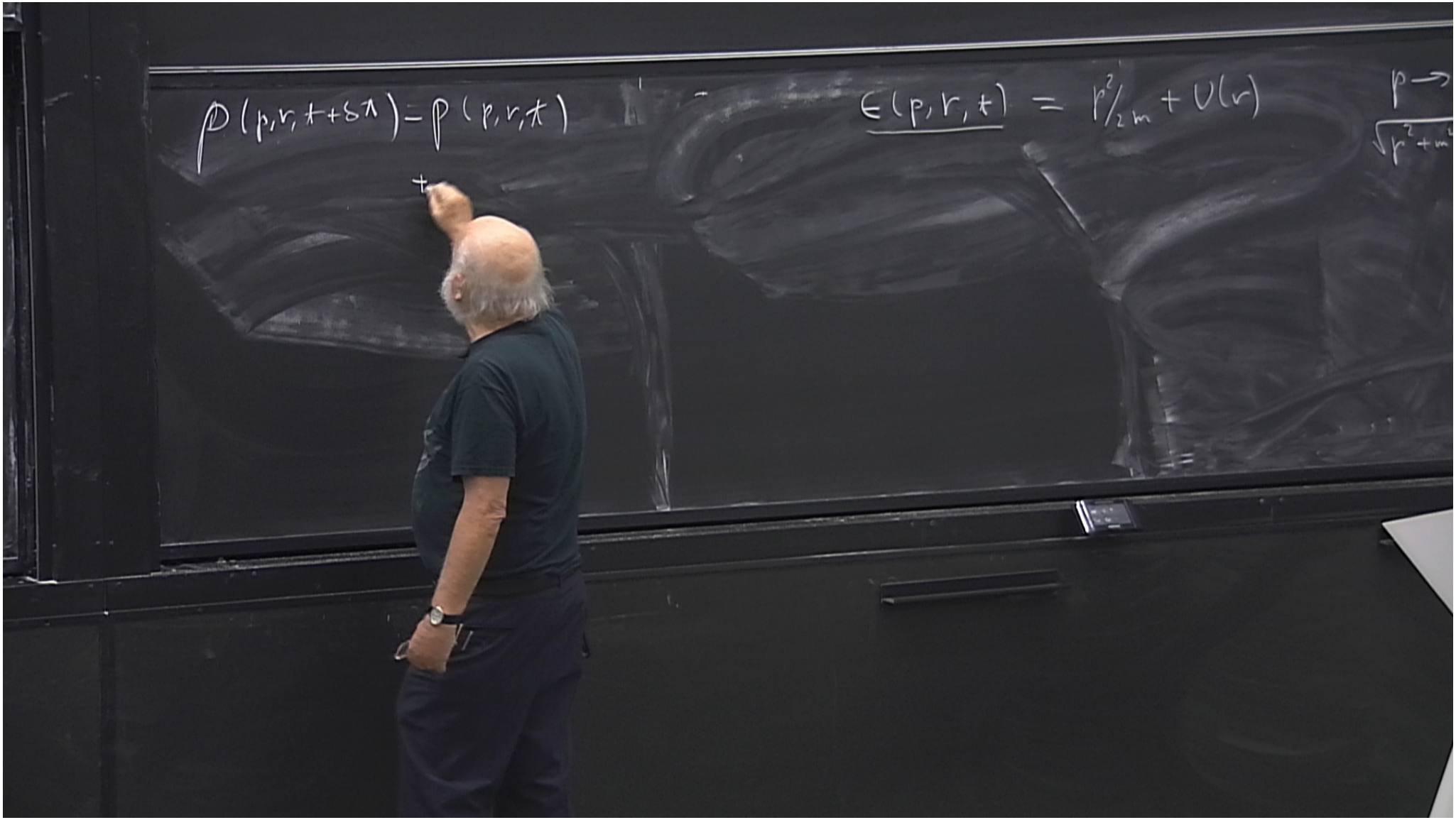

fluctuation
averaged over initial conditions
average over $\eta(u)$
$0 \le u \le t$
average over $\eta(u)$
$t < u < t + \delta t$

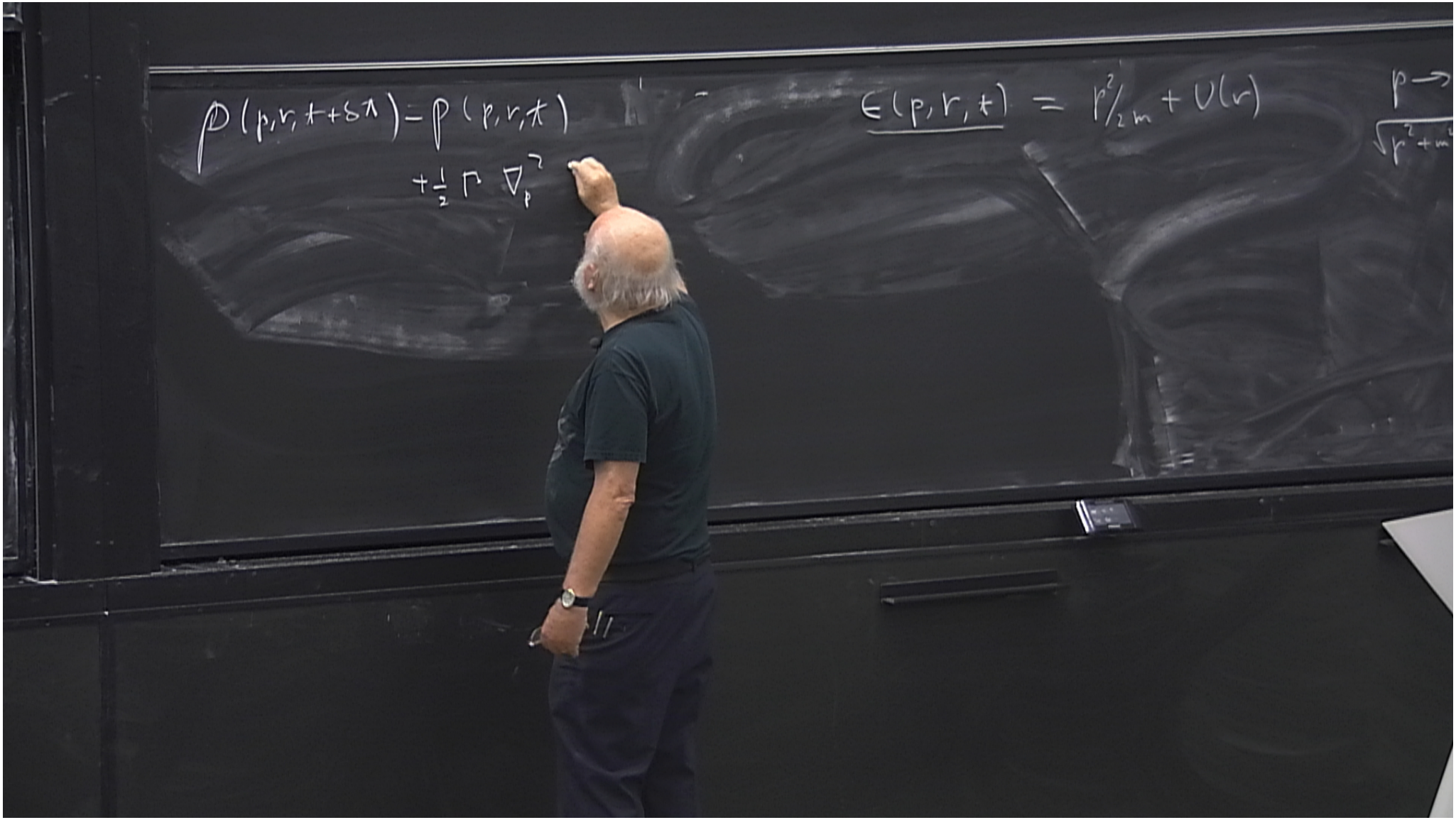

averaged over
initial conditions
average over η(u)
0 ≤ u ≤ t
average over η(u)
t < u < t + δt

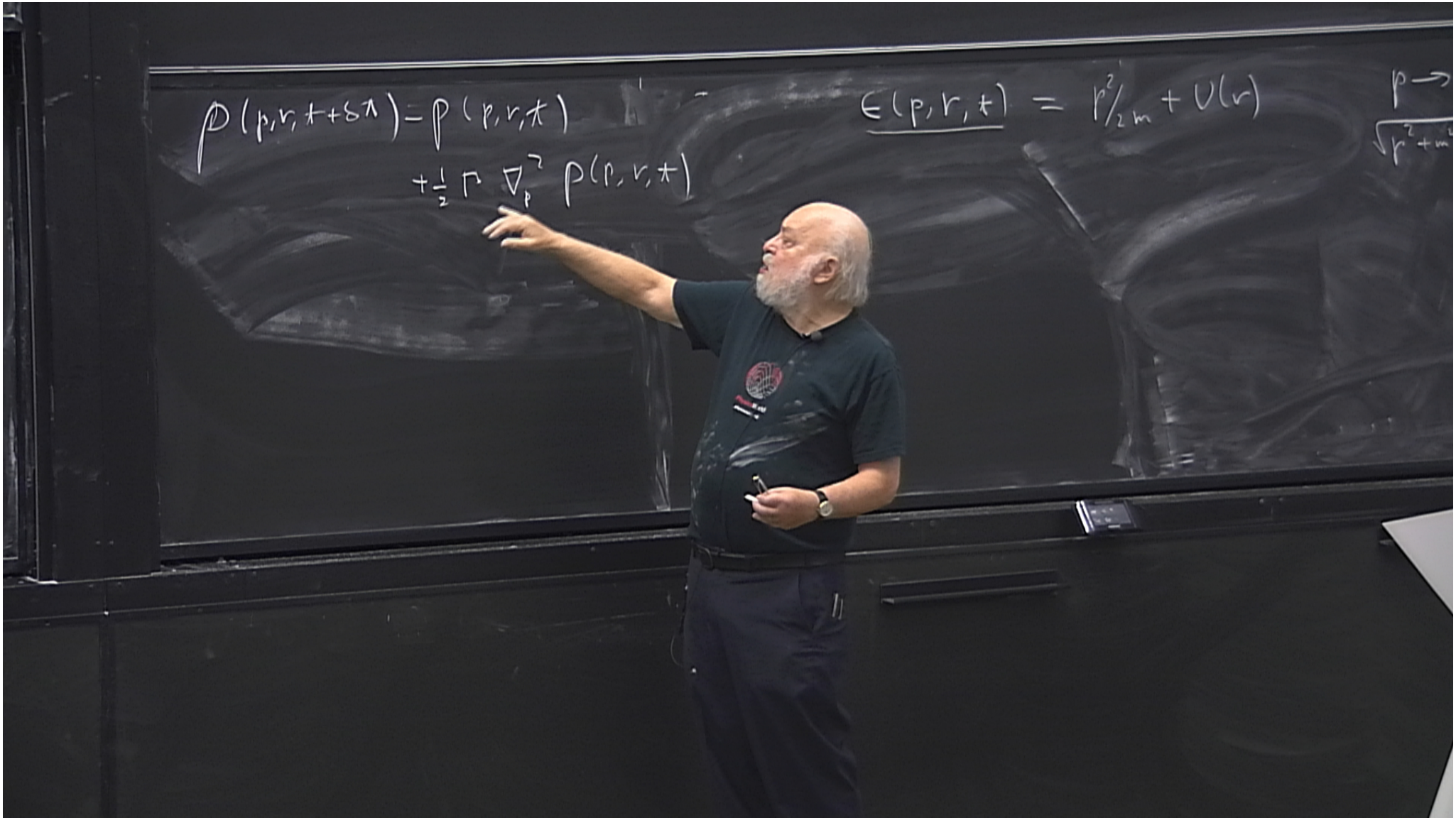

averaged over
initial condition
average over
$0 \le u \le t$
average over
$t < u < t+$

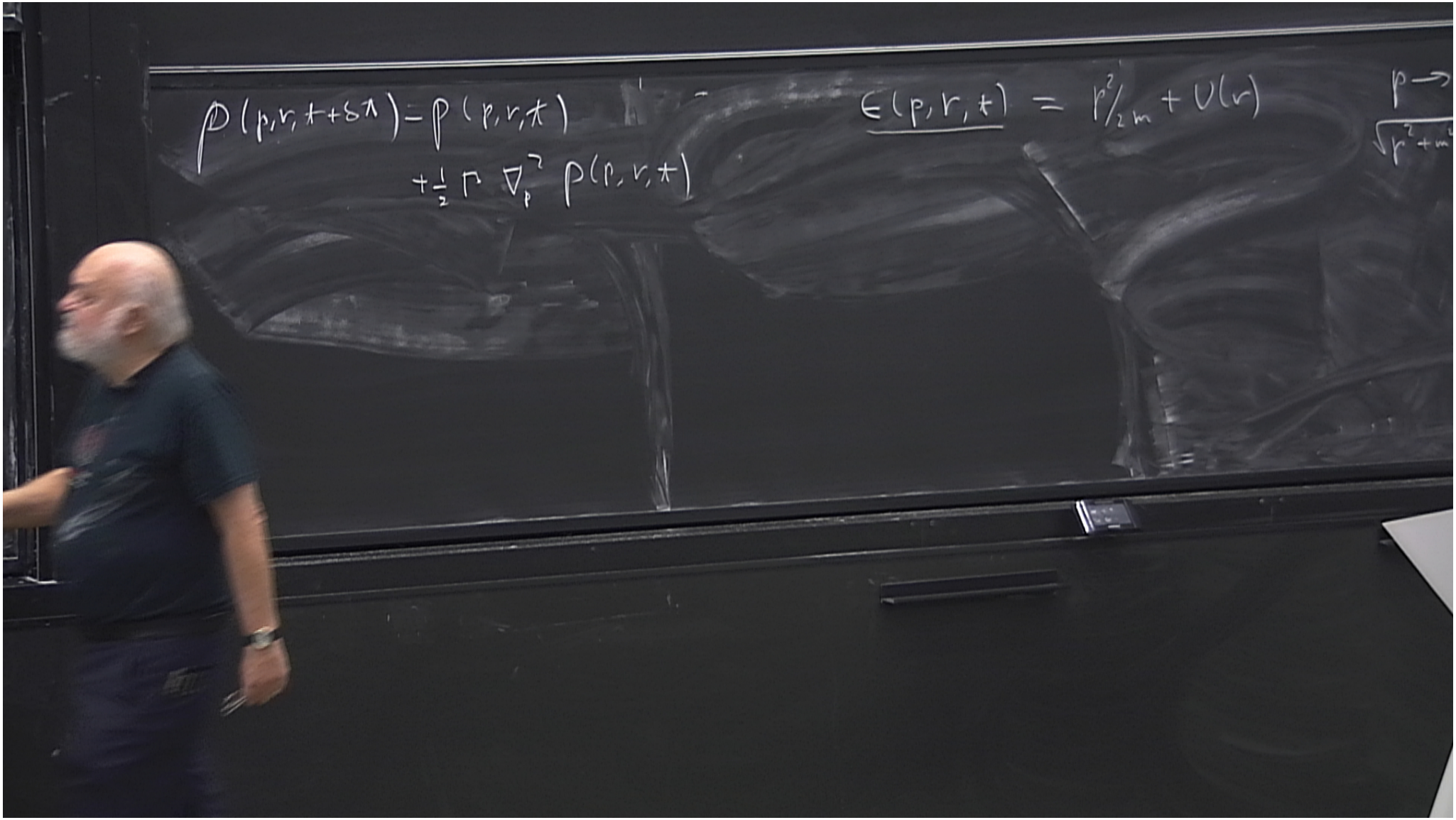

$\rho(p,r,t+\delta t)$
$\epsilon(p,r,t) = p^2/2m + U(r)$
$p \rightarrow$

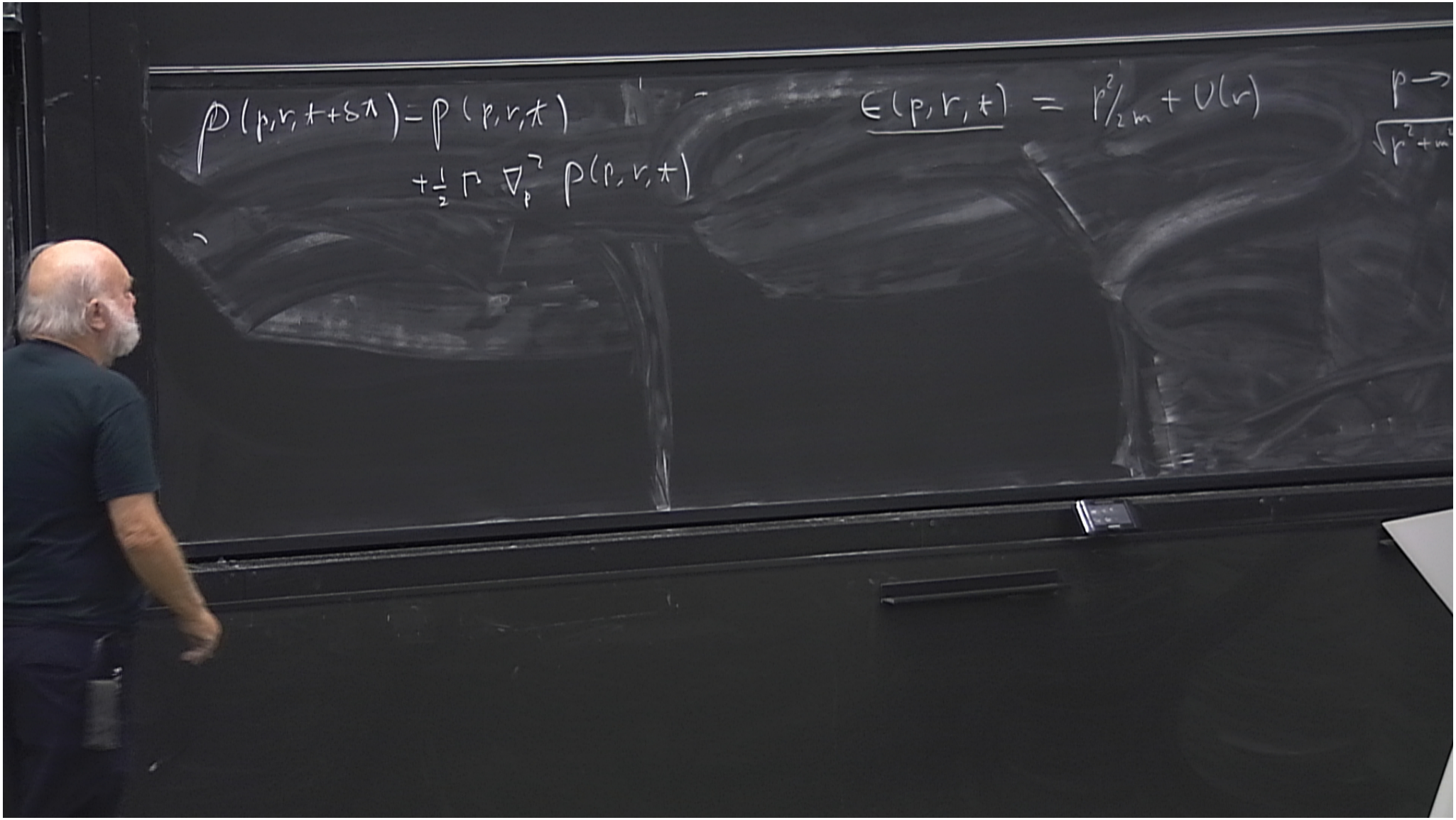

$\rho(p,r,t+\delta t) = \rho(p,r,t)$
$\epsilon(p,r,t) = p^2/2m + U(r)$
$p \rightarrow$

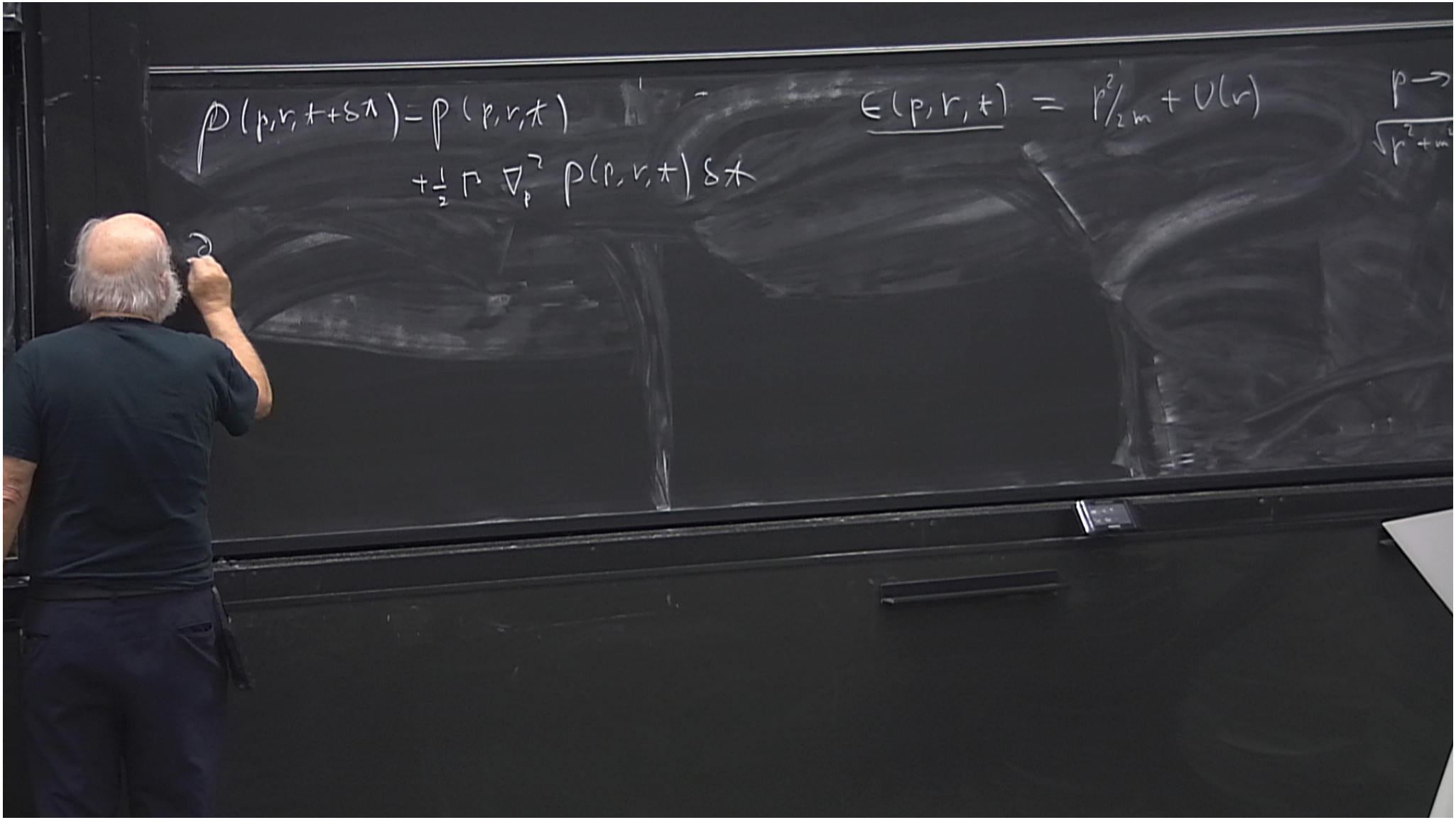

$\rho(p,r,t+\delta t) = \rho(p,r,t)$
$\epsilon(p,r,t) = p^2/2m + U(r)$

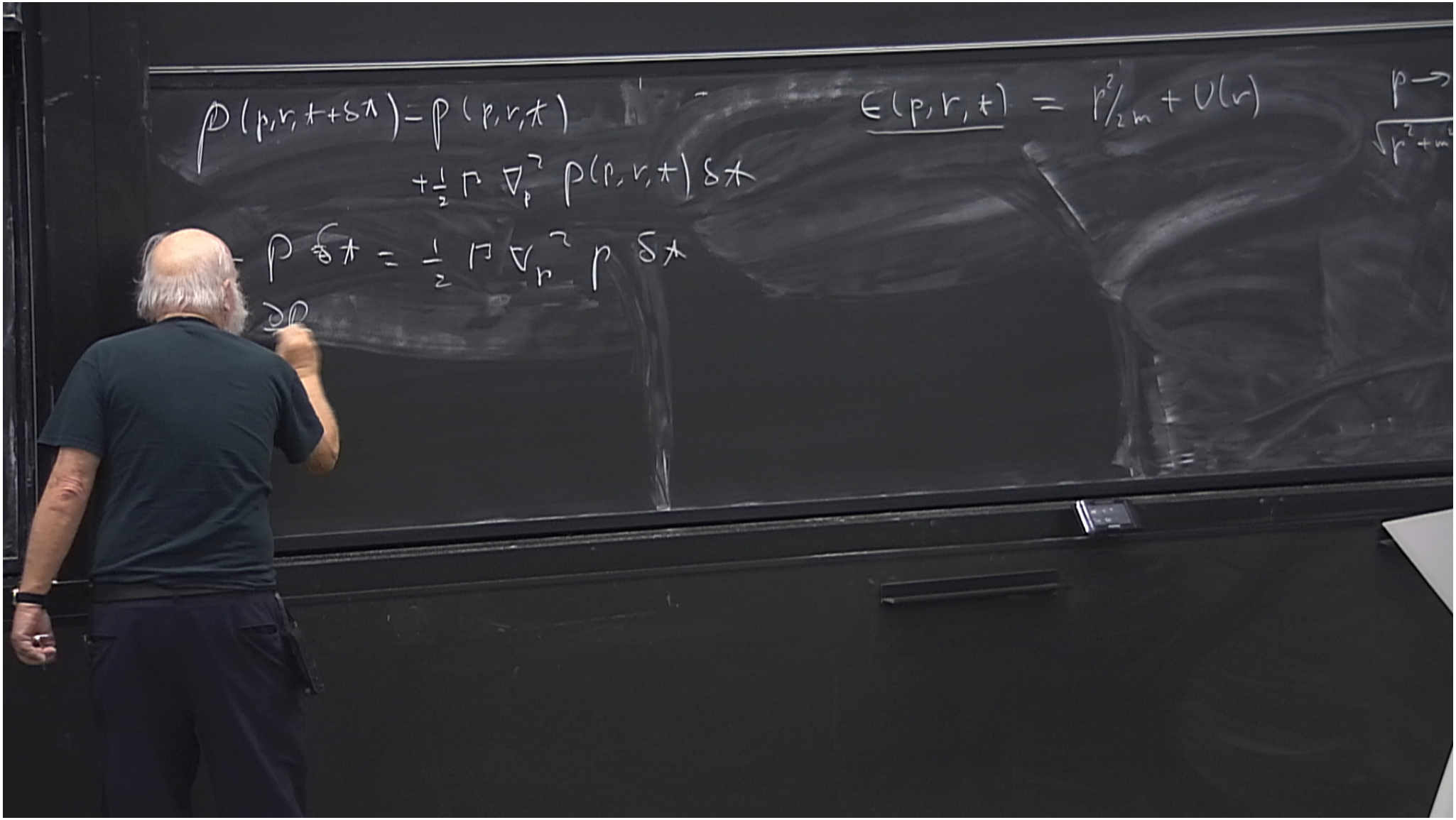

$\rho(p,r,t+\delta t) = \rho(p,r,t)$
$+\frac{1}{2}\Gamma\,\nabla_p^2$
$\epsilon(p,r,t) = p^2/2m + U(r)$
$p \to$

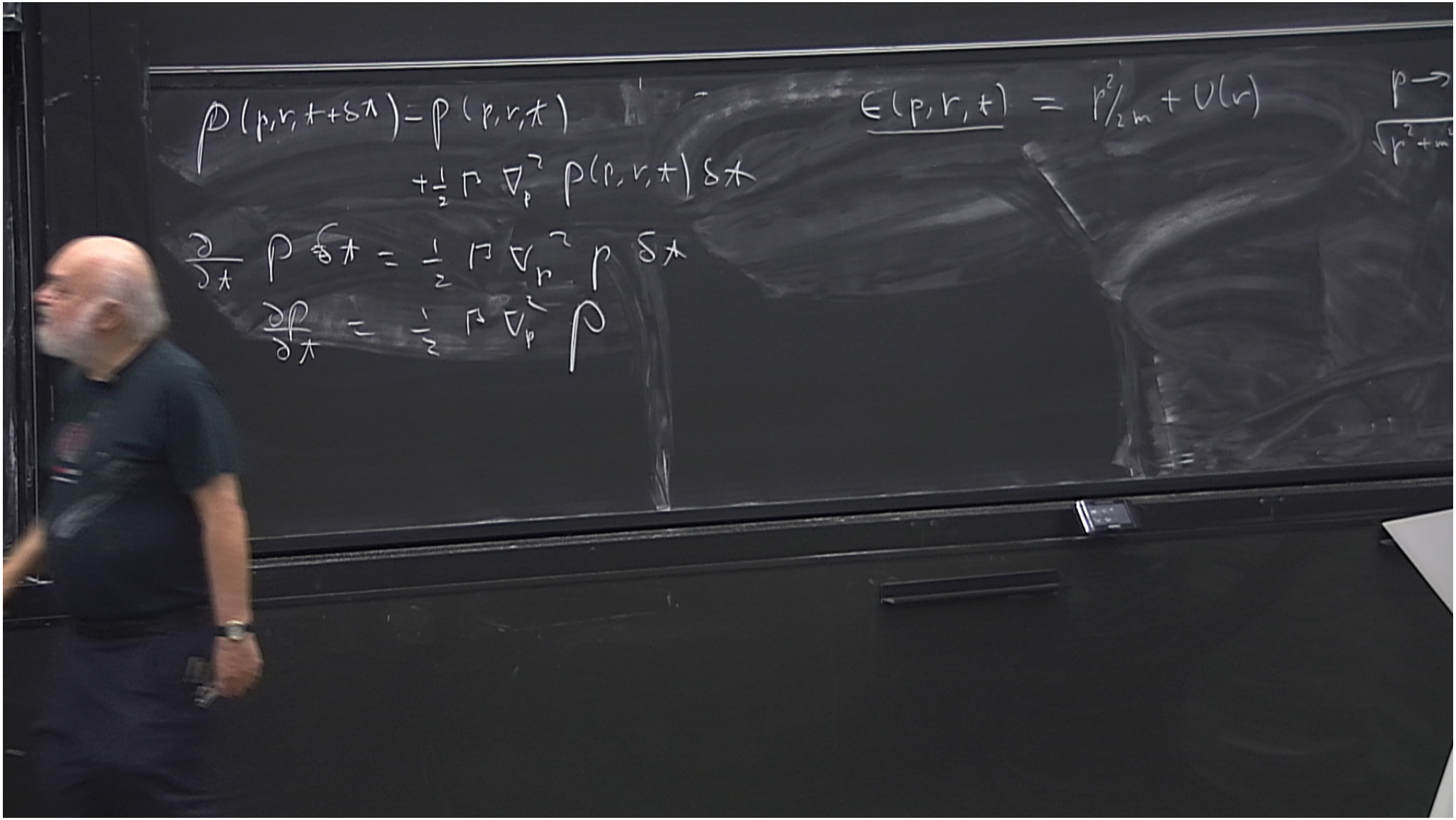

$\rho(p,r,t+\delta t) = \rho(p,r,t)$
$+\frac{1}{2}\Gamma\,\nabla_p^2\,\rho(p,r,t)$
$\epsilon(p,r,t) = p^2/2m + U(r)$
$p \rightarrow$

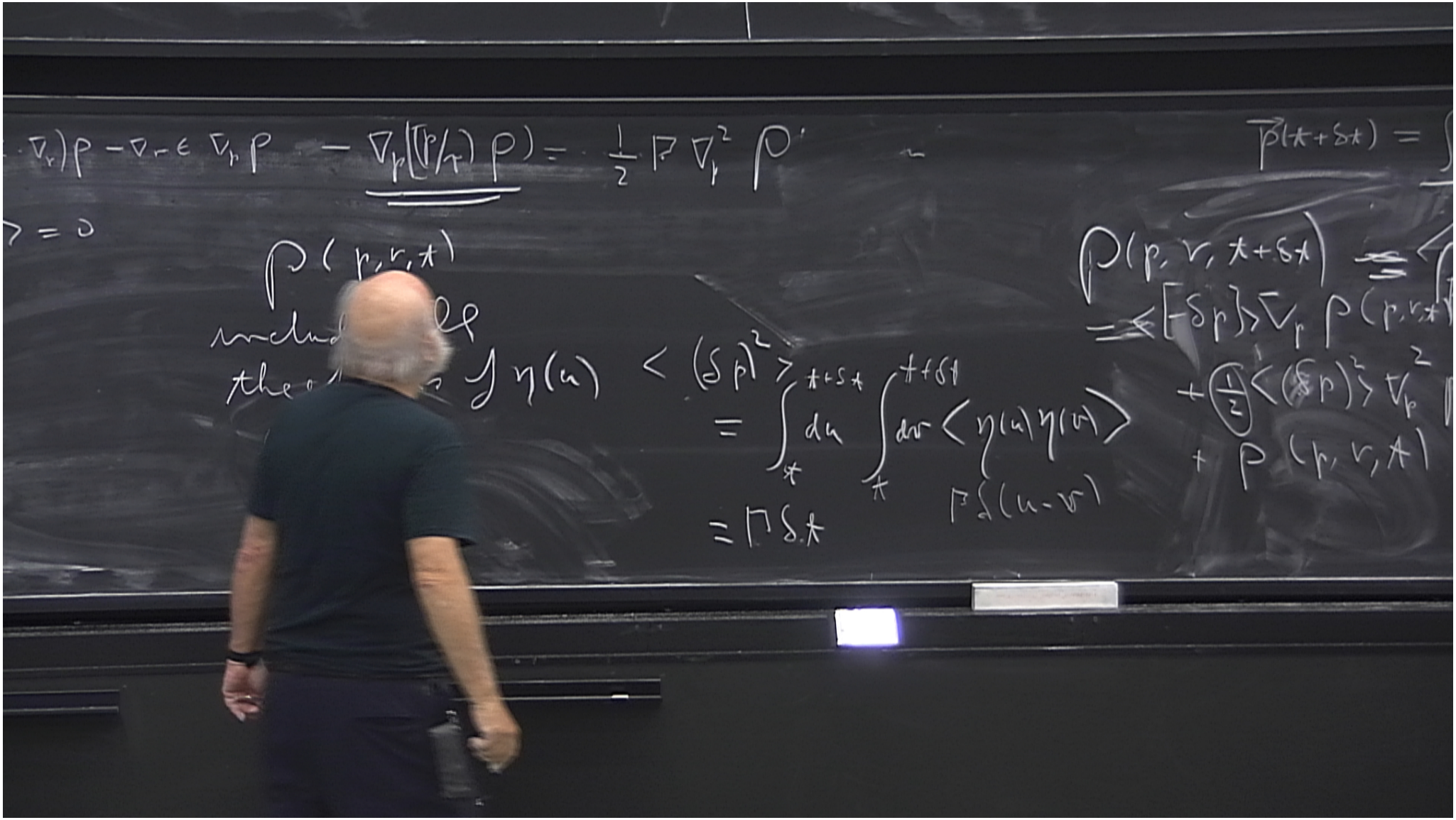

$\rho(p,r,t+\delta t) = \rho(p,r,t)$
$+\frac{1}{2}\Gamma\,\nabla_p^2\,\rho(p,r,t)$
$\epsilon(p,r,t) = p^2/2m + U(r)$
$p \rightarrow$
$\sqrt{r^2+m}$

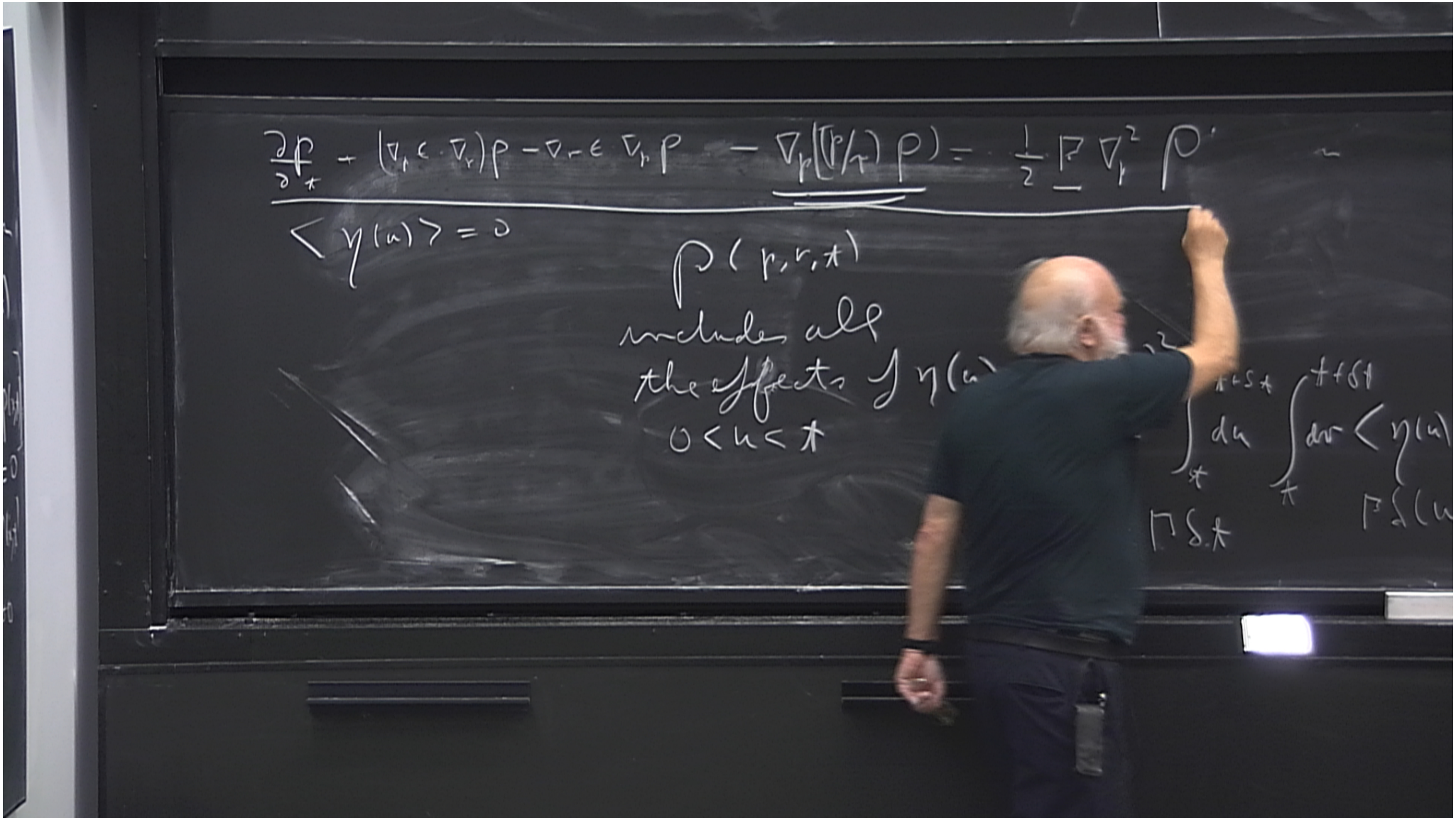

$\rho(p,r,t+\delta t) = \rho(p,r,t) + \frac{1}{2}\Gamma \nabla_p^2 \rho(p,r,t)$
$\epsilon(p,r,t) = p^2/2m + U(r)$
$p \rightarrow$
$\sqrt{r^2+m}$


$\rho(p,r,t+\delta t) = \rho(p,r,t)$
$+\frac{1}{2}\Gamma\,\nabla_p^2\,\rho(p,r,t)\,\delta t$
$\epsilon(p,r,t) = p^2/2m + U(r)$
$p \rightarrow$


$\rho(p,r,t+\delta t) = \rho(p,r,t)$
$+\frac{1}{2}\Gamma\nabla_p^2\rho(p,r,t)\delta t$
$\epsilon(p,r,t) = p^2/2m + U(r)$
$\frac{\partial\rho}{\partial t} = \frac{1}{2}\Gamma\nabla_r^2\rho\,\delta t$
$\frac{\partial\rho}{\partial}$


$\epsilon(p,r,t) = p^2/2m + U(r)$


$\rho(p, r, t)$
$\rho(p, r, t+\delta t)$
$= \Gamma \delta t$


$\langle \eta(u) \rangle = 0$
$\rho(r, v, t)$
includes all
the effects
$0 < u < t$

calculation of the effect of dp/dt=.....+ η(t) continued

$\rho(p,t+\delta t) - \rho(p,t) = < \rho(p-\delta p,t) - \rho(p,t) >$ $\qquad \delta p = \int_t^{t+\delta t} ds\ \eta(s)$

The average (<...>) in this equation is over values of η(s) for s between t and t+δt. All other averaging is included in ρ(p,t).

Now expand in δt. (the result is particularly simple because δp does not depend upon p (or q)).

$\delta t\ \partial_t \rho(p,t) = <\delta p>\ \partial_p \rho(p,t) + <\delta p^2>\ (\partial_p)^2 \rho(p,t)/2$

The average of δp is zero, and the rest gives

$\partial_t \rho(p,t) = [\ <\delta p^2>/(2\delta t)\ ]\ (\partial_p)^2 \rho(p,t)$

The average has the value:

$$[\delta p]^2 = \int_t^{t+\delta t} du \int_t^{t+\delta t} ds < \eta(u)\eta(s) > = \int_t^{t+\delta t} du \int_t^{t+\delta t} ds\ \Gamma\delta(u-s) = \Gamma\delta t$$

(recall that $<\eta(t)\ \eta(s)> = \Gamma\delta(t-s)$) so that we end up with the result

$\partial_t \rho(p,t) =$.....+ $(\Gamma/2)\ (\partial_p)^2 \rho($

This describes a diffusion in [illegible] momentum space.

(Notice that because it is diffusion [illegible] it cannot be followed backward in time!

Notice also that we had to go [illegible] order in our calculation of the effect of the random kicks because the second order [illegible] linear in δt and we were holding on to all linear terms in δt.)

calculation of the effect of dp/dt=.....+ η(t) continued

$\rho(p,t+\delta t) - \rho(p,t) = < \rho(p-\delta p,t) - \rho(p,t) >$ $\qquad \delta p = \int_t^{t+\delta t} ds\ \eta(s)$

The average (<...>) in this equation is over values of η(s) for s between t and t+δt. All other averaging is included in ρ(p,t).

Now expand in δt. (the result is particularly simple because δp does not depend upon p (or q)).

$\delta t\ \partial_t \rho(p,t) = <\delta p>\ \partial_p \rho(p,t) + <\delta p^2>\ (\partial_p)^2 \rho(p,t)/2$

The average of δp is zero, and the rest gives

$\partial_t \rho(p,t) = [\ <\delta p^2>/(2\delta t)\ ]\ (\partial_p)^2 \rho(p,t)$

The average has the value:

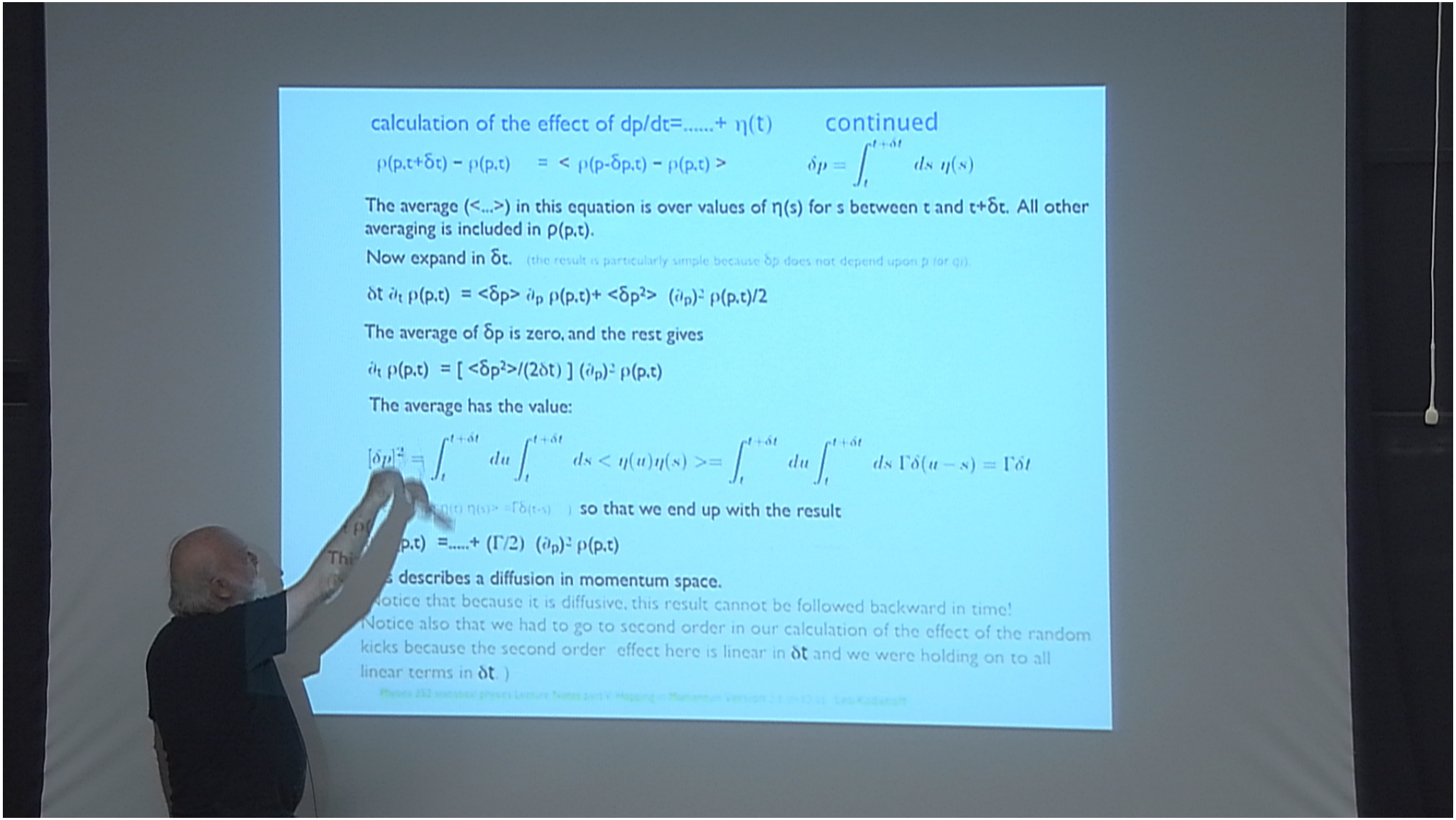
$$[\delta p]^2 = \int_t^{t+\delta t} du \int_t^{t+\delta t} ds < \eta(u)\eta(s) >= \int_t^{t+\delta t} du \int_t^{t+\delta t} ds\ \Gamma\delta(u-s) = \Gamma\delta t$$

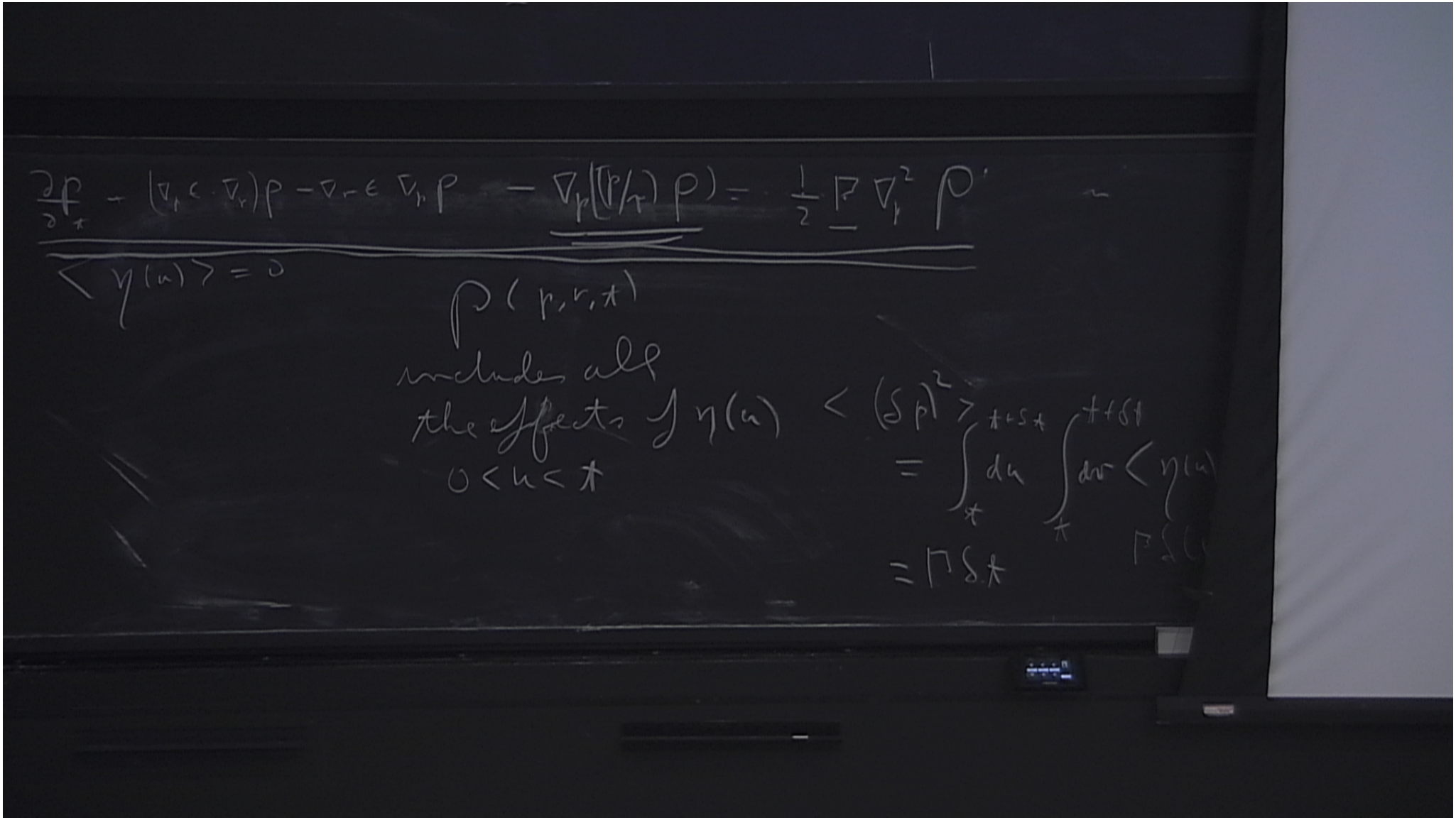
(recall that $<\eta(t)\,\eta(s)> = \Gamma\delta(t-s)$) so that we end up with the result

$\partial_t \rho(p,t) = ...+ (\Gamma/2)\ (\partial_p)^2 \rho(p,t)$

This describes a diffusion in momentum space.

(Notice that because it is diffusive, this result cannot be followed backward in time! Notice also that we had to go to second order in our calculation of the effect of the random kicks because the second order effect here is linear in δt and we were holding on to all linear terms in δt.)


calculation of the effect of dp/dt=.....+ η(t) continued
ρ(p,t+δt) − ρ(p,t) = < ρ(p-δp,t) − ρ(p,t) >
$\delta p = \int_t^{t+\delta t} ds\ \eta(s)$
The average (<...>) in this equation is over values of η(s) for s between t and t+δt. All other averaging is included in ρ(p,t).
Now expand in δt. (the result is particularly simple because δp does not depend upon p (or q)).
δt ∂t ρ(p,t) = <δp> ∂p ρ(p,t)+ <δp²> (∂p)² ρ(p,t)/2
The average of δp is zero, and the rest gives
∂t ρ(p,t) = [<δp²>/(2δt)] (∂p)² ρ(p,t)
The average has the value:
$[\delta p]^2 = \int_t^{t+\delta t} du \int_t^{t+\delta t} ds < \eta(u)\eta(s) > = \int_t^{t+\delta t} du \int_t^{t+\delta t} ds\ \Gamma\delta(u-s) = \Gamma\delta t$
so that we end up with the result
p,t) =.....+ (Γ/2) (∂p)² ρ(p,t)
s describes a diffusion in momentum space.
Notice that because it is diffusive, this result cannot be followed backward in time!
Notice also that we had to go to second order in our calculation of the effect of the random kicks because the second order effect here is linear in δt and we were holding on to all linear terms in δt.)


$\frac{\partial f}{\partial t} + (v_i \cdot \nabla_r) f - \nabla_r \cdot \epsilon \nabla_v f - \nabla_v \left[\left(\frac{F}{m} \right) f \right] = \frac{1}{2} \Gamma \nabla_v^2 f$
$\langle \eta(u) \rangle = 0$
$f(r, v, t)$
includes all
the effects of $\eta(u)$
$0 < u < t$
$\langle (\delta p)^2 \rangle = \int_t^{t+\delta t} du \int_t^{t+\delta t} dw \langle \eta(u)$
$= \Gamma \delta t$

Effect of Brownian motion: toward a unique probability distribution

We put together our two different pieces of the Brownian time derivative equation and get:

$$\partial_t \rho(p,x,t) = \text{.....} + (\Gamma/2)\ (\partial_p)^2 \rho(p,x,t) + \partial_p [c\ (\partial_p \varepsilon)\ \rho(p,x,t)]$$

The ....'s refer to terms which might come from usual Hamiltonian mechanics. We shall put them aside for a moment. An equation, like the one here, obtained from averaging the stochastic mechanics of a Langevin equation, is called a Fokker-Planck equation.

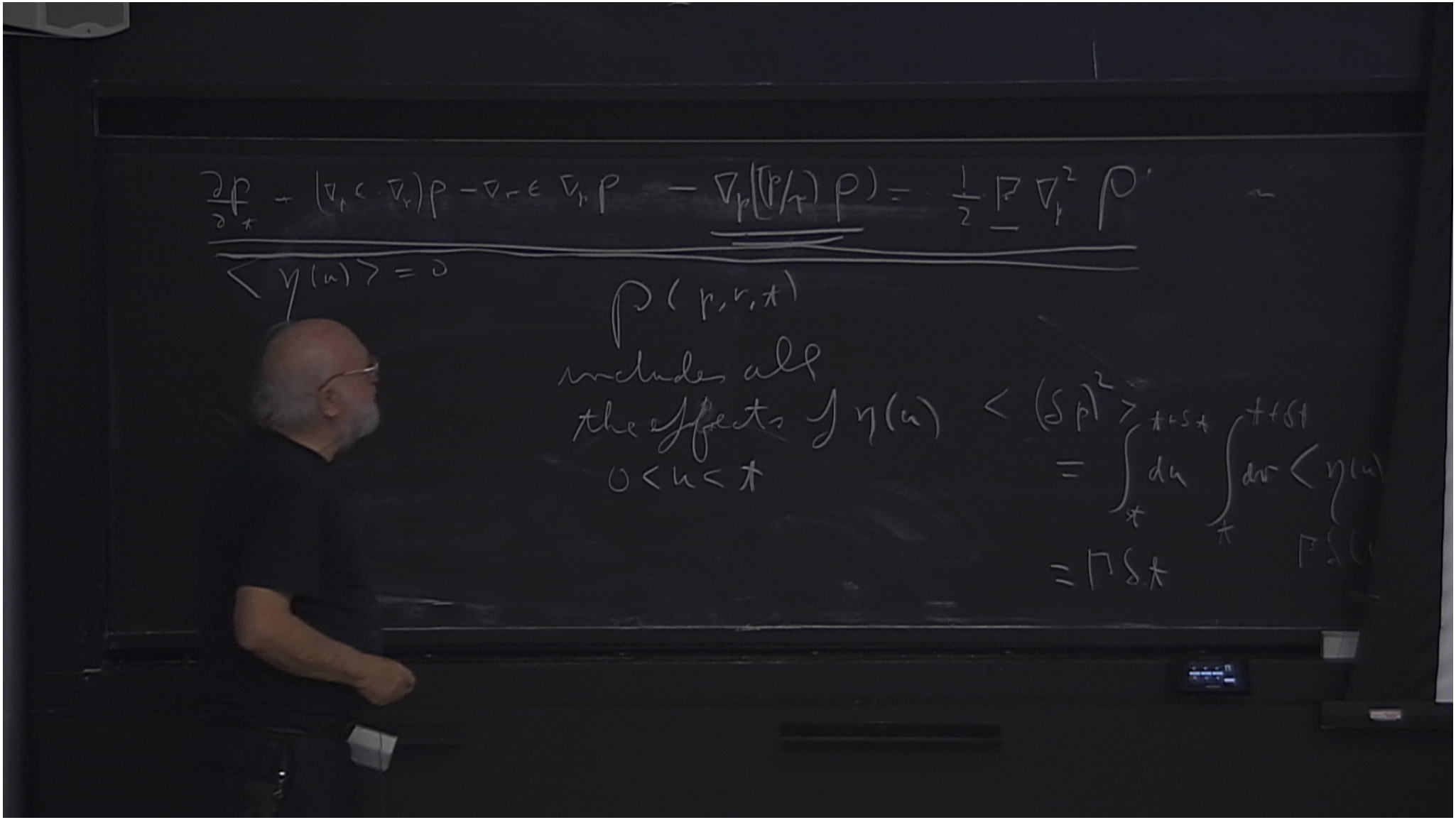
We look for a time-independent zero of this part of the equation to tell us about the equilibrium behavior. Write $\rho(p,x,t)=\exp[-Q(p,x)+C]$ and find that zeroing this part gives

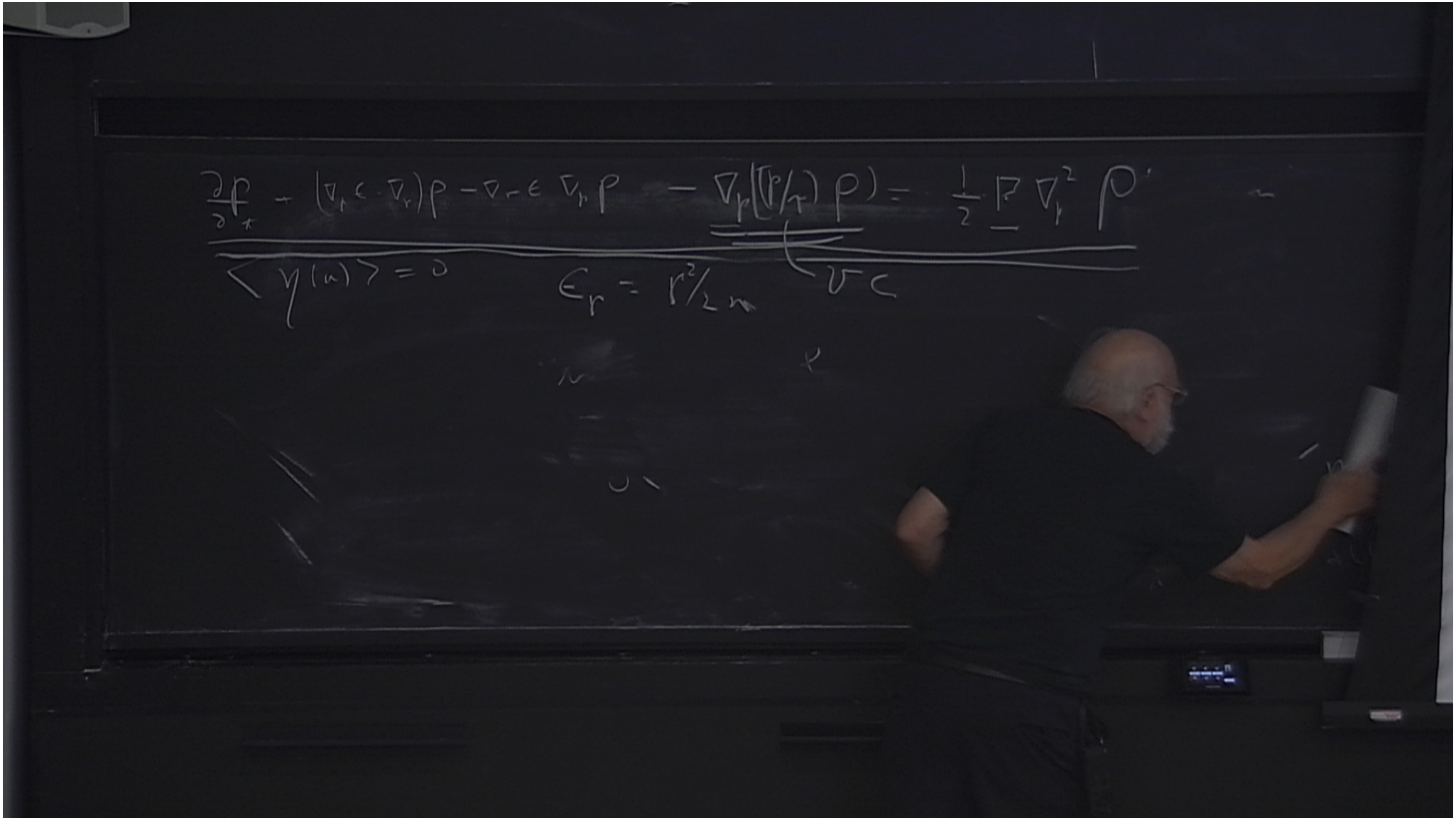
$$0 = \partial_p \{\exp[-Q(p,x)][\ (\Gamma/2)(-\partial_p Q) + \partial_p \varepsilon(p,r,t)\ c]\ \}$$ v.12

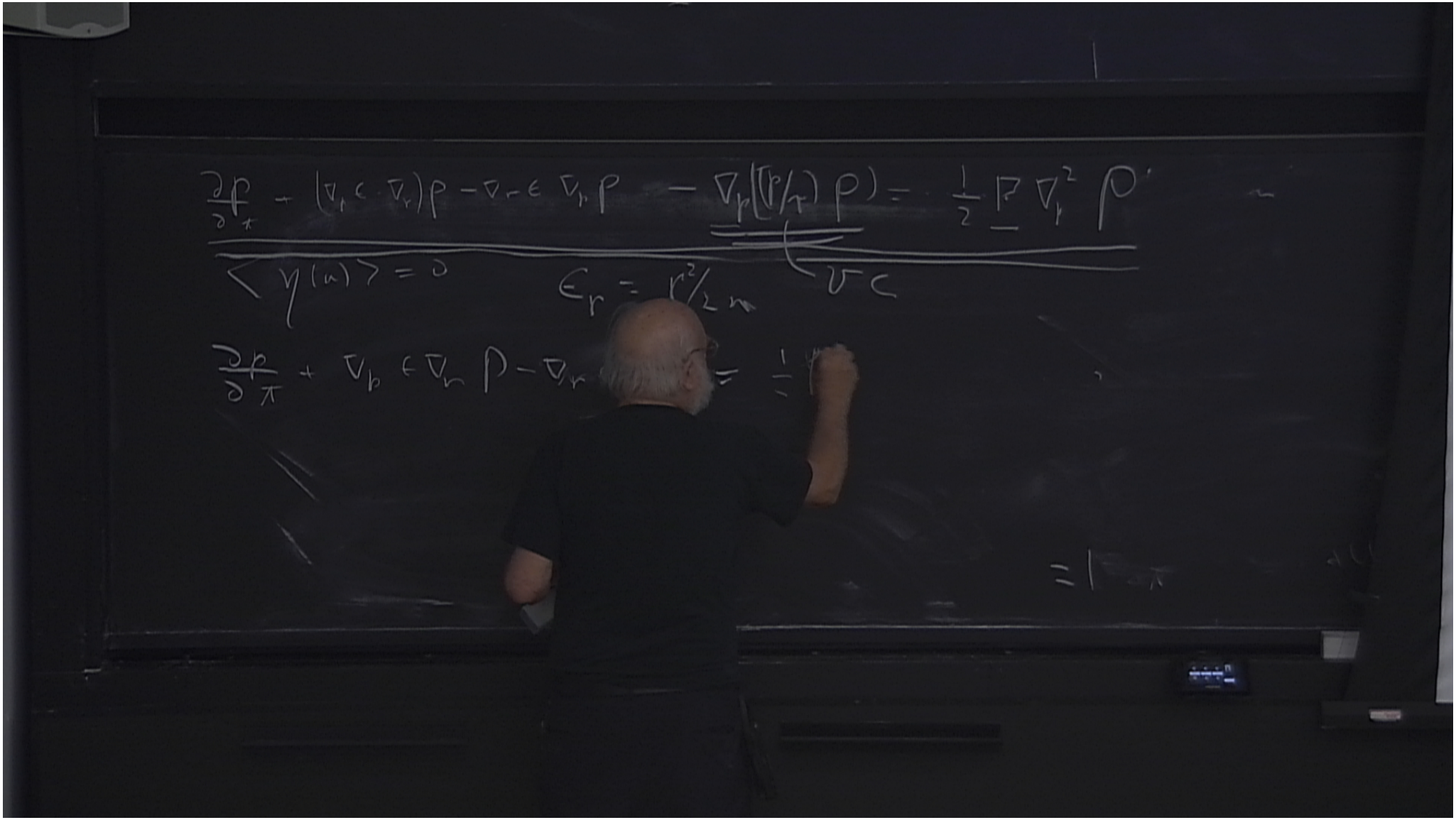
which has a solution $Q= \varepsilon(p,r,t)\ c$. To get the right resusult pick $c=1/(k_B T)$ and pick the other constant of integration so that $\exp(C)=1/z$,

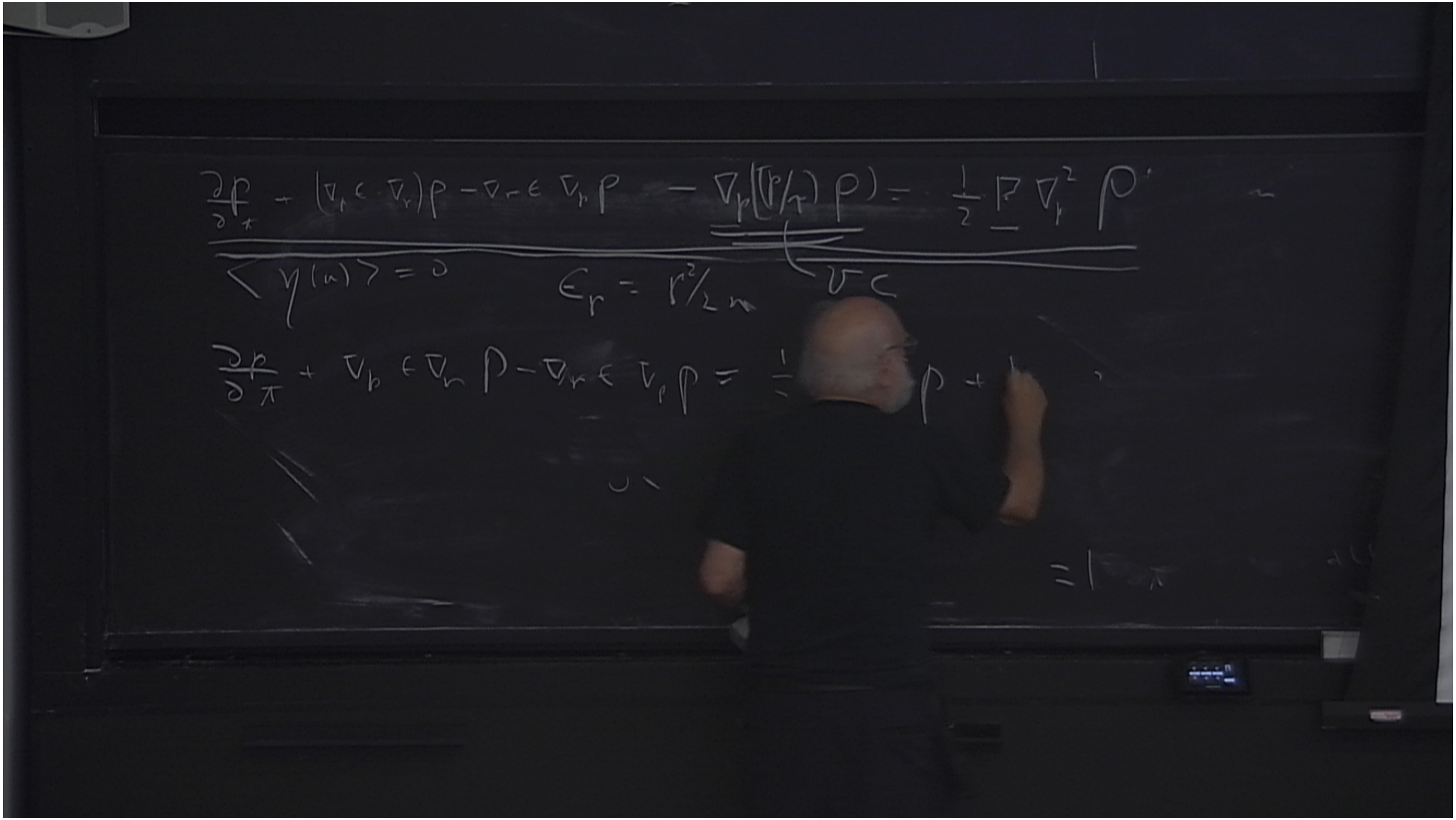
which then gives us the right normalization, with $\rho = [exp(-\varepsilon(p,r,t)/(k_B T)]/z$

nstein's calculation shows that, in order to get the Maxwell-Boltzmann equilibrium ult, as a unique result, we have to go beyond Hamiltonian mechanics and include some chastic behavior. This is a surprise.


$\frac{\partial f}{\partial t} + (v_1 \cdot \nabla_1) f - v_1 \cdot \epsilon \nabla_{v_1} f - \nabla_{v_1}\left[\left(\frac{F}{m}\right) f\right] = \frac{1}{2} \Gamma \nabla_{v_1}^2 f$
$\langle \eta(u) \rangle = 0$
$f(r, v, t)$
includes all
the effects of $\eta(u)$
$0 < u < t$
$\langle (\delta f)^2 \rangle$
$= \int_t^{t+\delta t} du \int_t^{t+\delta t} dw \langle \eta(u)$
$= \Gamma \delta t$


$\frac{\partial f}{\partial t} + (v_i \cdot \nabla_r) f - \nabla_r \epsilon \nabla_v f - \nabla_v \left[\left(F/\tau \right) f \right] = \frac{1}{2} D \nabla_v^2 f$
$\langle \eta(t) \rangle = 0$
$\epsilon_r = r^2/2m$
v_c


$\frac{\partial f}{\partial t} + (v_i \cdot \nabla_r) f - \nabla_r \epsilon \nabla_v f - \nabla_v \left[(F/m) f \right] = \frac{1}{2} D \nabla_v^2 f$
$\langle \eta(t) \rangle = 0$
$\epsilon_r = r^2/2m$
$v \cdot c$
$\frac{\partial f}{\partial t} + v_i \epsilon \nabla_r f - \nabla_r$


$\frac{\partial f}{\partial t} + (v_i \cdot \nabla_r) f - \nabla_r \epsilon \nabla_v f - \nabla_v [(F/m) f] = \frac{1}{2} D \nabla_v^2 f$
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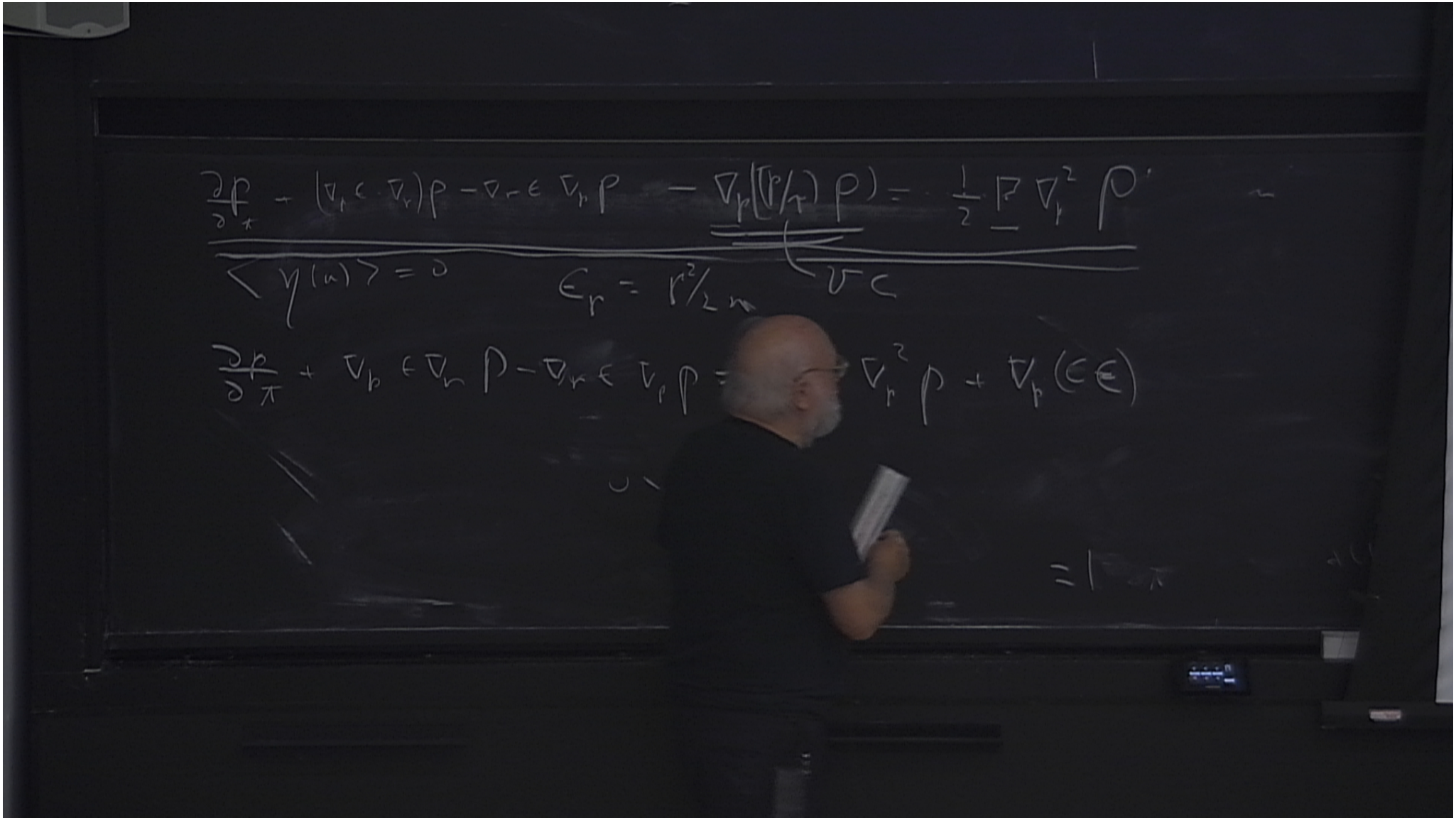
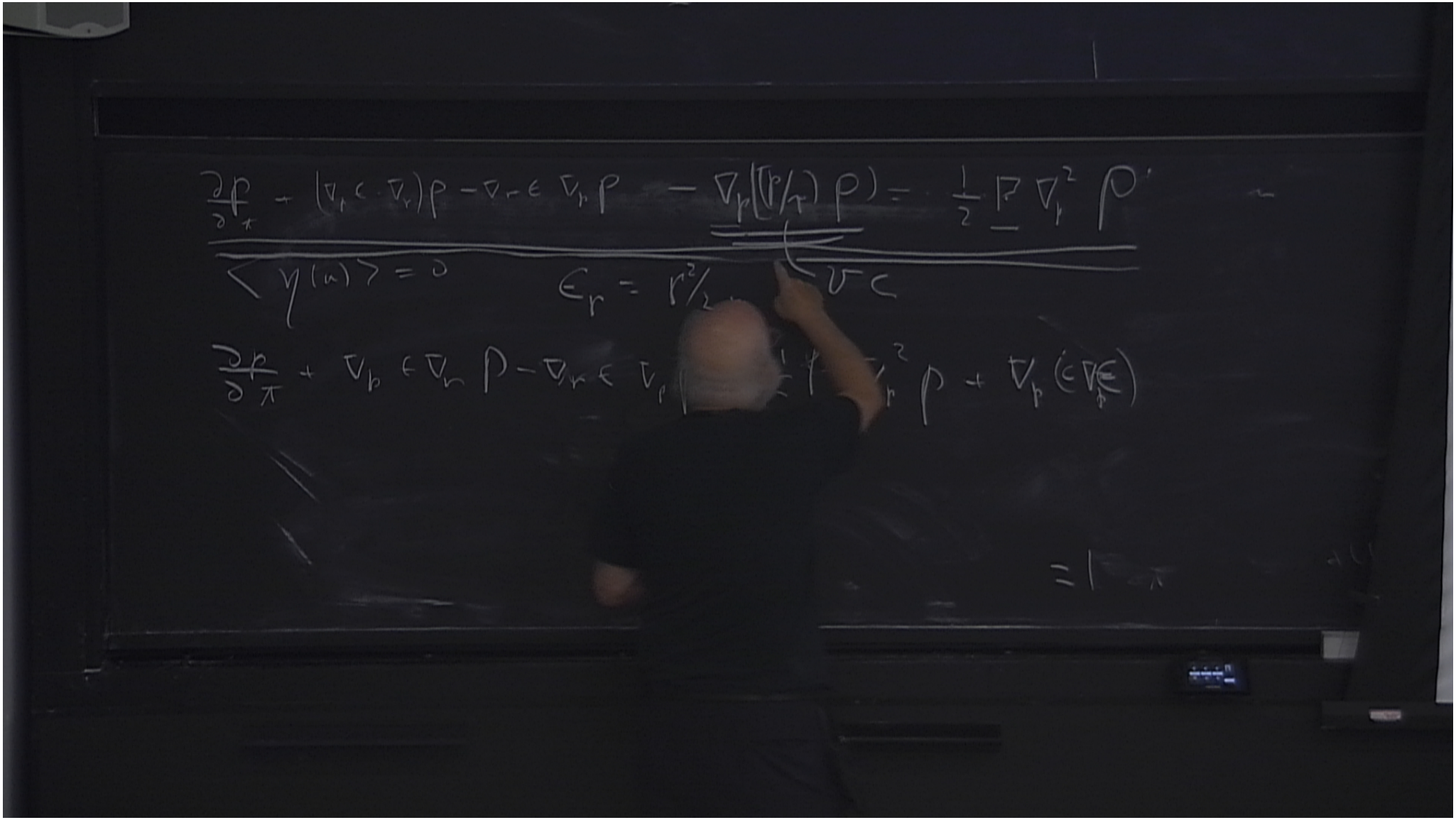
$\frac{\partial f}{\partial t} + v_i \cdot \nabla_r f - \nabla_r \epsilon \nabla_v f = \frac{1}{2} D \nabla_v^2 f$


$\frac{\partial f}{\partial t} + (v_i \cdot \nabla_r) f - \nabla_r \epsilon \nabla_v f - \nabla_v \left(\frac{F}{m} f \right) = \frac{1}{2} D \nabla_v^2 f$
$\langle \eta(t) \rangle = 0$
$\epsilon_r = r^2/2$
$\frac{\partial f}{\partial t} + v_k \epsilon \nabla_k f - \nabla_k \epsilon \nabla_i f = \nabla_i^2 f + \nabla_p (\epsilon \epsilon)$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$\langle \eta(t) \rangle = 0$
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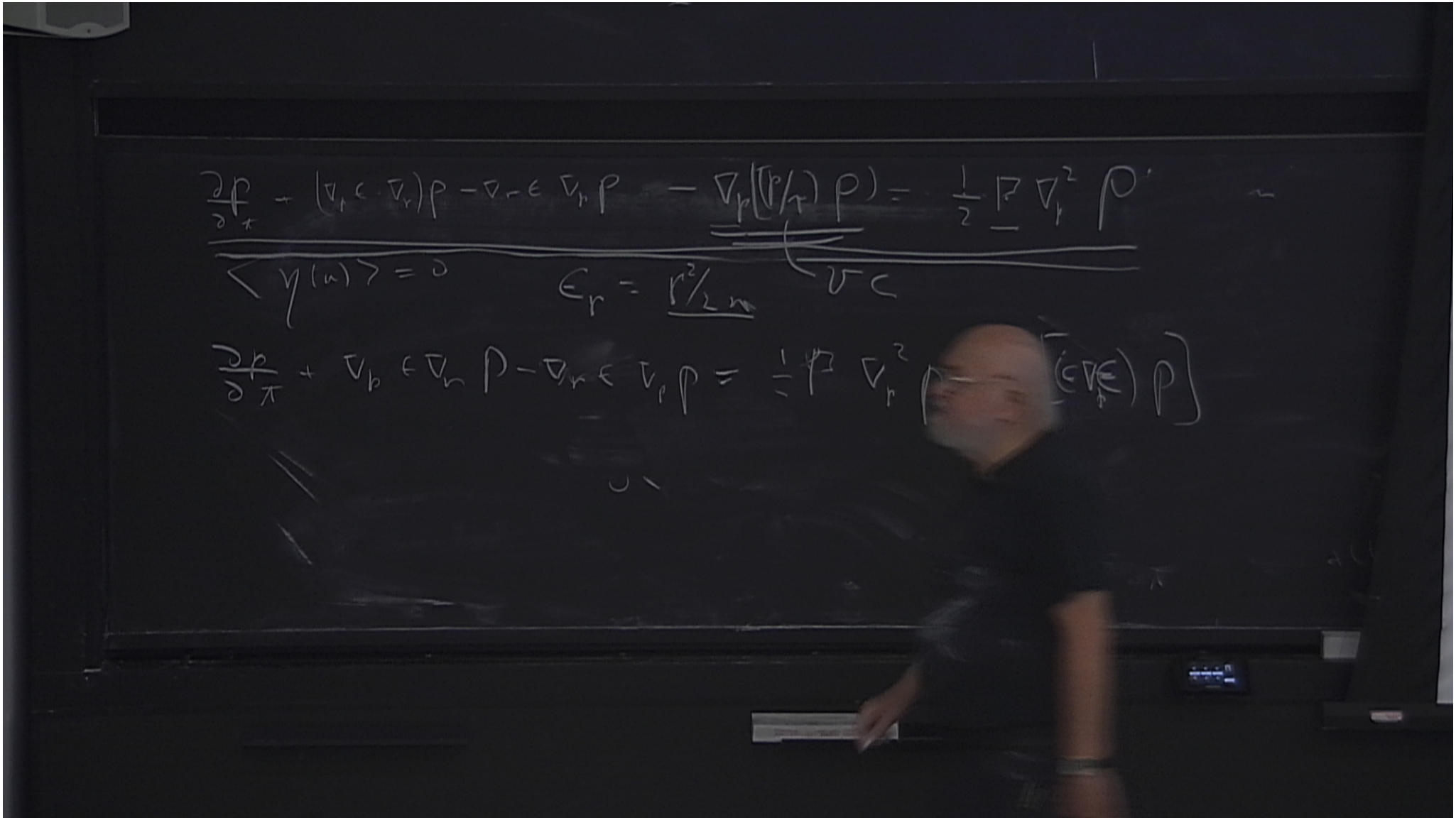
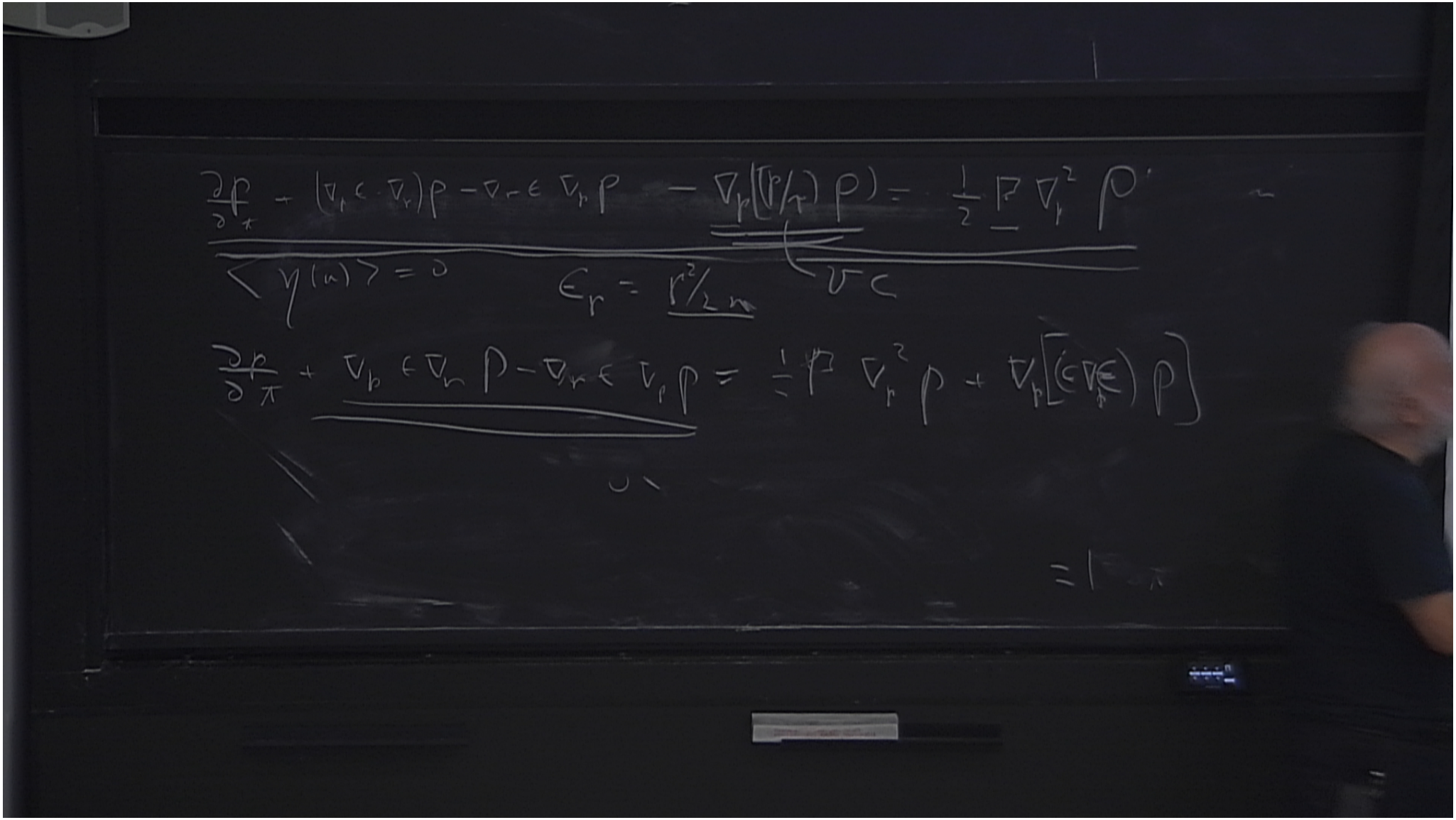
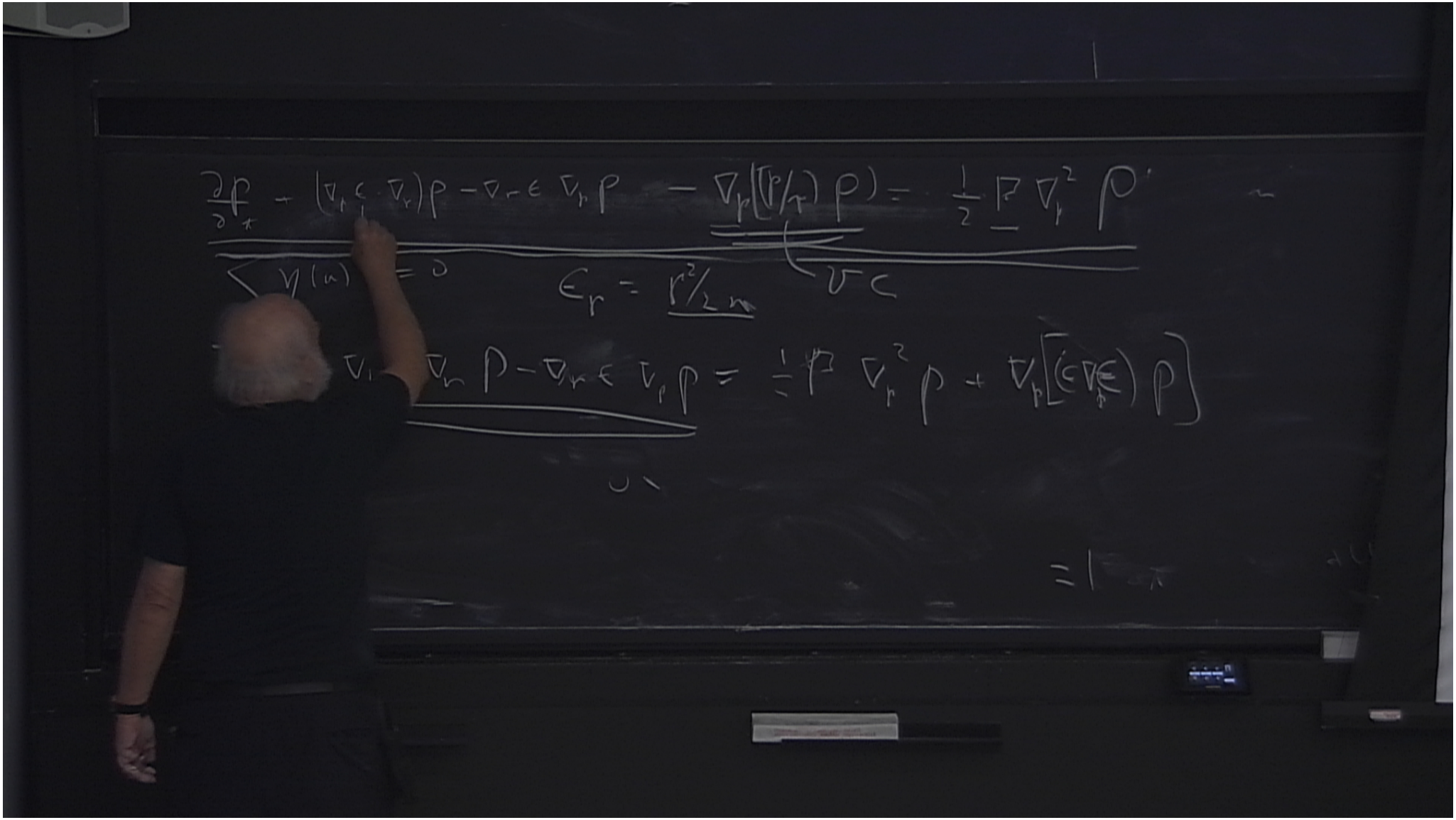
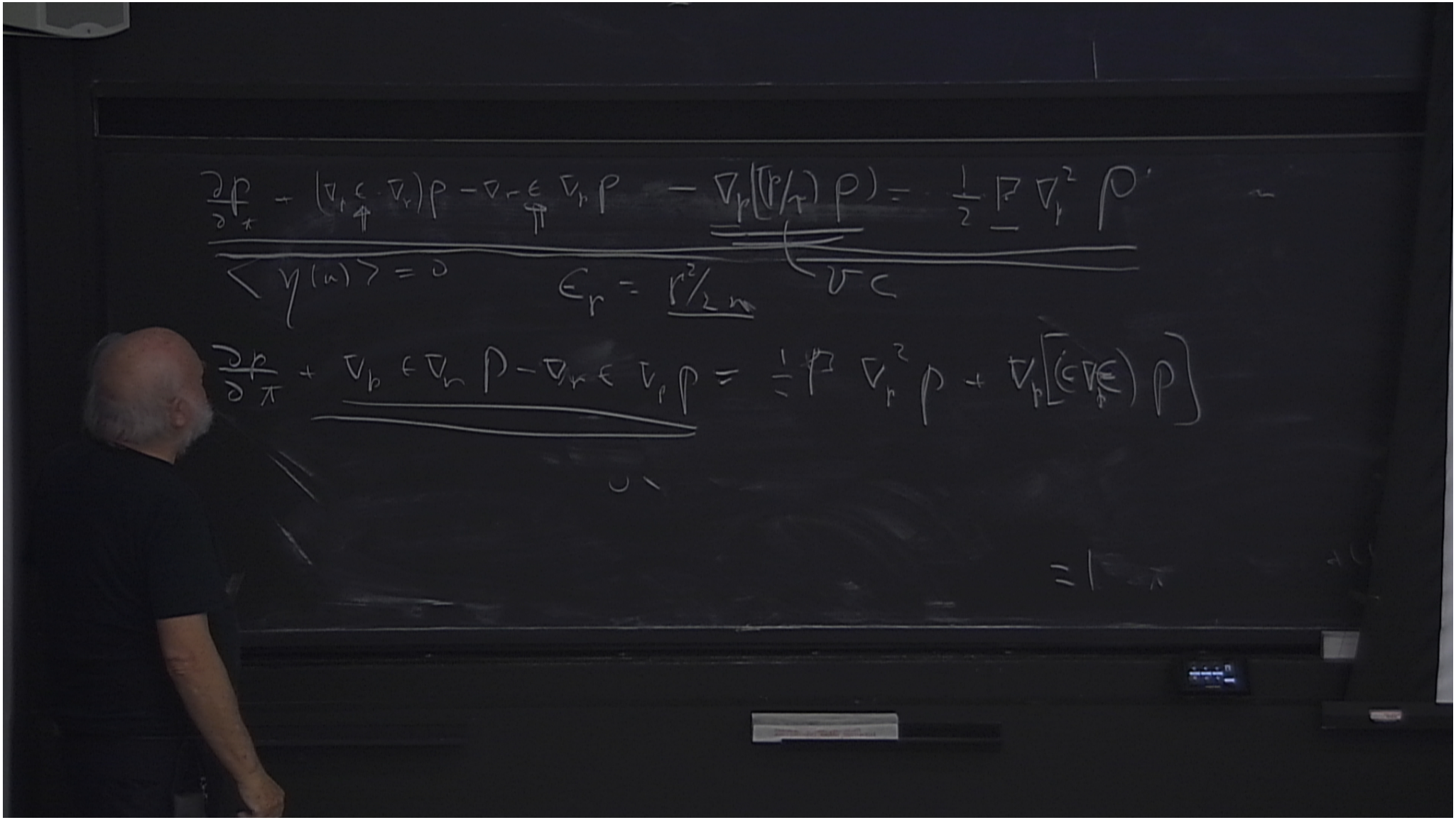
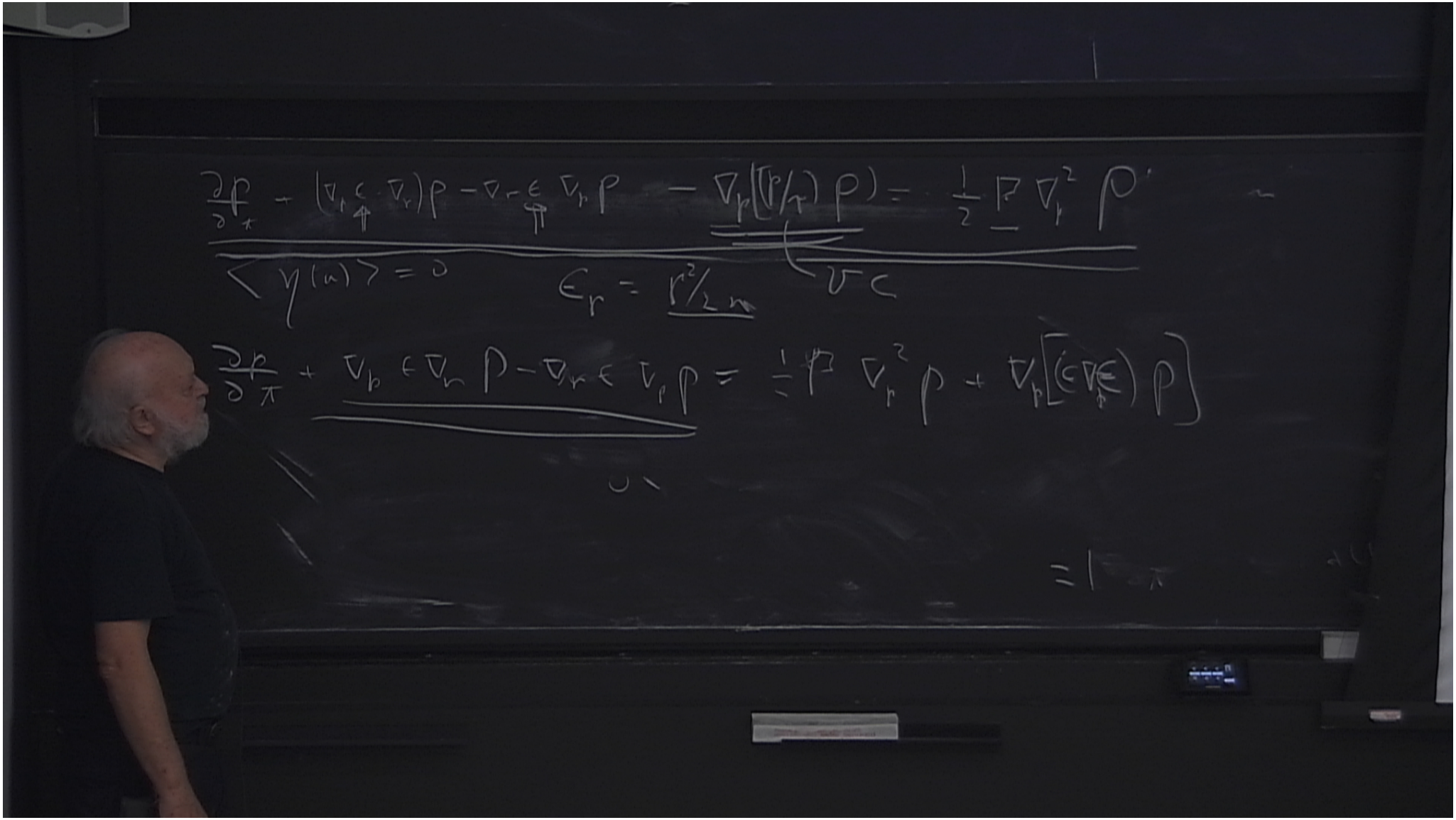
$\frac{\partial f}{\partial t} + v_i \epsilon \nabla_r f - \nabla_r \epsilon \nabla_v f = \frac{1}{2} D \nabla_v^2 f + \nabla_v (\epsilon \nabla \epsilon)$

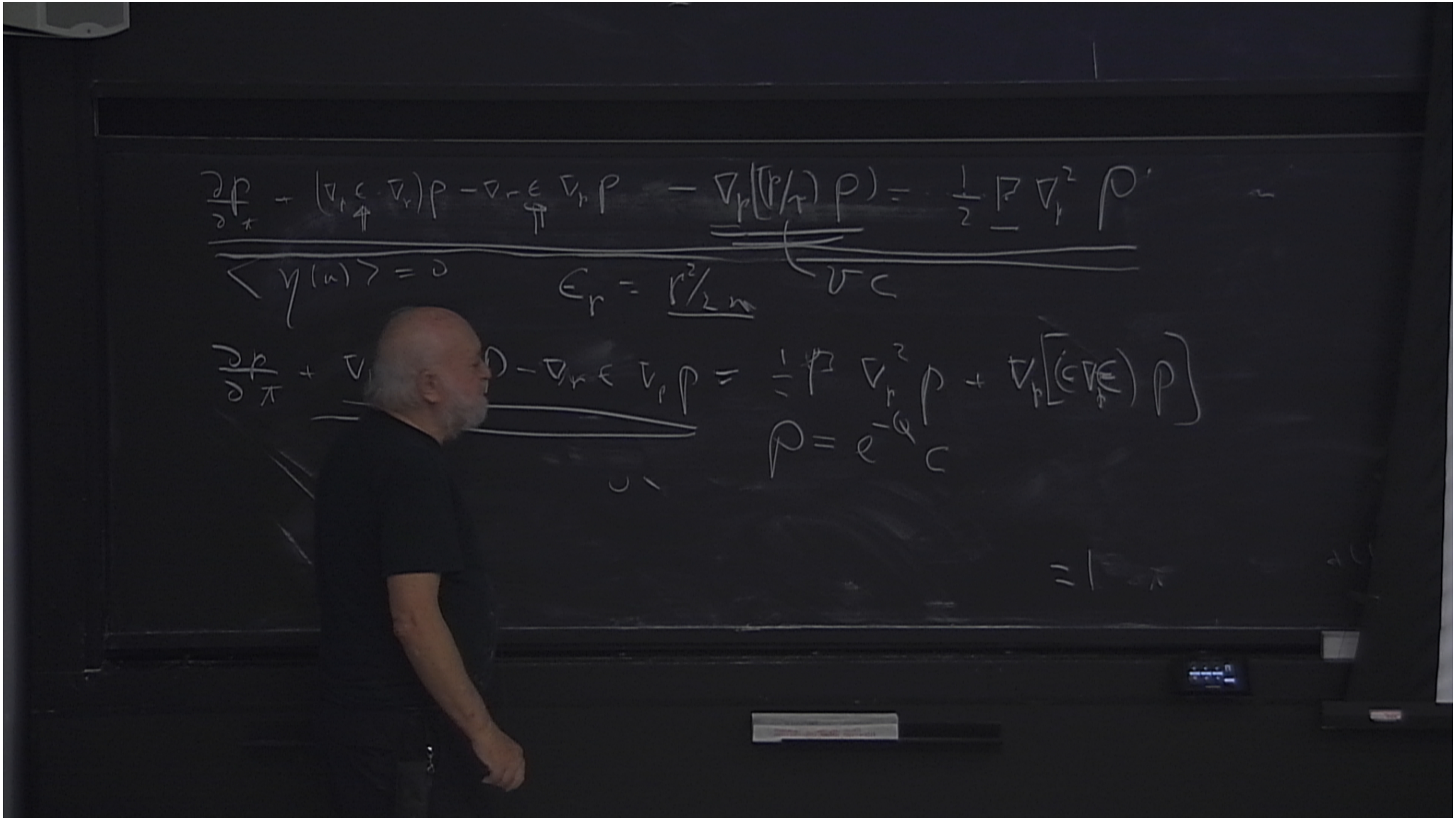

$\frac{\partial f}{\partial t} + (v_i \cdot \nabla_r) f - \nabla_r \epsilon \nabla_v f - \nabla_v [(F/m) f] = \frac{1}{2} D \nabla_v^2 f$
$\langle \eta(t) \rangle = 0$
$\epsilon_r = r^2/2m$
∇C
$\frac{\partial f}{\partial t} + v_i \epsilon \nabla_r f - \nabla_r \epsilon \nabla_v f = \frac{1}{2} D \nabla_v^2 f$


$\frac{\partial f}{\partial t} + (v_i \cdot \nabla_r) f - \nabla_r \epsilon \nabla_v f - \nabla_v \left[\left(\frac{F}{m} \right) f \right] = \frac{1}{2} D \nabla_v^2 f$
$\langle \eta(t) \rangle = 0$
$\epsilon_r = r^2 / 2m$
$v c$
$\frac{\partial f}{\partial t} + v_i \epsilon \nabla_r f - \nabla_r \epsilon \nabla_v f = \frac{1}{2} D \nabla_v^2 f + \nabla_v \left[\left(\epsilon \nabla \epsilon \right) f \right]$
$= 1$


$\epsilon_r = r^2/2m$


$\frac{\partial f}{\partial t} + (v_i \cdot \nabla_r) f - \nabla_r \epsilon \cdot \nabla_v f - \nabla_v [(F/m) f] = \frac{1}{2} D \nabla_v^2 f$
$\langle \eta(t) \rangle = 0$
$\epsilon_r = r^2/2m$
$\frac{\partial f}{\partial t} + v_i \cdot \nabla_r f - \nabla_r \epsilon \cdot \nabla_v f = \frac{1}{2} D \nabla_v^2 f + \nabla_v [(\epsilon \nabla \epsilon) f]$


$\frac{\partial f}{\partial t} + (v_i \cdot \nabla_r) f - v \cdot \frac{\epsilon}{m} \nabla_v f - \nabla_v \left[\left(\frac{F}{m} \right) f \right] = \frac{1}{2} D \nabla_v^2 f$
$\langle \eta(t) \rangle = 0$
$\epsilon_r = r^2 / 2m$
$v \cdot c$
$\frac{\partial f}{\partial t} + v_i \cdot \nabla_r f - \nabla_r \epsilon \cdot \nabla_v f = \frac{1}{2} D \nabla_v^2 f + \nabla_v \left[\left(\epsilon \nabla \frac{\epsilon}{m} \right) f \right]$


$\frac{\partial f}{\partial t} + (v_i \cdot \nabla_r) f - \nabla_r \cdot \frac{F}{m} \nabla_v f - \nabla_v \left[\left(\frac{F}{m} \right) f \right] = \frac{1}{2} D \nabla_v^2 f$
$\langle \eta(t) \rangle = 0$
$\epsilon_r = r^2 / 2m$
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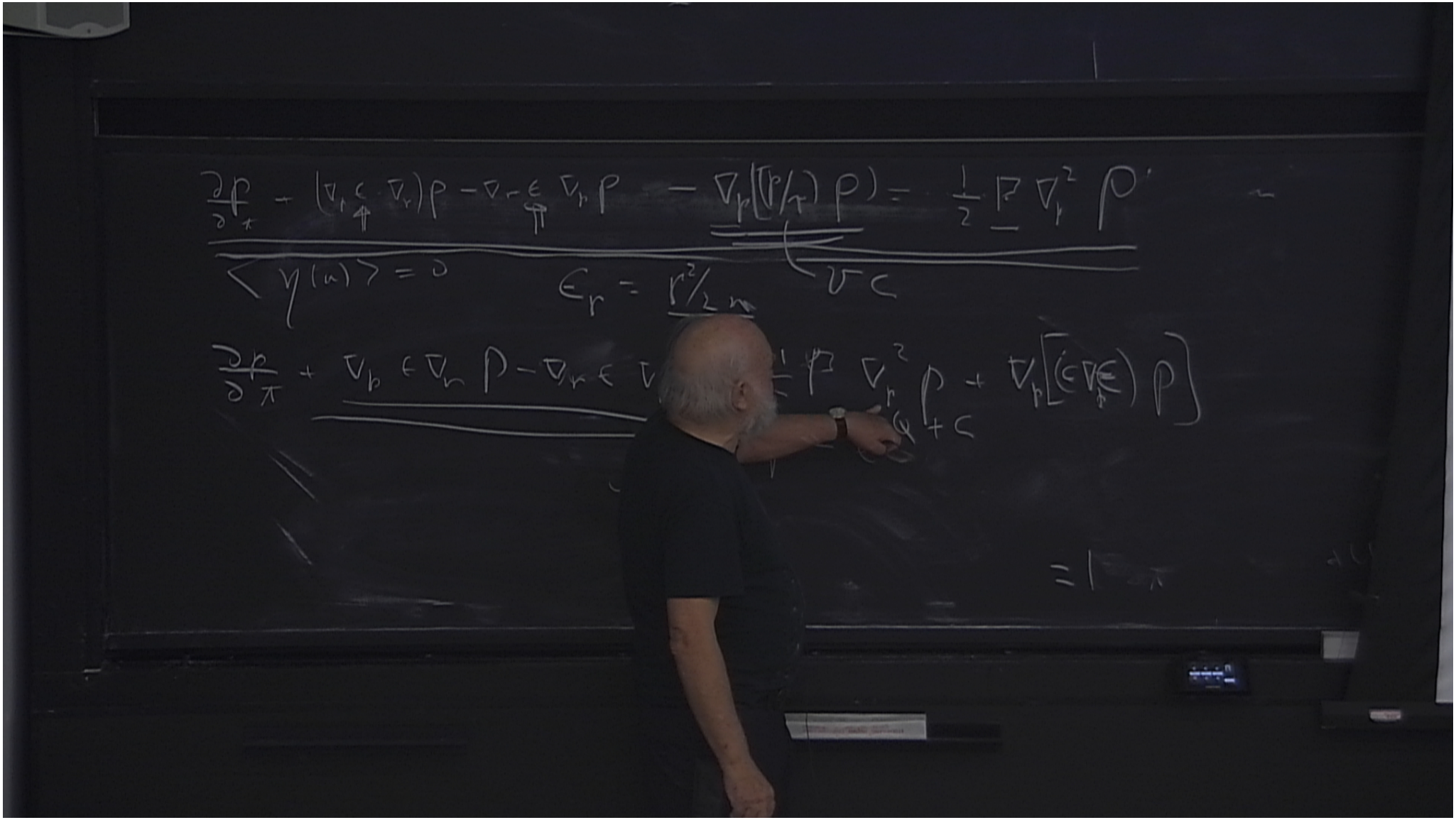
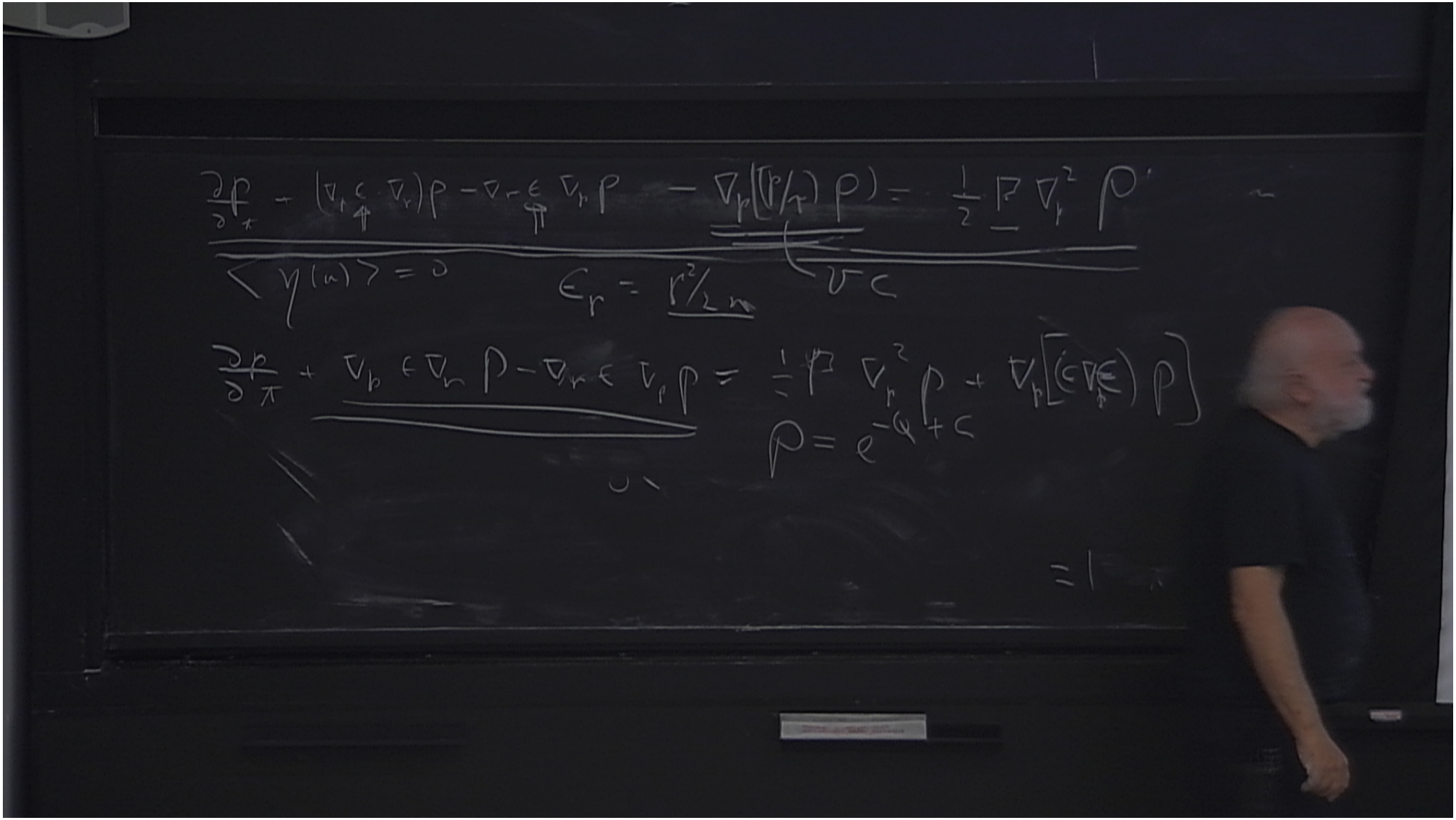
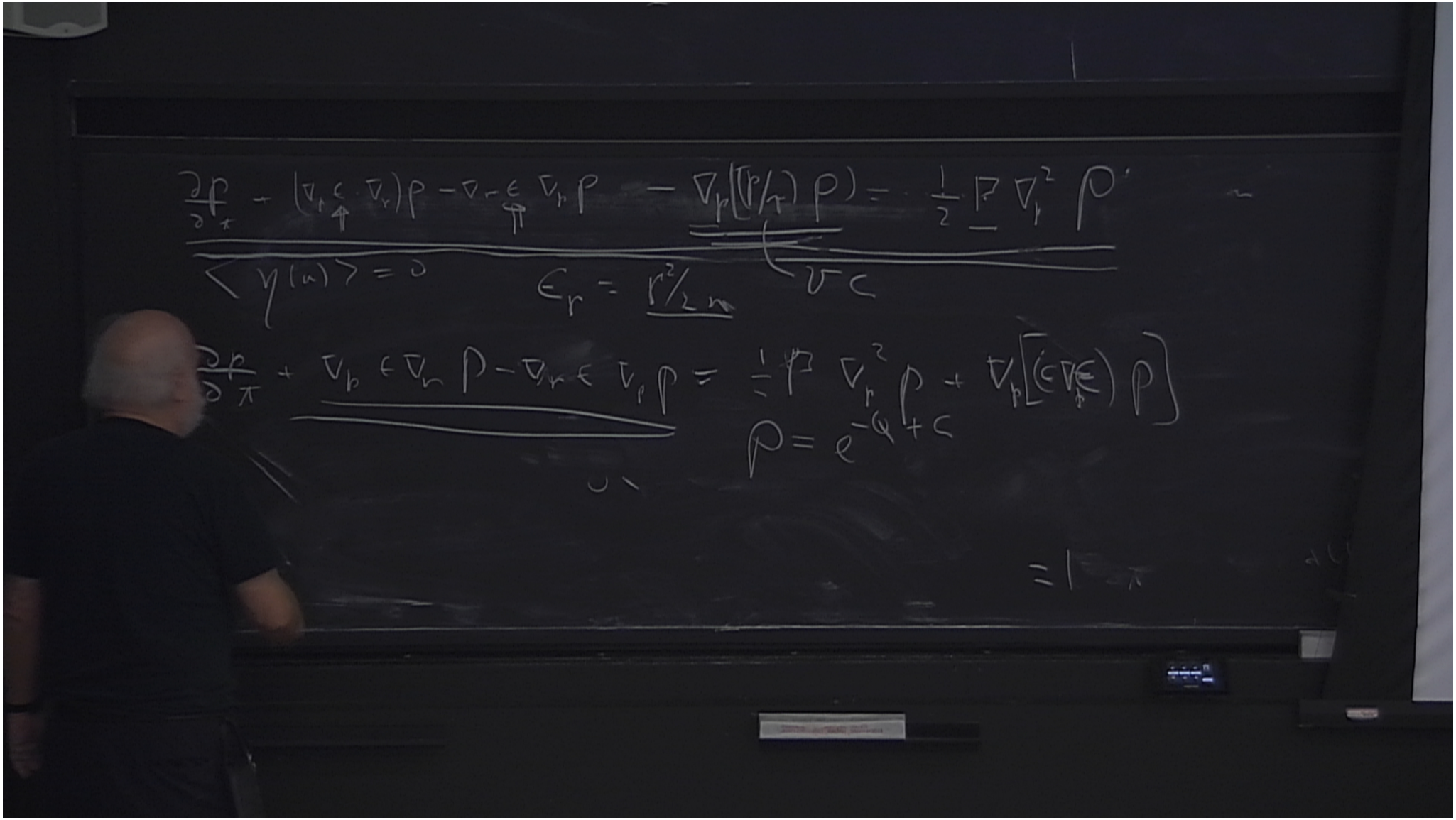
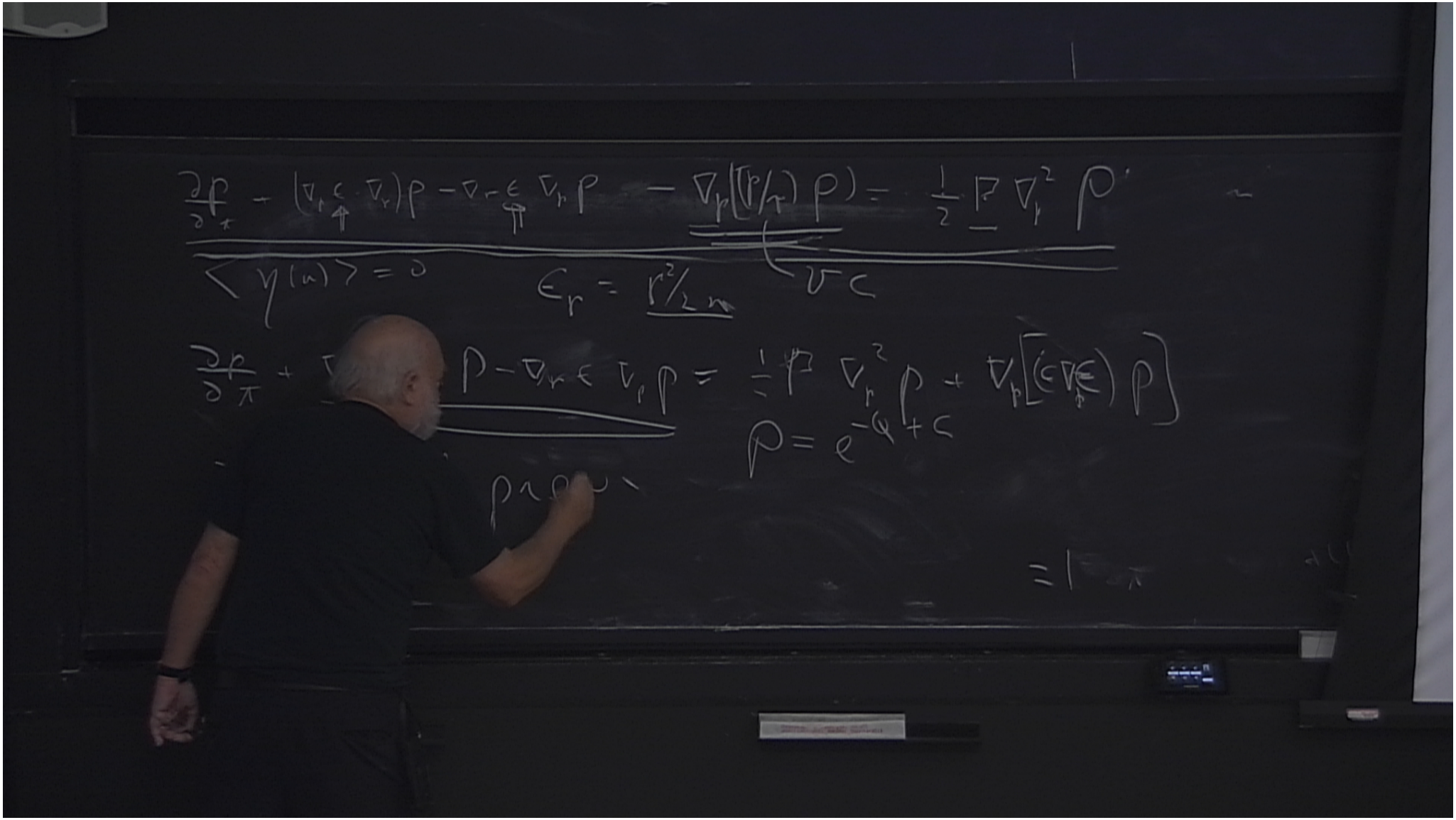
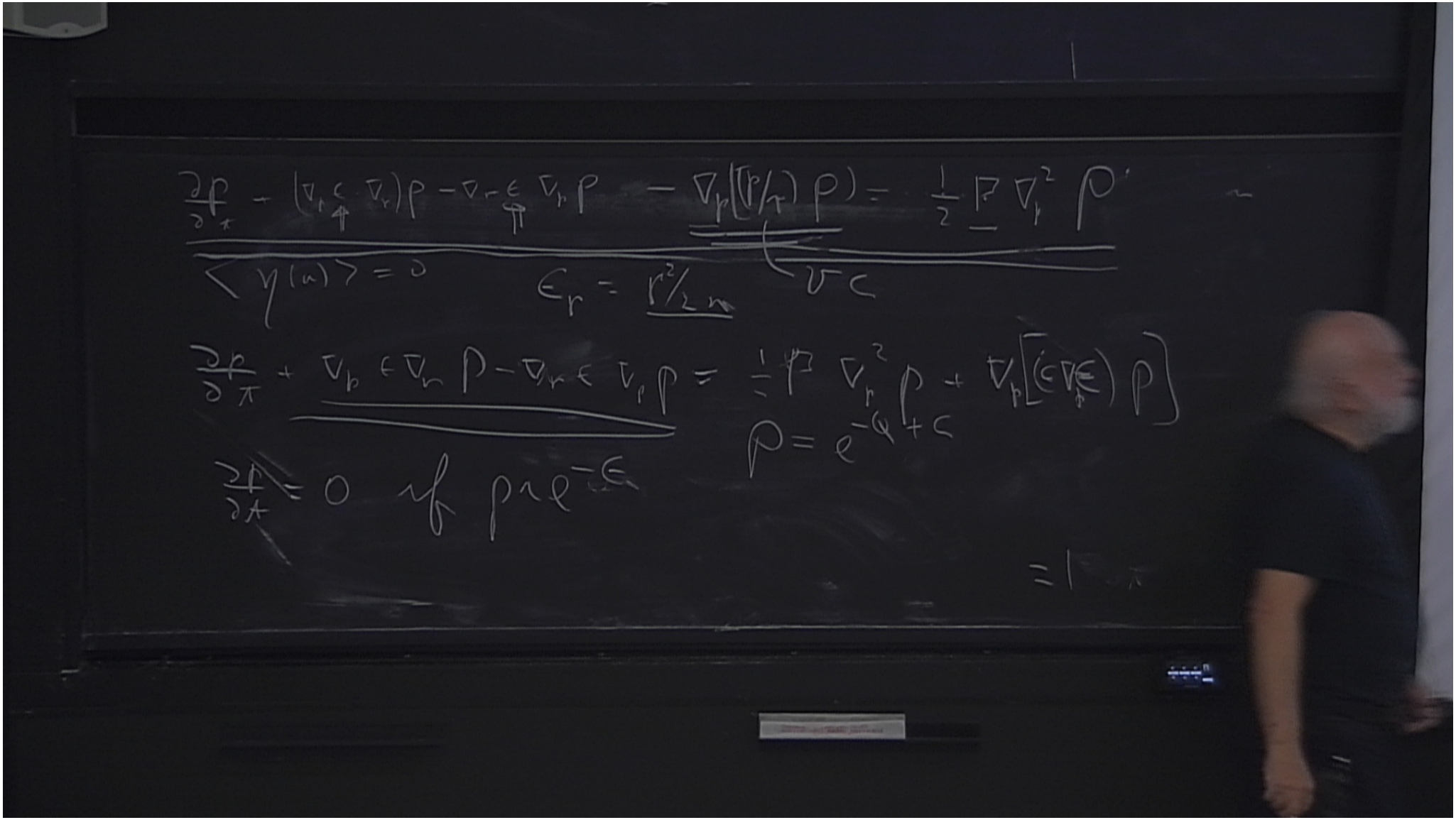
$\frac{\partial f}{\partial t} + v_i \cdot \nabla_r f - \nabla_r \epsilon \nabla_v f = \frac{1}{2} D \nabla_v^2 f + \nabla_v \left[\left(\epsilon \nabla \epsilon \right) f \right]$
$f = e^{-\epsilon / c}$
$= 1$

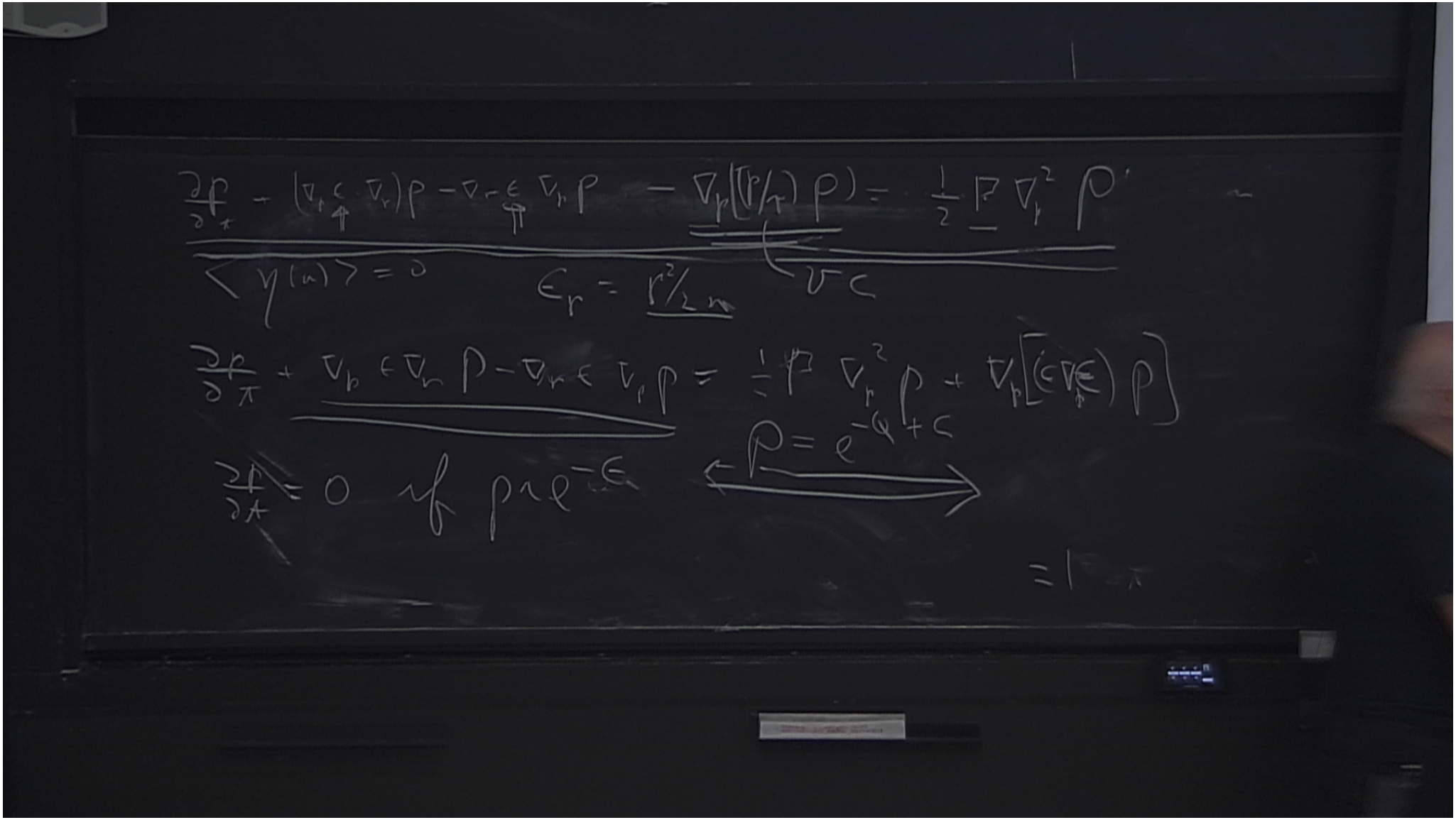

$\frac{\partial f}{\partial t} + (v_i \cdot \nabla_r) f - \nabla_r \frac{\epsilon}{m} \nabla_v f - \nabla_v \left[\left(F/m \right) f \right] = \frac{1}{2} F^2 \nabla_v^2 f$
$\langle \eta(t) \rangle = 0$
$\epsilon_r = p^2/2m$
$\frac{\partial f}{\partial t} + v_i \cdot \nabla_r f - \nabla_r \epsilon \cdot \nabla_p f = \frac{1}{2} F^2 \nabla_p^2 f + \nabla_p \left[\left(\epsilon \nabla \frac{\epsilon}{m} \right) f \right]$
$= 1$


$\frac{\partial f}{\partial t} + (v_i \cdot \nabla_r) f - \nabla \cdot \frac{e}{m} \nabla_v f - \nabla_v \left[\left(\frac{F}{m} \right) f \right] = \frac{1}{2} F \nabla_v^2 f$
$\langle \eta(u) \rangle = 0$
$\epsilon_r = r^2 / 2m$
σc
$\frac{\partial f}{\partial t} + v_k \epsilon \nabla_r f - \nabla_r \epsilon v_i f = \frac{1}{2} F \nabla_v^2 f + \nabla_v \left[\left(\epsilon \nabla \frac{\epsilon}{m} \right) f \right]$
$f = e^{-\varphi + c}$
$= 1$


$\frac{\partial f}{\partial t} - (v_i \cdot \nabla_r) f - \nabla_r \cdot \frac{\epsilon}{m} \nabla_v f - \nabla_v \left[\left(\frac{F}{m} \right) f \right] = \frac{1}{2} D \nabla_v^2 f$
$\langle \eta(t) \rangle = 0$
$\epsilon_r = r^2 / 2n$
$\frac{\partial f}{\partial t} + v_i \epsilon \nabla_r f - \nabla_r \epsilon \nabla_v f = \frac{1}{2} D \nabla_v^2 f + \nabla_v \left[\left(\epsilon \nabla \epsilon \right) f \right]$
$f = e^{-\phi} + c$


$\frac{\partial f}{\partial t} - (v_i \cdot \nabla_r) f - \nabla_r \cdot \frac{\epsilon}{m} \nabla_v f - \nabla_v \left[\left(\frac{F}{m} \right) f \right] = \frac{1}{2} \frac{F}{m} \nabla_v^2 f$
$\langle \eta(t) \rangle = 0$
$\epsilon_r = r^2 / 2m$
$\rho = e^{-\epsilon + c}$


$\frac{\partial P}{\partial t} - (v_i c \nabla_i) P - v_i \frac{\epsilon}{\uparrow\uparrow} \nabla_i P - \nabla_p \left[(P/\tau) P \right] = \frac{1}{2} P \nabla_i^2 P$
$\langle \eta(u) \rangle = 0$
$\epsilon_r = r^2/2n$
$\frac{\partial P}{\partial t} + v_b \epsilon \nabla_b P - v_r \epsilon \nabla_i P = \frac{1}{2} P \nabla_i^2 P + \nabla_p \left[(\epsilon \nabla \epsilon) P \right]$
$\frac{\partial f}{\partial t} = 0$ if $P \sim e^{-\epsilon}$
$P = e^{-\epsilon} + c$
$= 1$


$\frac{\partial \rho}{\partial t} - (v_i c \nabla_i)\rho - v_i \epsilon \nabla_i \rho - \nabla_p[(\nu/\tau)\rho] = \frac{1}{2}\nu \nabla_i^2 \rho$
$\langle \eta(n) \rangle = 0$
$\epsilon_r = r^2/2n$
νc
$\frac{\partial \rho}{\partial t} + v_i \epsilon \nabla_i \rho - v_i \epsilon \nabla_i \rho = \frac{1}{2}\nu \nabla_i^2 \rho + \nabla_p[(c\nabla\epsilon)\rho]$
$\rho = e^{-q} + c$
$\frac{\partial f}{\partial t} = 0$ if $\rho \propto e^{-\epsilon}$
$= 1$

Effect of Brownian motion: toward a unique probability distribution

We put together our two different pieces of the Brownian time derivative equation and get:

$$\partial_t \rho(p,x,t) = \ldots\ldots + (\Gamma/2)\ (\partial_p)^2 \rho(p,x,t) + \partial_p [c\ (\partial_p \varepsilon)\ \rho(p,x,t)]$$

The ....'s refer to terms which might come from usual Hamiltonian mechanics. We shall put them aside for a moment. An equation, like the one here, obtained from averaging the stochastic mechanics of a Langevin equation, is called a Fokker-Planck equation.

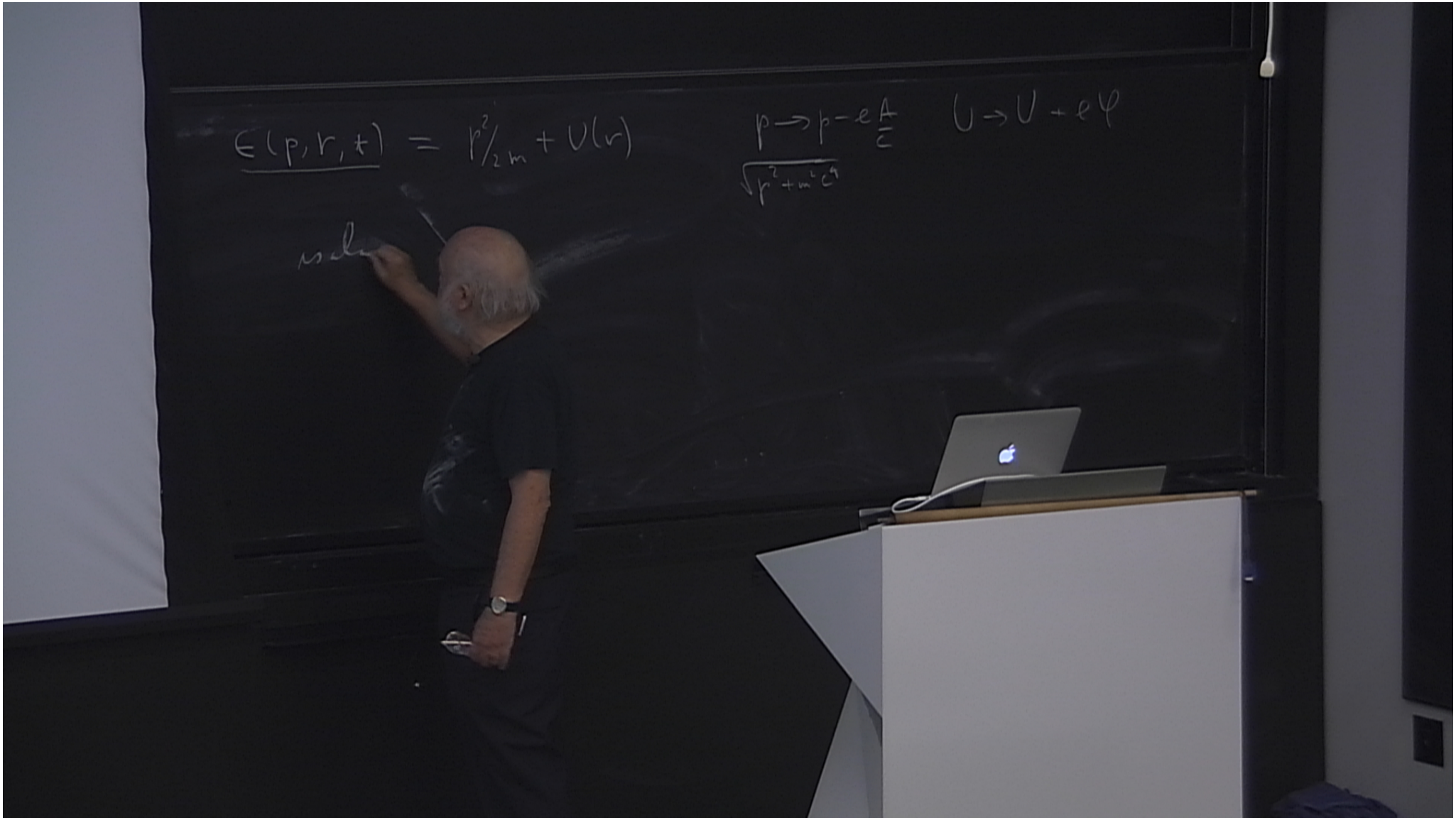
We look for a time-independent zero of this part of the equation to tell us about the equilibrium behavior. Write $\rho(p,x,t)=\exp[-Q(p,x)+C]$ and find that zeroing this part gives

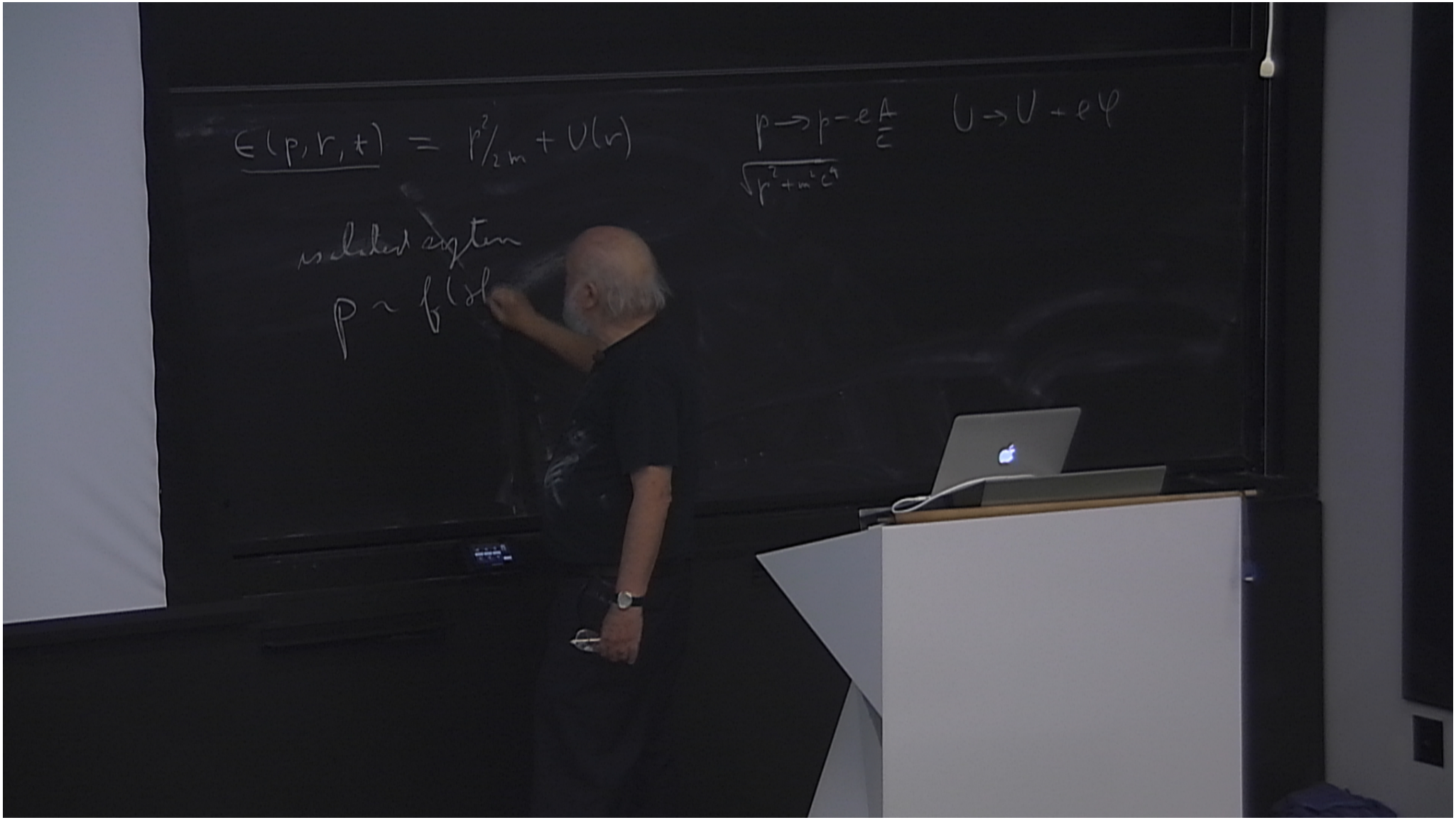
$$0 = \partial_p \{\exp[-Q(p,x)][\ (\Gamma/2)(-\partial_p Q) + \partial_p \varepsilon(p,r,t)\ c]\ \} \qquad \text{v.12}$$

which has a solution $Q= \varepsilon(p,r,t)\ c$. To get the right resusult pick $c=1/(k_B T)$ and pick the other constant of integration so that $\exp(C)=1/z$,

which then gives us the right normalization, with $\rho = [\exp(-\varepsilon(p,r,t)/(k_B T)]/z$

Einstein's calculation shows that, in order to get the Maxwell-Boltzmann equilibrium result, as a unique result, we have to go beyond Hamiltonian mechanics and include some stochastic behavior. This is a surprise.


$\epsilon(p,r,t) = p^2/2m + U(r)$
$p \to p - e\frac{A}{c}$
$\sqrt{p^2 + m^2c^4}$
$U \to U + e\varphi$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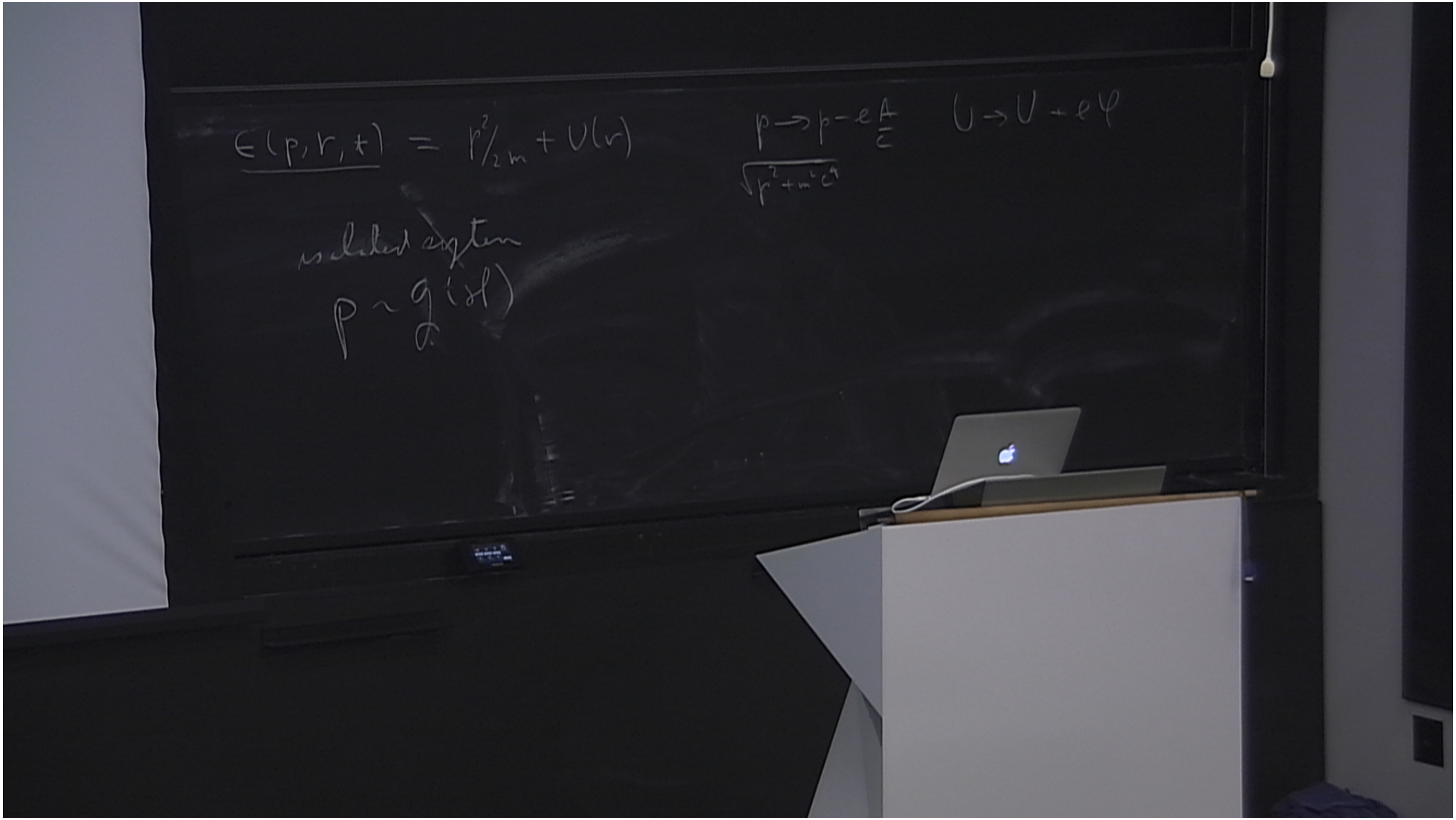
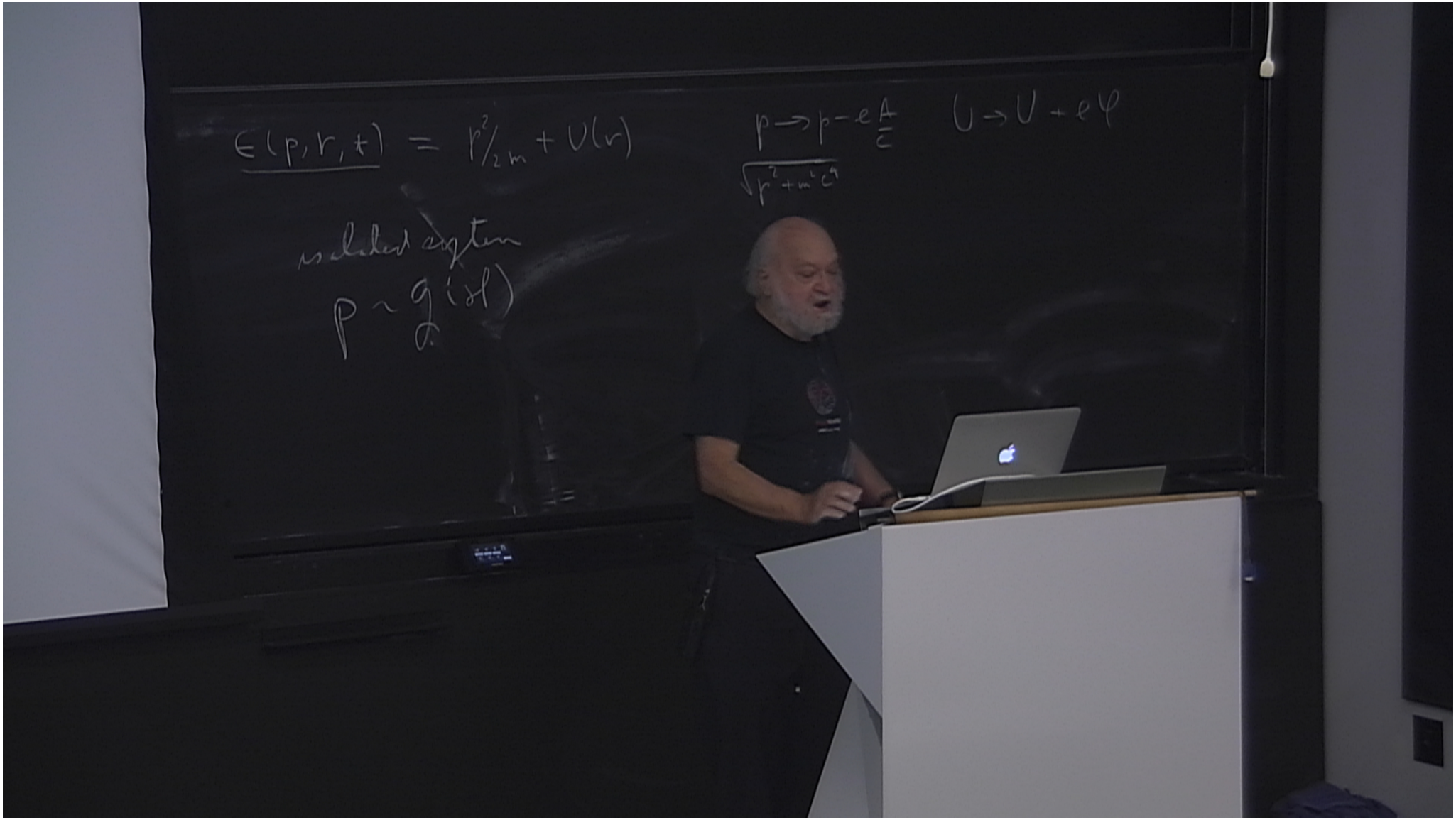

$\epsilon(p,r,t) = p^2/2m + U(r)$
$p \to p - e\frac{A}{c}$
$U \to U + e\varphi$
$\sqrt{p^2 + m^2c^2}$
isolated system
$p \sim g(s,t)$


$\epsilon(p,r,t) = p^2/2m + U(r)$
$p \to p - e\frac{A}{c}$
$U \to U + e\varphi$
$\sqrt{p^2 + m^2c^2}$
isolated system
$p \sim g(r)$

Effect of Brownian motion: toward a unique probability distribution

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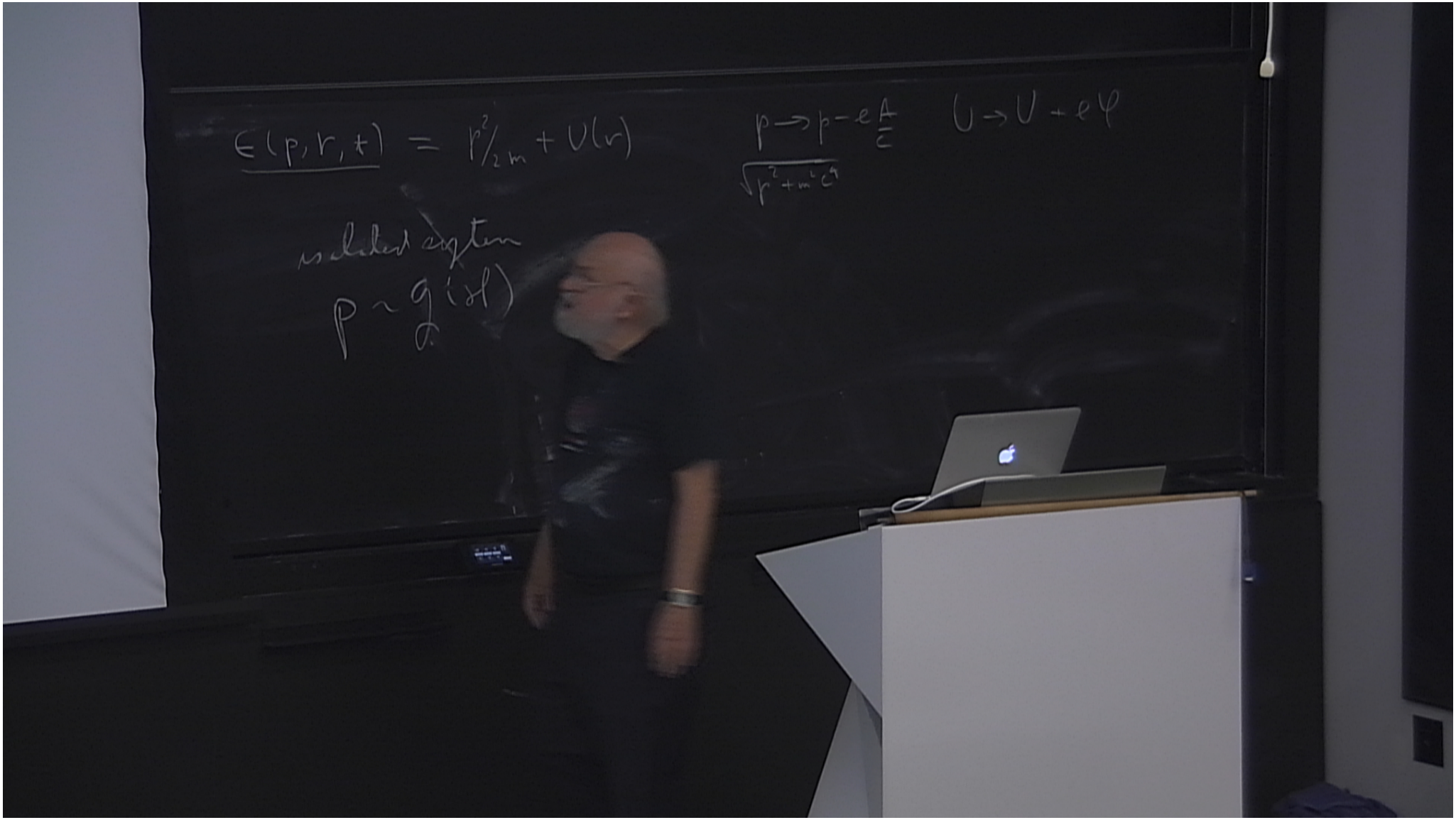
which has a solution $Q= \varepsilon(p,r,t)\ c$. To get the right resusult pick $c=1/(k_B T)$ and pick the other constant of integration so that $\exp(C)=1/z$,

which then gives us the right normalization, with $\rho = [\exp(-\varepsilon(p,r,t)/(k_B T)]/z$

Einstein's calculation shows that, in order to get the Maxwell-Boltzmann equilibrium result, as a unique result, we have to go beyond Hamiltonian mechanics and include some stochastic behavior. This is a surprise.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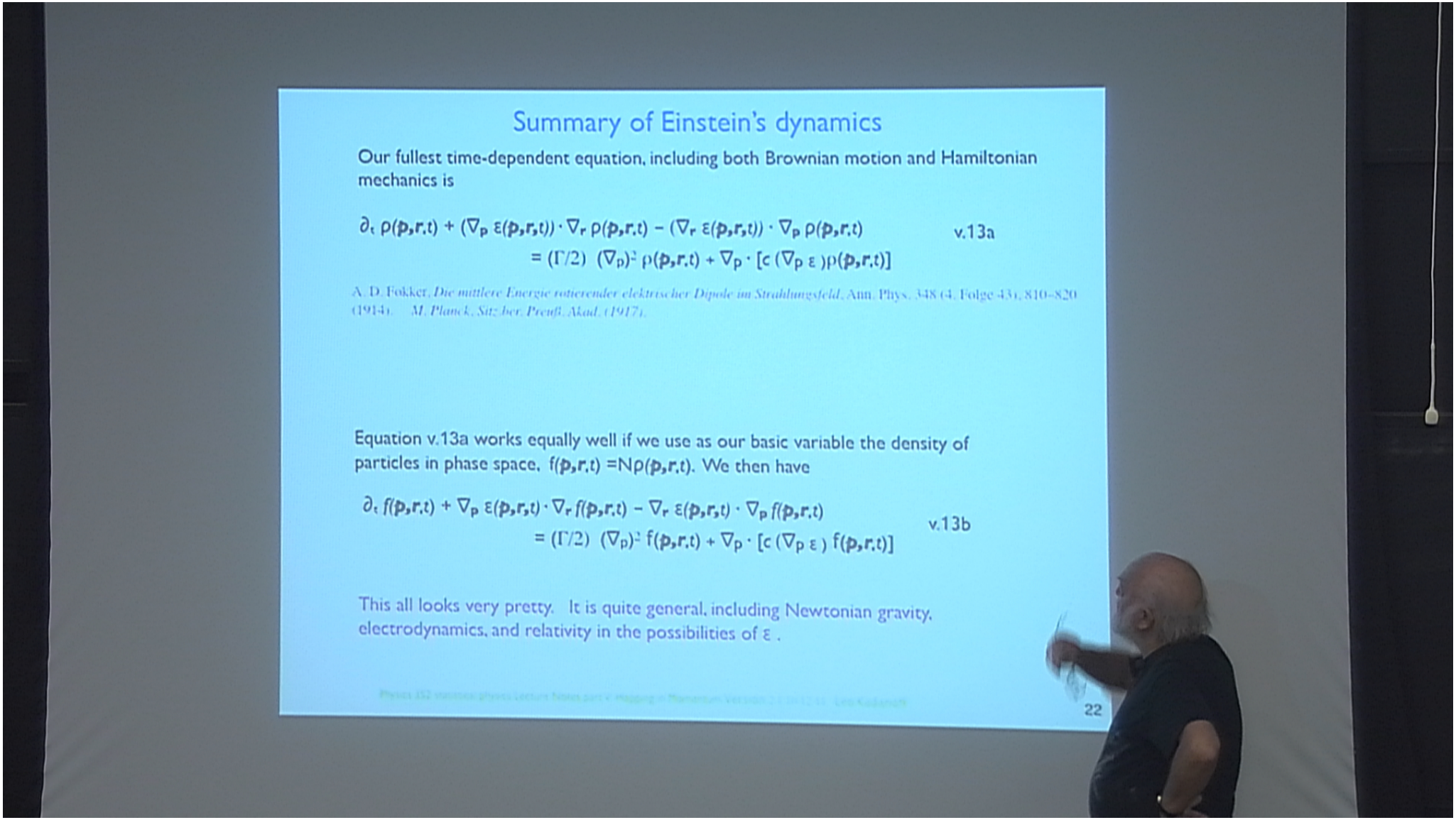
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A. D. Fokker, Die mittlere Energie rotierender elektrischer Dipole im Strahlungsfeld, Ann. Phys. 348 (4. Folge 43), 810–820 (1914). M. Planck, Sitzber. Preuß. Akad. (1917).
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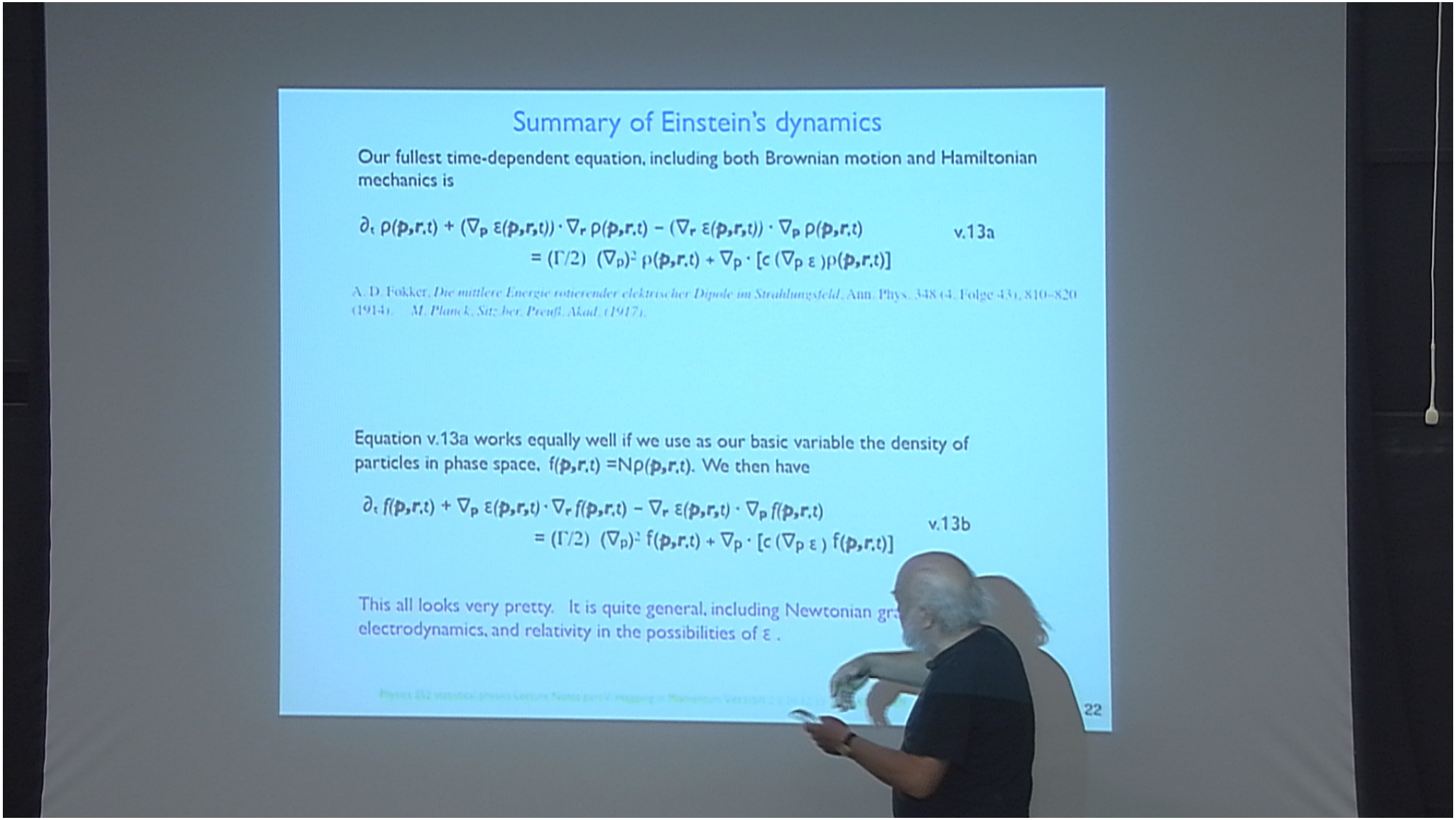
This all looks very pretty. It is quite general, including Newtonian gravity, electrodynamics, and relativity in the possibilities of ε .
22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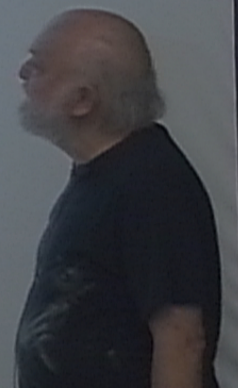

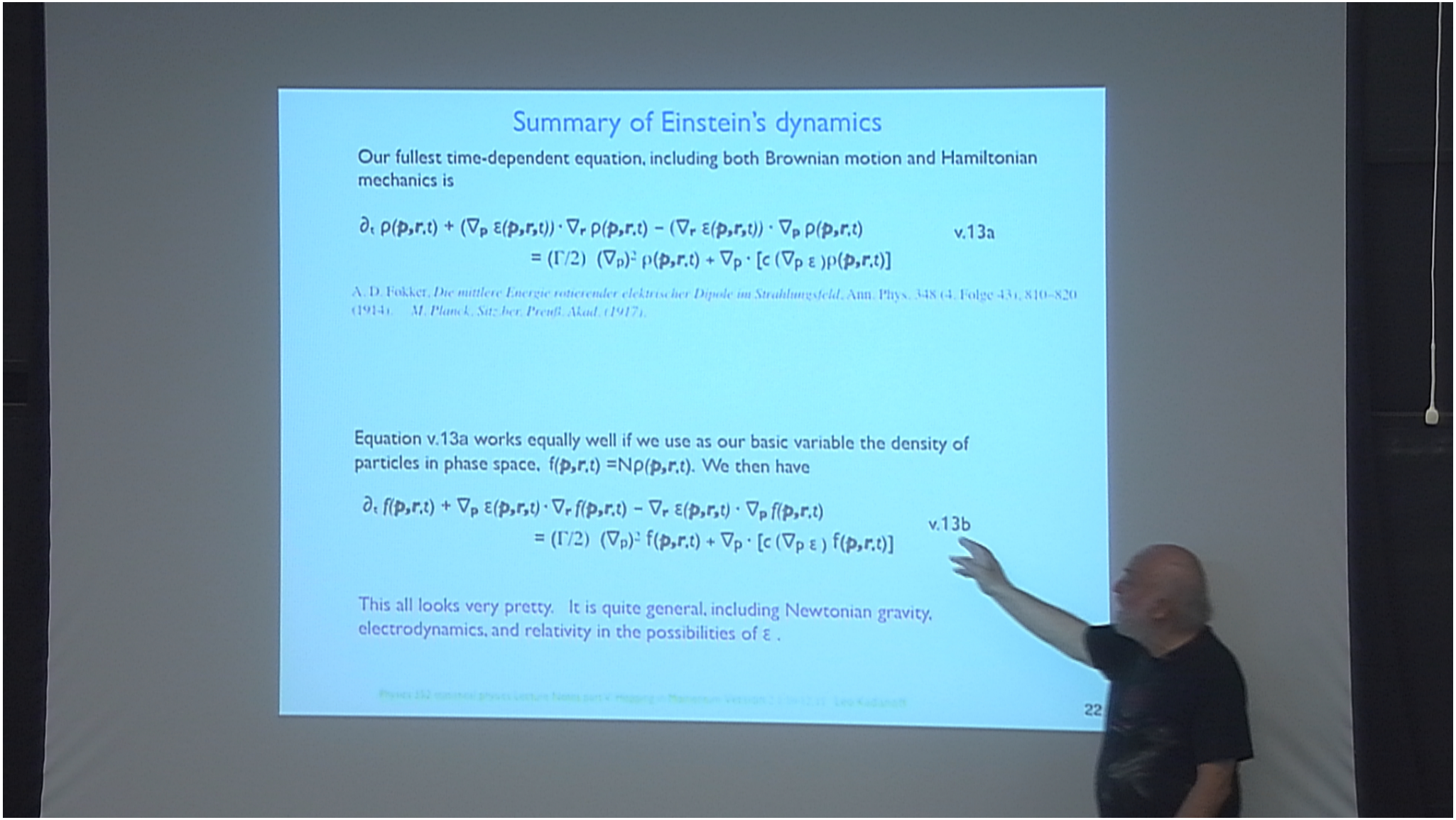
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Long before Einstein's work, Boltzmann had derived the Maxwell-Boltzmann distribution by using a more brute force approach than the one applied by Einstein. Like Einstein, Boltzmann looked at collisions of particles mostly described as moving independently. Therefore these particles could be well-characterized by the free particle Hamiltonian, $p^2/(2m) + U(r)$. In addition, however, Boltzmann imagined that these particles would, occasionally, come close to one another and collide, thereby substantially changing their momentum. So he started from an equation describing first the free particle motion, and then included the effect of collisions. The basic variable in this approach was $f(\mathbf{p},\mathbf{r},t)$, which would then obey an equation of the form

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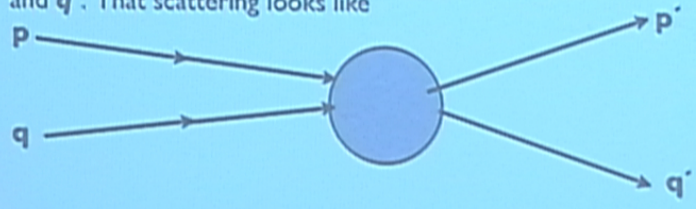
The collisional effects are best shown in a pair of pictures. The number of particles described by $(\mathbf{p},\mathbf{r})$ will diminish because particles with momentum $\mathbf{p}$ scatter against particles with momentum $\mathbf{q}$, producing particles with respective momentum $\mathbf{p}'$ and and $\mathbf{q}'$. That scattering looks like



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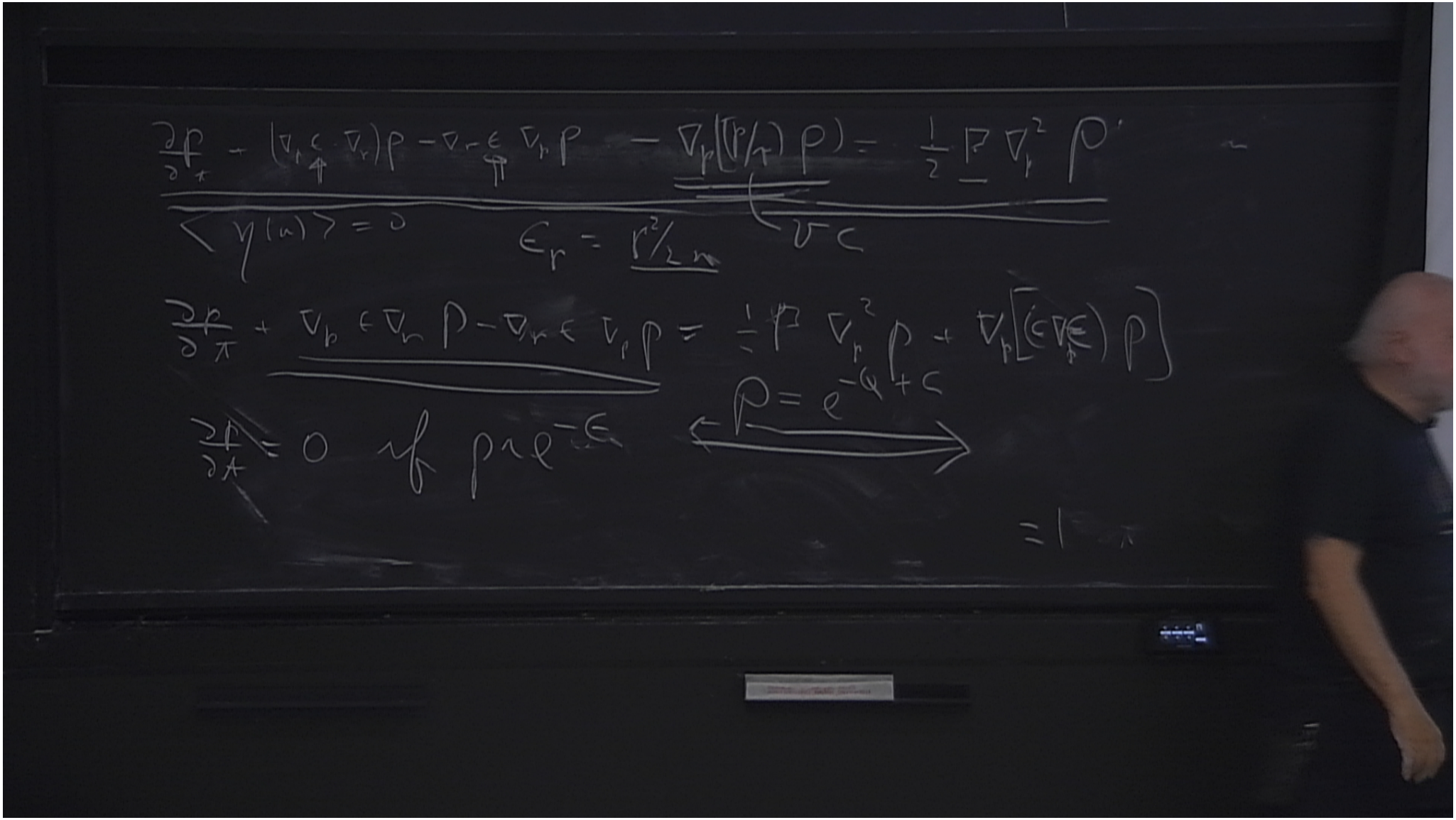
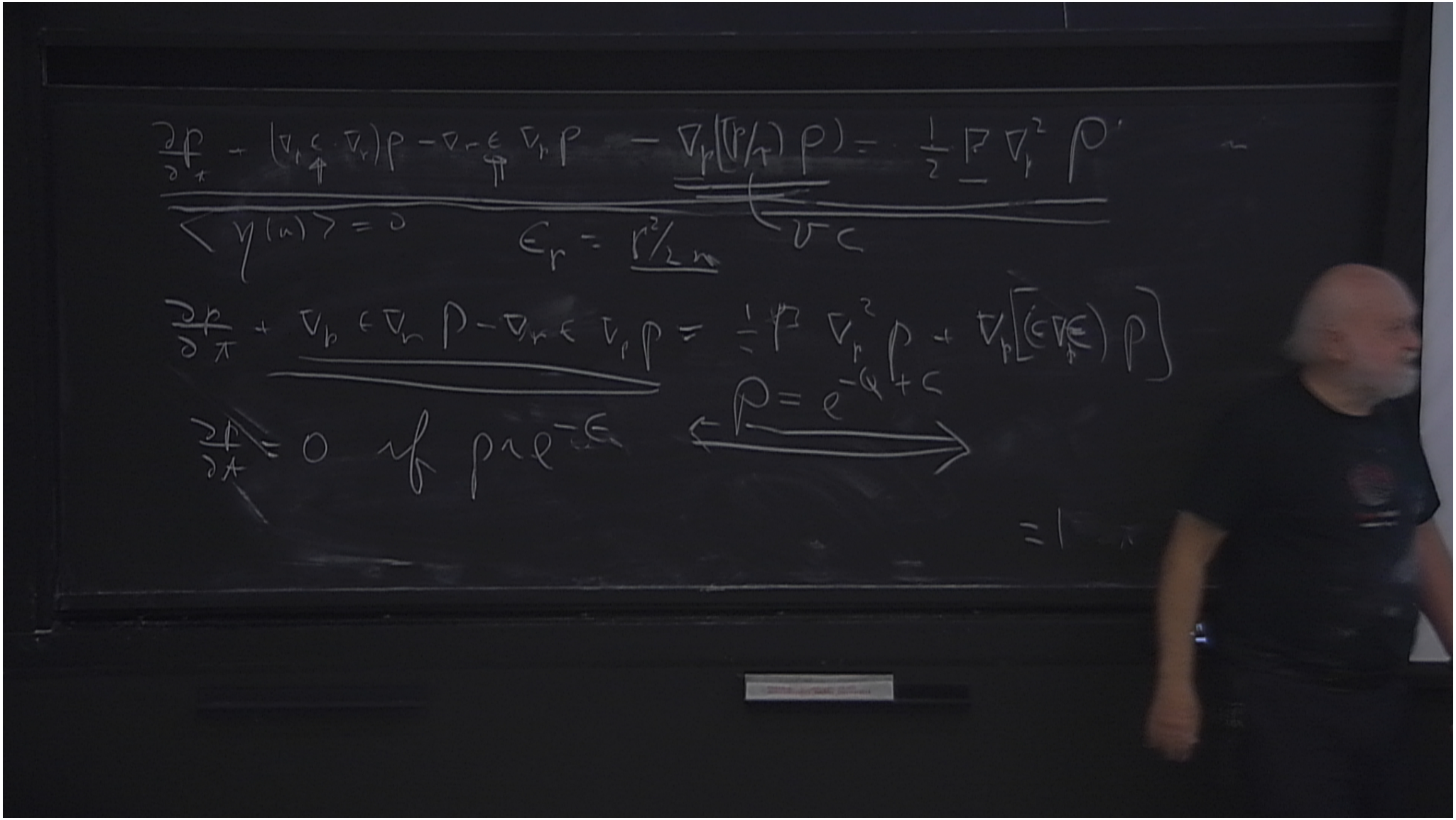

$\frac{\partial f}{\partial \tau} + (v_i \cdot \nabla_r) f - \nabla_r \epsilon \cdot \nabla_v f - \nabla_v [(F/m) f] = \frac{1}{2} F \nabla_v^2 f$
$\langle \eta(u) \rangle = 0$
$\epsilon_r = r^2/2$
$v \ll c$
$\frac{\partial f}{\partial \tau} + v_i \cdot \nabla_r f - \nabla_r \epsilon \cdot \nabla_v f = \frac{1}{2} F \nabla_v^2 f + \nabla_v [(\nabla \epsilon) f]$
$\frac{\partial f}{\partial \tau} = 0$ if $f \propto e^{-\epsilon}$
$f = e^{-\epsilon + c}$
$= 1$


$\frac{\partial \rho}{\partial \tau} - (v_i \epsilon \nabla_i)\rho - \nabla \cdot \epsilon \nabla_i \rho - \nabla_i[(\nabla/\tau)\rho] = \frac{1}{2} \nabla_i^2 \rho$
$\langle \eta(\tau) \rangle = 0$
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$v c$
$\frac{\partial \rho}{\partial \tau} + v_i \epsilon \nabla_i \rho - \nabla_i \epsilon v_i \rho = \frac{1}{2} \nabla_i^2 \rho + \nabla_i[(\epsilon \nabla \epsilon)\rho]$
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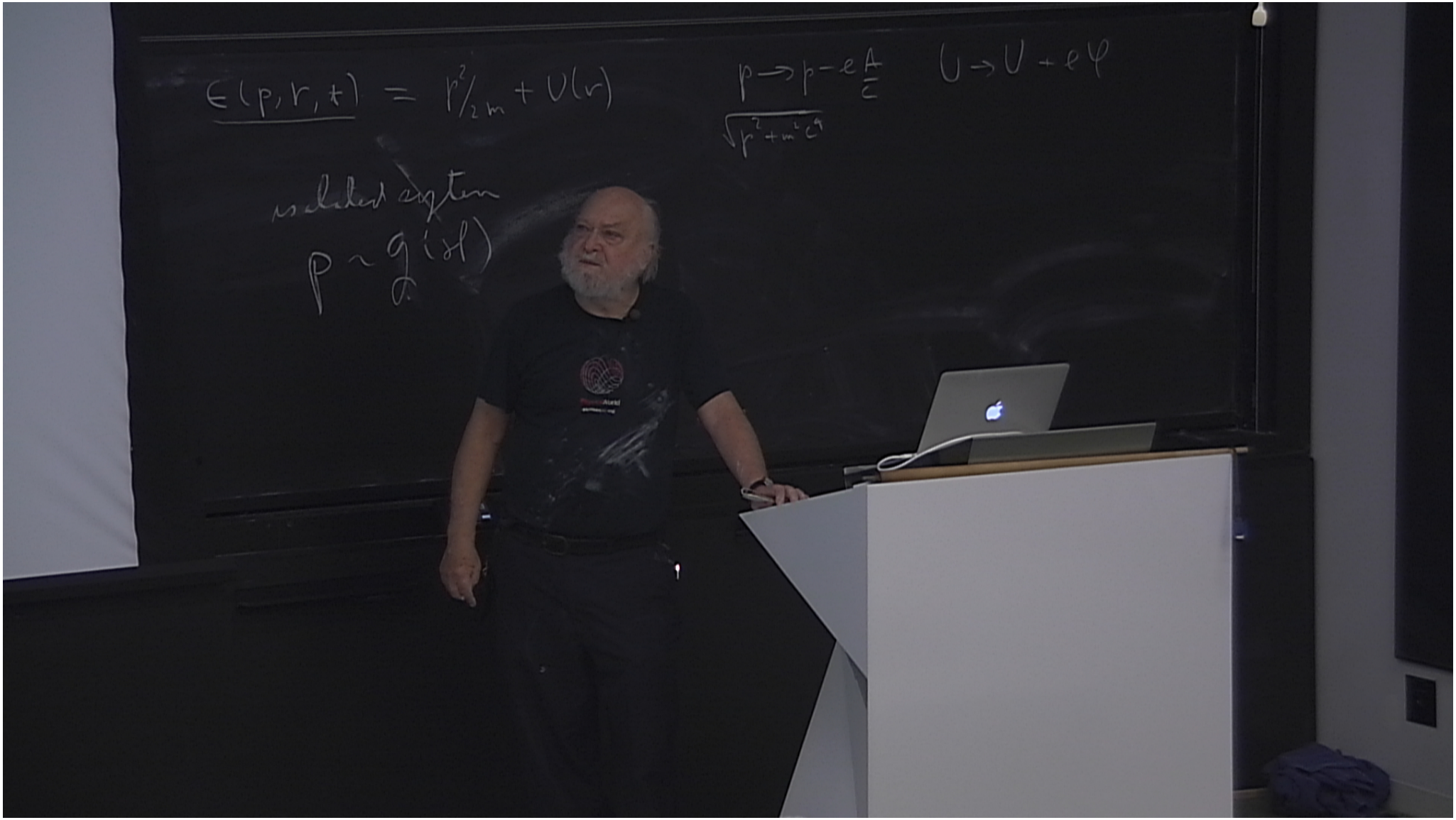
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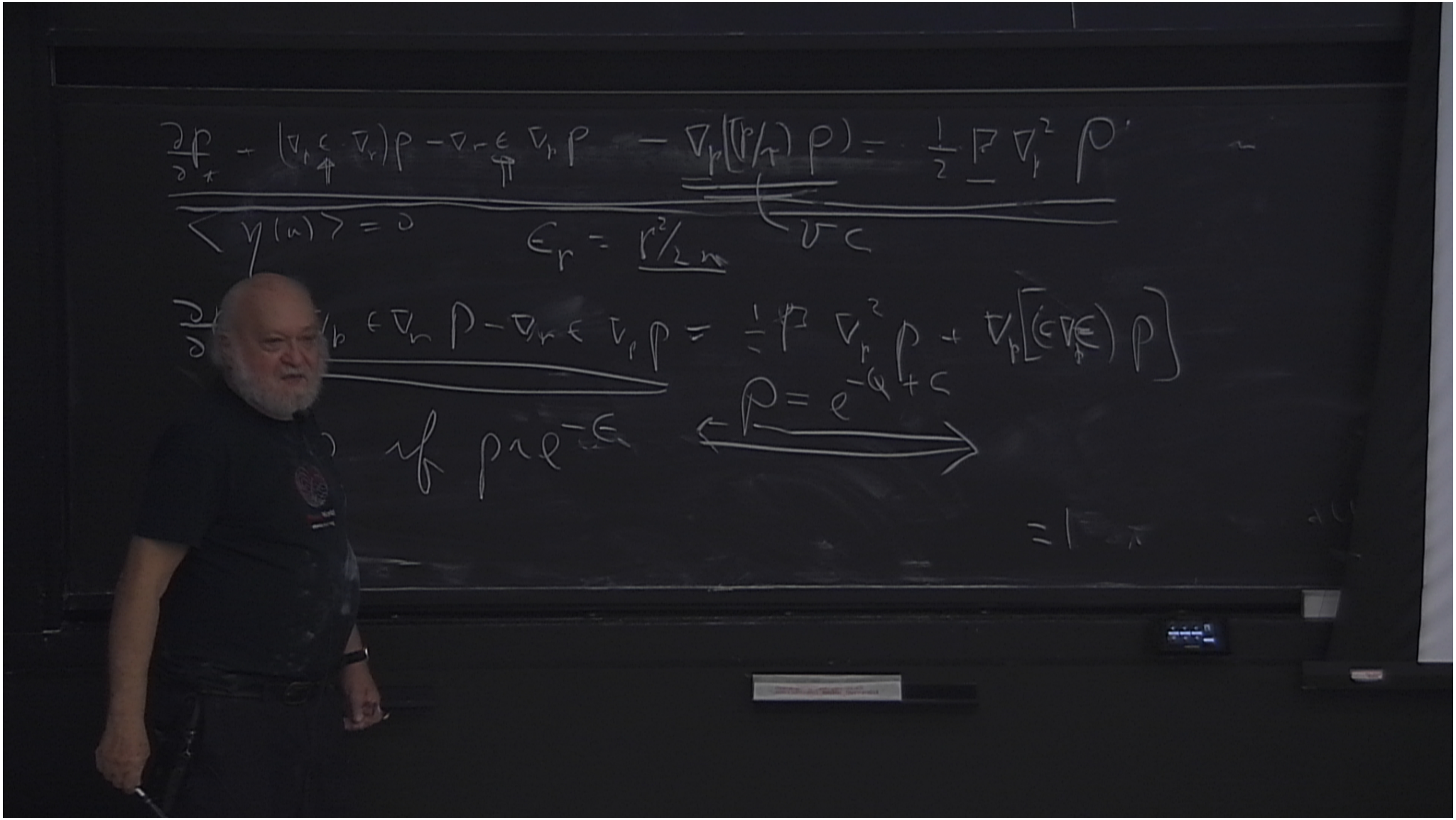
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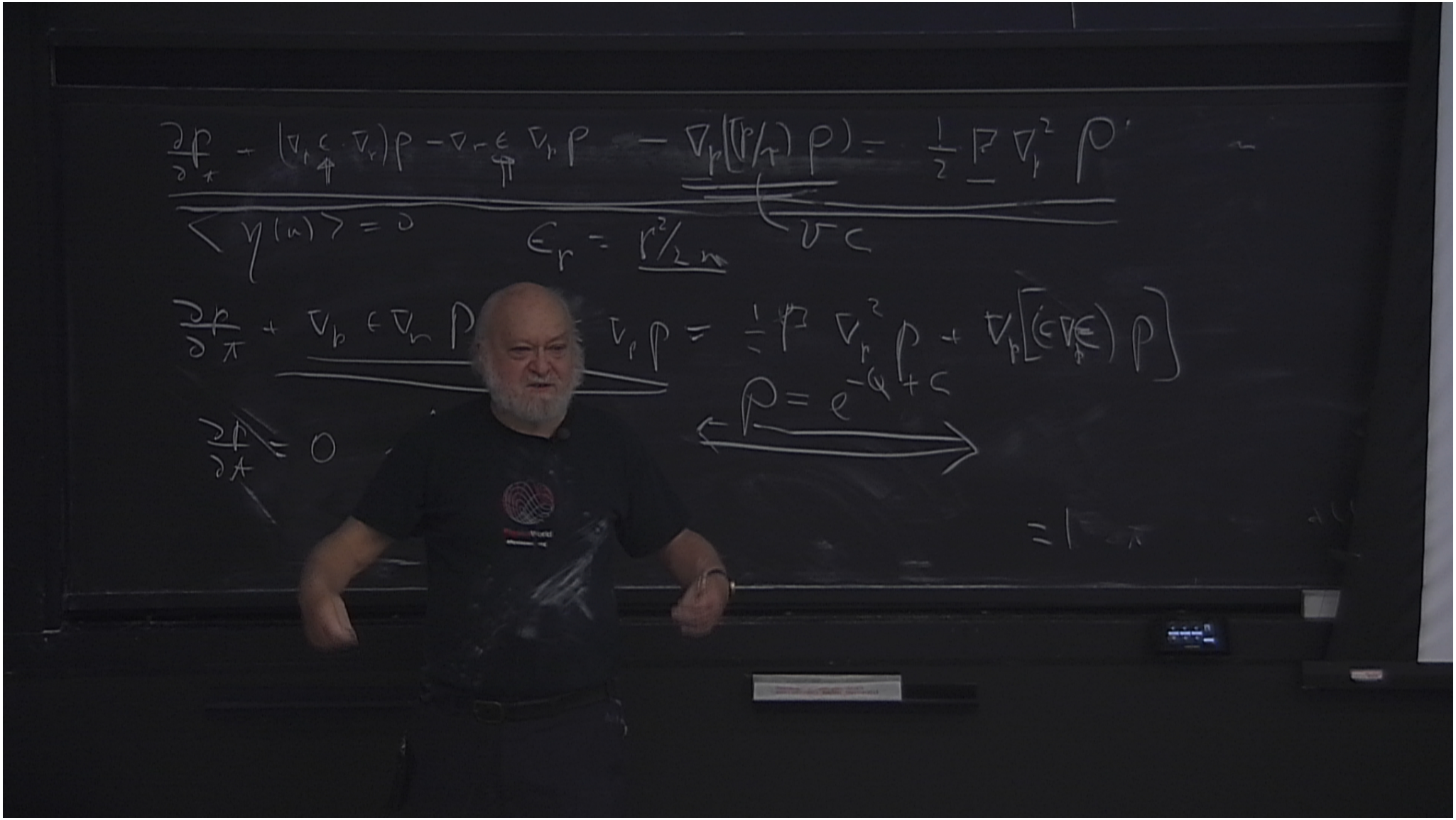
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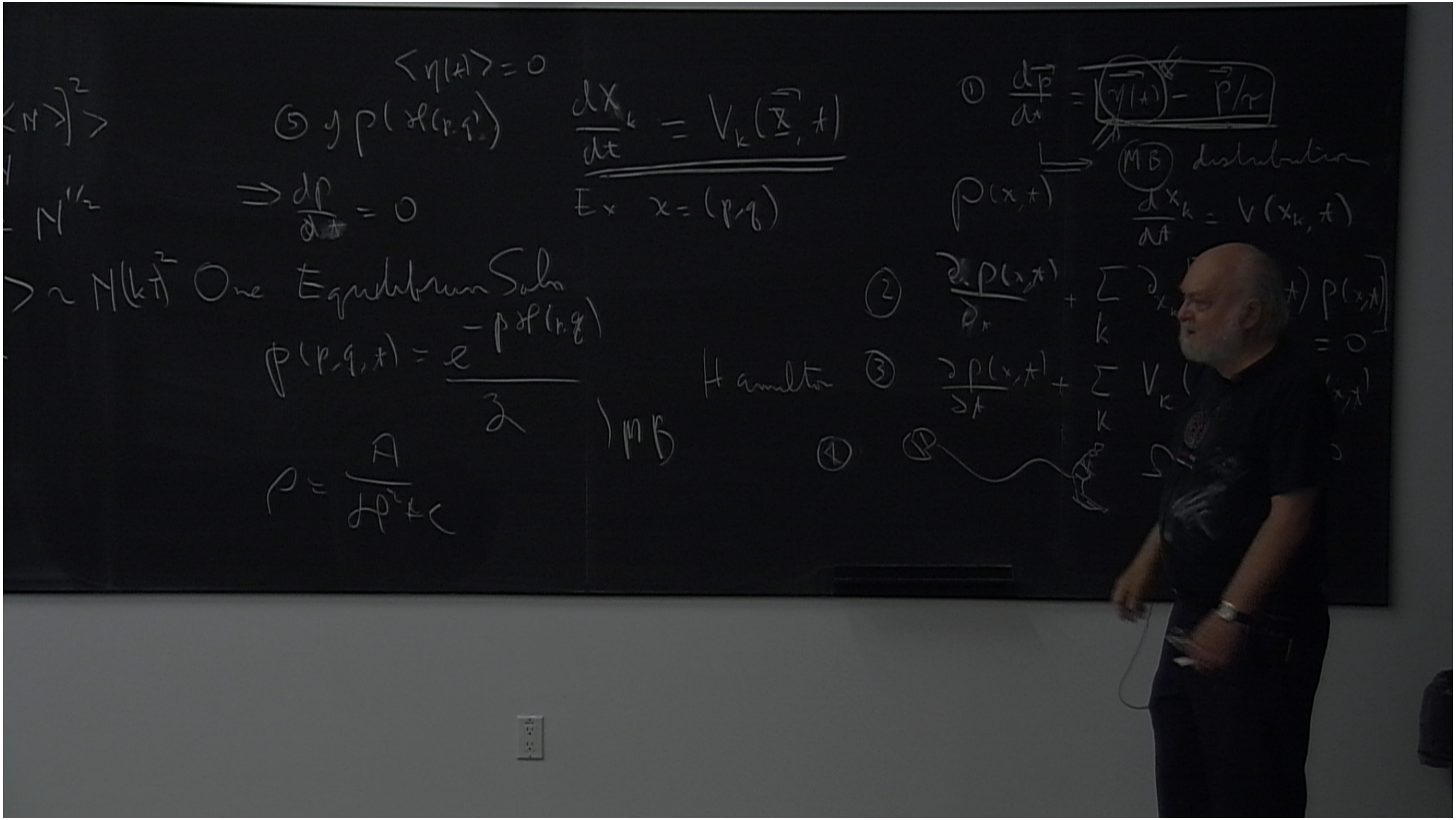
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MB distribution
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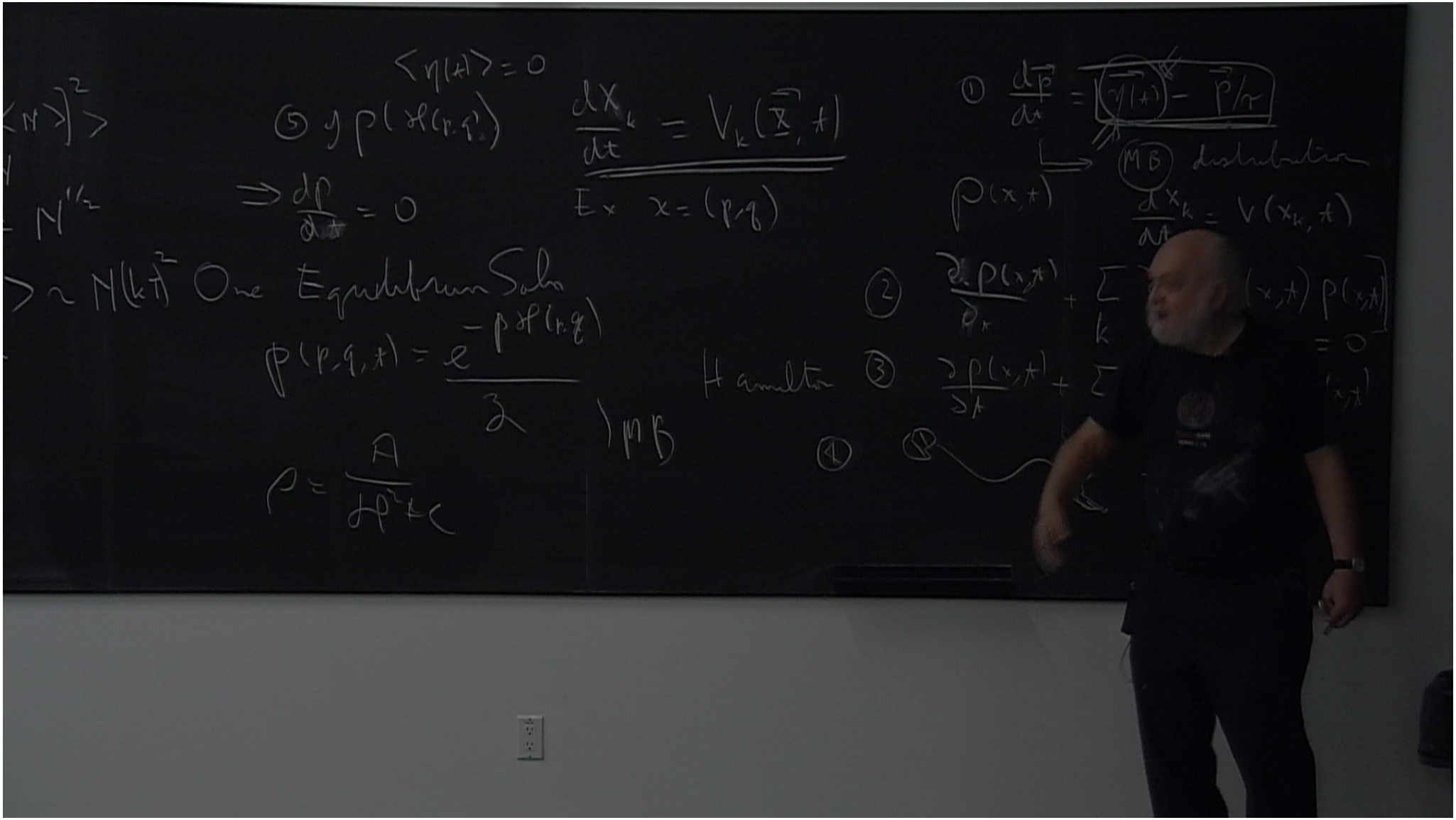
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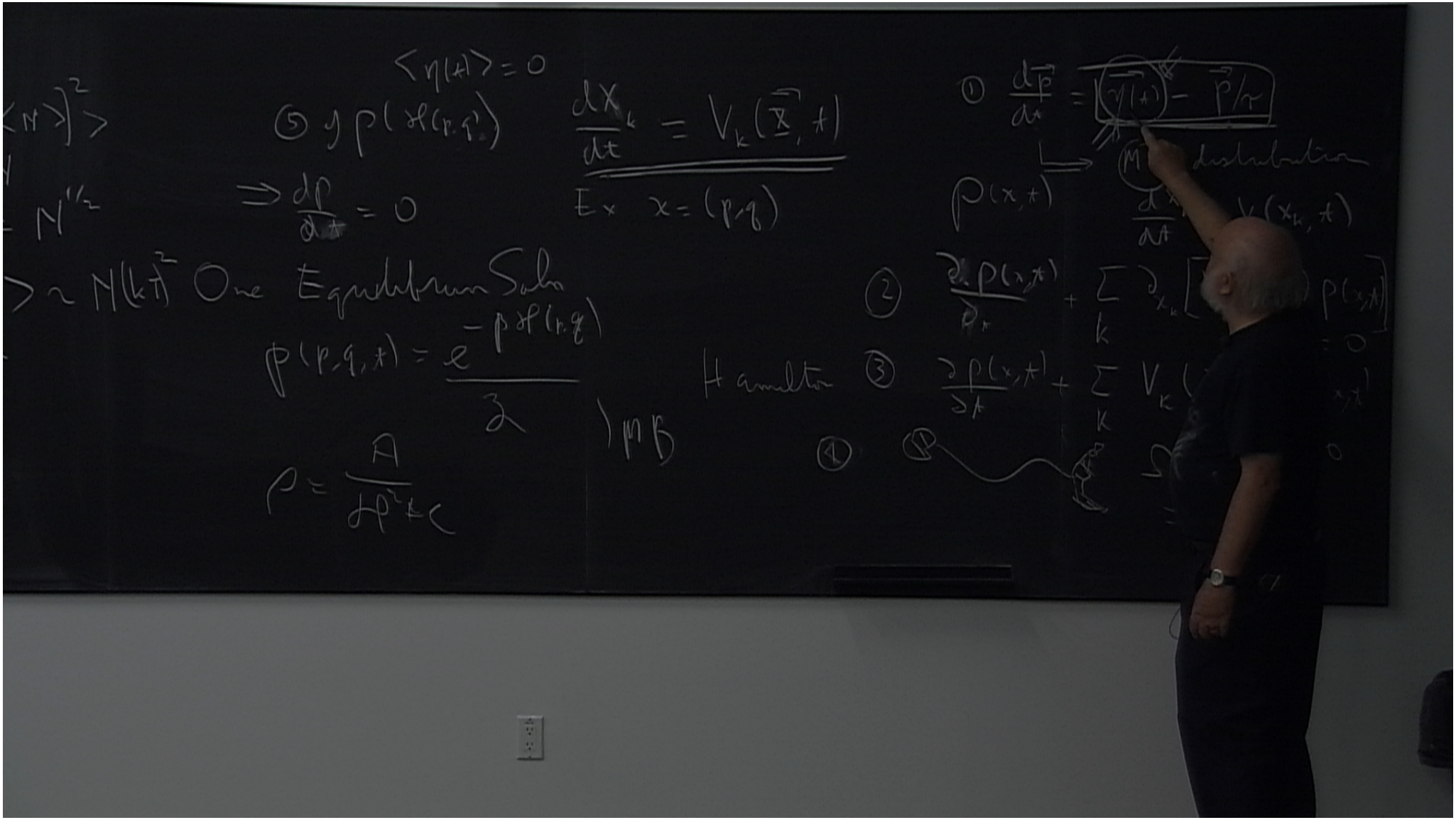
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distribution

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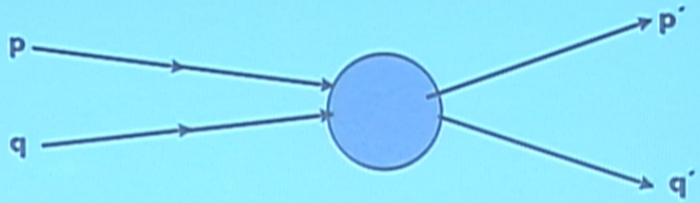


particles scatter out of **p**.

Given that there are particles available with the appropriate initial momentum, the scattering rate into a volume element of final momentum $dp' dq'$ can be written as $dp' dq' \; Q(p,q \to p',q')$

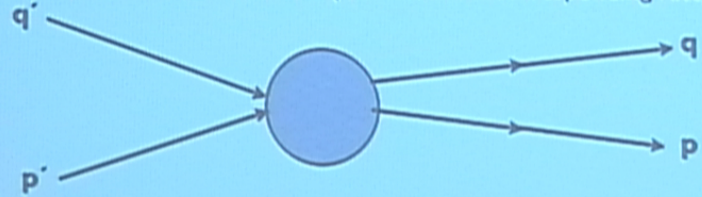
The probability that we could get the particles we need for the scattering produce a factor

$f(p,r,t) \; dq \; f(q,r,t)$, so that the total scattering rate for this process is

$$f(p,r,t) \int dq \, f(q,r,t) \, dp' \, dq' \; Q(p,q \to p',q')$$

The process *reduces* the number described by p,r.

In addition, there is an inverse process, and a corresponding rate of increase of $f(p,r,t)$:



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$$\int dq \; dp' \, f(p',r,t) \; dq' \, f(q',r,t) \; Q(p',q' \to p,q)$$