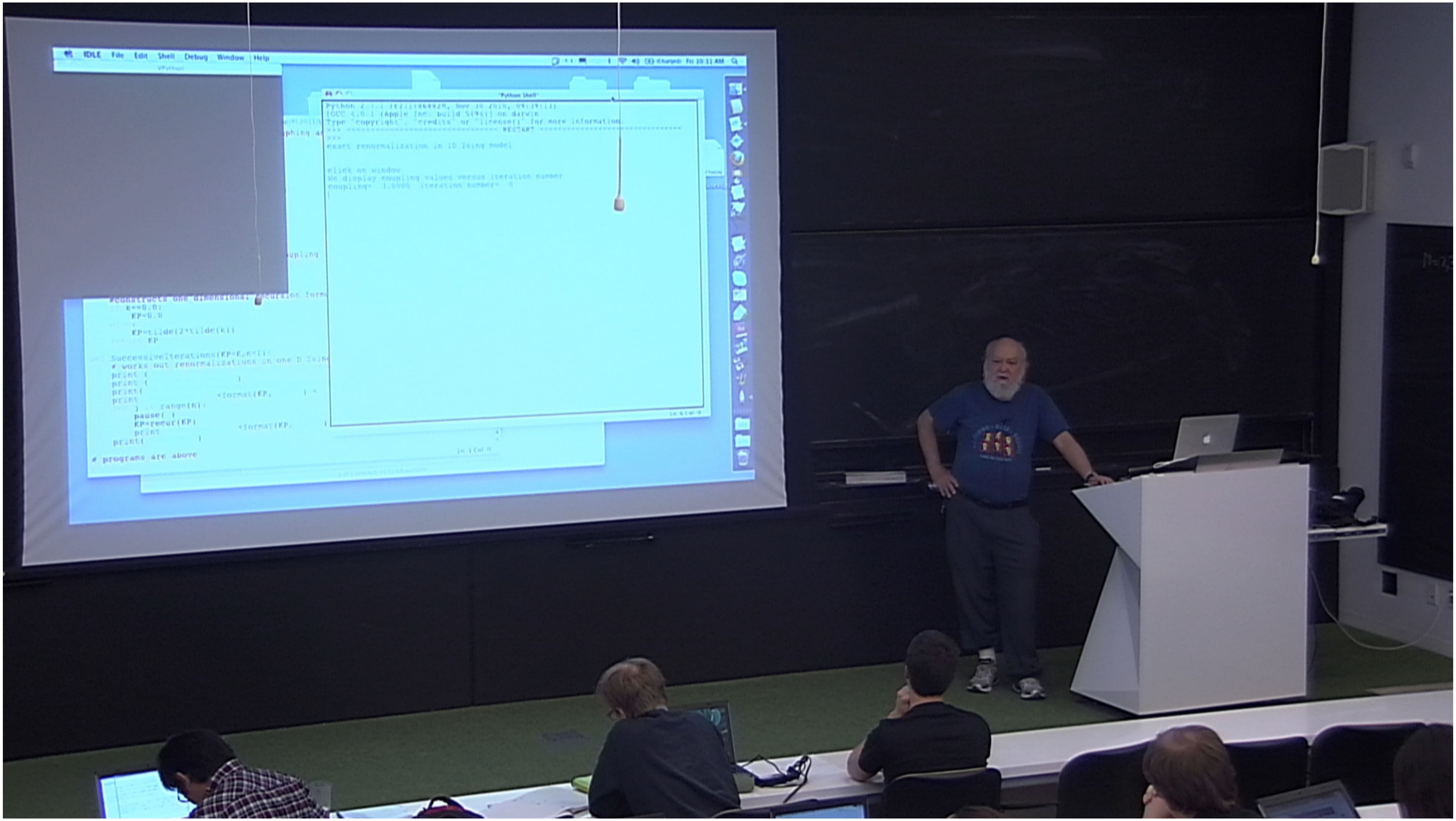
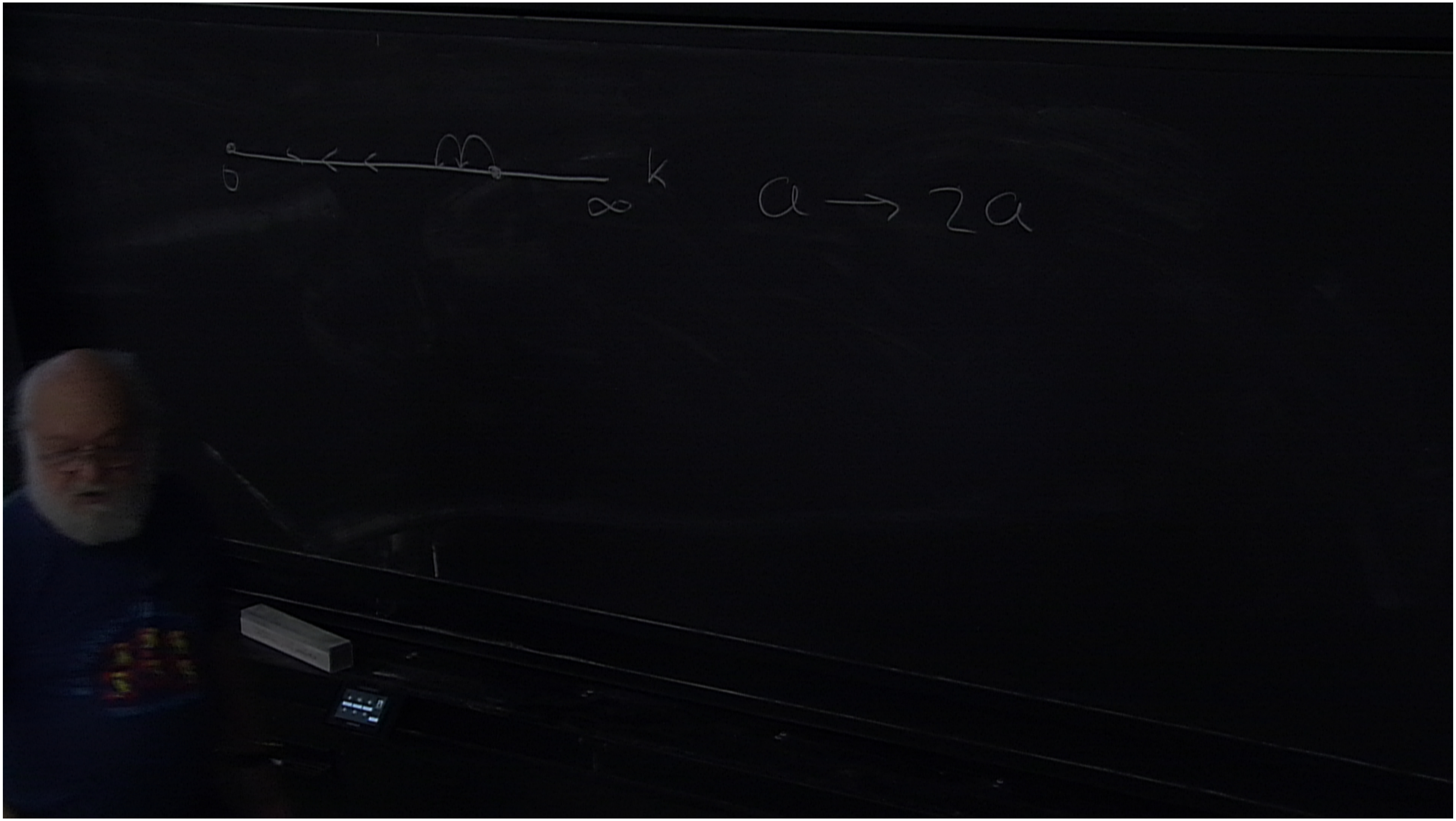
Title: Statistical Mechanics - Lecture 5

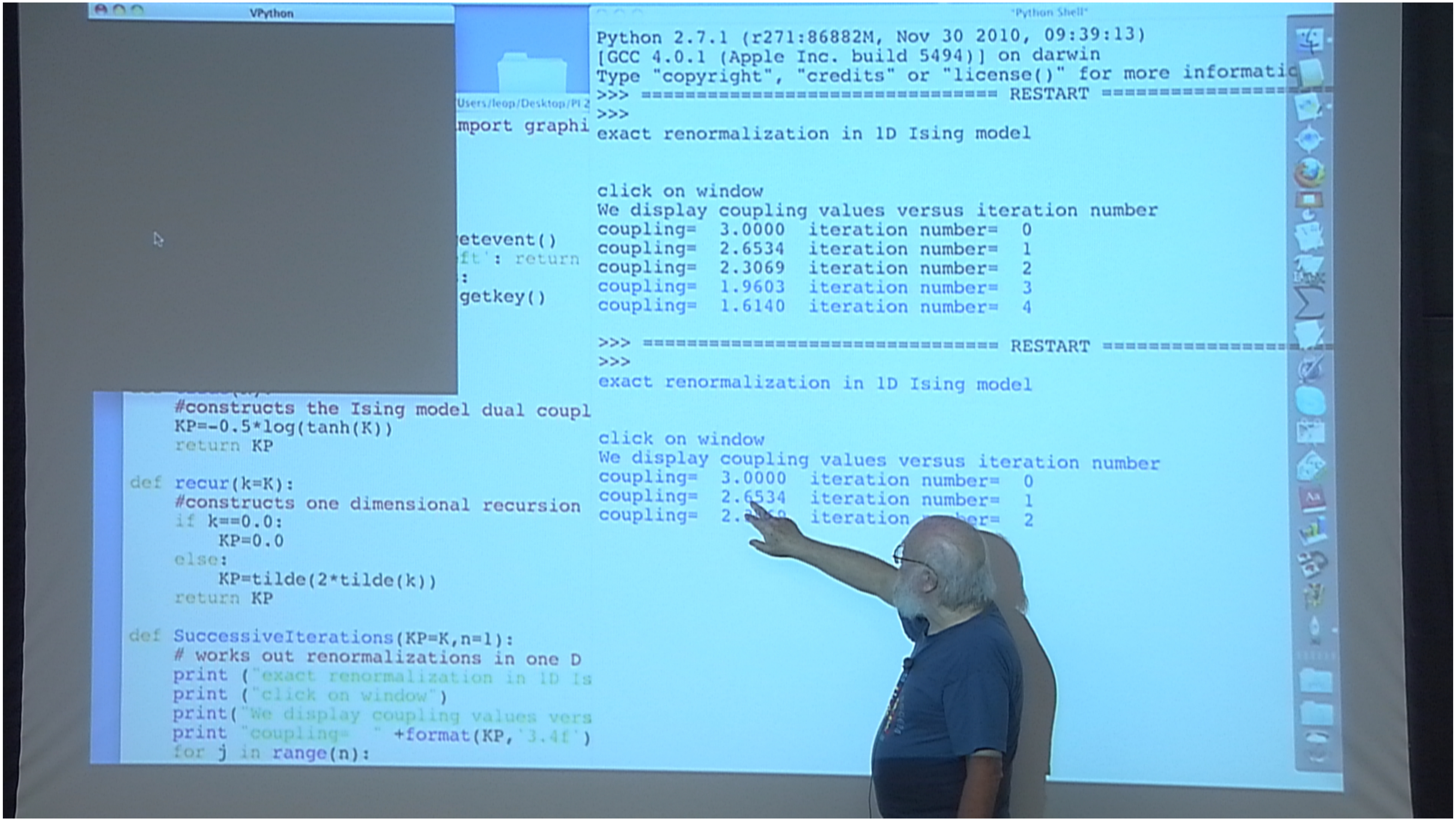
Date: Oct 07, 2011 10:30 AM

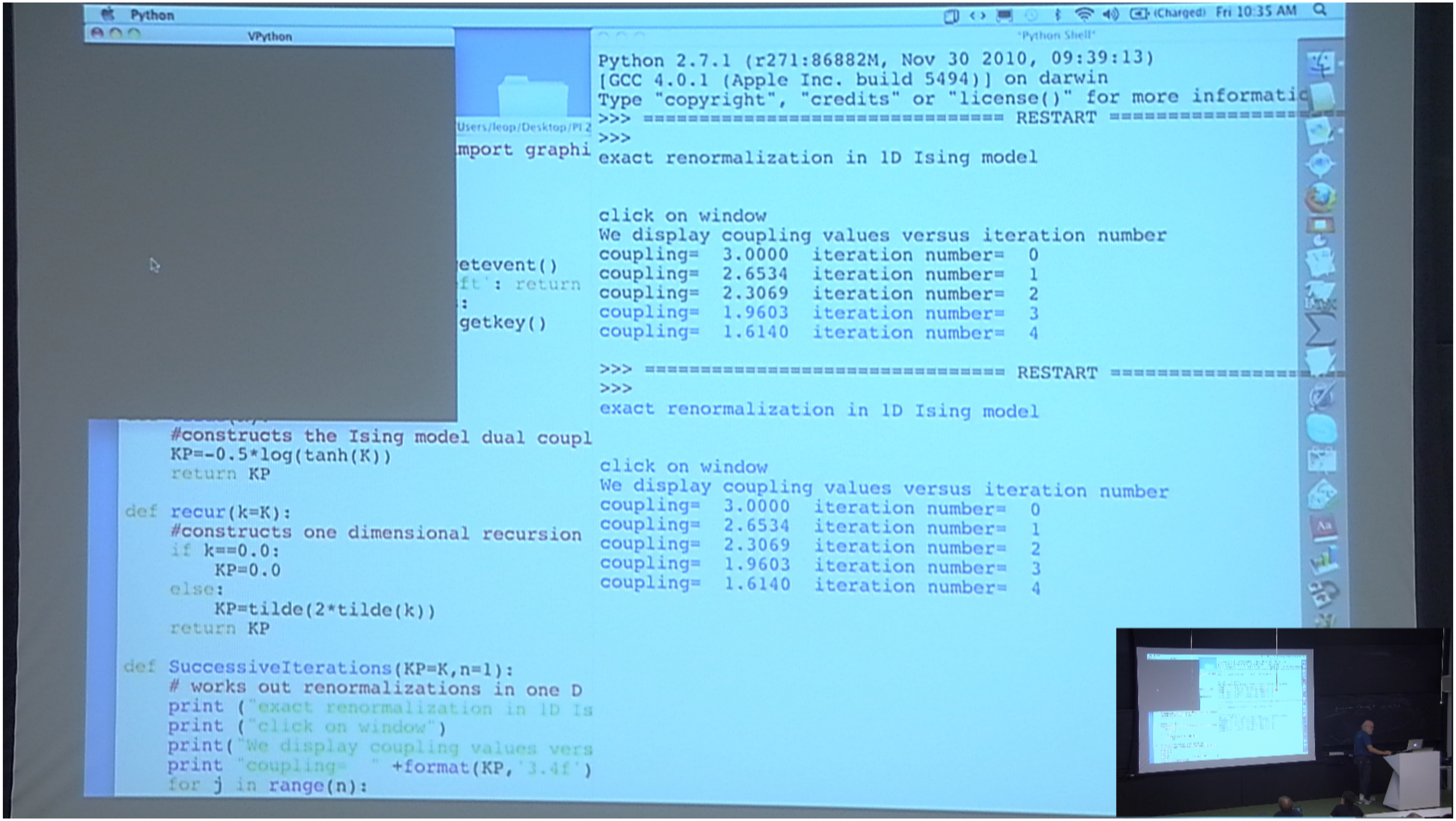
URL: http://pirsa.org/11100029

Abstract:


IDLE File Edit Shell Debug Window Help
Python Shell
Type "copyright", "credits" or "license()" for more information.
RESTART
exact renormalization in 1D Ising model
click on window
We display coupling values versus iteration number
KP=0.0
KP=tilde(2*tilde(K))
# works out renormalizations in one D Ising
pause( )
KP=recur(KP)
# programs are above


0
∞
k
a → 2a


VPython
Python 2.7.1 (r271:86882M, Nov 30 2010, 09:39:13)
[GCC 4.0.1 (Apple Inc. build 5494)] on darwin
Type "copyright", "credits" or "license()" for more informatio
>>> ================================ RESTART ================
>>>
exact renormalization in 1D Ising model
click on window
We display coupling values versus iteration number
coupling= 3.0000 iteration number= 0
coupling= 2.6534 iteration number= 1
coupling= 2.3069 iteration number= 2
coupling= 1.9603 iteration number= 3
coupling= 1.6140 iteration number= 4
>>> ================================ RESTART ================
>>>
exact renormalization in 1D Ising model
click on window
We display coupling values versus iteration number
coupling= 3.0000 iteration number= 0
coupling= 2.6534 iteration number= 1
#constructs the Ising model dual coupl
KP=-0.5*log(tanh(K))
return KP
def recur(k=K):
#constructs one dimensional recursion
if k==0.0:
KP=0.0
else:
KP=tilde(2*tilde(k))
return KP
def SuccessiveIterations(KP=K,n=1):
# works out renormalizations in one D
print ("exact renormalization in 1D Is
print ("click on window")
print("We display coupling values vers
print "coupling= " +format(KP,'3.4f')
for j in range(n):


Python
VPython
Fri 10:35 AM
Python 2.7.1 (r271:86882M, Nov 30 2010, 09:39:13)
[GCC 4.0.1 (Apple Inc. build 5494)] on darwin
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Users/leop/Desktop/PI 2
mport graphi
etevent()
ft': return
:
getkey()
#constructs the Ising model dual coupl
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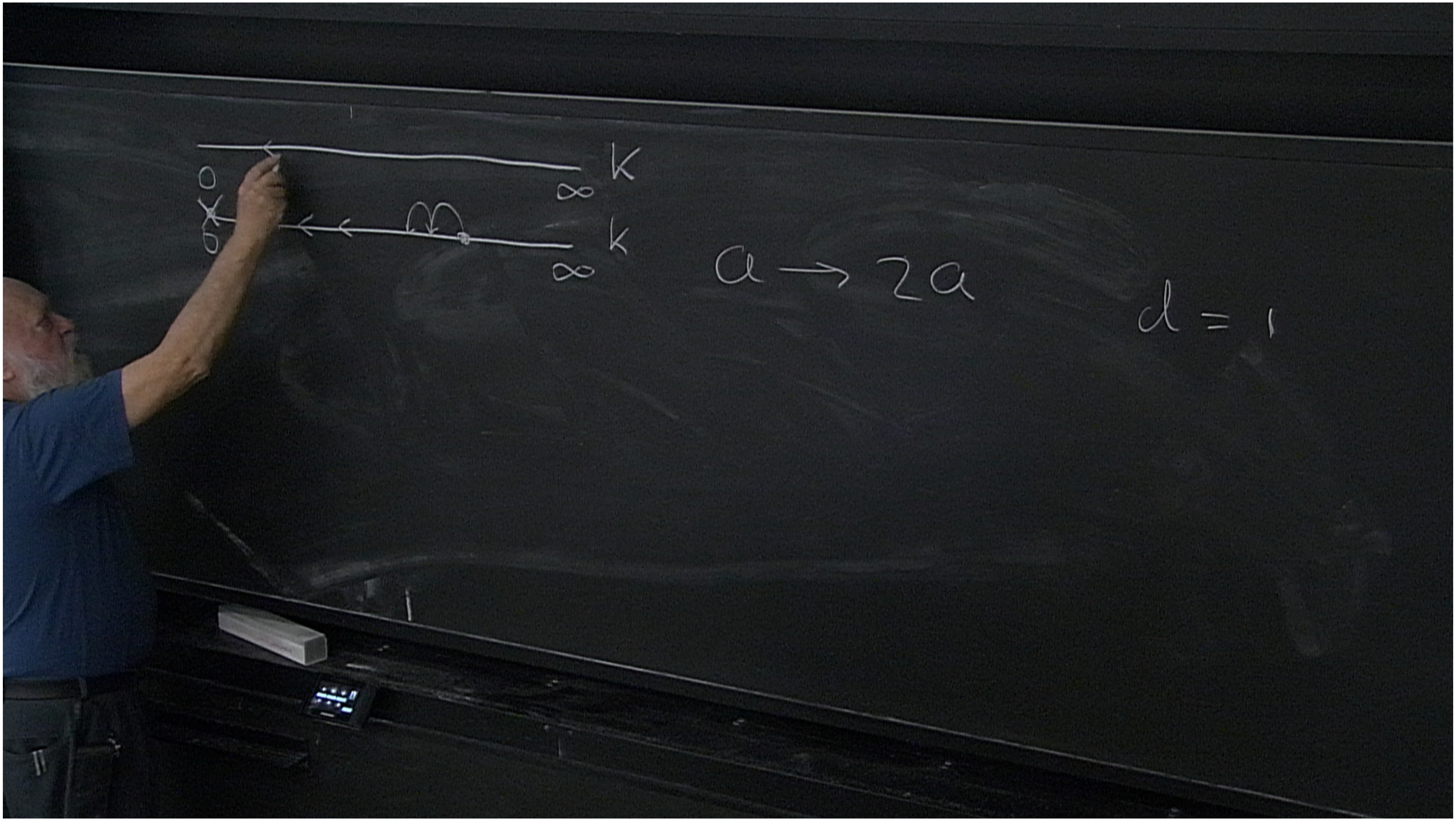
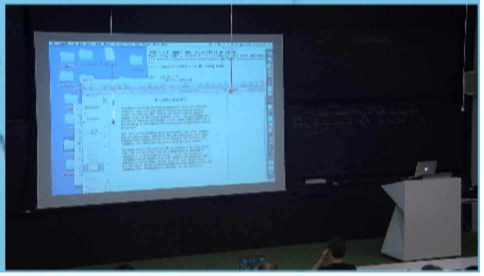

K
0
∞
k
0
∞
a → 2a
d = 1

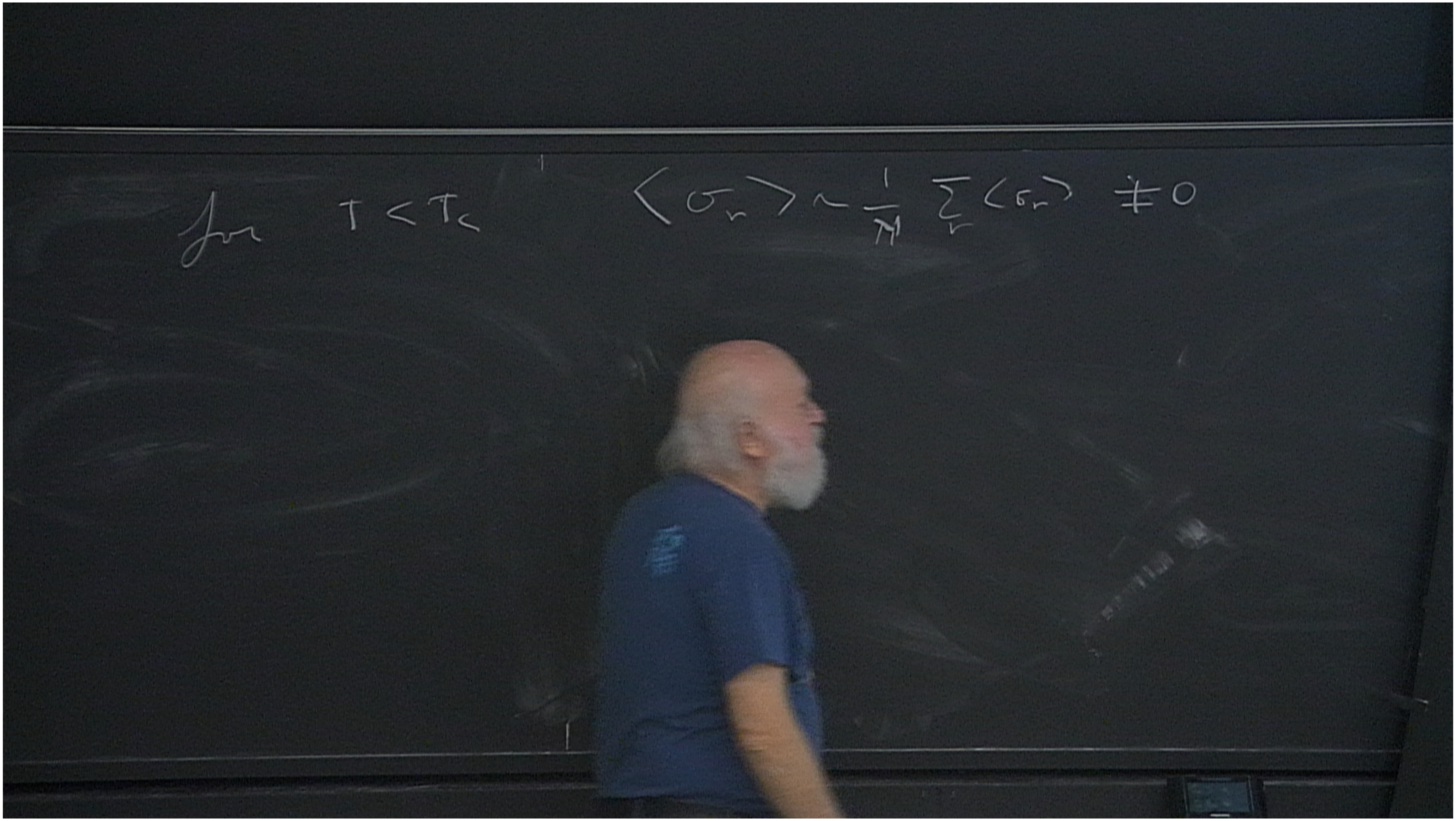
Ising Model in d=2

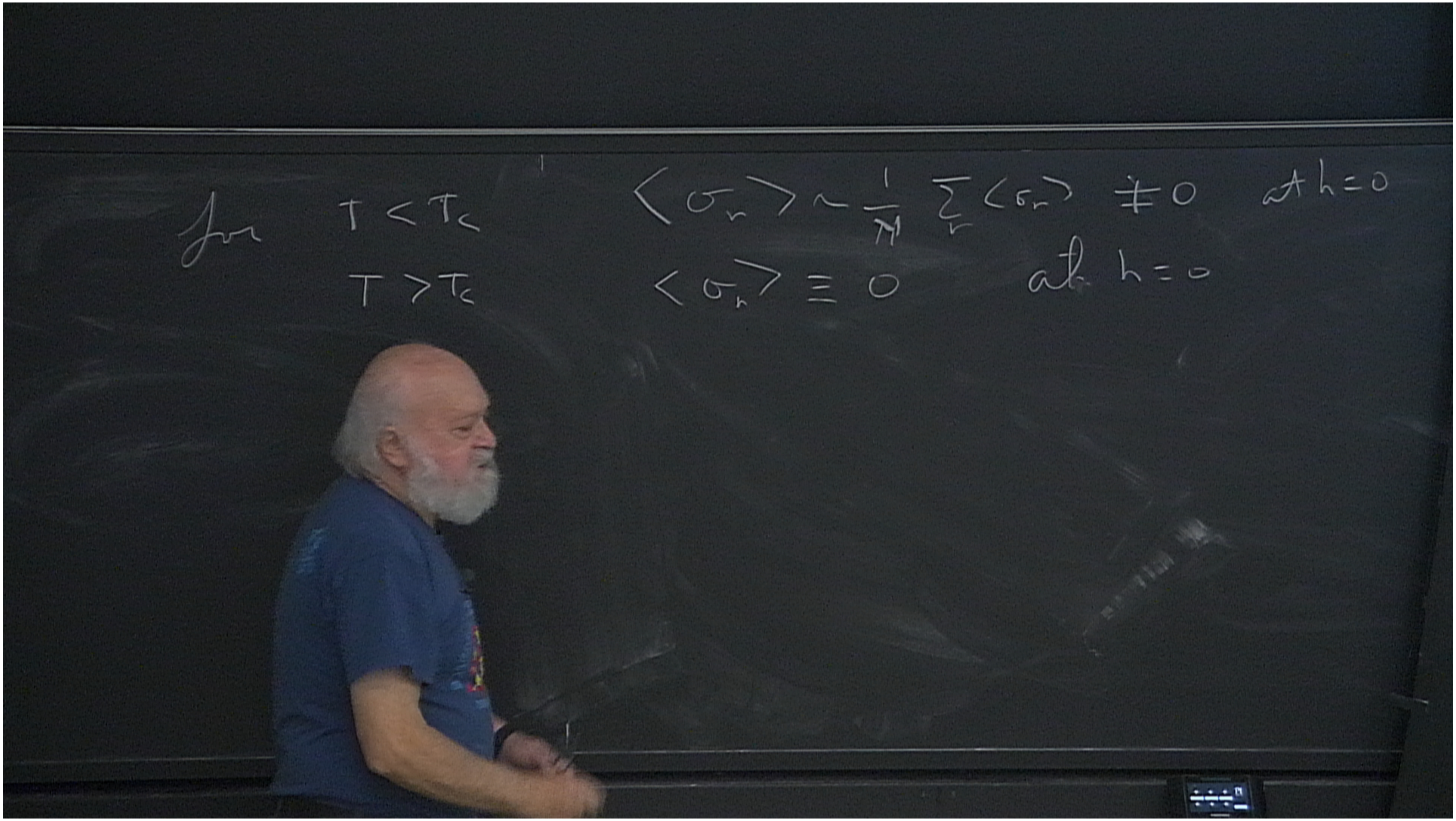
As we have seen, the results from calculating the statistical sum for the one-dimensional Ising model were relatively disappointing, basically because nothing much happens in that model or that solution. Quite the opposite situation holds for the two-dimensional Ising model. There are several phase transitions in that model, and a long list of authors have added substantially to our knowledge of statistical physics by obtaining exact or approximate solutions to this model.

One reason that this model has proved so interesting is that phase transitions exist on surfaces of real *3-d* materials. These two-dimensional models show behavior that is qualitatively similar to the behavior of real systems on real surfaces, at least in the region of parameters near the phase transition.

The earliest events in the discussion of these transitions in this model were a proof by Rudolph Peierls that there was a phase transition, followed by a calculation of the critical coupling value by H. Kramers and G. Wannier, followed by an exact solutions of the model (that is an exact calculation of the statistical sum for the partition function) by Lars Onsager. T.D. Lee and C.N. Yang then argued that this model described the liquid-gas phase transition. W shall discuss the critical temperature calculation later on in this section.




for T < T_c   ⟨σ_r⟩ ~ 1/N Σ_r ⟨σ_r⟩ ≠ 0


for T < T_c ⟨σ_r⟩ ~ 1/N Σ_r ⟨σ_r⟩ ≢ 0 at h=0
T > T_c ⟨σ_r⟩ ≡ 0 at h=0

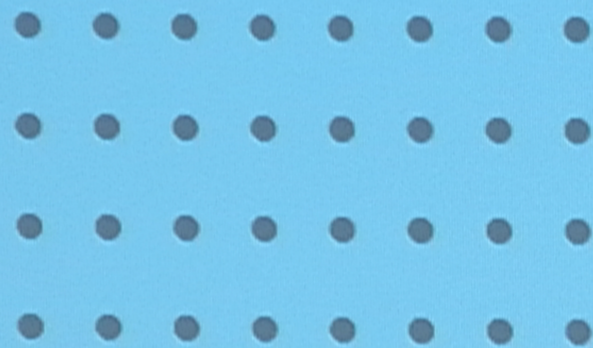
for $T < T_c$ $\quad \langle \sigma_r \rangle \sim \frac{1}{N} \sum_r \langle \sigma_r \rangle \not\equiv 0$ at $h=0$

$T > T_c$ $\quad \langle \sigma_r \rangle \equiv 0$ at $h=0$

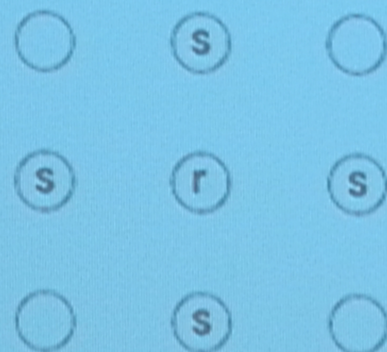
$\sinh(2K_c) = 1$

$K_c \simeq 0.44\ldots$

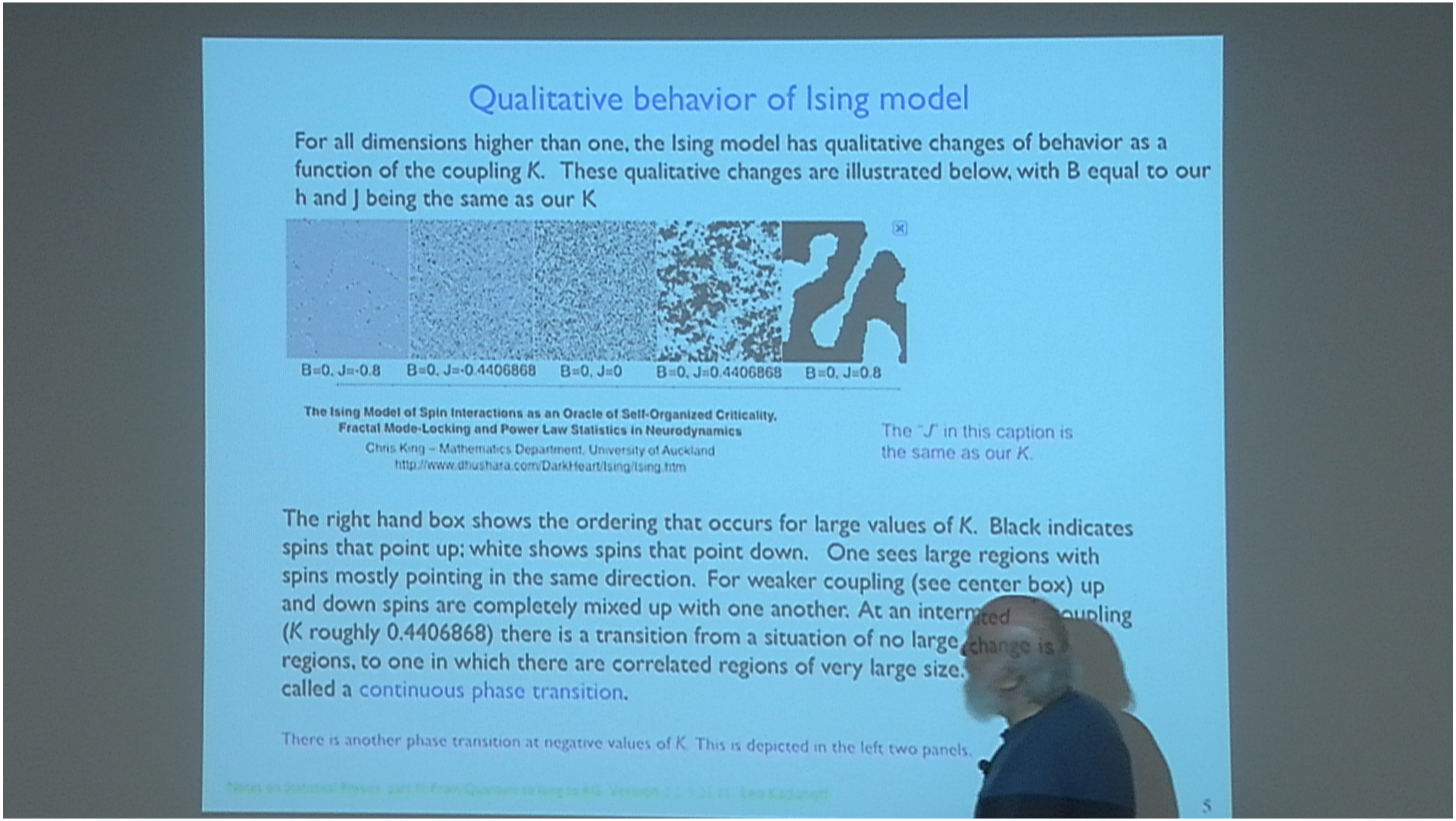
square lattice



Onsager calculated partition function and phase transition for this situation

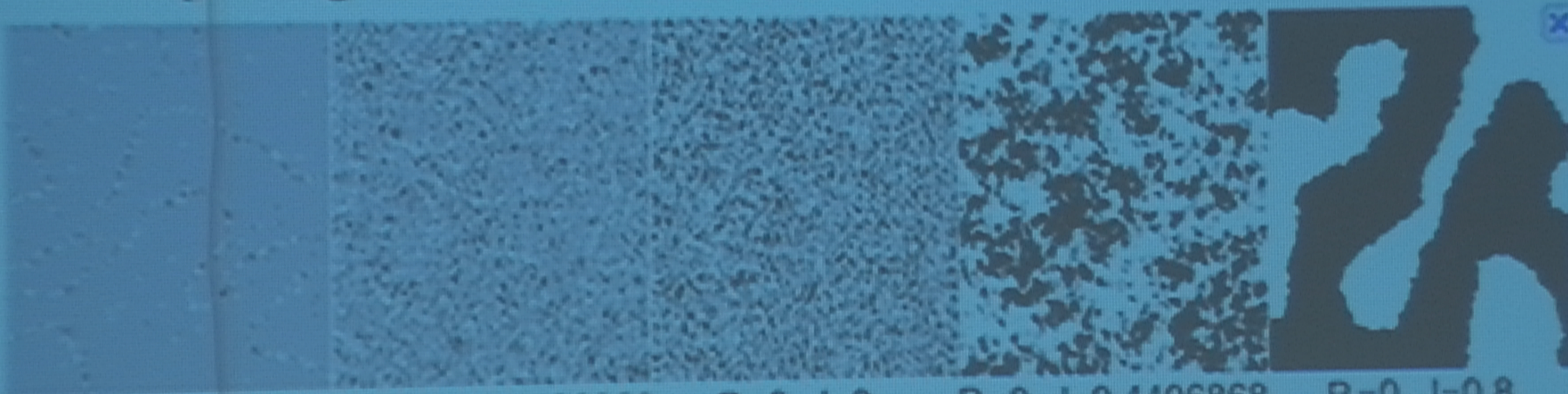


Nearest neighbor structure
s's are nearest neighbors to r
Bonds=exp($K\sigma\sigma'$) connect nearest neighbors. To simplify further calculations, we assume that the lattice is defined with periodic boundary conditions.



Qualitative behavior of Ising model
For all dimensions higher than one, the Ising model has qualitative changes of behavior as a function of the coupling K. These qualitative changes are illustrated below, with B equal to our h and J being the same as our K
B=0, J=-0.8
B=0, J=-0.4406868
B=0, J=0
B=0, J=0.4406868
B=0, J=0.8
The Ising Model of Spin Interactions as an Oracle of Self-Organized Criticality, Fractal Mode-Locking and Power Law Statistics in Neurodynamics
Chris King – Mathematics Department, University of Auckland
http://www.dhushara.com/DarkHeart/Ising/Ising.htm
The "J" in this caption is the same as our K.
The right hand box shows the ordering that occurs for large values of K. Black indicates spins that point up; white shows spins that point down. One sees large regions with spins mostly pointing in the same direction. For weaker coupling (see center box) up and down spins are completely mixed up with one another. At an interm
(K roughly 0.4406868) there is a transition from a situation of no large
regions, to one in which there are correlated regions of very large size.
called a continuous phase transition.
There is another phase transition at negative values of K. This is depicted in the left two panels.
5

For all dimensions higher than one, the Ising model has qualitative ch
function of the coupling *K*. These qualitative changes are illustrated
h and J being the same as our K



The Ising Model of Spin Interactions as an Oracle of Self-Organized Criticality, Fractal Mode-Locking and Power Law Statistics in Neurodynamics

Chris King – Mathematics Department, University of Auckland

http://www.dhushara.com/DarkHeart/Ising/Ising.htm

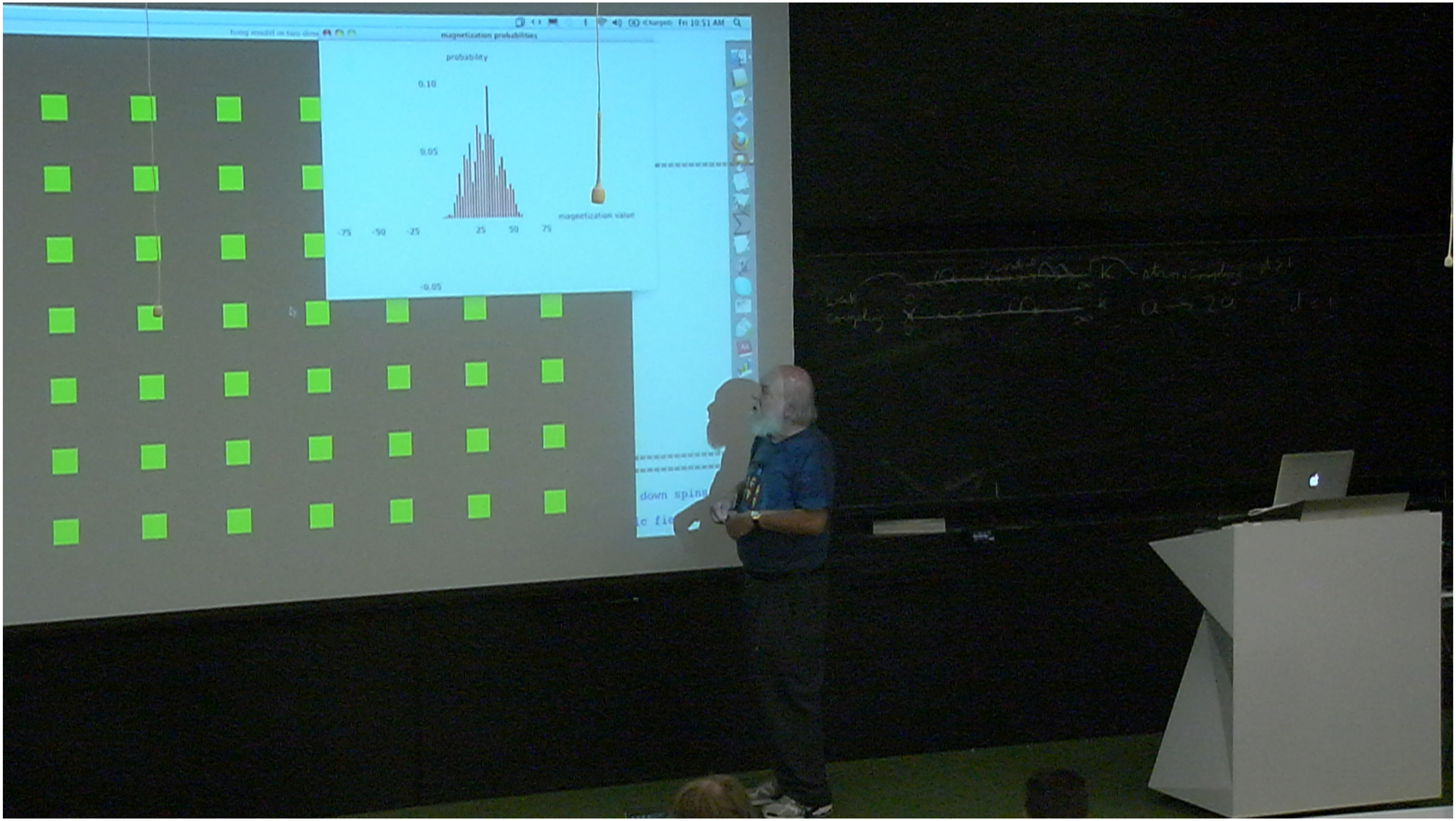
The "J" i
the sam

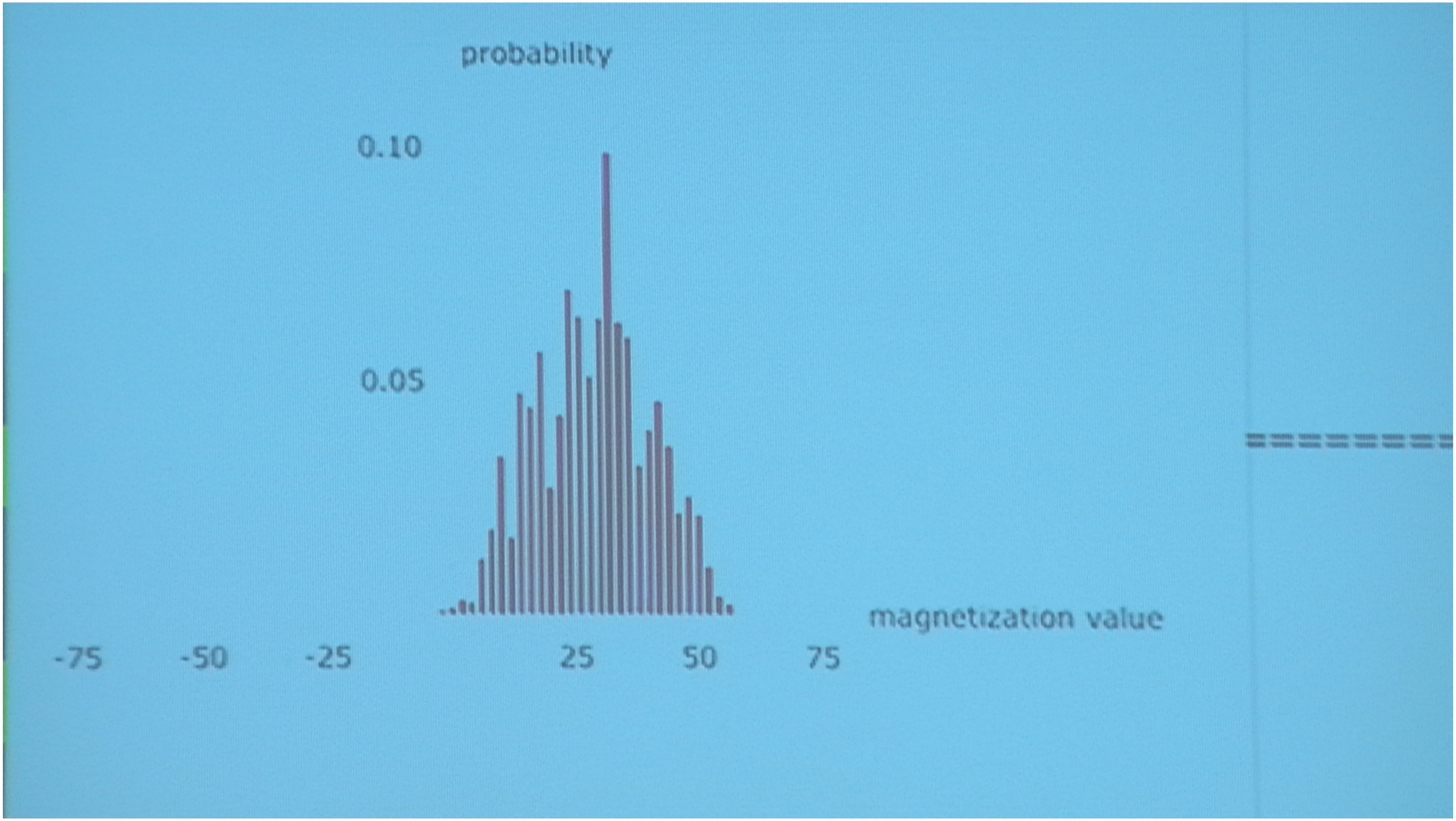
Why is 2D different from 1D

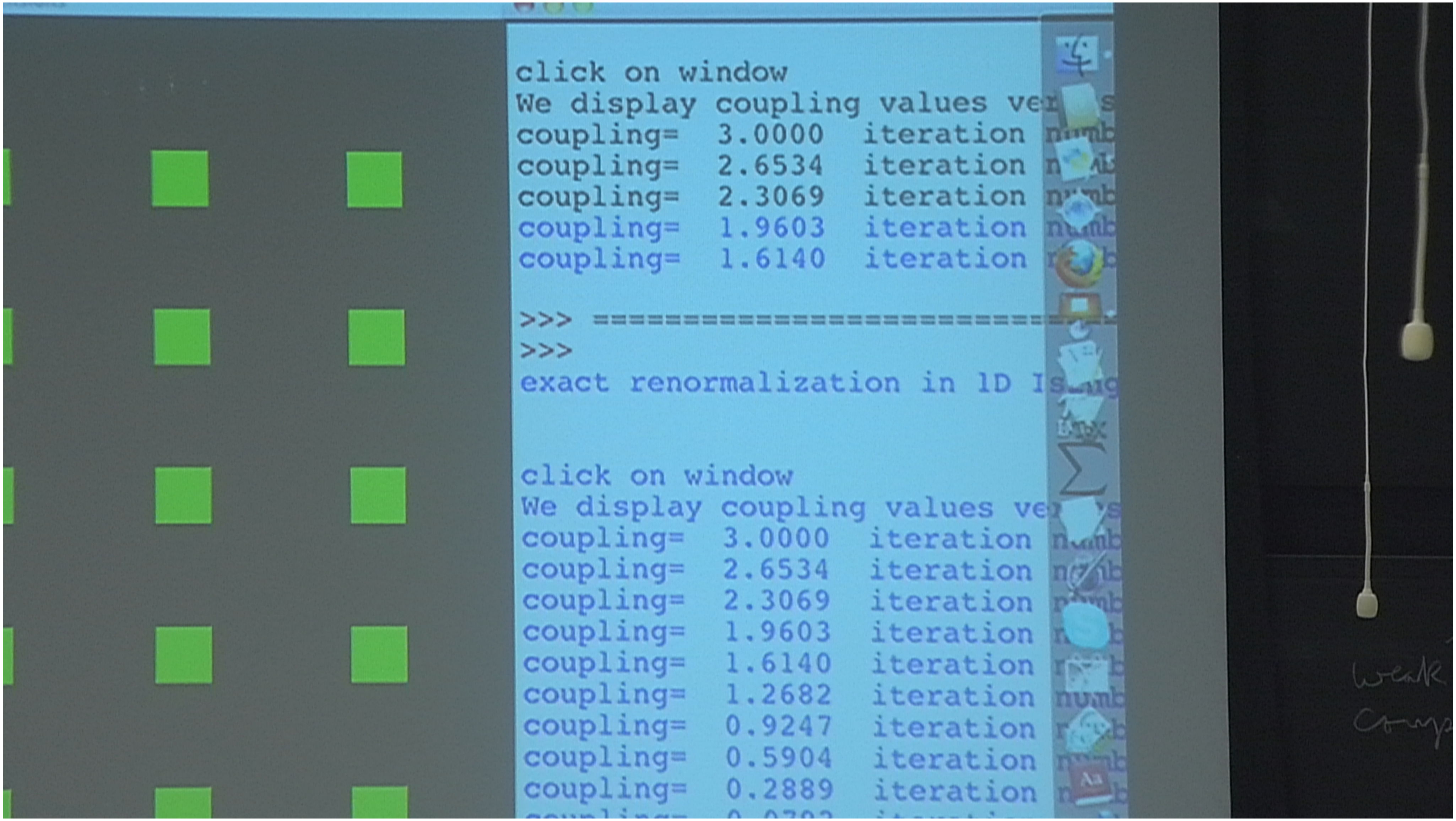
In 1D a single Bloch wall, cost exp(-2K) can break the coherence between differenr regions of the system. In a system of size L there will typically be (L/a) exp(-2K) of these walls, killing all long-range coherence. On the other hand in an L by L Ising model to break long-range order we we have to break at least L bonds, cost exp(-2KL/a). Very roughly the number of such walls across the entire system will be exp(-2KL/a) (L/a). That goes to zero as L goes to infinity.

Since walls can be placed in a huge variety of ways, I have underestimated the number of such walls. Pieirls and Griffiths estimated this probability correctly and shoiwed that at sufficiently low temperature, you would not have such wall and hence you will have long-range order. You can see this from running a Monte Carlo simulation of the two-dimensional Ising model.

Monte Carlo


magnetization probabilities
probability
0.10
0.05
-0.05
-75
-50
-25
25
50
75
magnetization value
down spins


probability
0.10
0.05
magnetization value
-75
-50
-25
25
50
75


click on window
We display coupling values
coupling= 3.0000 iteration
coupling= 2.6534 iteration
coupling= 2.3069 iteration
coupling= 1.9603 iteration
coupling= 1.6140 iteration
>>> ==========
>>>
exact renormalization in 1D
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coupling= 3.0000 iteration
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coupling= 1.6140 iteration
coupling= 1.2682 iteration
coupling= 0.9247 iteration
coupling= 0.5904 iteration
coupling= 0.2889 iteration

IsingTwoD.1.py - /Users/leop/Desktop/PI 2011/Lecture 3/IsingD=2py/IsingTwoD.

```
#Monte Carlo analysis for d=2 Ising Model

from visual.graph import * # import graphing and mathematical
from Depict import *  # This workspace contains the routines
                      #  the Ising model system.
from Record import *  # This workspace contains the routines
                      #  the Ising model system.  None of th

#global variables with default values
#LMax=(8,10)    #size of the Lattice
LMax=(8,10)    #size of the Lattice
K =0.15     #describes the state of the spin system here s is
s= []     #describes the state of the spin system here s is plus
          #eventually it will be a matrix with entries plus and m
          # note that s is and will be mutable
          # we shall write (s[k])[j]  where j and k are row and c
h=log(1)   # here h is a dimensionless magnetic field determi
Records=[ ]  # a record in which we will keep results, mutabl
mmax=3000  # number of Monte Caro Steps to be performed

def MonteCarloStep(s,j,k):
    global Magnetization
    #does one step of Monte Carlo running for our spin
    EnergyChange= deltaEnergy(s,j,k) # Calculates change in energ
    DoFlip=FlipFunction(EnergyChange)  # Determines whether
```

```
        # note that s is and will be mutable
        # we shall write (s[k])[j]  where j and k are row and c
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Records=[ ]  # a record in which we will keep results, mutabl
mmax=3000  # number of Monte Caro Steps to be performed

def MonteCarloStep(s,j,k):
    global Magnetization
    #does one step of Monte Carlo running for our spin
    EnergyChange= deltaEnergy(s,j,k) # Calculates change in energ
    DoFlip=FlipFunction(EnergyChange)  # Determines whether sp
    if DoFlip:
        change(j,k)  #does what we should do if spin flips
        Magnetization=Magnetization+2*spin(j,k)
    else:
        NoChange(j,k)  #updates quantities when spin does not fli
    AddToRecord(Magnetization)

def MonteCarlo():
    Start()  # Constructs the Initial spin variable, starts g
    m=0   #number of Monte Carlo Steps
    while m<=mmax:
        rate(500)
```

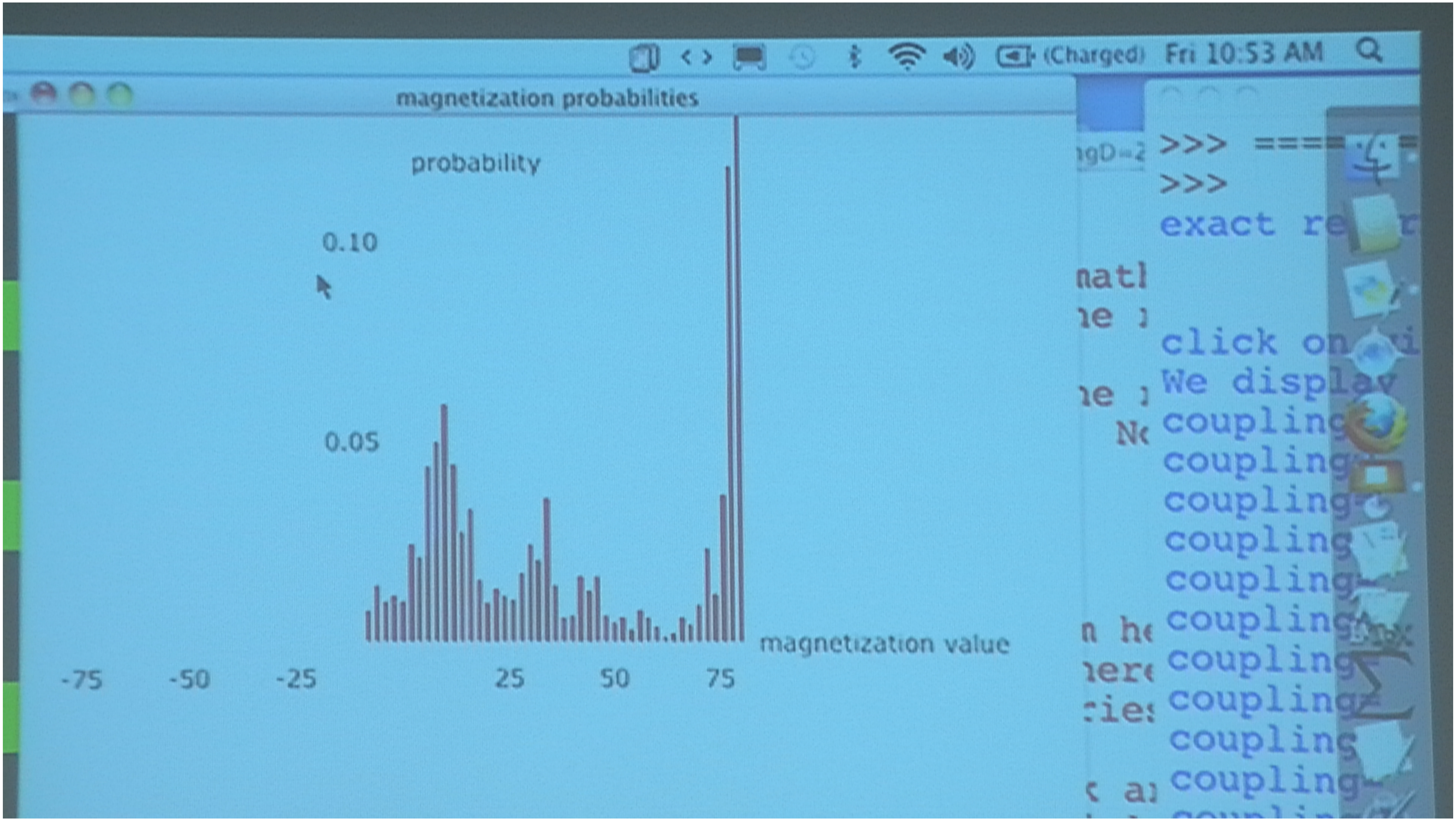
Fri 10:53 AM
magnetization probabilities
probability
0.10
0.05
magnetization value
-75
-50
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>>>
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exact re
click on
We display
coupling
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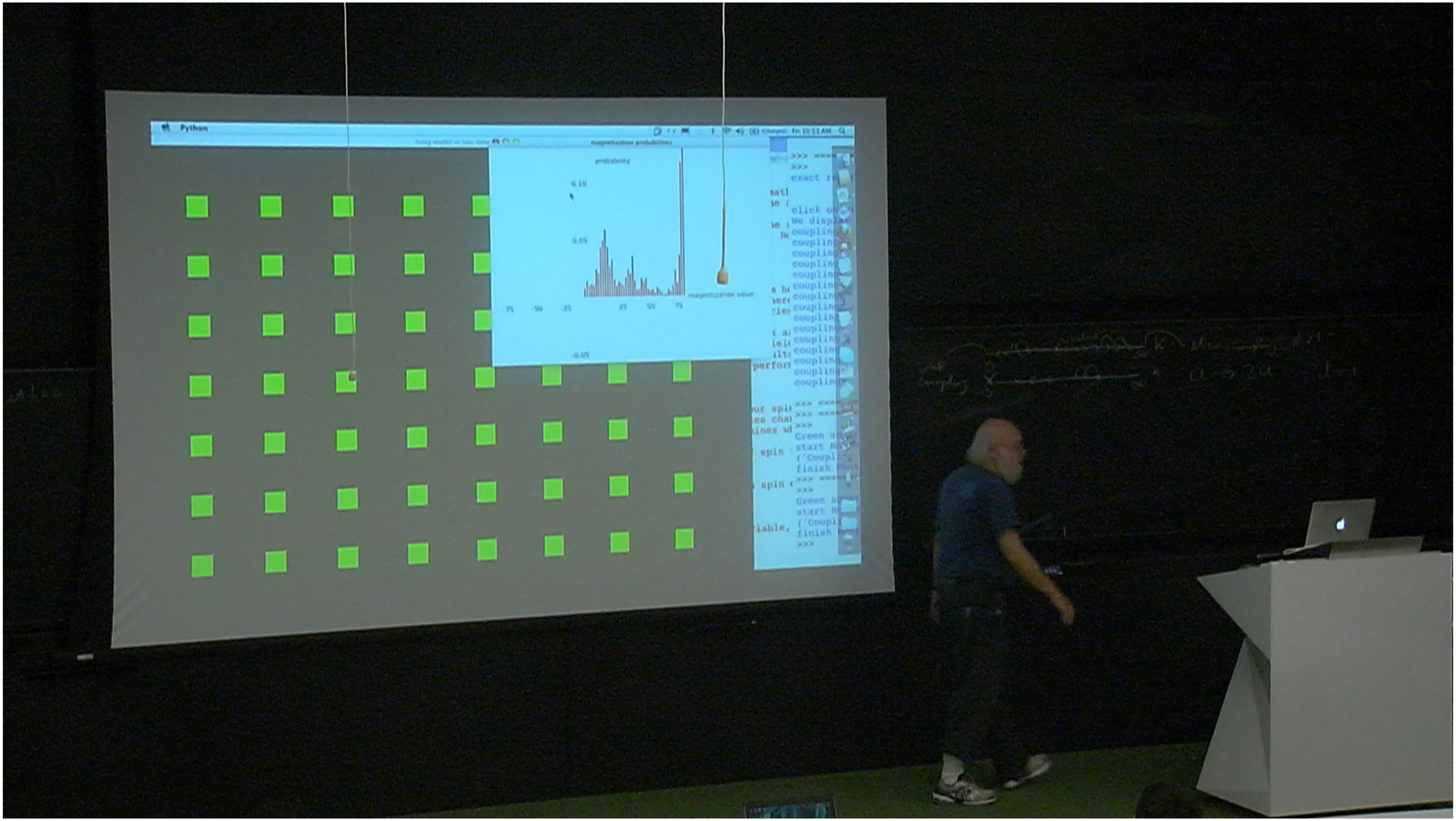
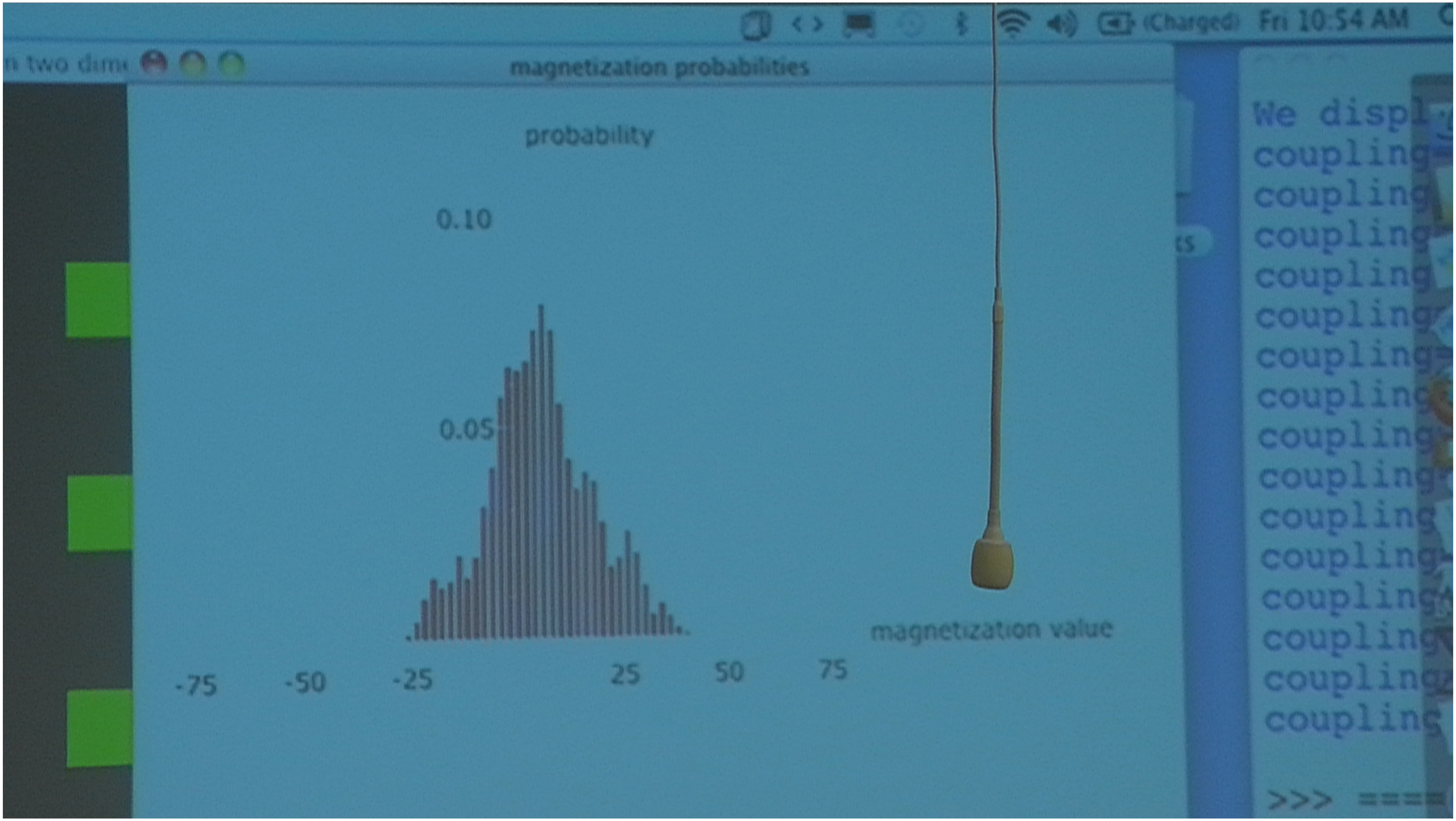

Python
probability
magnetization value

"IsingTwoD.1.py - /Users/leop/Desktop/PI 2011/Lecture 3/IsingD=2py/IsingTwoD.1.py"

```
    kp=mod(k,LMax[1])
    return (s[kp])[jp]

def deltaEnergy(s,j,k):
    # change in exponent for one spin when that is flipped.  negative ChangeInEx
    ChangeInExponent=spin(j+1,k)+spin(j-1,k)+spin(j,k+1)+spin(j,k-1)+h
    ChangeInExponent= -2*ChangeInExponent*spin(j,k)*K
    return ChangeInExponent

def change(j,k):
    global s
    #changes s
    OldSpin=spin(j,k)   #find old spin value
    ChangeSpinColor(OldSpin,j,k)   #calls routine in Visual namespace to change s
    (s[k])[j]=-OldSpin #spin direction changed right here



def NoChange(j,k):
    #holds s unchanged. dummy.  Does nothing now
    j=j


def PlotGraph(Records,m):
    #dummy
    m=m

# ***************
#*************program goes below here
#*************

LMax=(8,10)    #size of the Lattice
K =0.2    #describes the state of the spin system here s is plus or minus 1
h=log(1)   # here h is a dimensionless magnetic field determining probability di
           #for spins
mmax=2000  # number of Monte Caro Steps to be performed


MonteCarlo()
```



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magnetization probabilities
probability
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0.05
-75
-50
-25
25
50
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magnetization value
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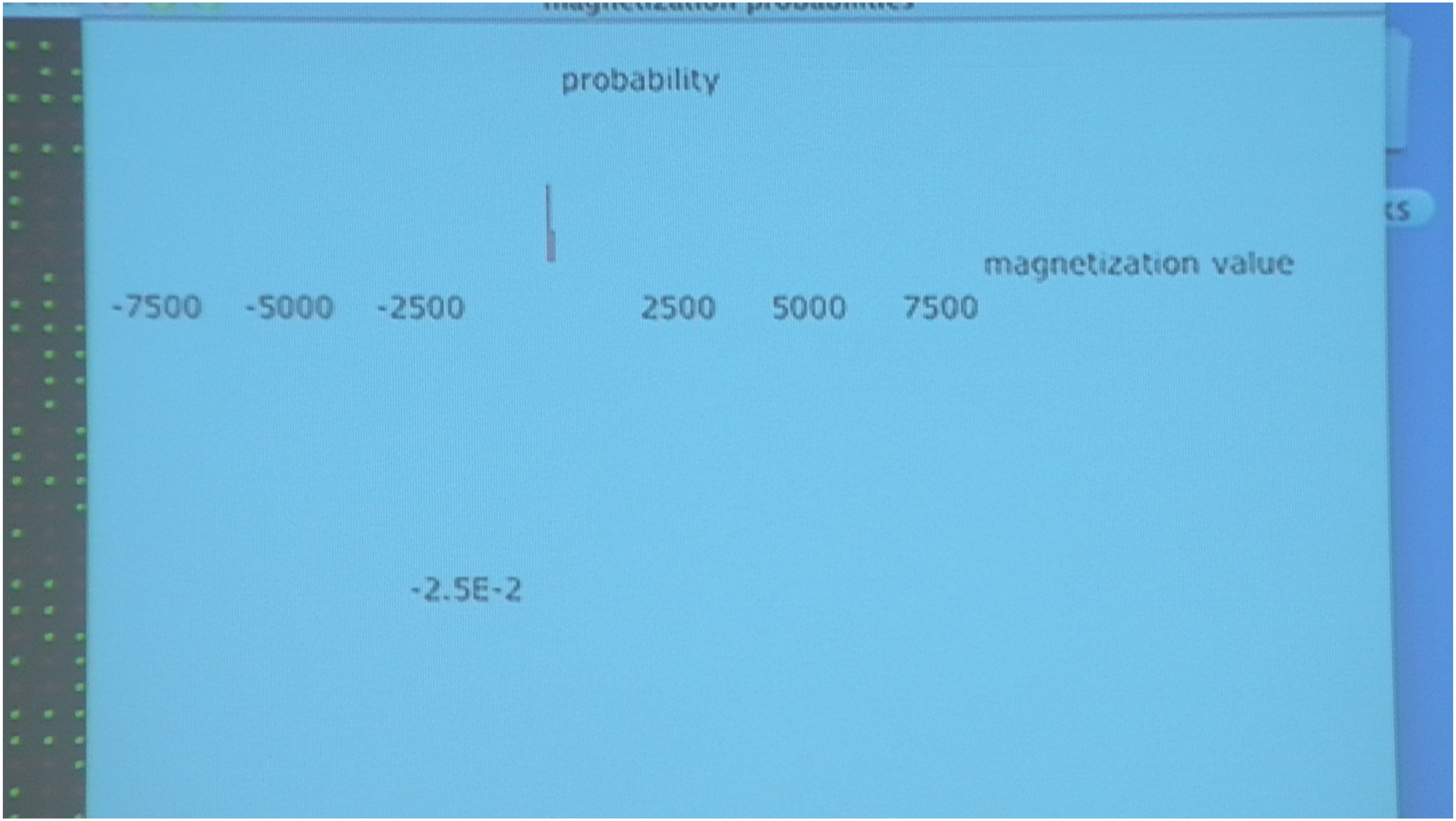
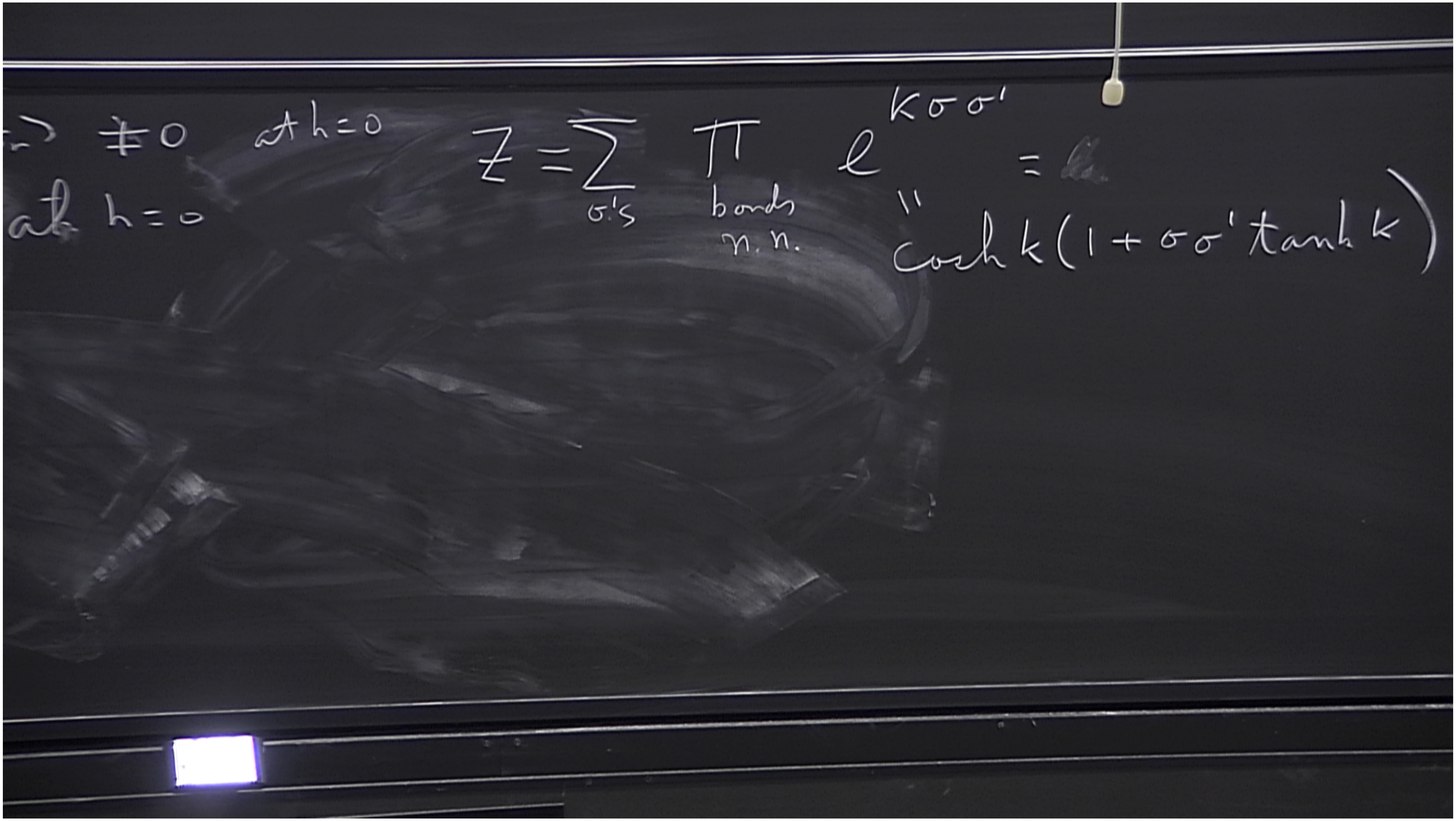

probability
magnetization value
-7500
-5000
-2500
2500
5000
7500
-2.5E-2

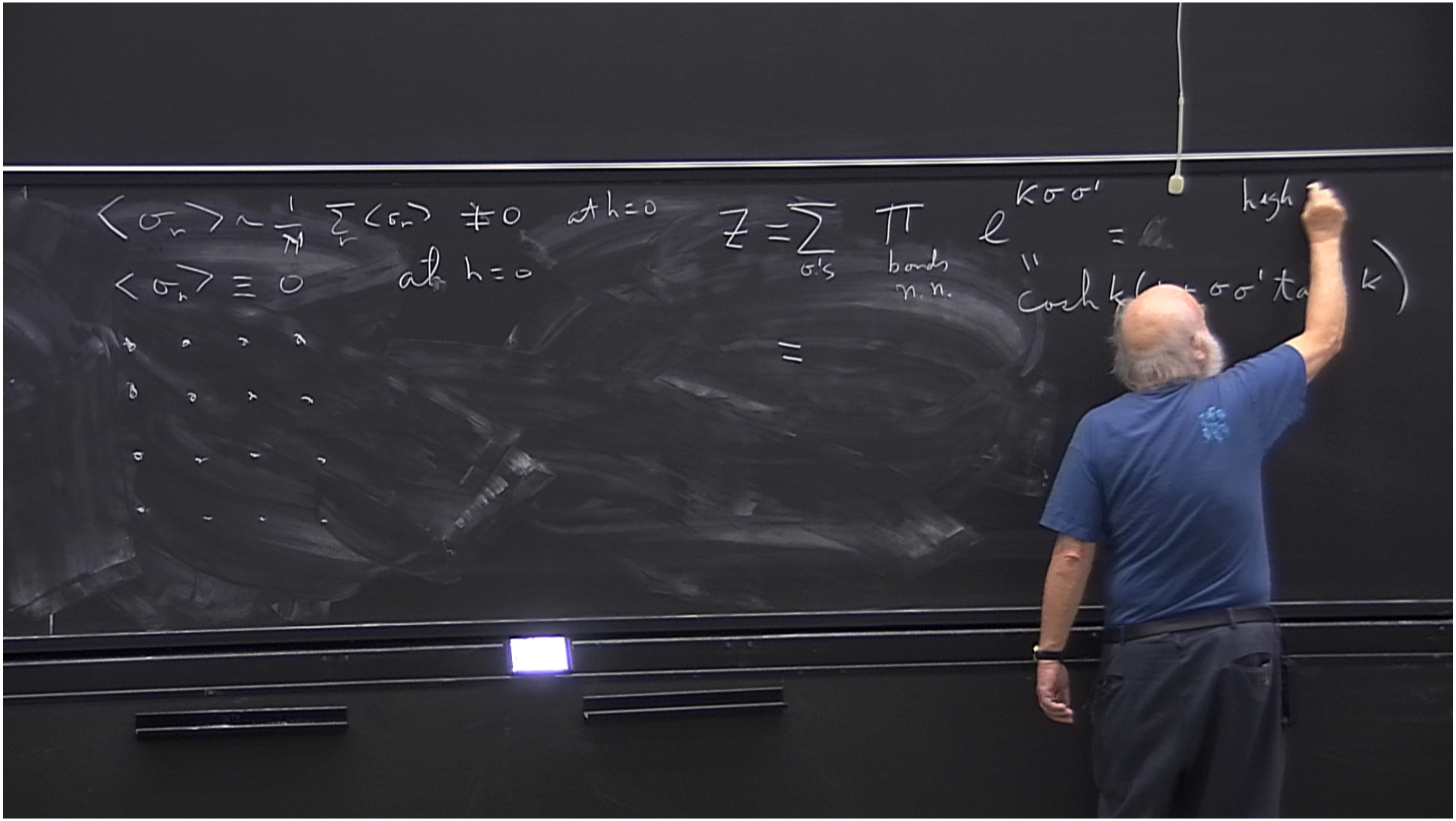

≠ 0 at h=0
at h=0
Z = Σ_{σ's} Π_{bonds n.n.} e^{Kσσ'}
= cosh K(1 + σσ' tanh K)

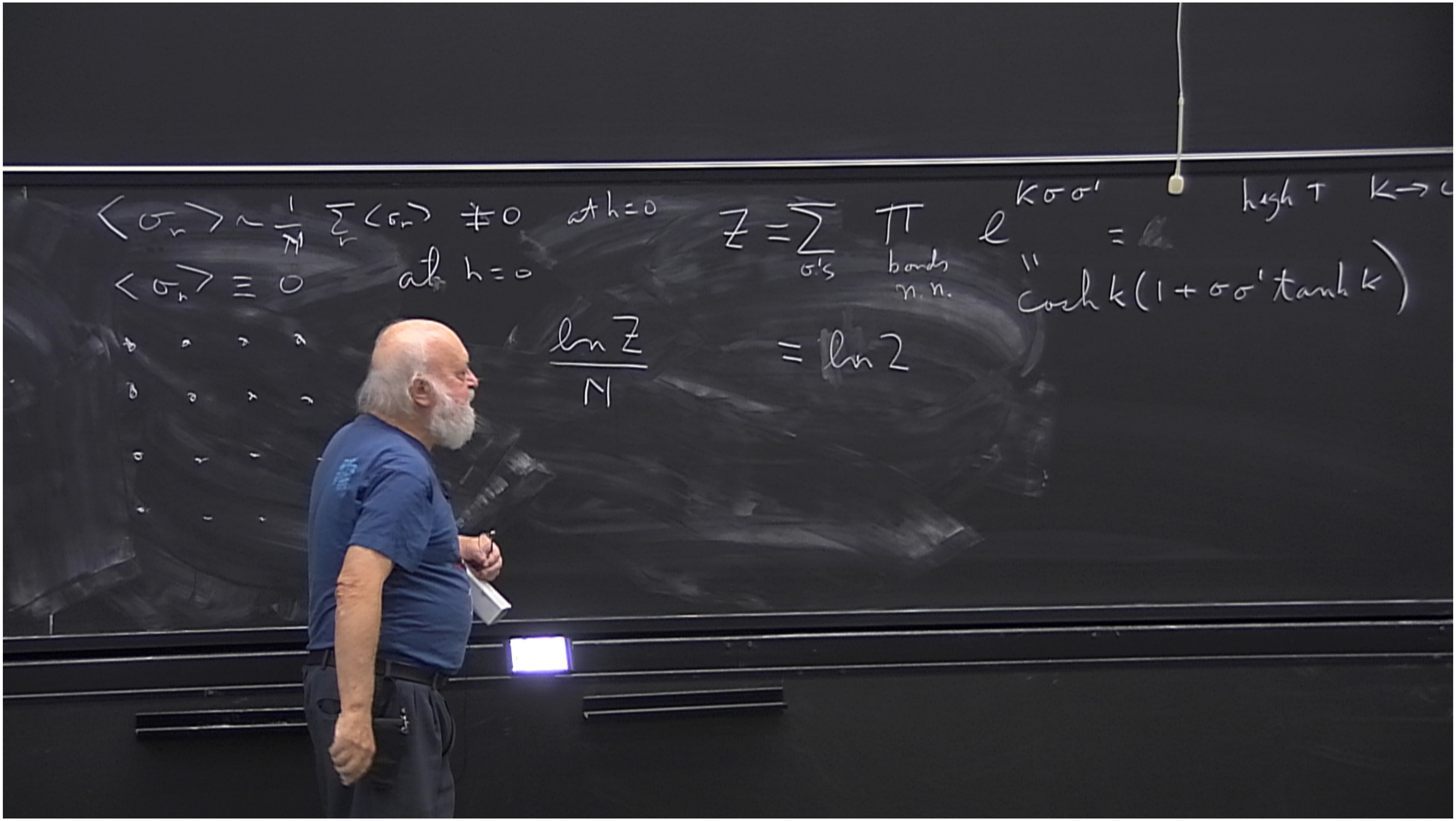
for $T < T_c$ $\quad \langle \sigma_r \rangle \sim \frac{1}{N} \sum_r \langle \sigma_r \rangle \not\equiv 0$ at $h=0$

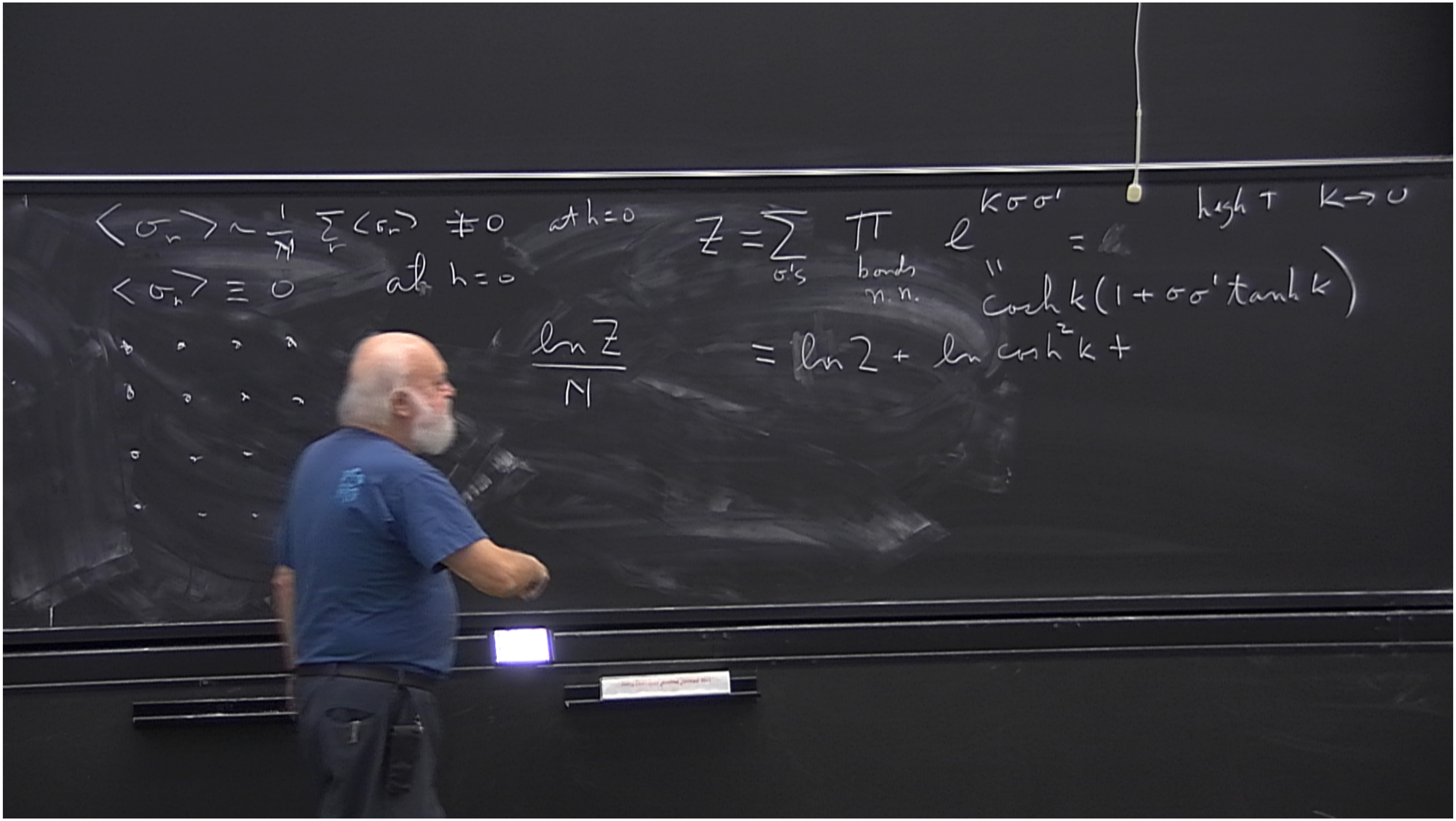
$T > T_c$ $\quad \langle \sigma_r \rangle \equiv 0$ at $h=0$

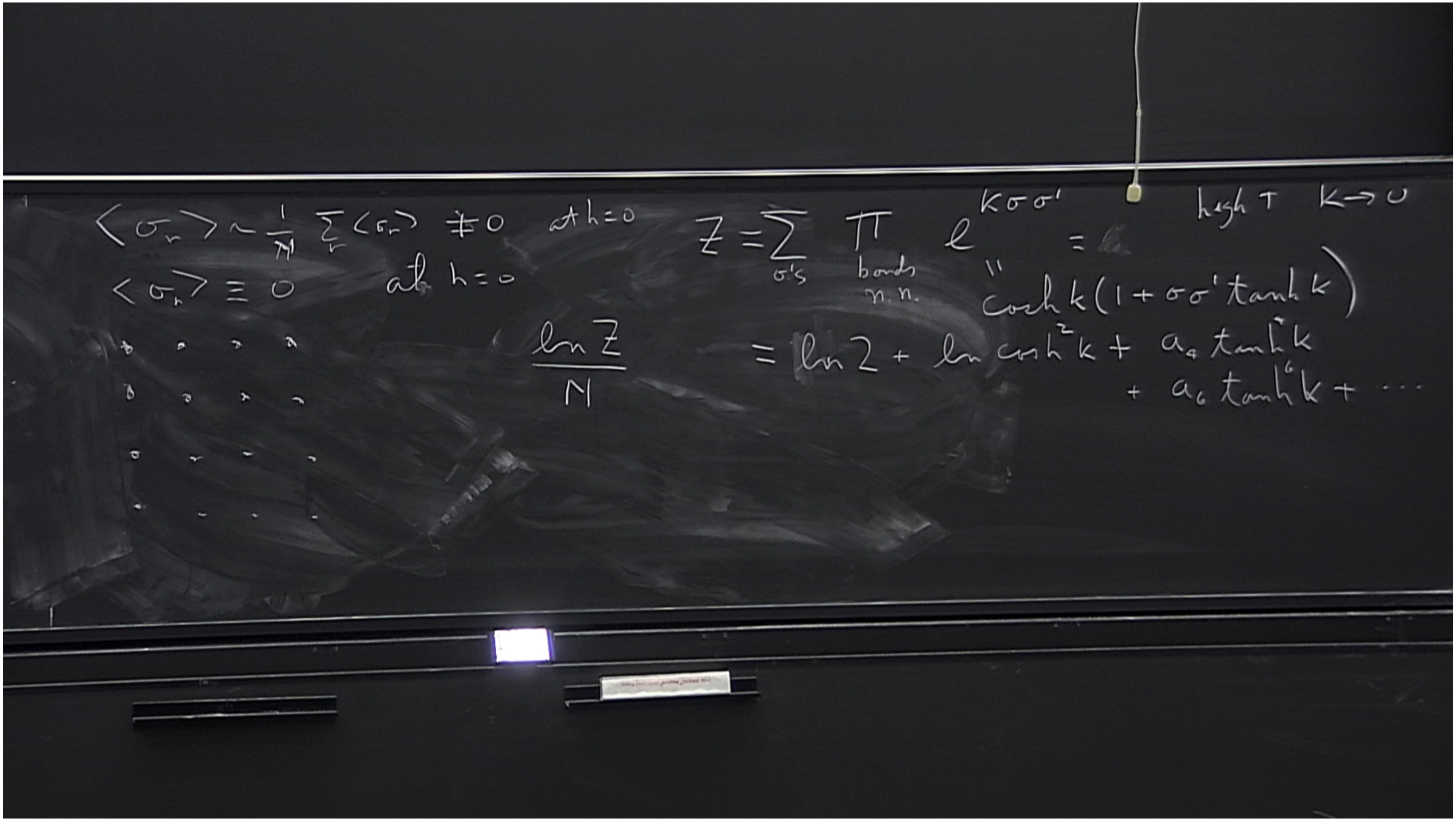
$\sinh(2K_c) = 1$

$K_c \simeq 0.44 \ldots$


at h=0
at h=0
σ's
bonds
n.n.
cosh


at h=0
at h=0
σ's
bonds
n.n.
high T
cosh k(1 + σσ' tanh k)
ln Z
N
= ln 2


⟨σn⟩ ~ 1/N Σr ⟨σr⟩ ≠ 0 at h=0
⟨σn⟩ ≡ 0 at h=0
Z = Σσ's Π bonds n.n. e^Kσσ' = cosh K (1 + σσ' tanh K)
high T K→0
ln Z / N = ln 2 + ln cosh² K +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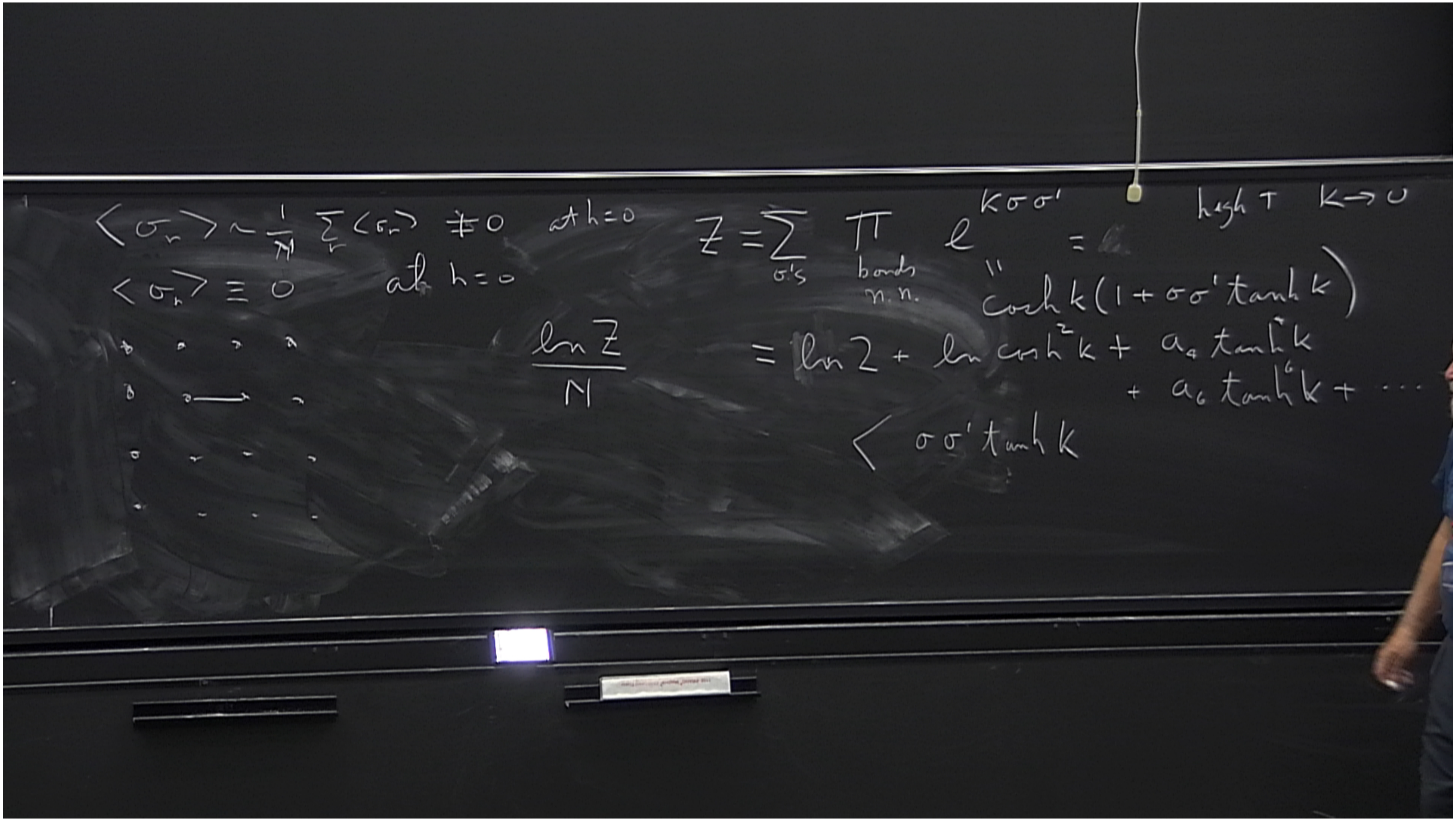
⟨σ_r⟩ ≡ 0 at h=0
Z = Σ_{σ's} Π_{bonds n.n.} e^{Kσσ'} = cosh K (1 + σσ' tanh K)
high T K → 0
ln Z / N = ln 2 + ln cosh² K + a₄ tanh⁴ K
+ a₆ tanh⁶ K + ...

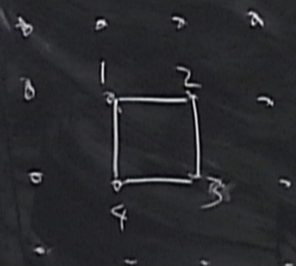

⟨σn⟩ ~ 1/N Σ ⟨σn⟩ ≠ 0 at h=0
⟨σn⟩ ≡ 0 at h=0
ln Z / N
Z = Σ σ's Π bonds n.n. e^{Kσσ'}
= cosh K (1 + σσ' tanh K)
high T K→0
= ln 2 + ln cosh² K + a4 tanh⁴ K
+ a6 tanh⁶ K + ...
σσ' tanh K

$$\langle \sigma_n \rangle \sim \frac{1}{N} \sum_r \langle \sigma_r \rangle \not\equiv 0 \quad \text{at } h=0$$

$$\langle \sigma_n \rangle \equiv 0 \quad \text{at } h=0$$

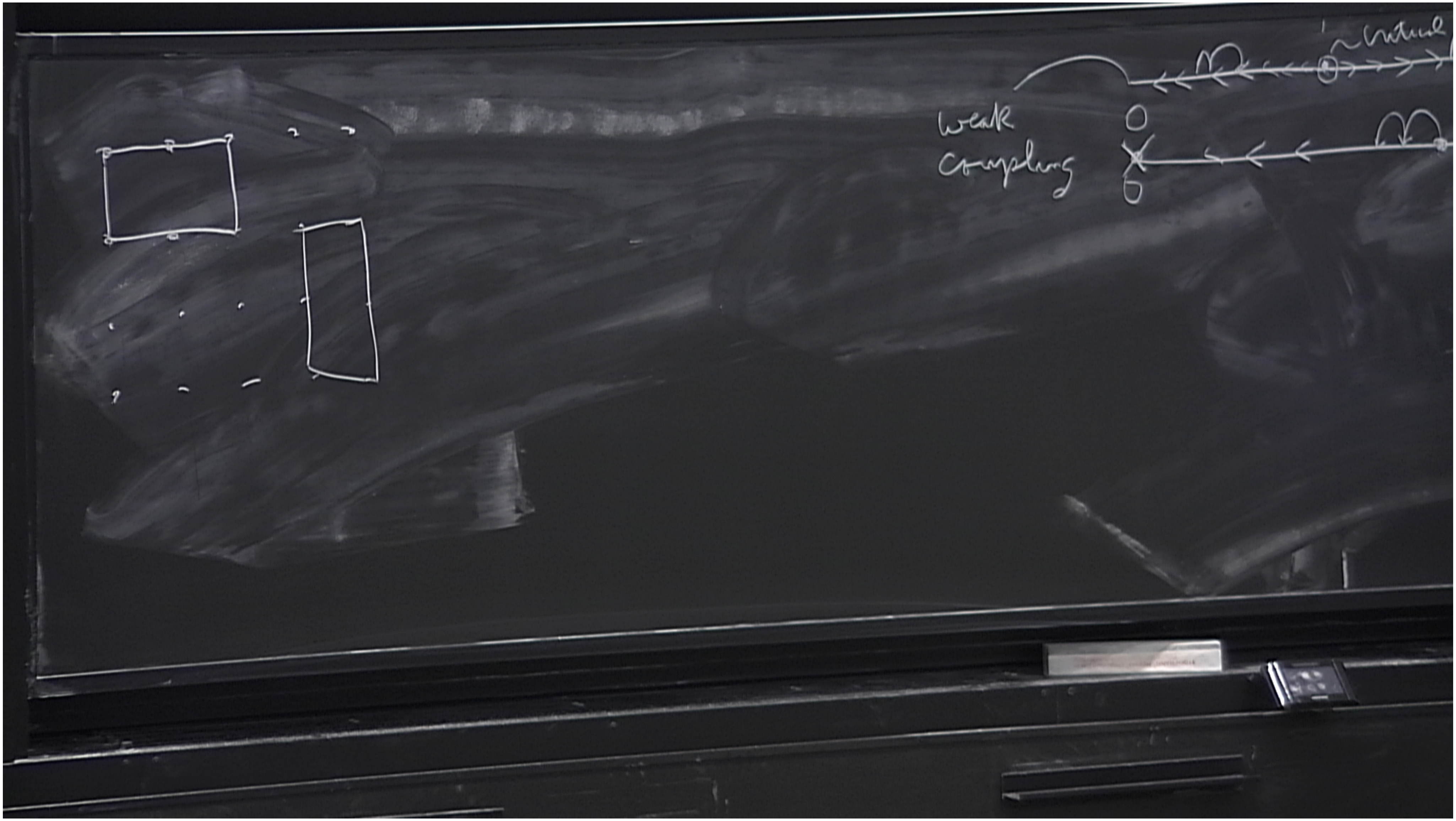
$$Z = \sum_{\sigma\text{'s}} \prod_{\substack{\text{bonds} \\ \text{n.n.}}} e^{K\sigma\sigma'} = \cosh K (1 + \sigma\sigma' \tanh K)$$

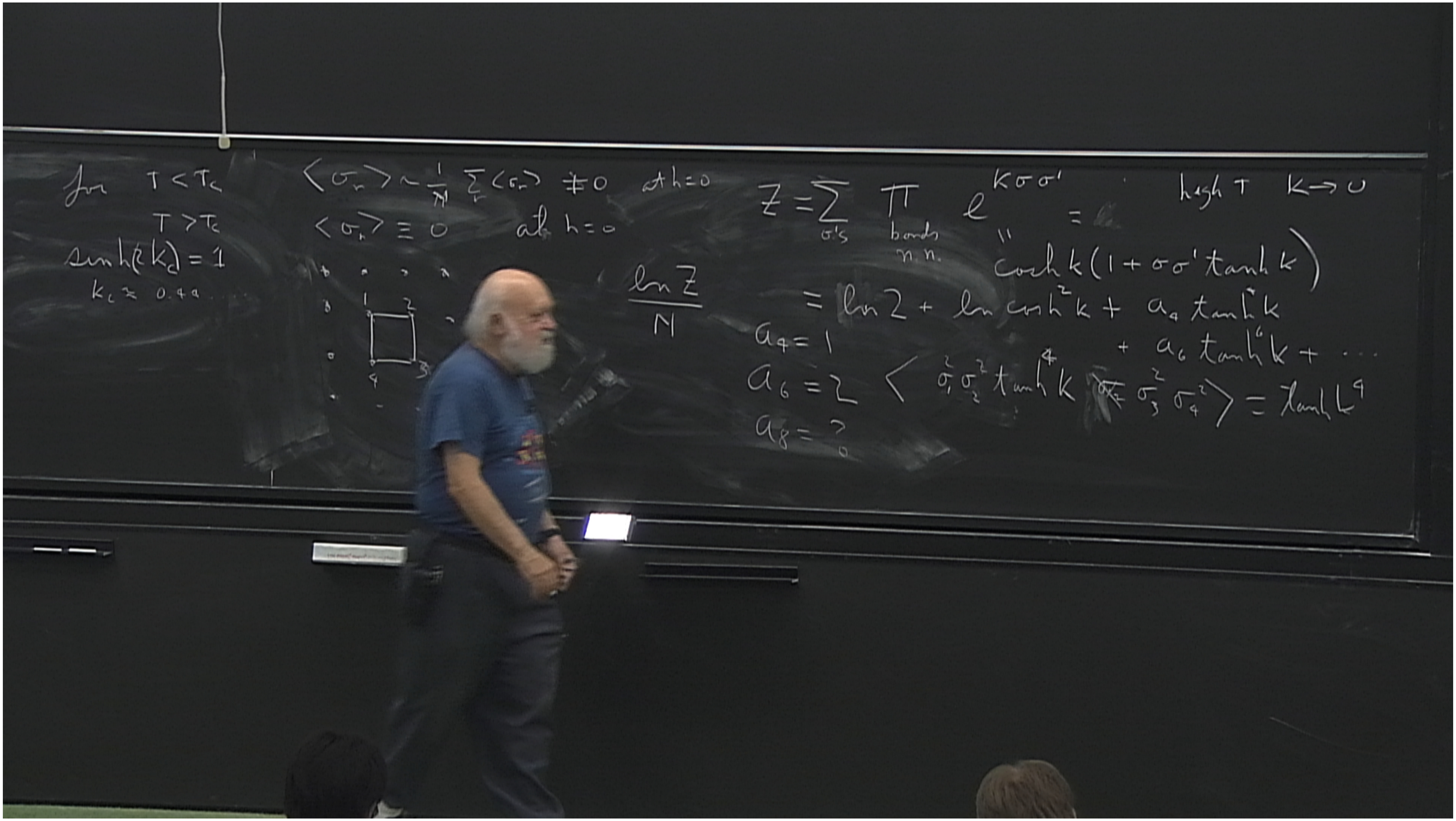
high T $\quad K \to 0$

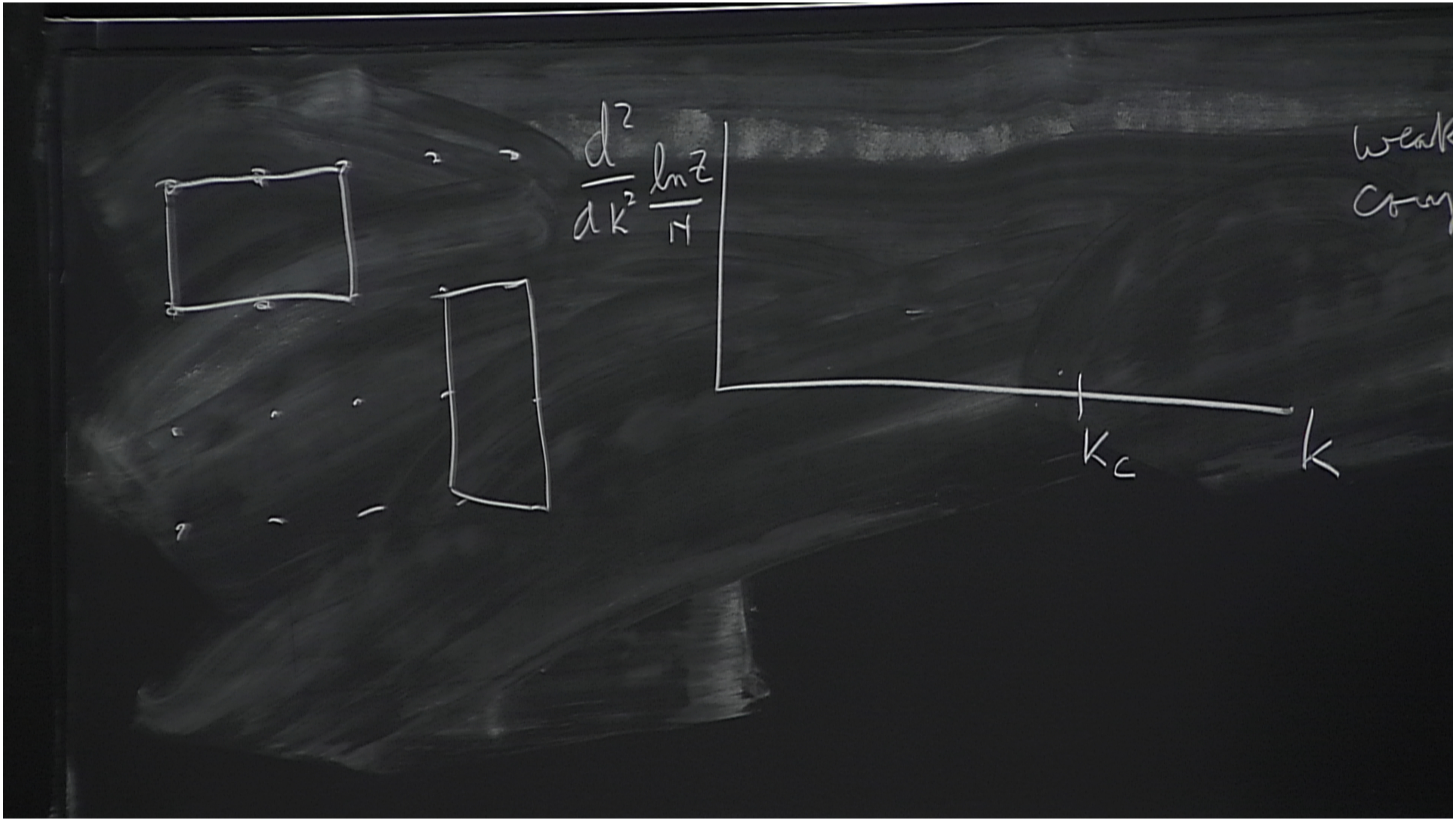


$$\frac{\ln Z}{N} = \ln 2 + \ln \cosh^2 K + a_4 \tanh^4 K + a_6 \tanh^6 K + \cdots$$

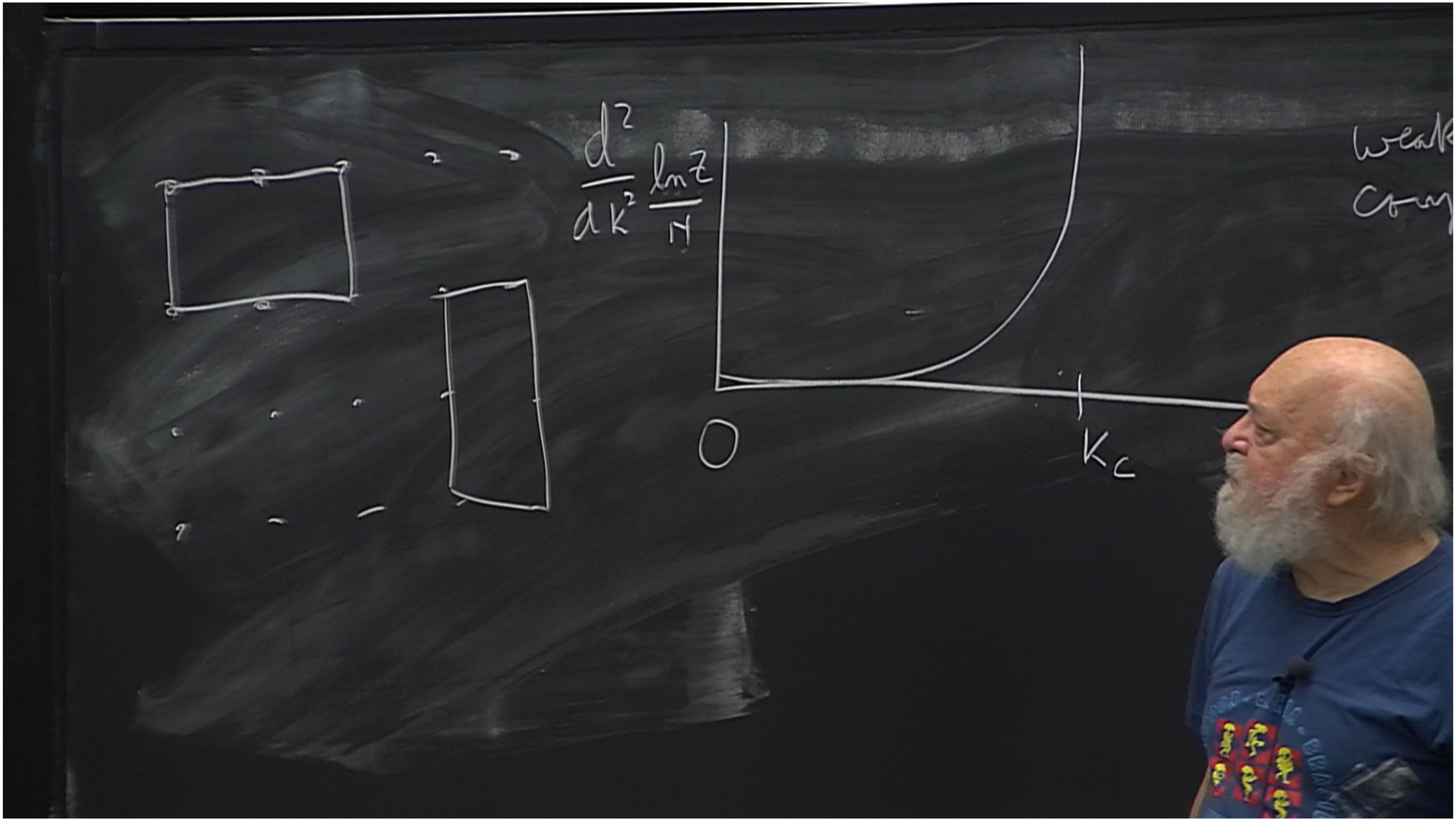
$$\langle \; \sigma_1 \sigma_2 \tanh^2 K \; \sigma_2 \sigma_3$$

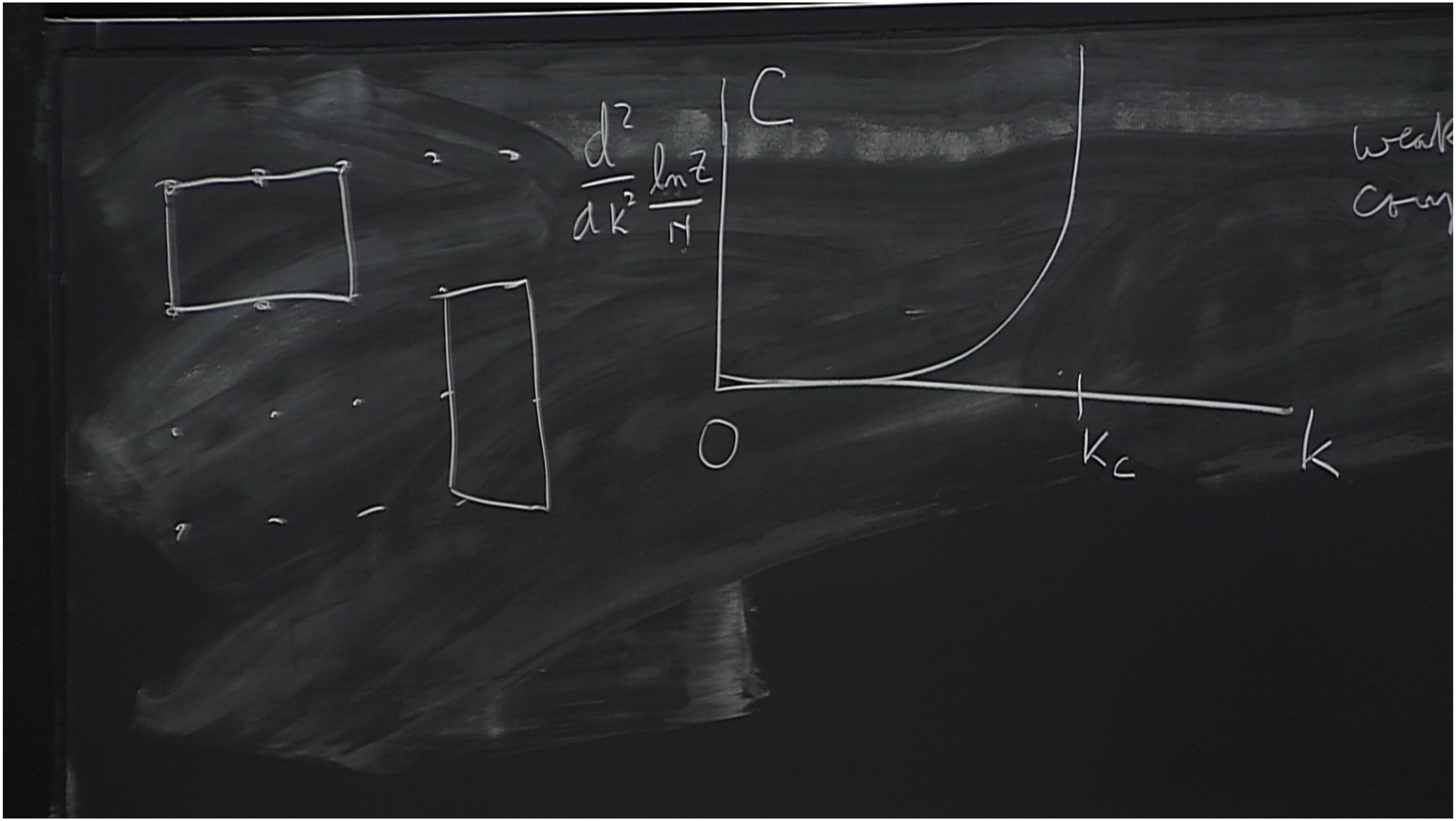

weak
coupling

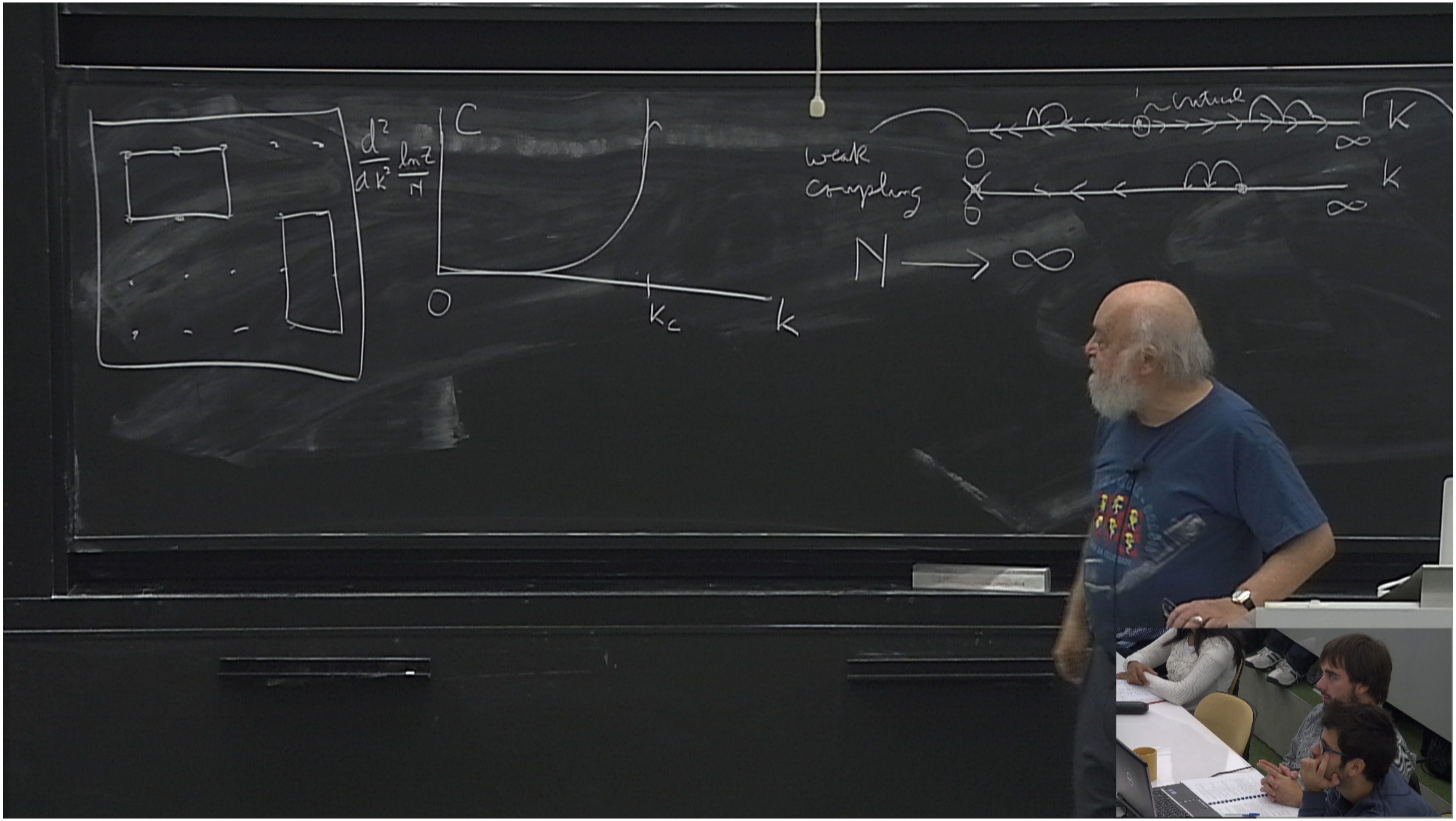




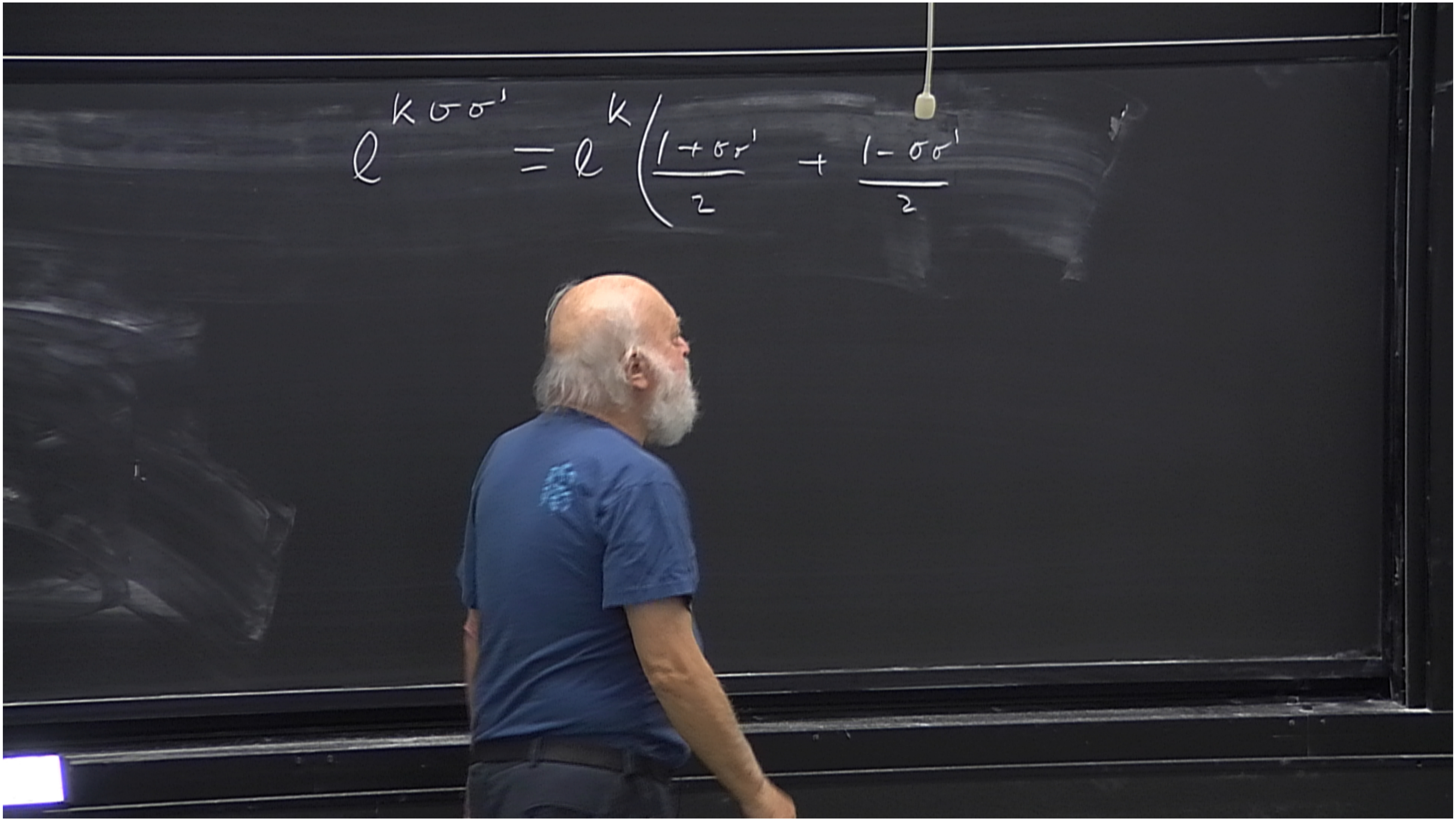
$\frac{d^2}{dK^2} \frac{\ln Z}{N}$
K_c
K


$\frac{d^2}{dK^2}\frac{\ln Z}{N}$
0
K_c
weak
coup


$\frac{d^2}{dK^2}\frac{\ln Z}{N}$
C
0
K_c
K

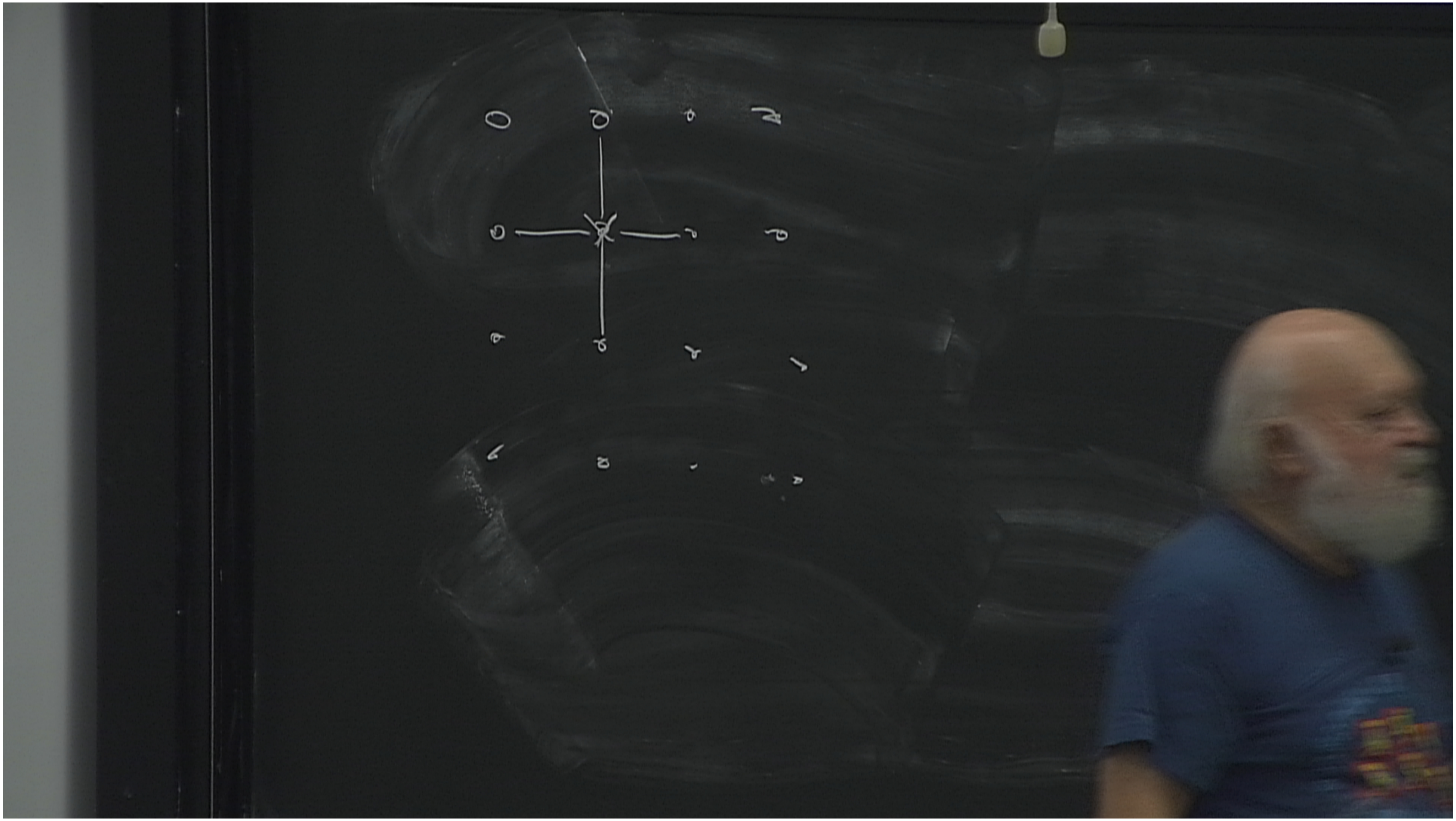


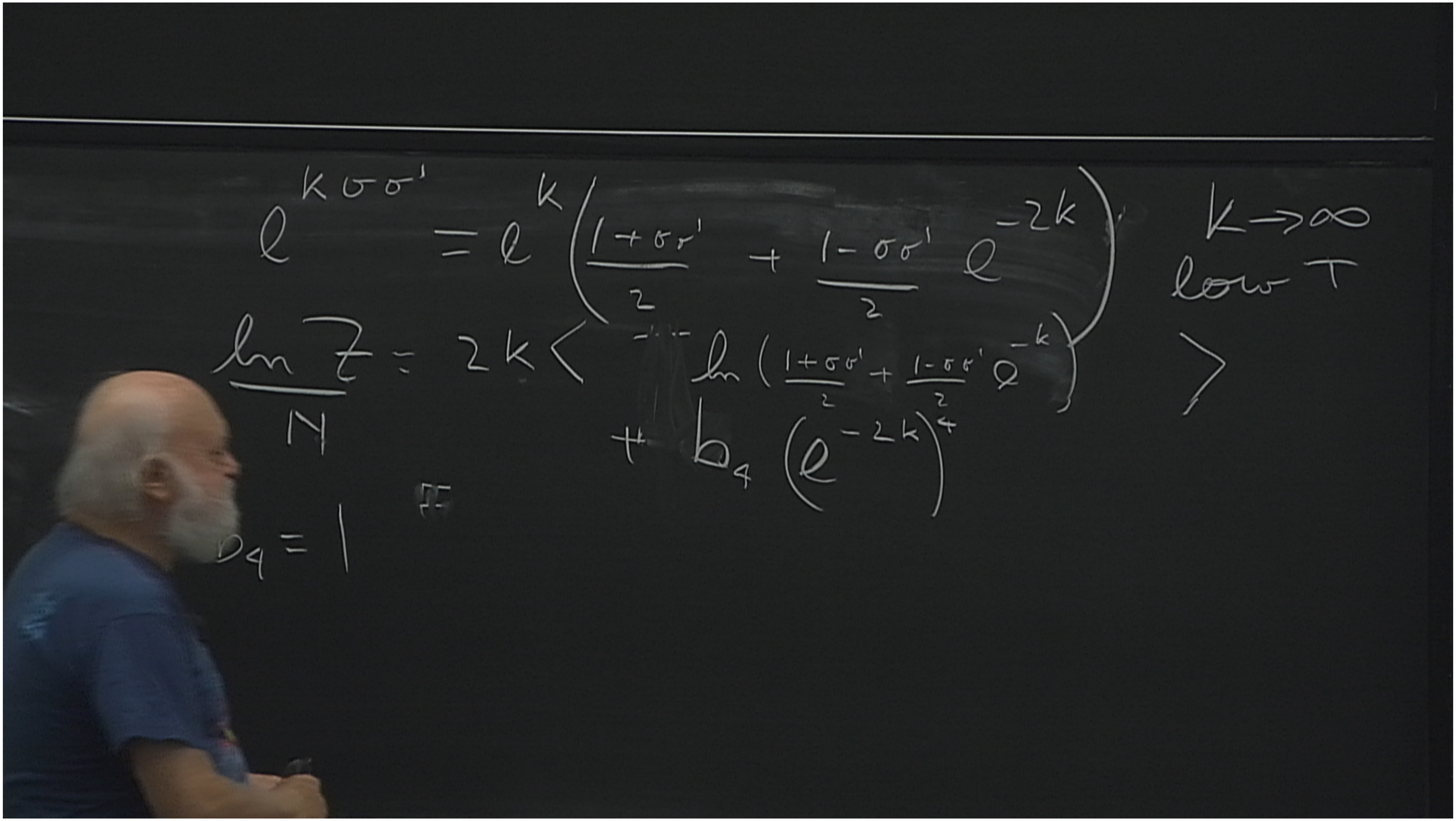
$\frac{d^2}{dk^2}\frac{\ln Z}{N}$
C
O
k_c
k
critical
K
∞
weak coupling
O
k
∞
N → ∞


e^{kσσ'} = e^k ((1+σσ')/2 + (1-σσ')/2

$$e^{k\sigma\sigma'} = e^{k}\left(\frac{1+\sigma\sigma'}{2} + \frac{1-\sigma\sigma'}{2}e^{-2k}\right)$$

$$\frac{\ln Z}{N} = 2k$$




e^{kσσ'} = e^k ((1+σσ')/2 + (1−σσ')/2 e^{−2k})
k → ∞
low T
ln Z / N = 2k < ln((1+σσ')/2 + (1−σσ')/2 e^{−k}) >
+ b4 (e^{−2k})^4
b4 = 1

$$e^{k\sigma\sigma'} = e^{k}\left(\frac{1+\sigma\sigma'}{2} + \frac{1-\sigma\sigma'}{2}e^{-2k}\right)$$

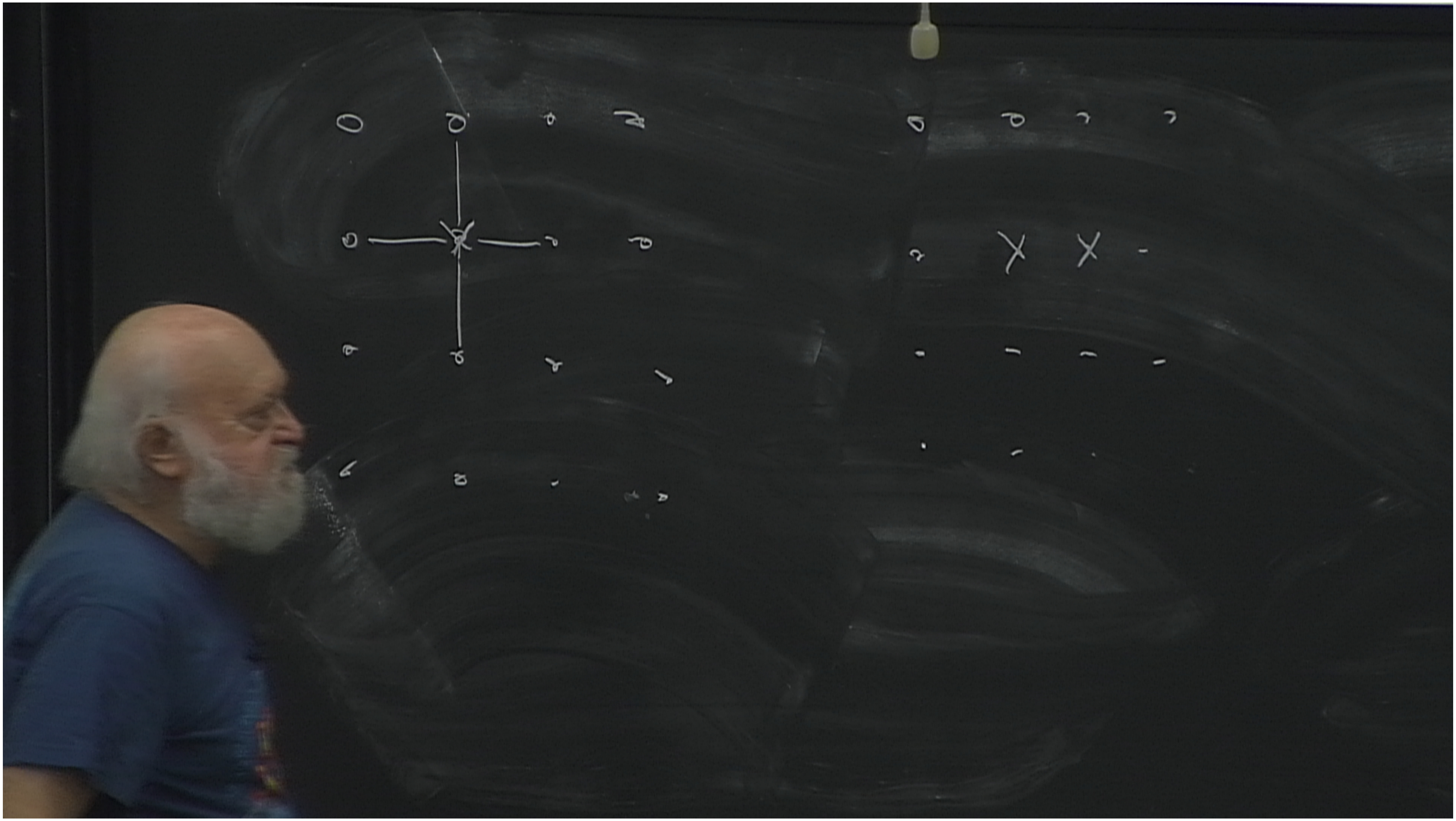
$k \to \infty$

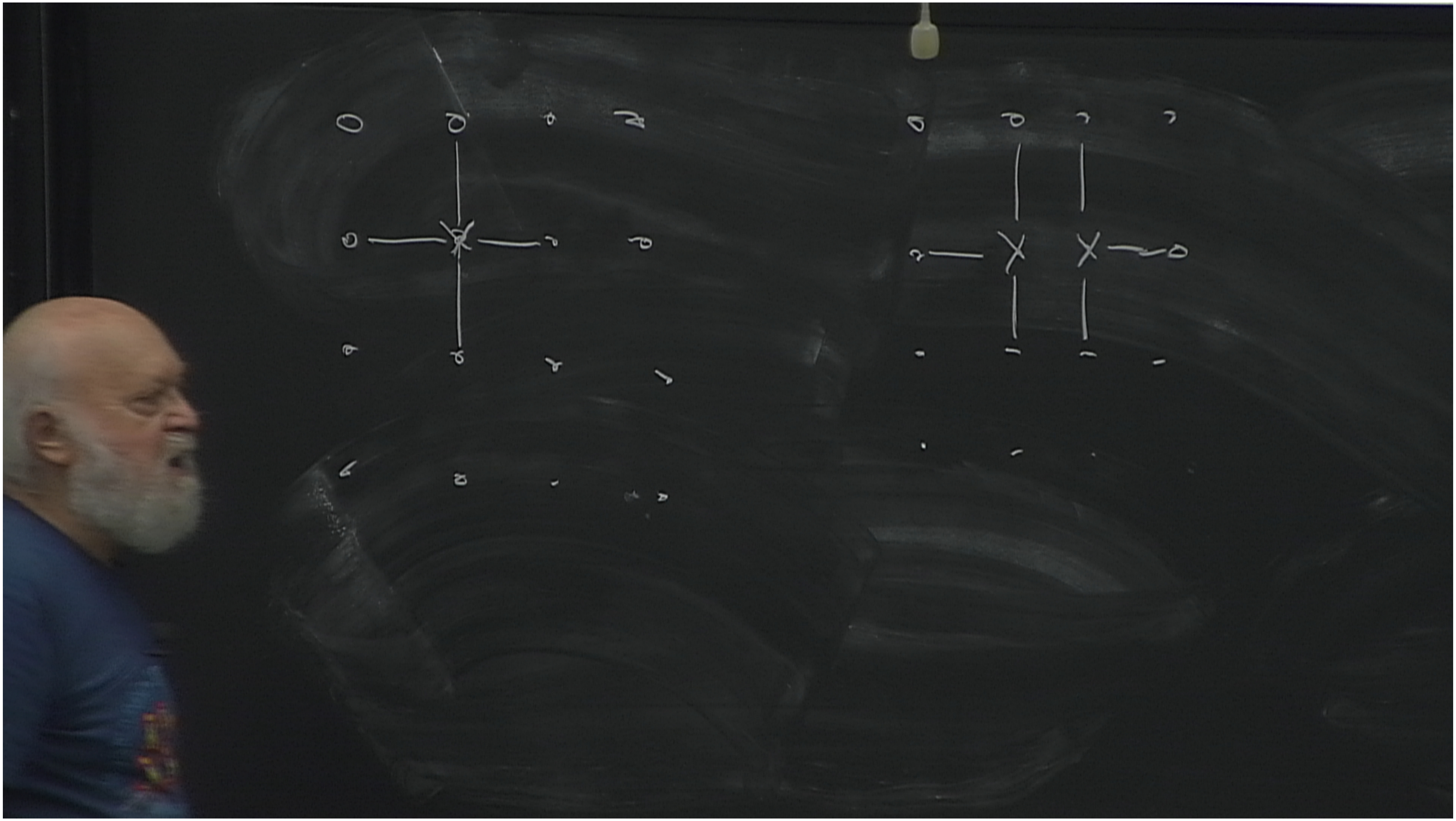
low T

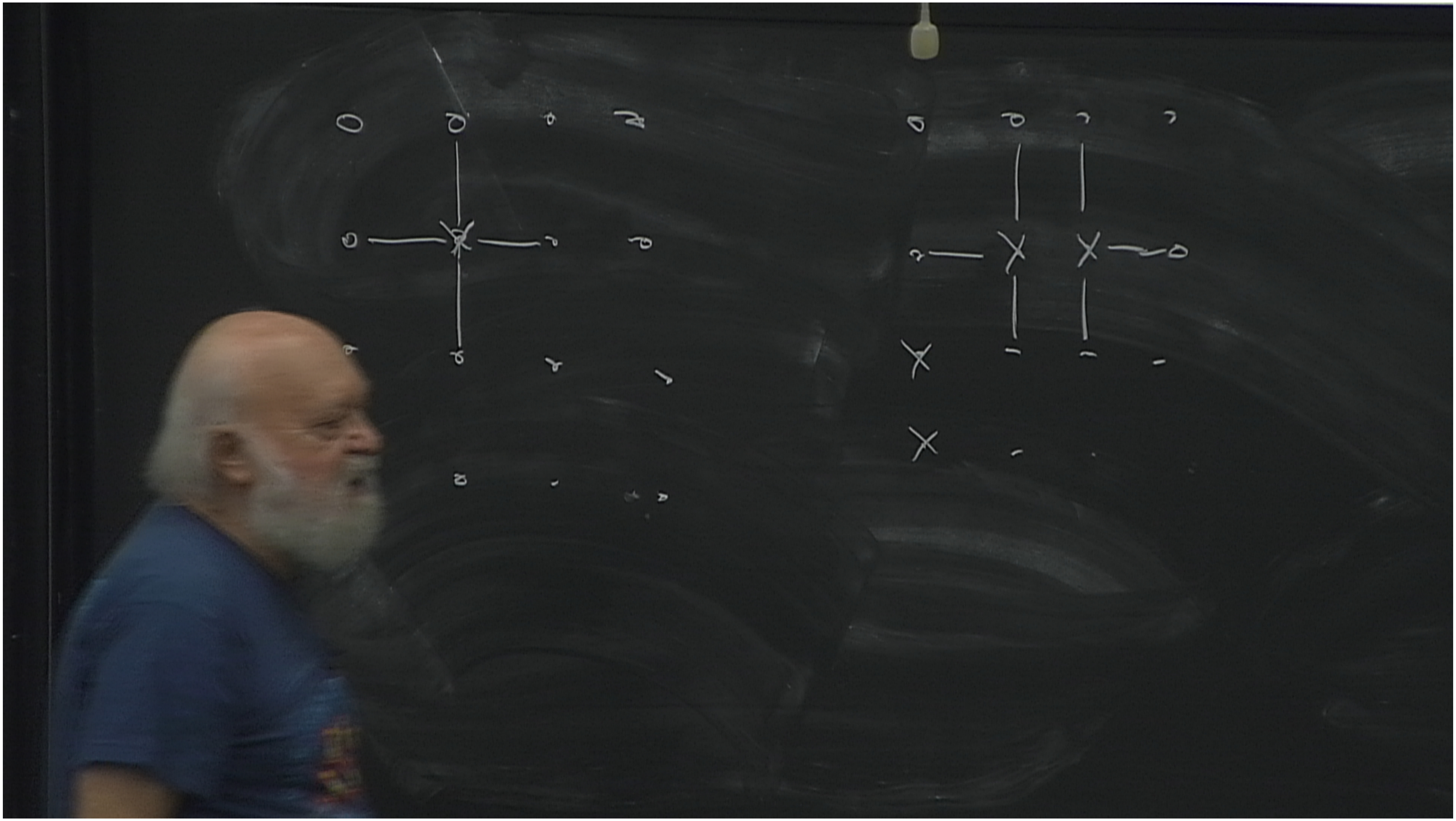
$$\frac{\ln Z}{N} = 2k \left\langle \ln\left(\frac{1+\sigma\sigma'}{2} + \frac{1-\sigma\sigma'}{2}e^{-k}\right) \right\rangle$$

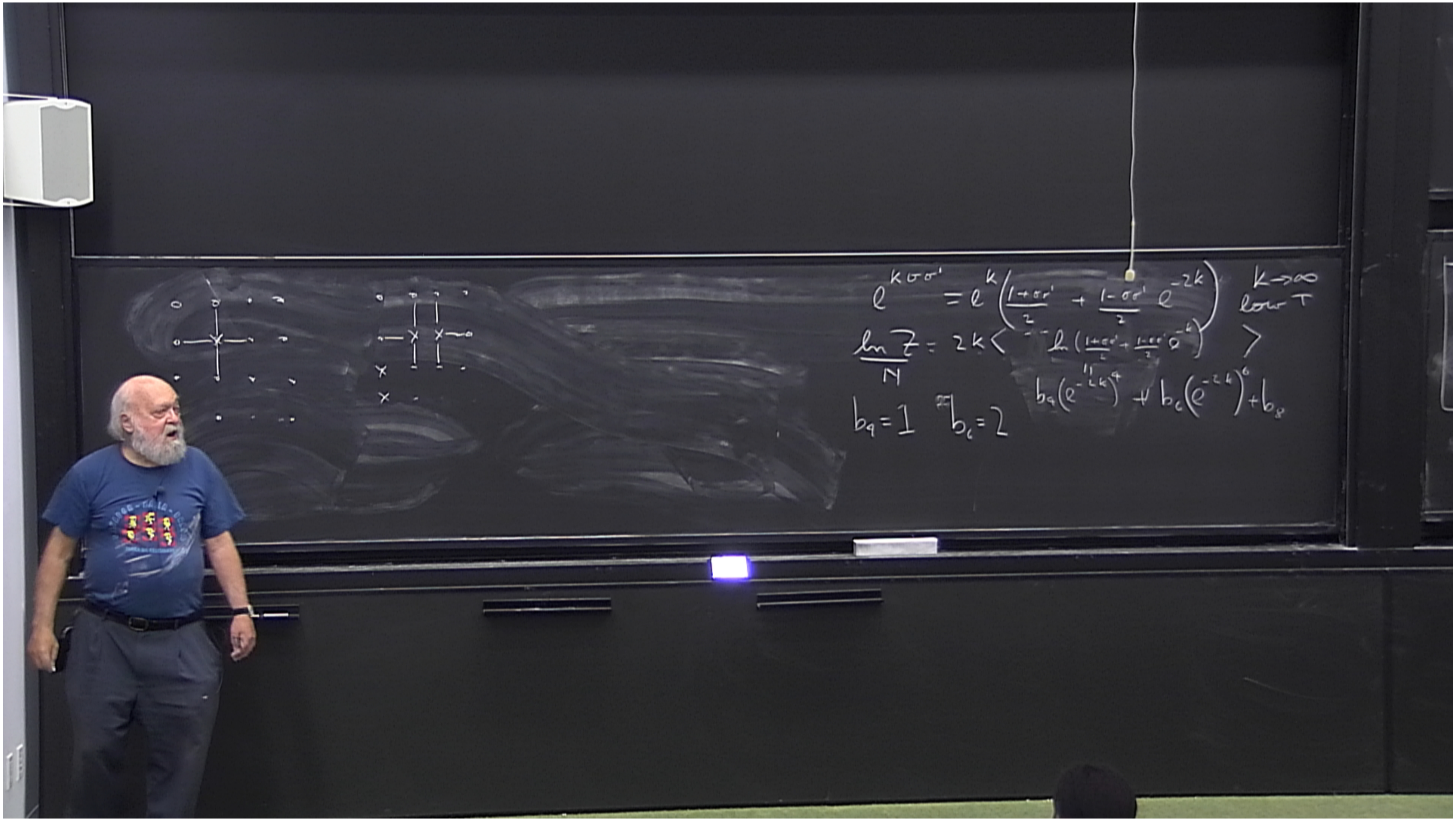
$$= b_4\left(e^{-2k}\right)^4 + b_6\left(e^{-2k}\right)^6 + b_8$$

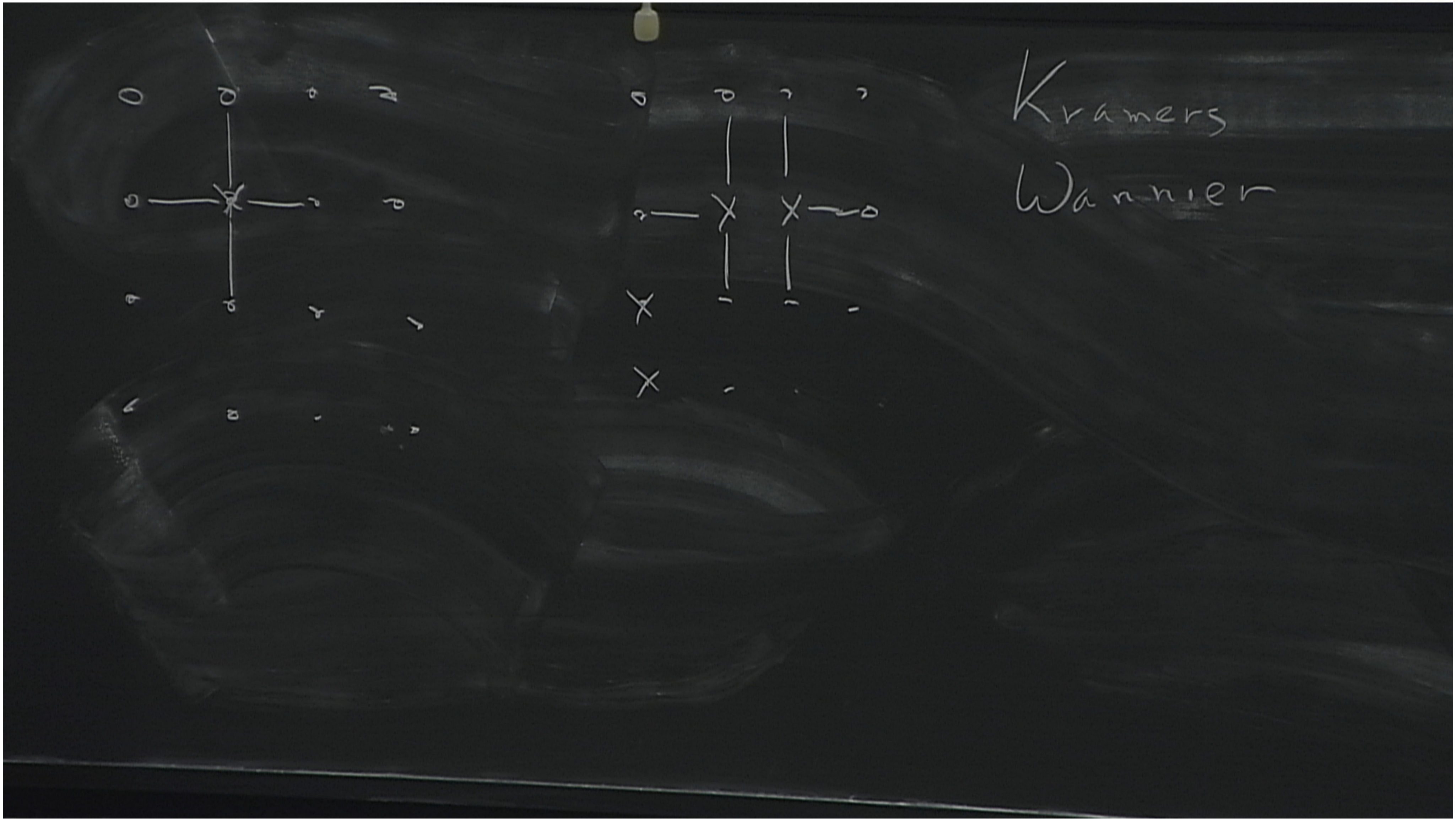
$b_4 = 1$

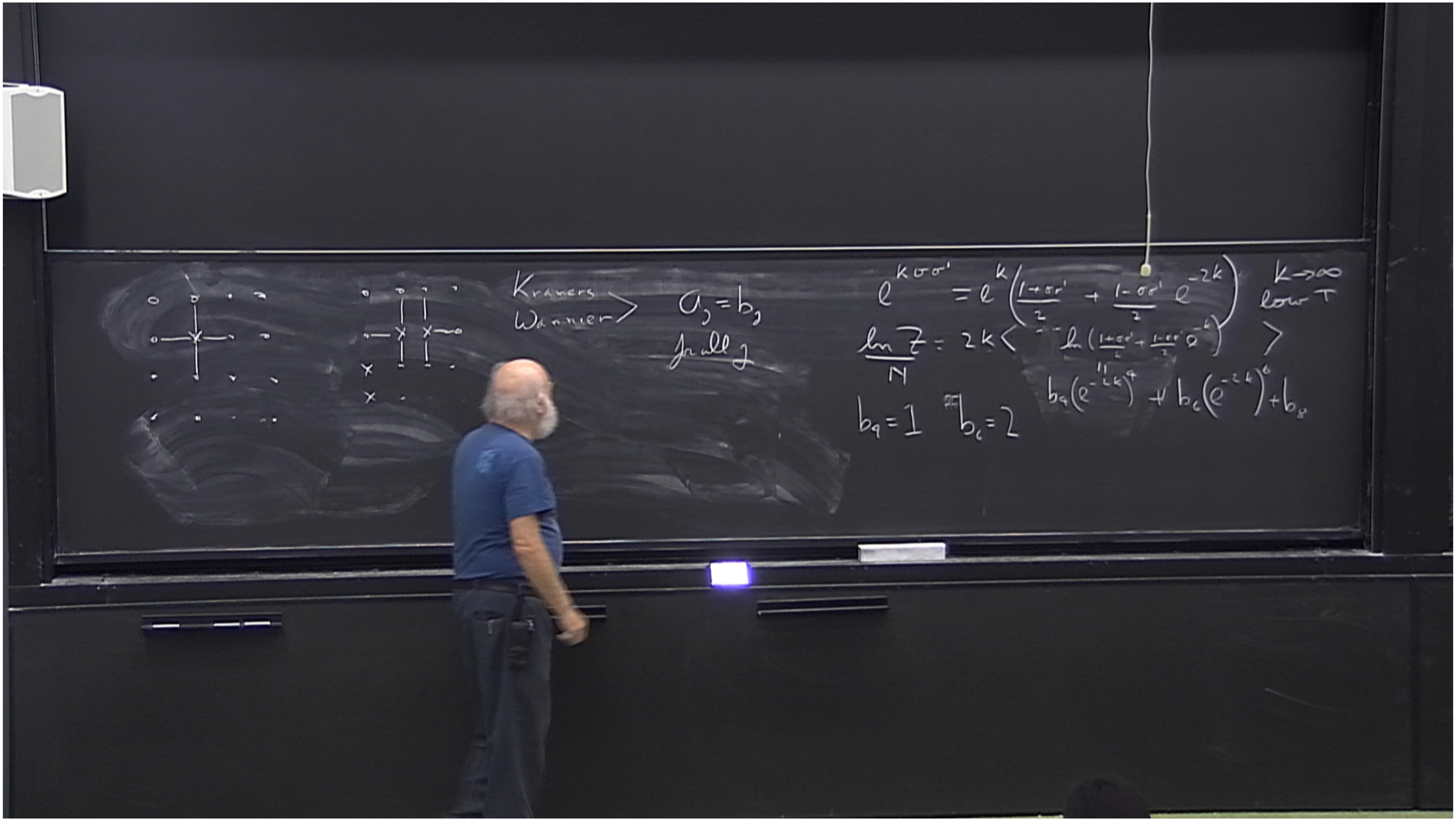


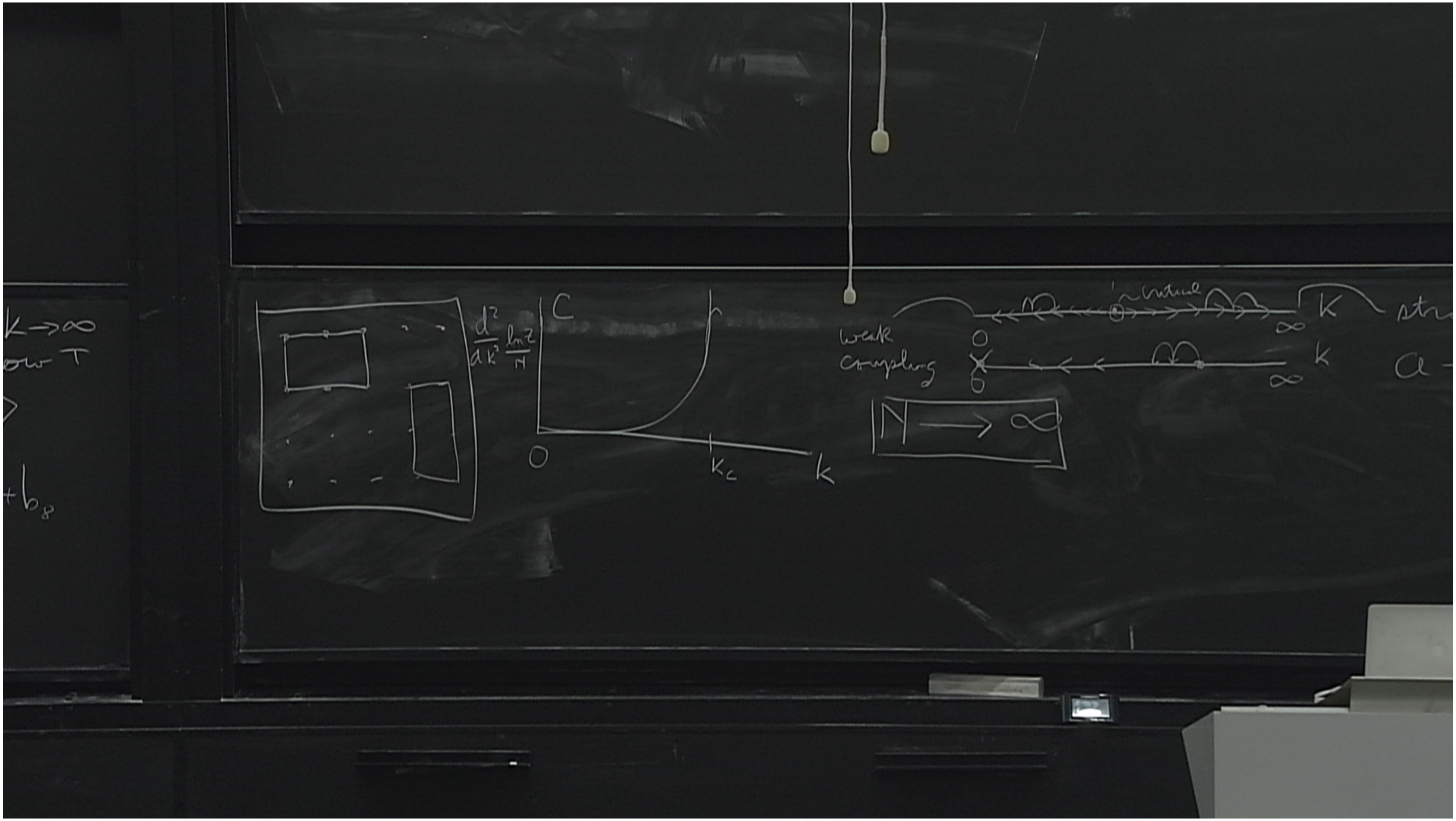


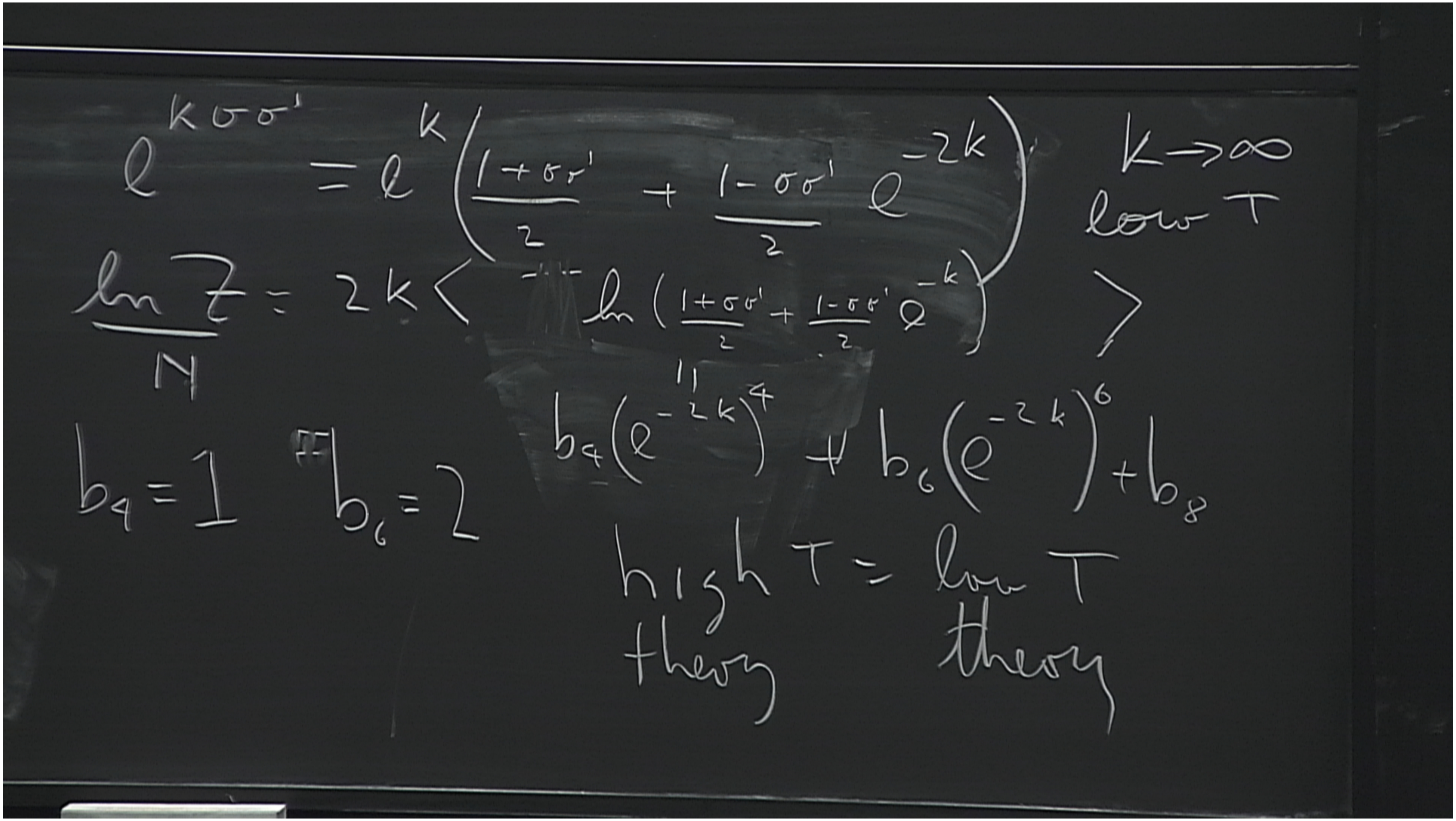



k→∞
low T
b4=1
b6=2

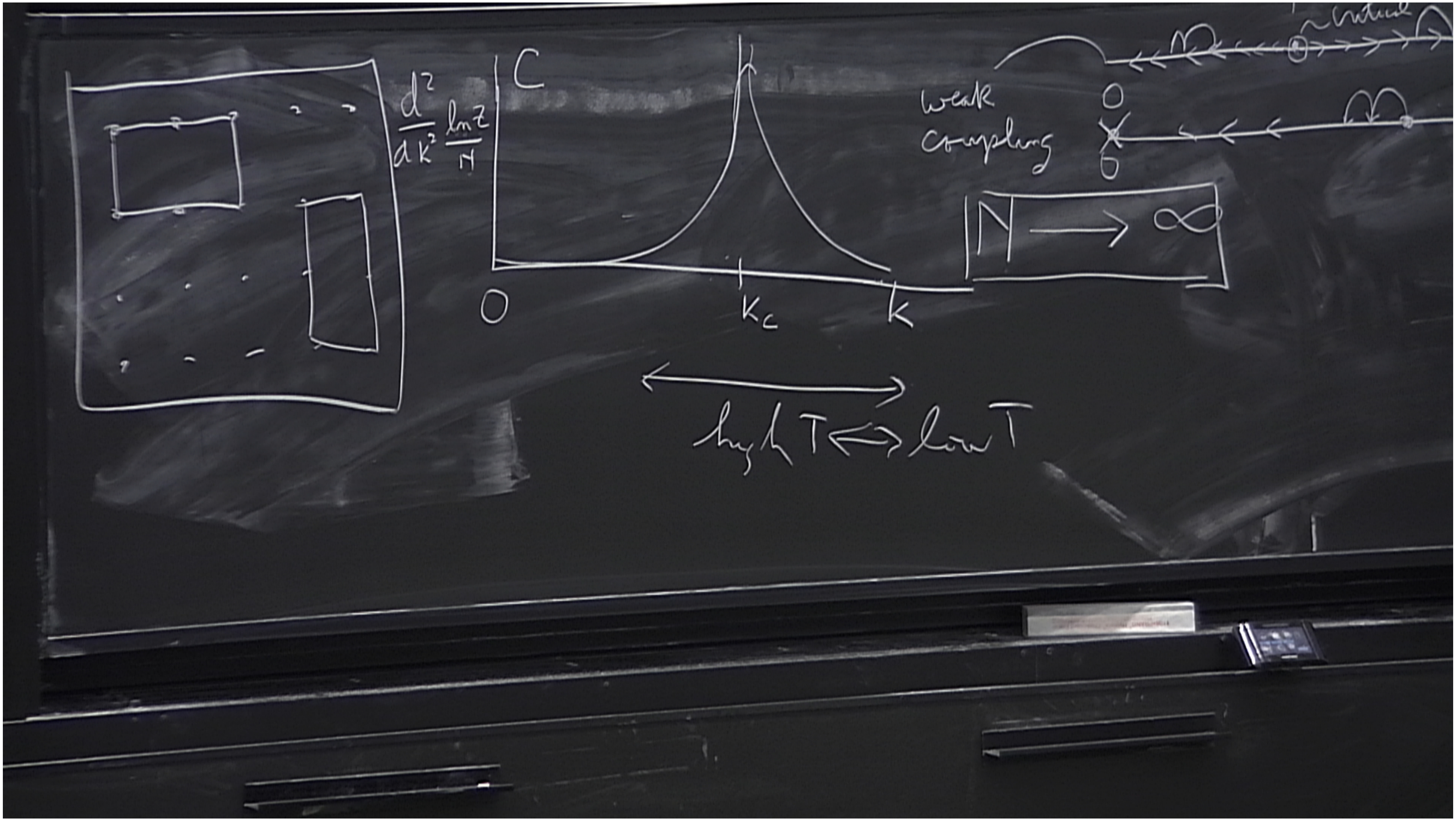

Kramers
Wannier

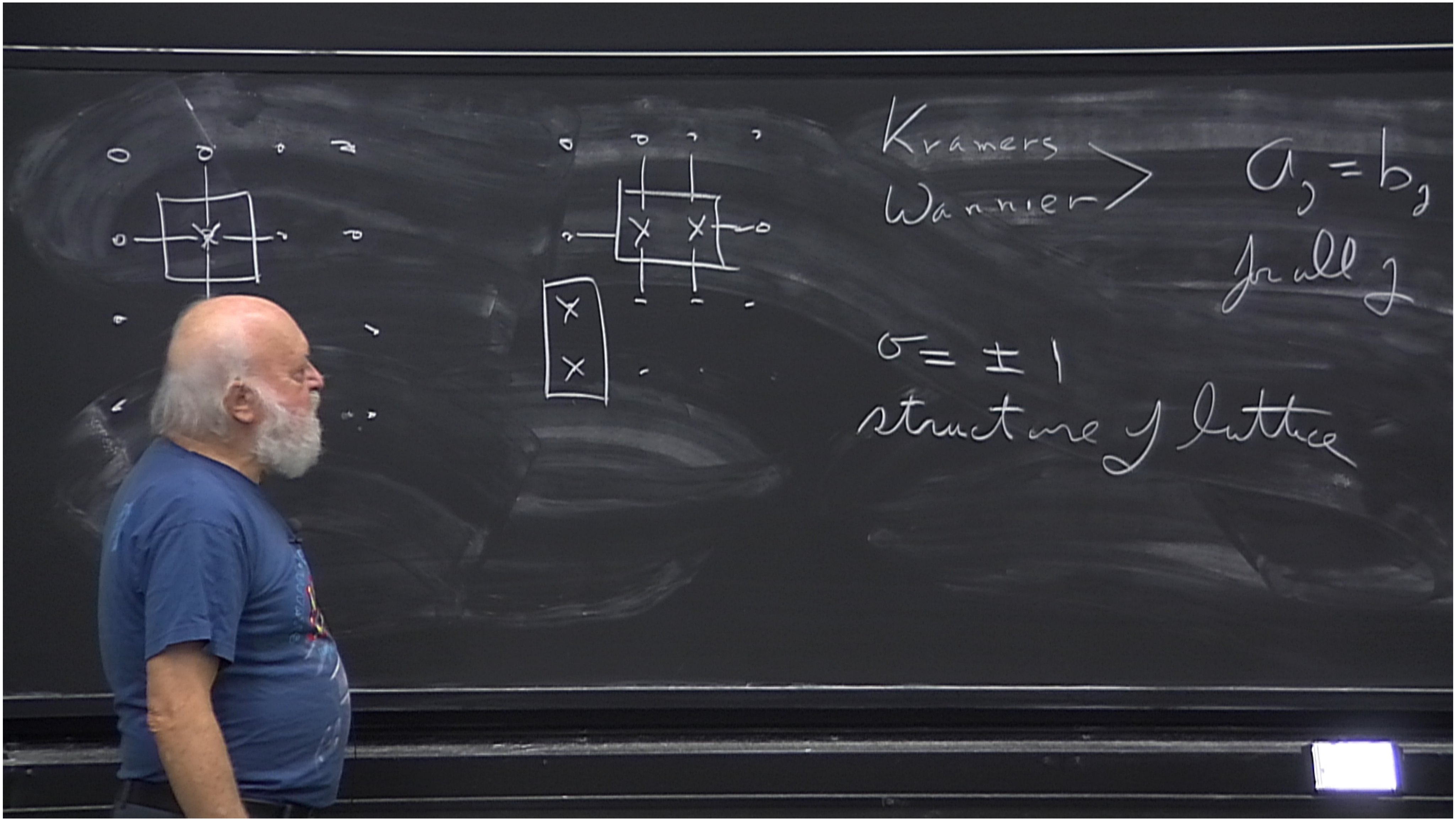

Kramers Wannier
k→∞
low T
b4 = 1
b6 = 2

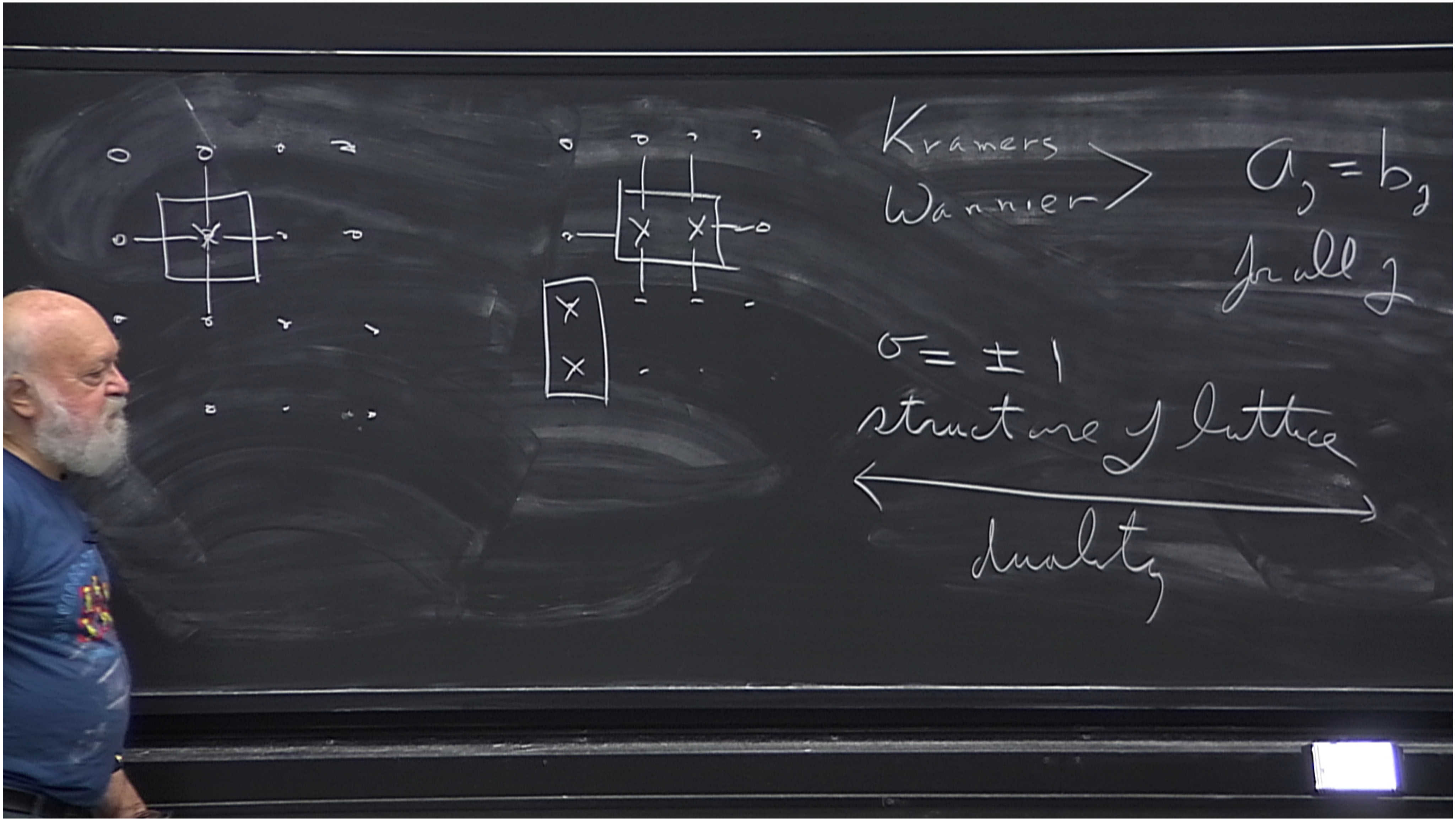

k → ∞
weak coupling
N → ∞
natural
C
0
k
∞

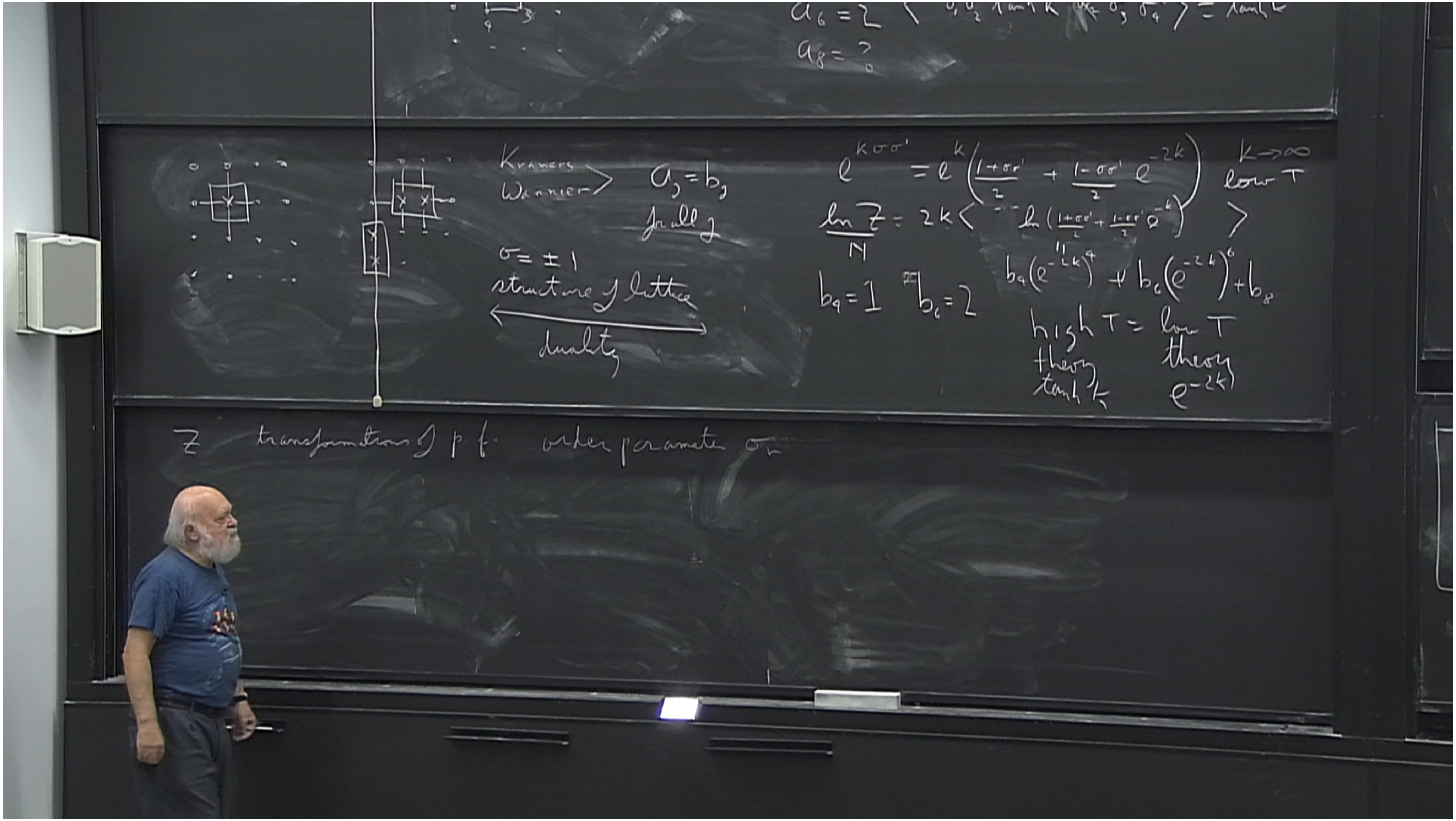


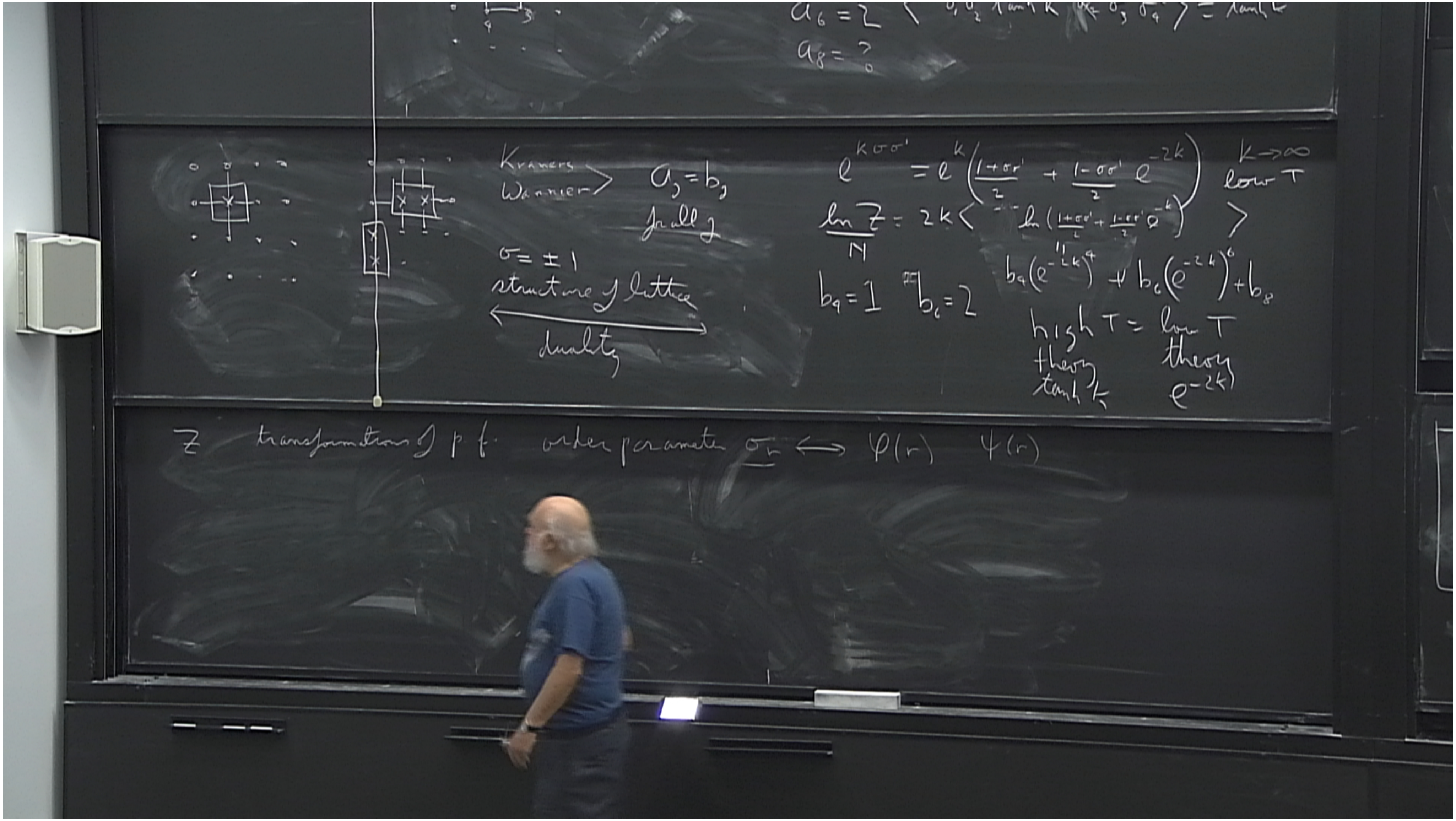
e^{kσσ'} = e^{k}((1+σσ')/2 + (1−σσ')/2 e^{−2k})
k → ∞
low T
ln Z / N = 2k < ln((1+σσ')/2 + (1−σσ')/2 e^{−k}) >
b_4 (e^{−2k})^4 + b_6 (e^{−2k})^6 + b_8
b_4 = 1
b_6 = 2
high T theory = low T theory

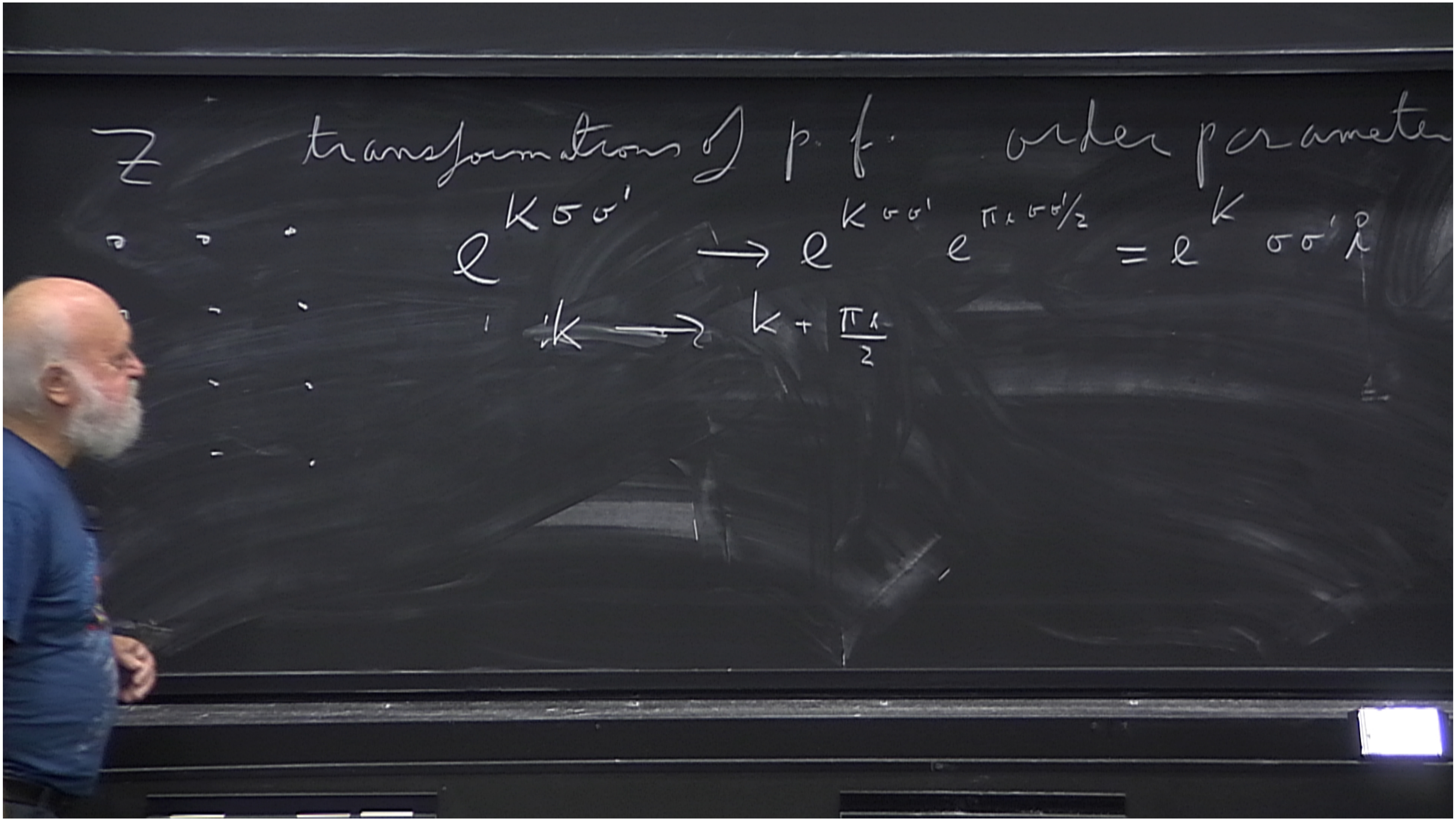

C
d²/dK² lnZ/N
0
K_c
K
weak coupling
N → ∞
high T ⟺ low T

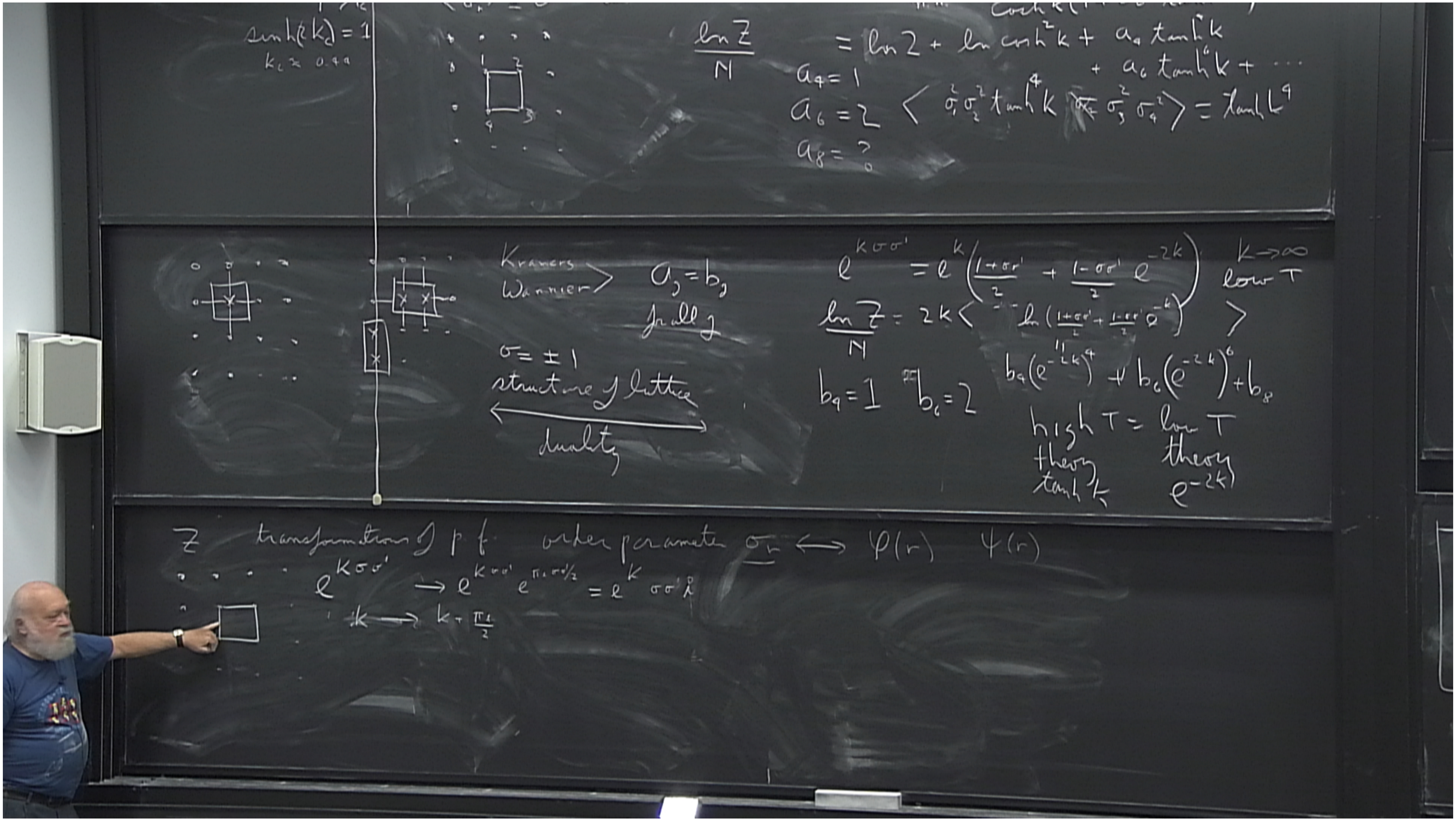

Kramers
Wannier
σ = ±1
structure of lattice

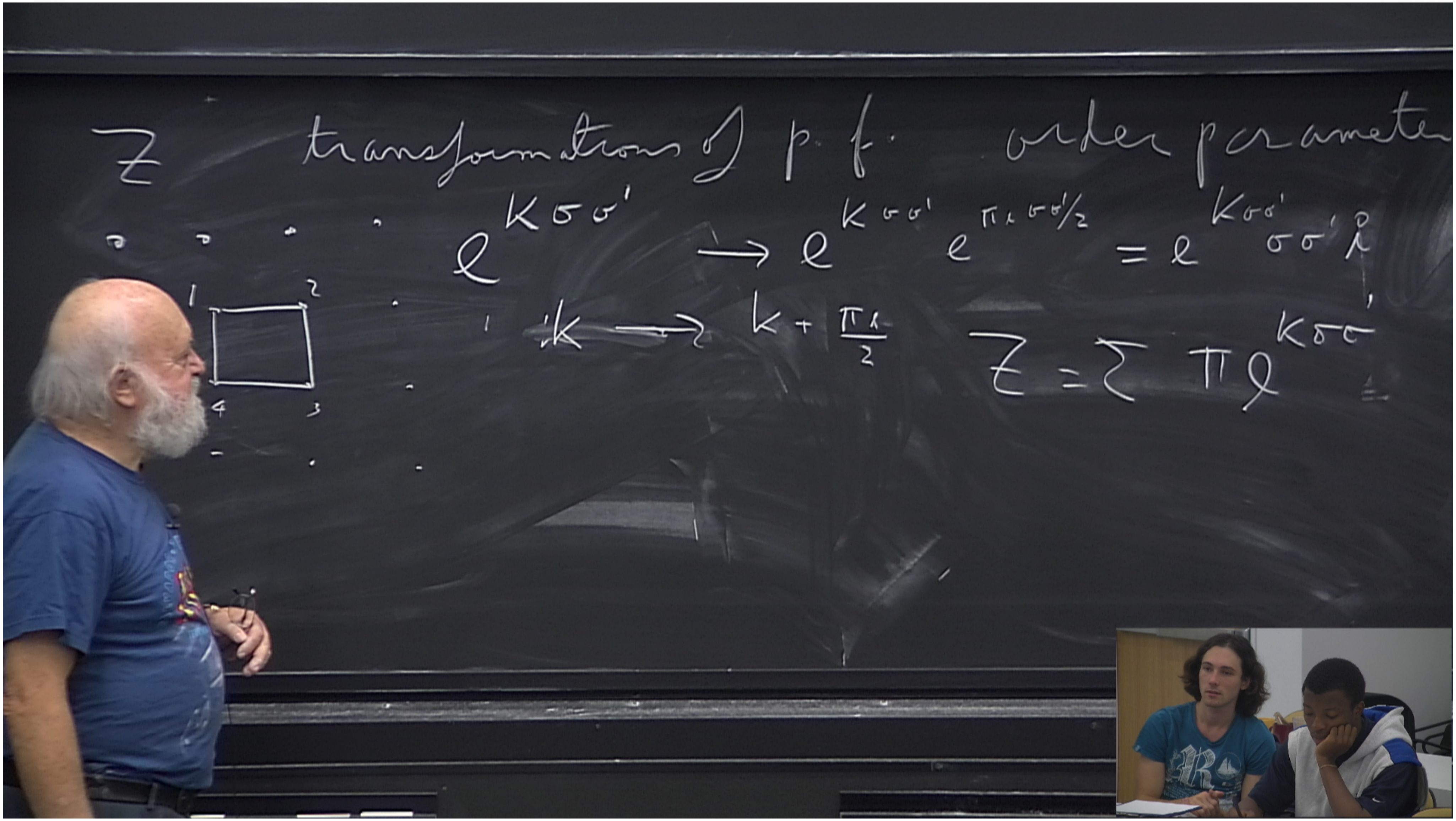

Kramers
Wannier
σ = ±1
structure of lattice
duality


Kramers
Wannier
σ = ±1
structure of lattice
duality
b4 = 1
b6 = 2
k → ∞
low T
high T = low T
theory theory
Z
transformations of p.f.
other parameters

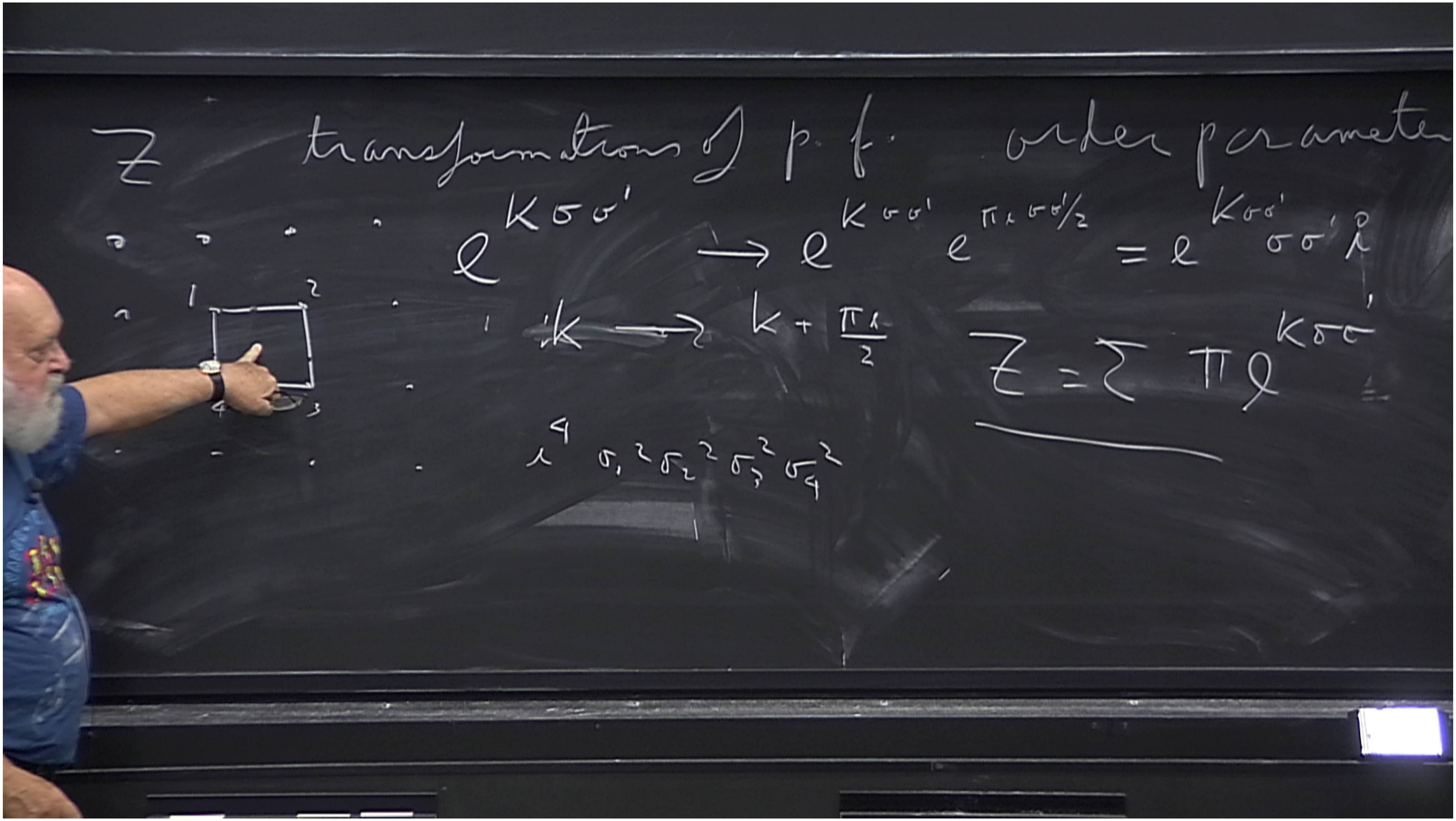

Kramers
Wannier
duality
σ = ±1
structure of lattice
duality
k→∞
low T
b4 = 1
b6 = 2
high T = low T
theory theory
Z
other parameter

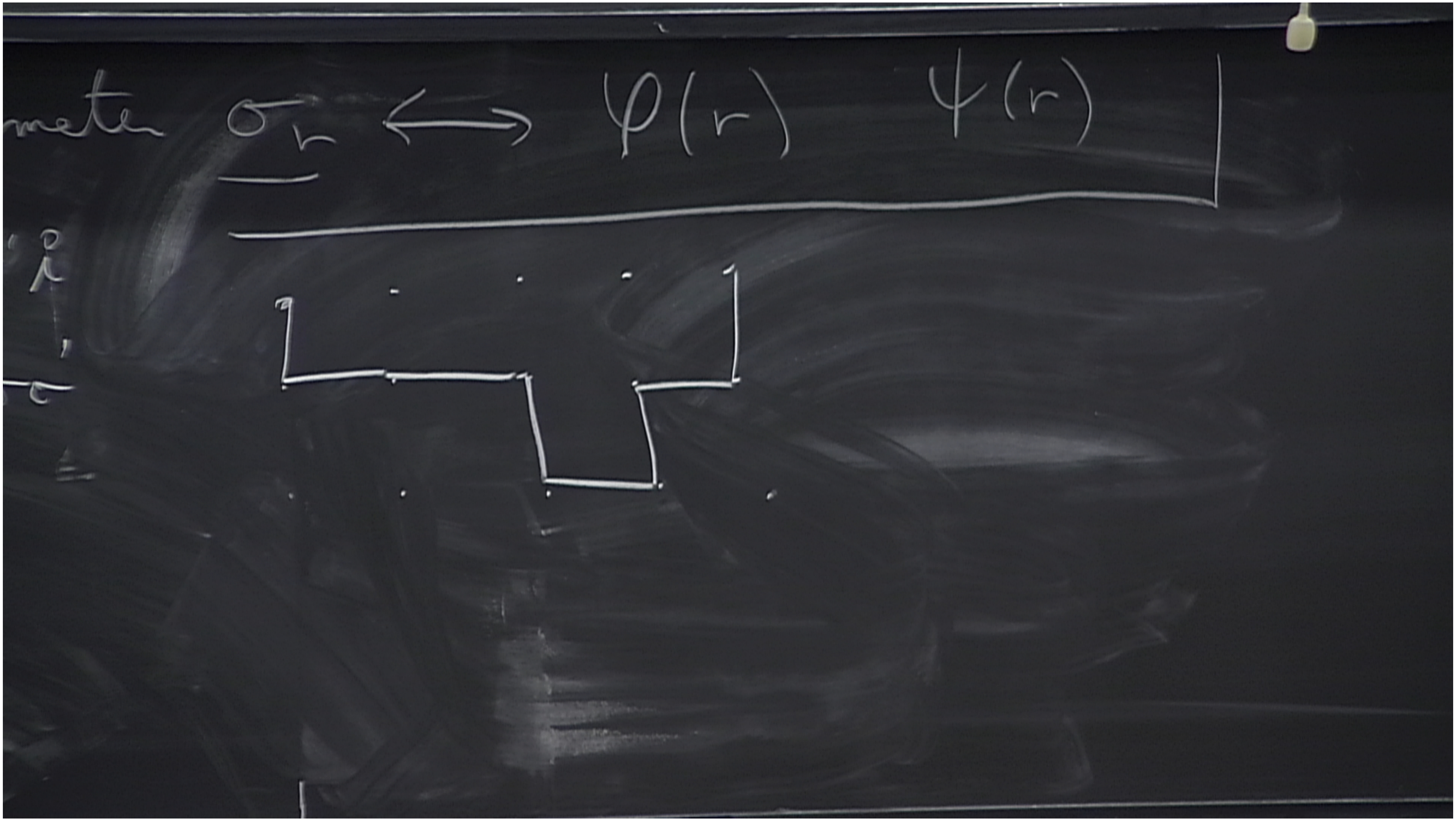

Z transformations of p.f. order parameter
e^{Kσσ'} → e^{Kσσ'} e^{πiσσ'/2} = e^{K σσ' i}
K → K + πi/2

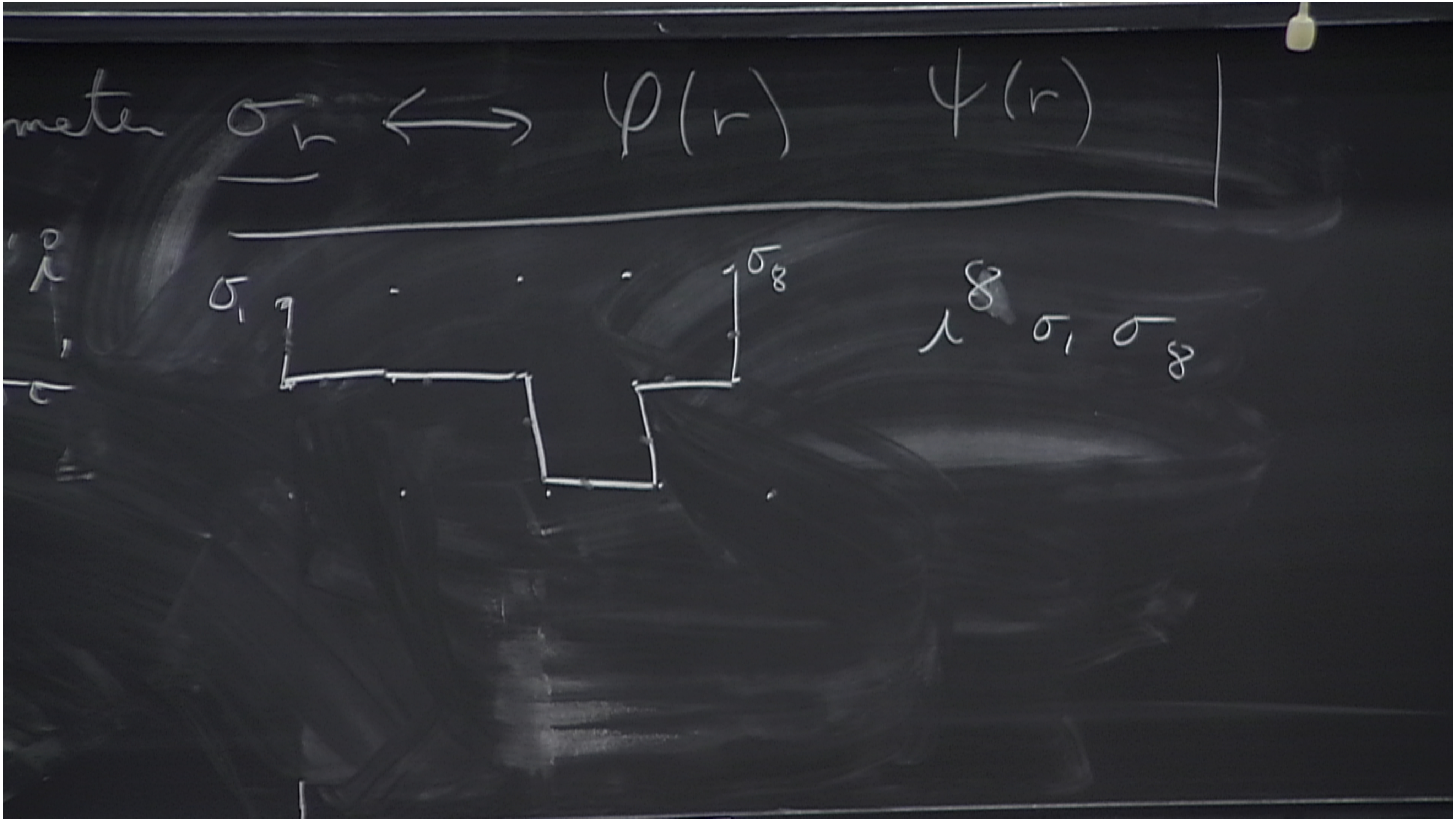

Kramers
Wannier
σ = ±1
structure of lattice
duality
low T
high T = low T
theory theory

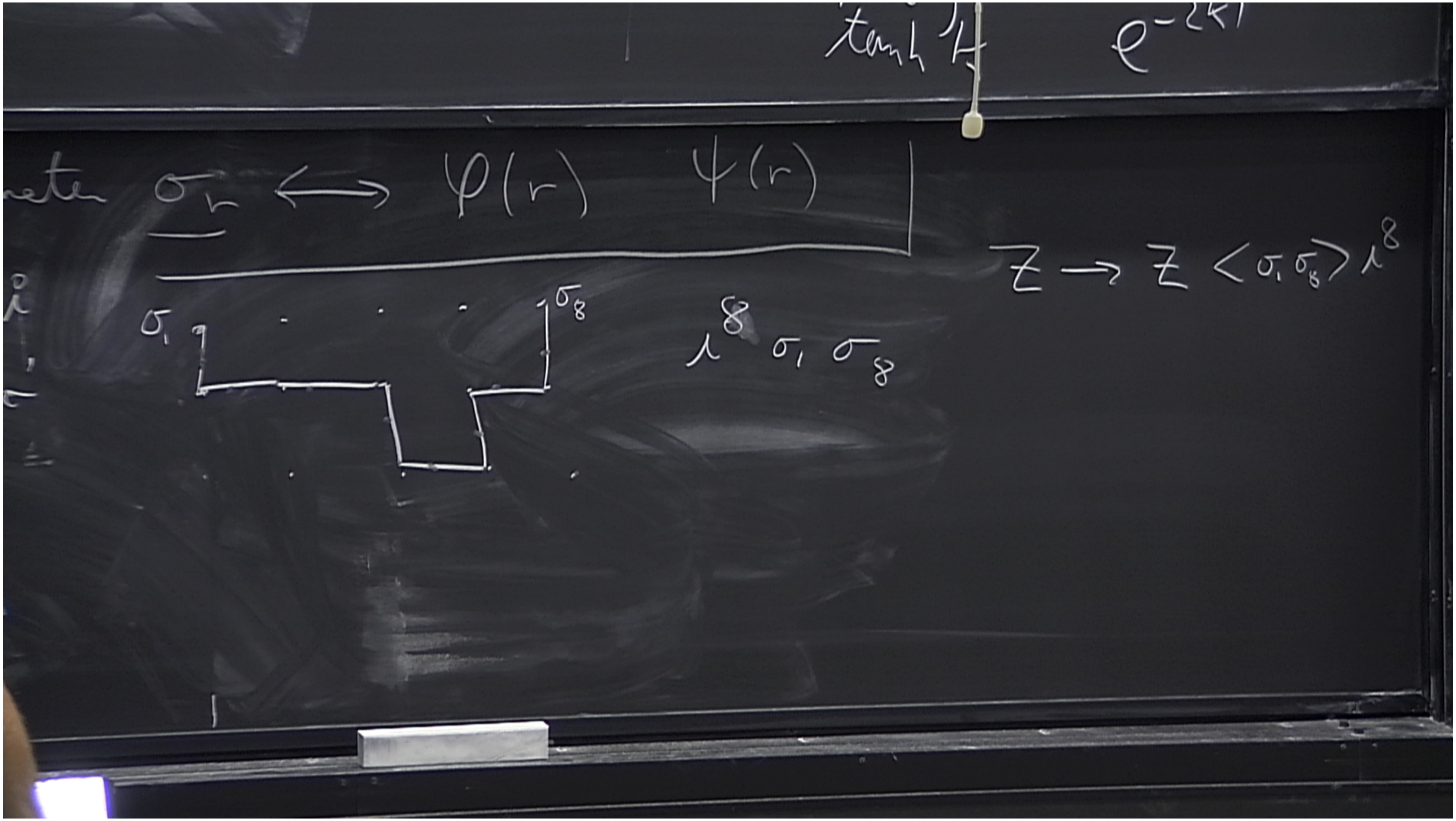


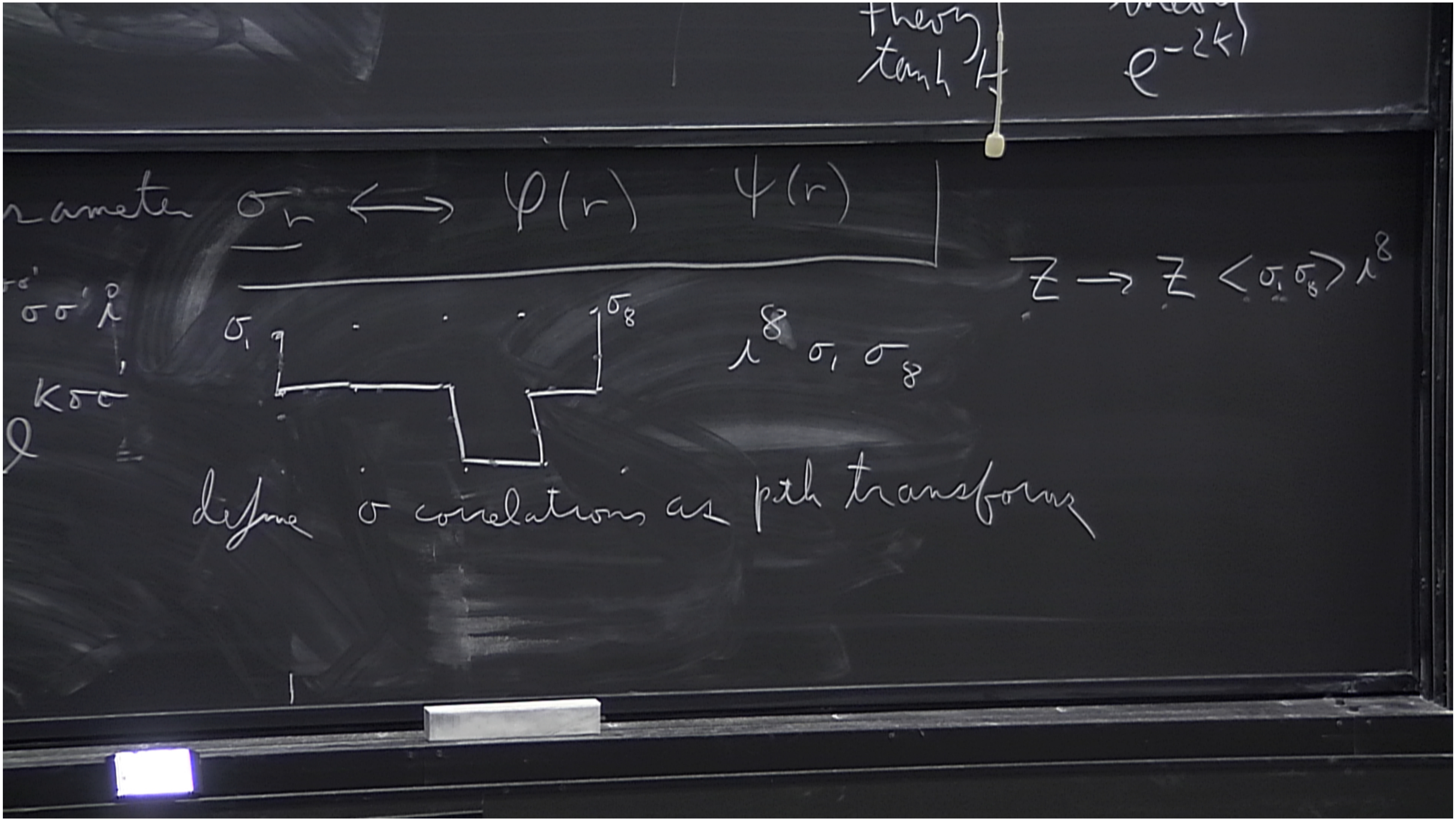
Z transformations of p.f. order parameter
e^{Kσσ'} → e^{Kσσ'} e^{πiσσ'/2} = e^{Kσσ'} σσ' i
K → K + πi/2
Z = Σ Π e^{Kσσ'}
1 2 4 3

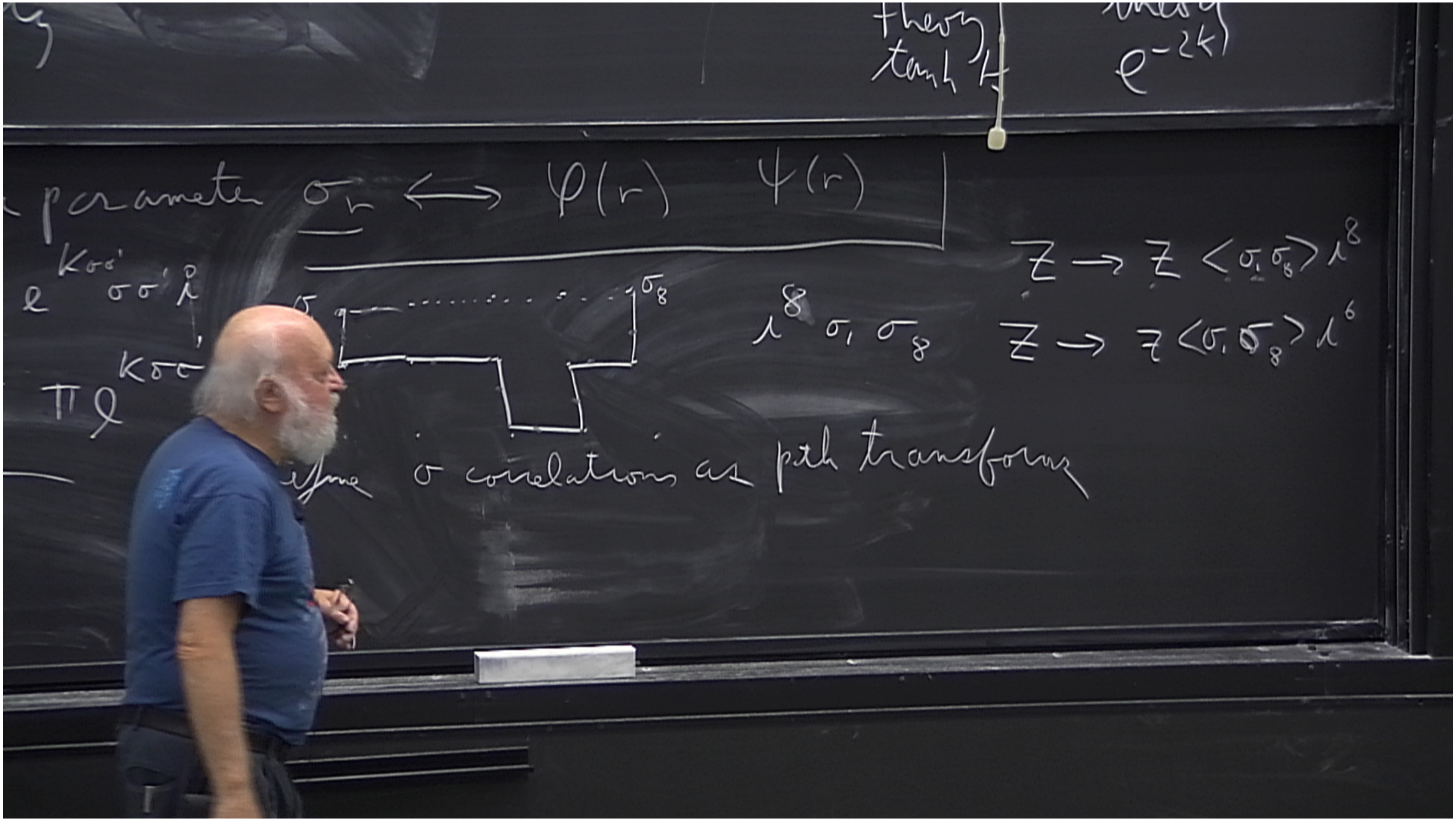

Z transformations of p.f. order parameter

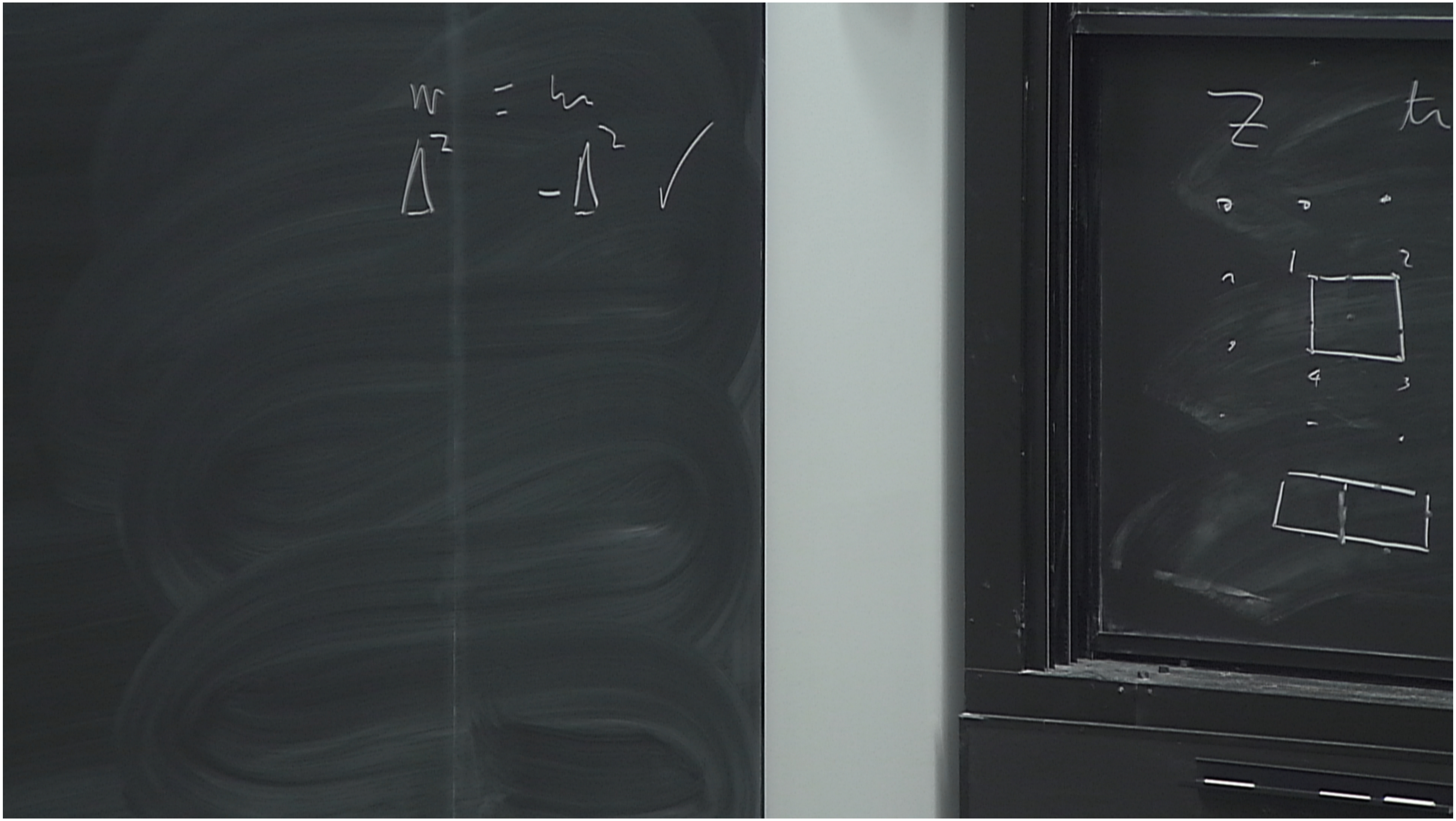

σr ⟷ φ(r) ψ(r)

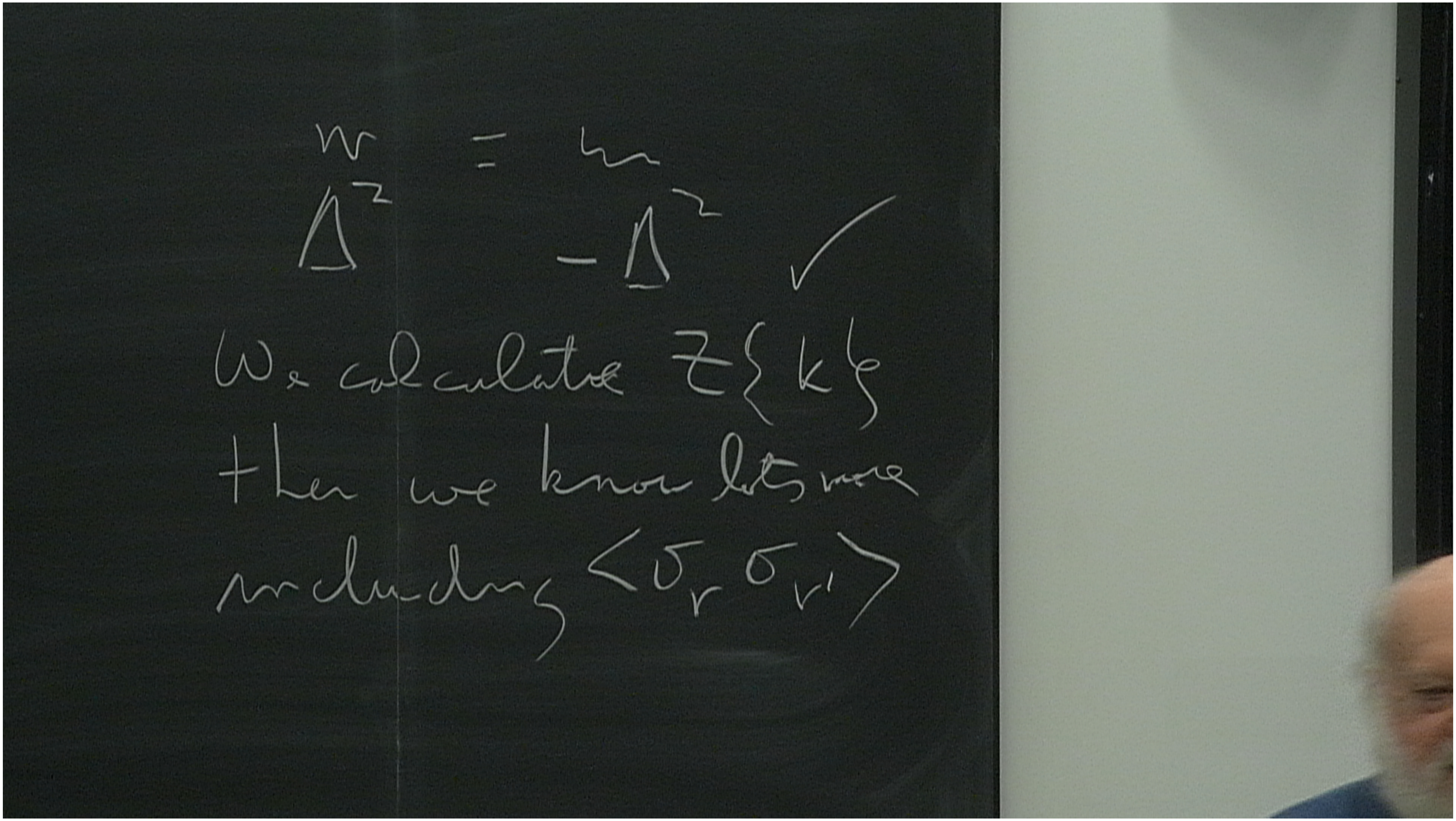

meter σ_r ⟷ φ(r) ψ(r)
σ_1
σ_8
λ^8 σ_1 σ_8

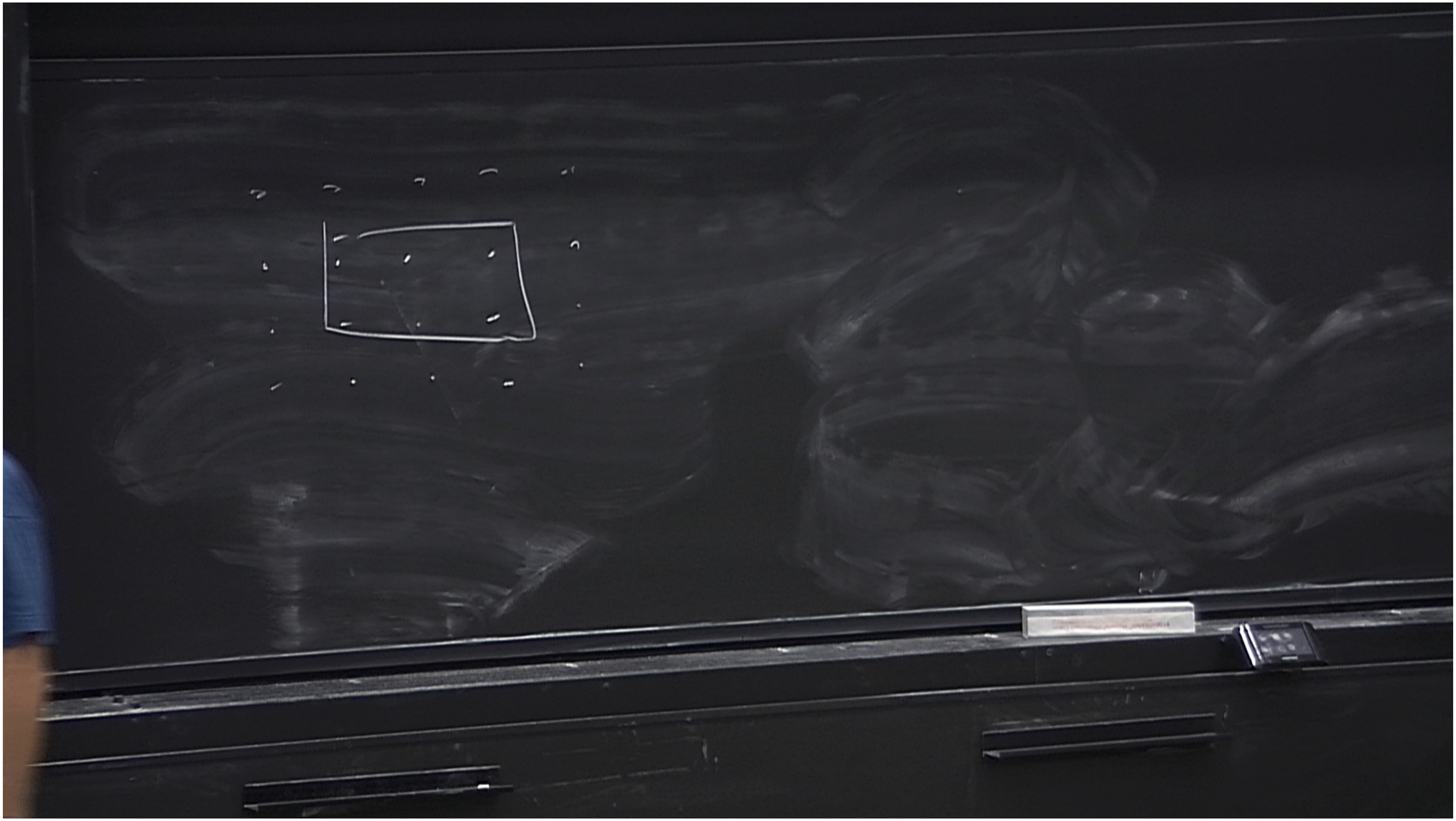

tanh
σr ⟷ φ(r) ψ(r)
σ1 σ8
λ8 σ1 σ8
Z → Z <σ1 σ8> λ8


tanh H
e^{-2k}
rameter σ_r ⟷ φ(r) ψ(r)
Z → Z <σ_1 σ_8> λ^8
σ_1 σ_8
λ^8 σ_1 σ_8
Kσσ'
define σ correlations as path transforms


theory
tanh H
e^-2K
parameter σ_r ⟷ φ(r) ψ(r)
Z → Z <σ1 σ8> λ^8
λ^8 σ1 σ8
Z → Z <σ1 σ8> λ^6
σ correlations as pth transform




We calculate Z{k}
then we know lots more
including <σr σr'>



$$i^{p} \langle \sigma_r \sigma_{r'} \rangle = \frac{Z\{k'\}}{Z\{k\}}$$

$$\frac{w}{\Delta^2} = \frac{w}{-\Delta^2} \quad \checkmark$$

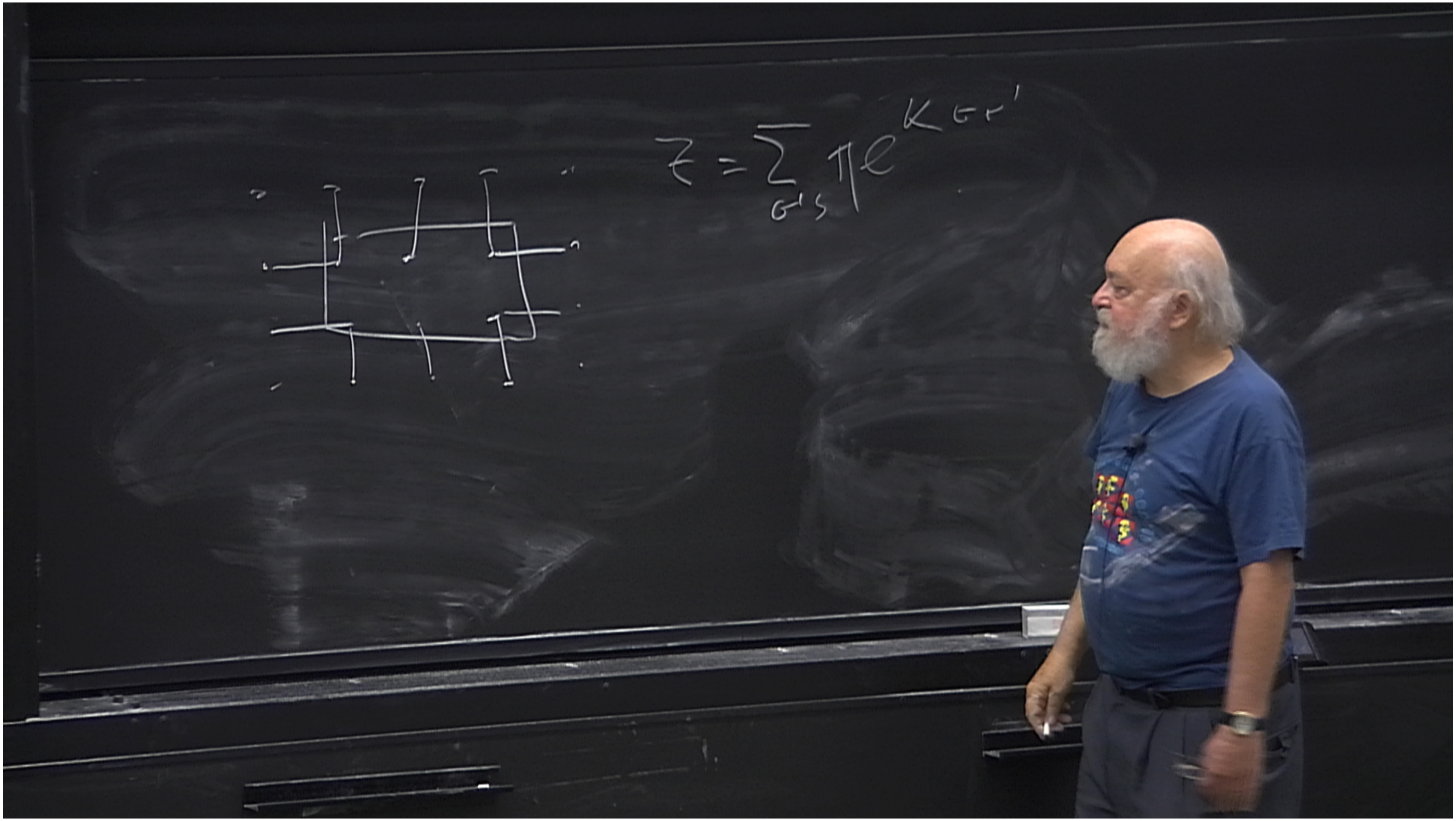
$K' = K$ almost everywhere

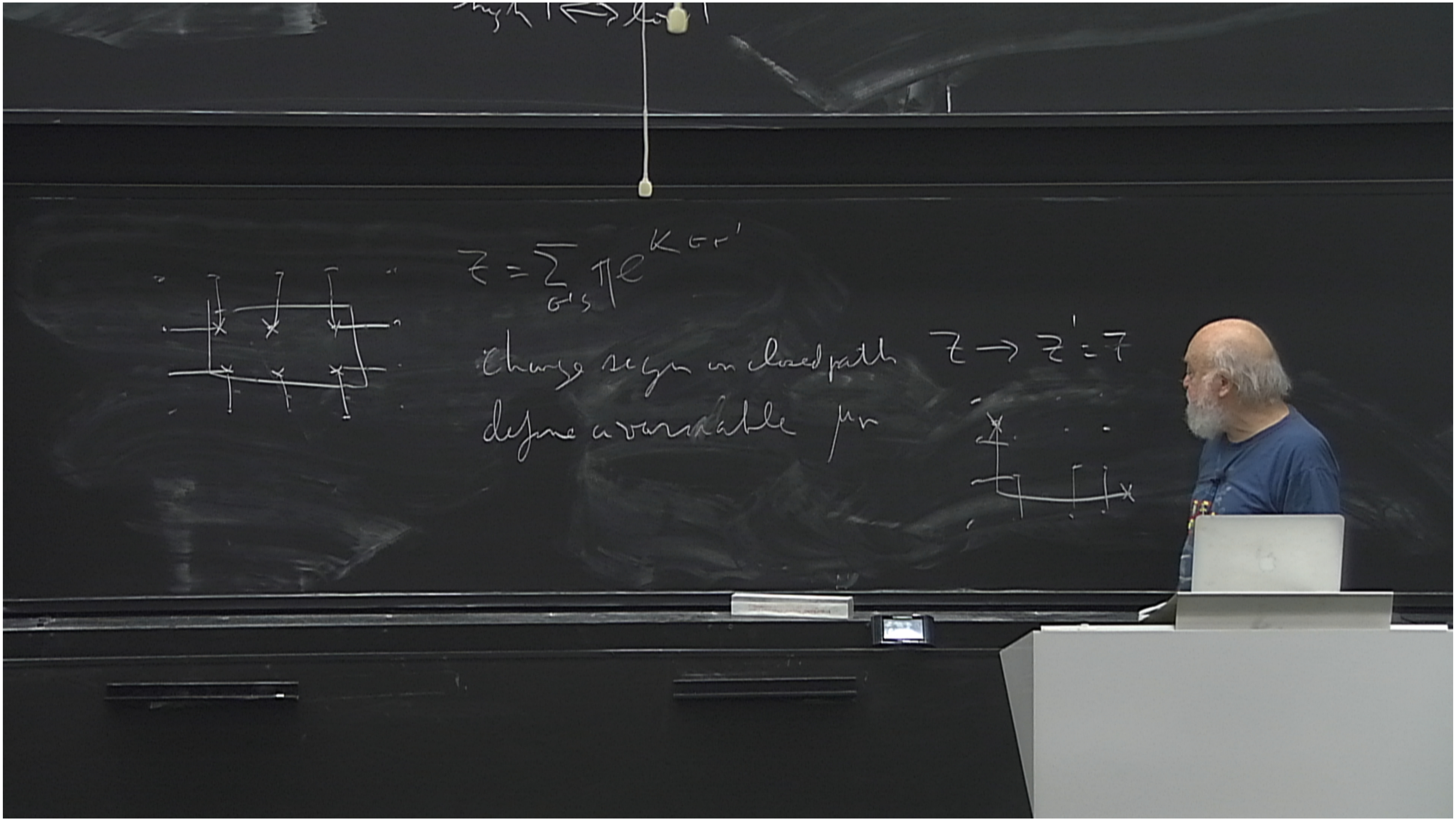
$k' \to -k$ on a path

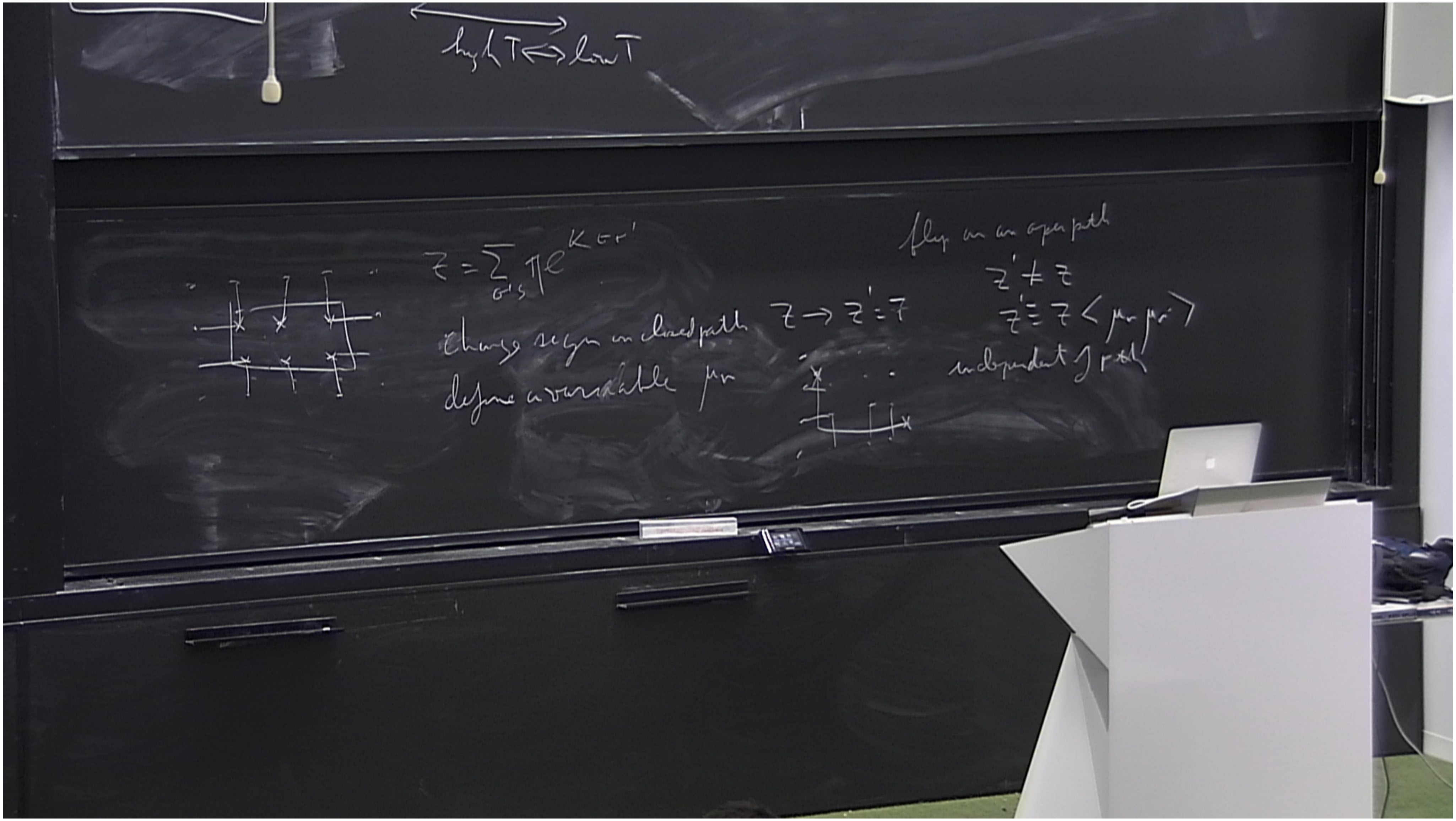
if $k = k'$ almost everywhere

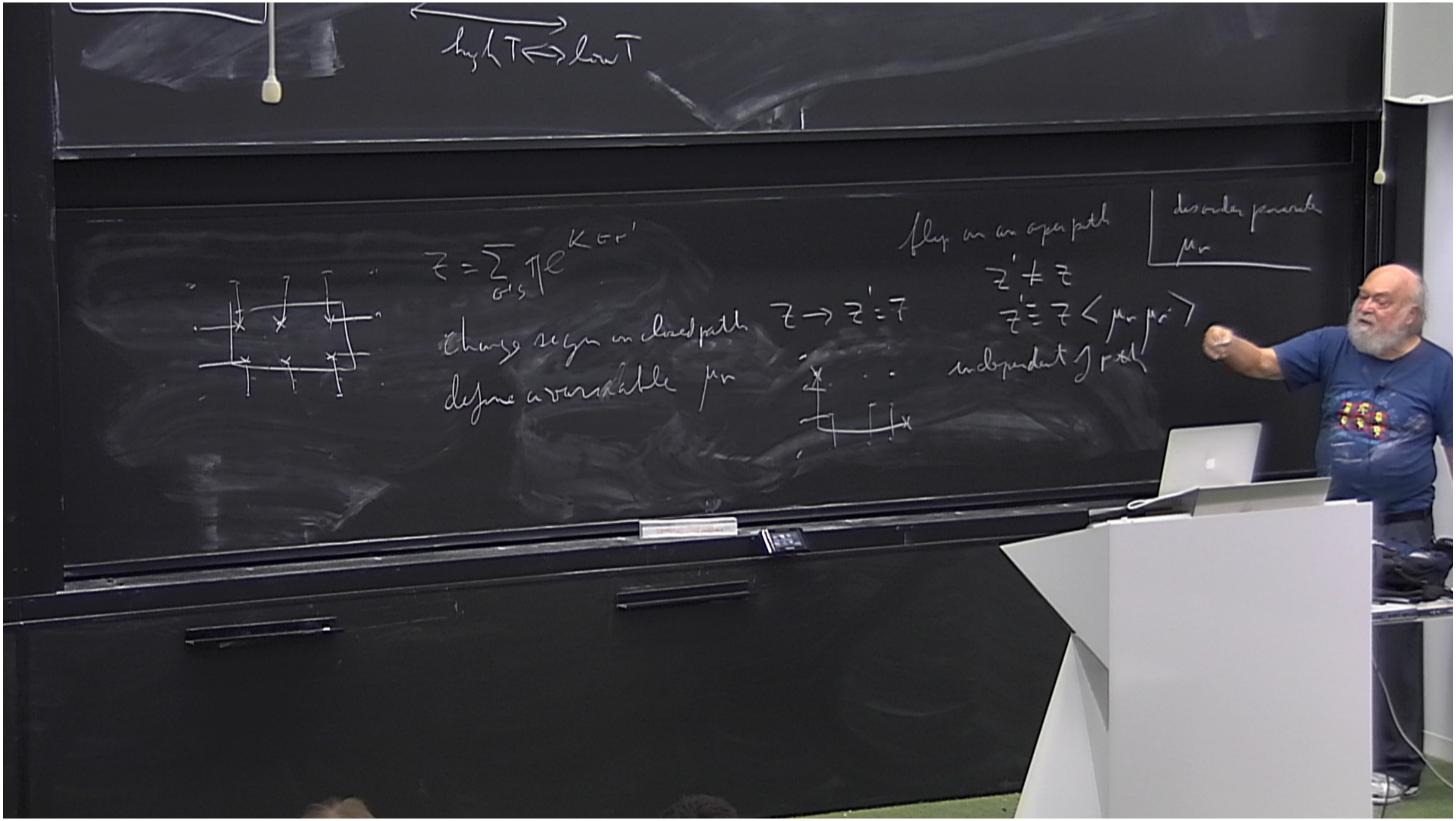
$K = K' + \frac{\pi i}{2}$ on a path connecting r & r'

We calculate $Z\{k\}$ then we know lots more including $\langle \sigma_r \sigma_{r'} \rangle$


Z = Σ Π e^{Kσσ'}
σ,s


Z = Σ Π e^{K σ σ'}
change sign on closed path Z → Z' = Z
define a variable μ


high T ⟺ low T
flip on an open path
change sign on closed path
define a variable μ
independent of path


high T ⟺ low T
flip on an open path
disorder parameter
μ
Z = Σ Π e^{K σ σ'}
change sign on closed path
define a variable μ
Z → Z' = Z
Z' ≠ Z
Z' = Z <μ μ'>
independent of path