Title: Quantum Field Theory I - Lecture 5a

Date: Oct 07, 2011 09:00 AM

URL: http://pirsa.org/11100010

Abstract:


Dirac equation
$(i\gamma^\mu \partial_\mu - m)\psi = 0$


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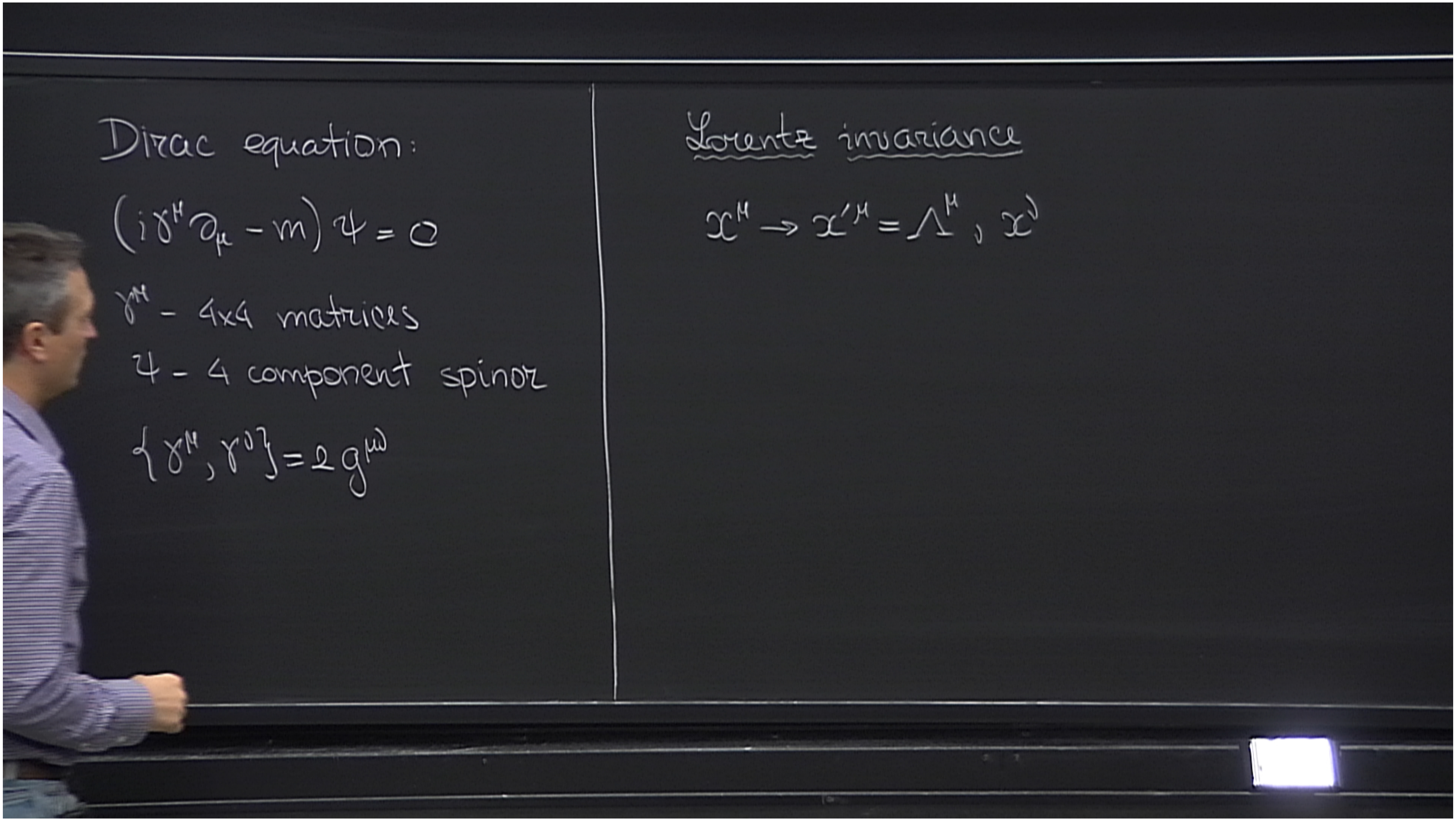
ψ - 4 component spinor
$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$


Dirac equation:
$(i\gamma^\mu \partial_\mu - m)\psi = 0$
γ^μ - 4x4 matrices
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Lorentz invariance
$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$

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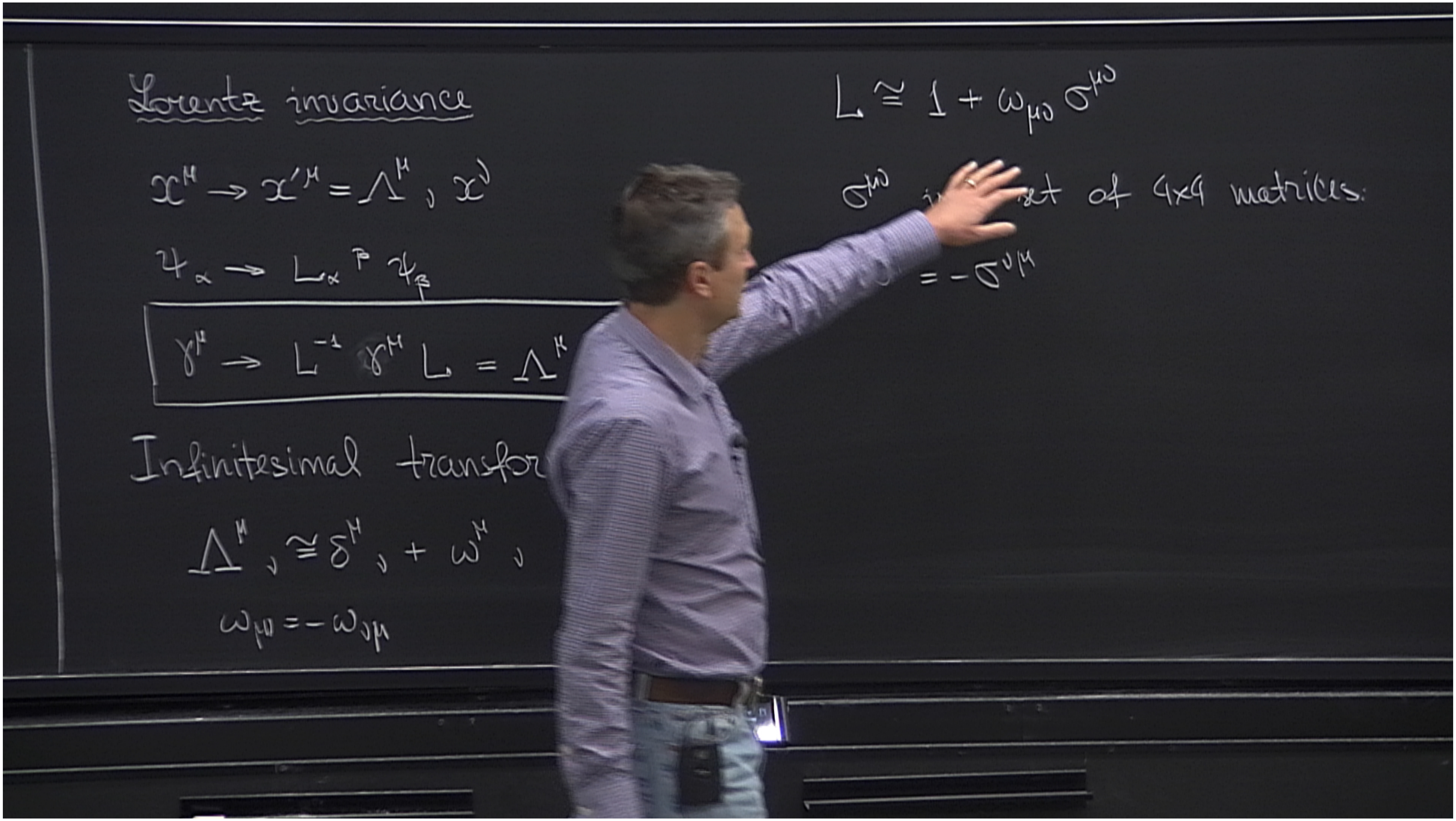
$$\boxed{\gamma^\mu \to L^{-1} \gamma^\mu L = \Lambda^\mu{}_\nu \, \gamma^\nu}$$


Lorentz invariance
$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu \, x^\nu$
$\psi_\alpha \to L_\alpha{}^\beta \psi_\beta$
$\gamma^\mu \to L^{-1} \gamma^\mu L = \Lambda^\mu$
Infinitesimal
$\Lambda^\mu{}_\nu \cong \delta^\mu{}_\nu + \omega^\mu{}_\nu$
$\omega_{\mu\nu} = -\omega_{\nu\mu}$
$L \cong 1 + \omega_{\mu\nu} \sigma^{\mu\nu}$
$\sigma^{\mu\nu}$
of 4x4 matrices:
$= -\sigma^{\nu\mu}$

Lorentz invariance

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$\sigma^{\mu\nu}$ is a set of 4x4 matrices:

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Lorentz invariance

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$$\boxed{\gamma^\mu \to L^{-1}\gamma^\mu L = \Lambda^\mu{}_\nu\, \gamma^\nu} \implies \boxed{[\sigma^{\mu\nu}, \gamma^\lambda] = g^{\nu\lambda}\gamma^\mu - g^{\mu\lambda}\gamma^\nu}$$

Infinitesimal transformations:

$$\Lambda^\mu{}_\nu \cong \delta^\mu{}_\nu + \omega^\mu{}_\nu$$

$$\omega_{\mu\nu} = -\omega_{\nu\mu}$$

$$L \cong 1 + \tfrac{1}{2}\omega_{\mu\nu}\sigma^{\mu\nu}$$

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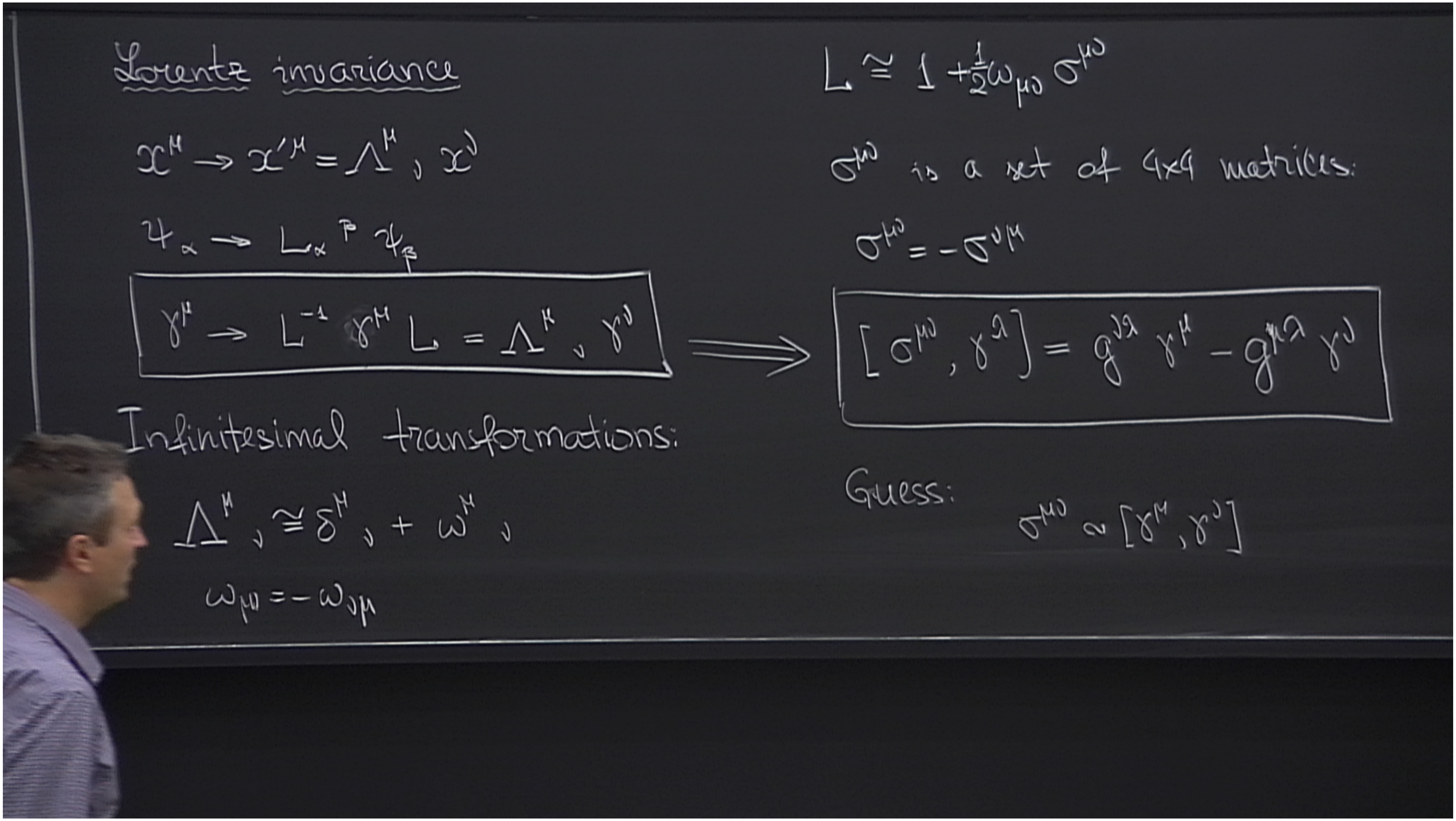
Lorentz invariance
$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu \, x^\nu$
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$\boxed{\gamma^\mu \to L^{-1} \gamma^\mu L = \Lambda^\mu{}_\nu \, \gamma^\nu} \Longrightarrow \boxed{[\sigma^{\mu\nu}, \gamma^\lambda] = g^{\nu\lambda} \gamma^\mu - g^{\mu\lambda} \gamma^\nu}$
Infinitesimal transformations:
$\Lambda^\mu{}_\nu \cong \delta^\mu{}_\nu + \omega^\mu{}_\nu$
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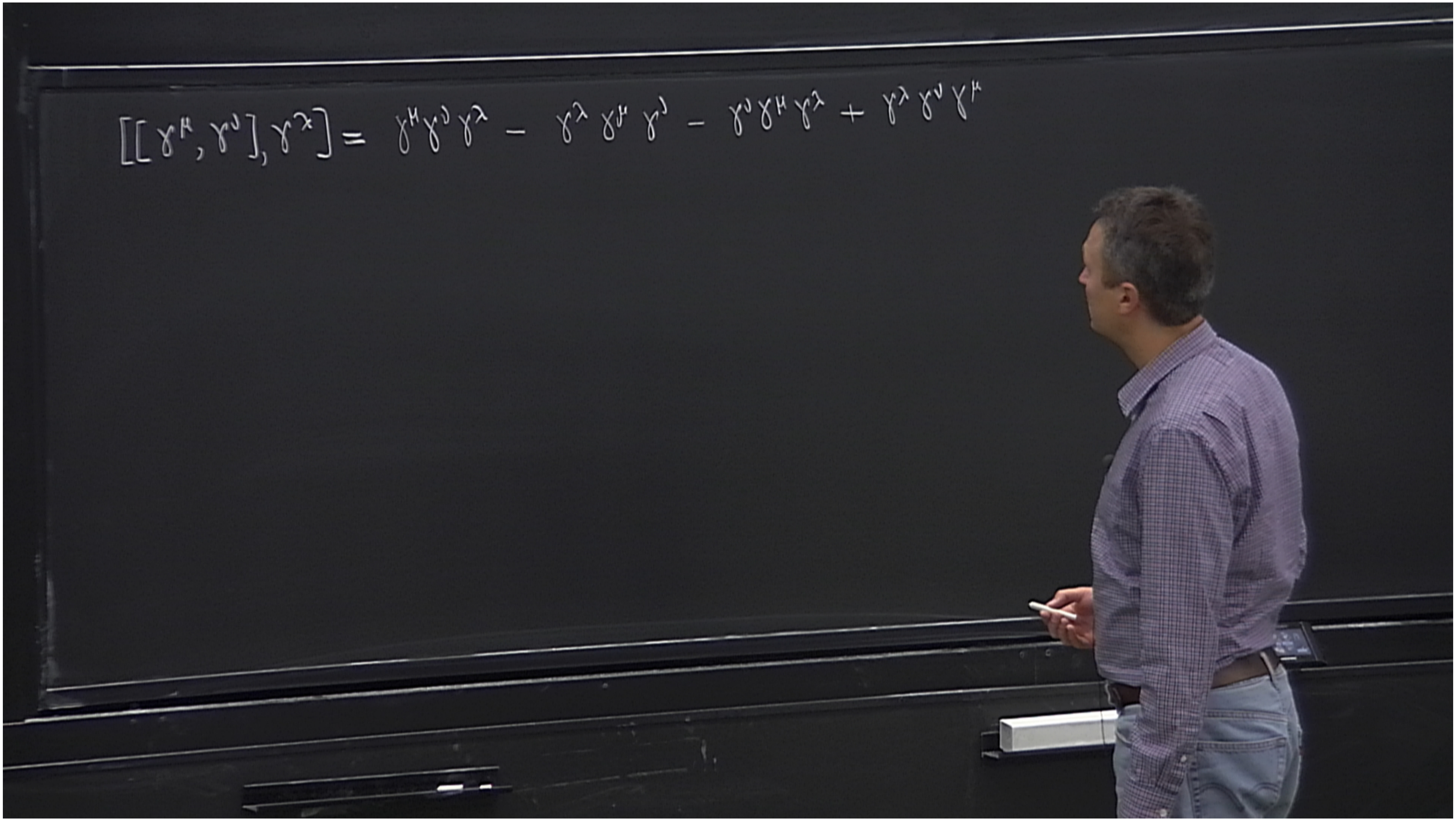
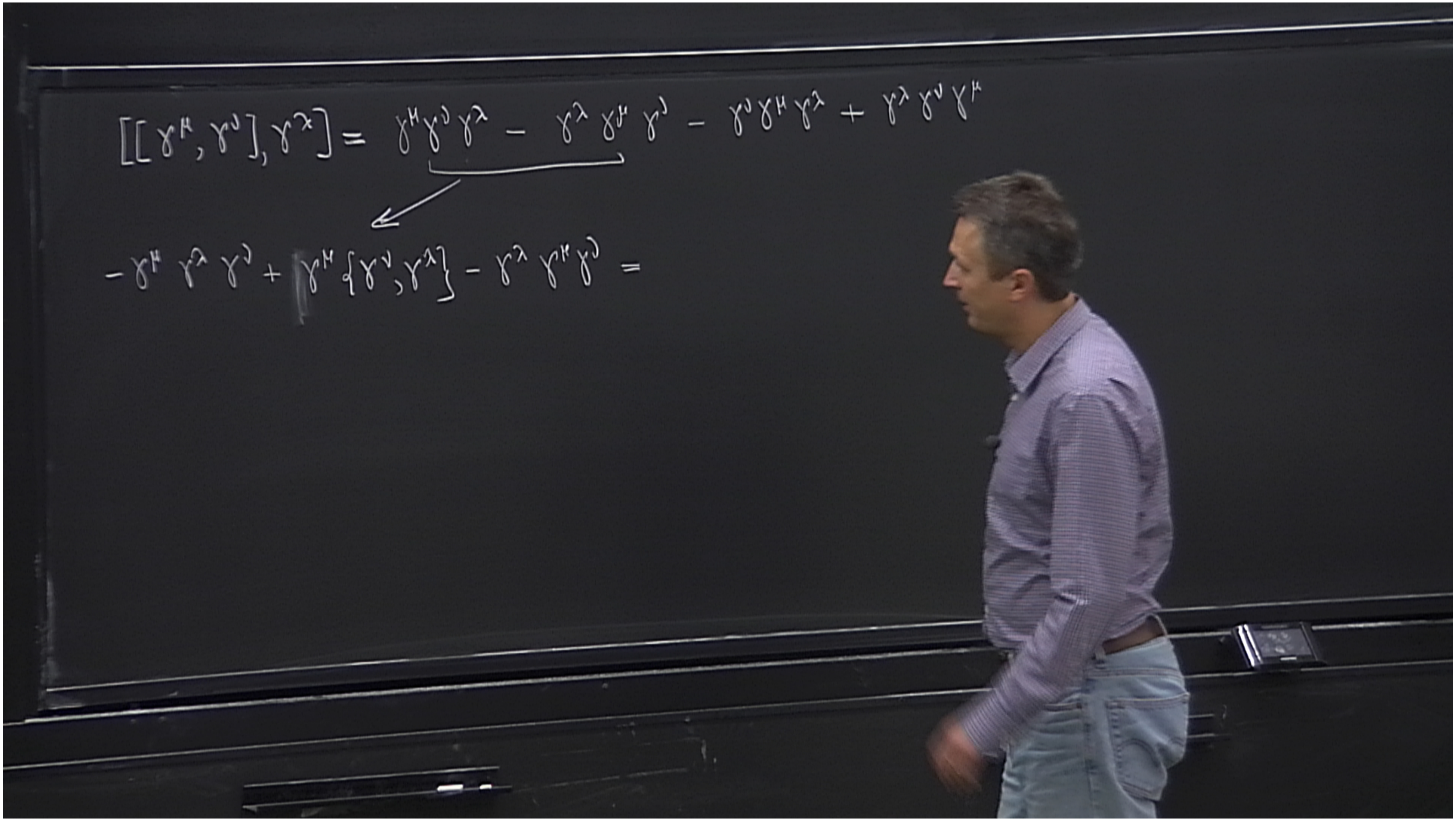
Guess:
$\sigma^{\mu\nu} \sim [\gamma^\mu, \gamma^\nu]$


$[[\gamma^\mu,\gamma^\nu],\gamma^\lambda] = \gamma^\mu\gamma^\nu\gamma^\lambda - \gamma^\lambda\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\gamma^\lambda + \gamma^\lambda\gamma^\nu\gamma^\mu$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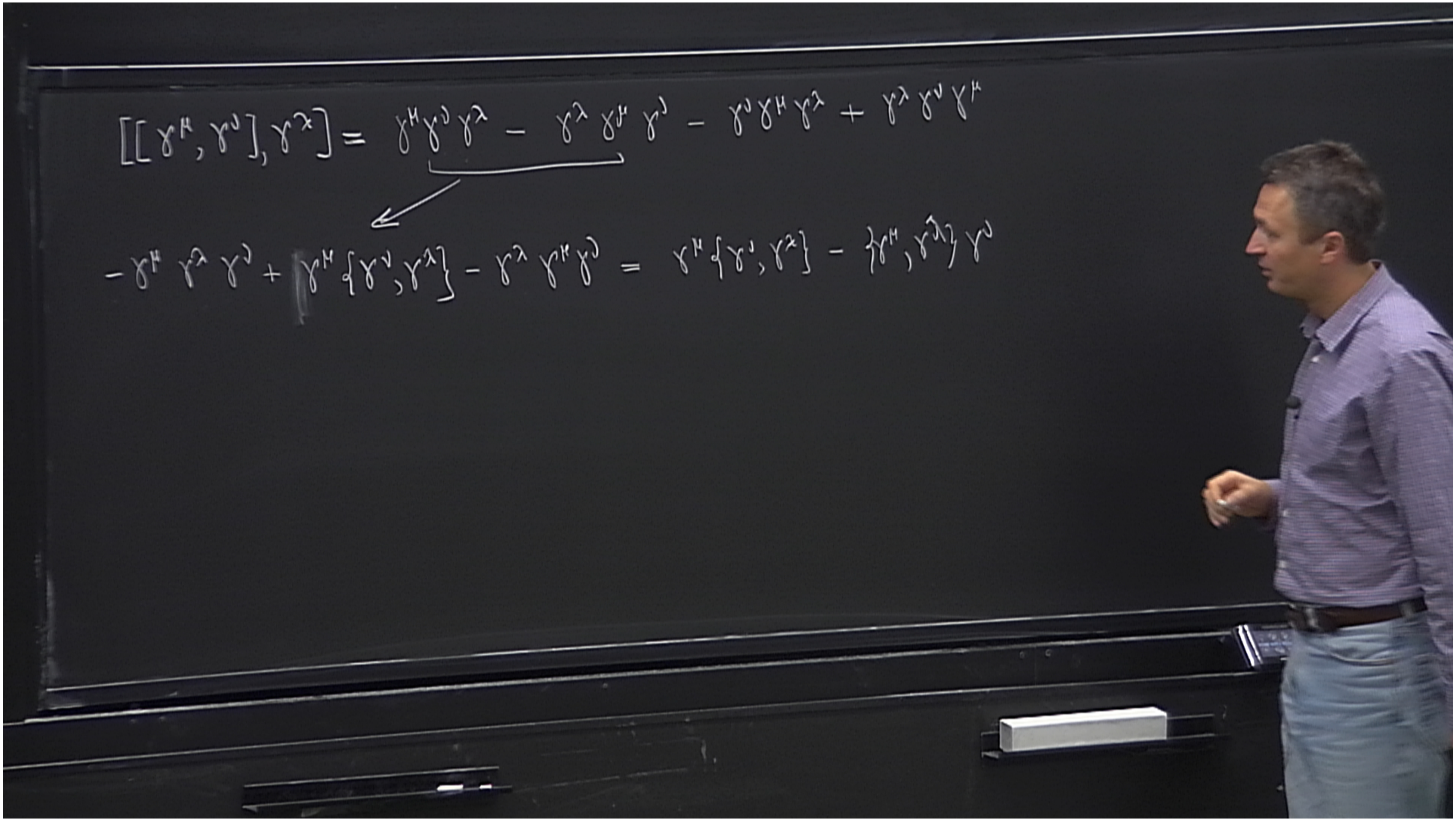
$-\gamma^\mu \gamma^\lambda \gamma^\nu + \gamma^\mu \{\gamma^\nu, \gamma^\lambda\} - \gamma^\lambda \gamma^\mu \gamma^\nu =$



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$-\gamma^\mu\gamma^\lambda\gamma^\nu + \gamma^\mu\{\gamma^\nu,\gamma^\lambda\} - \gamma^\lambda\gamma^\mu\gamma^\nu = \gamma^\mu\{\gamma^\nu,\gamma^\lambda\} - \{\gamma^\mu,\gamma^\lambda\}\gamma^\nu$

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$$2g^{\lambda\nu}\gamma^\mu - 2g^{\lambda\mu}\gamma^\nu$$

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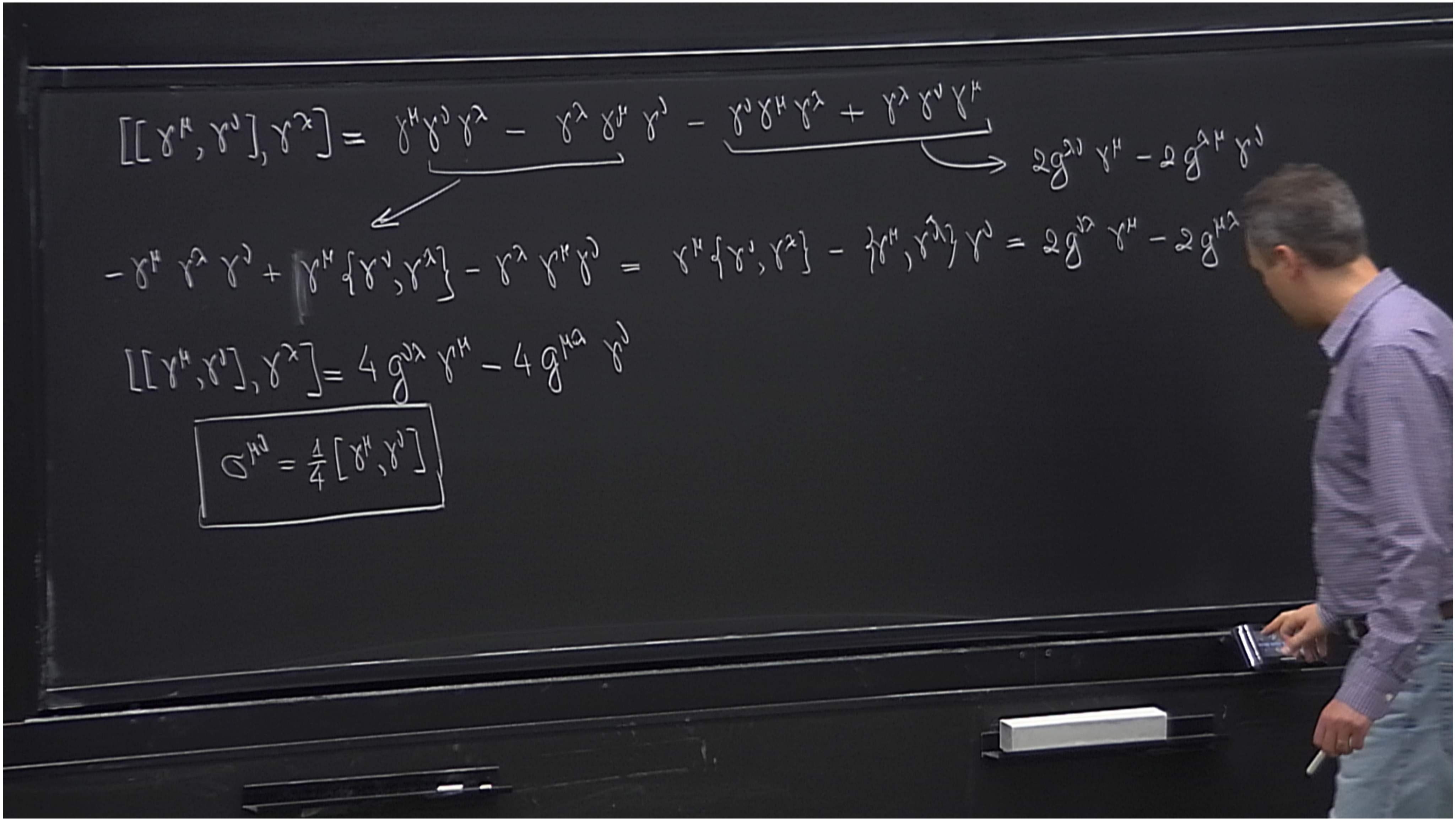
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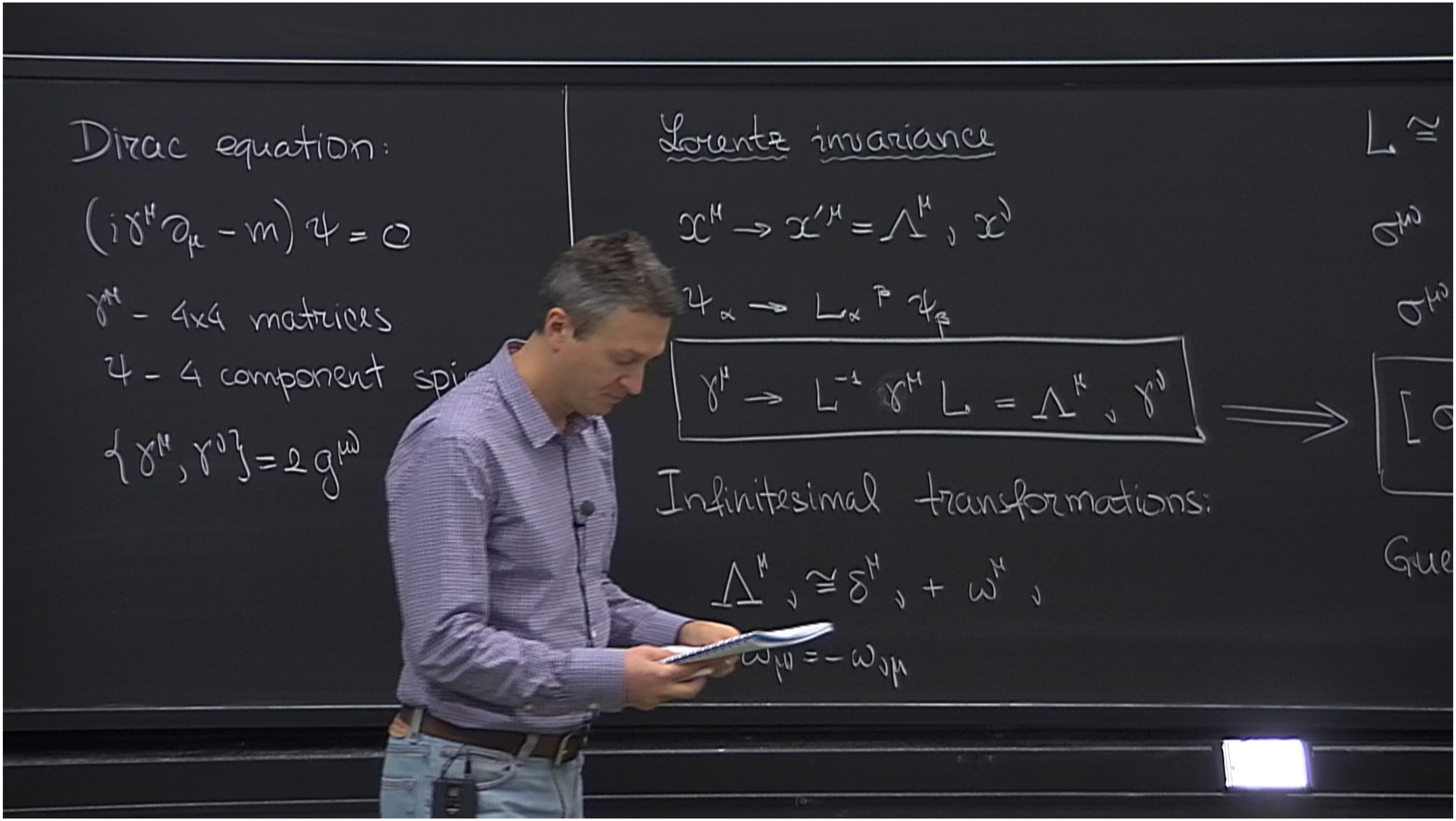
$2g^{\lambda\nu}\gamma^\mu - 2g^{\lambda\mu}\gamma^\nu$
$-\gamma^\mu\gamma^\lambda\gamma^\nu + \gamma^\mu\{\gamma^\nu,\gamma^\lambda\} - \gamma^\lambda\gamma^\mu\gamma^\nu = \gamma^\mu\{\gamma^\nu,\gamma^\lambda\} - \{\gamma^\mu,\gamma^\lambda\}\gamma^\nu = 2g^{\nu\lambda}\gamma^\mu - 2g^{\mu\lambda}$
$[[\gamma^\mu,\gamma^\nu],\gamma^\lambda] = 4g^{\nu\lambda}\gamma^\mu - 4g^{\mu\lambda}\gamma^\nu$
$\boxed{\sigma^{\mu\nu} = \frac{1}{4}[\gamma^\mu,\gamma^\nu]}$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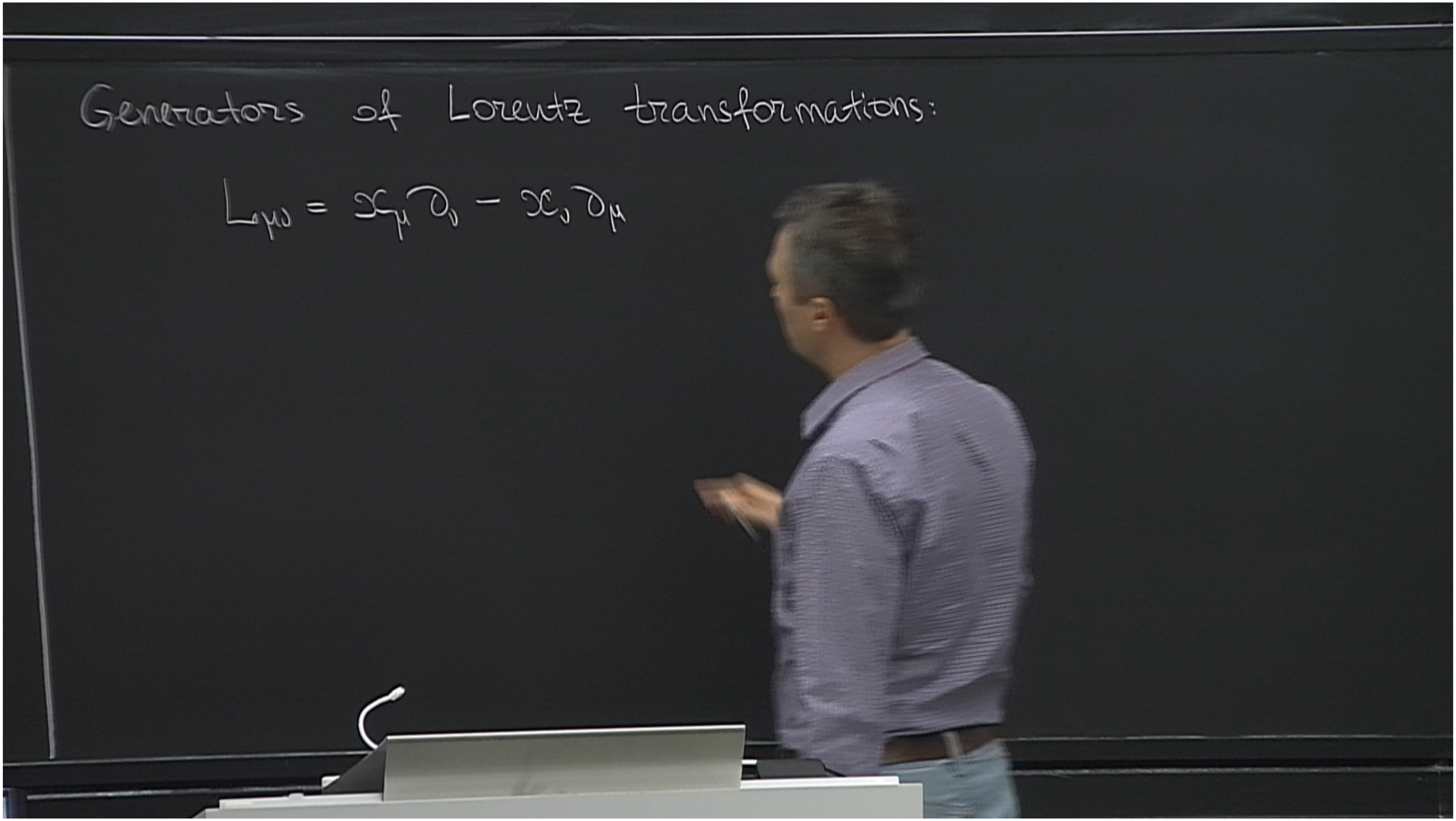
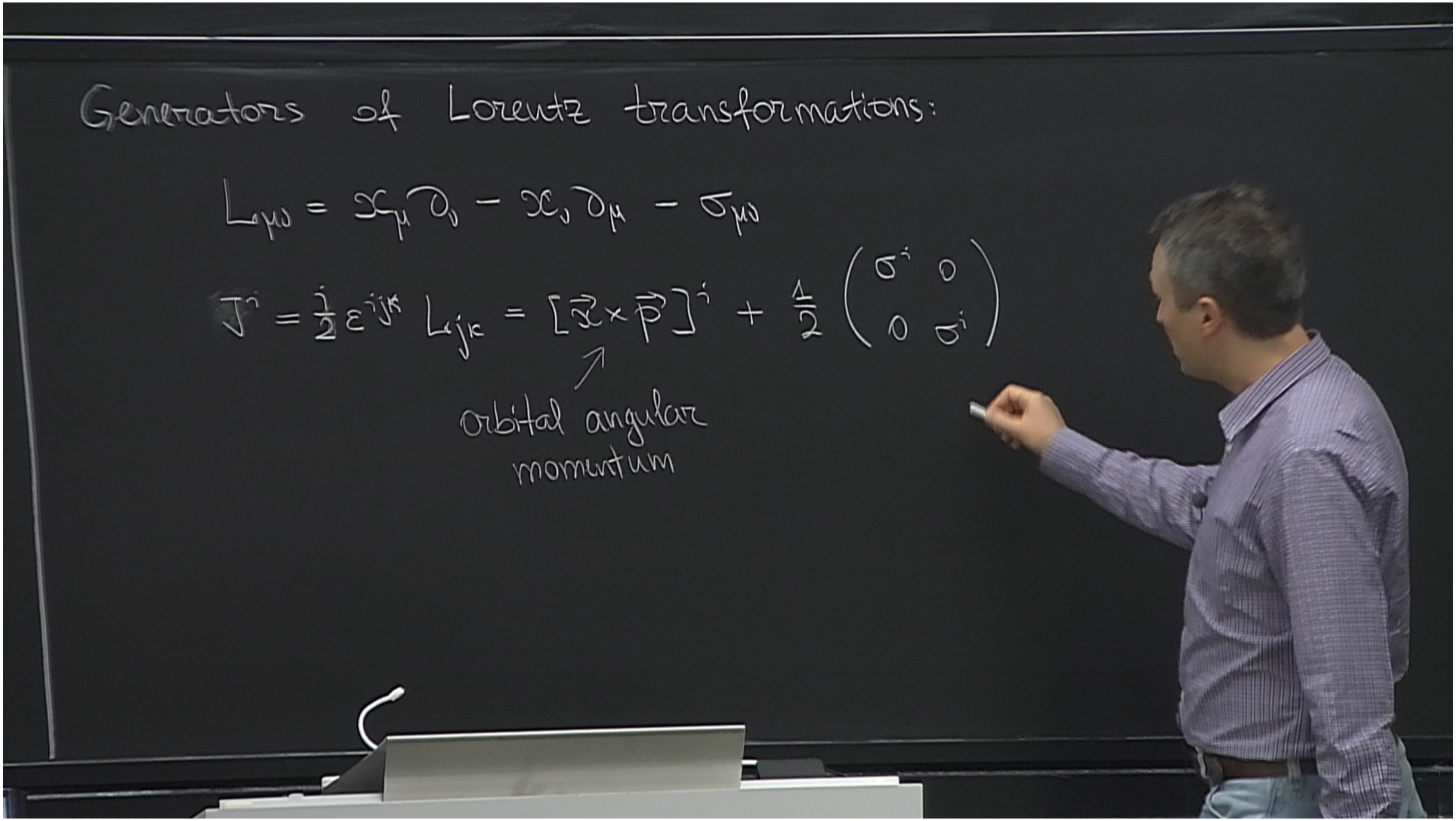
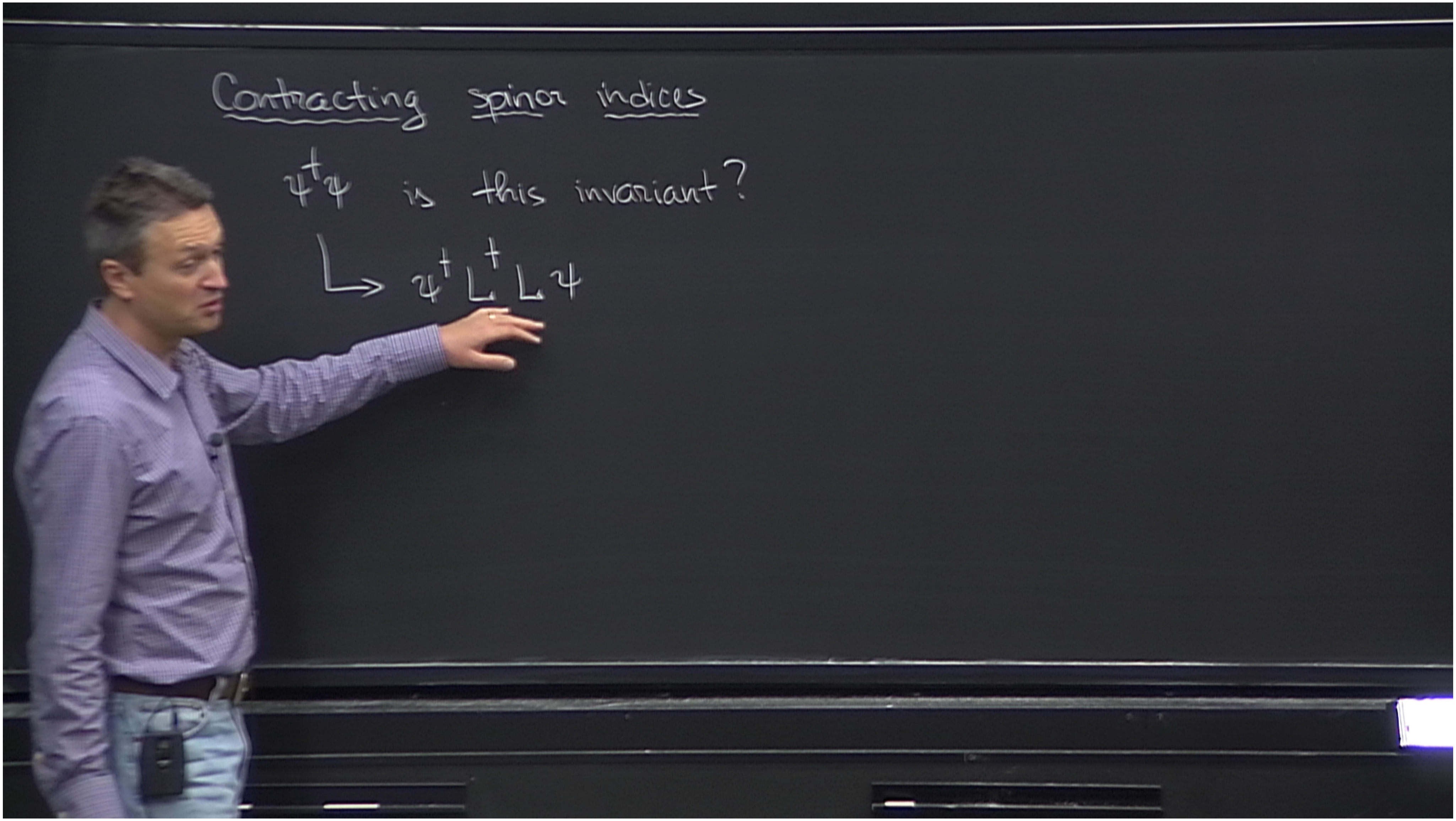

Generators of Lorentz transformations:
$L_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu$


Generators of Lorentz transformations:
$L_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu - \sigma_{\mu\nu}$
$J^i = \frac{i}{2} \varepsilon^{ijk} L_{jk} = [\vec{x} \times \vec{p}]^i + \frac{1}{2} \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}$
orbital angular momentum

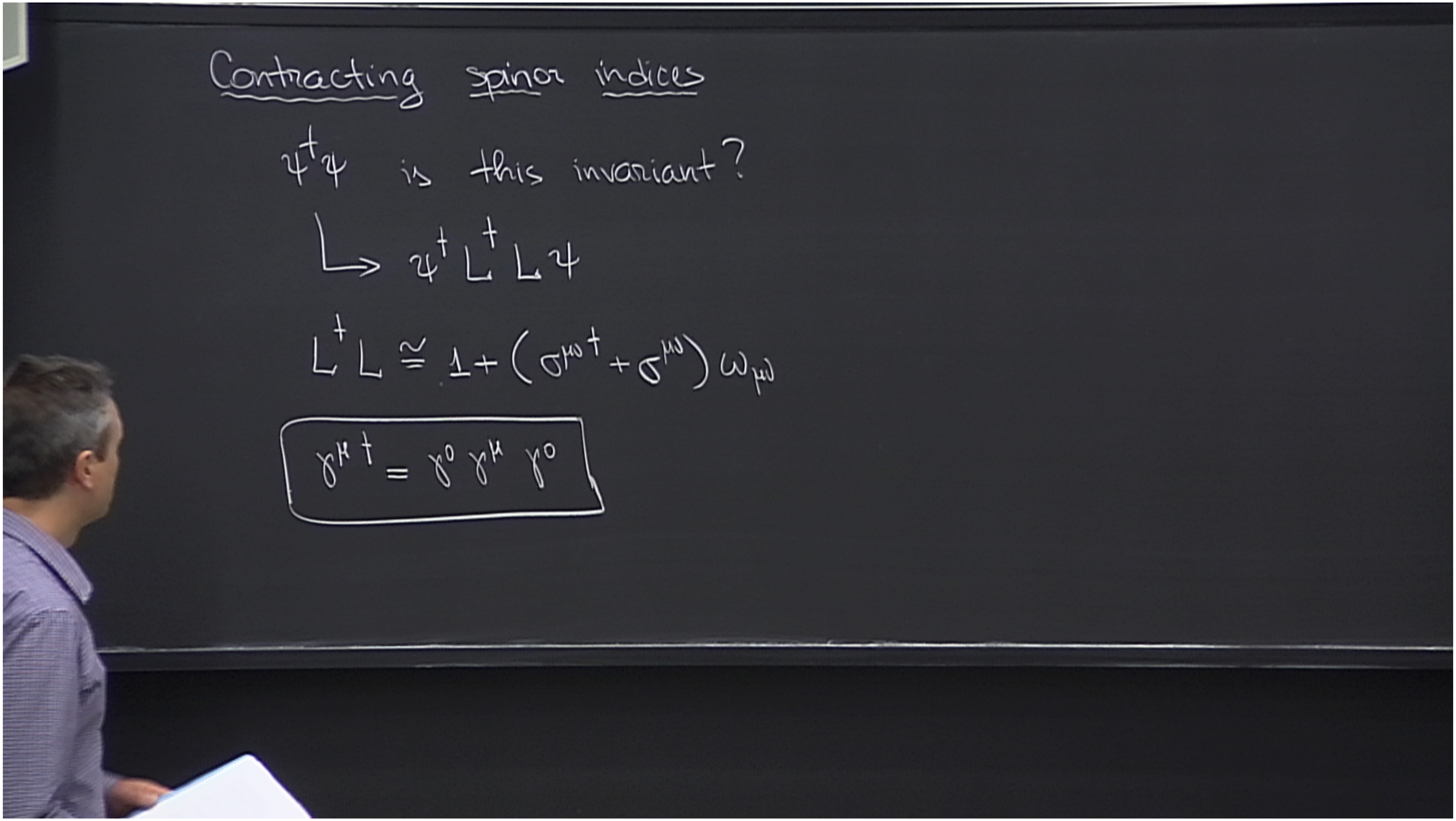

Contracting spinor indices
ψ†ψ is this invariant?
→ ψ† L† L ψ

Contracting spinor indices

$\psi^\dagger \psi$ is this invariant?

$\hookrightarrow \psi^\dagger L^\dagger L \psi$

$$L^\dagger L \simeq 1 + (\sigma^{\mu\nu\dagger} + \sigma^{\mu\nu})\,\omega_{\mu\nu}$$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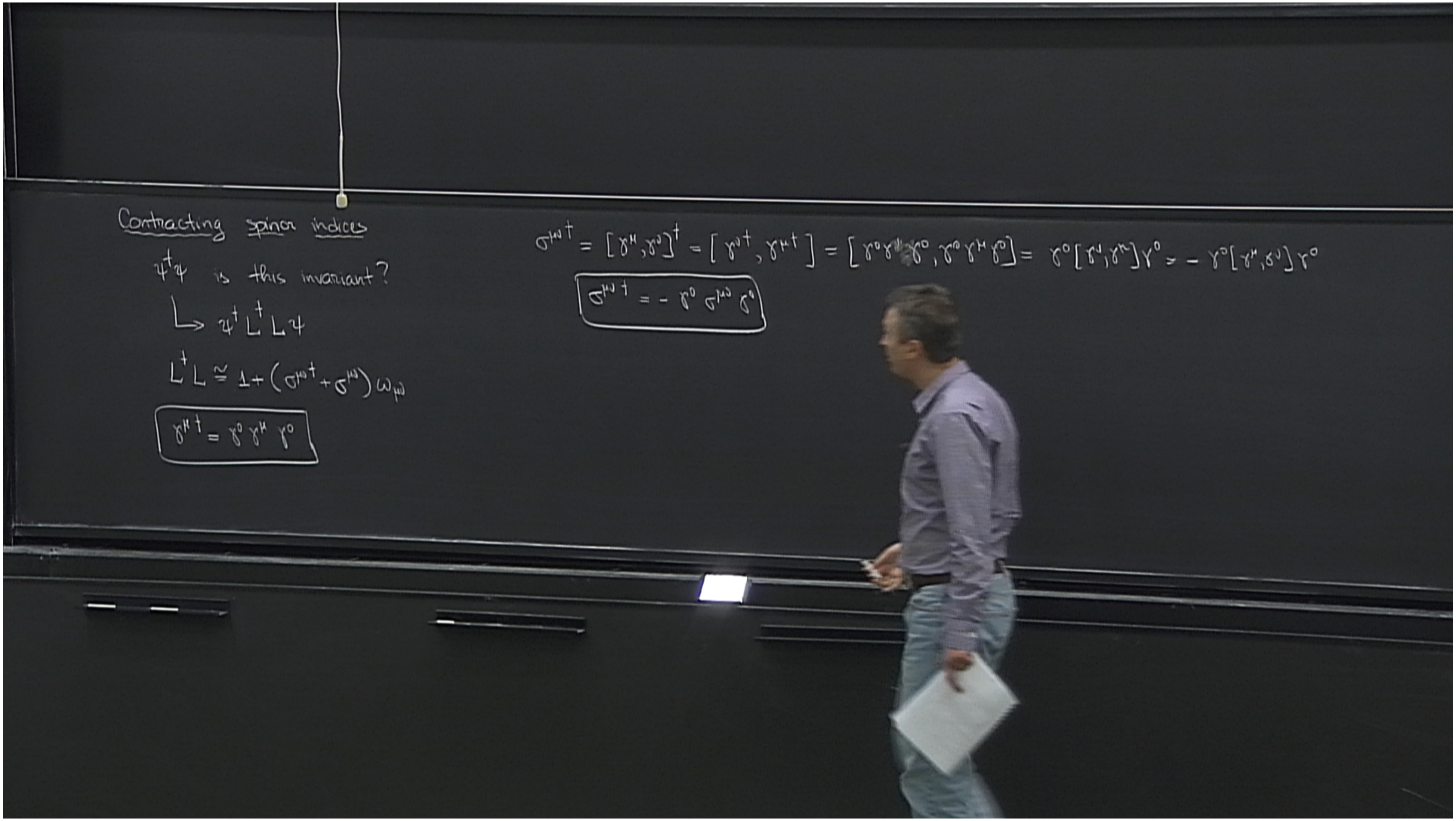
$$\sigma^{\mu\nu\dagger} = [\gamma^\mu,\gamma^\nu]^\dagger = [\gamma^{\nu\dagger},\gamma^{\mu\dagger}] = [\gamma^0\gamma^\nu\gamma^0,\gamma^0\gamma^\mu\gamma^0] = \gamma^0[\gamma^\nu,\gamma^\mu]\gamma^0 = -\gamma^0[\gamma^\mu,\gamma^\nu]\gamma^0$$

$$\boxed{\sigma^{\mu\nu\dagger} = -\gamma^0\sigma^{\mu\nu}\gamma^0}$$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$\boxed{\gamma^{\mu\dagger} = \gamma^0 \gamma^\mu \gamma^0}$
$\sigma^{\mu\nu\dagger} = [\gamma^\mu, \gamma^\nu]^\dagger = [\gamma^{\nu\dagger}, \gamma^{\mu\dagger}] = [\gamma^0\gamma^\nu\gamma^0, \gamma^0\gamma^\mu\gamma^0] = \gamma^0[\gamma^\nu, \gamma^\mu]\gamma^0 = -\gamma^0[\gamma^\mu, \gamma^\nu]\gamma^0$
$\boxed{\sigma^{\mu\nu\dagger} = -\gamma^0 \sigma^{\mu\nu} \gamma^0}$

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$$\boxed{\sigma^{\mu\nu\dagger} = -\gamma^0 \sigma^{\mu\nu} \gamma^0}$$

Dirac conjugation:

$$\boxed{\bar{\psi} = \psi^\dagger \gamma^0}$$

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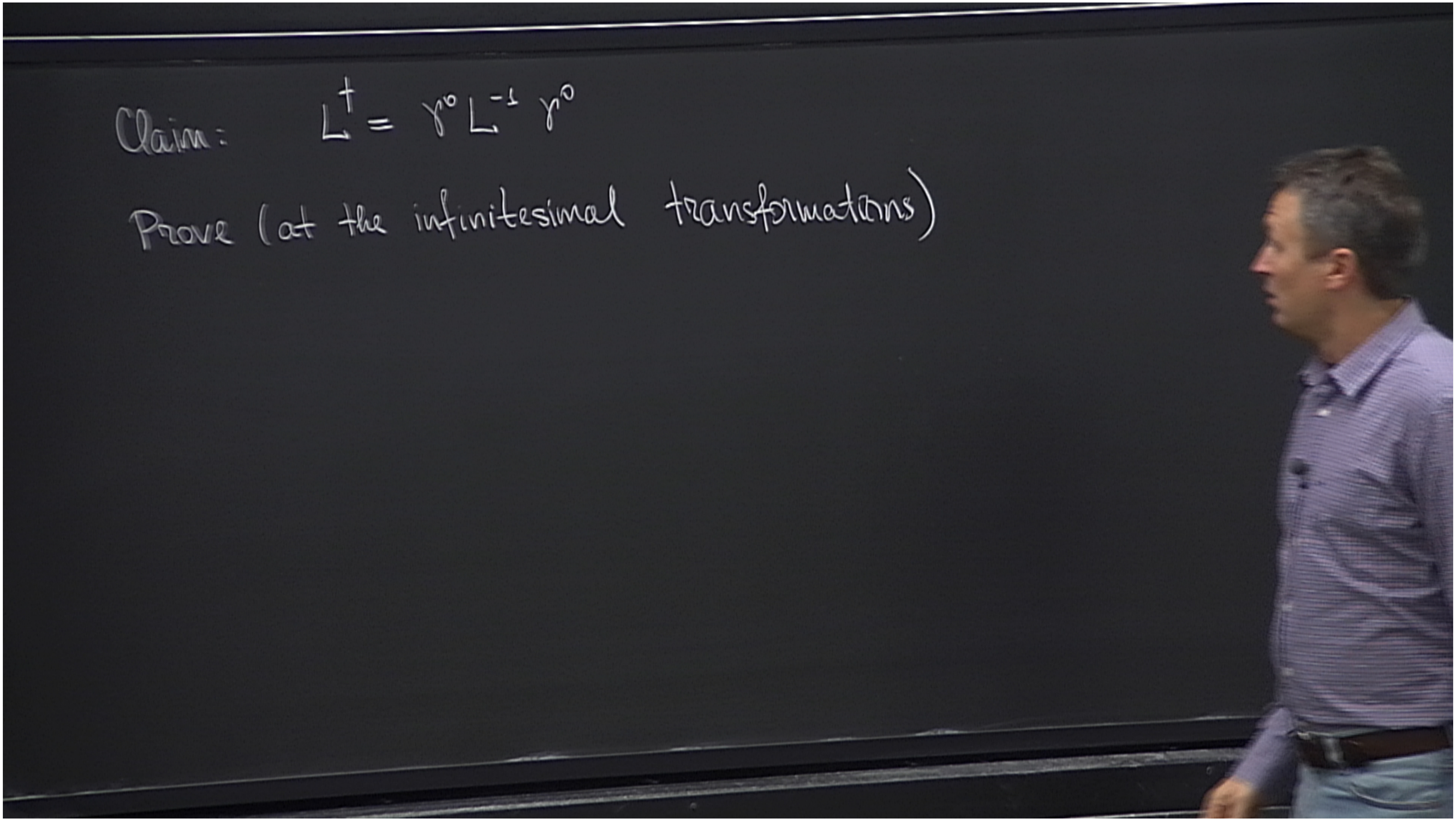
$$\bar{\psi}\psi = \psi^\dagger\gamma^0\psi \longrightarrow \psi^\dagger L^\dagger\gamma^0 L\psi$$


Claim: $L^{\dagger} = \gamma^0 L^{-1} \gamma^0$
Prove (at the infinitesimal transformations)

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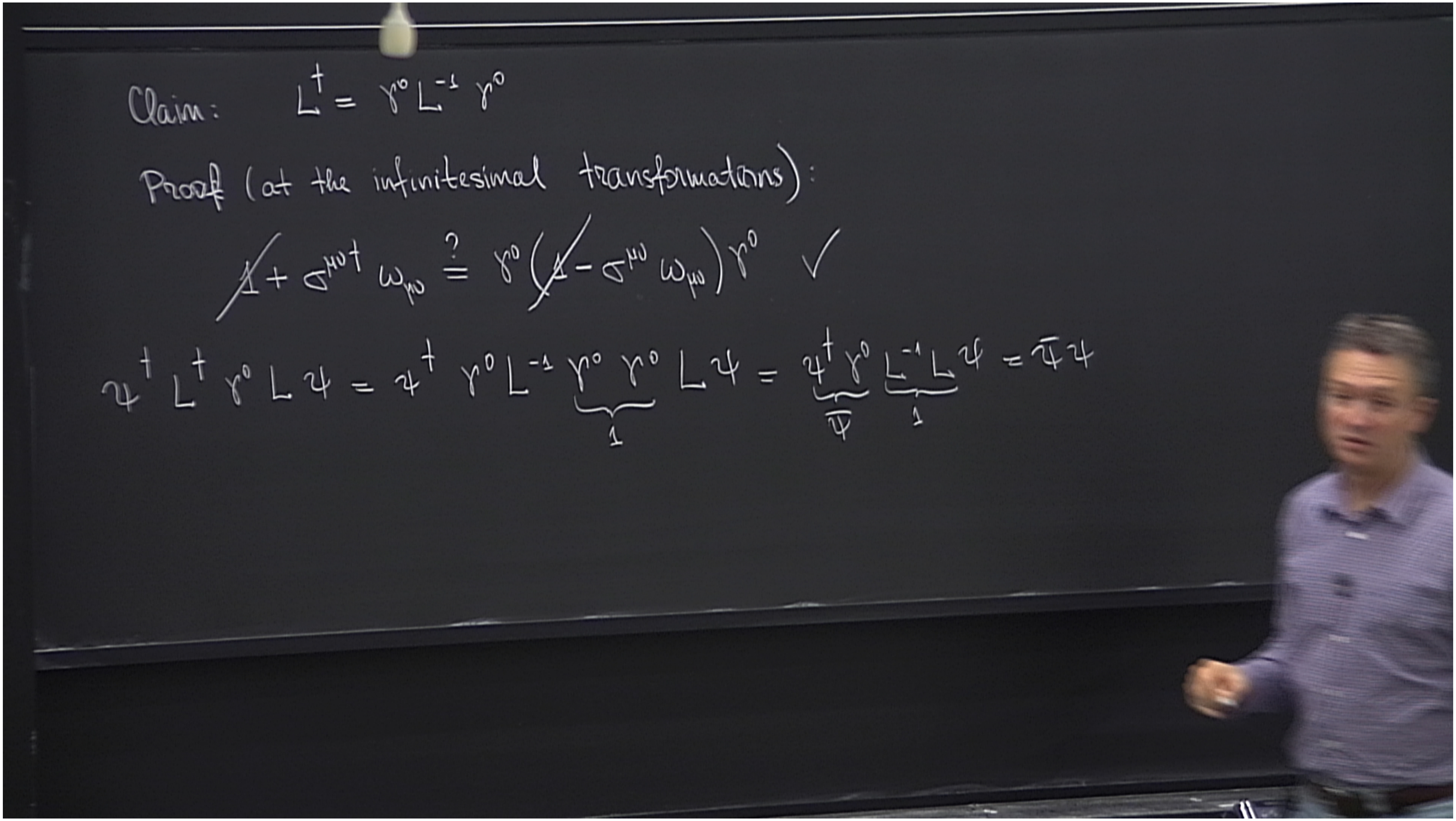
Proof (at the infinitesimal transformations):

$$1 + \sigma^{\mu\nu\dagger} \omega_{\mu\nu} \stackrel{?}{=} \gamma^0 \left(1 - \sigma^{\mu\nu} \omega_{\mu\nu}\right) \gamma^0$$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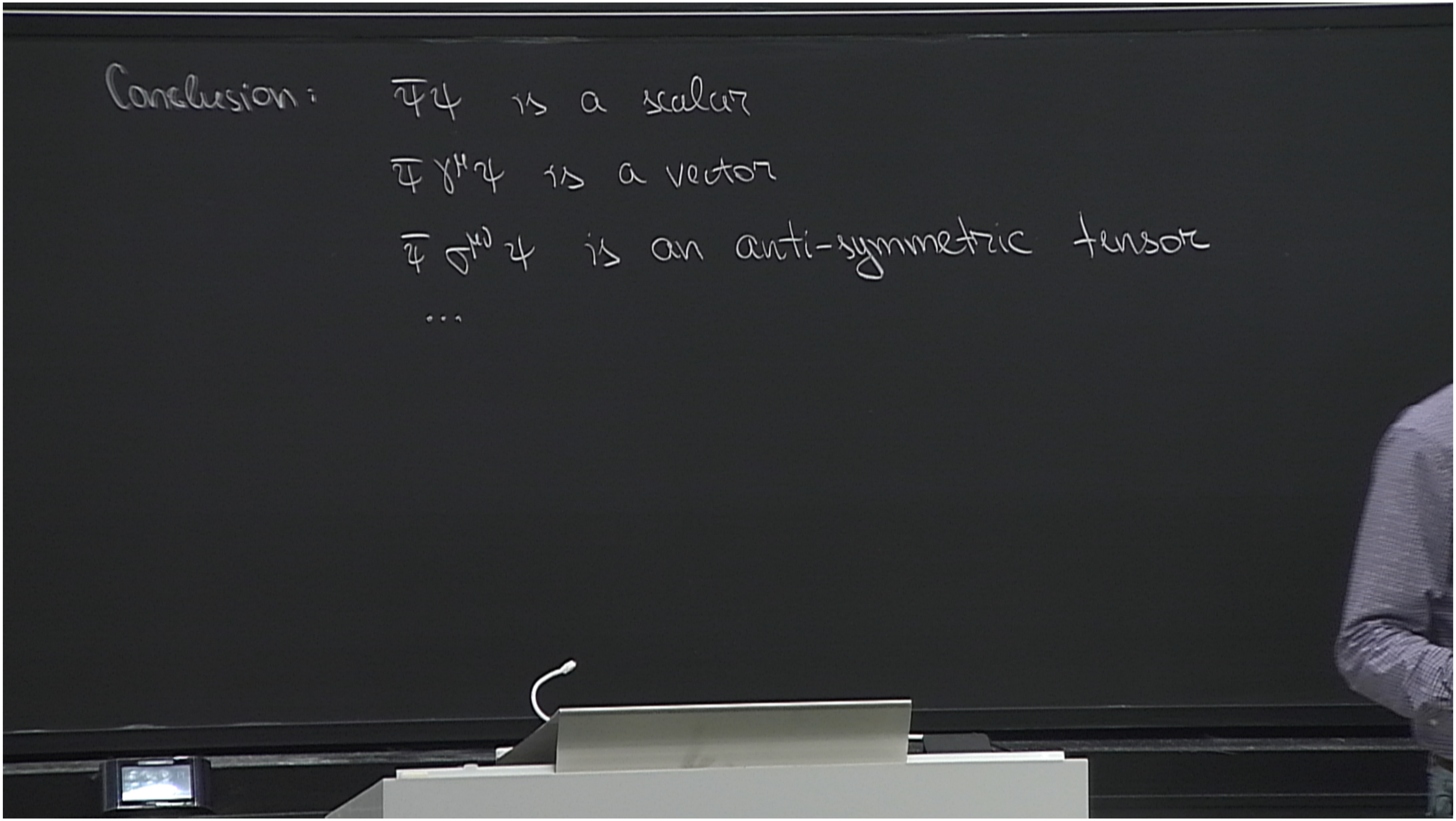
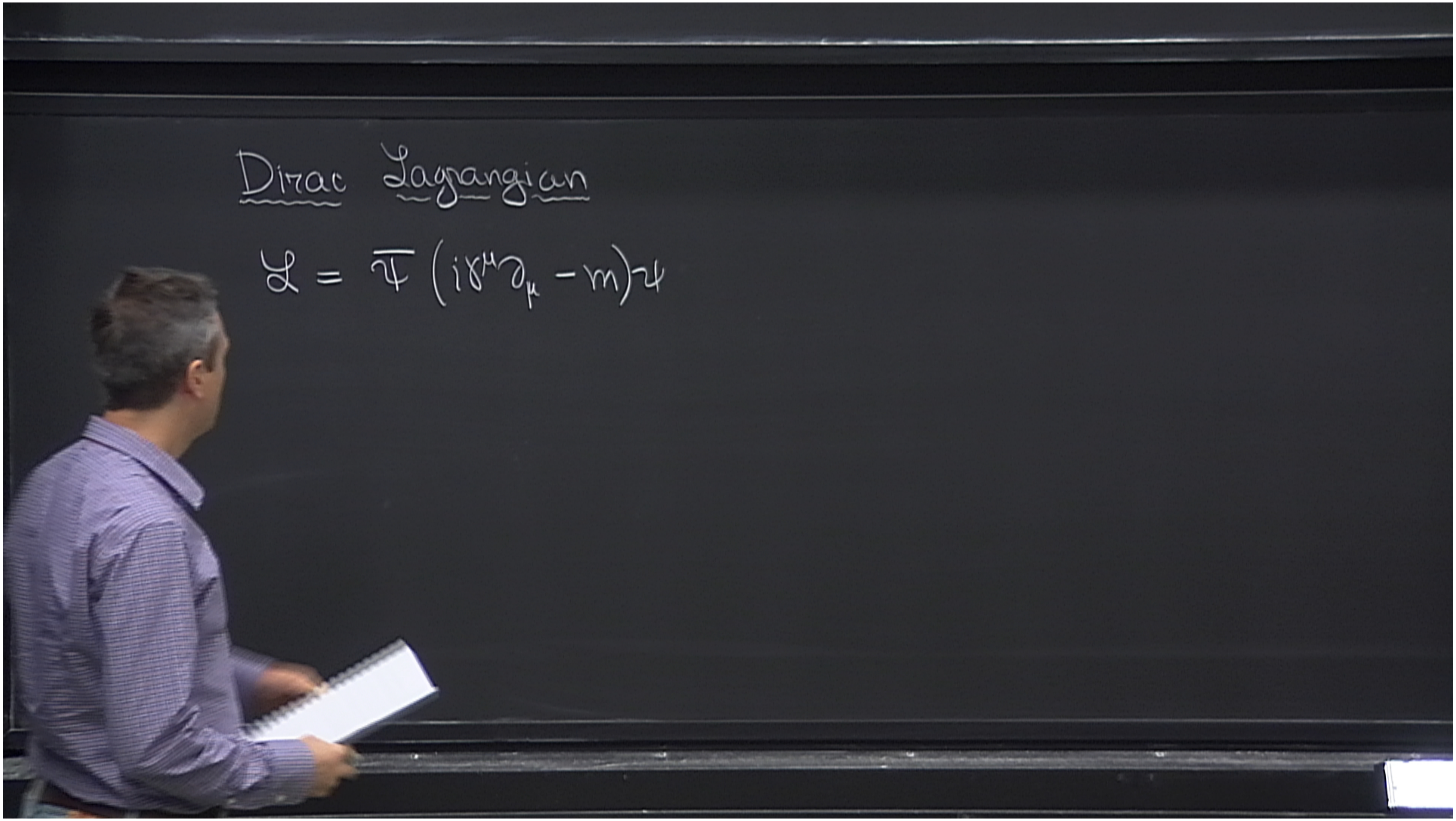
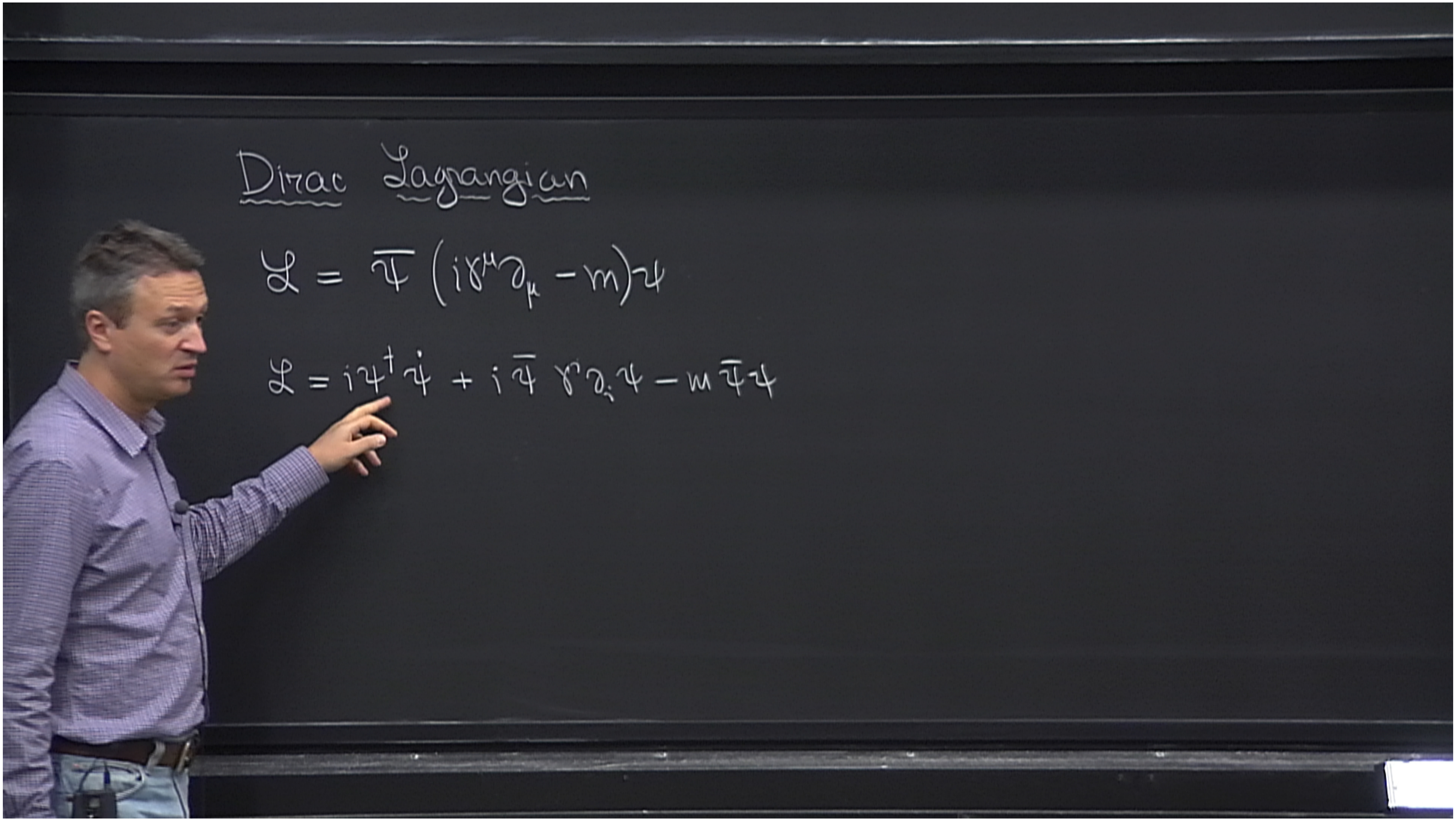
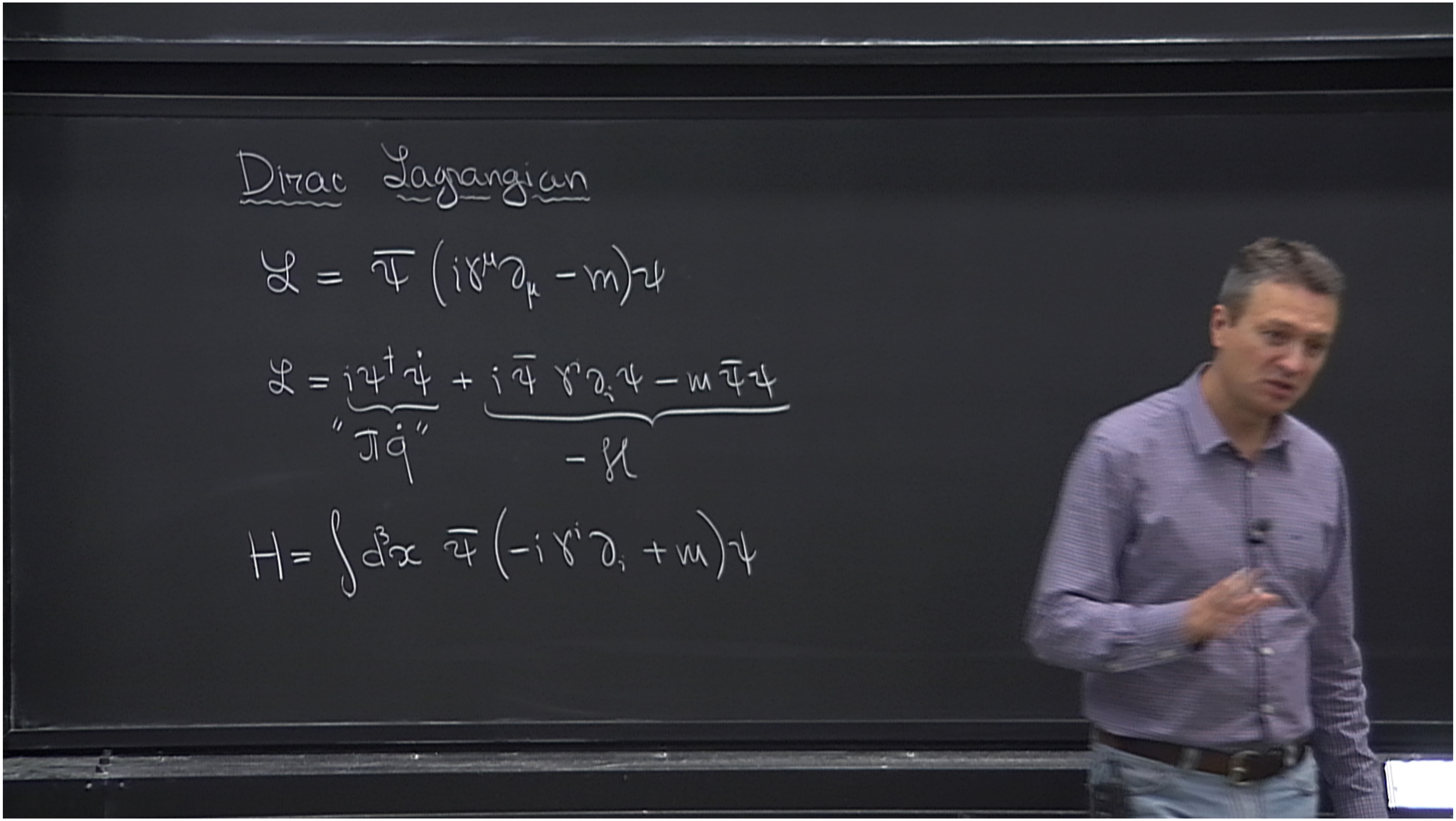
$\cancel{1} + \sigma^{\mu\nu\dagger}\omega_{\mu\nu} \stackrel{?}{=} \gamma^0 (\cancel{1} - \sigma^{\mu\nu}\omega_{\mu\nu}) \gamma^0$ ✓
$\psi^\dagger L^\dagger \gamma^0 L \psi = \psi^\dagger \gamma^0 L^{-1} \underbrace{\gamma^0 \gamma^0}_{1} L \psi = \underbrace{\psi^\dagger \gamma^0}_{\bar\psi} \underbrace{L^{-1} L}_{1} \psi = \bar\psi \psi$



Conclusion: $\bar{\Psi}\Psi$ is a scalar
$\bar{\Psi}\gamma^{\mu}\Psi$ is a vector
$\bar{\Psi}\sigma^{\mu\nu}\Psi$ is an anti-symmetric tensor
...


Dirac Lagrangian
$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$


Dirac Lagrangian
$\mathcal{L} = \bar{\psi}\,(i\gamma^\mu \partial_\mu - m)\psi$
$\mathcal{L} = i\psi^\dagger \dot{\psi} + i\,\bar{\psi}\,\gamma^i \partial_i \psi - m\,\bar{\psi}\psi$


Dirac Lagrangian
$\mathcal{L} = \bar{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi$
$\mathcal{L} = \underbrace{i\psi^\dagger \dot{\psi}}_{\text{"}\pi \dot{q}\text{"}} + \underbrace{i\bar{\psi}\gamma^i \partial_i \psi - m\bar{\psi}\psi}_{-\mathcal{H}}$
$H = \int d^3x \, \bar{\psi}\left(-i\gamma^i \partial_i + m\right)\psi$

Dirac Lagrangian

$$\mathcal{L} = \bar{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi$$

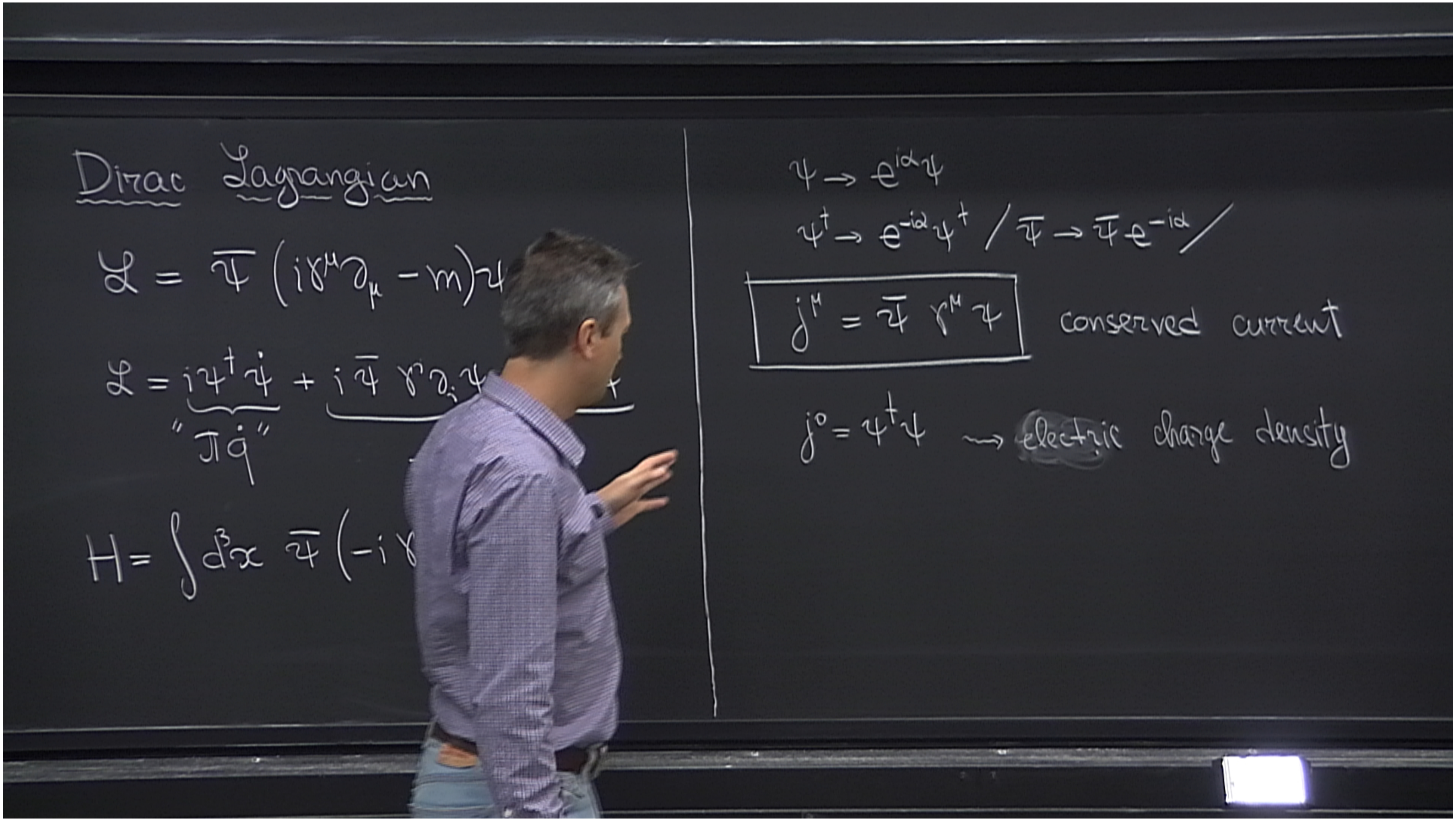
$$\mathcal{L} = \underbrace{i\psi^\dagger \dot{\psi}}_{``\pi \dot{q}"} + \underbrace{i\bar{\psi}\gamma^i \partial_i \psi - m\bar{\psi}\psi}_{-\mathcal{H}}$$

$$H = \int d^3x\, \bar{\psi}\left(-i\gamma^i \partial_i + m\right)\psi$$

$$\psi \to e^{i\alpha}\psi$$

$$\psi^\dagger \to e^{-i\alpha}\psi^\dagger \;/\; \bar{\psi} \to \bar{\psi} e^{-i\alpha} \;/$$

$$\boxed{j^\mu = \bar{\psi}\gamma^\mu \psi}$$ conserved current


Dirac Lagrangian
$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$
$\mathcal{L} = \underbrace{i\psi^\dagger\dot{\psi}}_{\text{"}\pi\dot{q}\text{"}} + i\bar{\psi}\gamma^i\partial_i\psi$
$H = \int d^3x\, \bar{\psi}(-i\gamma$
$\psi \to e^{i\alpha}\psi$
$\psi^\dagger \to e^{-i\alpha}\psi^\dagger$ / $\bar{\psi} \to \bar{\psi}e^{-i\alpha}$ /
$\boxed{j^\mu = \bar{\psi}\gamma^\mu\psi}$ conserved current
$j^0 = \psi^\dagger\psi \rightsquigarrow$ electric charge density