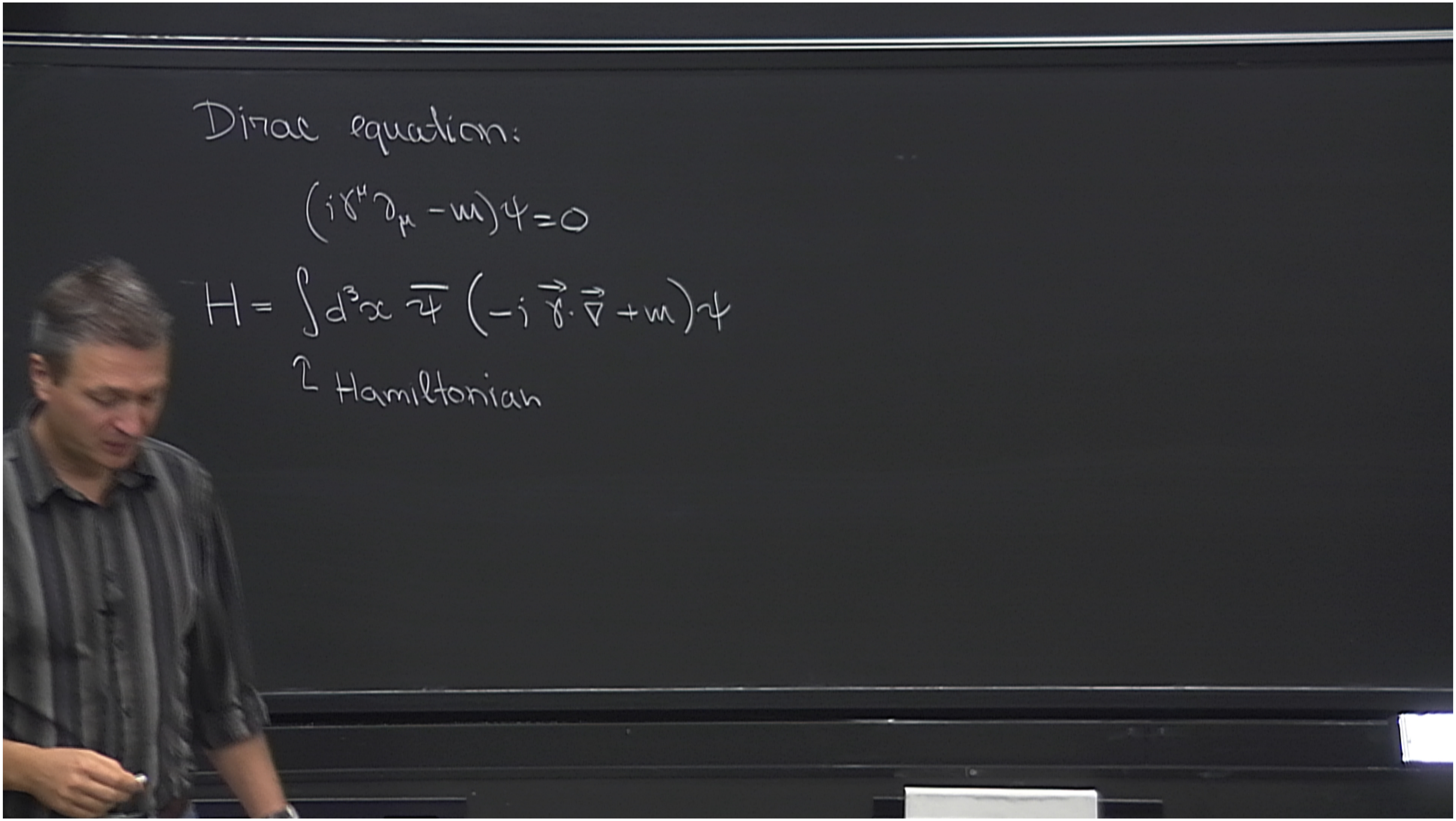
Title: Quantum Field Theory I - Lecture 6

Date: Oct 11, 2011 09:00 AM

URL: http://pirsa.org/11100014

Abstract:


Dirac equation:
$(i\gamma^\mu \partial_\mu - m)\psi = 0$
$H = \int d^3x \, \bar{\psi} \left(-i \vec{\gamma} \cdot \vec{\nabla} + m \right) \psi$
Hamiltonian

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↑ Hamiltonian

Electric charge:

$$Q = \int d^3x \, \psi^\dagger \psi$$

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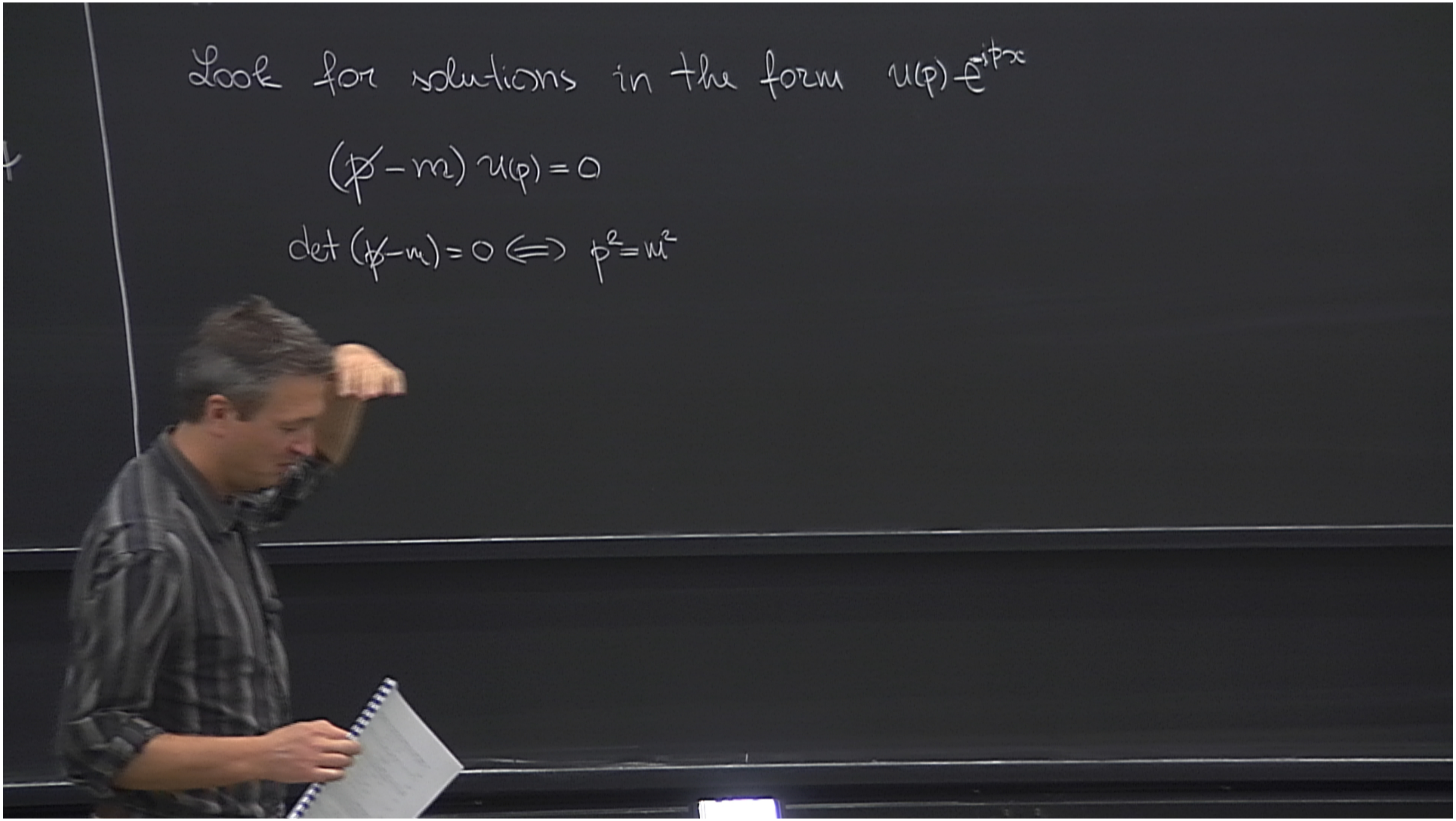
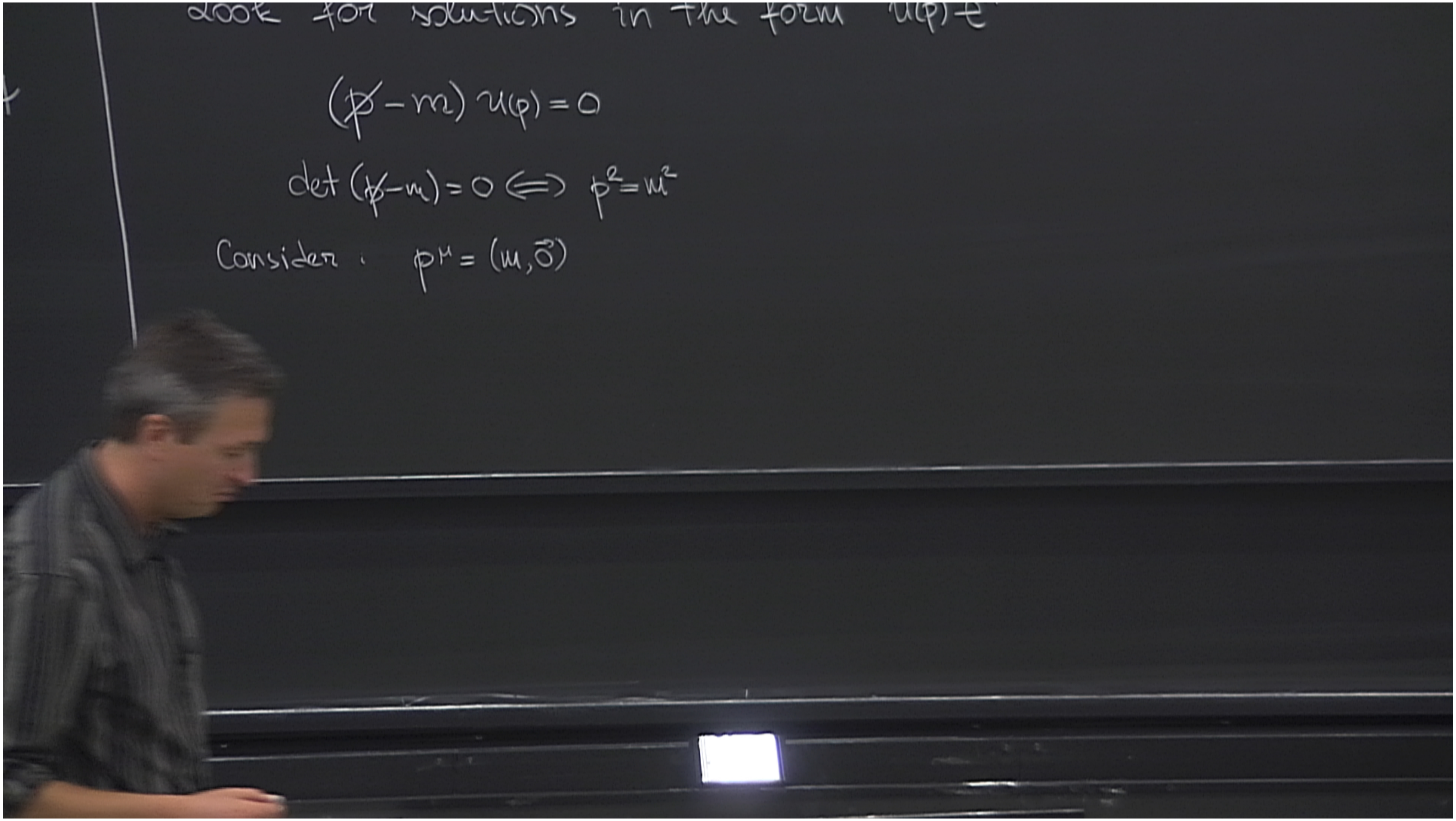
$$(\not{p} - m)\, u(p) = 0$$


Look for solutions in the form u(p) e^{-ipx}
(p̸ − m) u(p) = 0
det (p̸ − m) = 0 ⟺ p² = m²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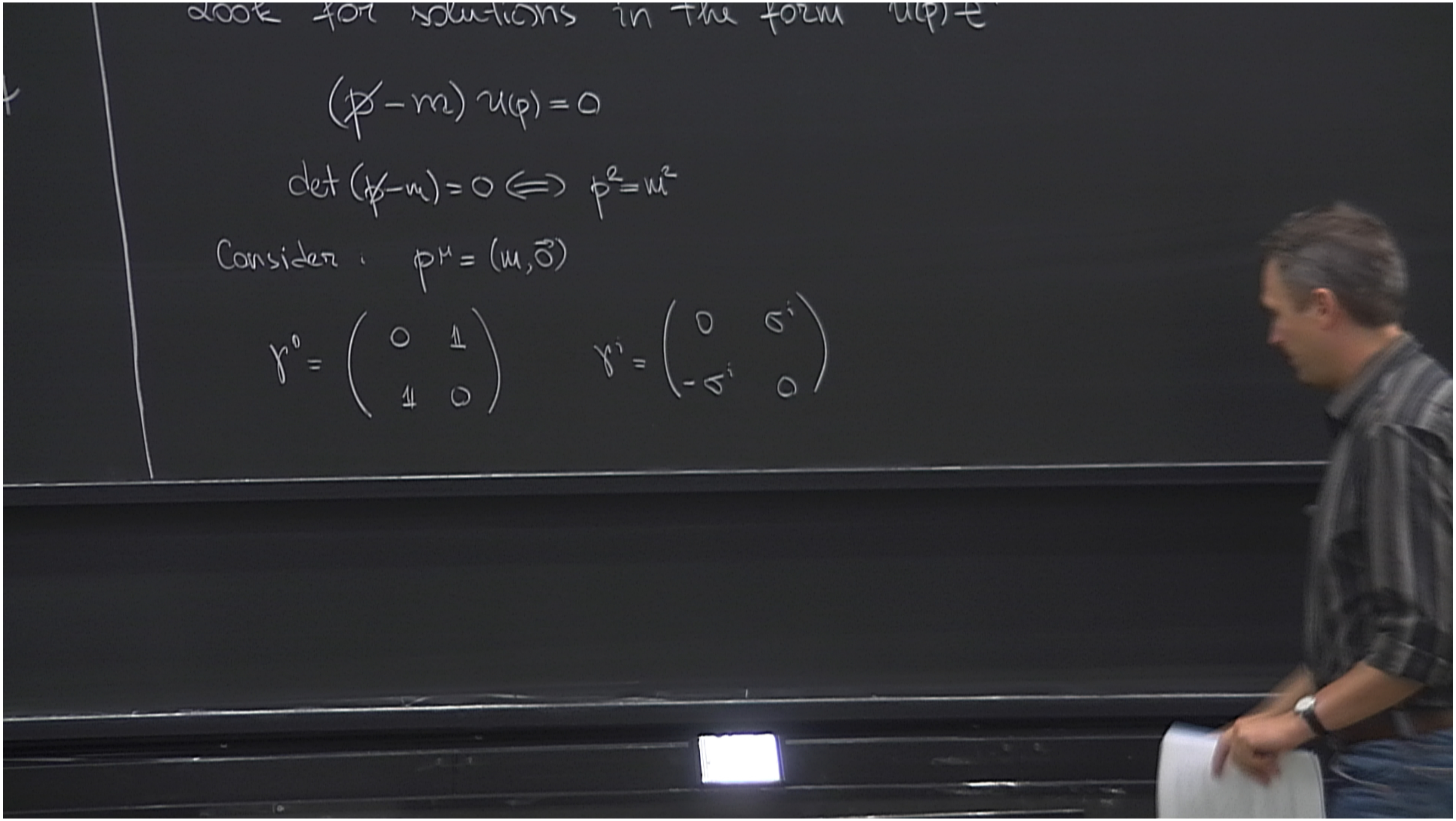
Consider: $p^\mu = (m, \vec{0})$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γ⁰ = (0 1 / 1 0)
γ^i = (0 σ^i / −σ^i 0)

Solutions of Dirac equation

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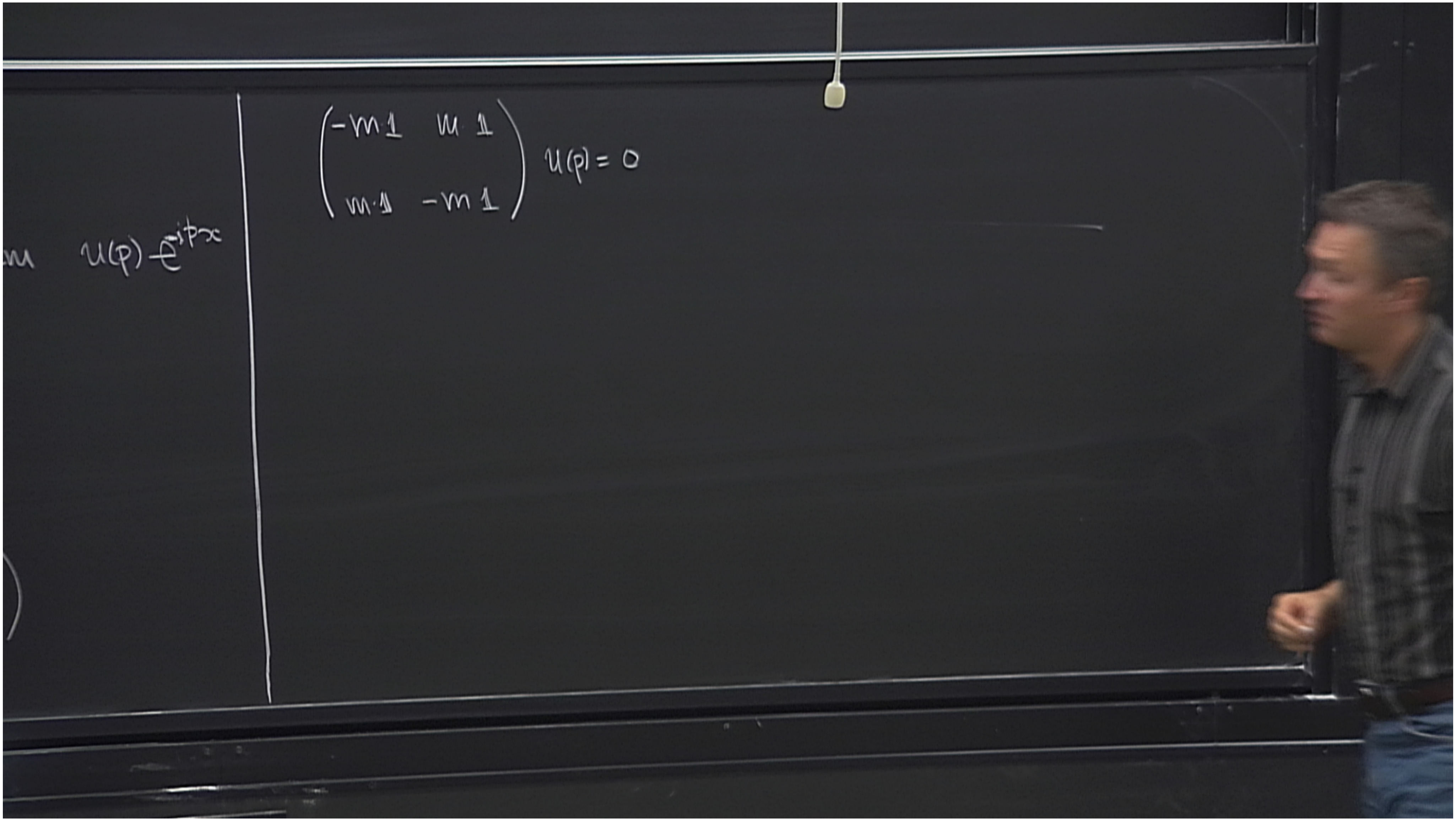
$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$


$\begin{pmatrix} -m\mathbb{1} & m\mathbb{1} \\ m\mathbb{1} & -m\mathbb{1} \end{pmatrix} u(p) = 0$
$u(p)\, e^{-ipx}$

$m \quad u(p)\, e^{-ipx}$

$$\begin{pmatrix} -m\mathbb{1} & m\cdot\mathbb{1} \\ m\cdot\mathbb{1} & -m\mathbb{1} \end{pmatrix} u(p) = 0$$

$$\Downarrow$$

$$u = \begin{pmatrix} u_L \\ u_R \end{pmatrix}$$ $u_{L,R}$ are 2-component spinors

$$u_L = u_R$$

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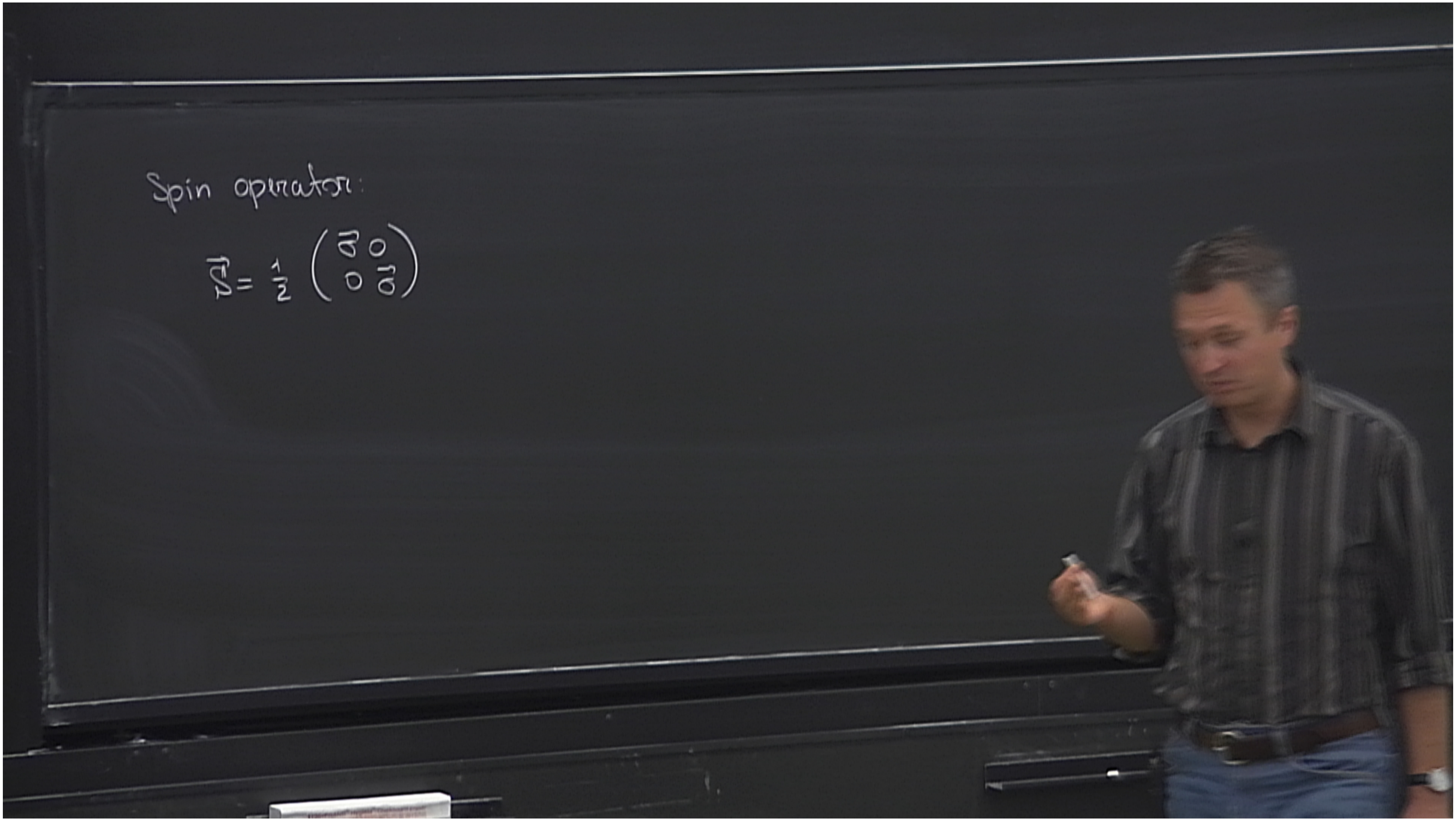
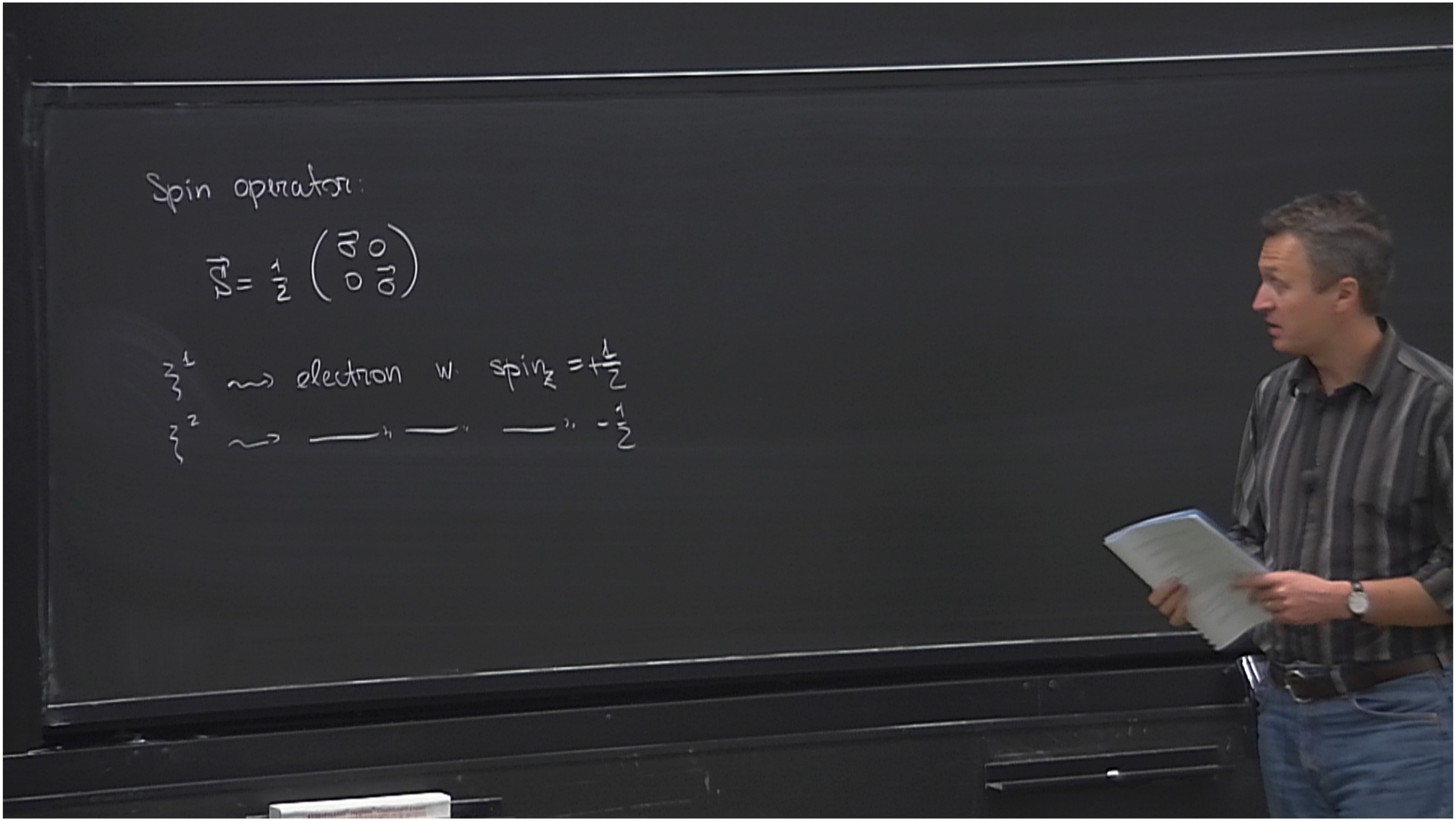
Basis of solutions:

$$u = \sqrt{m} \begin{pmatrix} \xi^s \\ \xi^s \end{pmatrix} \qquad \xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$


Spin operator:
$\vec{S} = \frac{1}{2} \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$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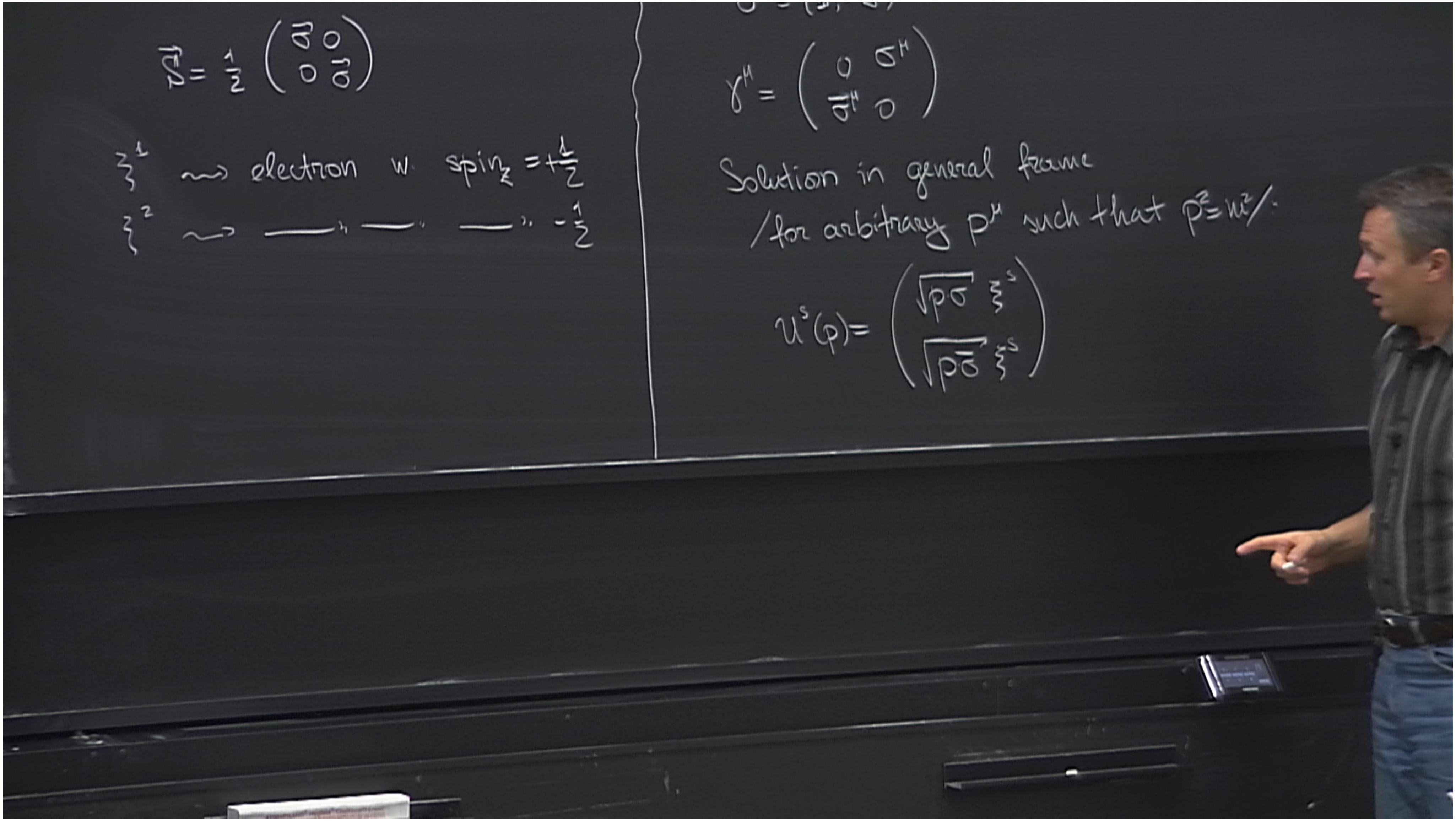
ξ^1 ⇝ electron w. $spin_z = +\frac{1}{2}$
ξ^2 ⇝ —"— —"— —"— $-\frac{1}{2}$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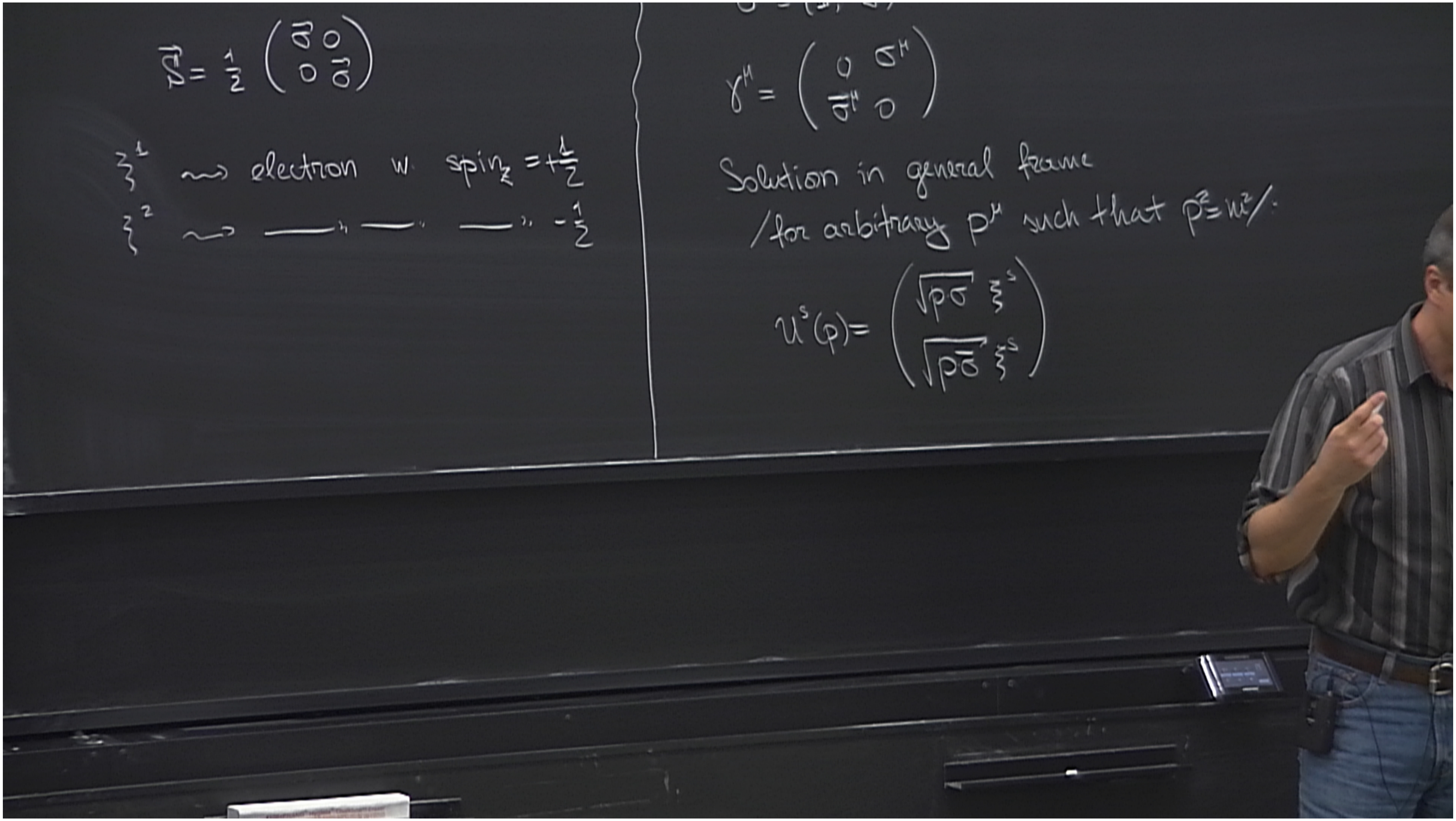
$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0\end{pmatrix}$
Solution in general frame
/for arbitrary p^μ such that $p^2 = m^2$/.
$u^s(p) = \begin{pmatrix}\sqrt{p\sigma}\,\xi^s \\ \sqrt{p\bar{\sigma}}\,\xi^s\end{pmatrix}$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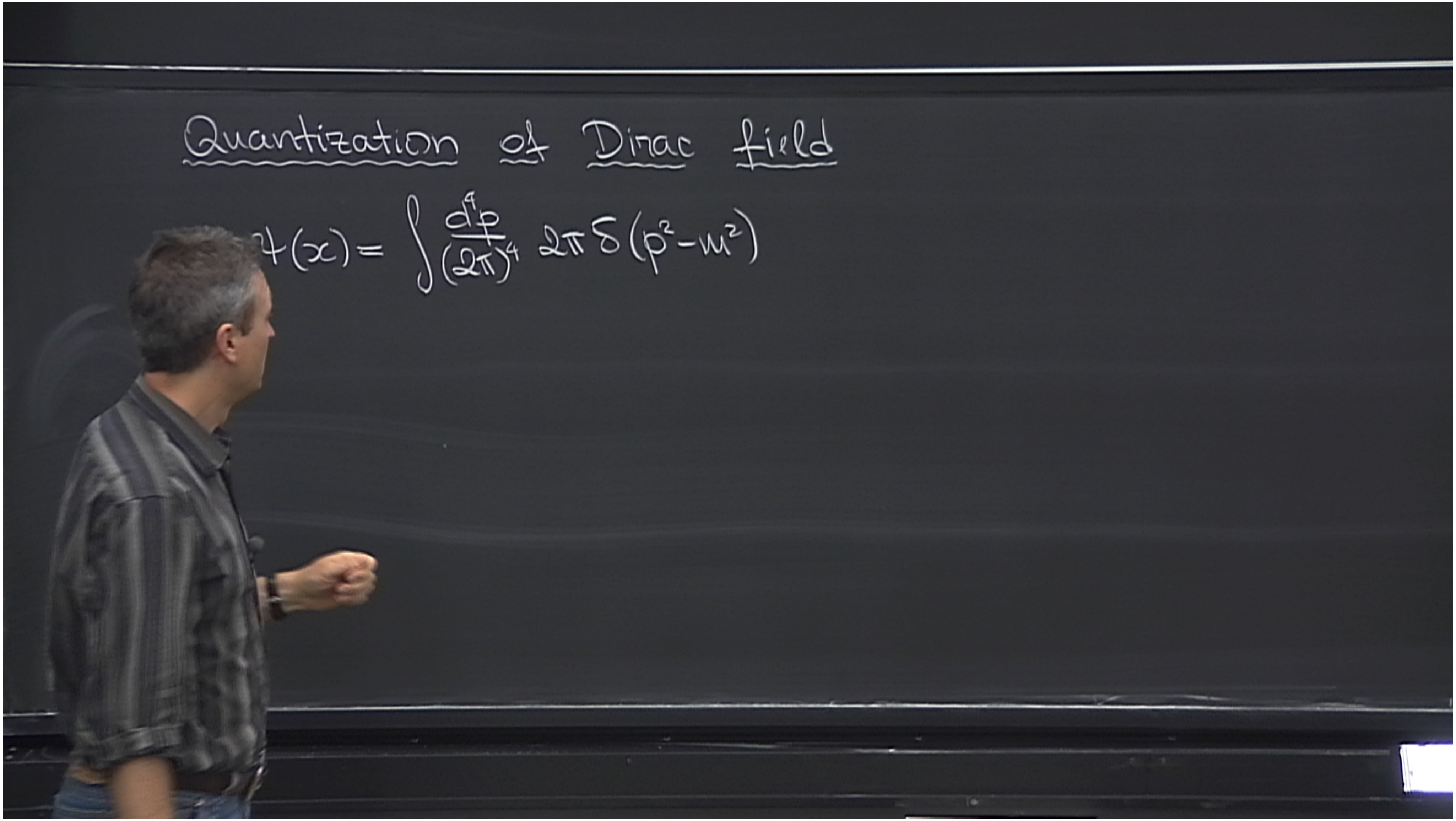
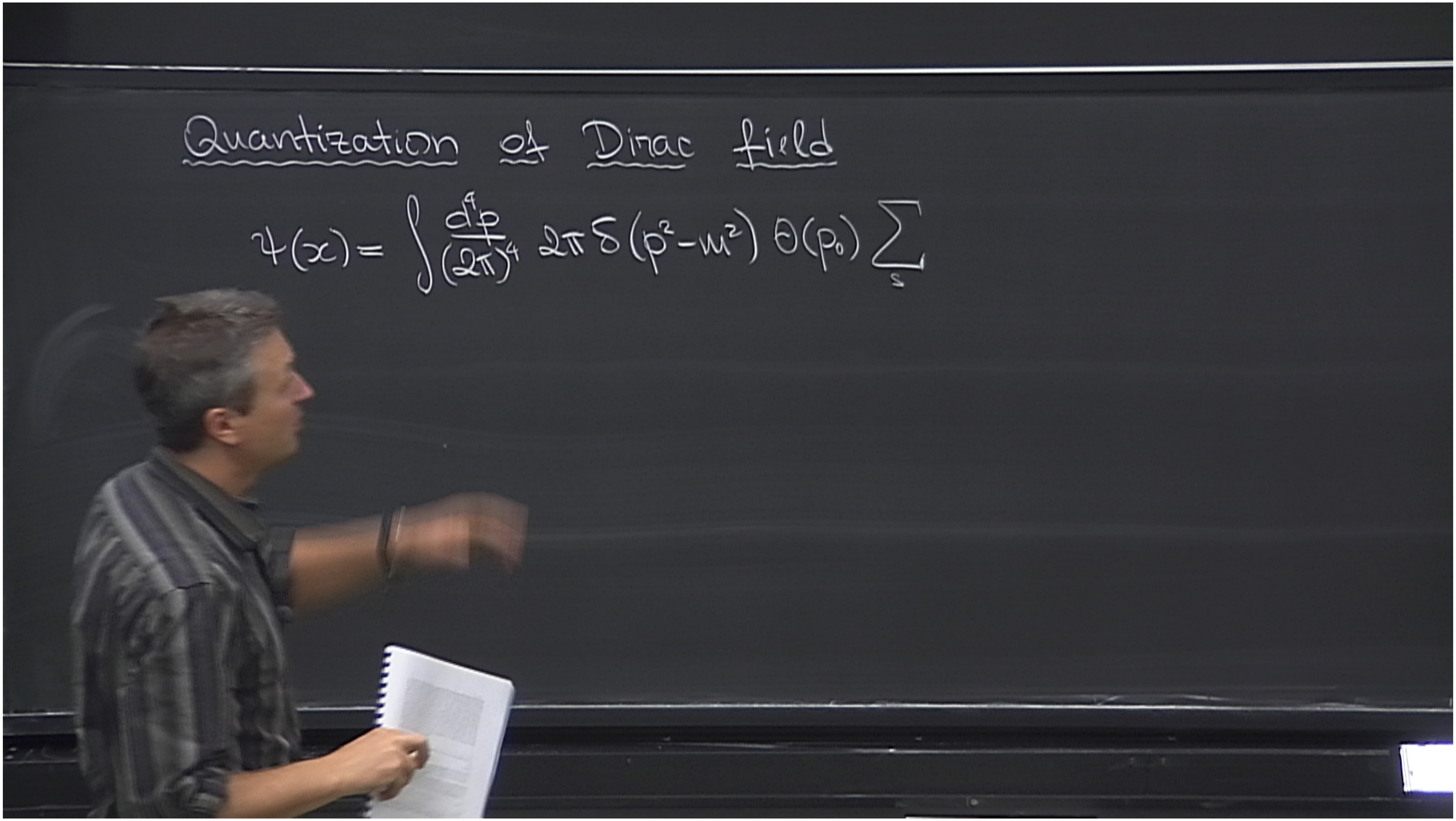

Quantization of Dirac field
$\psi(x) = \int \frac{d^4p}{(2\pi)^4} \, 2\pi \, \delta(p^2 - m^2)$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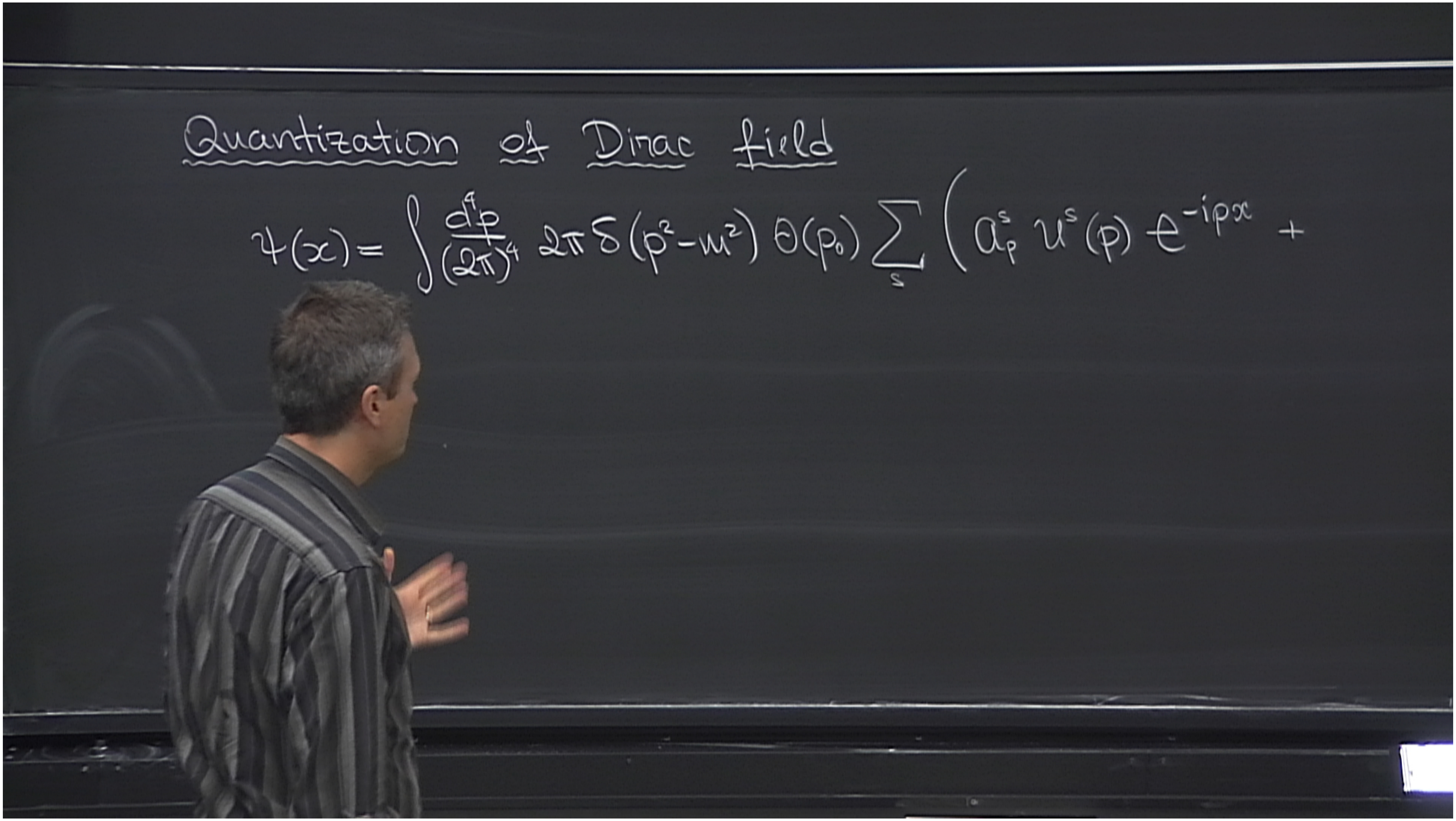
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Quantization of Dirac field

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$$\psi^\dagger(x) = \int \frac{d^4p}{(2\pi)^4}\, 2\pi\,\delta(p^2 - m^2)\,\theta(p_0) \sum_s \Big(a_p^{s\dagger}\, \bar{u}$$

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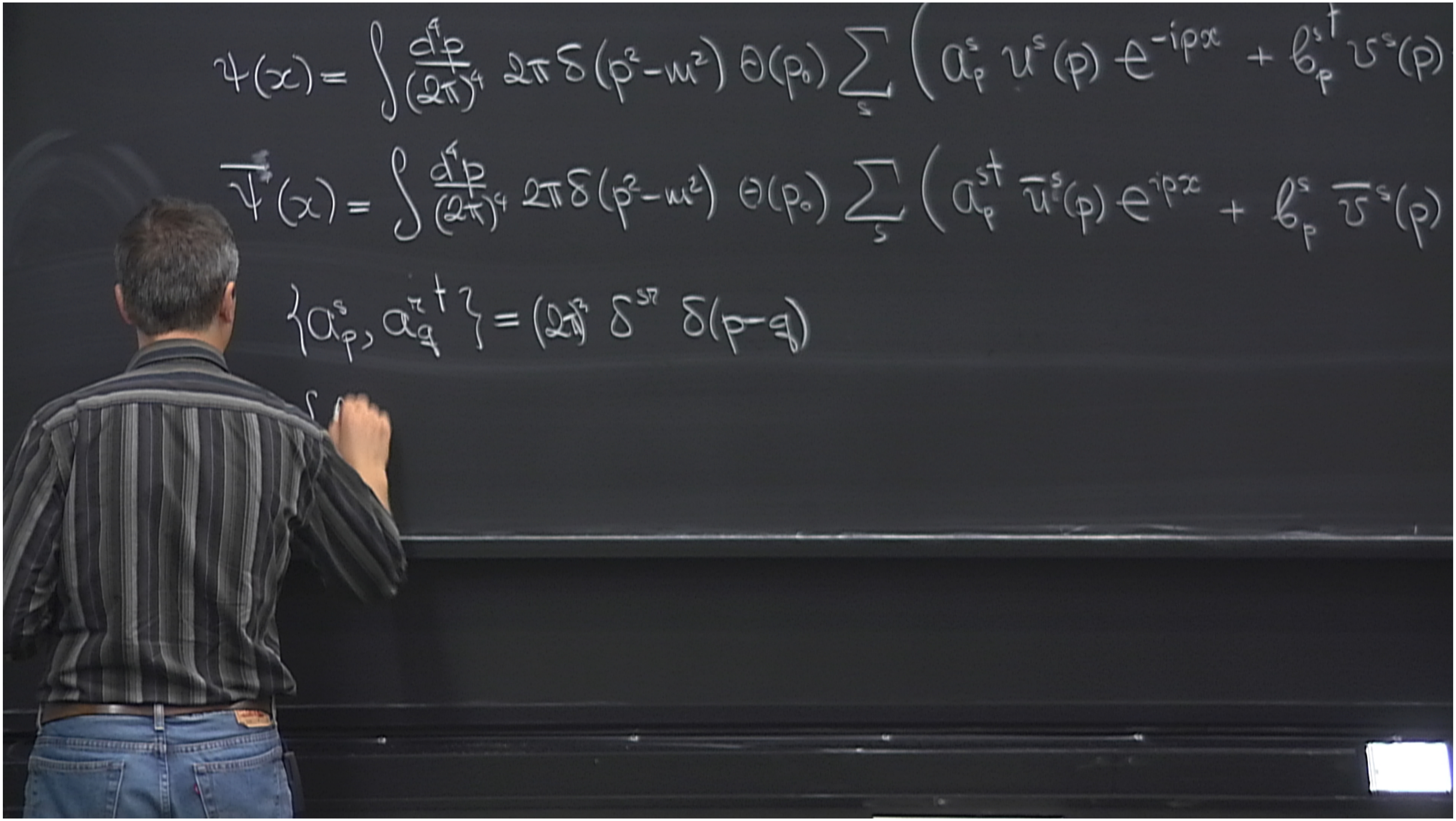
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$\{a_p^s, a_q^{r\dagger}\} = (2\pi)^3 \delta^{sr} \delta(p-q)$

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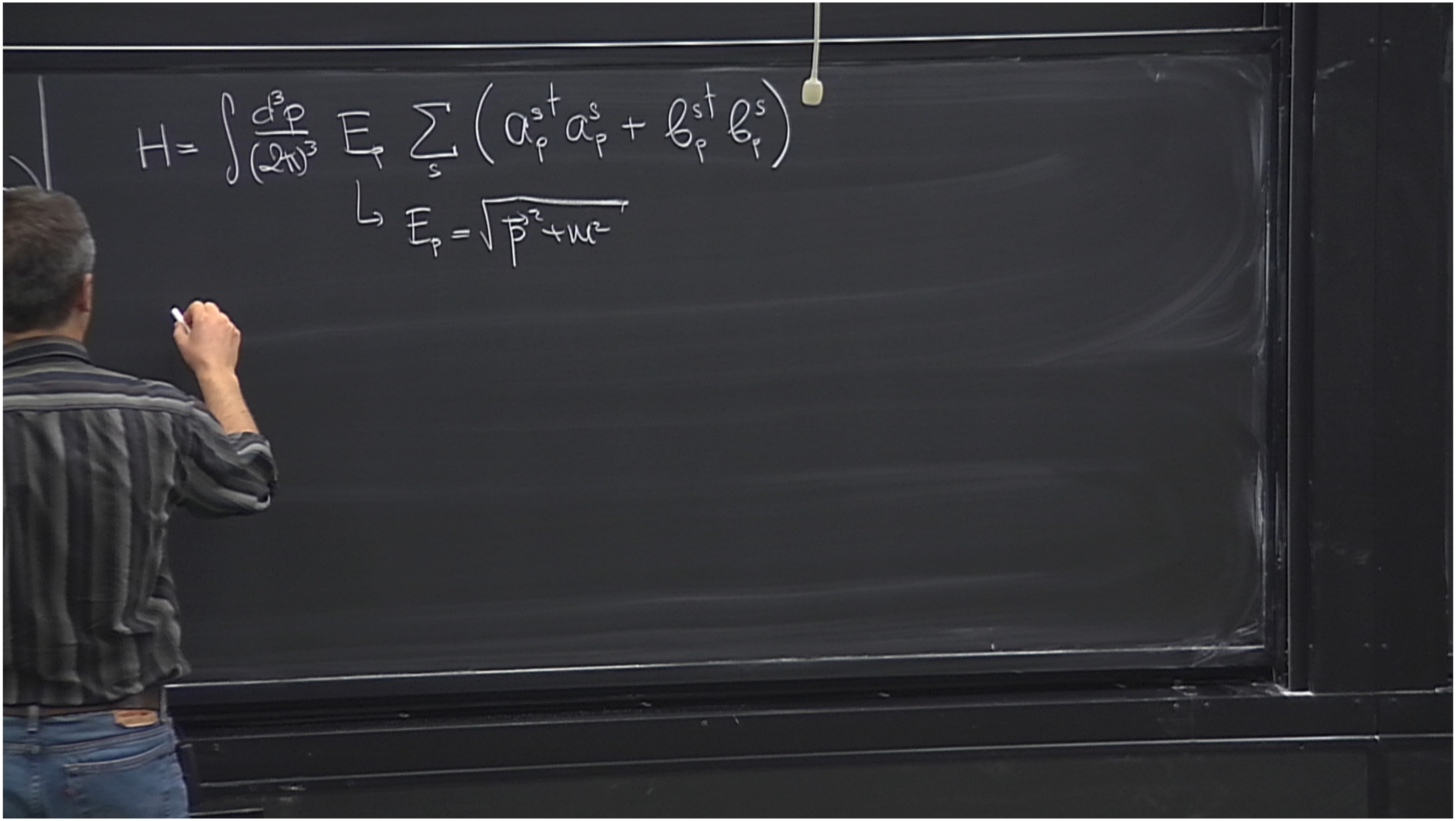
$H = \int \frac{d^3p}{(2\pi)^3} E_p \sum_s \left(a_p^{s\dagger} a_p^s + b_p^{s\dagger} b_p^s \right)$
$\hookrightarrow E_p = \sqrt{\vec{p}^2 + m^2}$

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up to infinite normal-ordering constants

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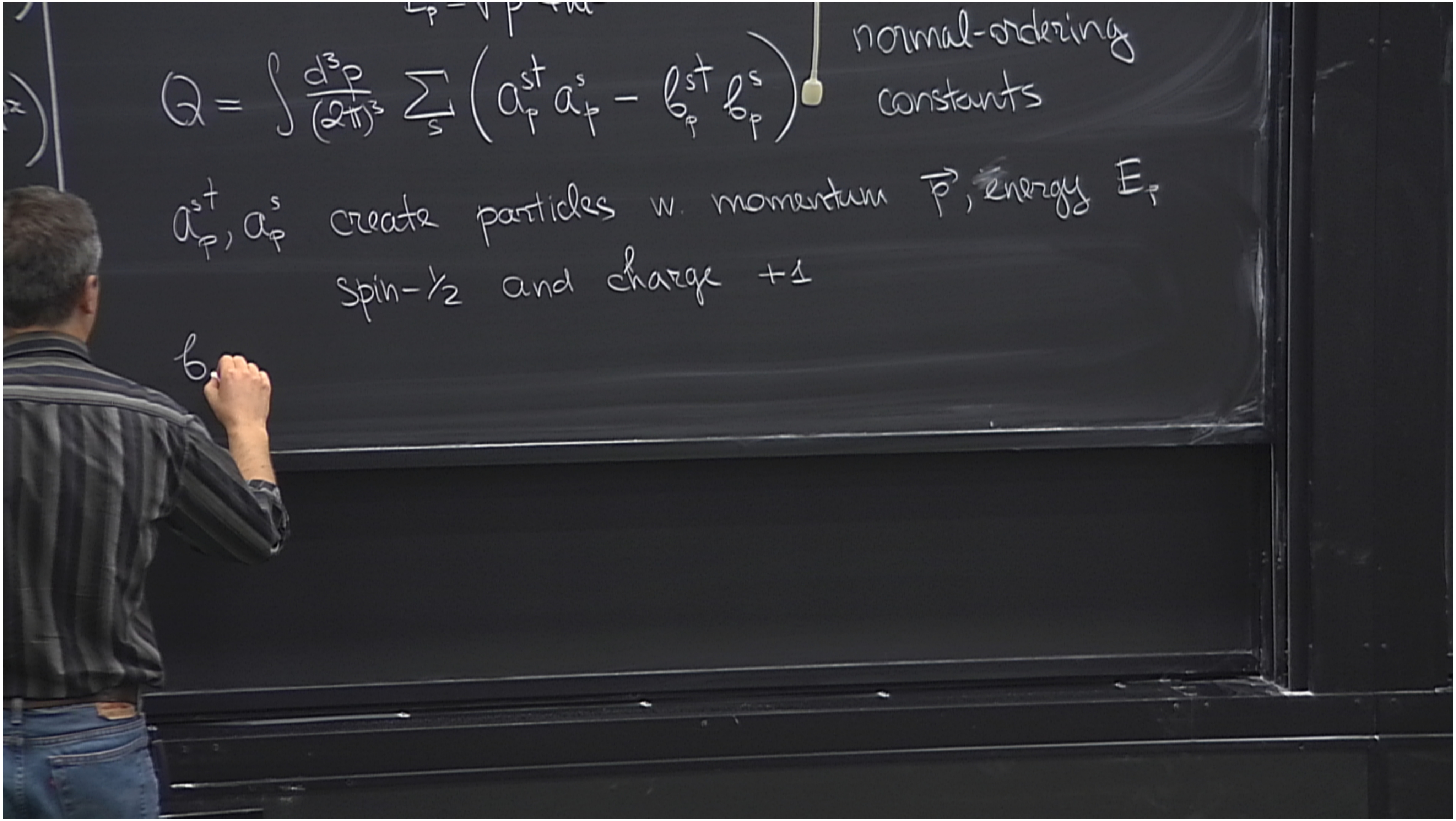

$Q = \int \frac{d^3p}{(2\pi)^3} \sum_s \left(a_p^{s\dagger} a_p^s - b_p^{s\dagger} b_p^s \right)$
normal-ordering constants
$a_p^{s\dagger}, a_p^s$ create particles w. momentum $\vec{p}$, energy E_p
spin-$\frac{1}{2}$ and charge $+1$

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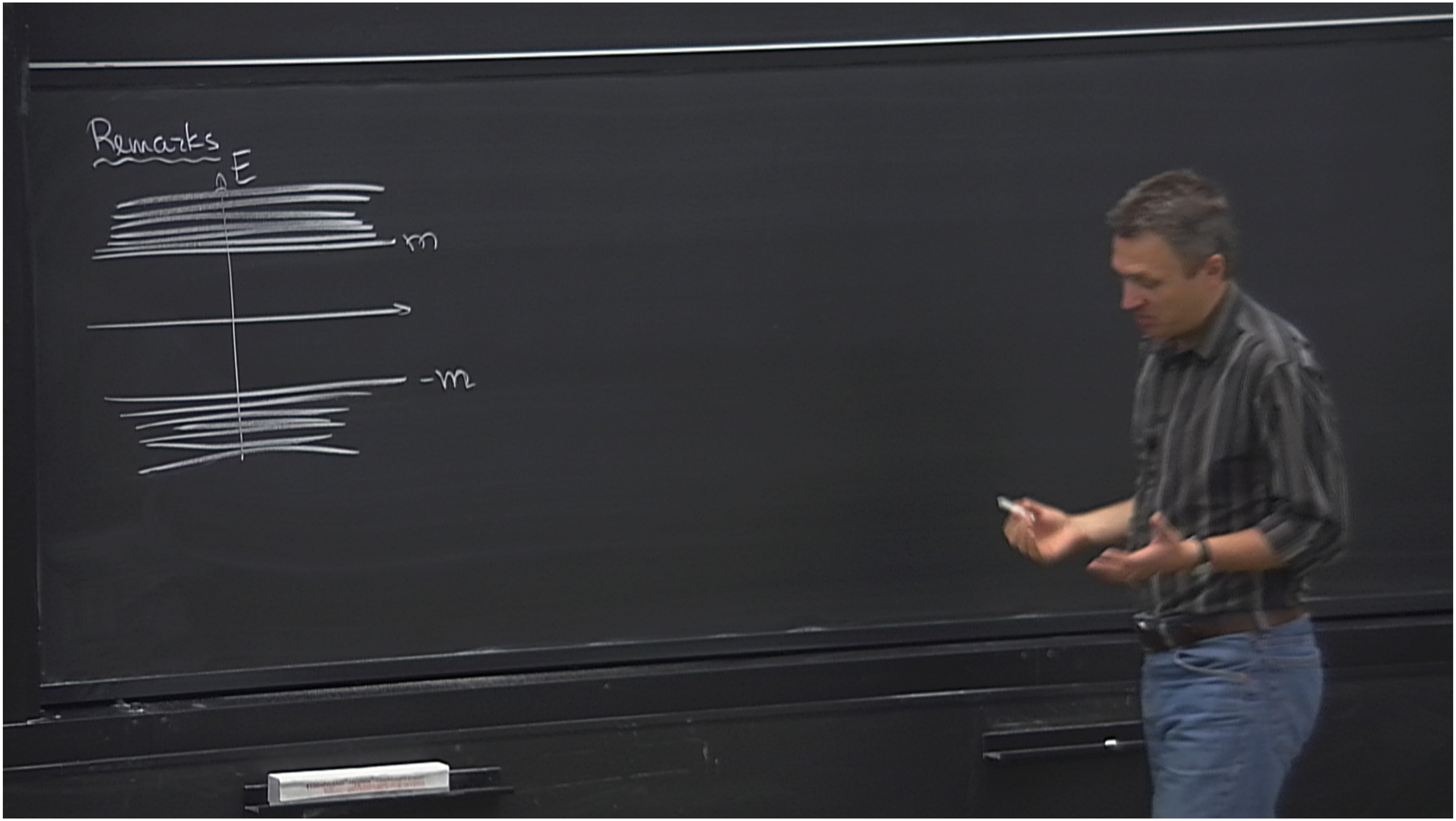
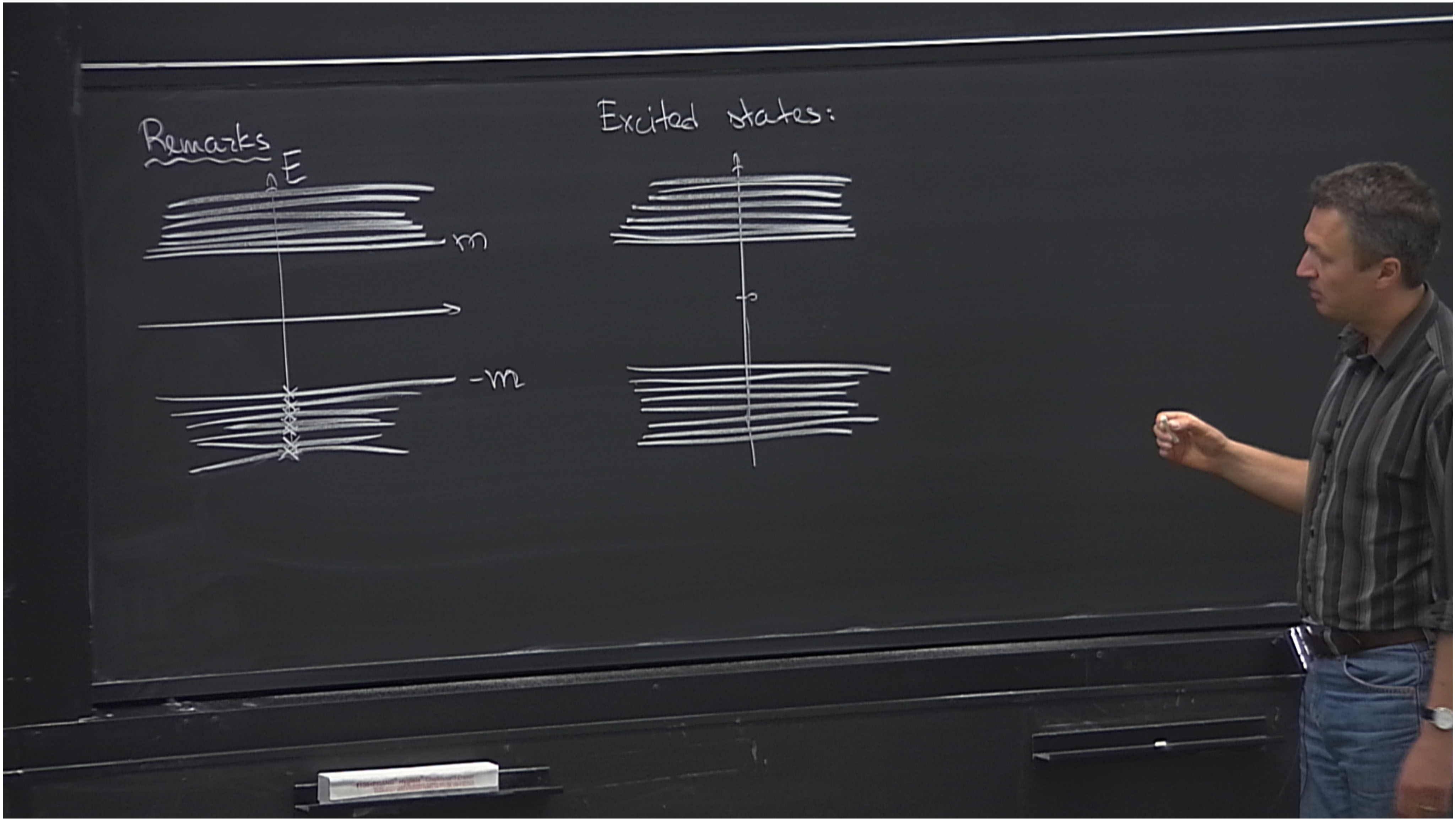
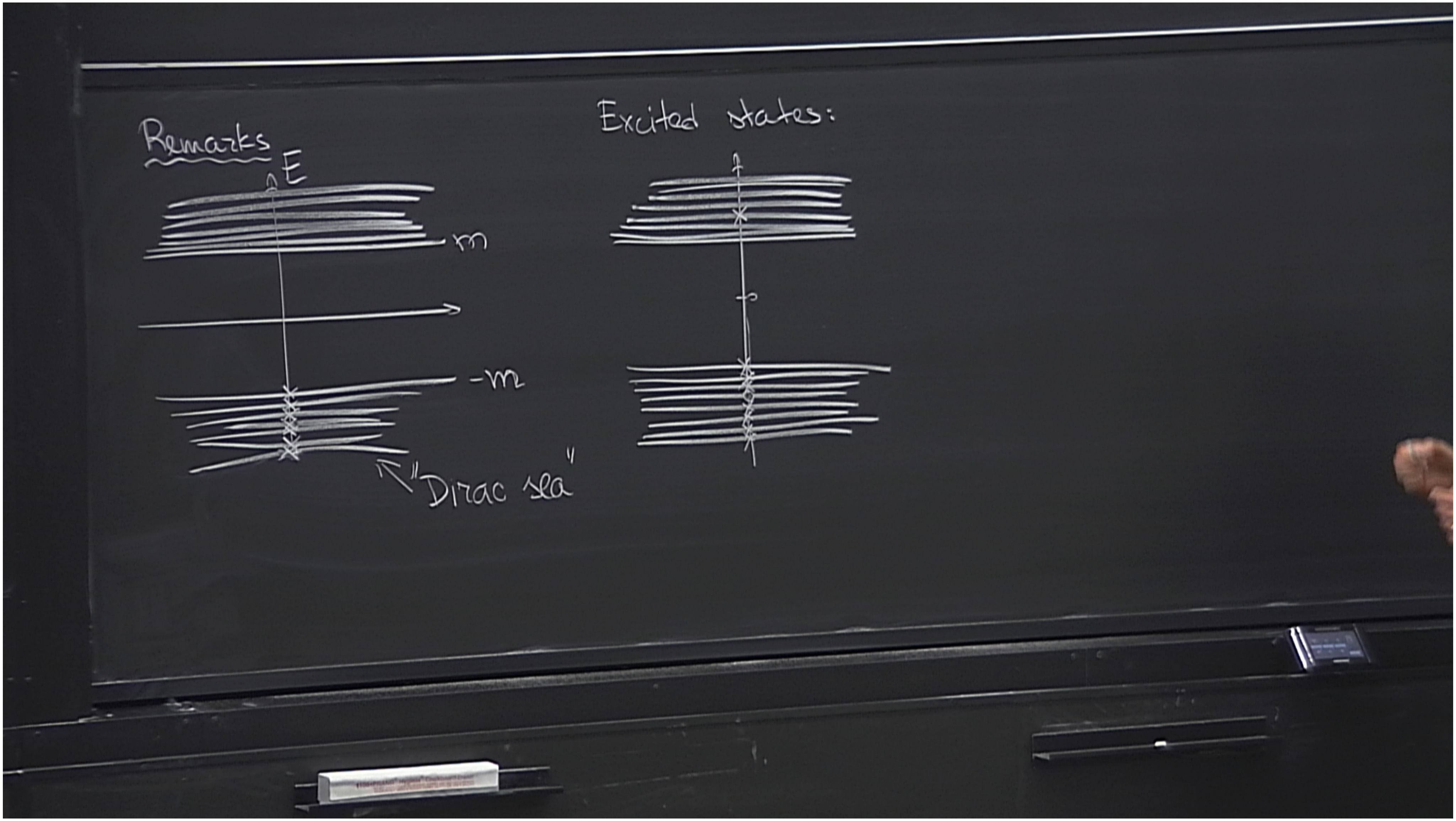
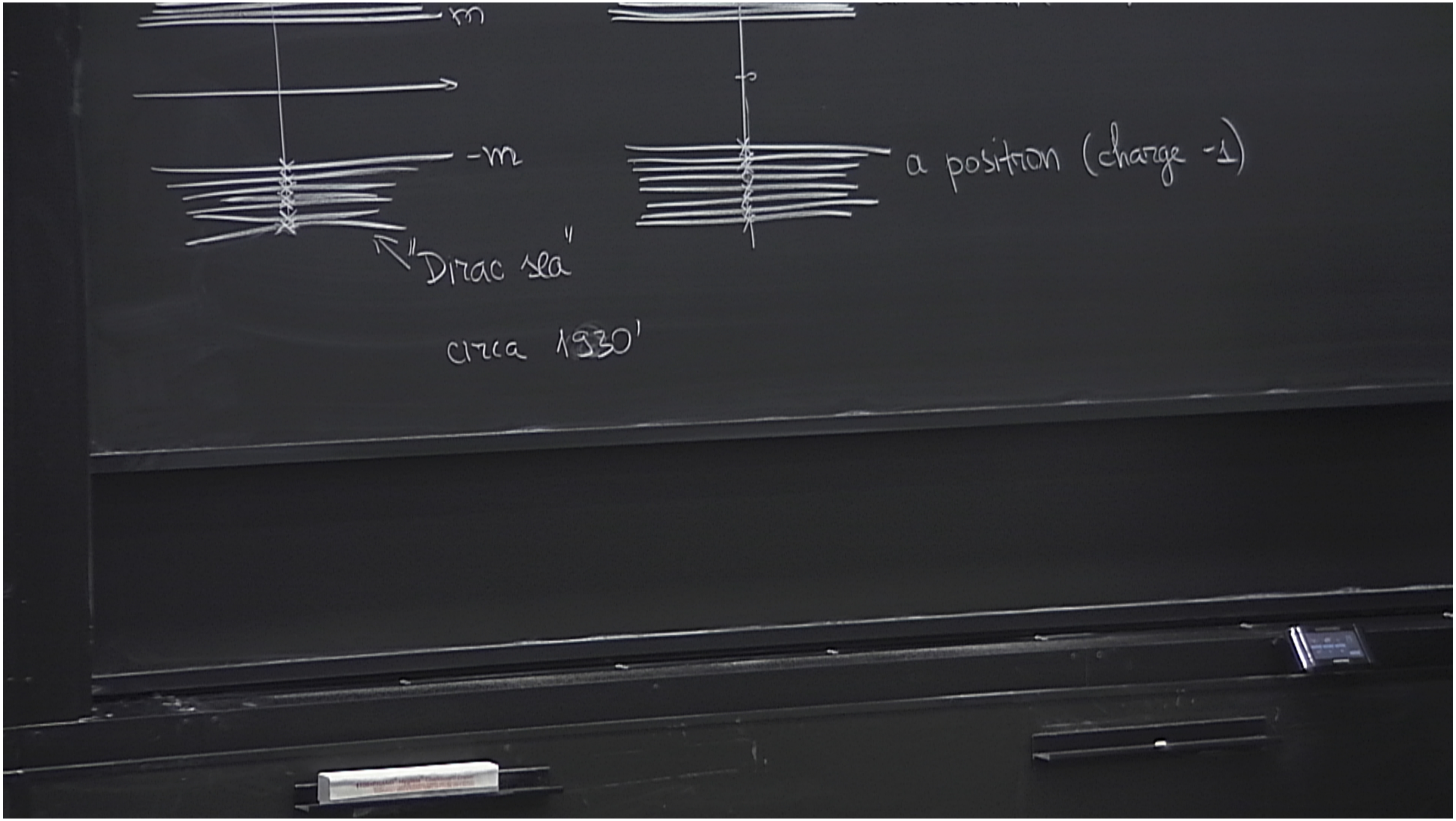
$b_p^{s\dagger}, b_p^s$ —— " —— " —— " —— " —— " —— " ——

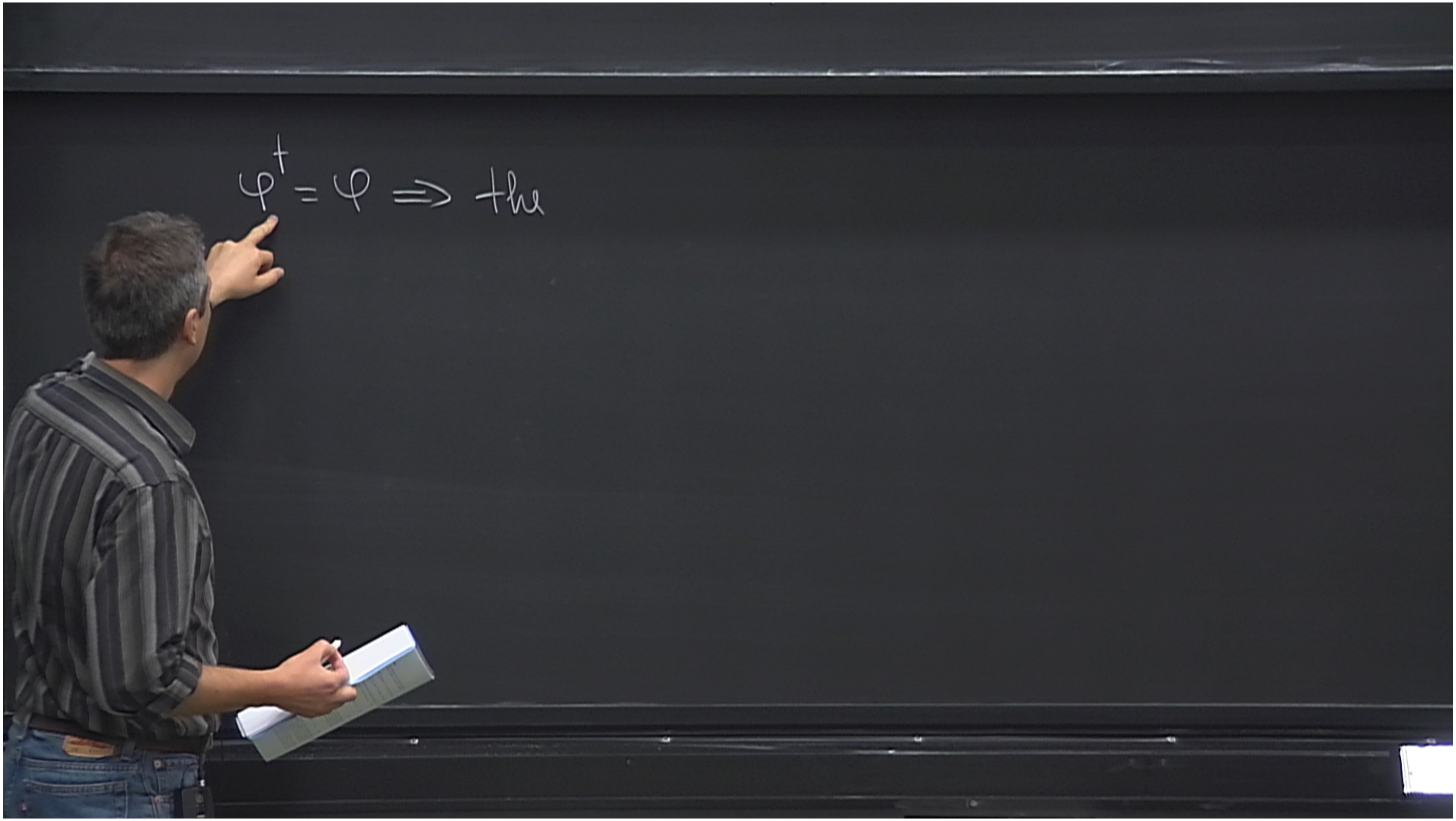
—— " —— " —— " —— -1

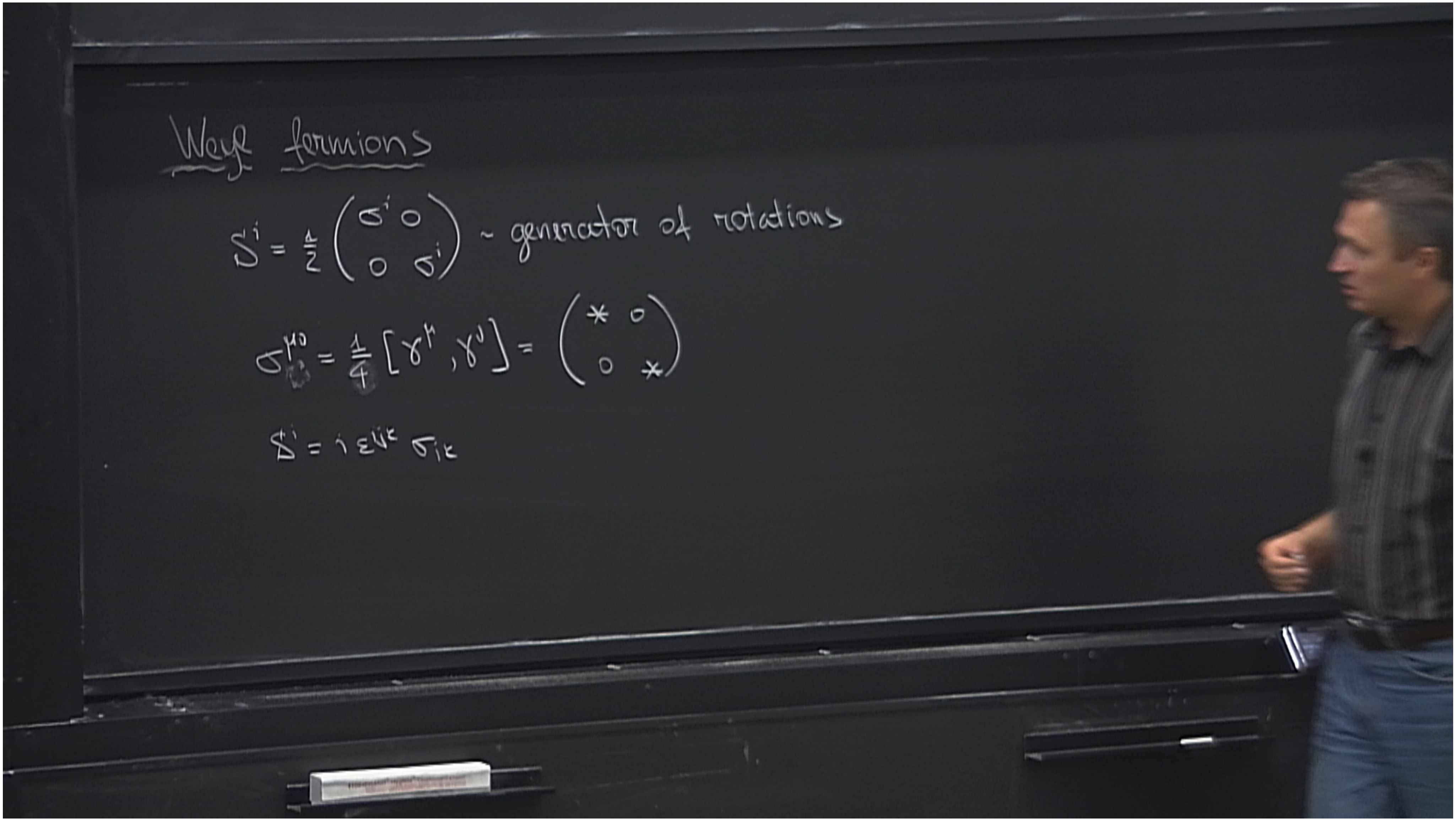

Remarks
E
m
-m

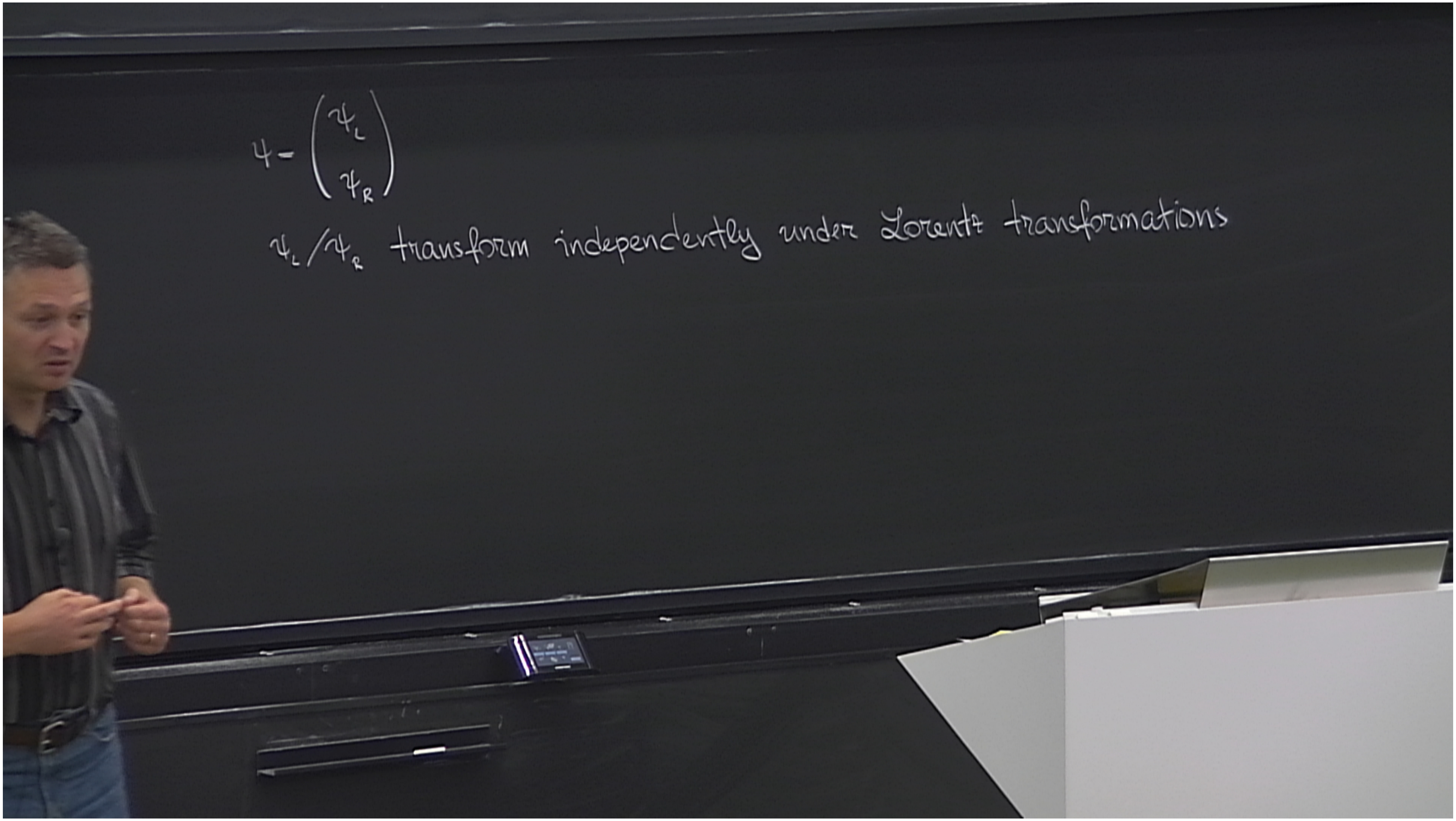

Remarks
E
m
-m
Excited states:

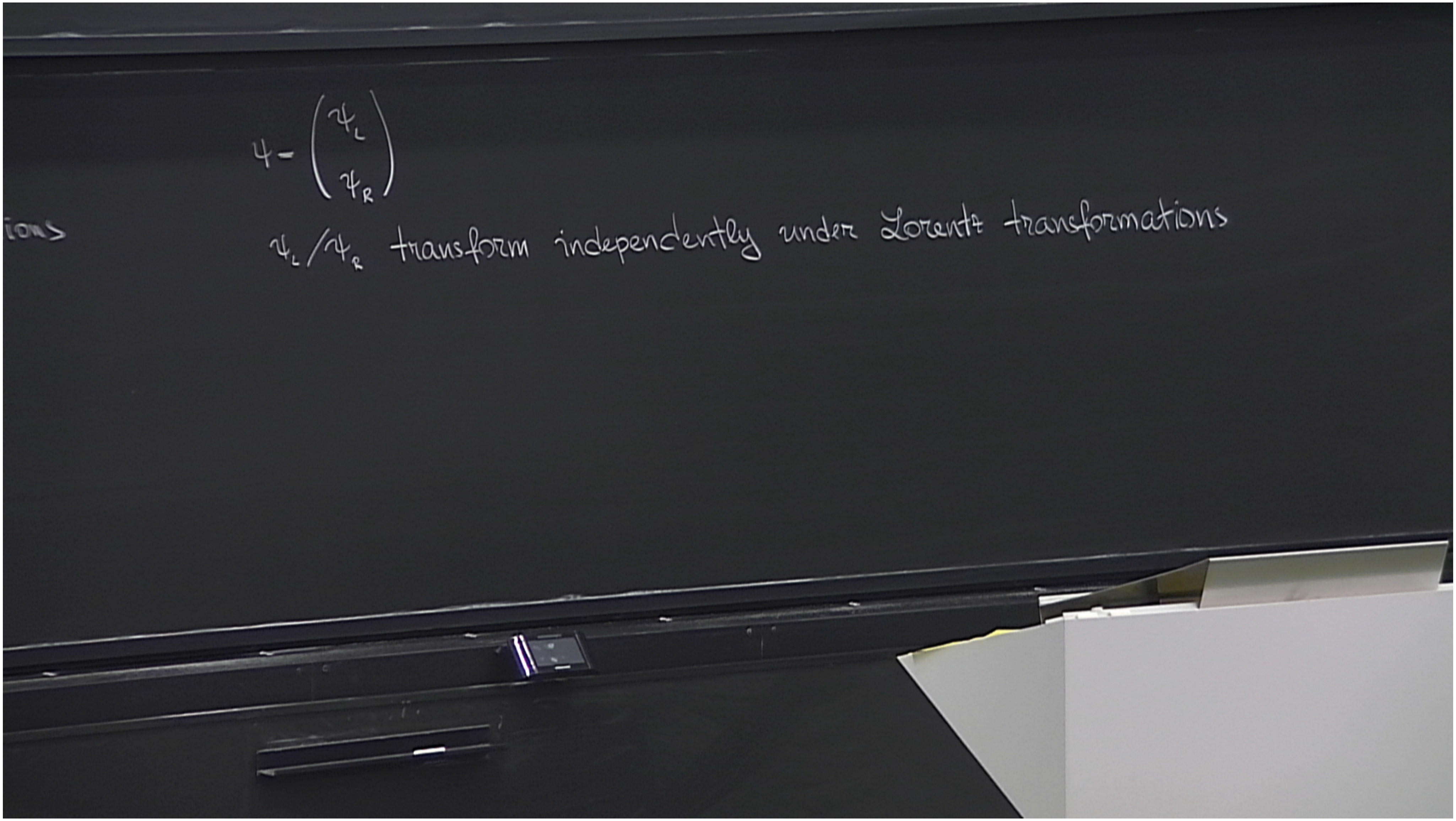

Remarks
E
m
-m
"Dirac sea"
Excited states:


m
-m
"Dirac sea"
circa 1930'
a positron (charge -1)


$\varphi^{\dagger} = \varphi \Rightarrow$ the


Weyl fermions
$S^i = \frac{1}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}$ ~ generator of rotations
$\sigma^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu] = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$
$S^i = i\,\varepsilon^{ijk}\sigma_{jk}$


$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$
ψ_L / ψ_R transform independently under Lorentz transformations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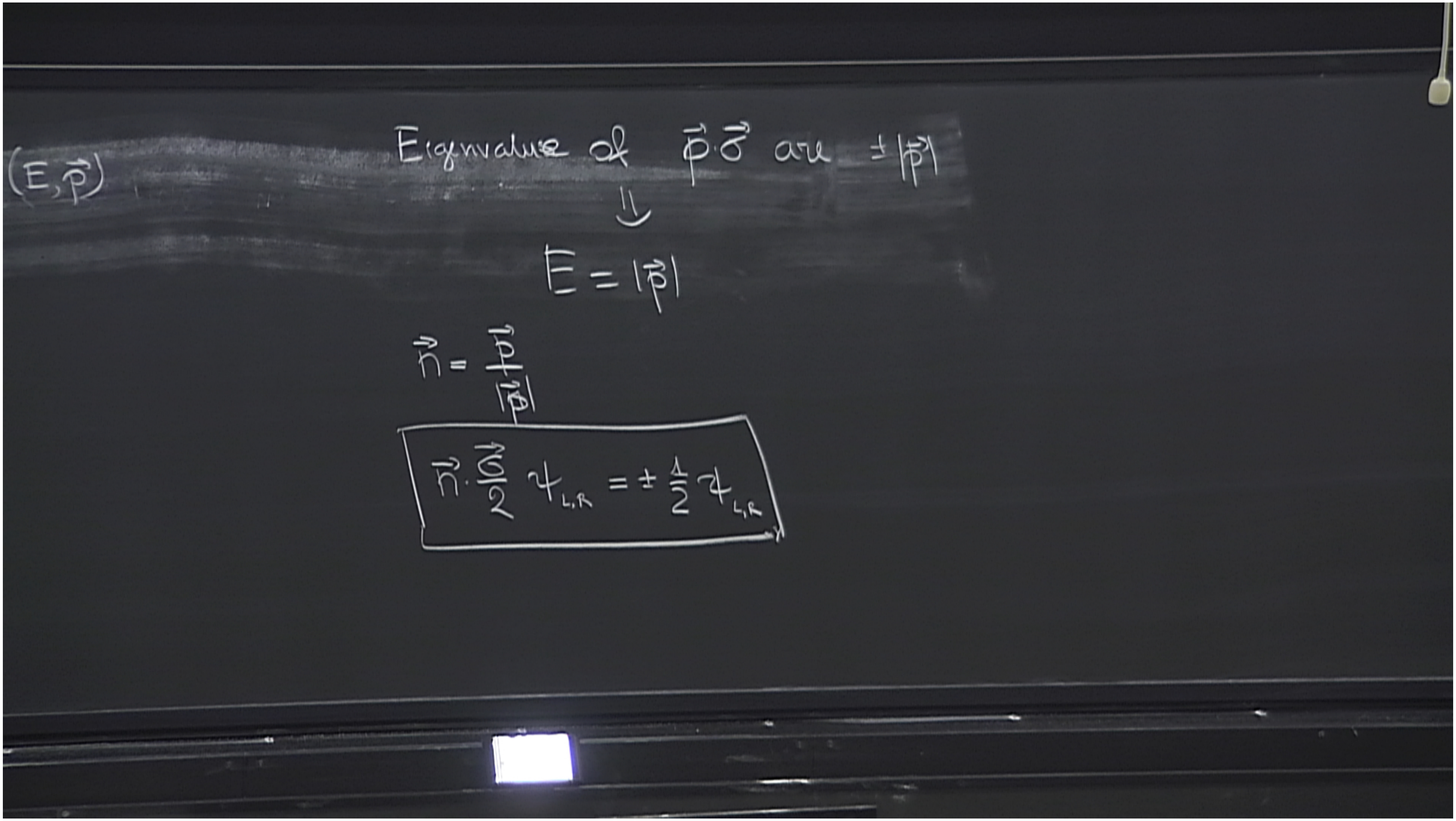
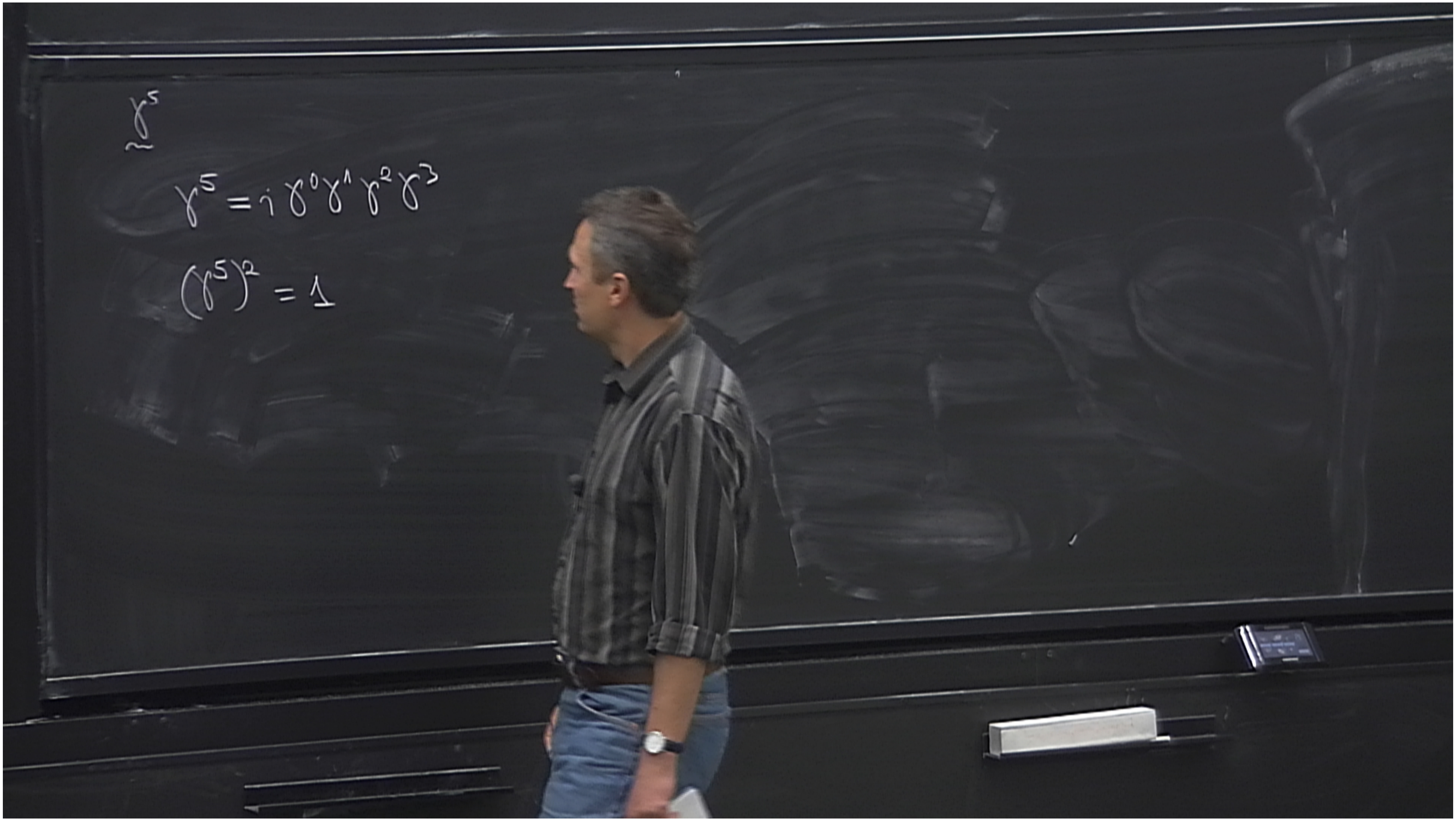
ψ_L / ψ_R transform independently under Lorentz transformations

Dirac equation for $m=0$: $p^\mu = (E, \vec{p})$

$$\begin{pmatrix} 0 & E + \vec{p}\cdot\vec{\sigma} \\ E - \vec{p}\cdot\vec{\sigma} & 0 \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0$$

$$\left(E \mp \vec{p}\cdot\vec{\sigma}\right)\psi_{L,R} = 0$$


$(E, \vec{p})$
Eignvalue of $\vec{p}\cdot\vec{\sigma}$ are $\pm|\vec{p}|$
$\Downarrow$
$E = |\vec{p}|$
$\vec{n} = \frac{\vec{p}}{|\vec{p}|}$
$\vec{n}\cdot\frac{\vec{\sigma}}{2}\psi_{L,R} = \pm\frac{1}{2}\psi_{L,R}$


γ^5
$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$
$(\gamma^5)^2 = 1$

$\underset{\sim}{\gamma^5}$

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

$$(\gamma^5)^2 = 1$$

$$\{\gamma^5, \gamma^\mu\} = 0$$

$$\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\dfrac{1 \mp \gamma^5}{2}$ are projection operators

$$\psi_{L,R} = \frac{1 \mp \gamma^5}{2}\psi$$