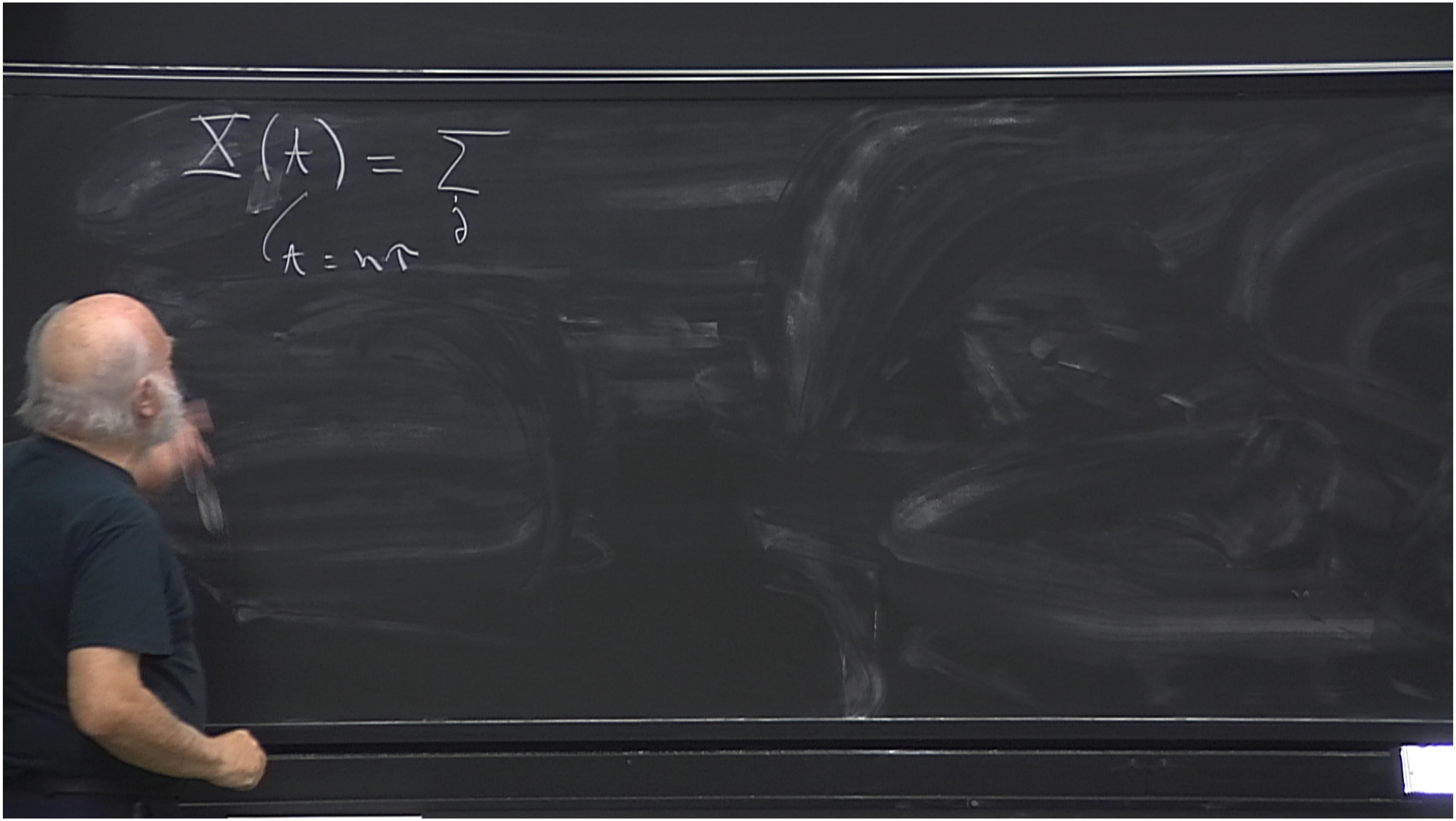
Title: Statistical Mechanics - Lecture 6

Date: Oct 11, 2011 10:30 AM

URL: http://pirsa.org/11100033

Abstract:


Ψ(t) = Σ j
t = nτ

$$X(t) = \sum_{j=1}^{n} \sigma_j a \qquad \sigma_j = \pm 1$$

$$(t = n\tau$$

$$X(t) = \sum_{j=1}^{n} \sigma_j a \qquad \sigma_j = \pm 1 \quad \text{iid}$$

$$t = n\tau$$

$$\langle X(t) \rangle = \sum_{j=1}^{n} \langle \sigma_j \rangle a = 0$$

$$\langle X^2(t) \rangle = \langle \sum_{jk} a^2 \sigma_j \sigma_k \rangle = a^2 \sum_{j=1}^{n} \langle \sigma_j^2 \rangle = a^2 n$$

$$\max X(t) = a n$$

$$\left(\langle X(t)^2 \rangle \right)^{1/2} = a\sqrt{n}$$

$$X(t) = \sum_{j=1}^{n} \sigma_j a \qquad \sigma_j = \pm 1 \quad \text{iid}$$

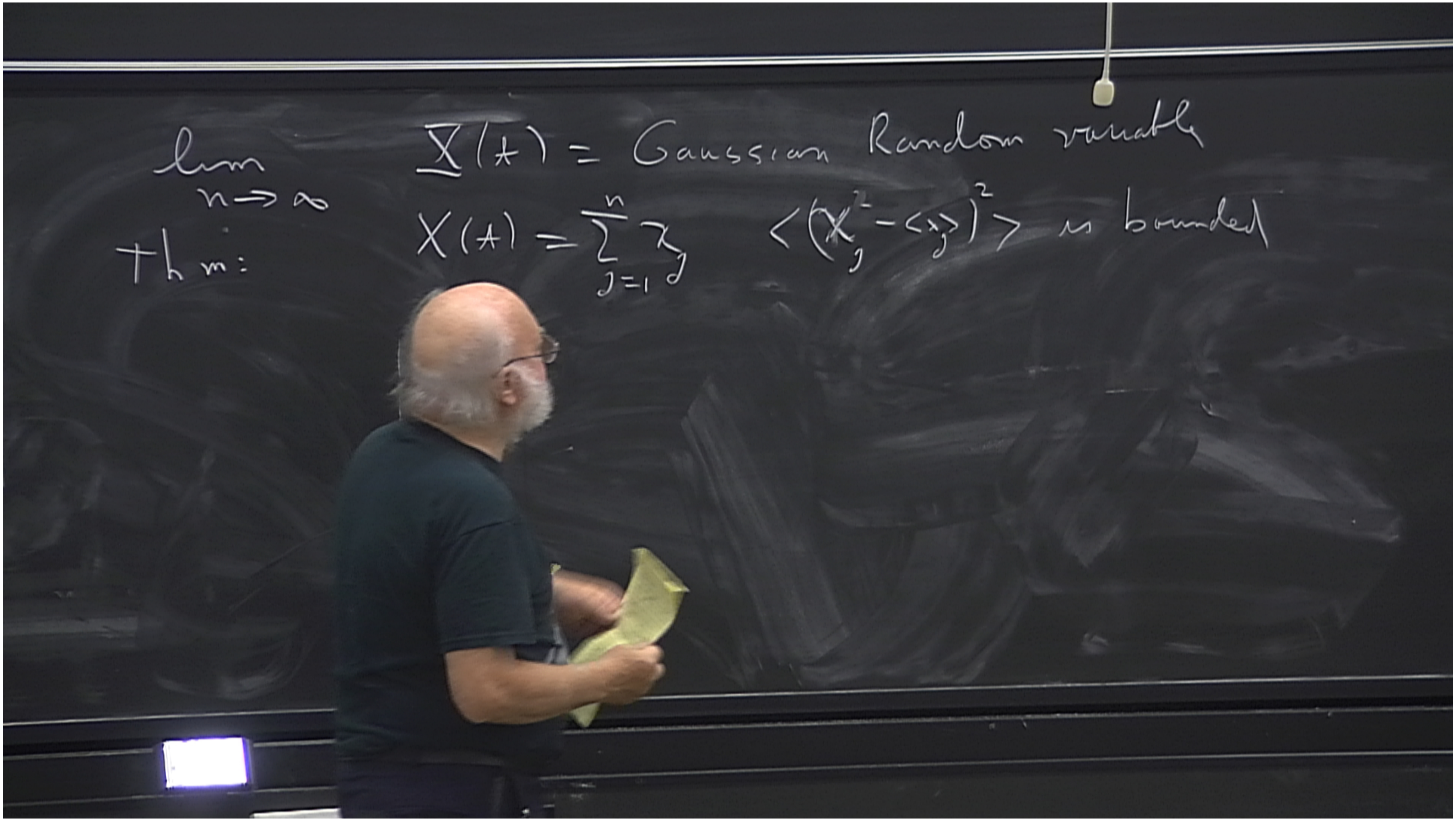
$t = n\tau$

$$\langle X(t) \rangle = \sum_{j=1}^{n} \langle \sigma_j \rangle a = 0$$

$$\langle X^2(t) \rangle = \langle \sum_{jk} a^2 \sigma_j \sigma_k \rangle = a^2 \sum_{j=1}^{n} \langle \sigma_j^2 \rangle = a^2 n$$

$$\max X(t) = a n \qquad \text{ex} \quad n = 10^6$$

$$\left(\langle X(t)^2 \rangle \right)^{1/2} = a\sqrt{n}$$


$\lim_{n\to\infty}$
$\underline{X}(t)$ = Gaussian Random variable
Thm:
$X(t) = \sum_{j=1}^{n} x_j$
$\langle (x_j^2 - \langle x_j \rangle)^2 \rangle$ is bounded

$$\lim_{n\to\infty} \underline{X}(t) = \text{Gaussian Random variable}$$

thm: $\underline{X}(t) = \sum_{j=1}^{n} x_j$ $\quad \langle (x_j^2 - \langle x_j \rangle)^2 \rangle$ is bounded

$$\rho(\underline{X}(t) = x) = \frac{e^{-x^2/2a^2 n}}{\sqrt{2\pi a^2 n}}$$

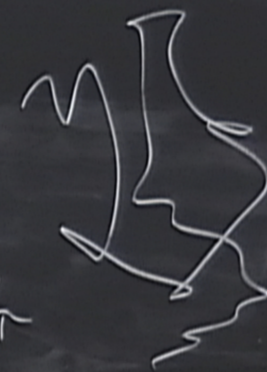
Gaussian!

$a^2 n$

$\underline{X}(t)$ = Gaussian Random variable

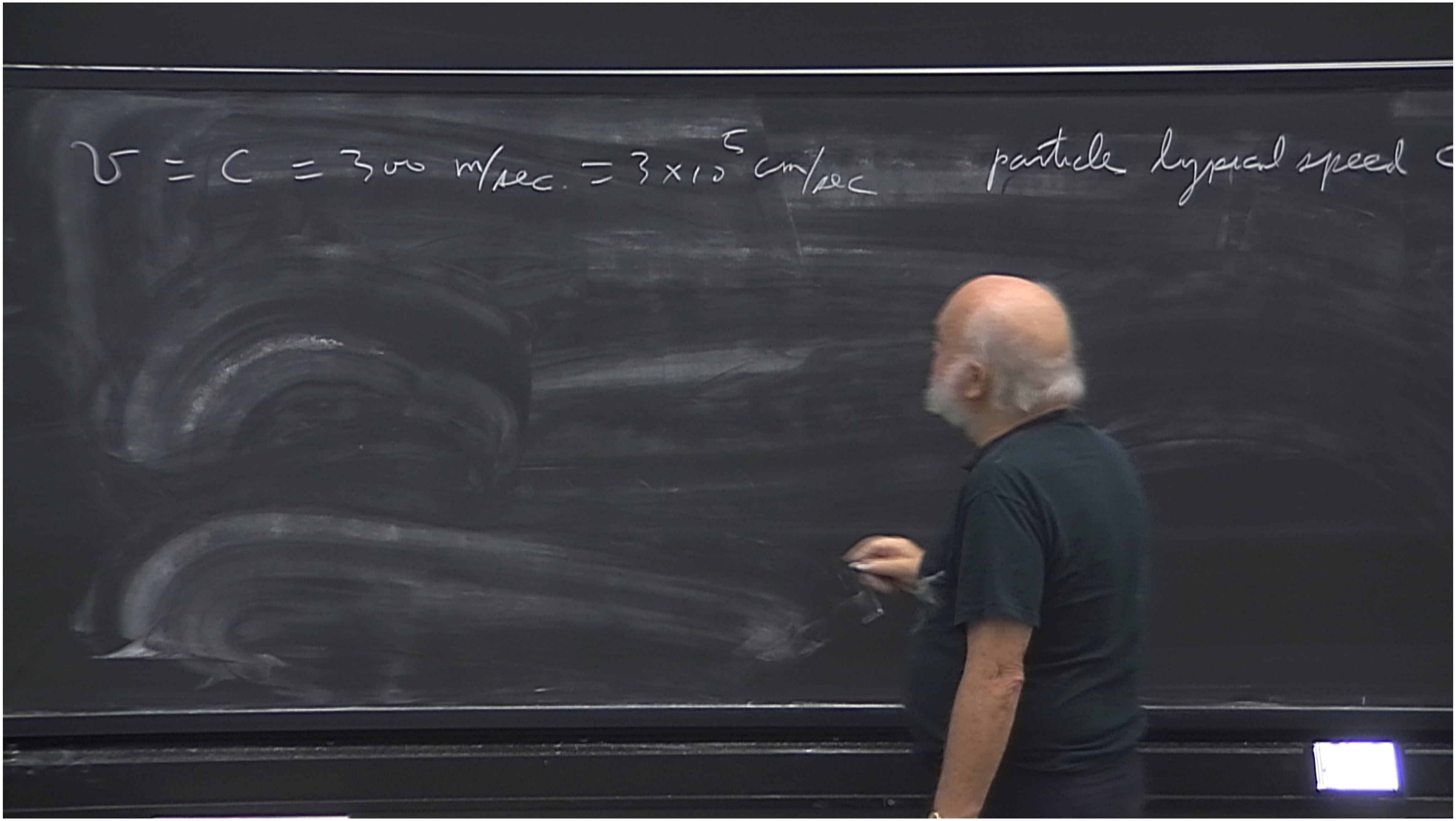
$\to \infty$

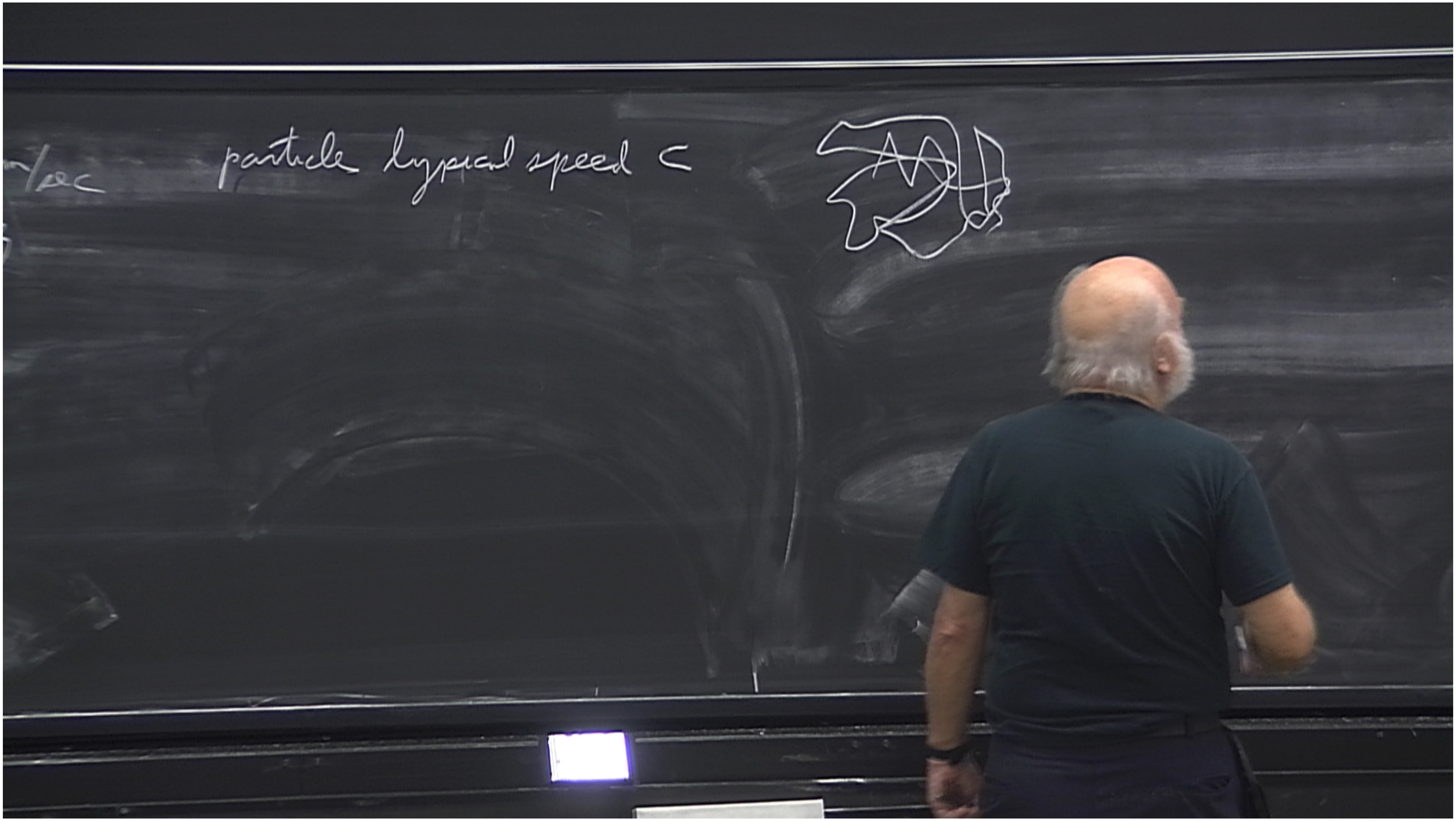
$\underline{X}(t) = \sum_{j=1}^{n} x_j \qquad \langle (x_j^2 - \langle x_j \rangle)^2 \rangle$ is bounded



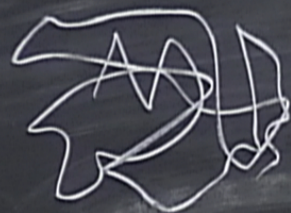
$) = x) = \dfrac{e^{-x^2/2a^2 n}}{\sqrt{2\pi a^2 n}}$ Gaussian! $= \dfrac{e^{-x^2/2a^2 t/\tau}}{(2\pi a^2 t/\tau)^{1/2}}$

Maxwell kinetic theory of gases


v = c = 300 m/sec. = 3 x 10^5 cm/sec
particle typical speed

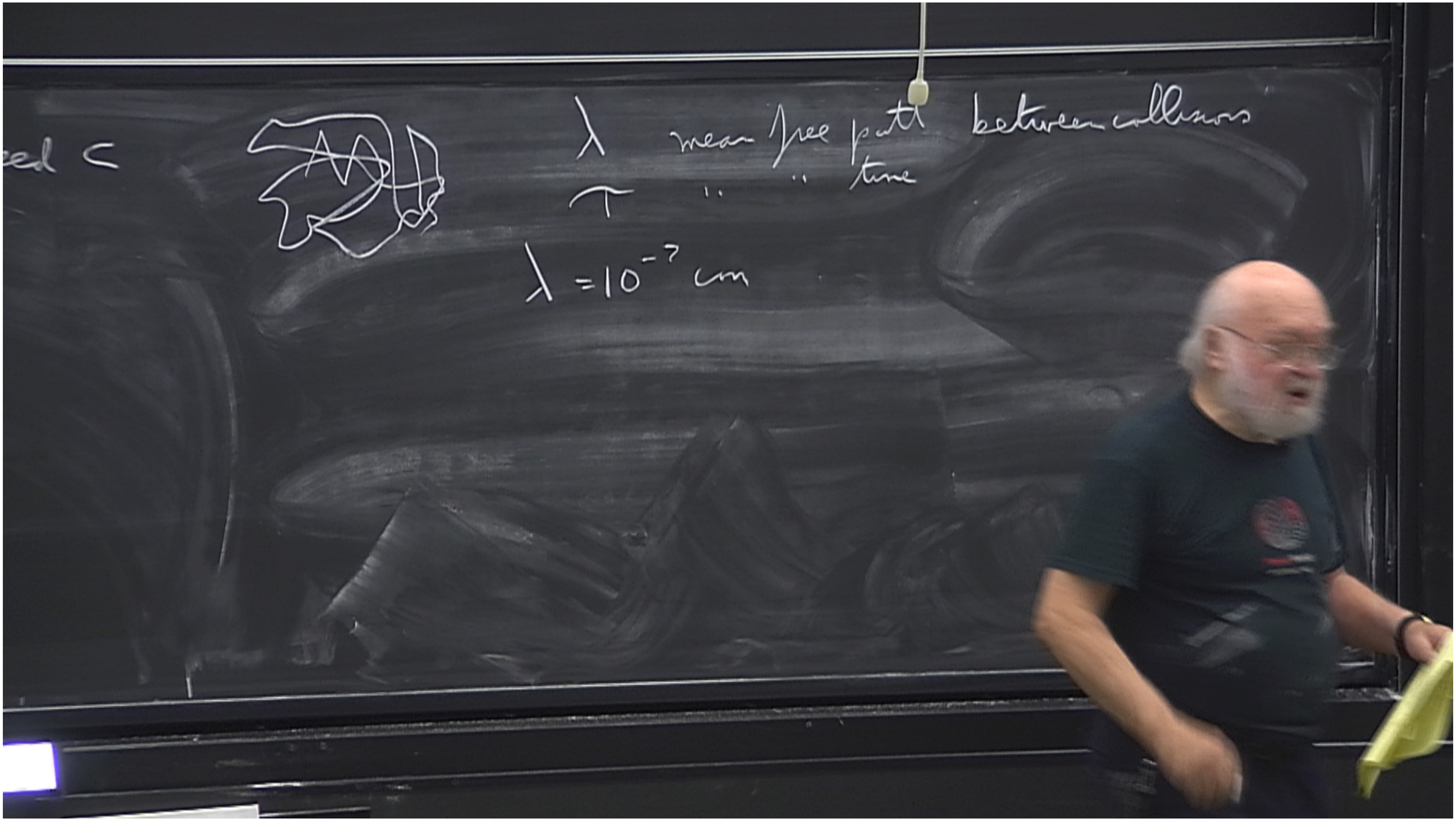

particle typical speed c

eed c

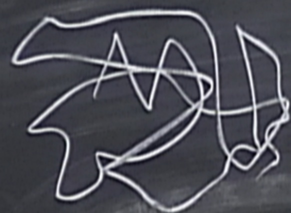


λ mean free path between collisions

τ " " time


eed c
λ mean free path between collisions
τ " " time
λ = 10⁻⁷ cm

eed c

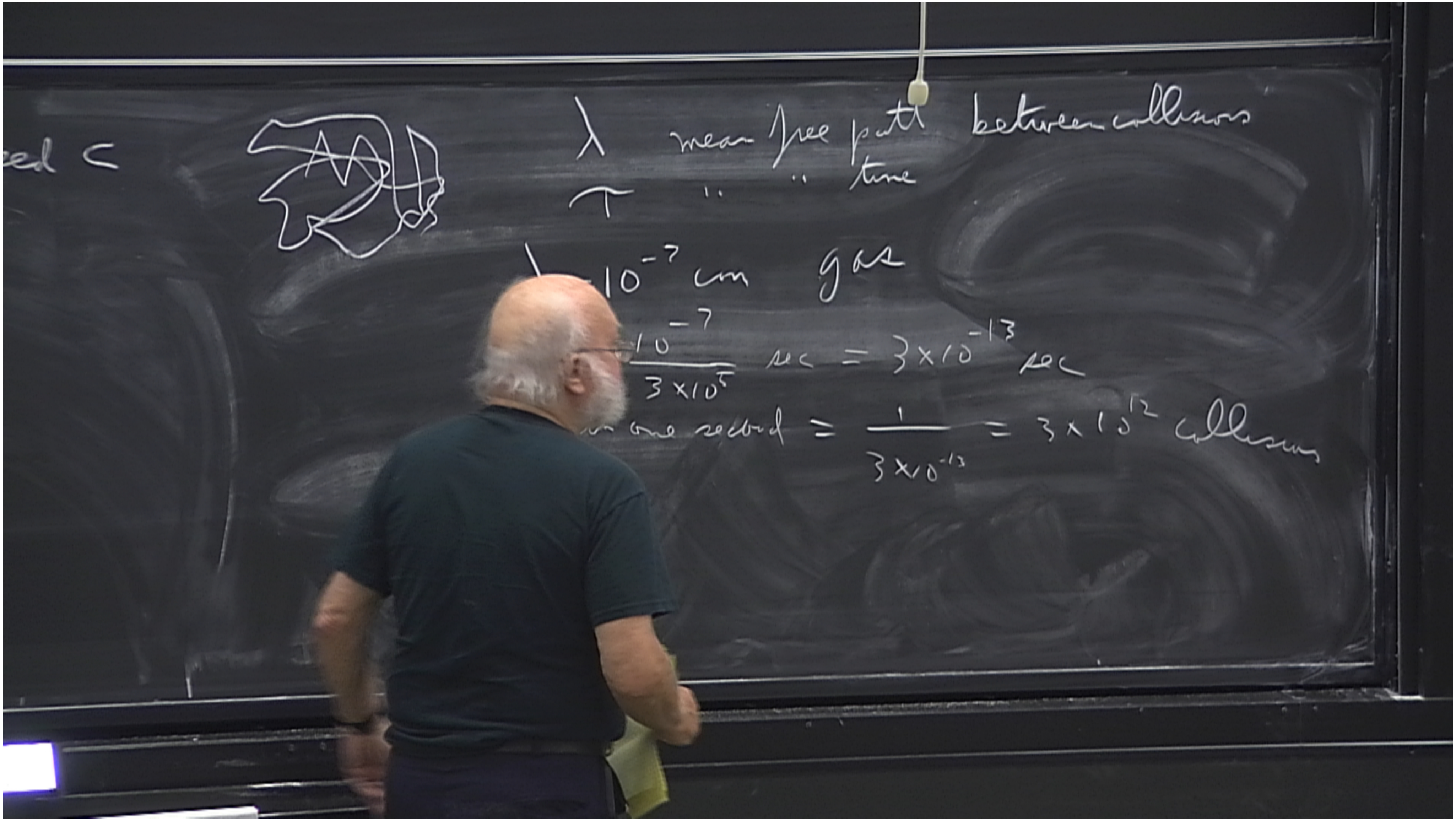


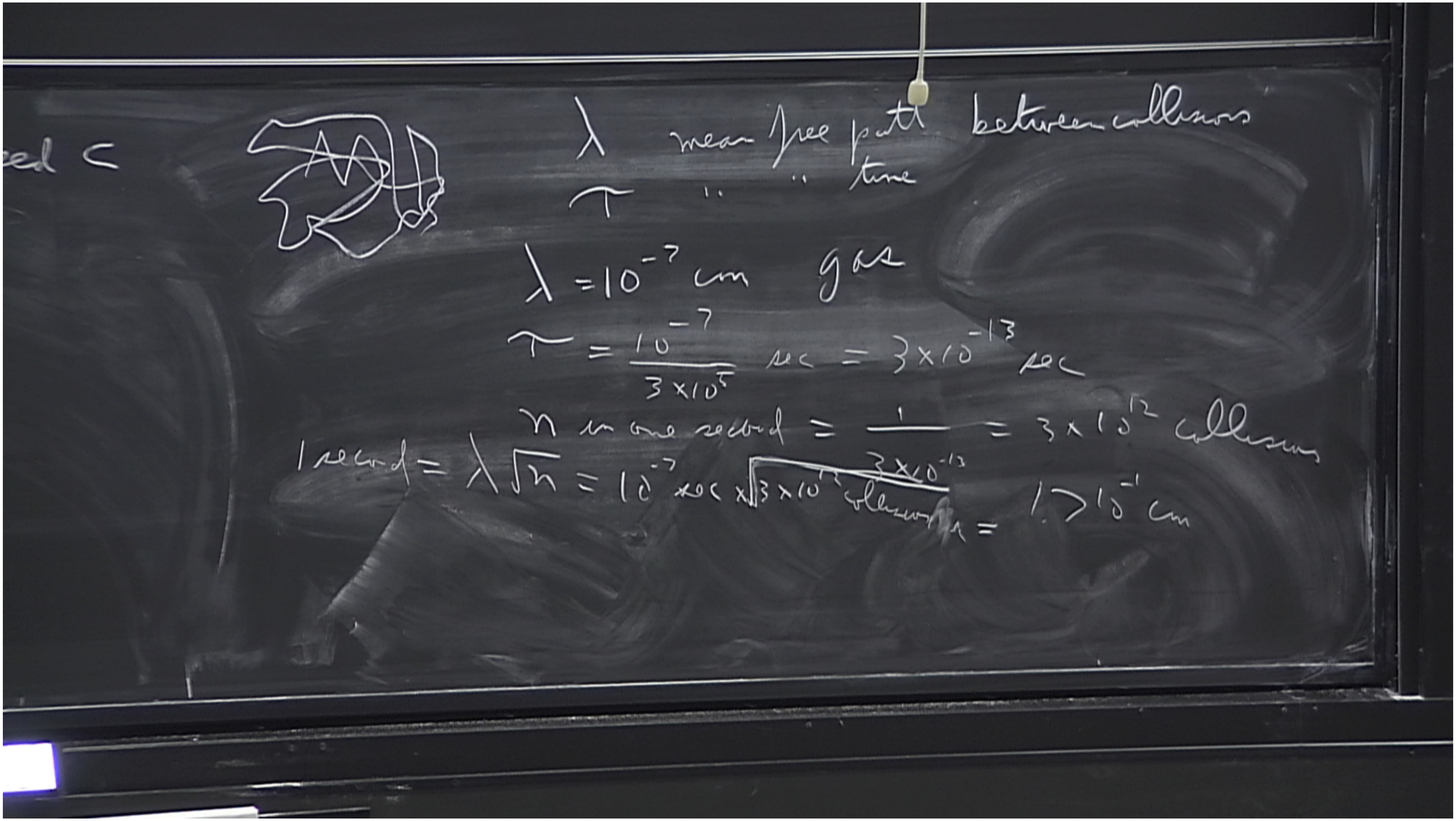
λ mean free path between collisions

τ " " time

$\lambda = 10^{-7}$ cm gas

$$\tau = \frac{10^{-7}}{3\times10^{5}} \text{ sec} = 3\times10^{-13} \text{ sec}$$


ed c
λ mean free path between collisions
τ " " time
10^-7 cm gas
10^-7 / 3×10^5 sec = 3×10^-13 sec
one second = 1 / 3×10^-13 = 3×10^12 collisions


ed c
λ mean free path between collisions
τ " " time
λ = 10^-7 cm gas
τ = 10^-7 / 3×10^5 sec = 3×10^-13 sec
n in one second = 1 / 3×10^-13 = 3×10^12 collisions
1 second = λ √n = 10^-7 sec × √(3×10^12 collisions) = 1.7×10^-1 cm

$v = c = 300$ m/sec. $= 3 \times 10^{4}$ cm/sec

particle typical spe

smell moves much more slowly

$$\underline{X}(t),\ \underline{Y}(t),\ Z(t) = \vec{R}(t)$$

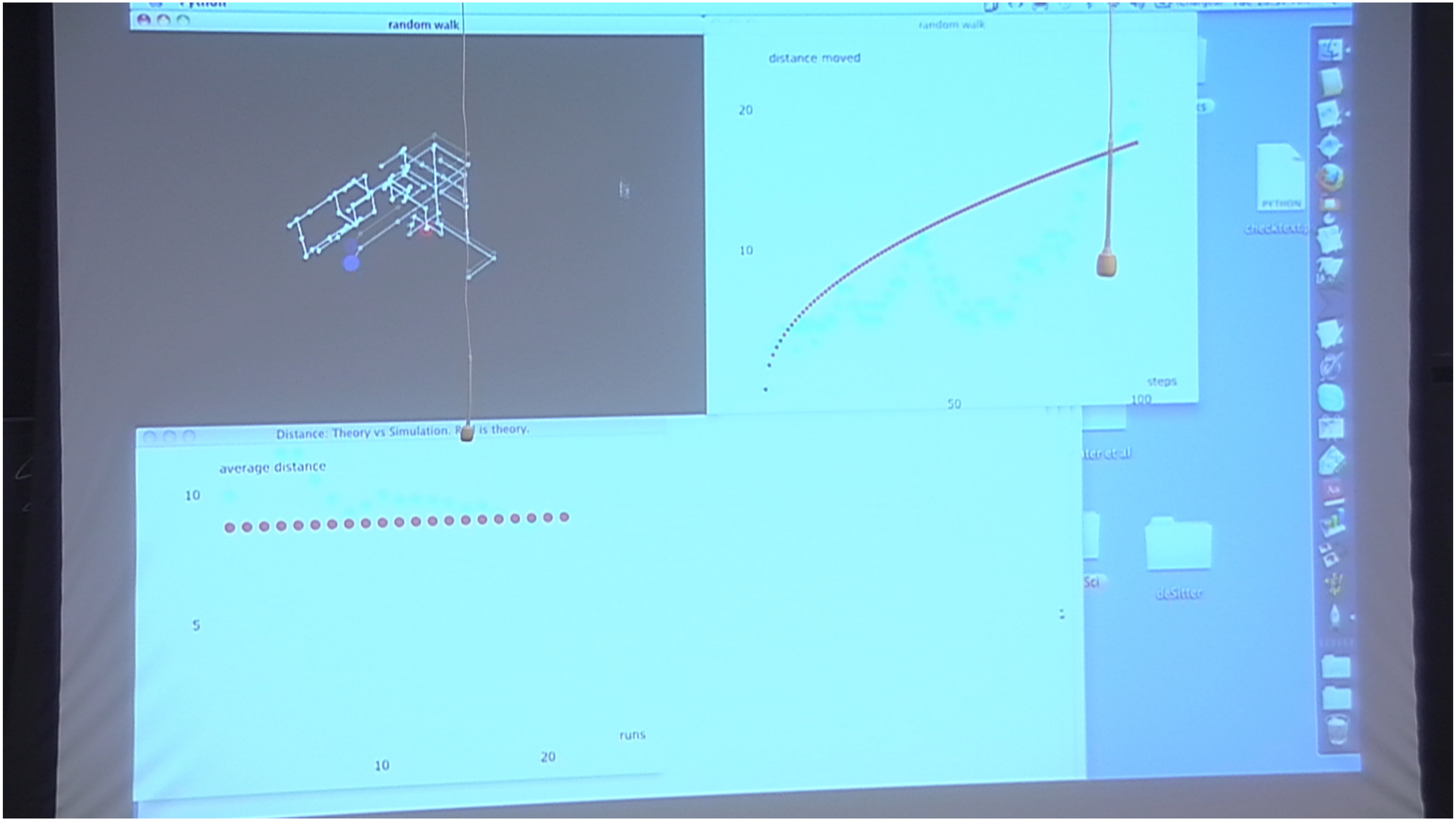
$v = c = 300$ m/sec. $= 3 \times 10^4$ cm/sec

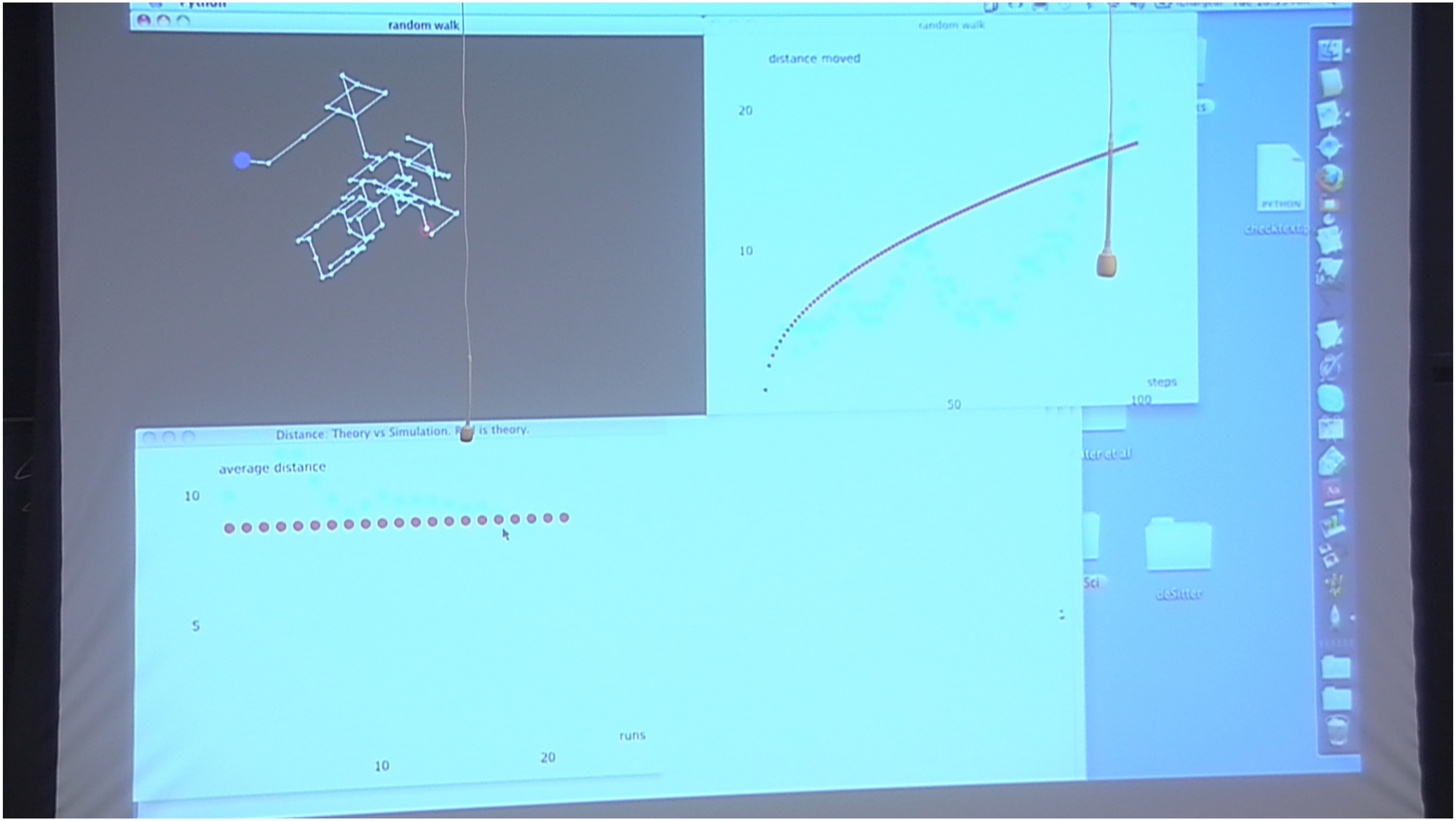
particle typical spe

smell moves much more slowly

$X(t), Y(t), Z(t) = \vec{R}(t)$

$$\rho(R(t) = \vec{r}) = \frac{e^{-r^2/2[a^2(t/\tau)]}}{(2\pi\, a^2(t/\tau))^{3/2}}$$


random walk
random walk
distance moved
20
10
steps
50
100
Distance: Theory vs Simulation. Red is theory.
average distance
10
5
runs
10
20
deSitter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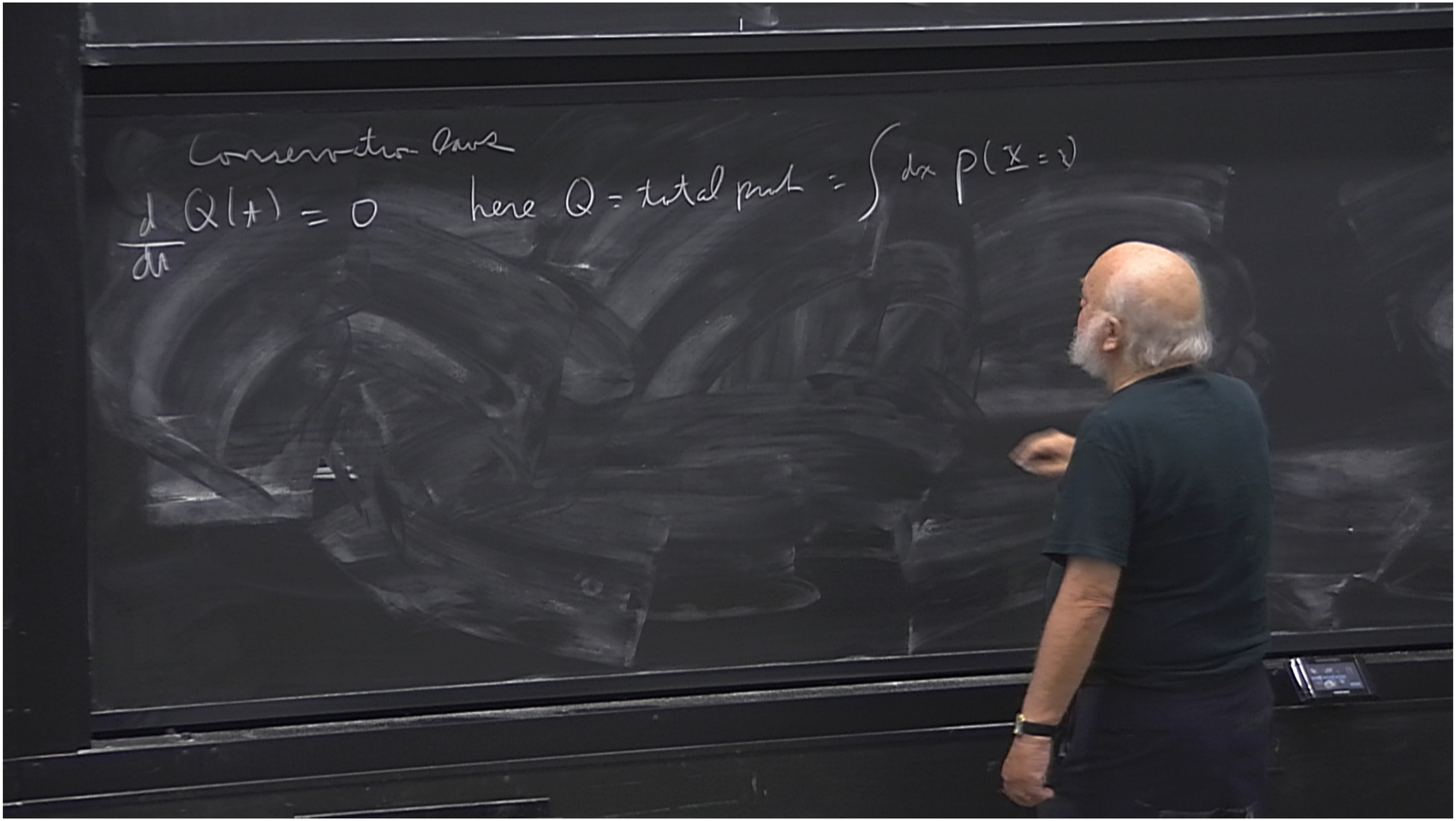
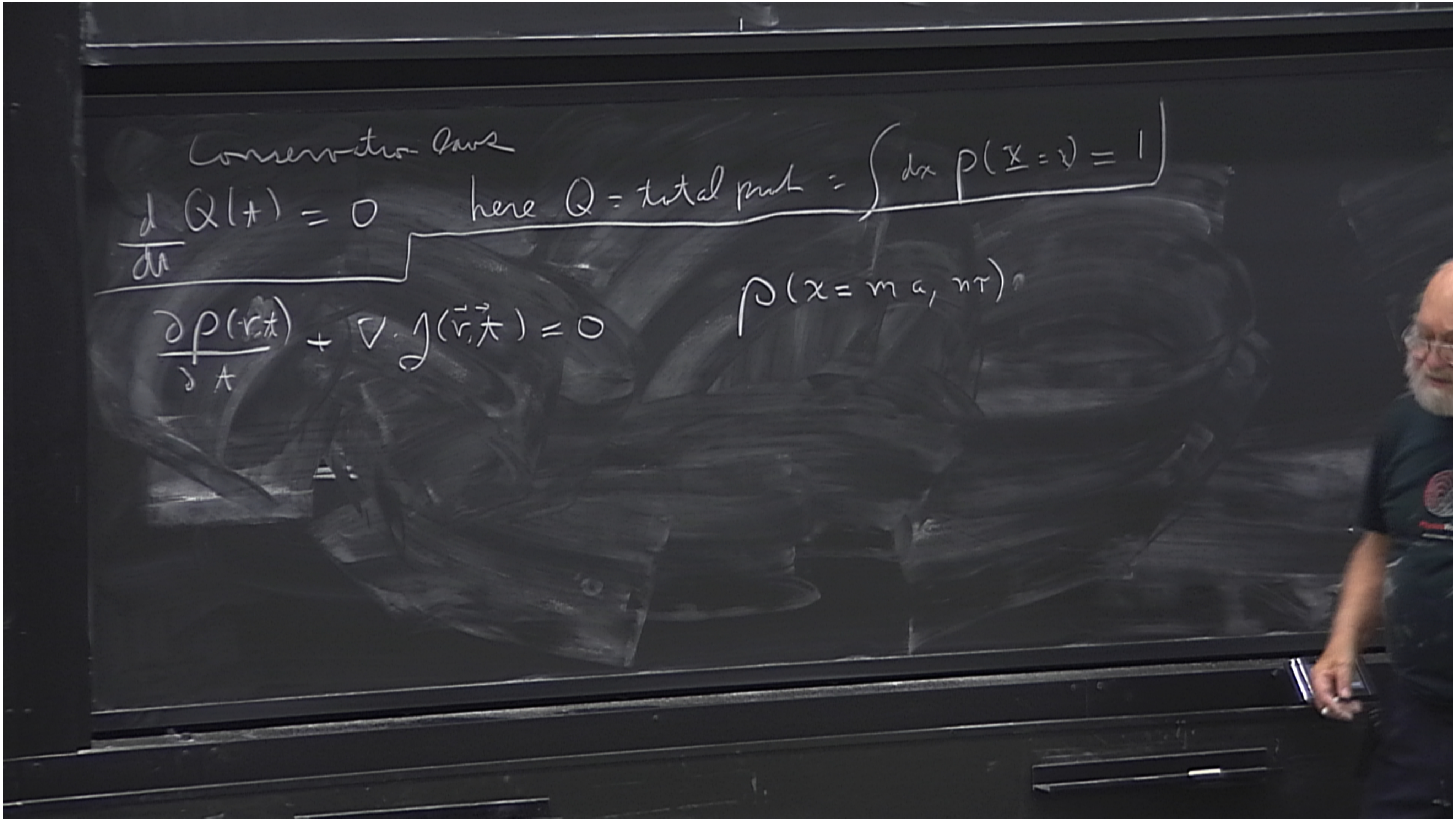

Conservation laws
d/dt Q(t) = 0
here Q = total prob = ∫ dx p(x = t)

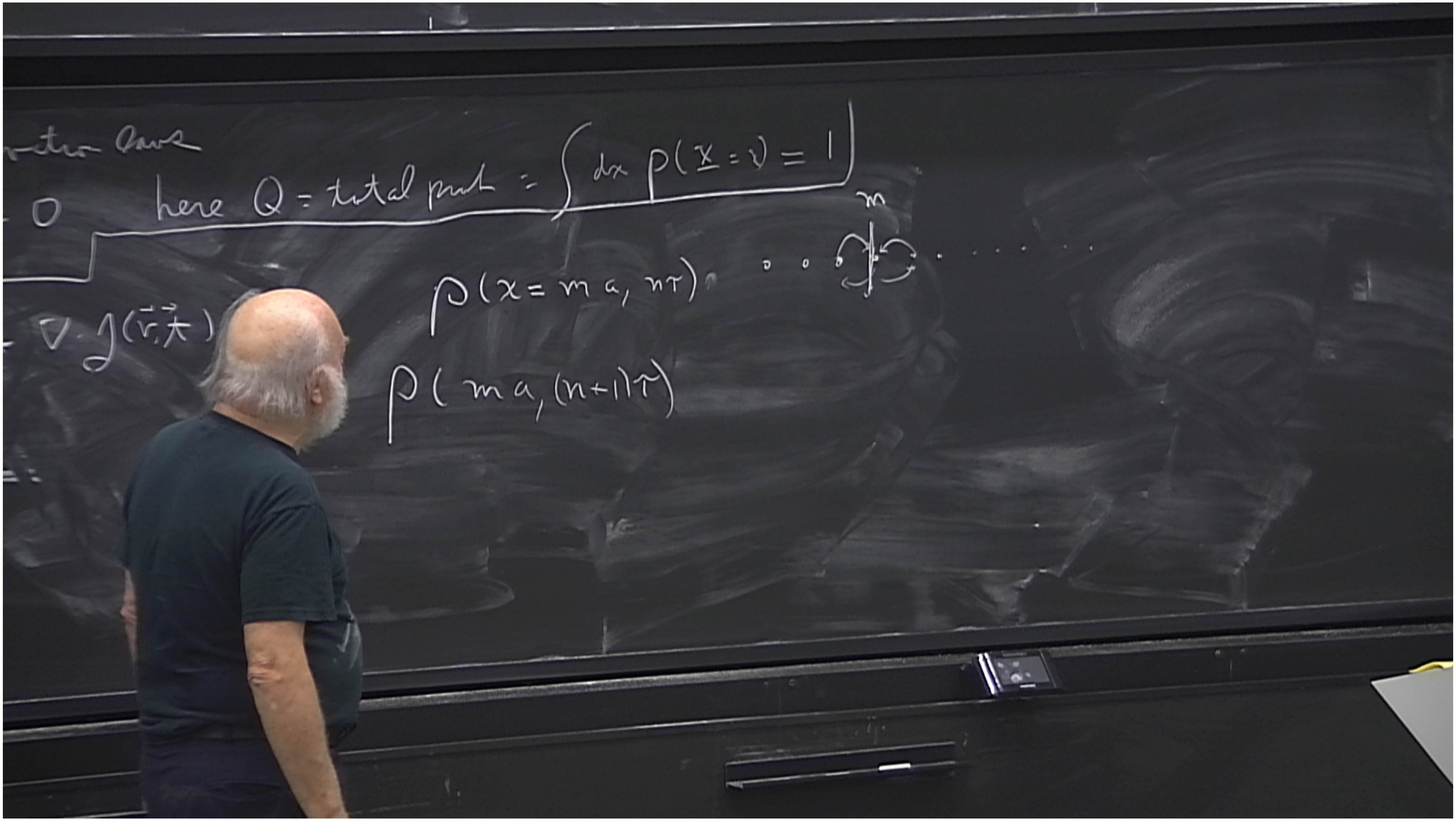
Conservation laws

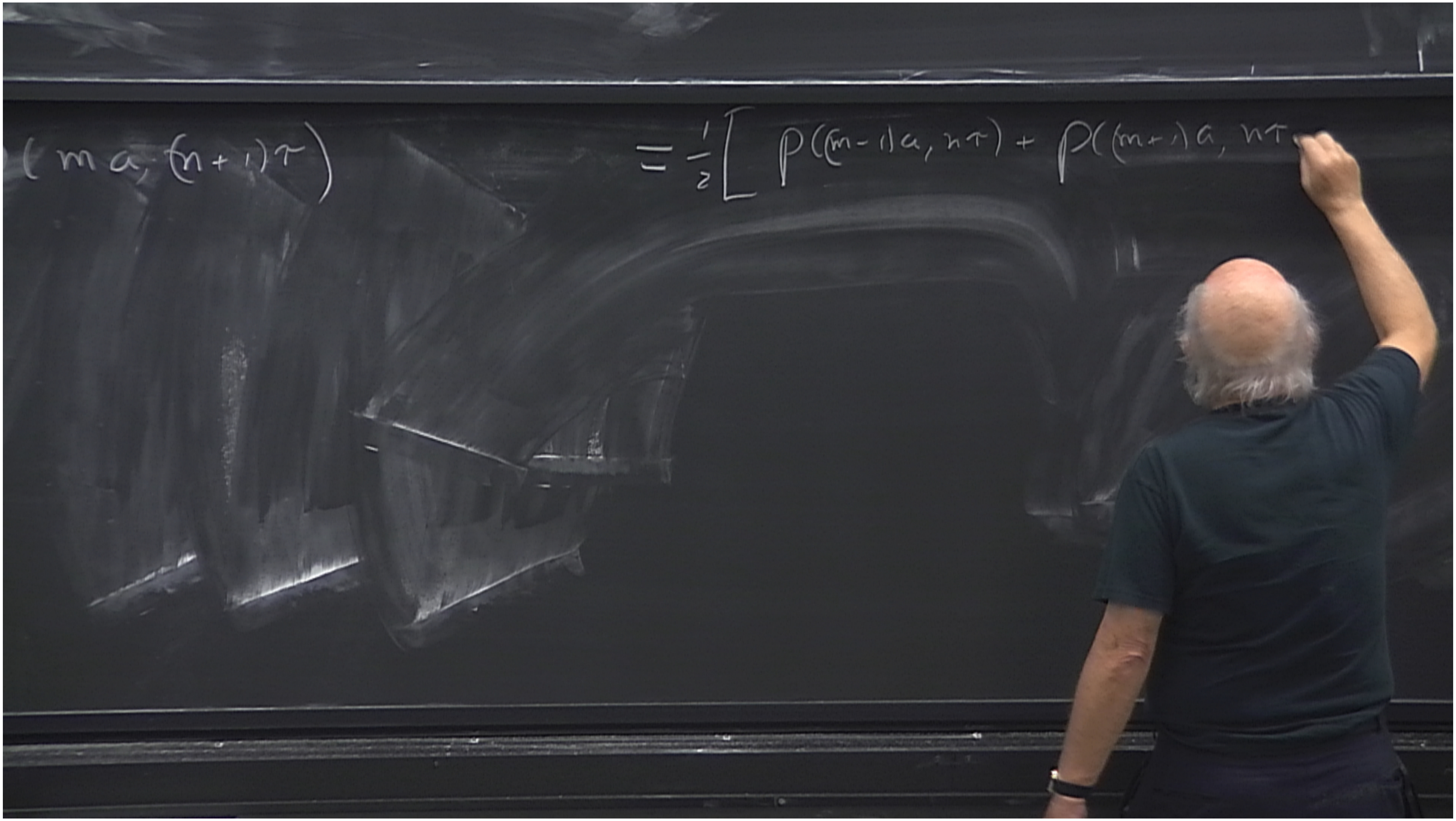
$$\frac{d}{dt} Q(t) = 0 \quad \text{here } Q = \text{total prob} = \int dx\, \rho(\underline{x}, t) = 1$$

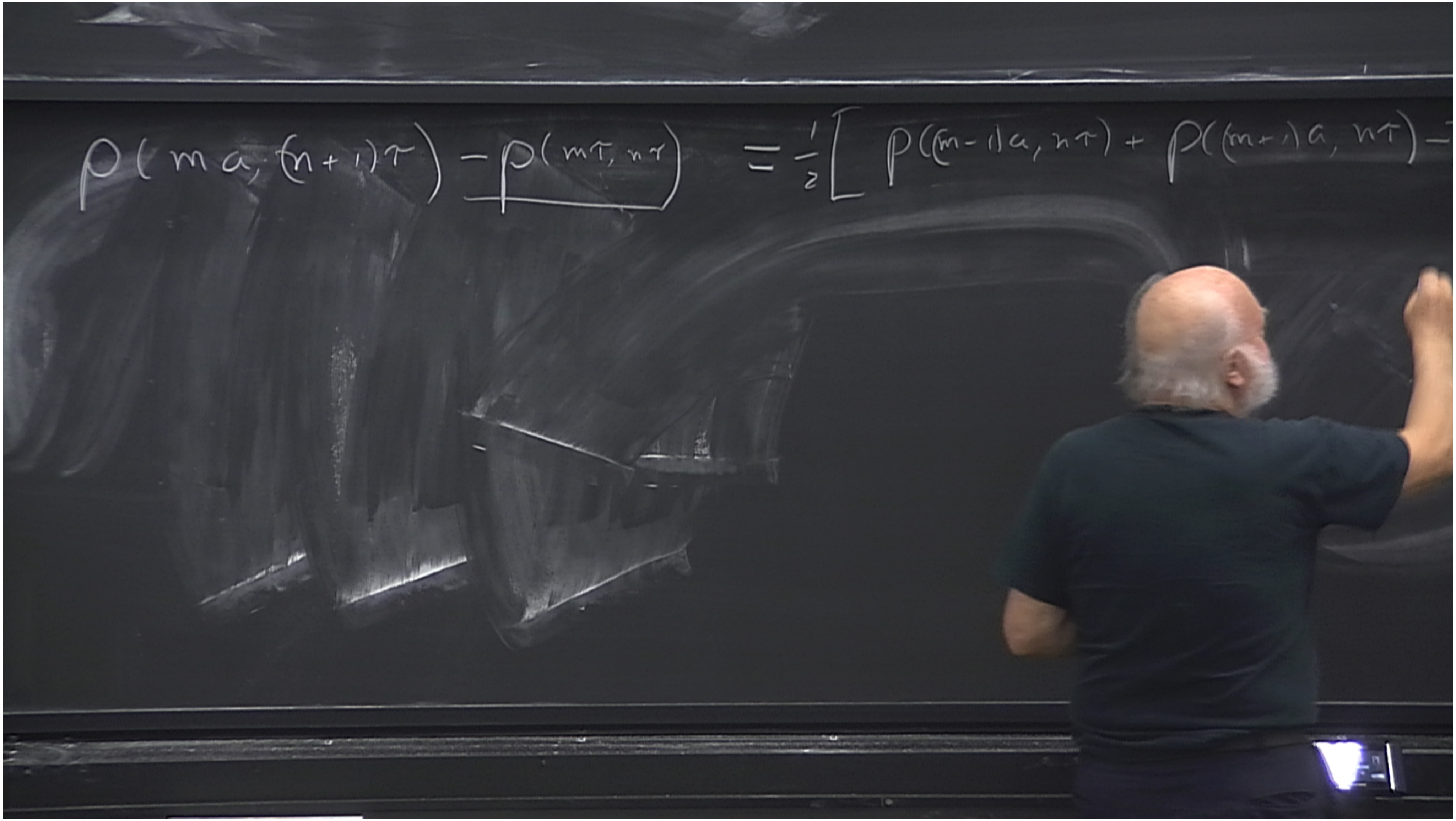
$$\frac{\partial \rho(\vec{r},t)}{\partial t} + \nabla \cdot \vec{j}(\vec{r},t) = 0$$

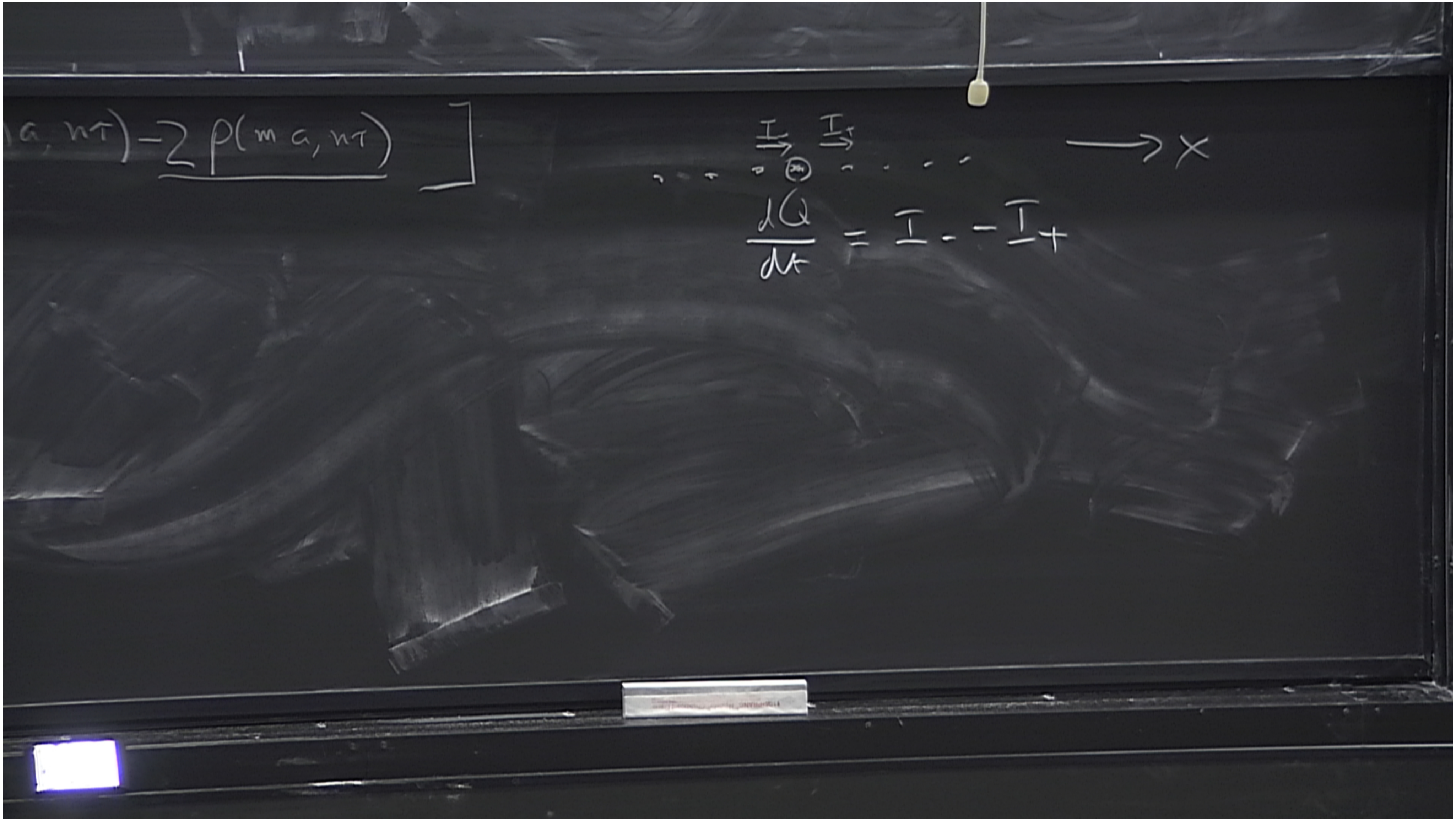
$\rho(x = r$

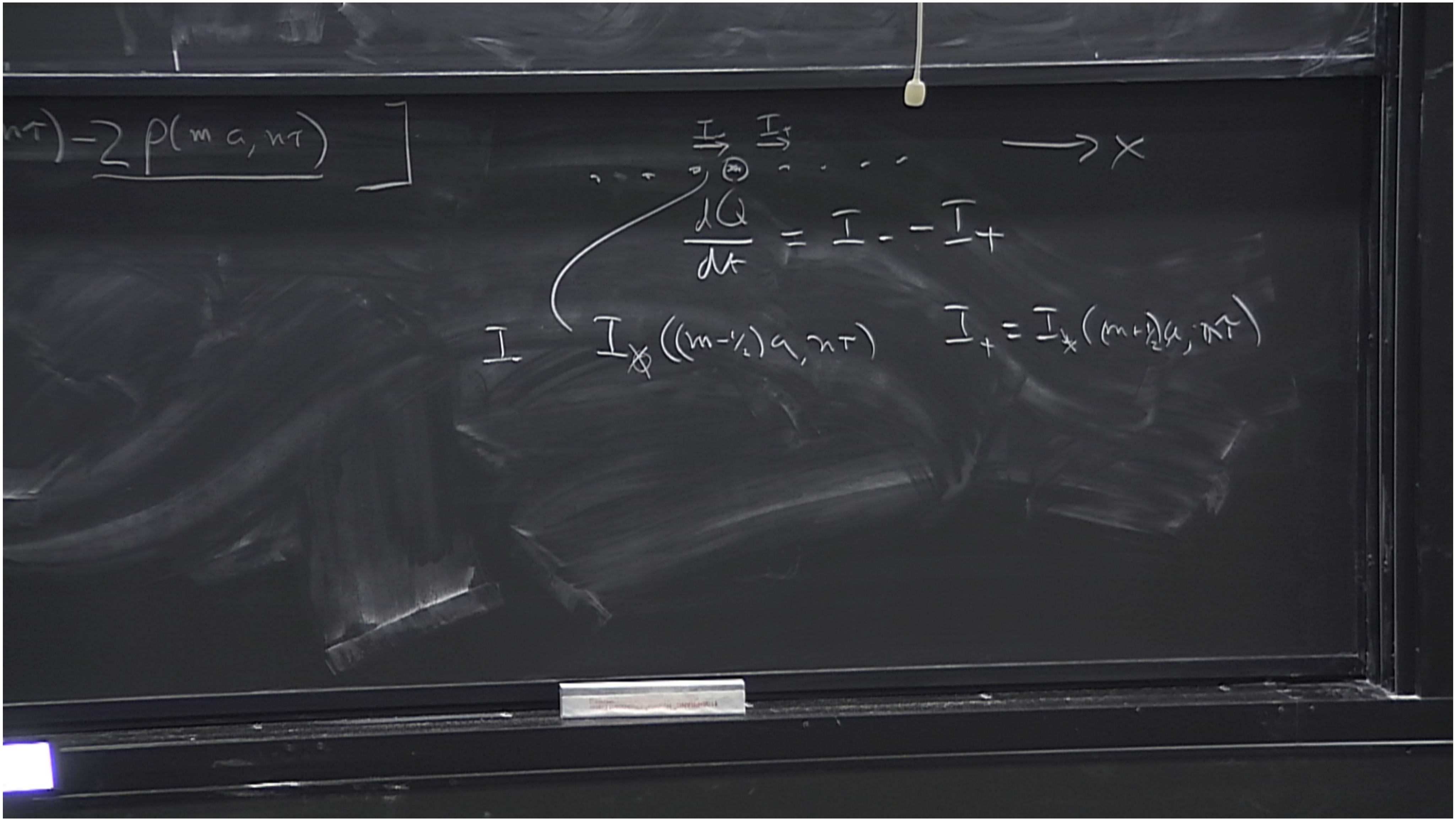

Conservation laws
d/dt Q(t) = 0
here Q = total prob = ∫ dx ρ(x, t) = 1
∂ρ(r, t)/∂t + ∇·j(r, t) = 0
ρ(x = ma, nτ)


here Q = total prob = ∫ dx ρ(x, t) = 1
ρ(x = m a, nτ)
ρ(m a, (n+1)τ)


(ma, (n+1)τ)
= 1/2 [P((m-1)a, nτ) + P((m+1)a, nτ


$\rho(ma,(n+1)\tau) - \rho(m\tau, n\tau) = \frac{1}{2}[\rho((m-1)a, n\tau) + \rho((m+1)a, n\tau) -$


a, nτ) − 2 ρ(m a, nτ)]
I₋ I₊
→ x
dQ/dt = I₋ − I₊


nτ) − Σ ρ(m a, nτ)]
I₋ I₊
→ x
dQ/dt = I₋ − I₊
I₋ = I_x((m−½)a, nτ)
I₊ = I_x((m+½)a, nτ)

$$\rho(ma, (n+1)\tau) - \rho(m\tau, n\tau) = \frac{1}{2}\Big[\rho((m-1)a, n\tau) + \rho((m+1)a,$$

$$\delta\rho(ma, n\tau) = I_{-}(\qquad) - I_{+}$$

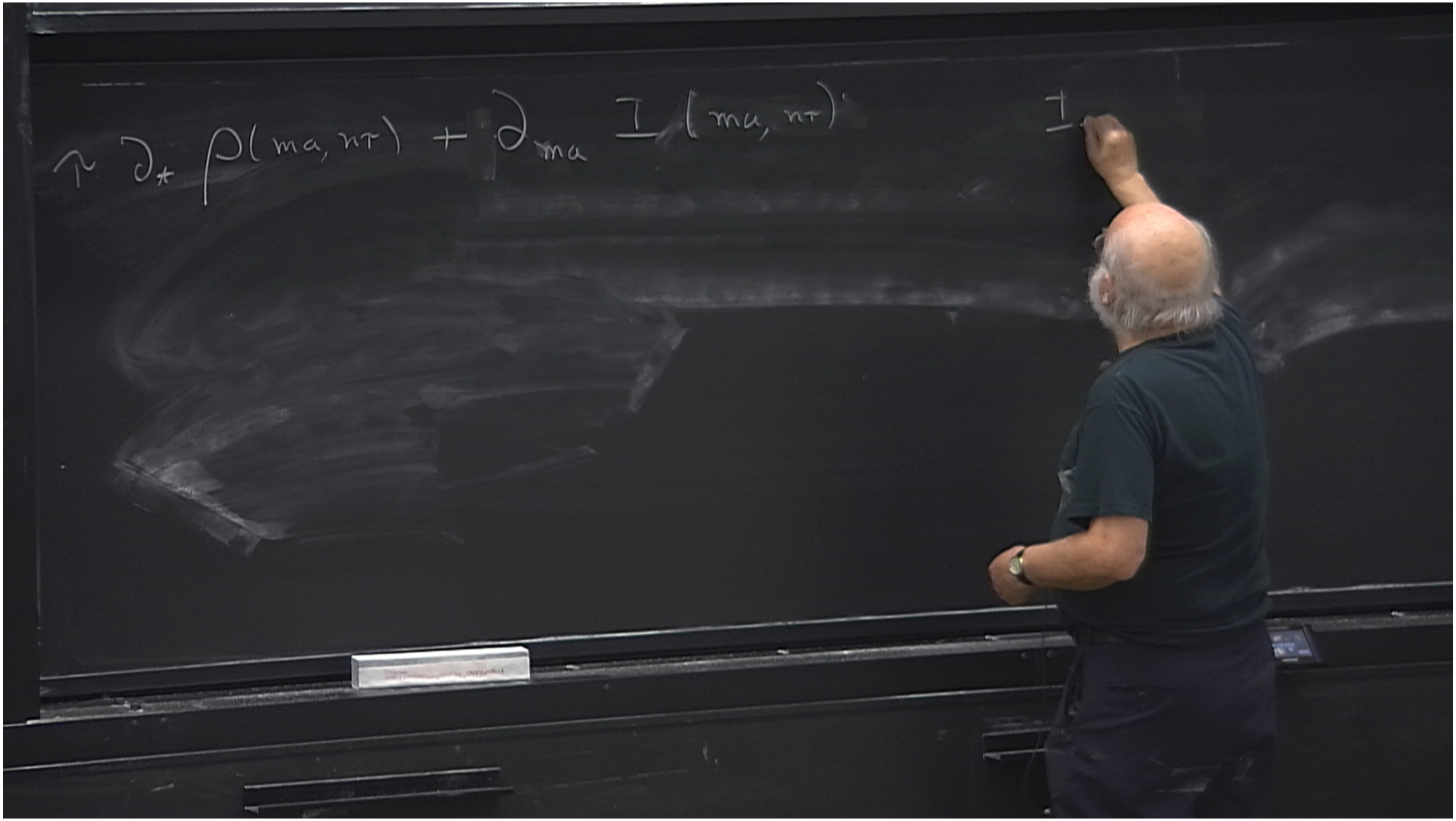
$$I((m-\tfrac{1}{2})a, n\tau) = \frac{1}{2}\Big[\rho((m-1)a, n\tau) -$$

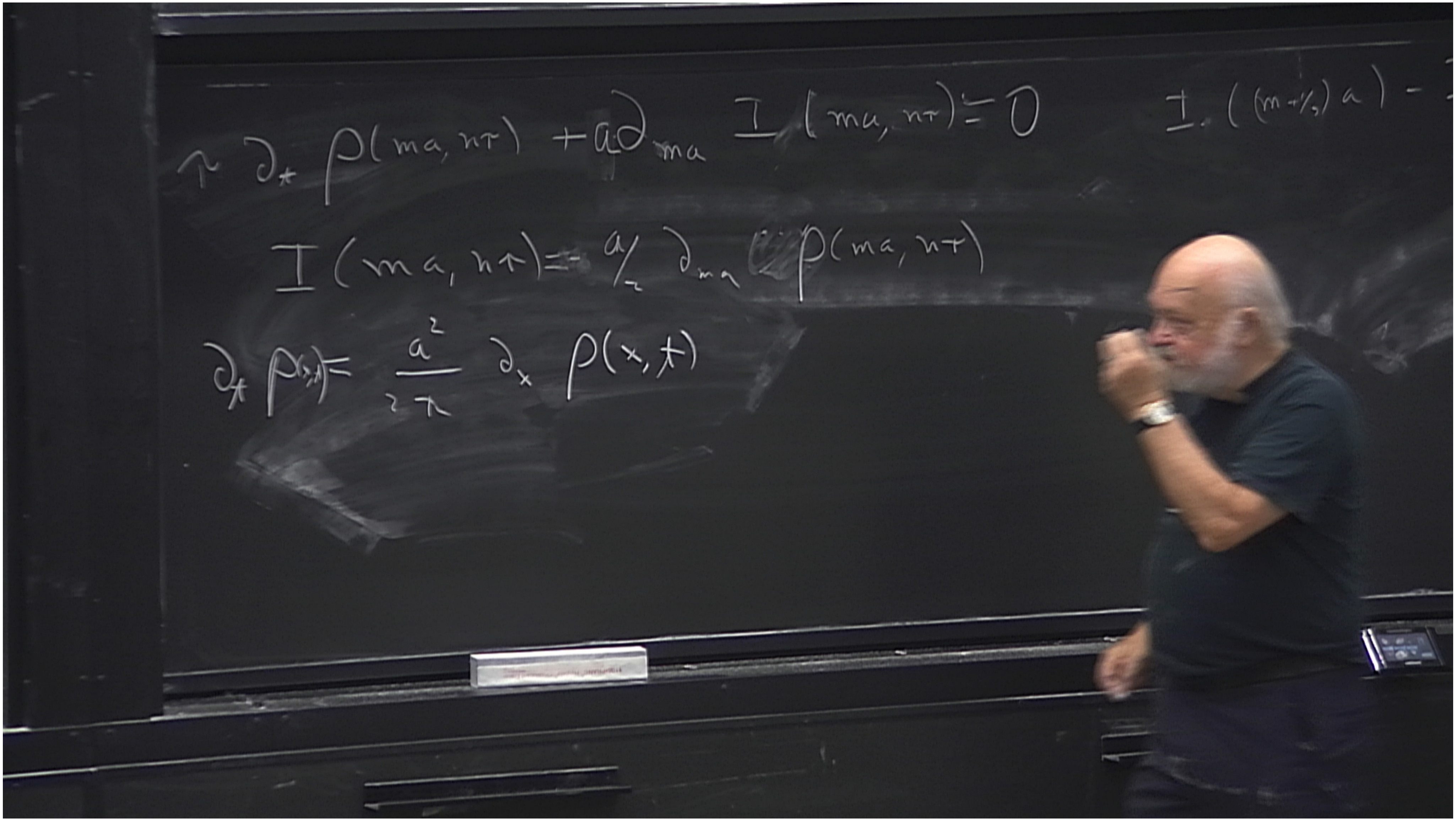
$$\rho(ma,(n+1)\tau) - \rho(m\tau, n\tau) = \frac{1}{2}\Big[\rho((m-1)a, n\tau) + \rho((m+1)a, n\tau$$

$$\delta\rho(ma, n\tau) = I_-(\qquad) - I_+$$

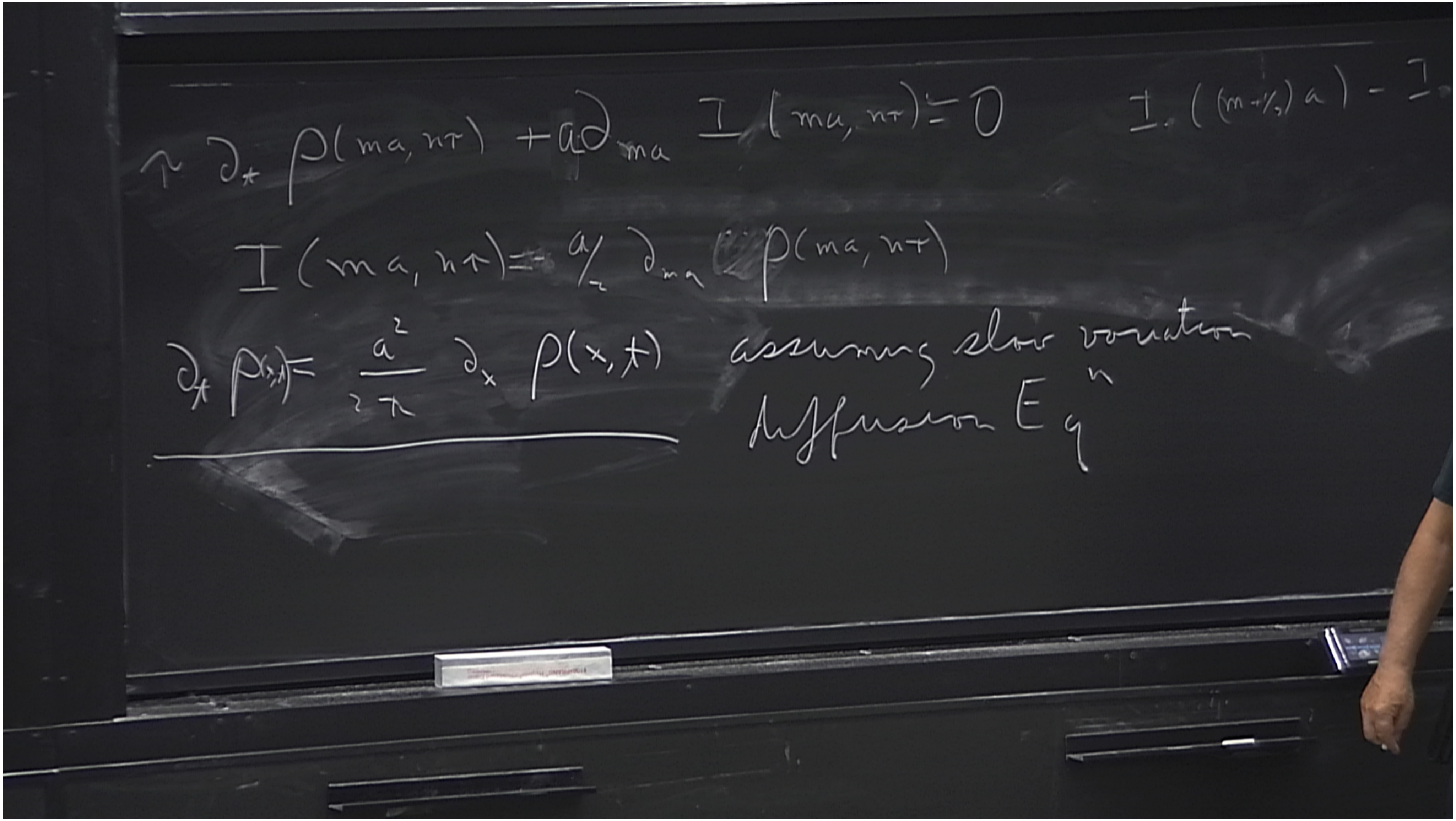
$$I_- = I((m-\tfrac{1}{2})a, n\tau) = \frac{1}{2}\Big[\rho((m-1)a, n\tau) - \rho(ma, n\tau)\Big]$$

$$I_+ = I((m+\tfrac{1}{2})a, n\tau) = \frac{1}{2}\Big[\big(\rho(ma, n\tau) - \rho((m+1)a, n\tau)\big)\Big]$$

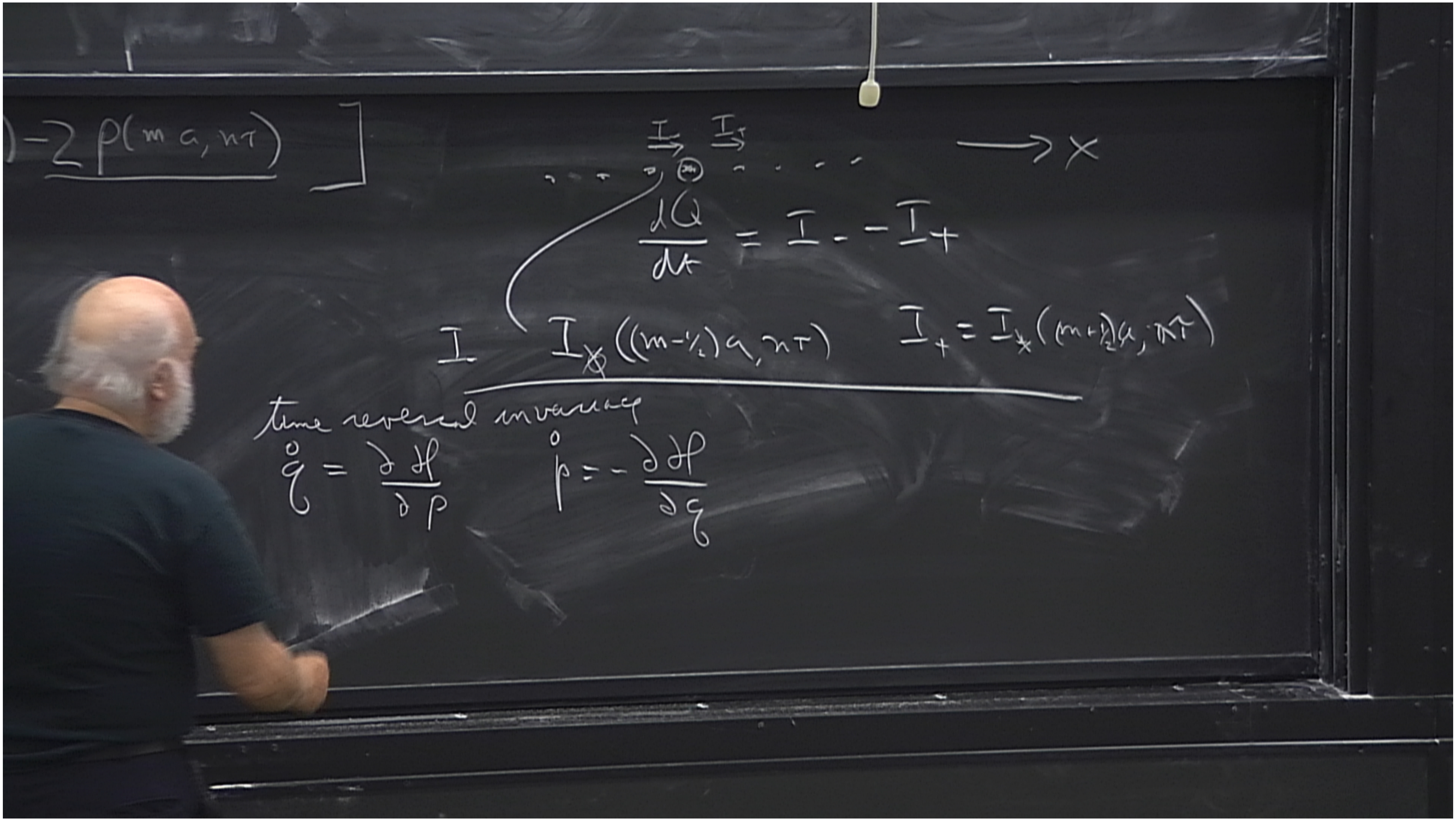

$\tau\,\partial_t\,\rho(ma, n\tau) + \partial_{ma}\,\underline{I}\,(ma, n\tau)$ $\underline{I}$



$\tau \partial_t \rho(ma, n\tau) + a\partial_{ma} I(ma, n\tau) = 0$
$I((m+1/2)a) -$
$I(ma, n\tau) = -a/2 \, \partial_{ma} \rho(ma, n\tau)$
$\partial_t \rho(x,t) = \frac{a^2}{2\tau} \partial_x \rho(x,t)$



$\tau \partial_t \rho(ma, n\tau) + a\partial_{ma} I(ma, n\tau) = 0$
$I((m+1/2)a) - I$
$I(ma, n\tau) = -a/2 \, \partial_{ma} \rho(ma, n\tau)$
$\partial_t \rho(x,t) = \frac{a^2}{2\tau} \partial_x \rho(x,t)$
assuming slow variation
Diffusion Eq^n


)−2ρ(ma, nτ)
→ x
dQ/dt = I₋ − I₊
I₋ = I_*((m−½)a, nτ)
I₊ = I_*((m+½)a, nτ)
time reversal invariance
q̇ = ∂H/∂p
ṗ = −∂H/∂q

$n\tau) - \sum \rho(ma, n\tau)$]

$\underrightarrow{I_-}$ $\underrightarrow{I_+}$ $\longrightarrow x$

$$\frac{dQ}{dt} = I_- - I_+$$

$I_- = I_x((m-\tfrac{1}{2})a, n\tau)$ $\quad$ $I_+ = I_x((m+\tfrac{1}{2})a, n\tau)$

time reversal invariance

$$\dot{q} = \frac{\partial H}{\partial p} \qquad \dot{p} = -\frac{\partial H}{\partial q}$$

$$\dot{q} \to -\dot{q} \qquad p \to -p$$

$$e^{-iHt/\hbar}$$

$n\tau) - 2\rho(ma, n\tau) \Big]$

$\overset{I_-}{\Rightarrow} \; \overset{I_+}{\Rightarrow}$

$\cdots \; \circ \; \cdots \longrightarrow x$

$$\frac{dQ}{dt} = I_- - I_+$$

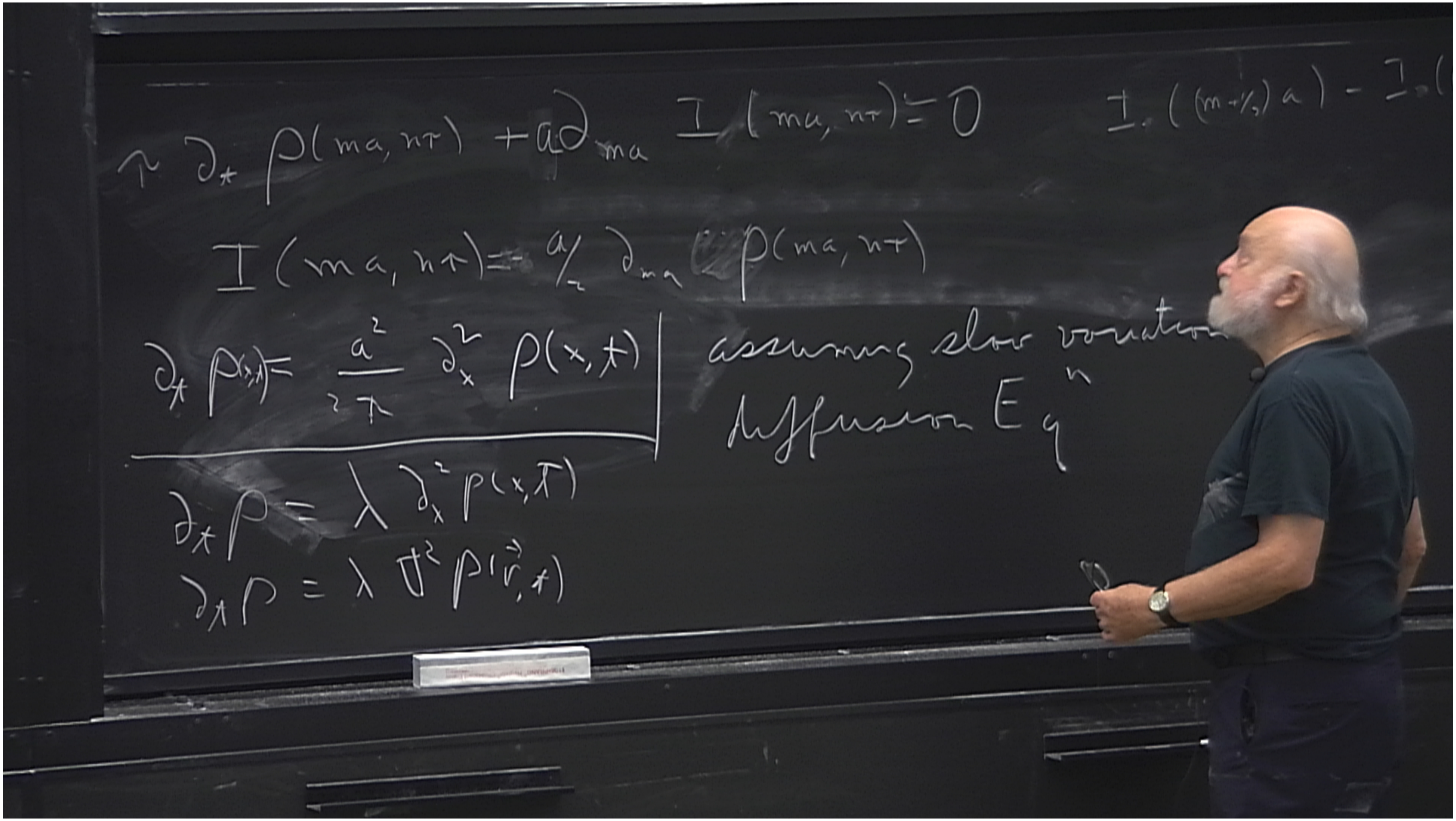
$I_- \quad I_\phi((m-\tfrac{1}{2})a, n\tau) \qquad I_+ = I_\phi((m+\tfrac{1}{2})a, n\tau)$

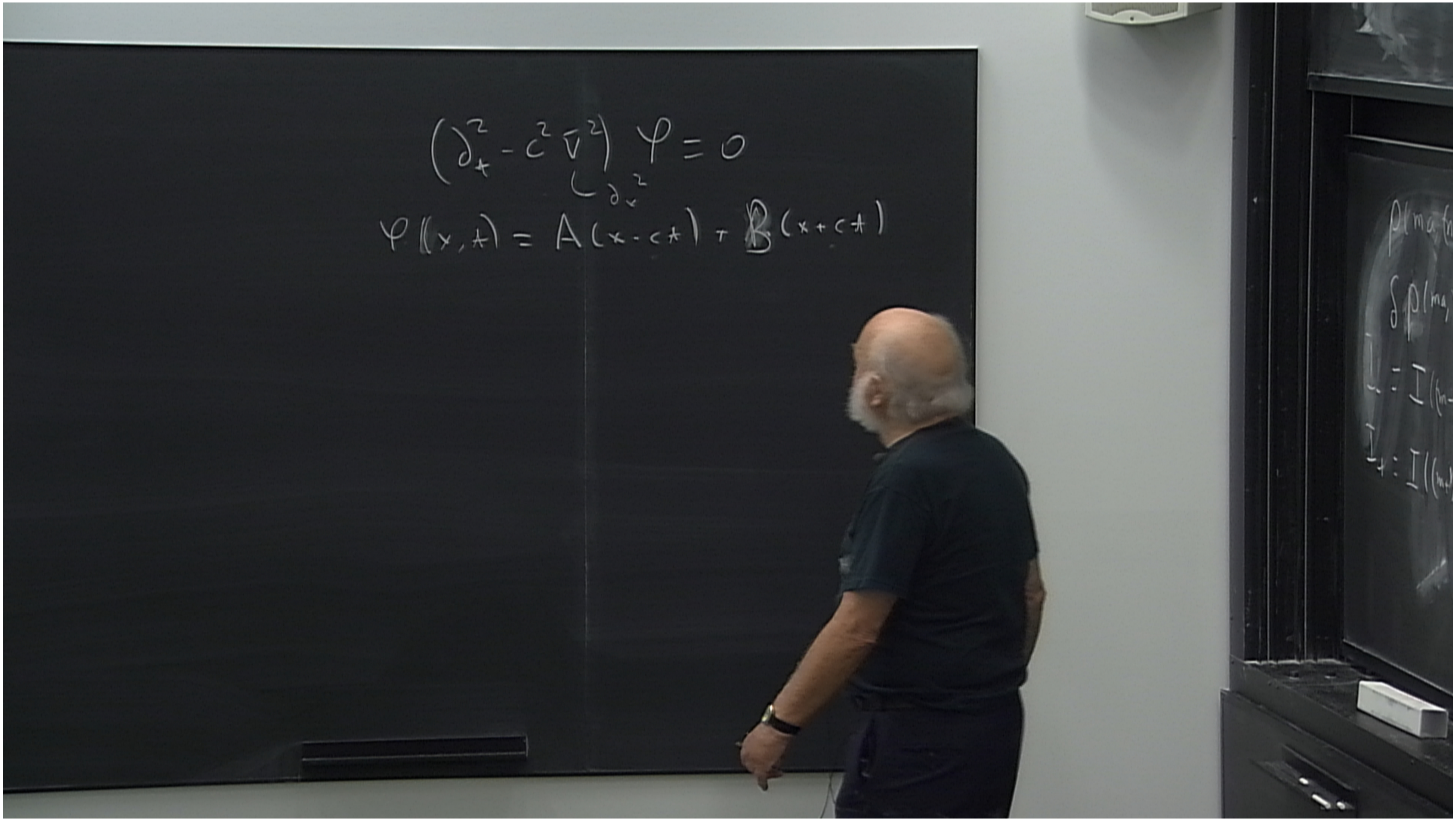
time reversal invariance

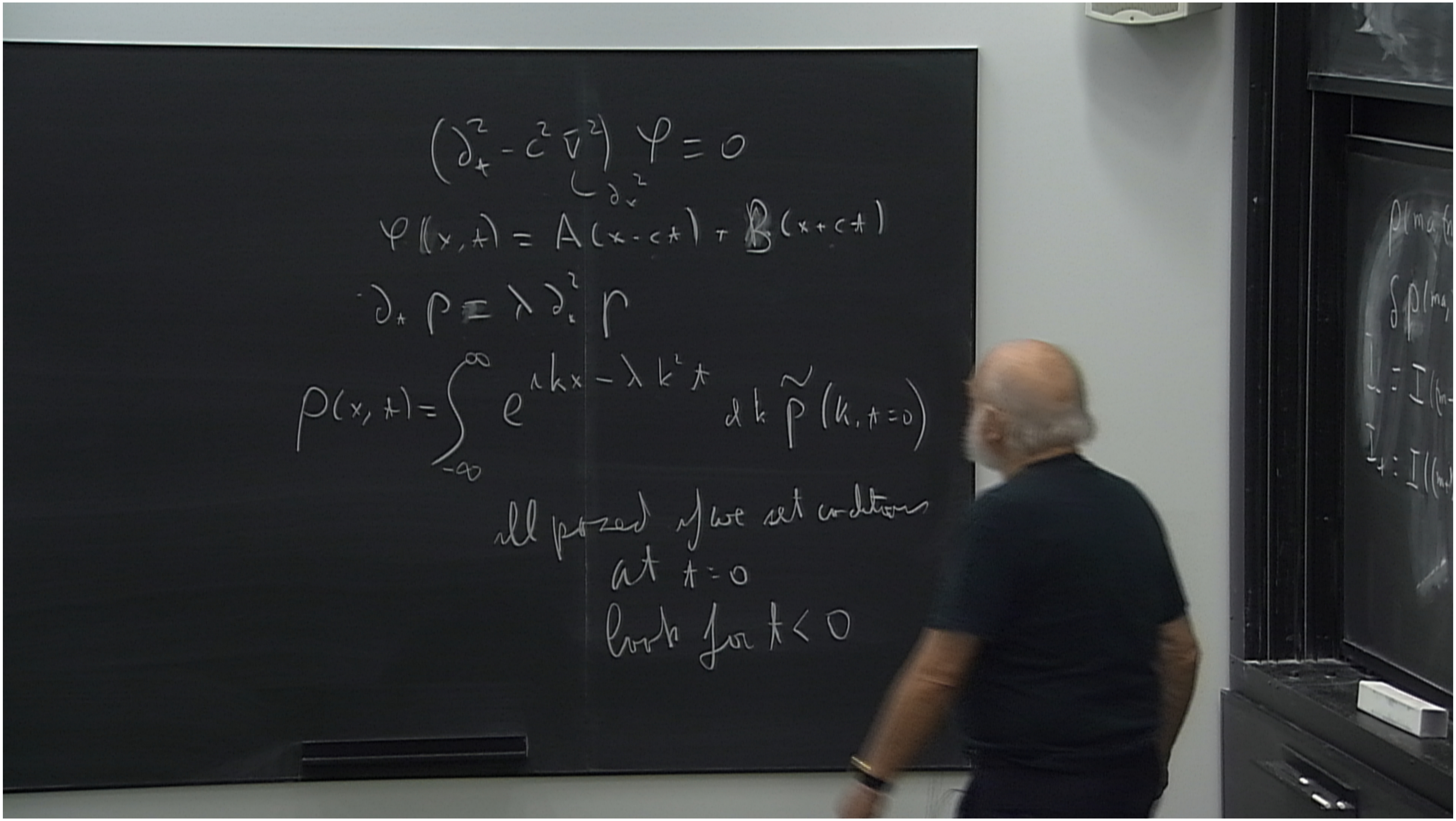
$$\dot{q} = \frac{\partial H}{\partial p} \qquad \dot{p} = -\frac{\partial H}{\partial q}$$

$$\dot{q} \to -\dot{q} \qquad p \to -p$$

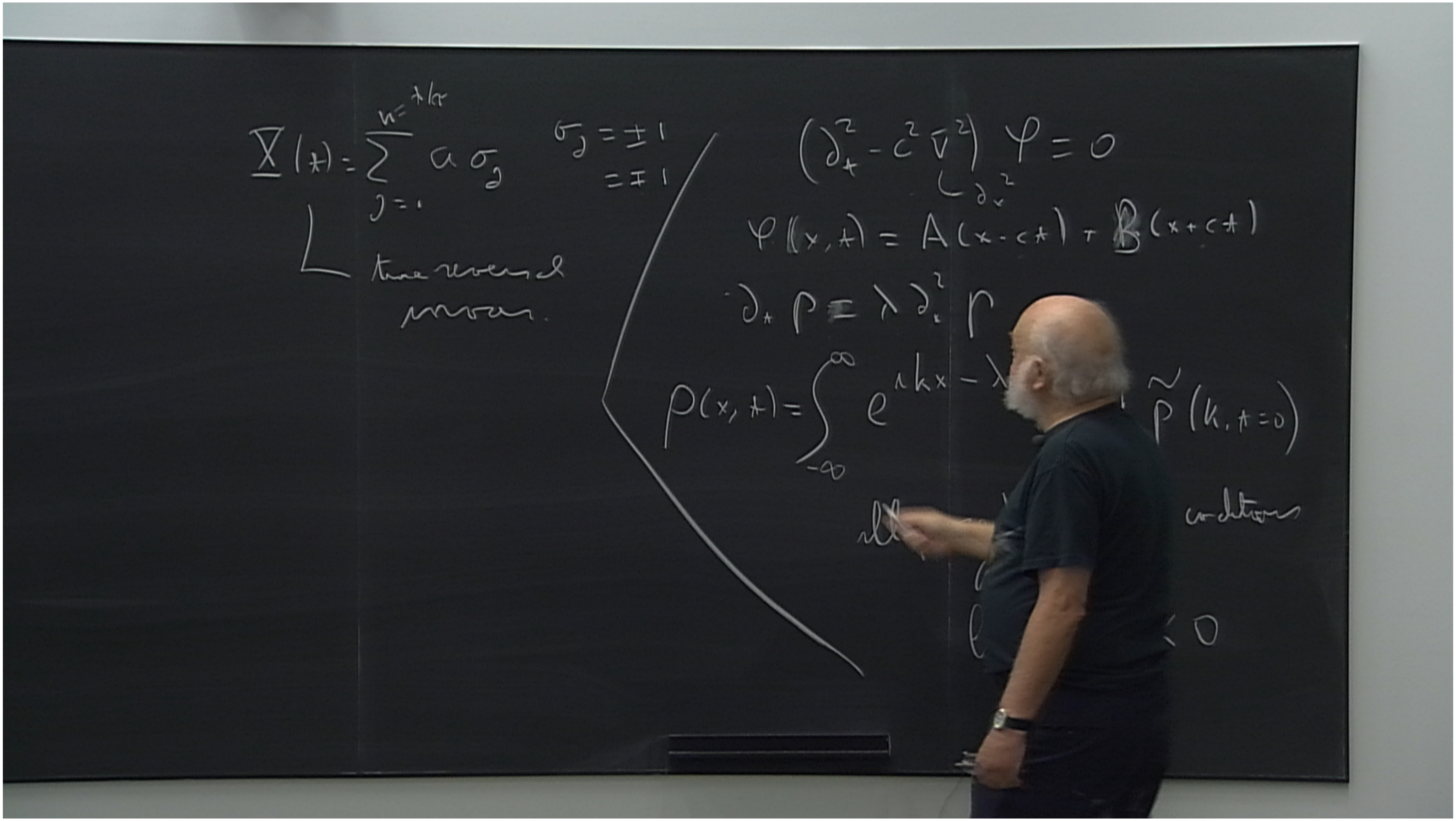
$e^{-i\bar{H}t/\hbar}$ Quantum

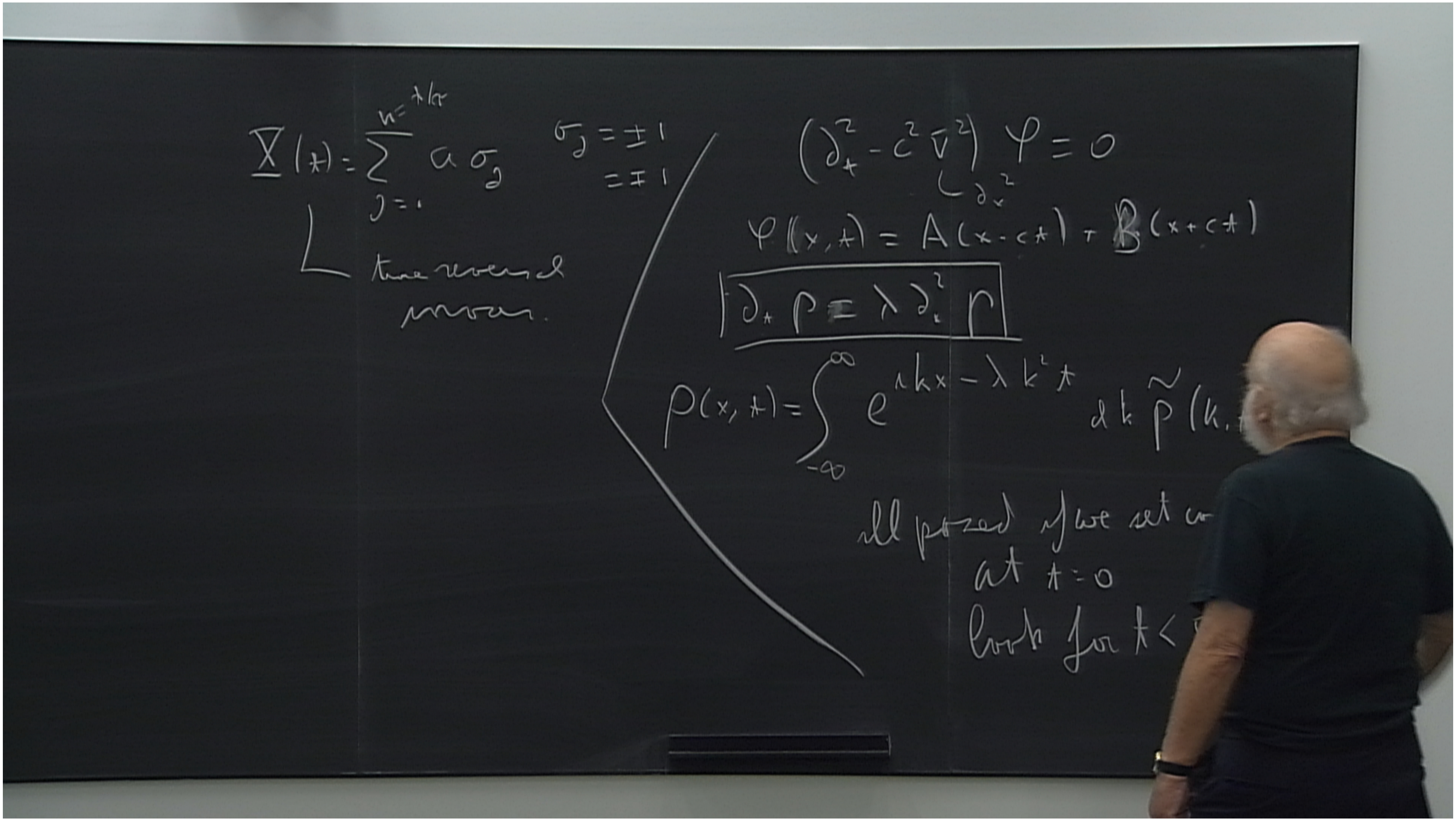

$\tau\, \partial_t \rho(ma, n\tau) + a\partial_{ma} I(ma, n\tau) = 0$
$I((m+1/2)a) - I($
$I(ma, n\tau) = -\frac{a}{2}\partial_{ma}\rho(ma, n\tau)$
$\partial_t \rho(x,t) = \frac{a^2}{2\tau}\partial_x^2 \rho(x,t)$
assuming slow variation
Diffusion Eq^n
$\partial_t \rho = \lambda \partial_x^2 \rho(x,t)$
$\partial_t \rho = \lambda \nabla^2 \rho(\vec{r},t)$

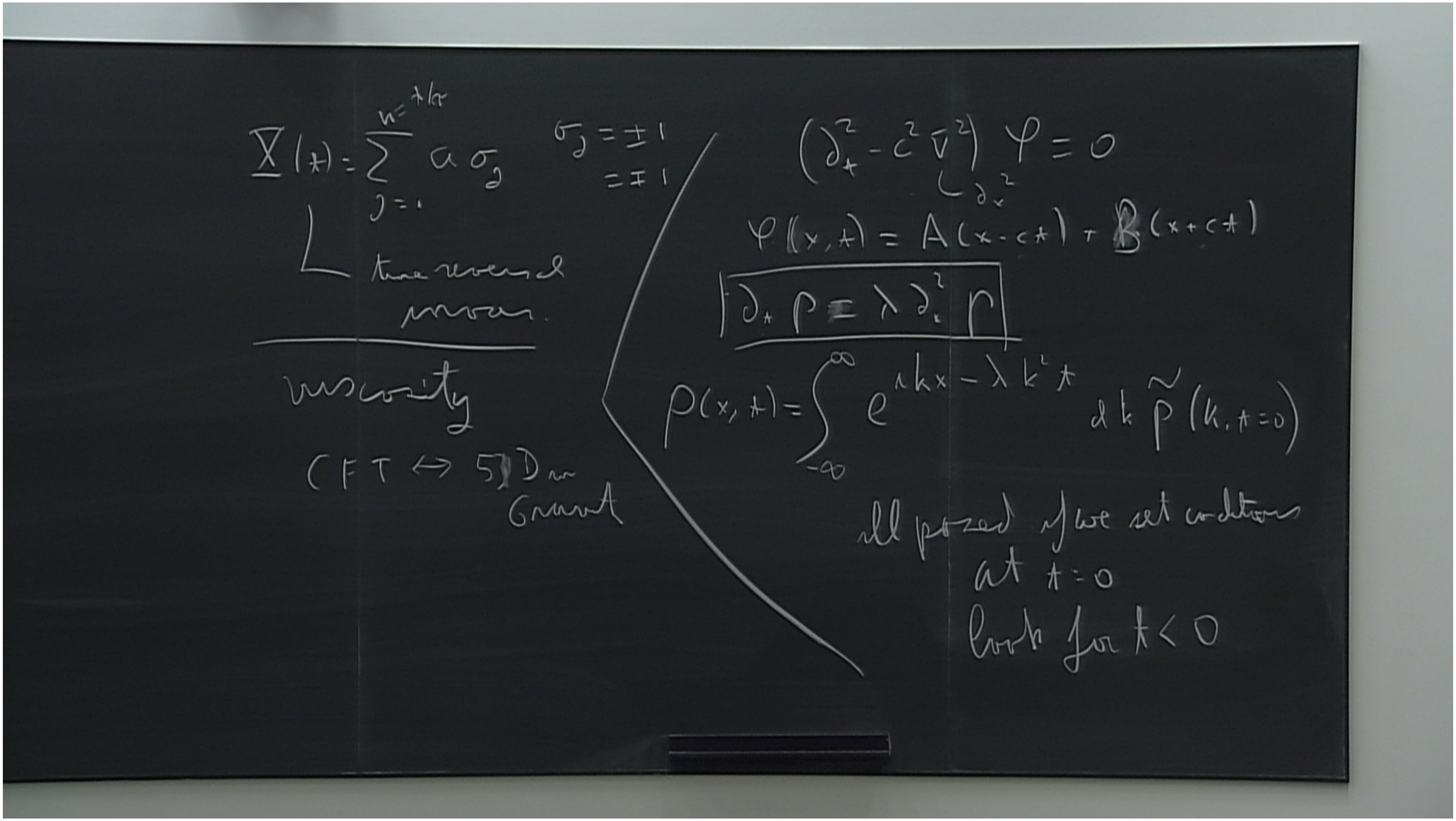

$(\partial_t^2 - c^2\nabla^2)\varphi = 0$
∂_x^2
$\varphi(x,t) = A(x-ct) + B(x+ct)$

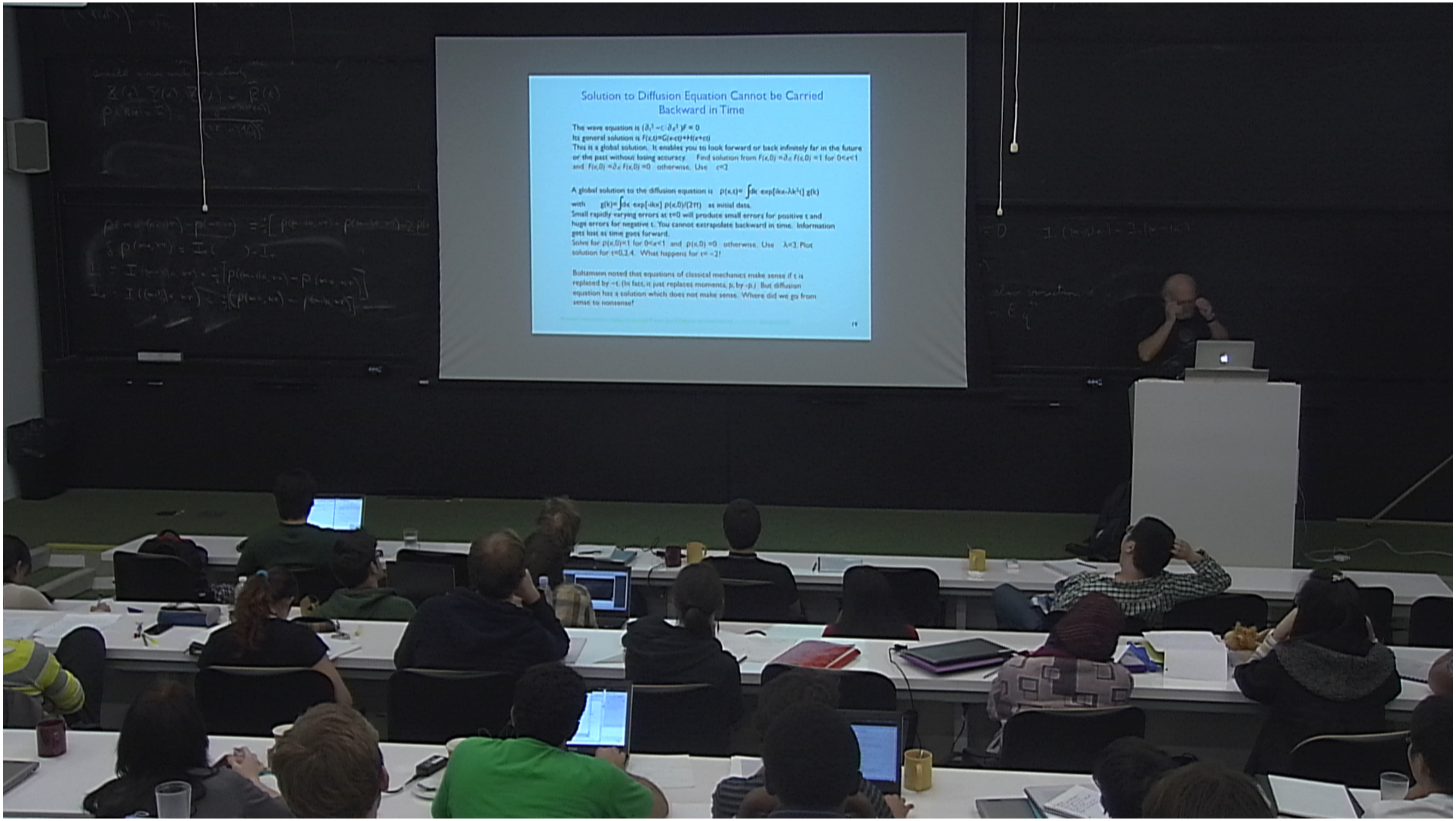


$(\partial_t^2 - c^2 \nabla^2)\,\varphi = 0$
∂_x^2
$\varphi(x,t) = A(x-ct) + B(x+ct)$
$\partial_t \rho = \lambda \partial_x^2 \rho$
$\rho(x,t) = \int_{-\infty}^{\infty} e^{ikx - \lambda k^2 t}\, dk\, \tilde{\rho}(k, t=0)$
ill posed if we set conditions
at $t=0$
look for $t<0$


$\Sigma(t) = \sum_{j=1}^{n=t/\Delta t} a\,\sigma_j$
$\sigma_j = \pm 1$
$= \mp 1$
time reversal
$(\partial_t^2 - c^2\nabla^2)\,\varphi = 0$
∂_x^2
$\varphi(x,t) = A(x-ct) + B(x+ct)$
$\partial_t \rho = \lambda\, \partial_x^2 \rho$
$\rho(x,t) = \int_{-\infty}^{\infty} e^{ikx - \lambda}$
$\tilde{\rho}(k, t=0)$
conditions


$\underline{X}(t) = \sum_{j=1}^{n=t/\tau} a\,\sigma_j$
$\sigma_j = \pm 1$
$= \mp 1$
time reversal
$(\partial_t^2 - c^2\nabla^2)\,\varphi = 0$
∂_x^2
$\varphi(x,t) = A(x-ct) + B(x+ct)$
$\boxed{\partial_t \rho = \lambda \partial_x^2 \rho}$
$\rho(x,t) = \int_{-\infty}^{\infty} e^{ikx - \lambda k^2 t}\, dk\, \tilde{\rho}$
ill posed if we set
at $t = 0$
look for $t <$


σ = ±1
= ∓1
time reversal
viscosity
CFT ↔ 5D Gravity
φ(x, t) = A(x − ct) + B(x + ct)
ill posed if we set conditions
at t = 0
look for t < 0


Solution to Diffusion Equation Cannot be Carried Backward in Time
The wave equation is $(\partial_t^2 - c^2\partial_x^2)F = 0$
Its general solution is $F(x,t)=G(x-ct)+H(x+ct)$
This is a global solution. It enables you to look forward or back infinitely far in the future or the past without losing accuracy. Find solution from $F(x,0) = \partial_x F(x,0) = 1$ for $0<x<1$ and $F(x,0) = \partial_x F(x,0) = 0$ otherwise. Use $c=2$
A global solution to the diffusion equation is $\rho(x,t)= \int dk\ \exp[ikx-\lambda k^2 t]\ g(k)$
with $g(k)= \int dx\ \exp[-ikx]\ \rho(x,0)/(2\pi)$ as initial data.
Small rapidly varying errors at $t=0$ will produce small errors for positive t and huge errors for negative t. You cannot extrapolate backward in time. Information gets lost as time goes forward.
Solve for $\rho(x,0)=1$ for $0<x<1$ and $\rho(x,0)=0$ otherwise. Use $\lambda=2$. Plot solution for $t=0,2,4$. What happens for $t=-2$!
Boltzmann noted that equations of classical mechanics make sense if t is replaced by −t. (In fact, it just replaces momenta, p, by −p.) But diffusion equation has a solution which does not make sense. Where did we go from sense to nonsense!

Solution to Diffusion Equation Cannot be Carried Backward in Time

The wave equation is $(\partial_t^2 - c^2 \partial_x^2)F = 0$
Its general solution is $F(x,t)=G(x-ct)+H(x+ct)$
This is a global solution. It enables you to look forward or back infinitely far in the future or the past without losing accuracy. Find solution from $F(x,0) = \partial_x F(x,0) = 1$ for [illegible]$<x<1$ and $F(x,0) = \partial_x F(x,0) = 0$ otherwise. Use $c=2$

A global solution to the diffusion equation is $\rho(x,t)= \int dk \; \exp[ikx-\lambda k^2 t] \; g(k)$
with $g(k)= \int dx \; \exp[-ikx] \; \rho(x,0)/(2\pi)$ as initial data.
Small rapidly varying errors at t=0 will produce small errors for positive t and huge errors for negative t. You cannot extrapolate backward in time. Information gets lost as time goes forward.
Solve for $\rho(x,0)=1$ for $0<x<1$ and $\rho(x,0) = 0$ otherwise. Use $\lambda=2$. Plot solution for $t=0,2,4$. What happens for $t= -2$?

Boltzmann noted that equations of classical mechanics make sense if t is replaced by −t. (In fact, it just replaces momenta, *p*, by -*p*.) But diffusion equation has a solution which does not make sense. Where did we go from sense to nonsense?

19

Applications: I. "DLA"

One of the first examples of a physical system being put forward as an algorithm. Question answered "How can you construct a fractal by a natural process?

Hastings and Levitov

10

T.A. Witten and L. Sander Diffusion-limited aggregation, a kinetic phenomenon. Phys. Rev. Lett. 47, 1400, (1981)

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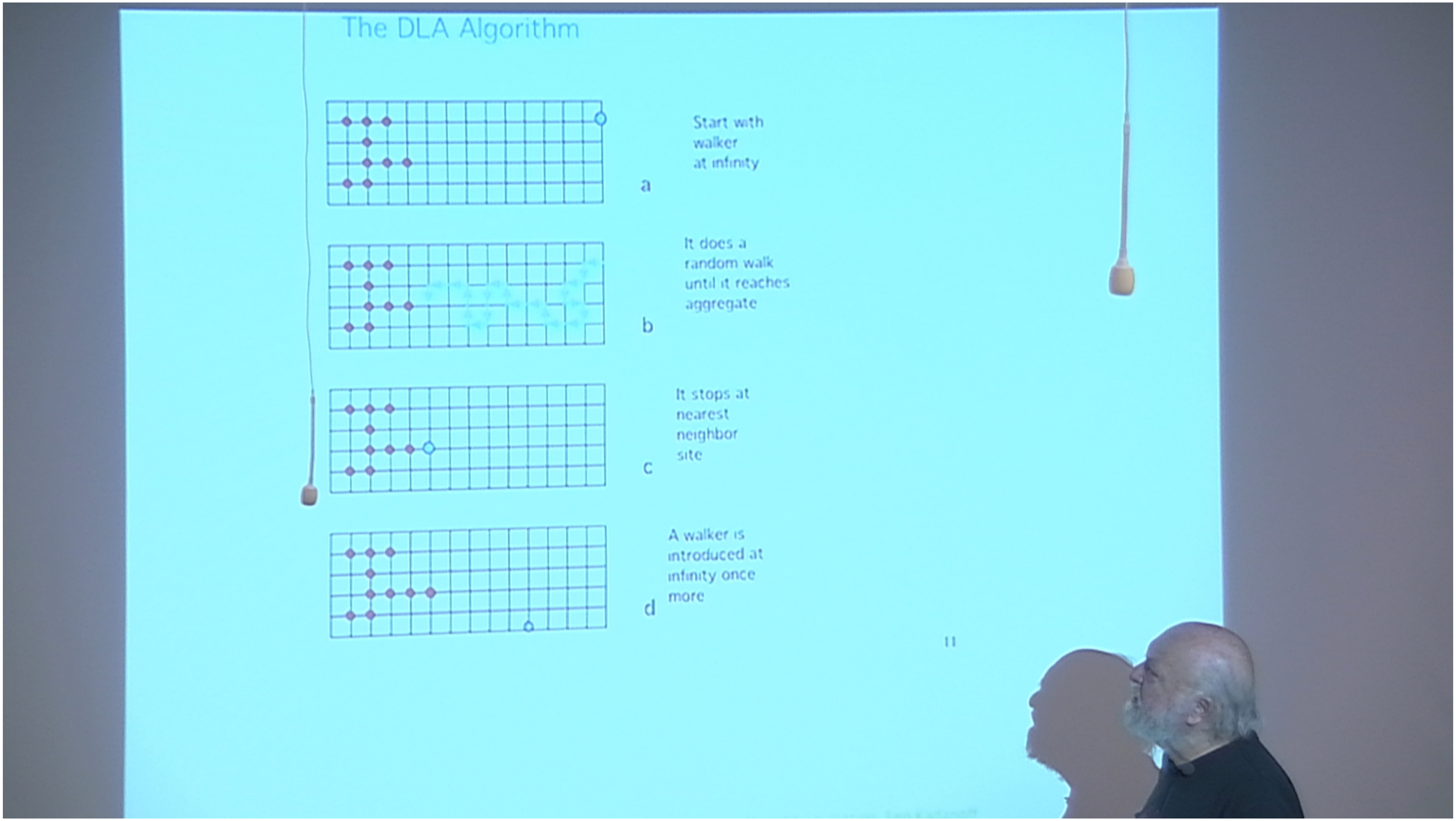

The DLA Algorithm
Start with walker at infinity
a
It does a random walk until it reaches aggregate
b
It stops at nearest neighbor site
c
A walker is introduced at infinity once more
d
11

Fractal… Fractal Dimension

Mandelbrot, B.B. (1982). *The Fractal Geometry of Nature*. W.H. Freeman and Company..
ISBN 0-7167-1186-9
Fractals : By Jens Feder. Plenum Press, 1989

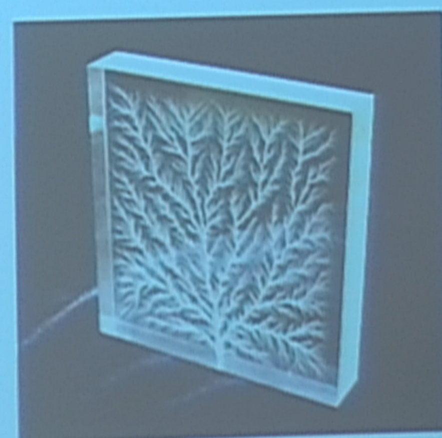
In d-dimensions, a cubic lattice with lattice constant a and side L contains $N=(L/a)^d$ vertices. A DLA cluster of size L contains a number of points which grows as $L^{1.65}$. Hence we say that the fractal dimension of DLA is $d_F=1.65$.

Our understanding of DLA is closely tied to diffusion processes and to electrostatics.
Random walk itself obeys $\nabla^2\rho=0$ with boundary condition $\rho=0$ on object and $\rho=\ln r$ at infinity. Actually it obeys discrete version of this equation:

$$\rho(x+1,y)+\rho(x-1,y)+\rho(x,y+1)+\rho(x+1,y-1)-4\rho(x,y)=0$$

Relative probability of landing is set by value of ρ just beside object, related to discrete normal derivative at surface.

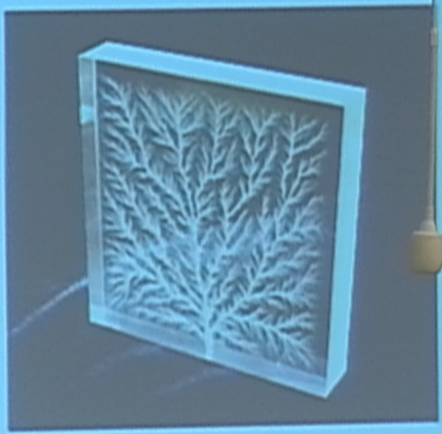
Thus Laplace's equation produces a fractal object. The process is called Laplacian growth.



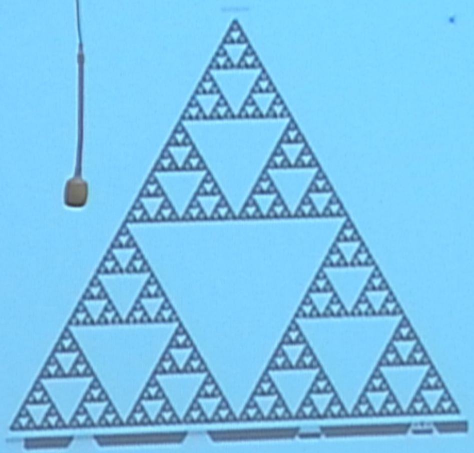
22

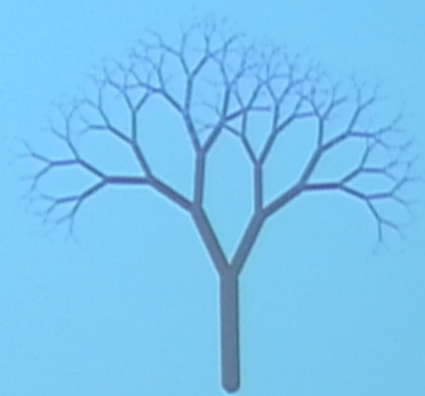
Fractal… Fractal Dimension

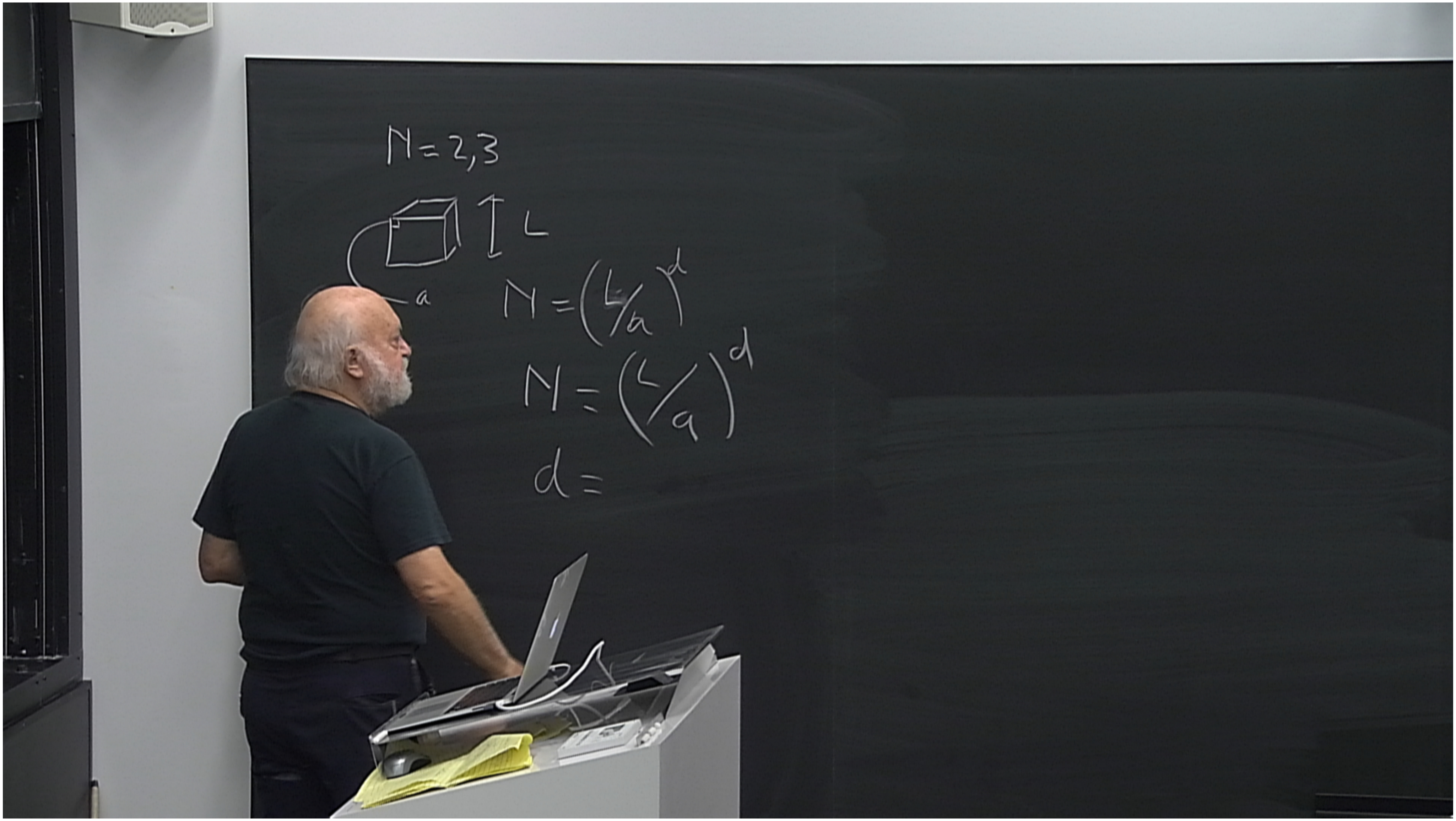
Mandelbrot, B.B. (1982). *The Fractal Geometry of Nature*. W.H. Freeman and Company..
ISBN 0-7167-1186-9
Fractals : By Jens Feder. Plenum Press, 1989

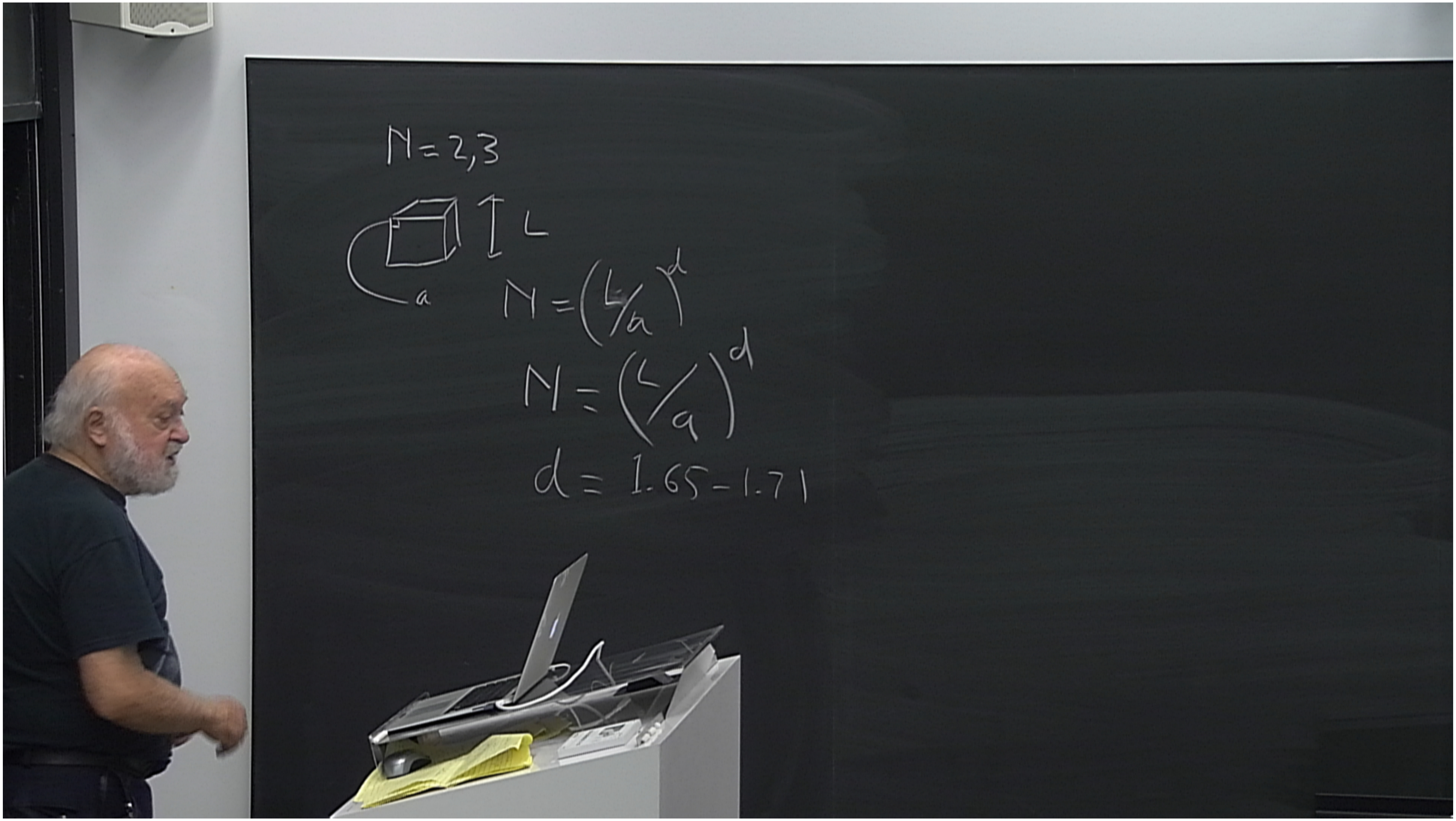


Nature, Math, and Physics are full of objects that can be described as fractals.






N = 2,3
L
a
N = (L/a)^d
N = (L/a)^d
d =


N=2,3
L
a
N=(L/a)^d
N=(L/a)^d
d = 1.65-1.71

Applications II: Markets

Many people assume that the price of a good, like a financial security can be modeled as

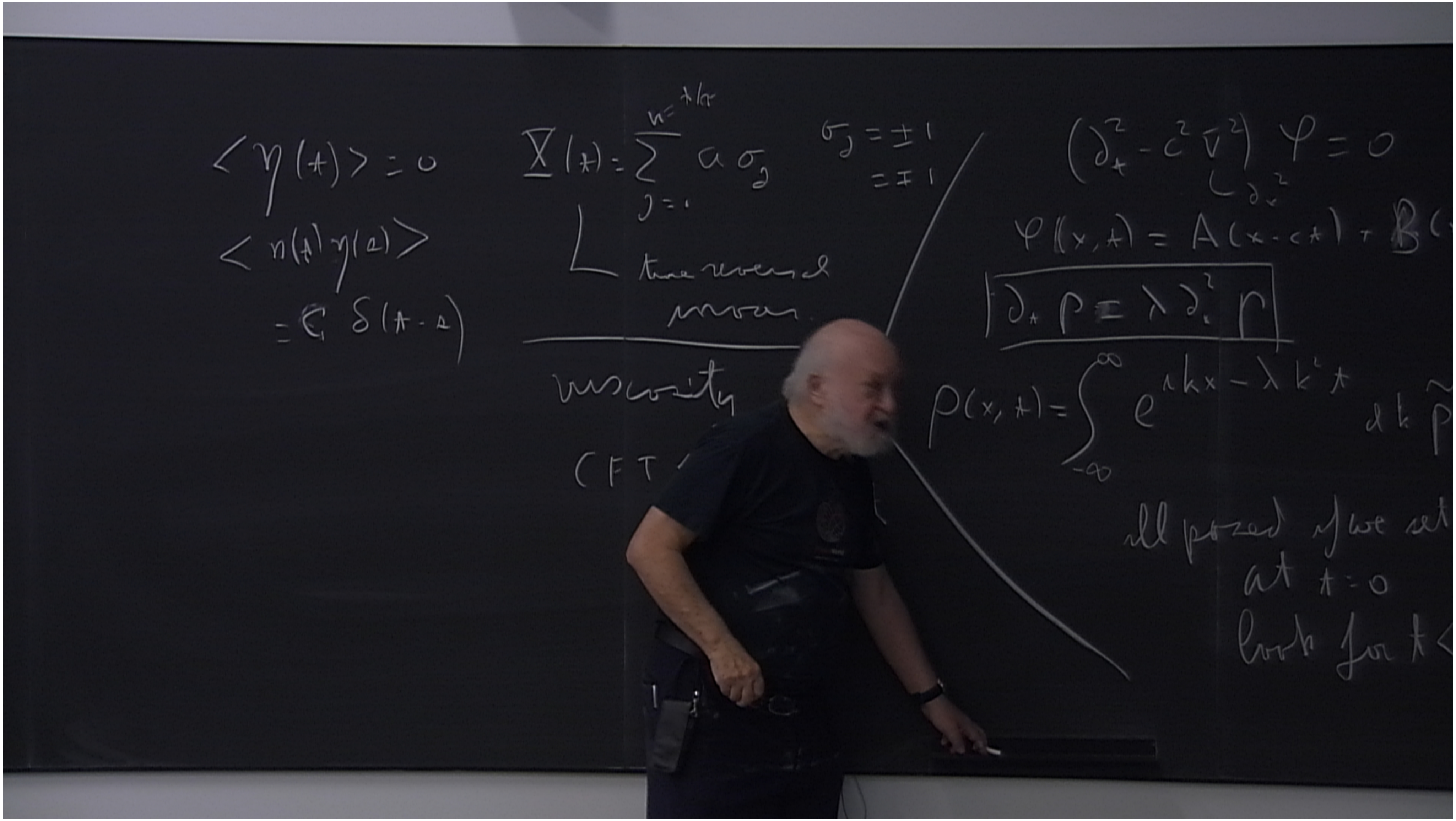
$d\{\ln [p(t)/p(0)]\}/dt = \alpha + \beta\, \eta(t)$, where α and β are constants depending upon the particular good and $\eta(t)$ is a random variable like the one we used for the random walk. Here α describes the average logarithmic growth rate of the good's value, while β describes the corresponding growth in time of a behavior of the log price that is believed to be random. It is called the volitility. One can connect with mathematics by saying that $\eta(t)$ is given as the time derivative of a Wiener process, W(t)….. except that W(t) is not really differentiable. Be that as it may, we know how to deal with $\eta(t)$, by setting average equal to zero and correlation equal to δ function . In particular, we can solve for the price as

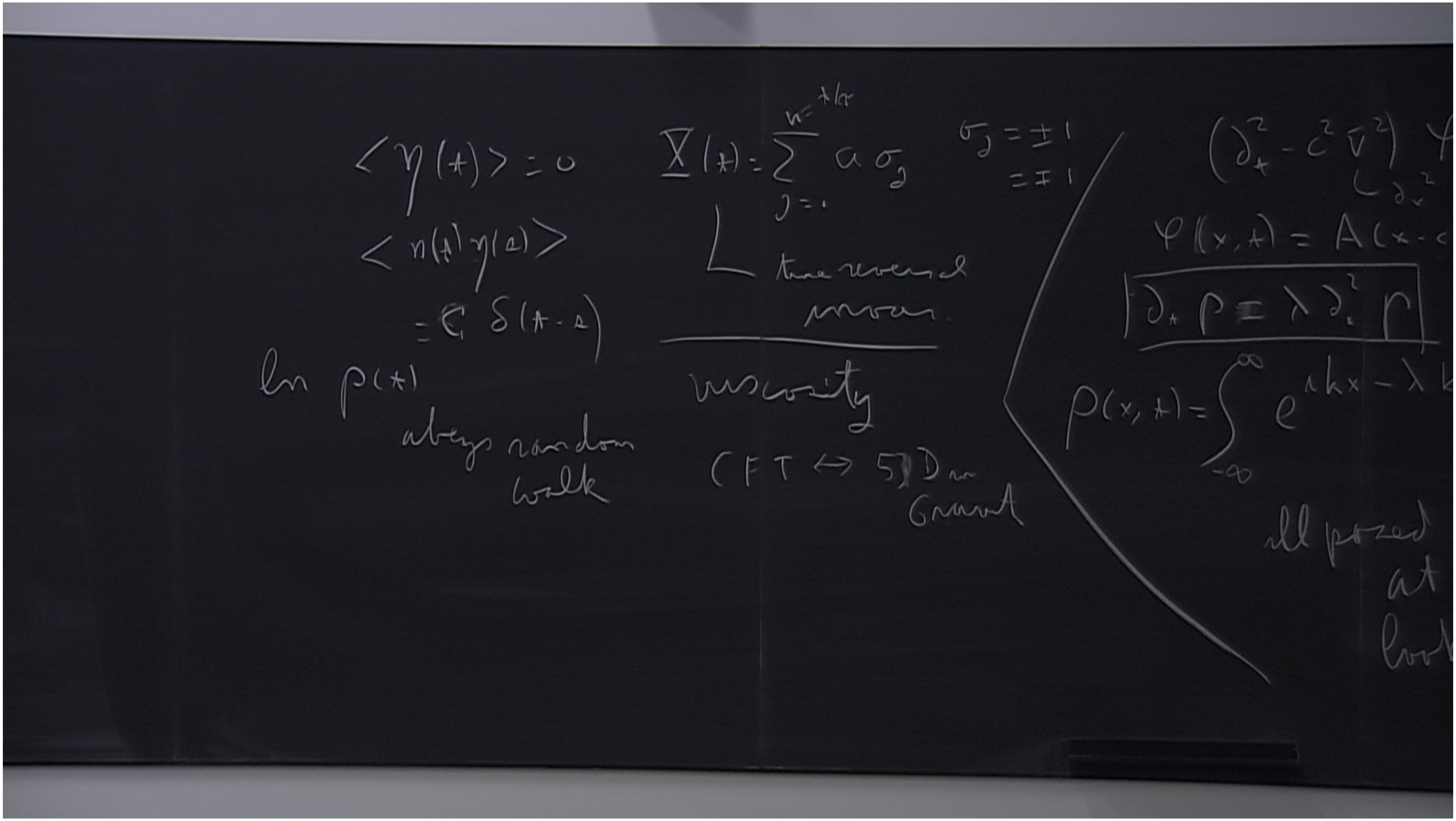
$$\ln[p(t)/p(0)] = \alpha t + \int^t ds\, \eta(s)$$

This is the basis of a theory of pricing of derivative securities called the Black-Scholes theory.

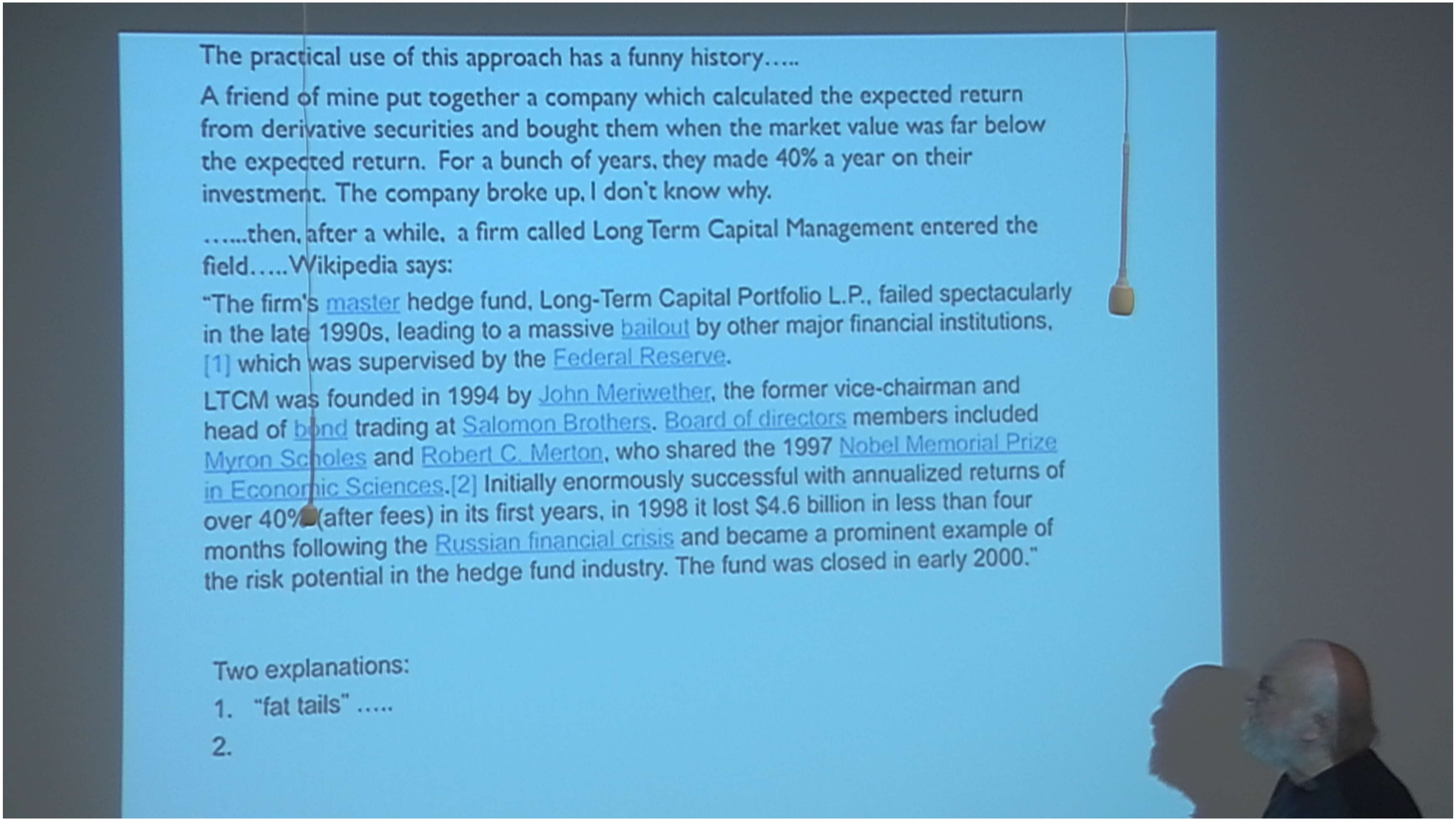
- Black, Fischer; Myron Scholes (1973). "The Pricing of Options and Corporate Liabilities". *Journal of Political Economy* **81** (3): 637–654. doi:10.1086/260062. [1] (Black and Scholes' original paper.)
- Merton, Robert C. (1973). "Theory of Rational Option Pricing". *Bell Journal of Economics and Management Science* (The RAND Corporation) 4 (1): 141–183. doi:10.2307/3003143. [2]
- Hull, John C. (1997). *"Options, Futures, and Other Derivatives"*. Prentice Hall. ISBN ISBN 0-13-601589-1.

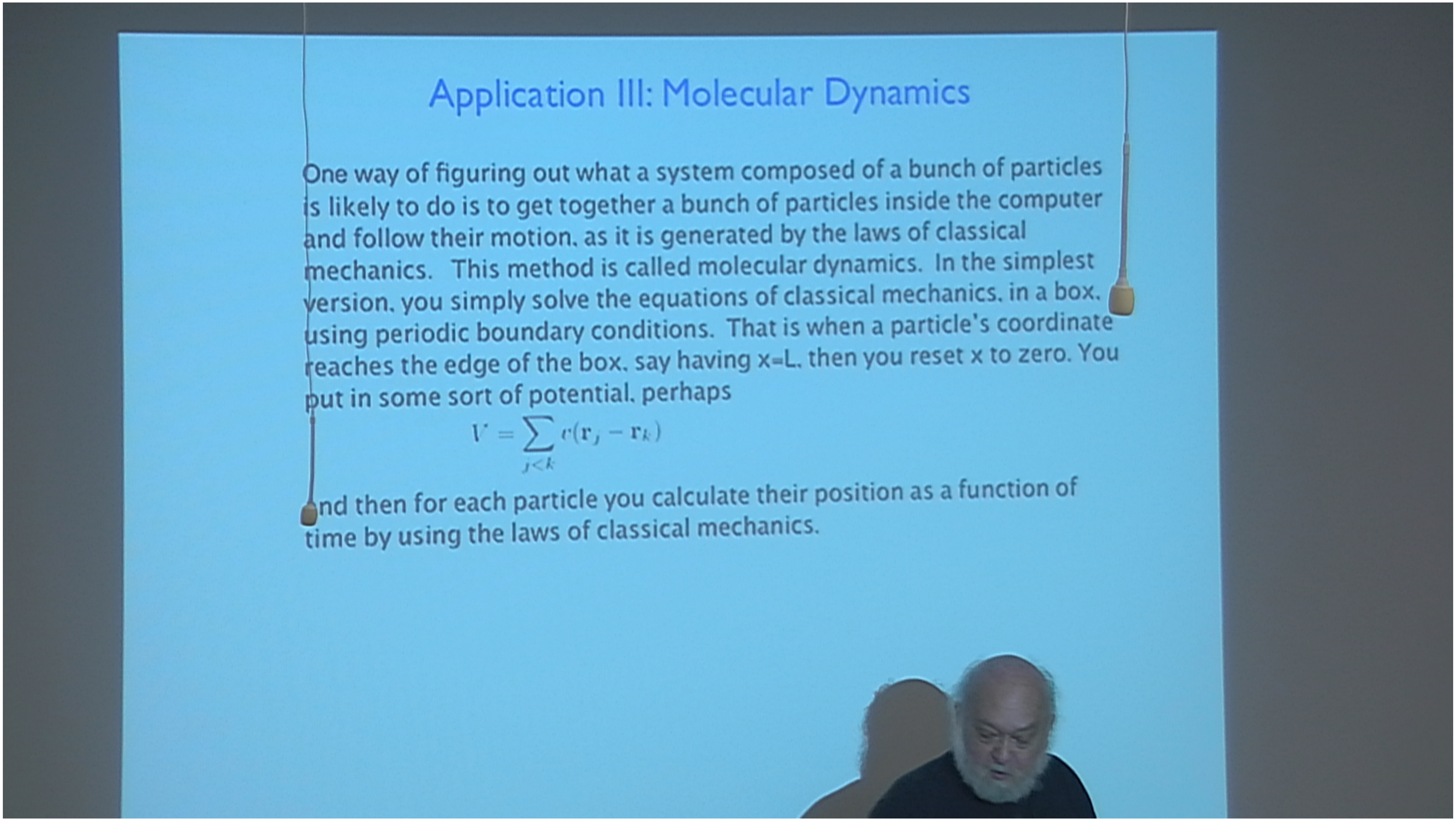
The practical use of this approach has a funny history…..


⟨η(t)⟩ = 0
⟨η(t) η(s)⟩ = ε δ(t − s)
σ_j = ±1
= ∓1
viscosity
(∂_t² − c²∇²) φ = 0
φ(x, t) = A(x − ct) + B(
∂_t ρ = λ ∂_x² ρ
ill posed if we set
at t = 0
look for t <



<η(t)> = 0
<η(t) η(s)>
= C δ(t - s)
ln ρ(t)
always random walk
X(t) = Σ a σ_j
σ_j = ±1
= ∓1
time reversal
viscosity
(CFT ↔ 5D
Gravity
φ(x, t) = A(x - c
∂_t ρ = λ ∂_x² ρ
ρ(x, t) = ∫ e
ill posed
at


The practical use of this approach has a funny history…..
A friend of mine put together a company which calculated the expected return from derivative securities and bought them when the market value was far below the expected return. For a bunch of years, they made 40% a year on their investment. The company broke up, I don't know why.
…...then, after a while, a firm called Long Term Capital Management entered the field…..Wikipedia says:
"The firm's master hedge fund, Long-Term Capital Portfolio L.P., failed spectacularly in the late 1990s, leading to a massive bailout by other major financial institutions, [1] which was supervised by the Federal Reserve.
LTCM was founded in 1994 by John Meriwether, the former vice-chairman and head of bond trading at Salomon Brothers. Board of directors members included Myron Scholes and Robert C. Merton, who shared the 1997 Nobel Memorial Prize in Economic Sciences.[2] Initially enormously successful with annualized returns of over 40% (after fees) in its first years, in 1998 it lost $4.6 billion in less than four months following the Russian financial crisis and became a prominent example of the risk potential in the hedge fund industry. The fund was closed in early 2000."
Two explanations:
1. "fat tails" …..
2.


Application III: Molecular Dynamics
One way of figuring out what a system composed of a bunch of particles is likely to do is to get together a bunch of particles inside the computer and follow their motion, as it is generated by the laws of classical mechanics. This method is called molecular dynamics. In the simplest version, you simply solve the equations of classical mechanics, in a box, using periodic boundary conditions. That is when a particle's coordinate reaches the edge of the box, say having x=L, then you reset x to zero. You put in some sort of potential, perhaps
$V = \sum_{j<k} v(\mathbf{r}_j - \mathbf{r}_k)$
and then for each particle you calculate their position as a function of time by using the laws of classical mechanics.

Alder and Wainright:

Berni Alder and Tom Wainright produced two great discoveries using the molecular dynamics method. They looked at N=600 particles inside a two-dimensional box of total area Ω. They used repulsive interactions described as "hard-sphere" interactions in which the potential is either zero or infinite depending upon whether the interparticle distance is larger or smaller than 2a, where a is the particle radius. Despite the purely repulsive interactions, they saw a phase transition from a fluid state into a solid one. That is, at some particular density of particles, $n=Na^2/\Omega$, the particles arranged themselves into a lattice with long-ranged order. (The actual ordering in two dimensions is in the directions of the links in the lattice, not the spacing of the particles.) This liquid to solid transition was not entirely a surprise. It had been predicted earlier by Kirkwood who used a method based on series expansion.

http://www.gasresources.net/Alder-WainwrightTransition.htm

B. J. Alder and T. E. Wainwright, "Phase transition for a hard sphere system," *J. Chem. Phys.*, 1957, **27**, 1208-1209.

J. G. Kirkwood, "Statistical mechanics of fluid mixtures," *J. Chem. Phys.*, 1935, **3**, 300-313.

http://focus.aps.org/story/v10/st5#author

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Phys. Rev. Lett. **89**, 07610