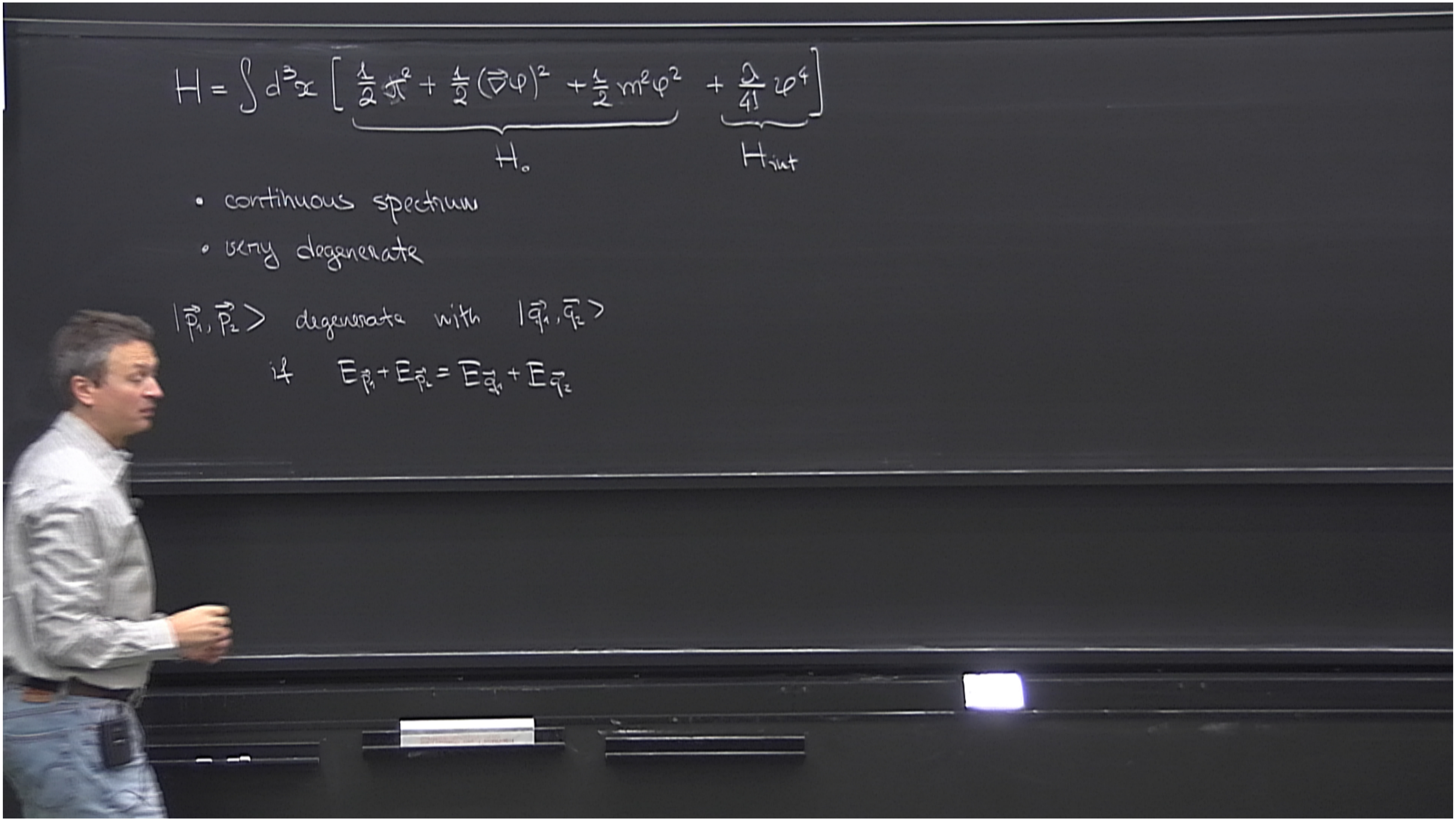
Title: Quantum Field Theory I - Lecture 10

Date: Oct 17, 2011 09:00 AM

URL: http://pirsa.org/11100020

Abstract:


$H = \int d^3x \left[\underbrace{\frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\varphi)^2 + \frac{1}{2}m^2\varphi^2}_{H_0} + \underbrace{\frac{\lambda}{4!}\varphi^4}_{H_{int}} \right]$
• continuous spectrum
• very degenerate
$|\vec{p}_1, \vec{p}_2\rangle$ degenerate with $|\vec{q}_1, \vec{q}_2\rangle$
if $E_{\vec{p}_1} + E_{\vec{p}_2} = E_{\vec{q}_1} + E_{\vec{q}_2}$

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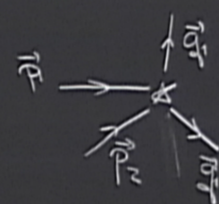
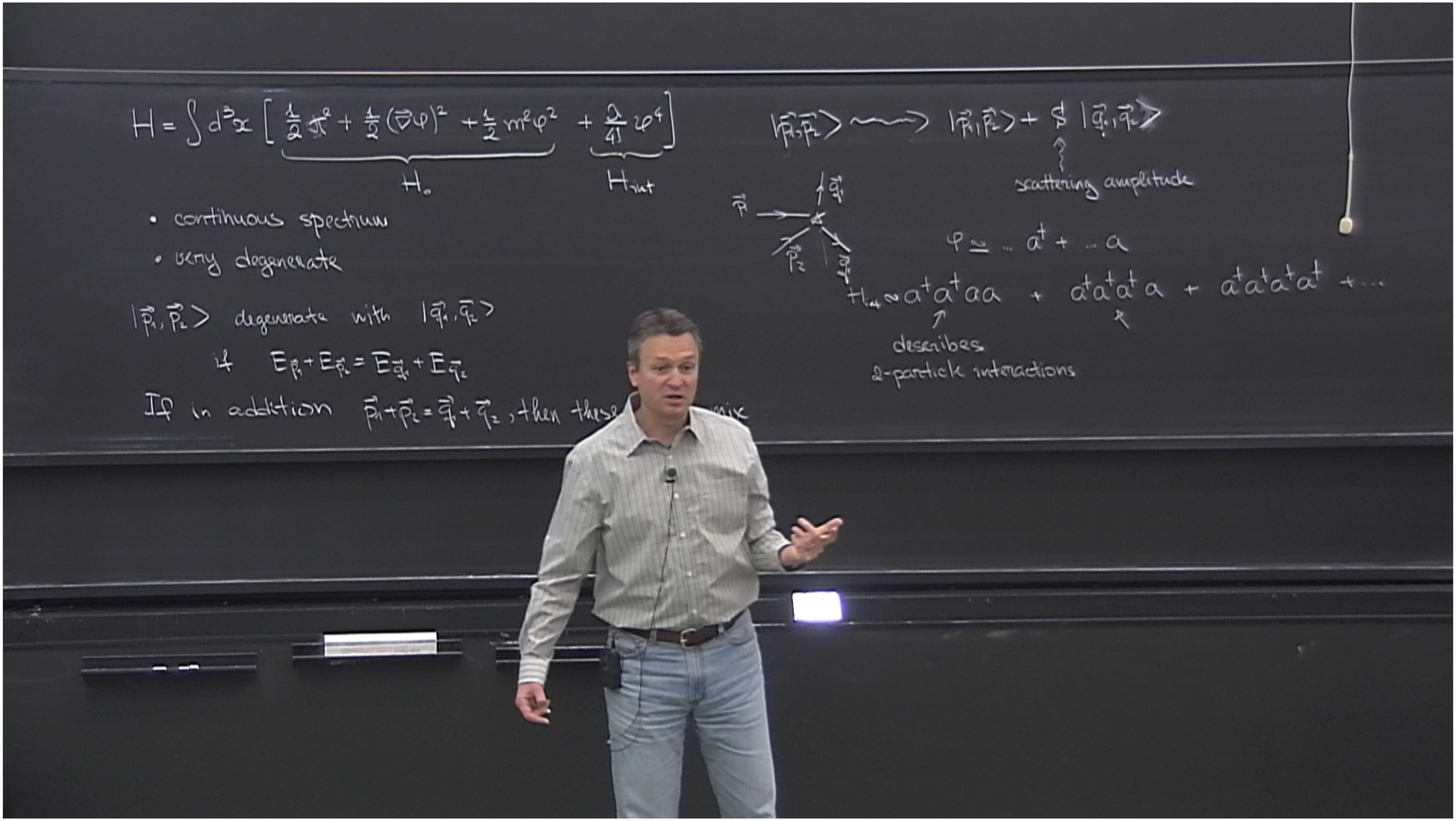
If in addition $\vec{p}_1 + \vec{p}_2 = \vec{q}_1 + \vec{q}_2$, then these states mix

$$|\vec{p}_1, \vec{p}_2\rangle \rightsquigarrow |\vec{p}_1, \vec{p}_2\rangle + \underset{\text{scattering amplitude}}{\$}\, |\vec{q}_1, \vec{q}_2\rangle$$

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$|\vec{p}_1, \vec{p}_2\rangle \rightsquigarrow |\vec{p}_1, \vec{p}_2\rangle + S\, |\vec{q}_1, \vec{q}_2\rangle$
scattering amplitude
$\vec{p}_1$ $\vec{p}_2$ $\vec{q}_1$ $\vec{q}_2$
$\varphi \simeq \dots a^\dagger + \dots a$
$H_{int} \sim a^\dagger a^\dagger a a + a^\dagger a^\dagger a^\dagger a + a^\dagger a^\dagger a^\dagger a^\dagger + \dots$
describes
2-particle interactions

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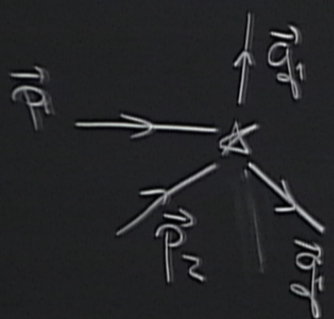

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describes 2-particle interactions

particle creation $(1 \to 3)$

s mix

$|\Omega\rangle$ - true ground state

is not $|0\rangle$

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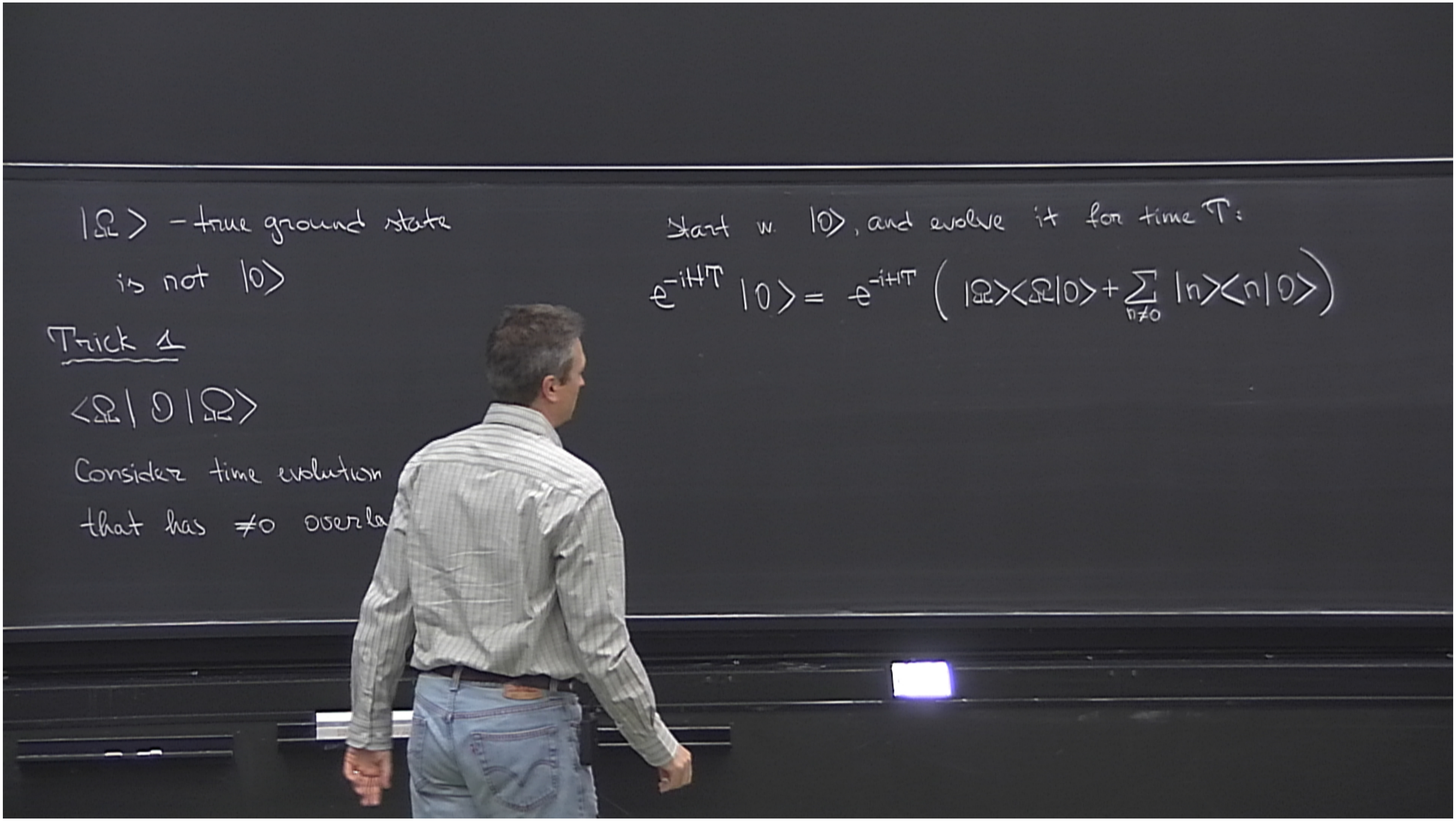
Trick 1

$\langle\Omega| \mathcal{O} |\Omega\rangle$

Consider time evolution of some state

that has $\neq 0$ overlap w. $|\Omega\rangle$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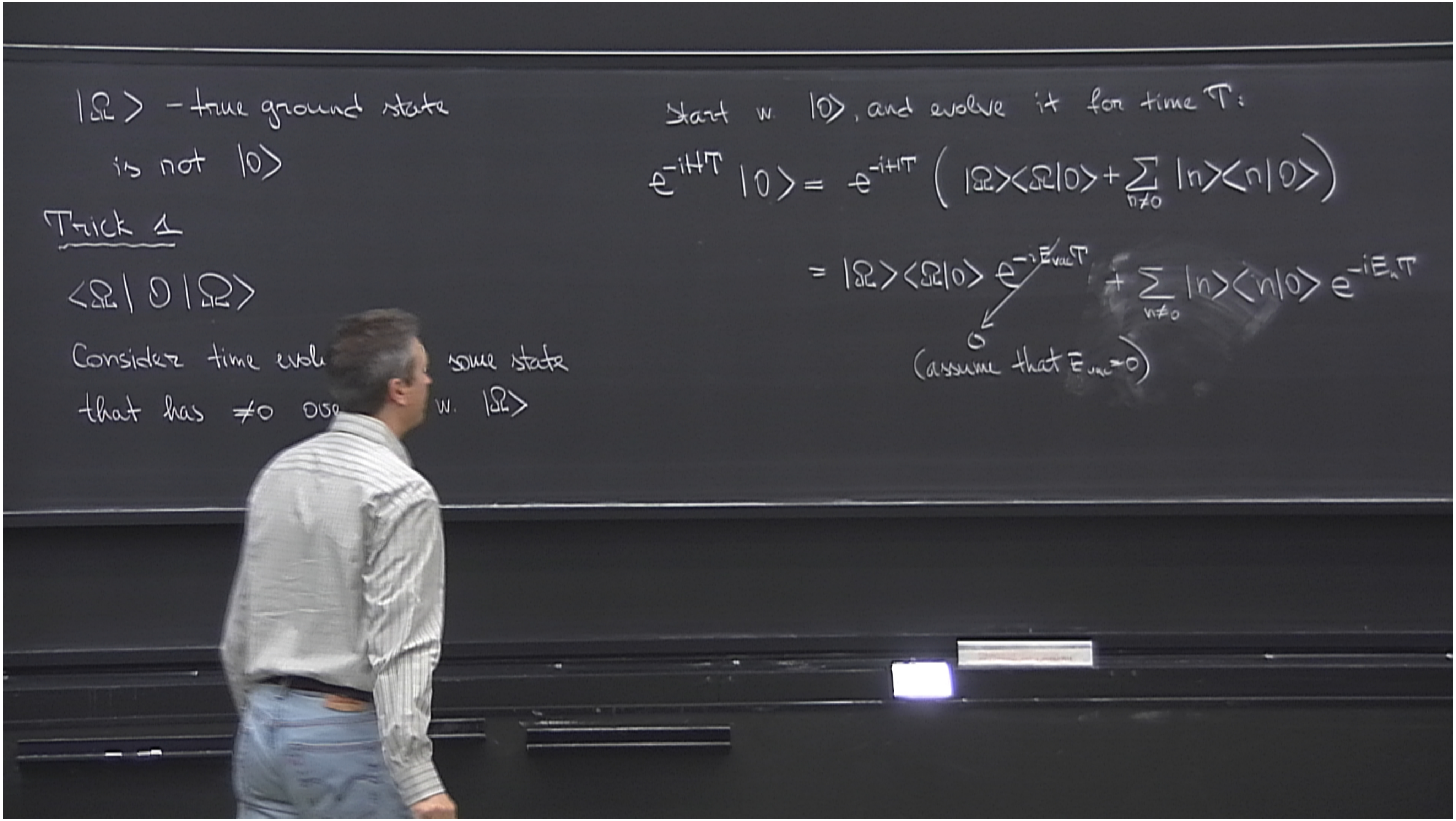
Start w. $|0\rangle$, and evolve it for time T:
$e^{-iHT}|0\rangle = e^{-iHT}\left(|\Omega\rangle\langle\Omega|0\rangle + \sum_{n\neq 0}|n\rangle\langle n|0\rangle\right)$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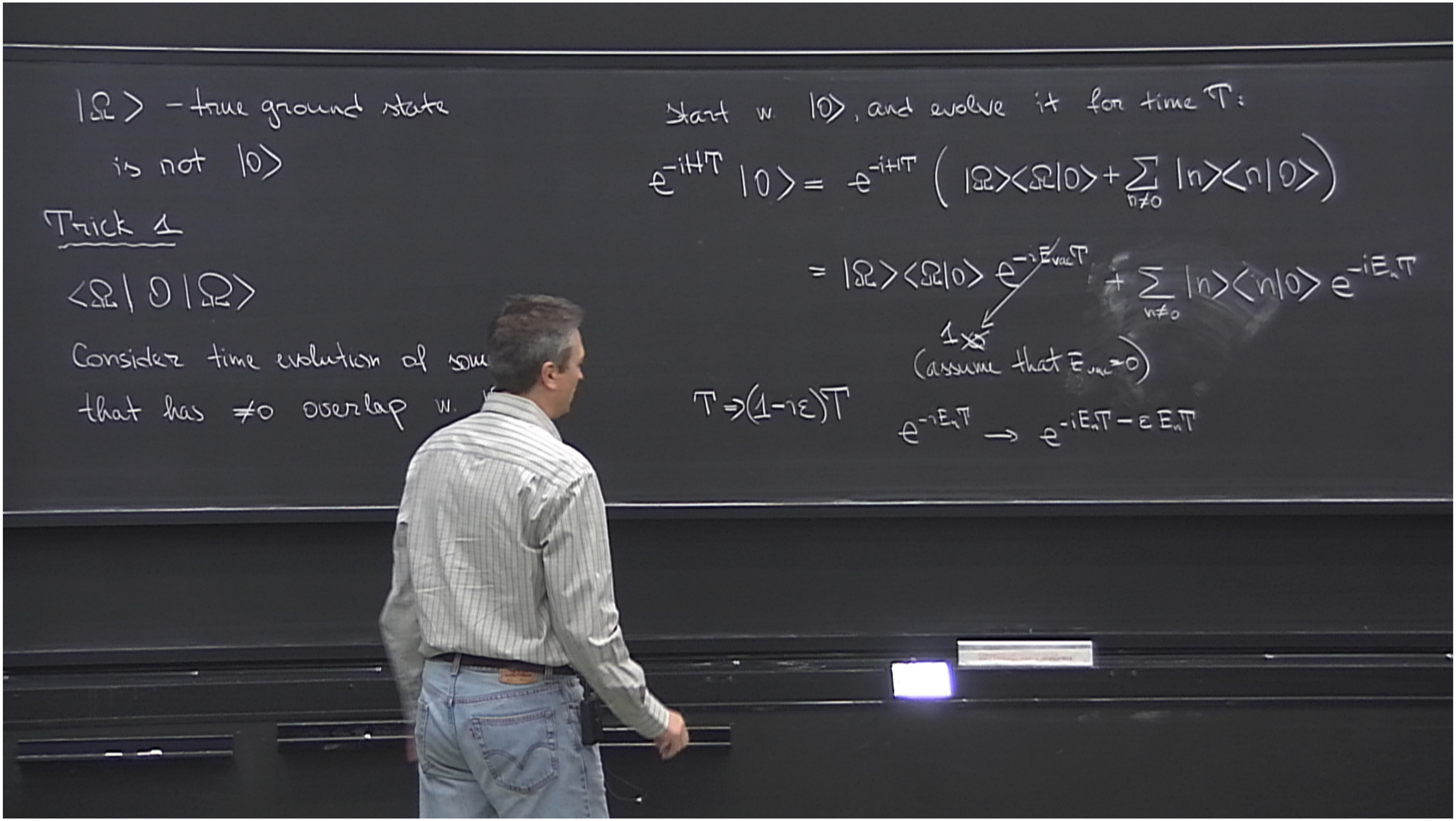
0
(assume that $E_{vac}=0$)



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(assume that $E_{vac}=0$)
$T \Rightarrow (1-i\varepsilon)T$
$e^{-iE_nT} \to e^{-iE_nT - \varepsilon E_nT}$

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1 (assume that $E_{vac}=0$)

$T \Rightarrow (1-i\varepsilon)T$ $\qquad e^{-iE_nT} \longrightarrow e^{-iE_nT-\varepsilon E_nT} \xrightarrow{T\to\infty} 0$

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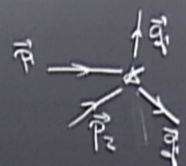

$$\varphi \simeq \ldots a^\dagger + \ldots a$$

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describes 2-particle interactions

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does not annihilate $|0\rangle$

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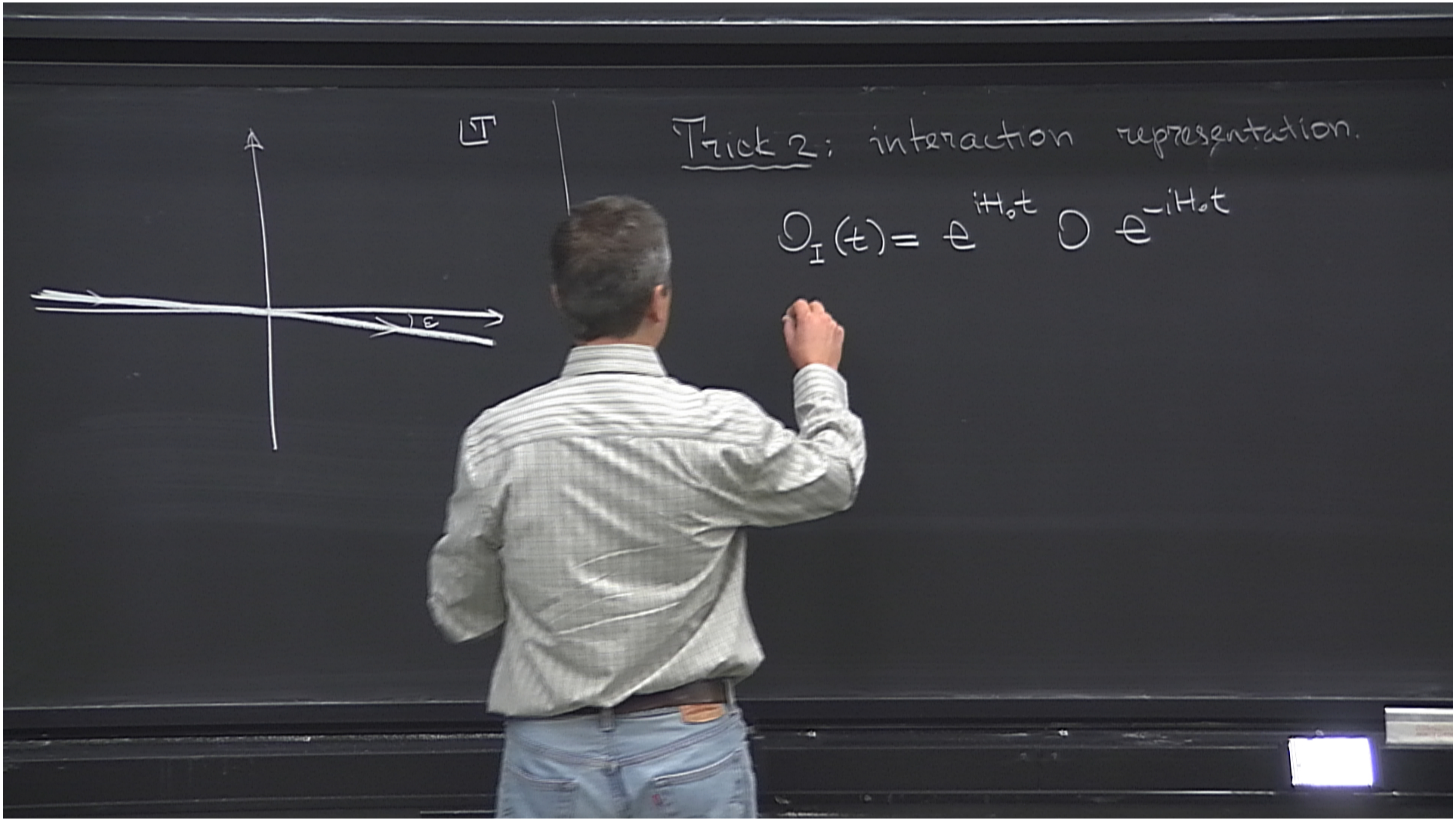
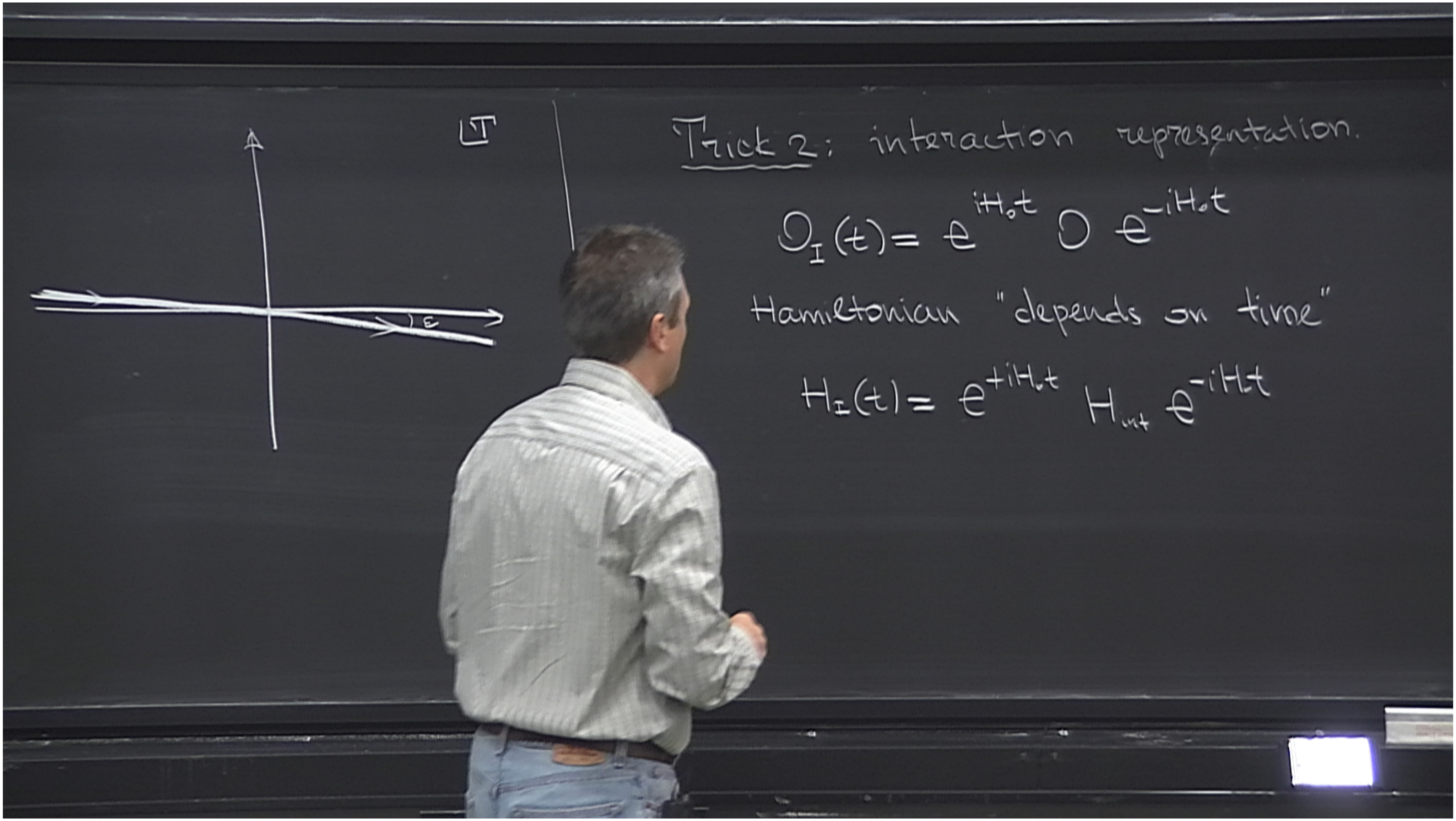

LT
Trick 2: interaction representation.
$O_I(t) = e^{iH_0 t}\, O\, e^{-iH_0 t}$
ϵ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Hamiltonian "depends on time"
$H_I(t) = e^{+iH_0 t}\, H_{int}\, e^{-iH_0 t}$

$$e^{-iE_n T} \longrightarrow e^{-iE_n T - \varepsilon E_n T} \longrightarrow 0$$

Trick 2: interaction representation.

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Hamiltonian "depends on time":

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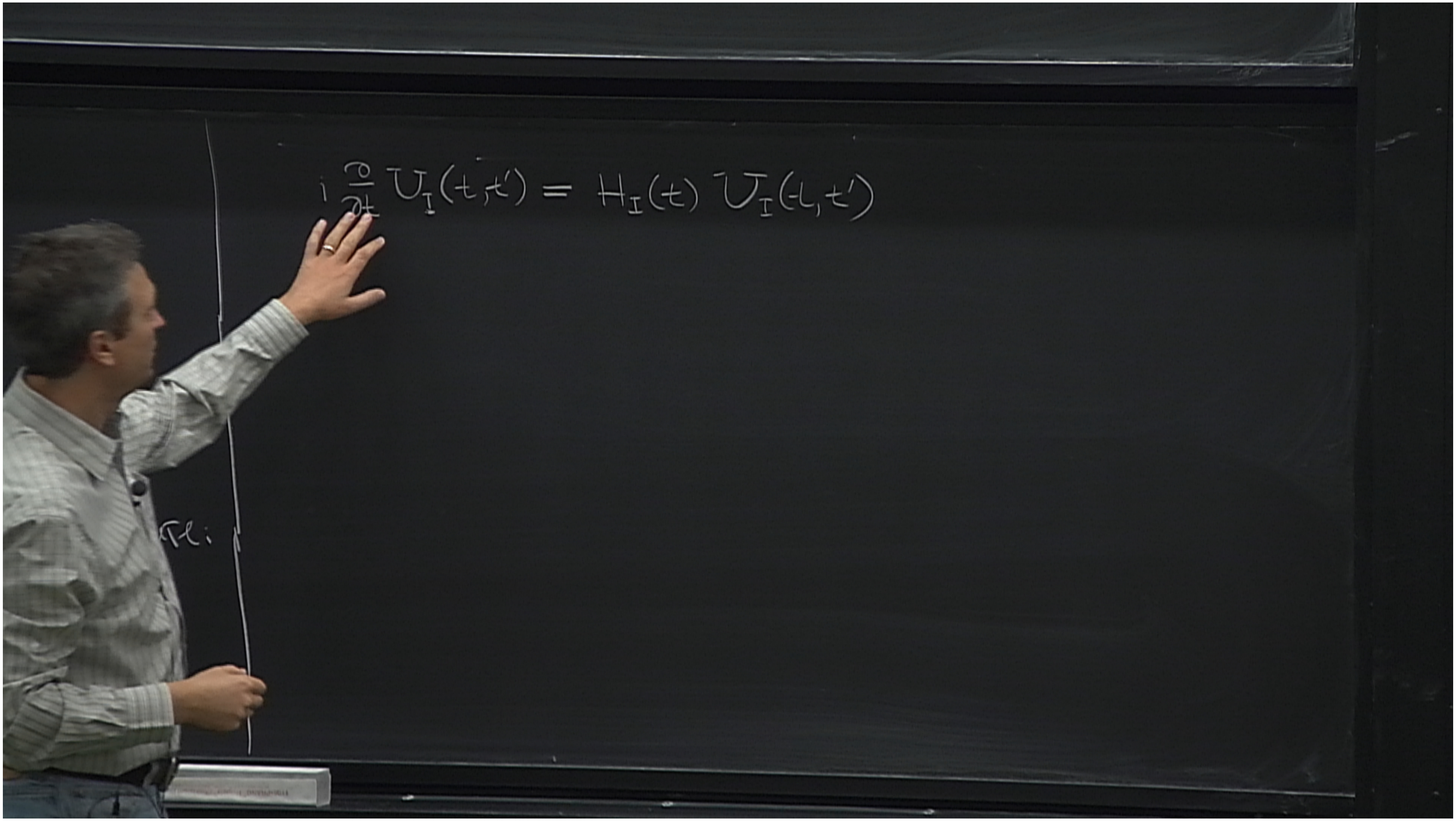
Evolution operator in the interaction picture:

$$U_I(t,t') = e^{iH_0 t}\, U(t-t')\, e^{-iH_0 t'}$$


$i \frac{\partial}{\partial t} U_I(t,t') = H_I(t)\, U_I(t,t')$

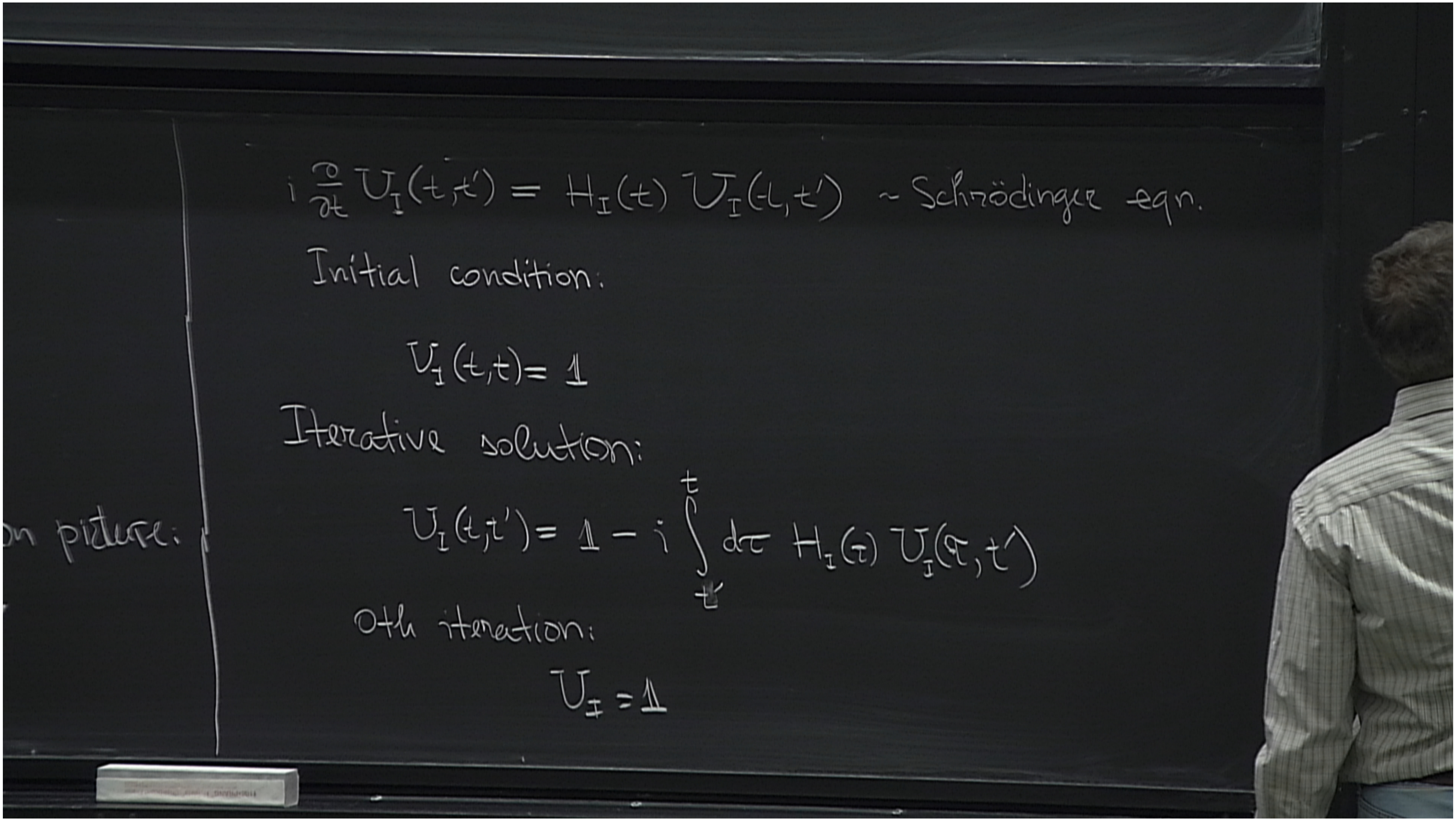
tion picture:

t'

$$i \frac{\partial}{\partial t} U_I(t,t') = H_I(t)\, U_I(t,t')$$ ~ Schrödinger eqn.

Initial condition:

$$U_I(t,t) = \mathbb{1}$$



$i \frac{\partial}{\partial t} U_I(t,t') = H_I(t)\, U_I(t,t')$ ~ Schrödinger eqn.
Initial condition:
$U_I(t,t) = \mathbb{1}$
Iterative solution:
$U_I(t,t') = \mathbb{1} - i \int_{t'}^{t} d\tau\, H_I(\tau)\, U_I(\tau,t')$
0th iteration:
$U_I = \mathbb{1}$
on picture:

1 iteration:

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2 iteration:

$$U_I(t,t') = \mathbb{1} - i \int_{t'}^{t} dt_1 \, H_I(t_1) + (-i)^2 \int_{t'}^{t} dt_1 \int_{t'}^{t_1} dt_2 \, H_I(t_1) H_I(t_2)$$

1 iteration:

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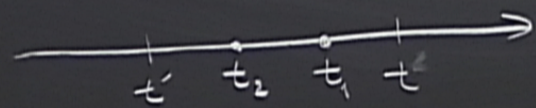

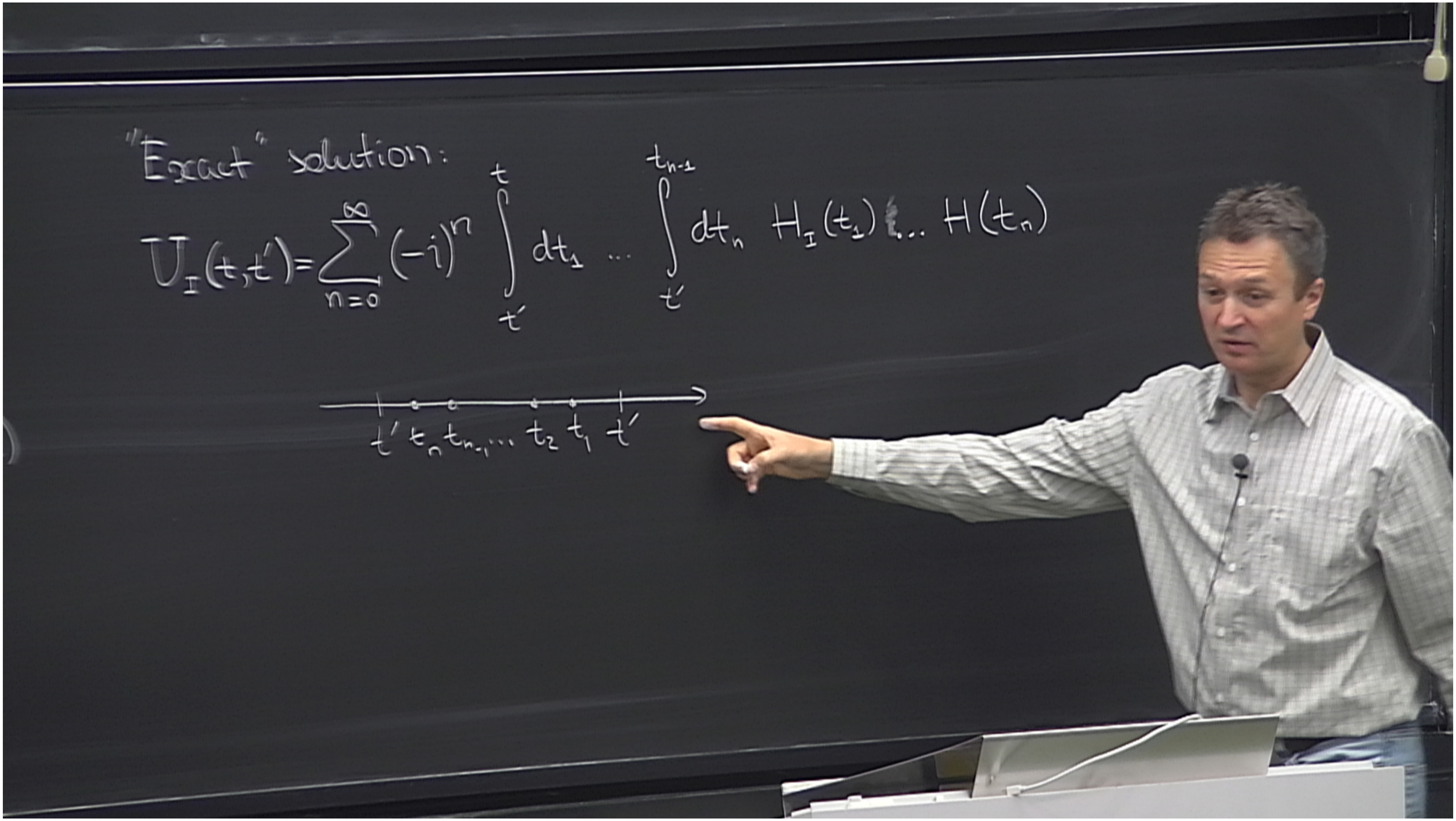
"Exact" solution:
$U_I(t,t') = \sum_{n=0}^{\infty} (-i)^n \int_{t'}^{t} dt_1 \ldots \int_{t'}^{t_{n-1}} dt_n \; H_I(t_1) \ldots H(t_n)$
$t' \; t_n \; t_{n-1} \ldots t_2 \; t_1 \; t'$

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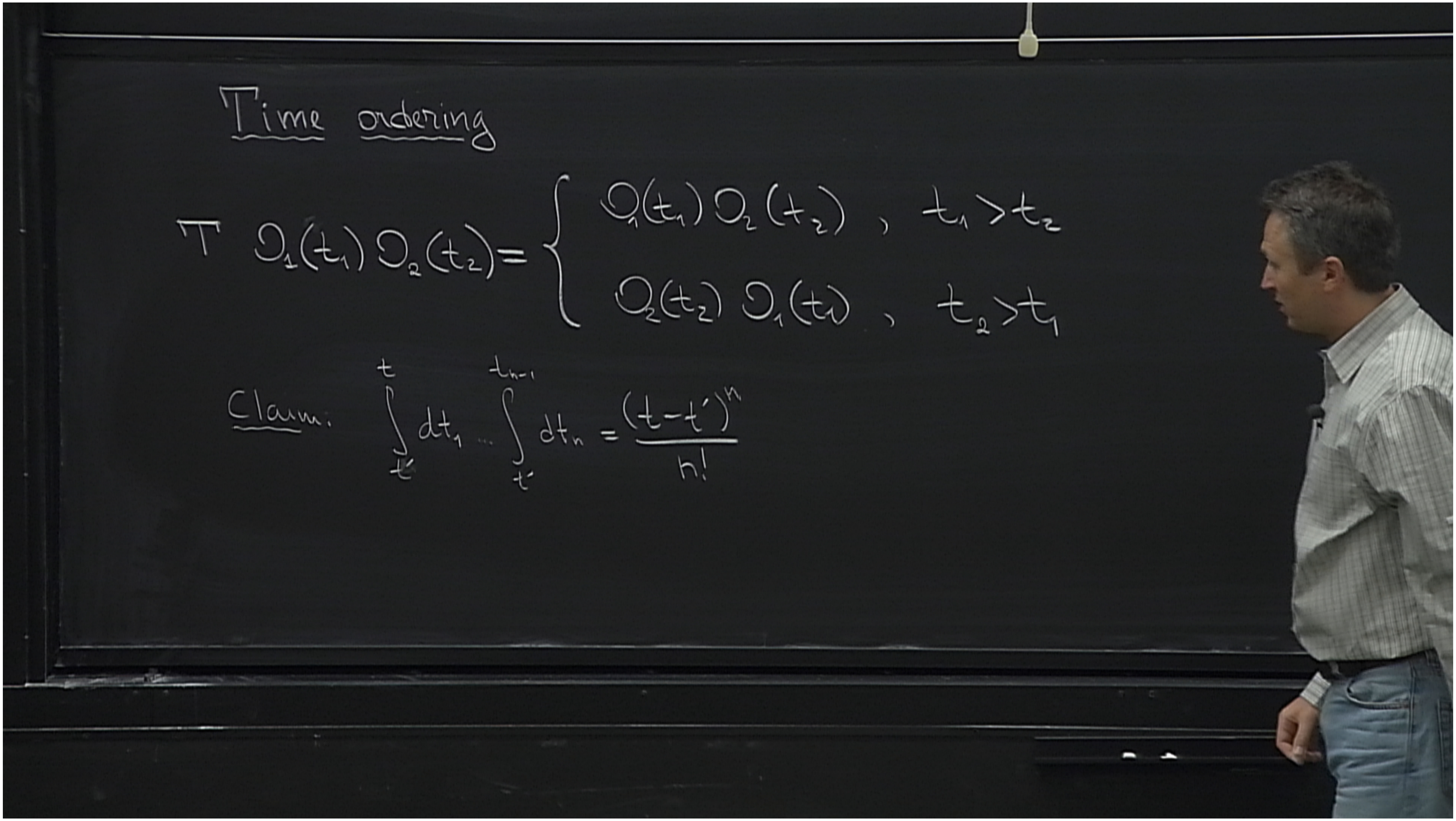
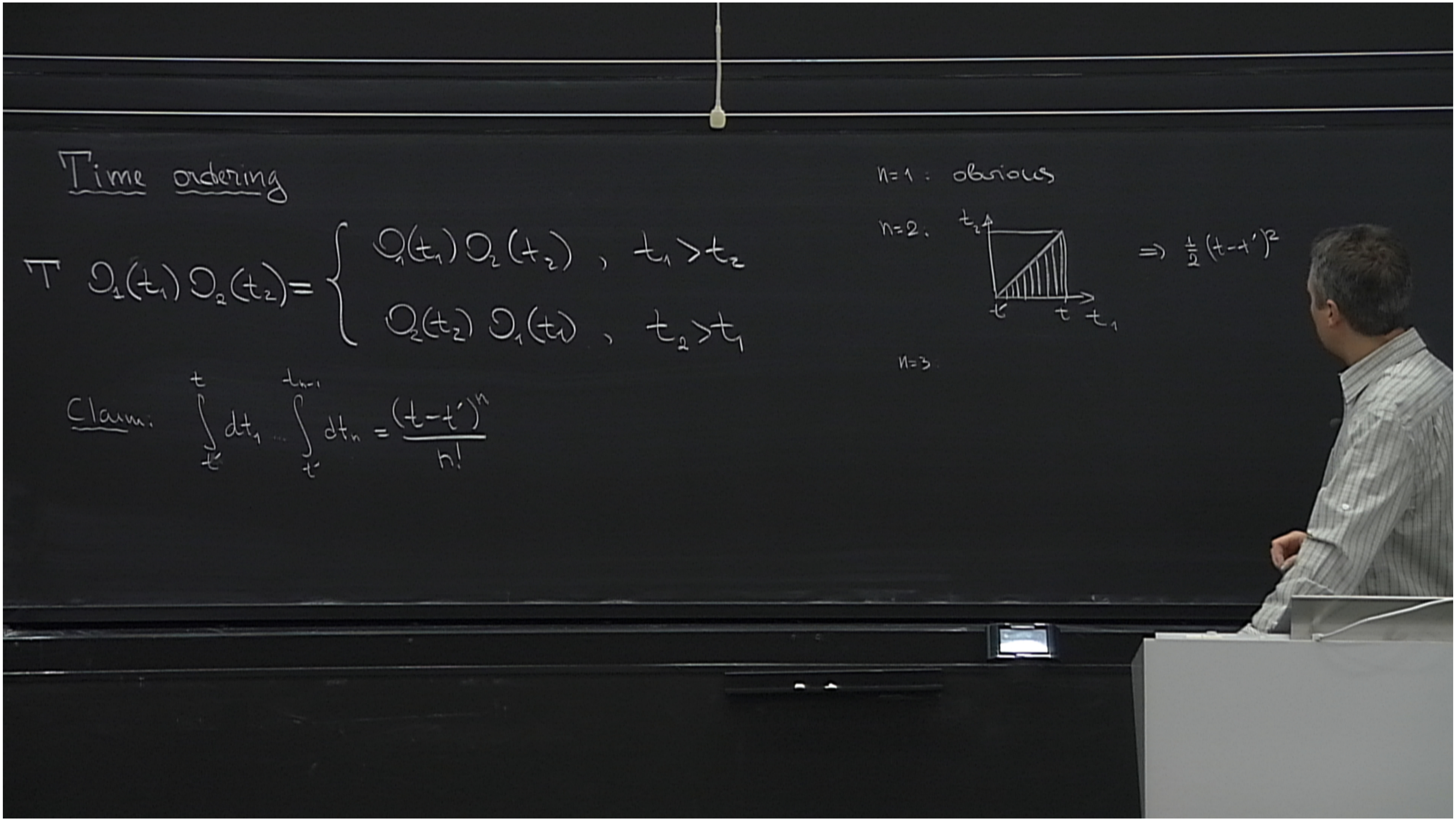
$$t' \quad t_n \quad t_{n-1} \dots \quad t_2 \quad t_1 \quad t'$$


Time ordering
$T\, O_1(t_1)\, O_2(t_2) = \begin{cases} O_1(t_1)\, O_2(t_2), & t_1 > t_2 \\ O_2(t_2)\, O_1(t_1), & t_2 > t_1 \end{cases}$
Claim: $\int_{t'}^{t} dt_1 \ldots \int_{t'}^{t_{n-1}} dt_n = \frac{(t-t')^n}{n!}$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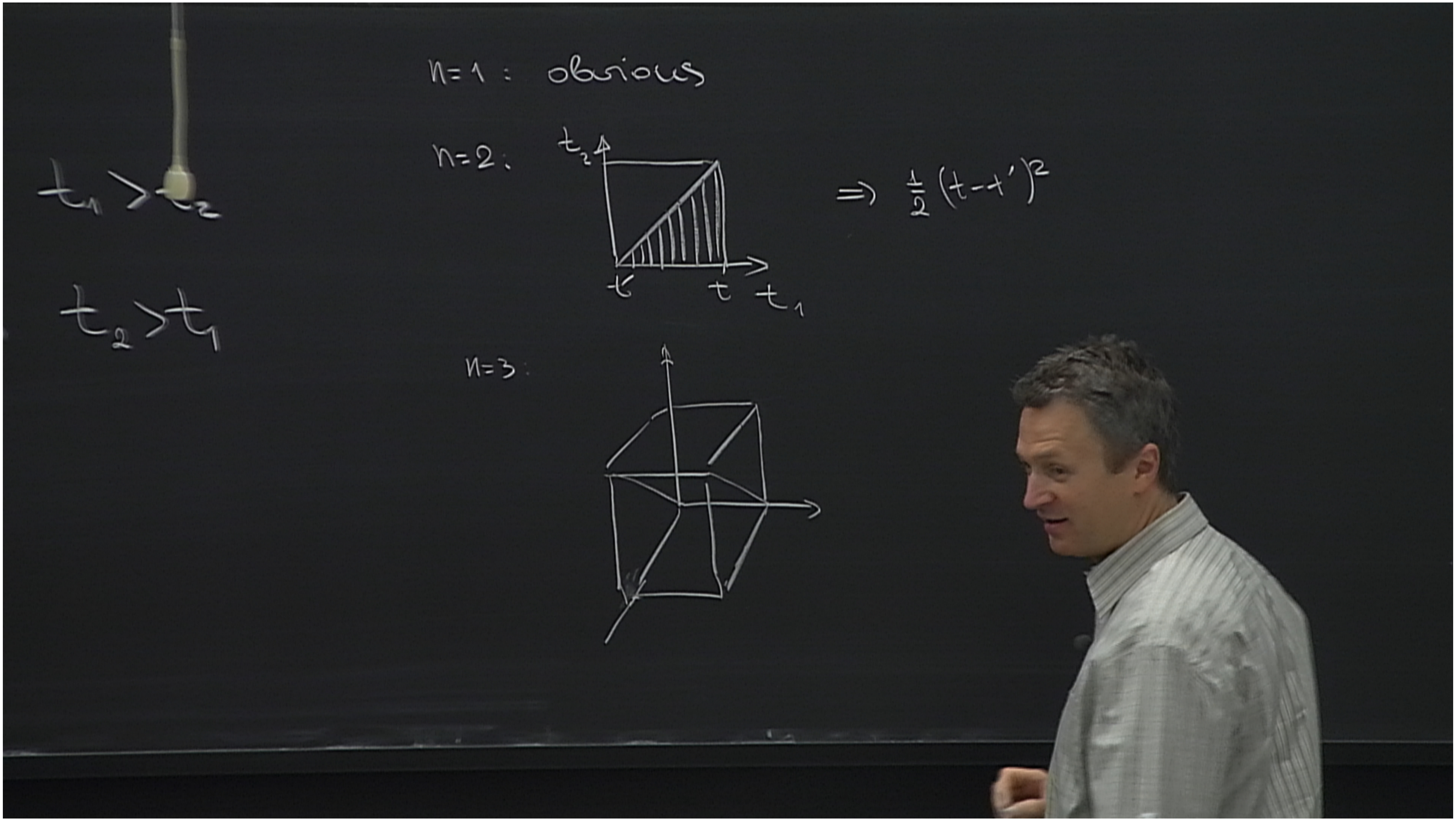
n=1 : obvious
n=2 :
t_2
t'
t
t_1
$\Rightarrow \frac{1}{2}(t-t')^2$
n=3

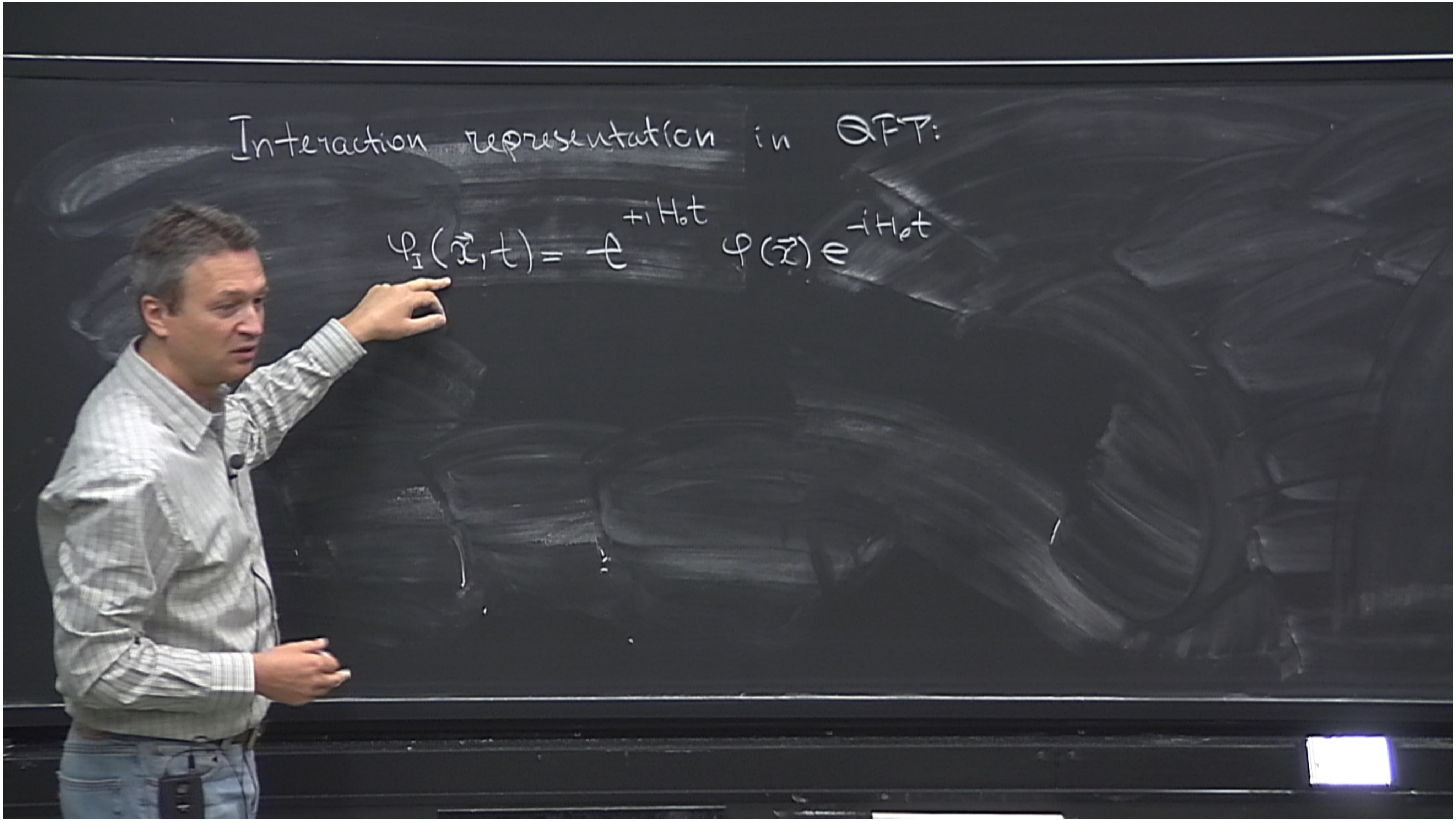

n=1 : obvious
n=2 :
t2
t'
t
t1
⇒ 1/2 (t−t')^2
n=3 :
t1 > t2
t2 > t1

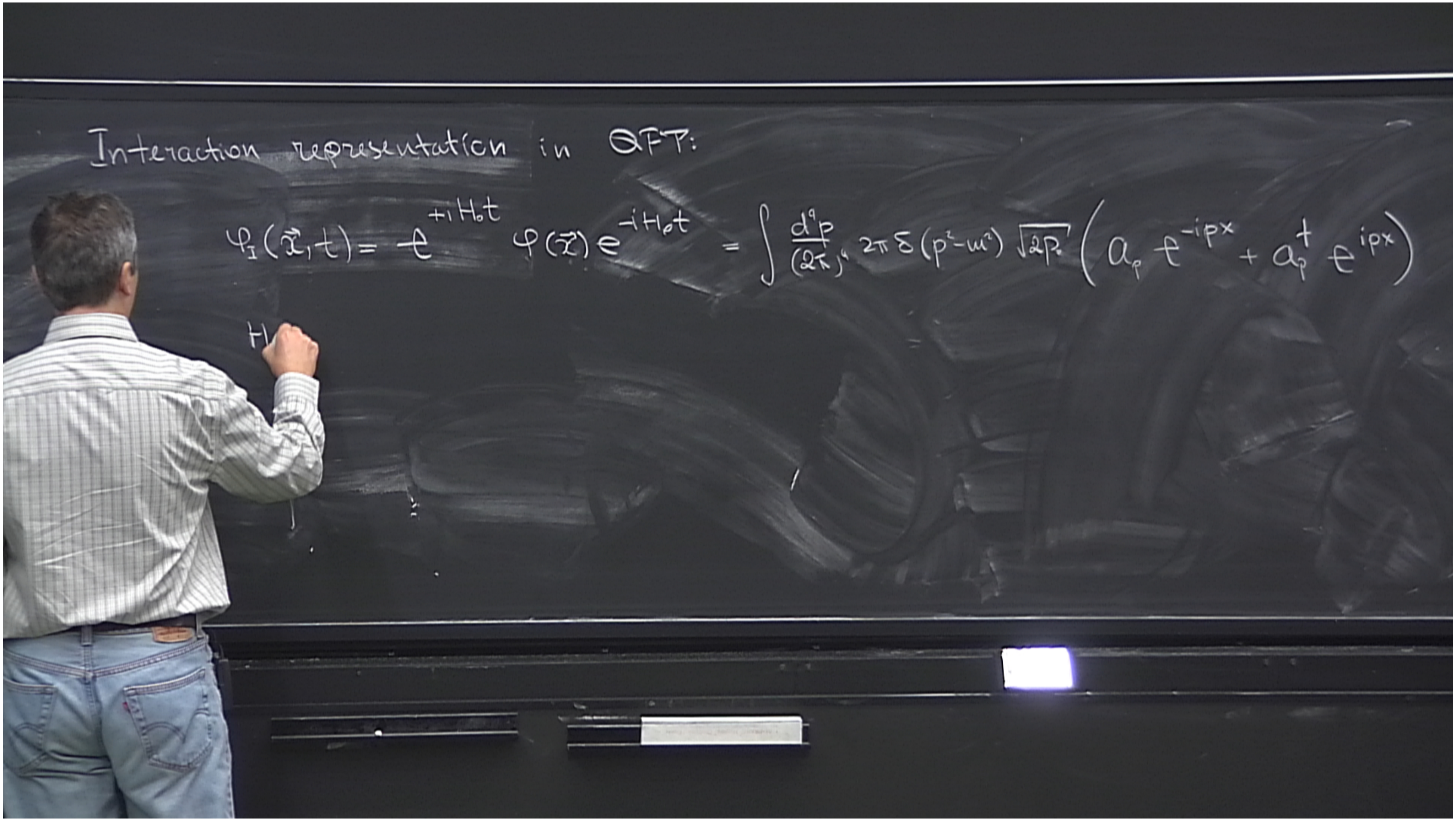
Claim 2:

$$U_I(t,t') = \sum_{n=0}^{\infty} (-i)^n \frac{1}{n!} \int_{t'}^{t} dt_1 \cdots \int_{t'}^{t} dt_n \, T \, H_I(t_1) \cdots H_I(t_n)$$

Claim 2: $U_I(t,t') = \sum_{n=0}^{\infty} (-i)^n \frac{1}{n!} \int_{t'}^{t} dt_1 \dots \int_{t'}^{t} dt_n \, T \, H_I(t_1) \dots H_I(t_n)$

$$\boxed{U_I(t,t') = T \exp\left(-i \int_{t'}^{t} d\tau \, H_I(\tau)\right)}$$


Interaction representation in QFT:
$\varphi_I(\vec{x},t) = e^{+iH_0 t}\,\varphi(\vec{x})\,e^{-iH_0 t}$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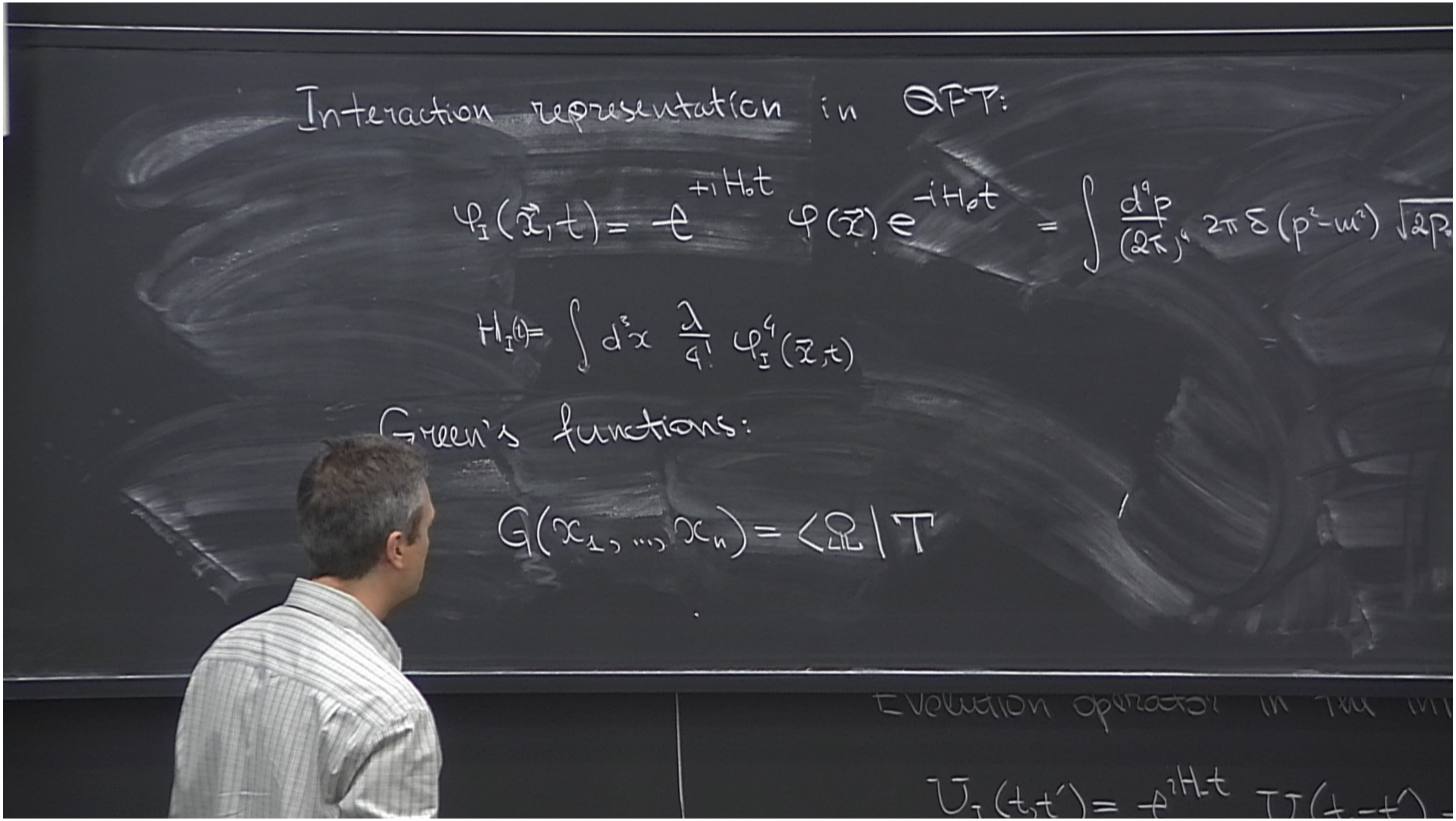

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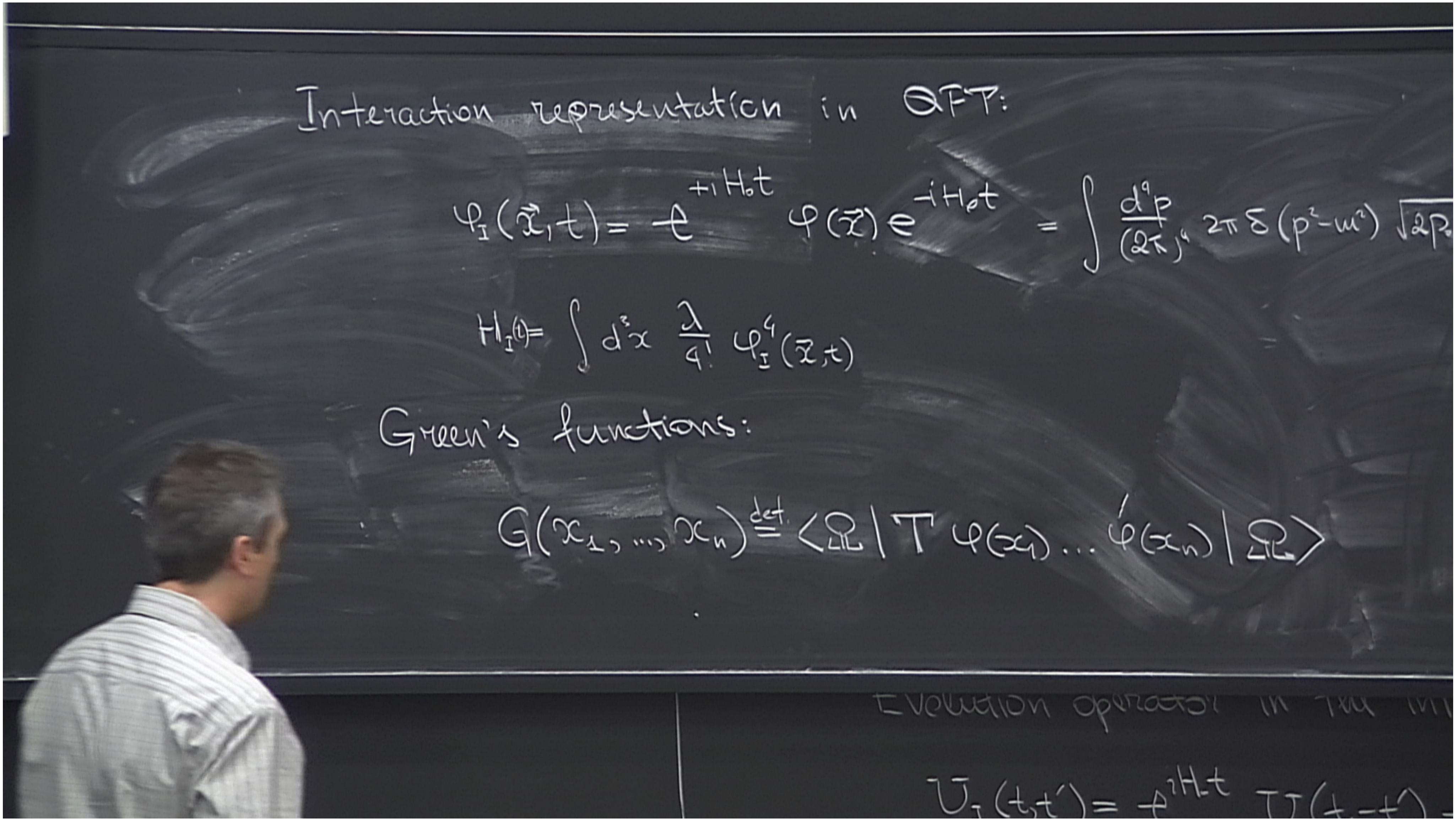
$H_I(t) = \int d^3x\, \frac{\lambda}{4!}\, \varphi_I^4(\vec{x},t)$
Green's functions:
$G(x_1,\dots,x_n) = \langle\Omega| T$
Evolution operator in the
$U_I(t,t') = e^{iH_0 t}$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