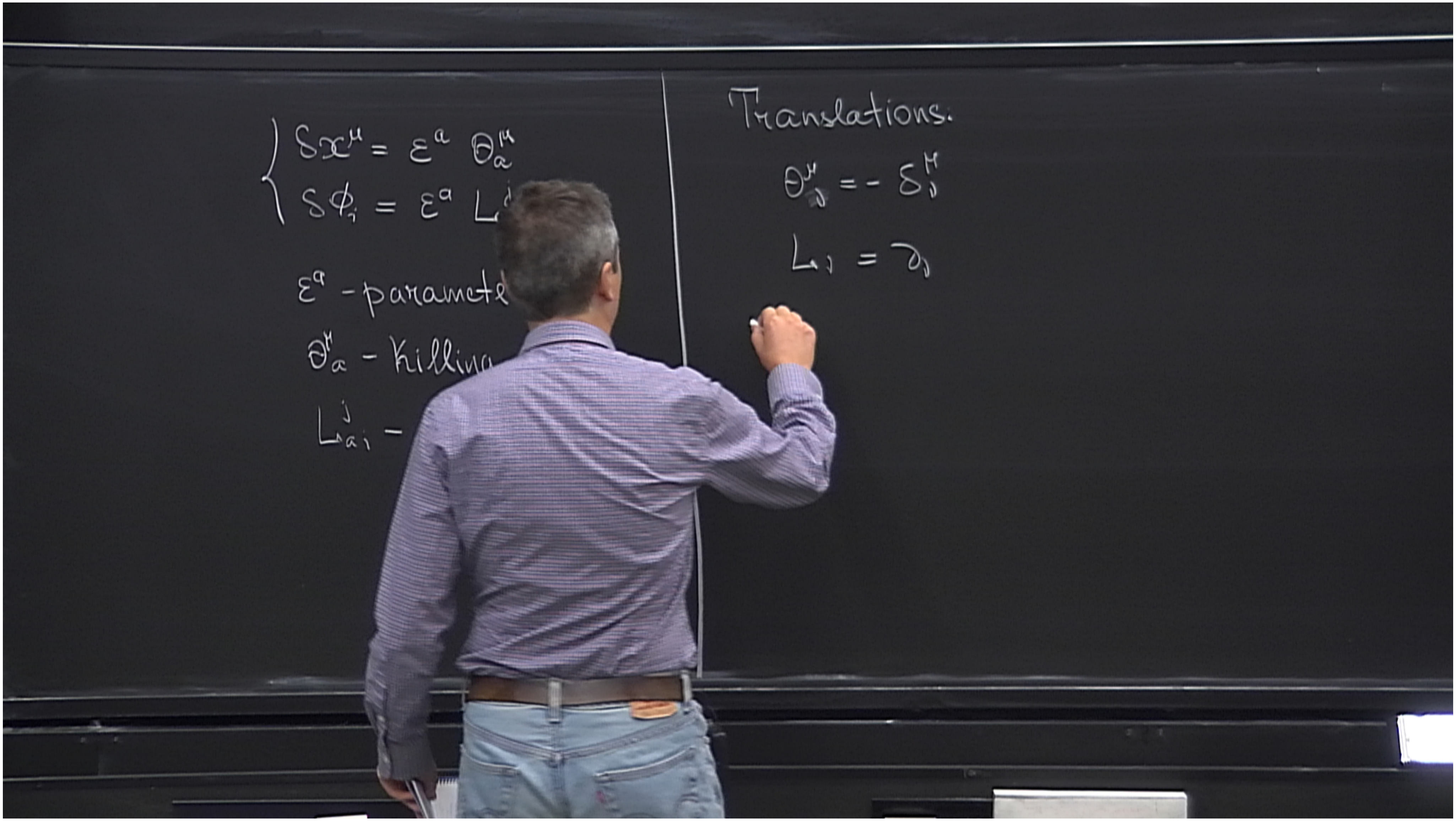
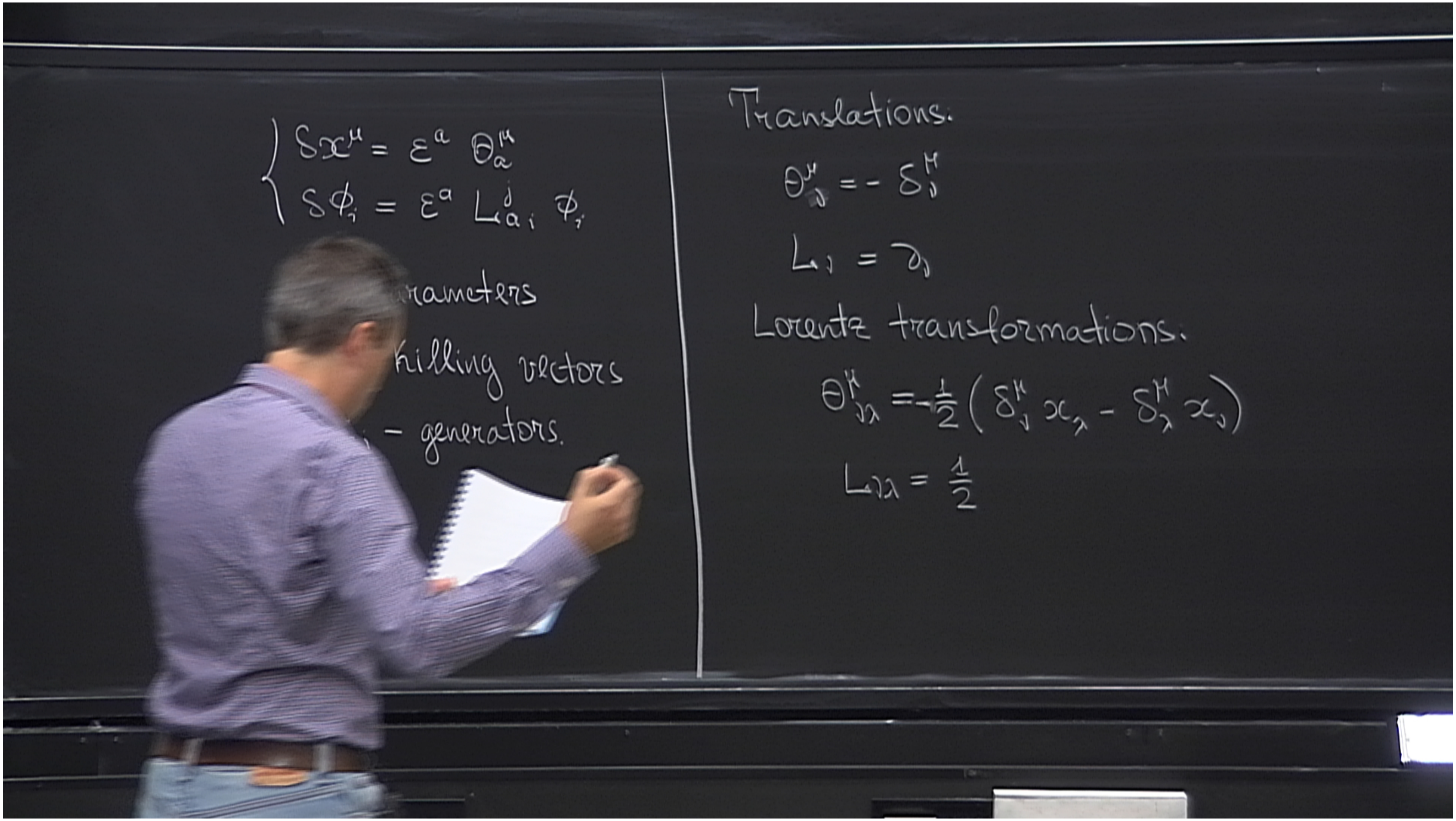
Title: Quantum Field Theory I - Lecture 4

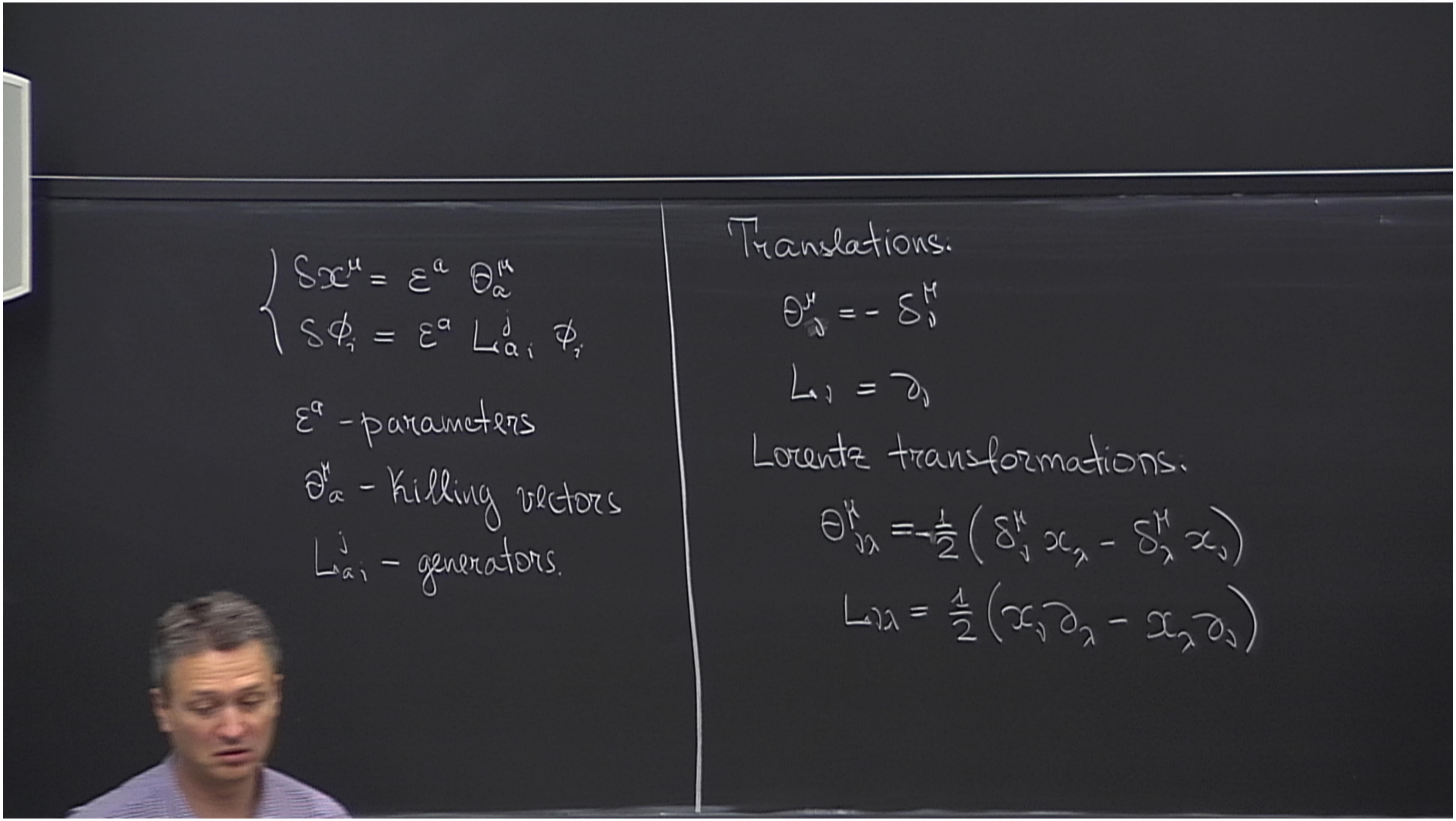
Date: Oct 06, 2011 09:00 AM

URL: http://pirsa.org/11100009

Abstract:


$\delta x^\mu = \varepsilon^a \Theta^\mu_a$
Translations.
$\Theta^\mu_\nu = - \delta^\mu_\nu$
$L_\nu = \partial_\nu$


$\delta x^\mu = \varepsilon^a \Theta_a^\mu$
$\delta\phi_i = \varepsilon^a L_{a\,i}^{\ j}\, \phi_j$
rameters
killing vectors
– generators.
Translations:
$\Theta^\mu_\nu = -\delta^\mu_\nu$
$L_\nu = \partial_\nu$
Lorentz transformations:
$\Theta^\mu_{\nu\lambda} = -\frac{1}{2}\left(\delta^\mu_\nu x_\lambda - \delta^\mu_\lambda x_\nu\right)$
$L_{\nu\lambda} = \frac{1}{2}$


$\delta x^\mu = \varepsilon^a \,\Theta_a^\mu$
$\delta \phi_i = \varepsilon^a L_{a\,i}^{\,j}\, \phi_j$
ε^a - parameters
Θ_a^μ - Killing vectors
$L_{a\,i}^{\,j}$ - generators.
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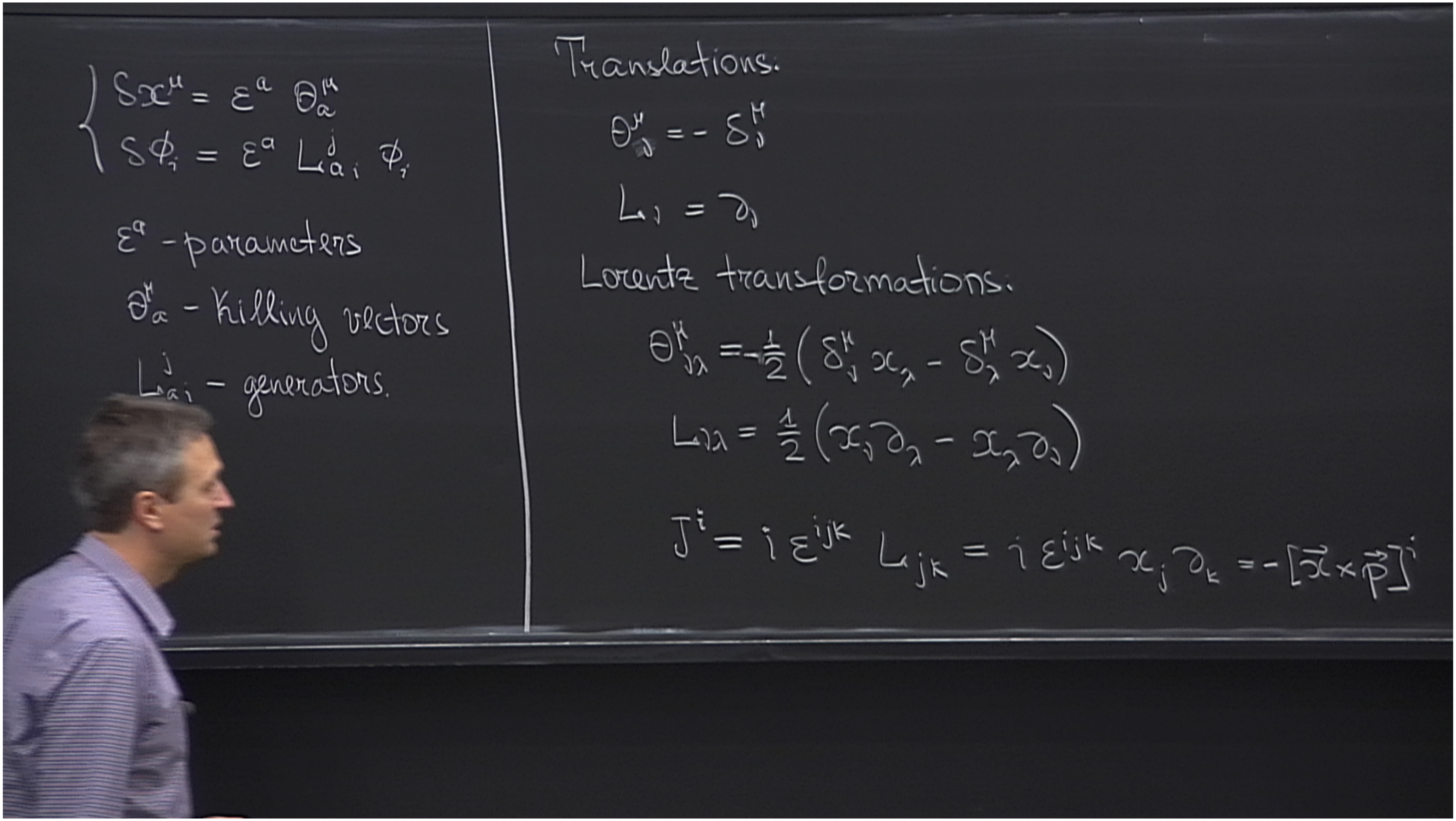
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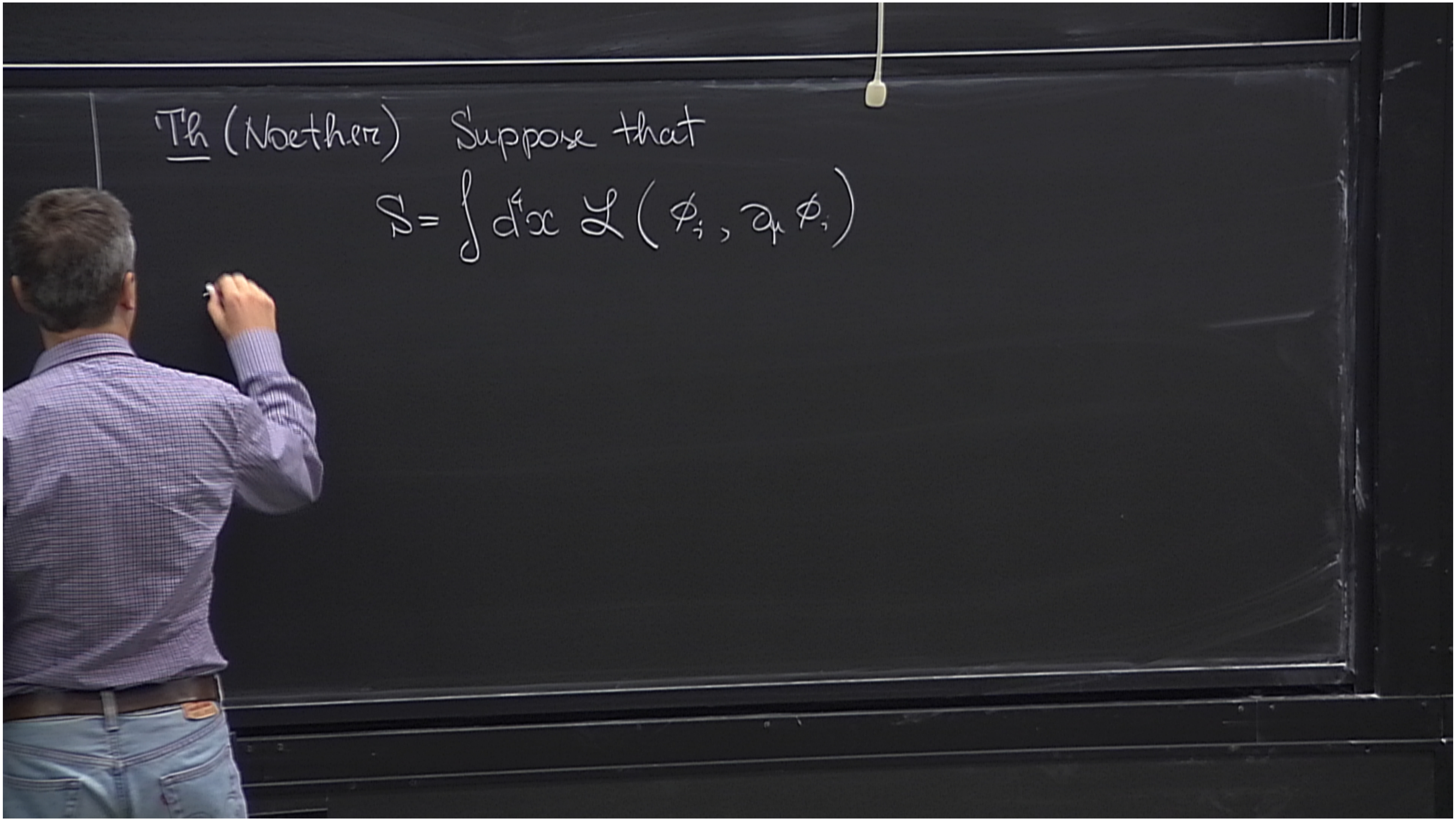
$J^i = i\, \varepsilon^{ijk} L_{jk} = i\, \varepsilon^{ijk} x_j \partial_k = -[\vec{x} \times \vec{p}]^i$


Th (Noether) Suppose that
$S = \int d^4x \, \mathcal{L}(\phi_i, \partial_\mu \phi_i)$

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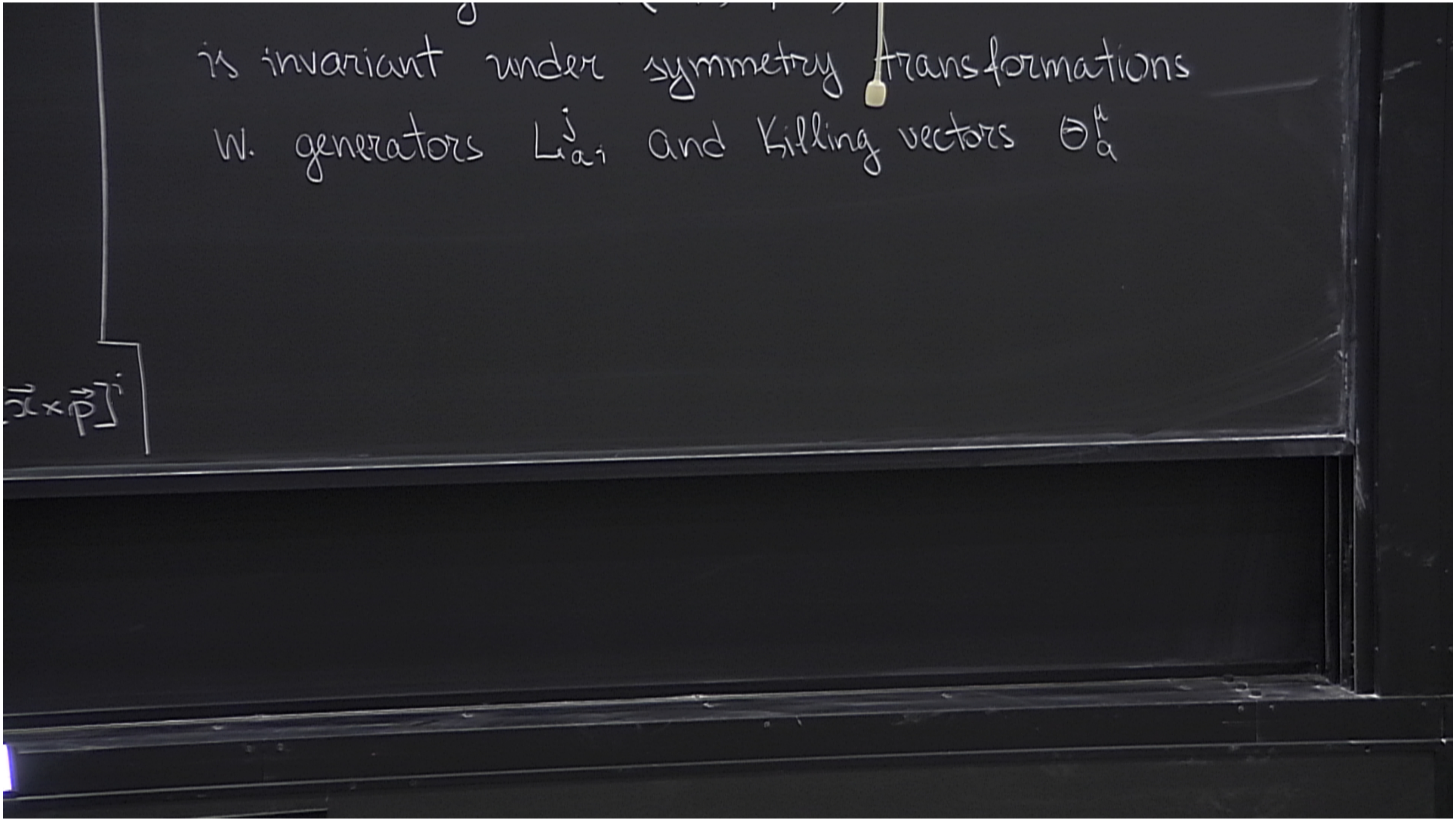
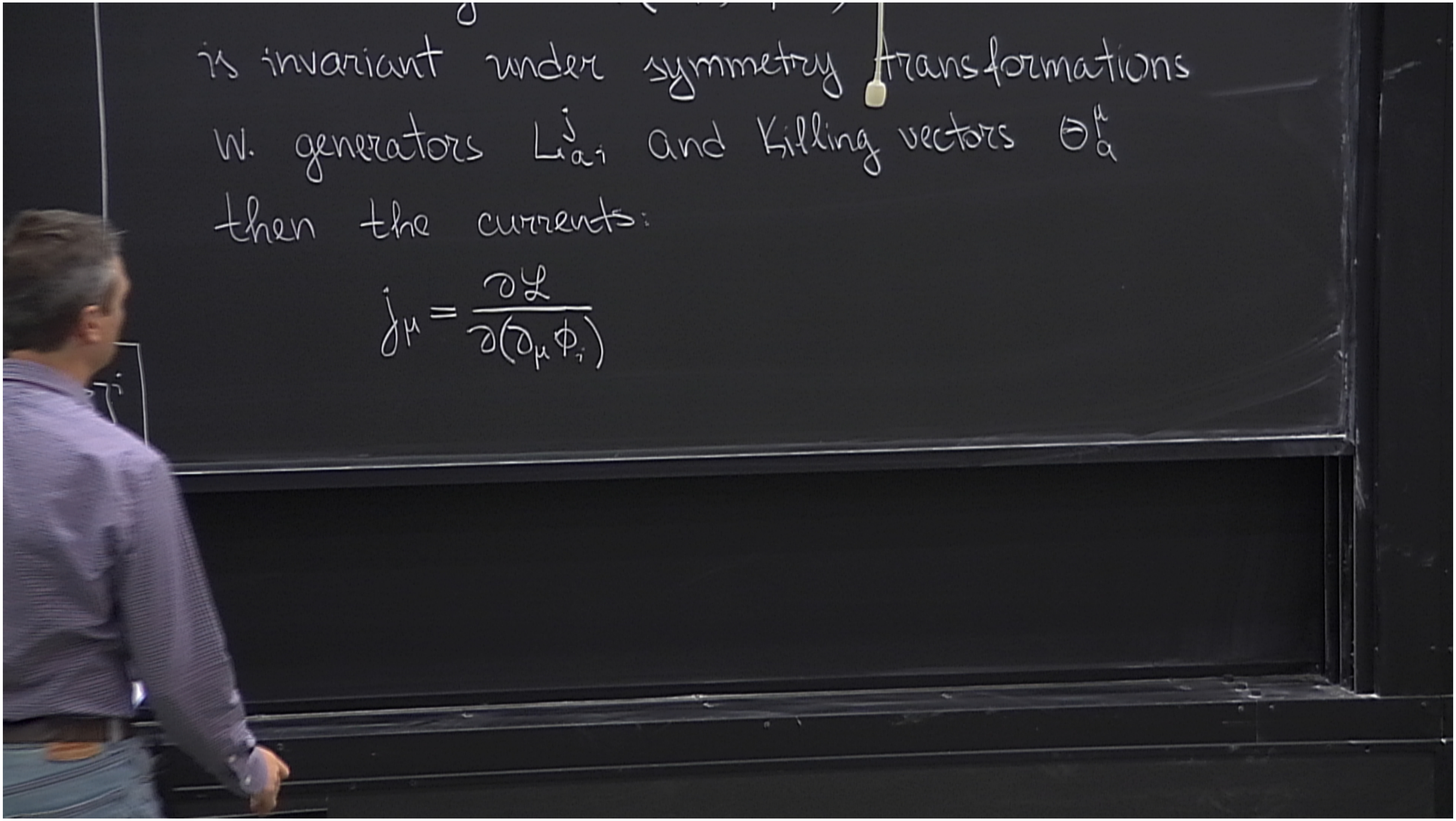
is invariant under symmetry transformations w. generators L_{ai}^{j} and Killing vectors Θ_a^μ

$[\vec{x} \times \vec{p}]^i$

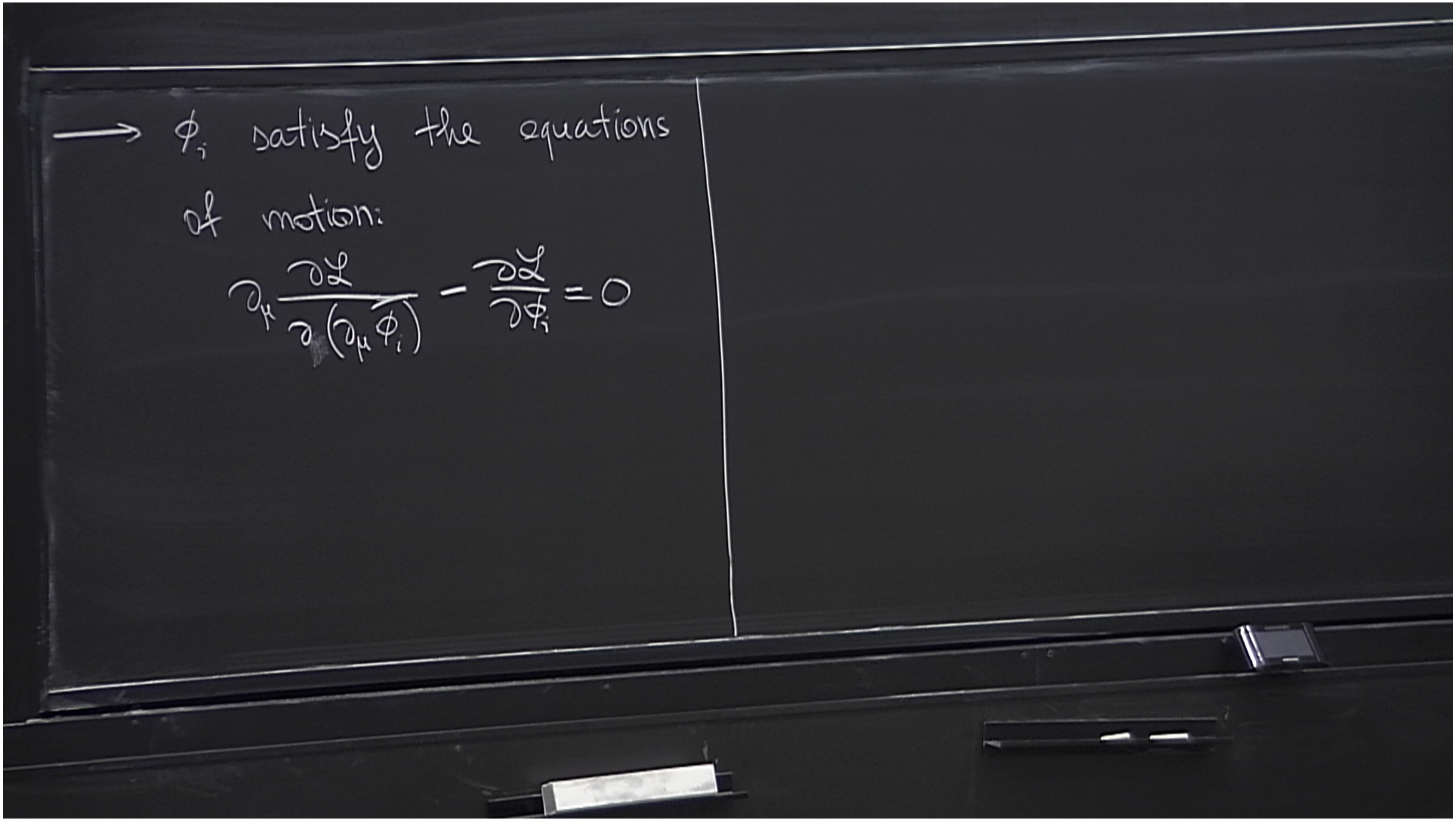

is invariant under symmetry transformations
w. generators $L^{j}_{a\,i}$ and Killing vectors Θ^{μ}_{a}
$[\vec{x}\times\vec{p}]^{i}$


is invariant under symmetry transformations
w. generators L^{j}_{ai} and Killing vectors Θ^{μ}_{a}
then the currents:
$j_\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}$

is invariant under symmetry transformations w. generators $L_{a\,i}^{\;j}$ and Killing vectors Θ_a^μ then the currents:

$$j_a^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} L^i_{\;aj}\,\phi^j + \Theta_a^\mu \mathcal{L}$$

$[\vec{x} \times \vec{p}]^i$


→ ϕ_i satisfy the equations
of motion:
$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} - \frac{\partial \mathcal{L}}{\partial \phi_i} = 0$

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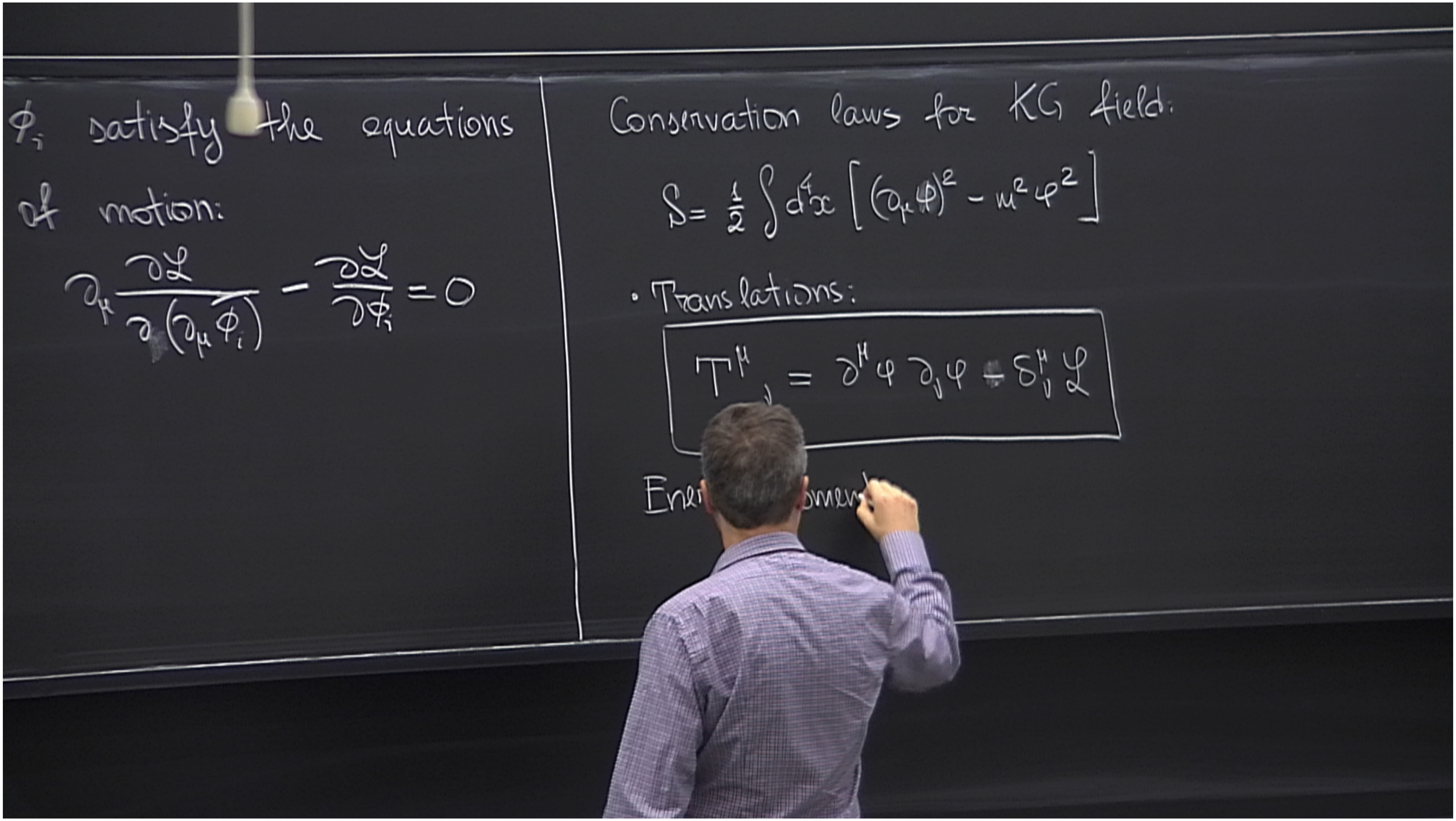
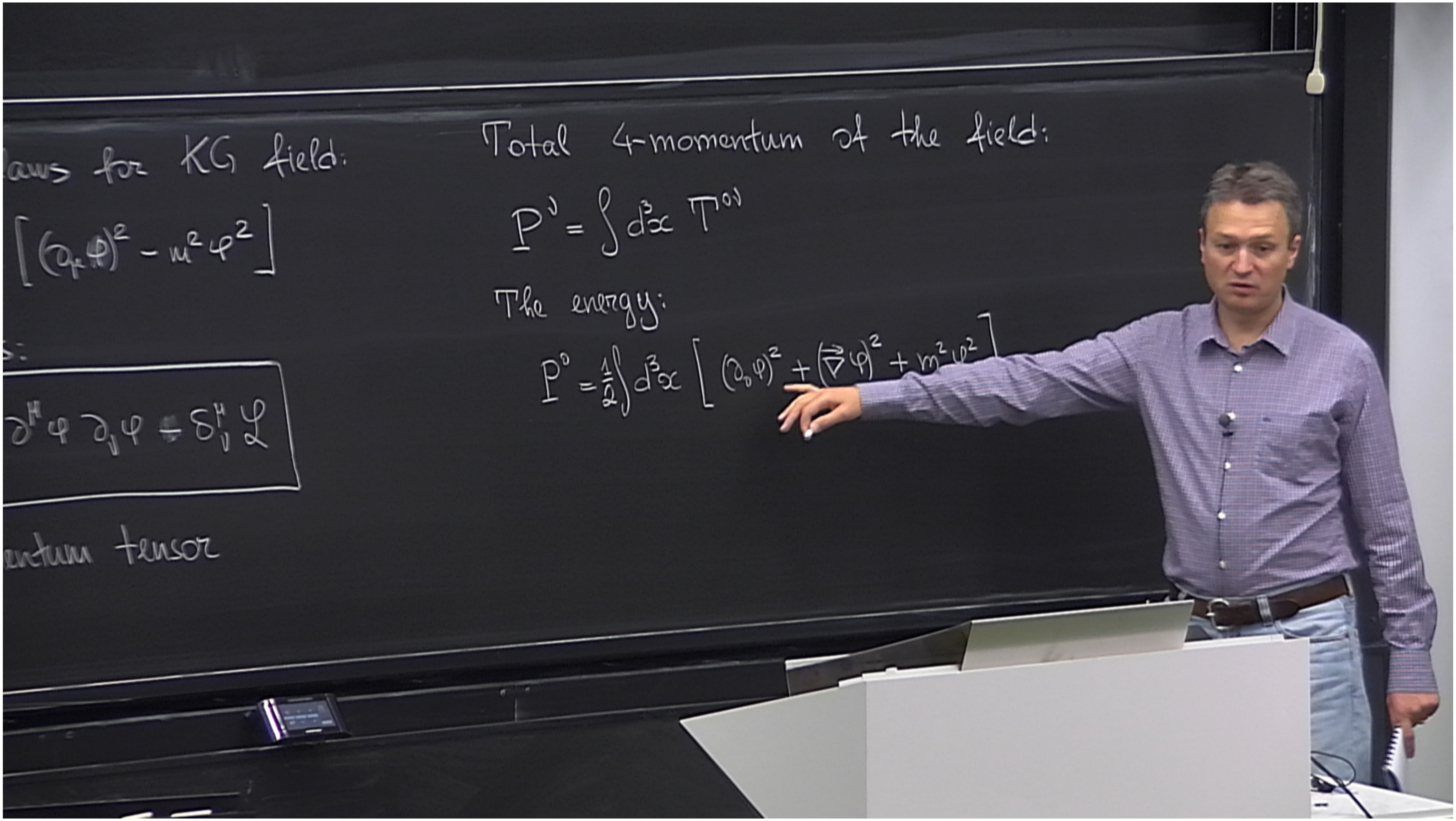
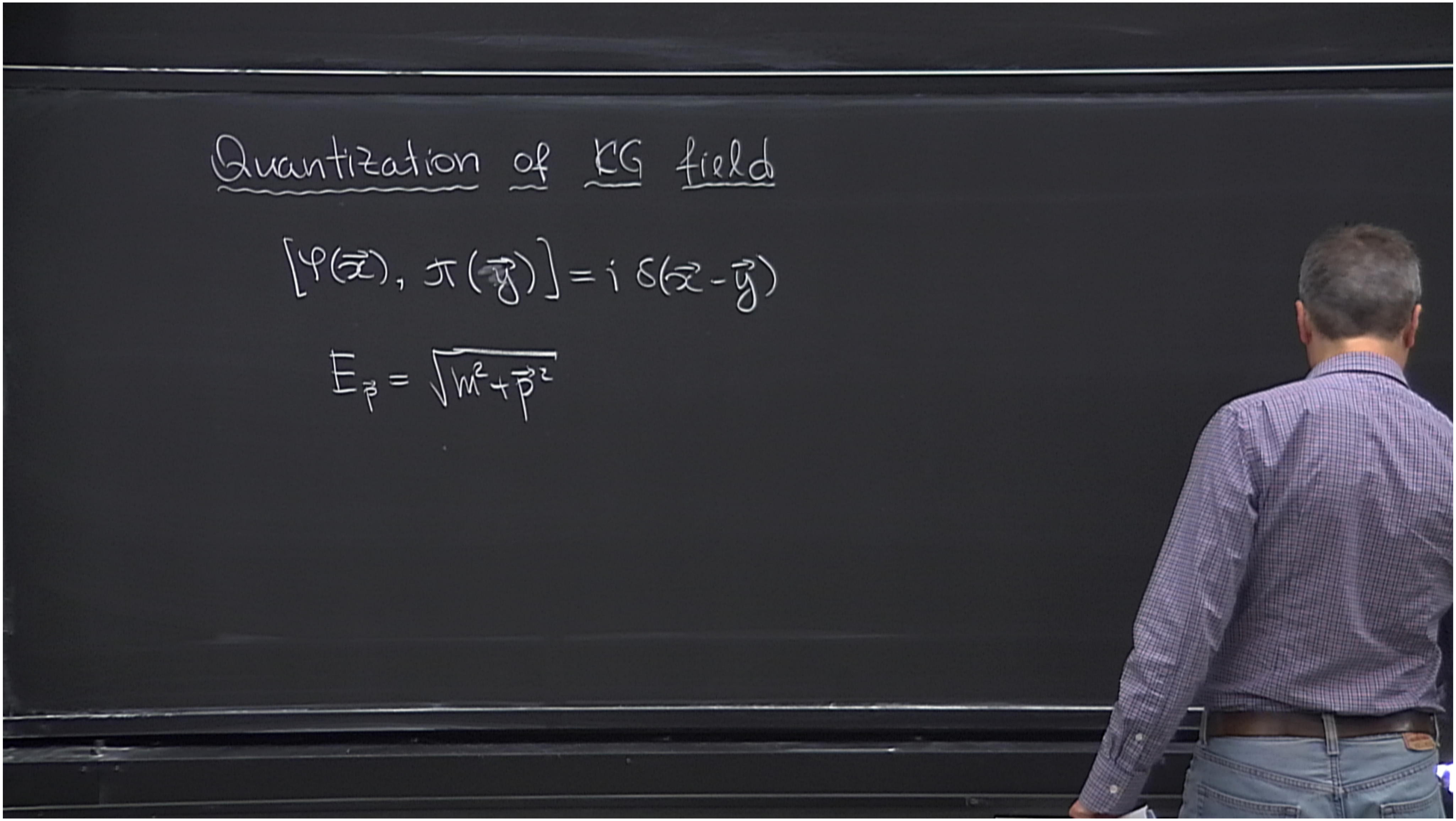
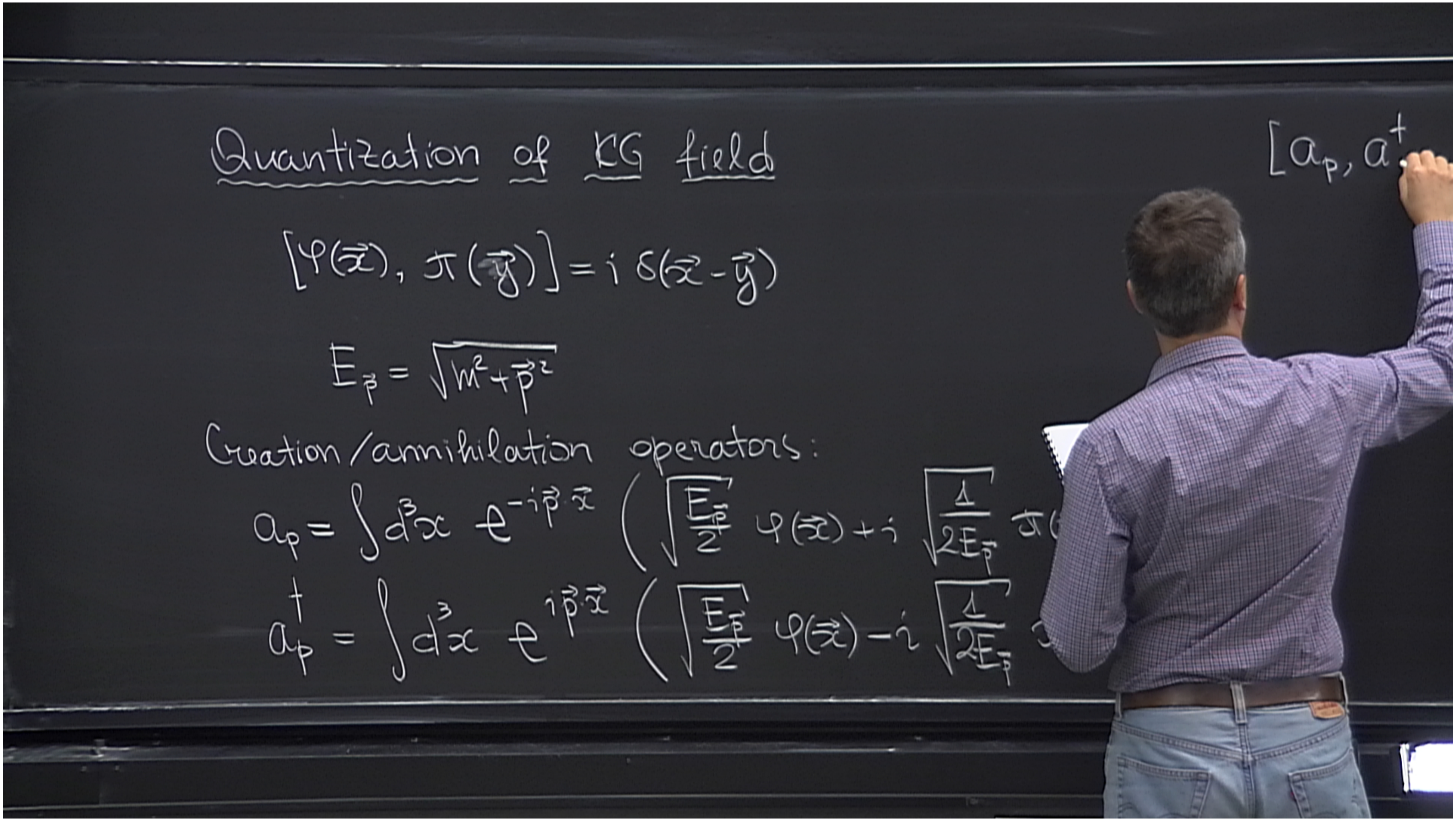
Conservation laws for KG field:

$$S = \frac{1}{2}\int d^4x \left[(\partial_\mu \phi)^2 - m^2 \phi^2 \right]$$


φ_i satisfy the equations
of motion:
∂_μ ∂L/∂(∂_μ φ_i) − ∂L/∂φ_i = 0
Conservation laws for KG field:
S = 1/2 ∫ d^4x [(∂_μ φ)^2 − m^2 φ^2]
• Translations:
T^μ_ν = ∂^μ φ ∂_ν φ − δ^μ_ν L


laws for KG field:
Total 4-momentum of the field:
$P^\nu = \int d^3x \, T^{0\nu}$
The energy:
$P^0 = \frac{1}{2}\int d^3x \left[(\partial_0\varphi)^2 + (\vec\nabla\varphi)^2 + m^2\varphi^2 \right]$
entum tensor


Quantization of KG field
$[\varphi(\vec{x}), \pi(\vec{y})] = i\,\delta(\vec{x}-\vec{y})$
$E_{\vec{p}} = \sqrt{m^2+\vec{p}^{\,2}}$



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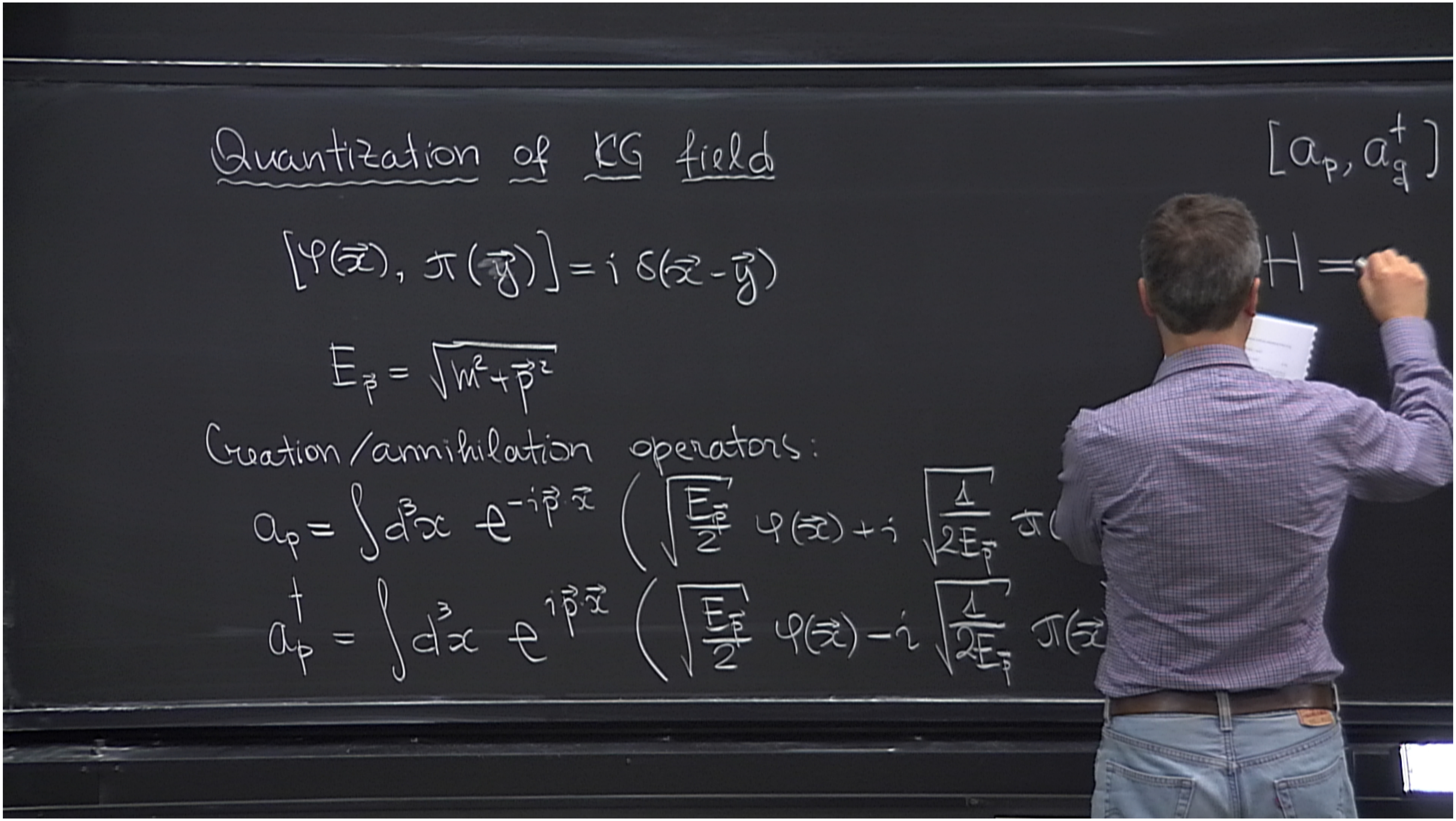
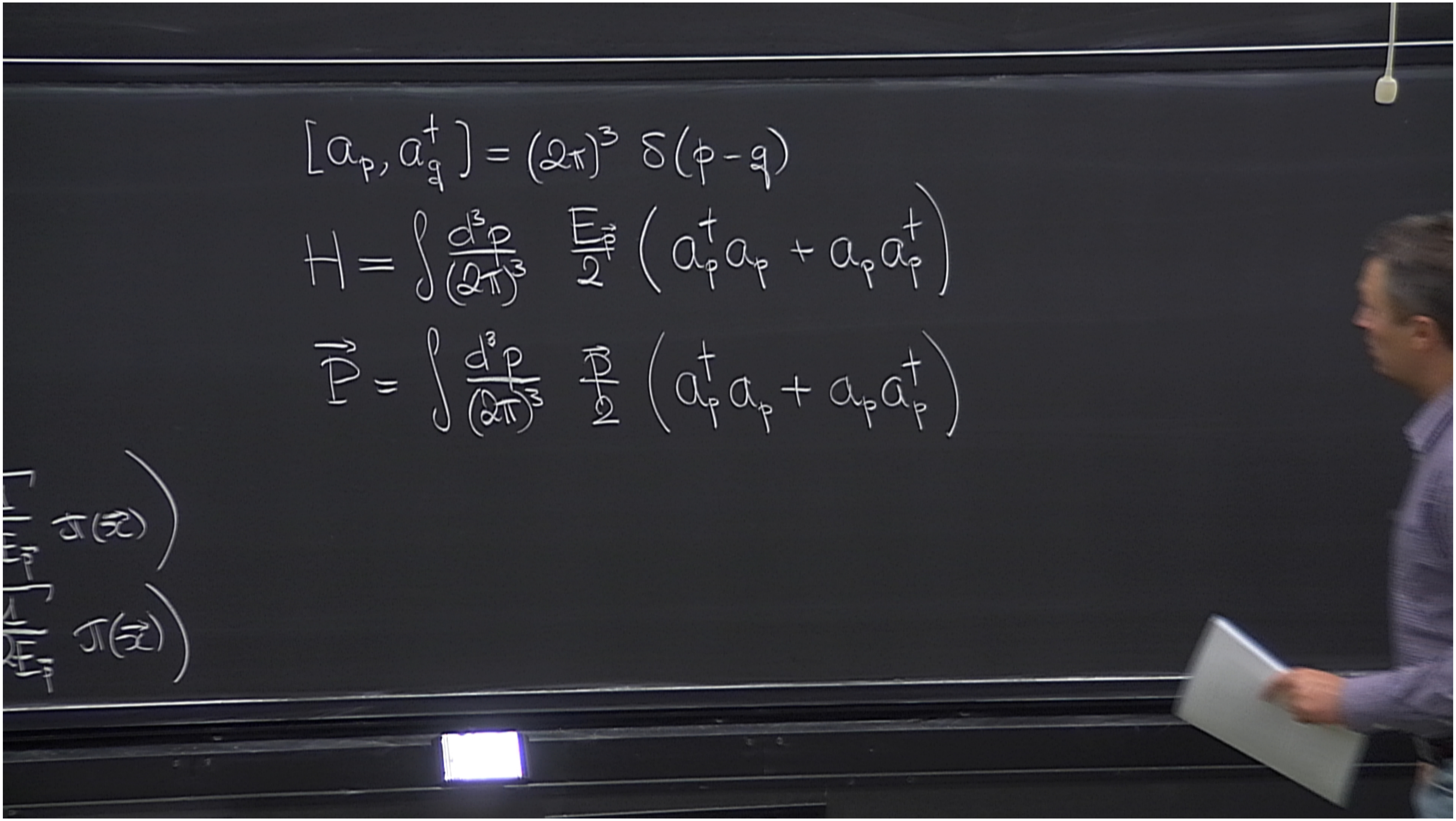
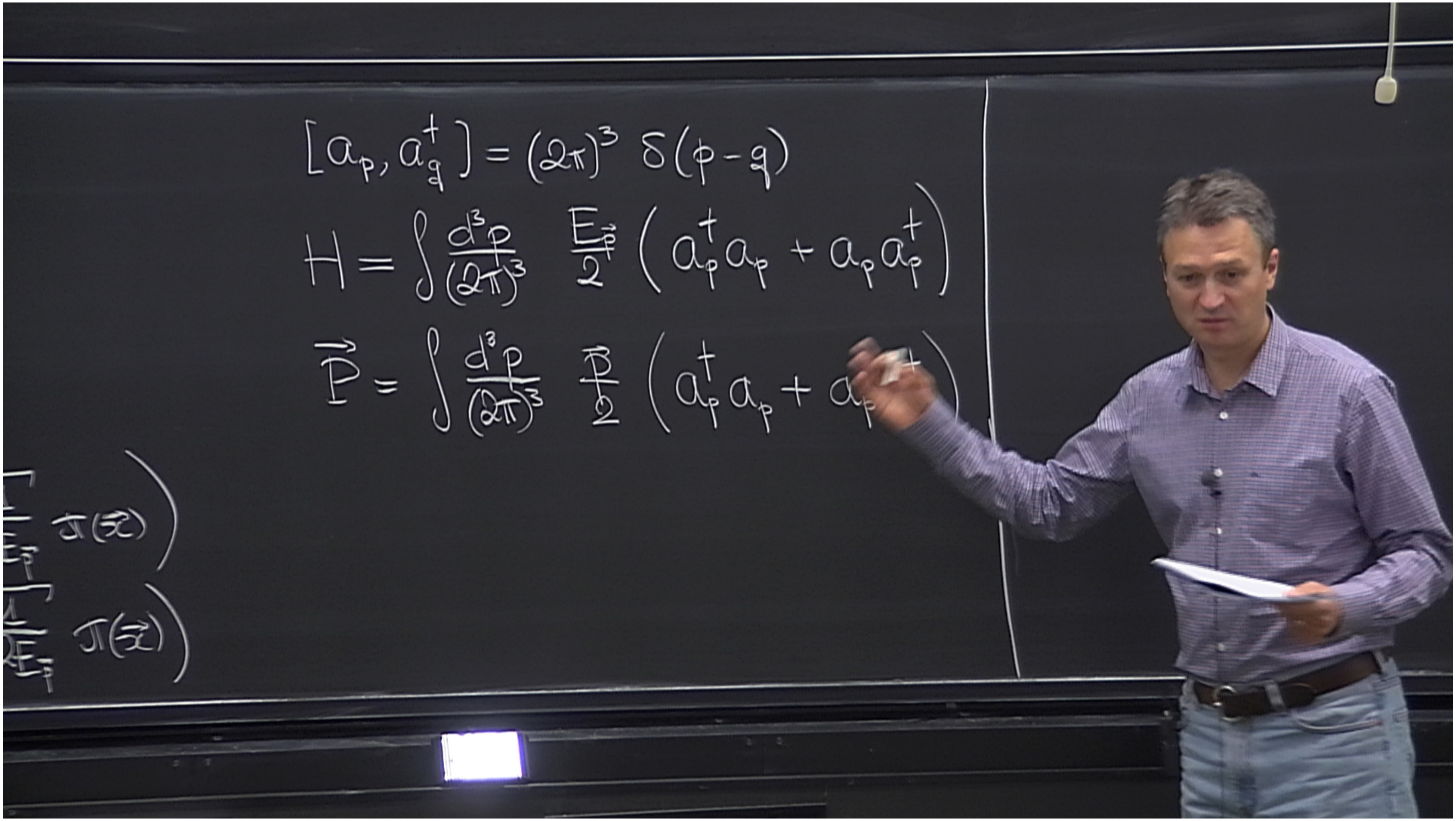
$E_{\vec{p}} = \sqrt{m^2+\vec{p}^{\,2}}$
Creation/annihilation operators:
$a_p = \int d^3x\, e^{-i\vec{p}\cdot\vec{x}} \left(\sqrt{\frac{E_{\vec{p}}}{2}}\,\varphi(\vec{x}) + i\sqrt{\frac{1}{2E_{\vec{p}}}}\,\pi(\right.$
$a_p^\dagger = \int d^3x\, e^{i\vec{p}\cdot\vec{x}} \left(\sqrt{\frac{E_{\vec{p}}}{2}}\,\varphi(\vec{x}) - i\sqrt{\frac{1}{2E_{\vec{p}}}} \right.$
$[a_p, a^\dagger$


Quantization of KG field
[φ(x⃗), π(y⃗)] = i δ(x⃗ - y⃗)
E_p⃗ = √(m² + p⃗²)
Creation/annihilation operators:
a_p = ∫d³x e^{-ip⃗·x⃗} (√(E_p⃗/2) φ(x⃗) + i √(1/2E_p⃗) π(
a_p† = ∫d³x e^{ip⃗·x⃗} (√(E_p⃗/2) φ(x⃗) - i √(1/2E_p⃗) π(x⃗
[a_p, a_q†]
H =


$[a_p, a_q^\dagger] = (2\pi)^3 \delta(p-q)$
$H = \int \frac{d^3p}{(2\pi)^3} \frac{E_{\vec{p}}}{2} \left(a_p^\dagger a_p + a_p a_p^\dagger \right)$
$\vec{P} = \int \frac{d^3p}{(2\pi)^3} \frac{\vec{p}}{2} \left(a_p^\dagger a_p + a_p a_p^\dagger \right)$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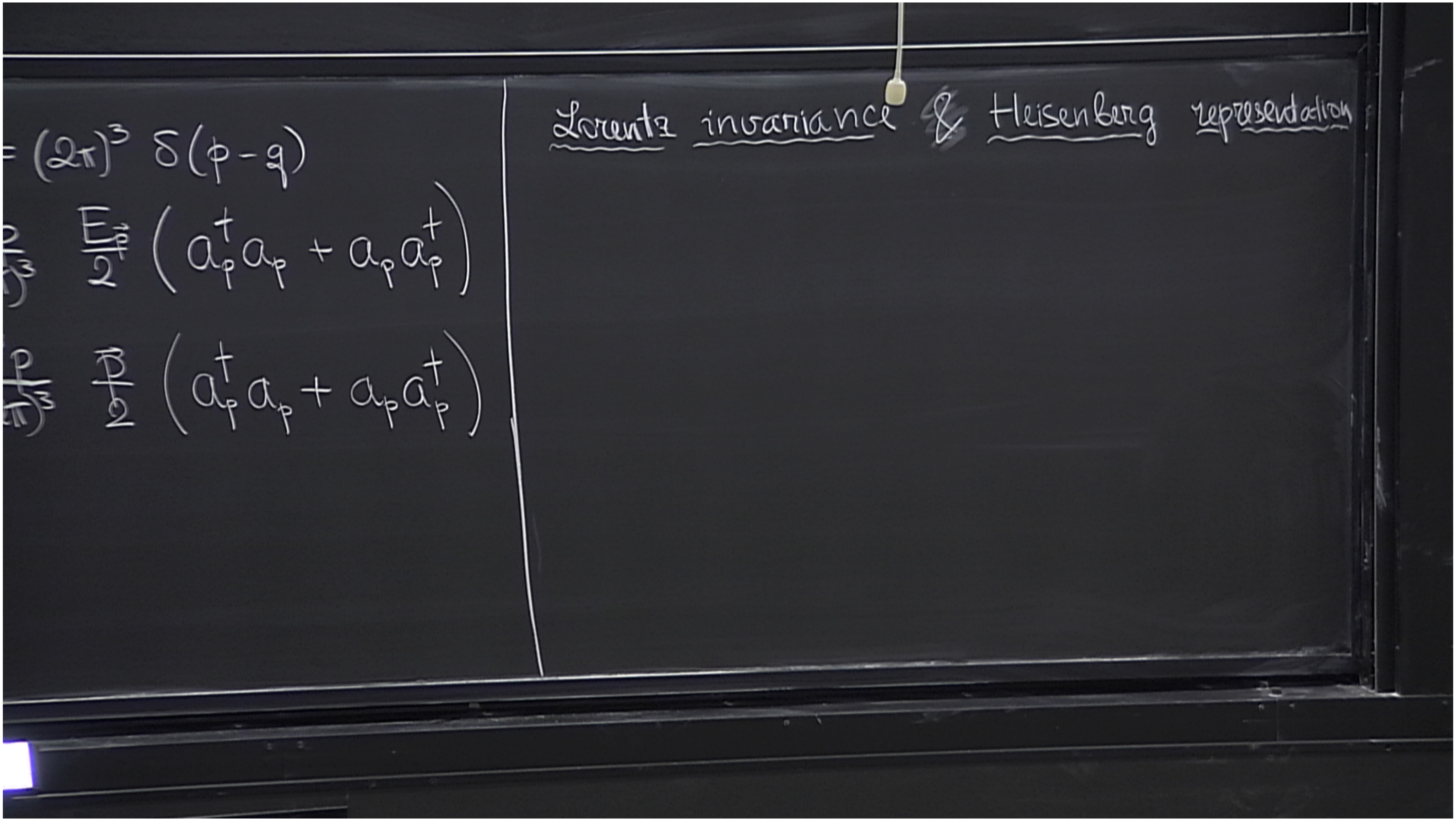

Lorentz invariance & Heisenberg representation
$= (2\pi)^3 \delta(p-q)$
$\frac{E_{\vec{p}}}{2} \left(a_p^\dagger a_p + a_p a_p^\dagger \right)$
$\frac{\vec{p}}{2} \left(a_p^\dagger a_p + a_p a_p^\dagger \right)$

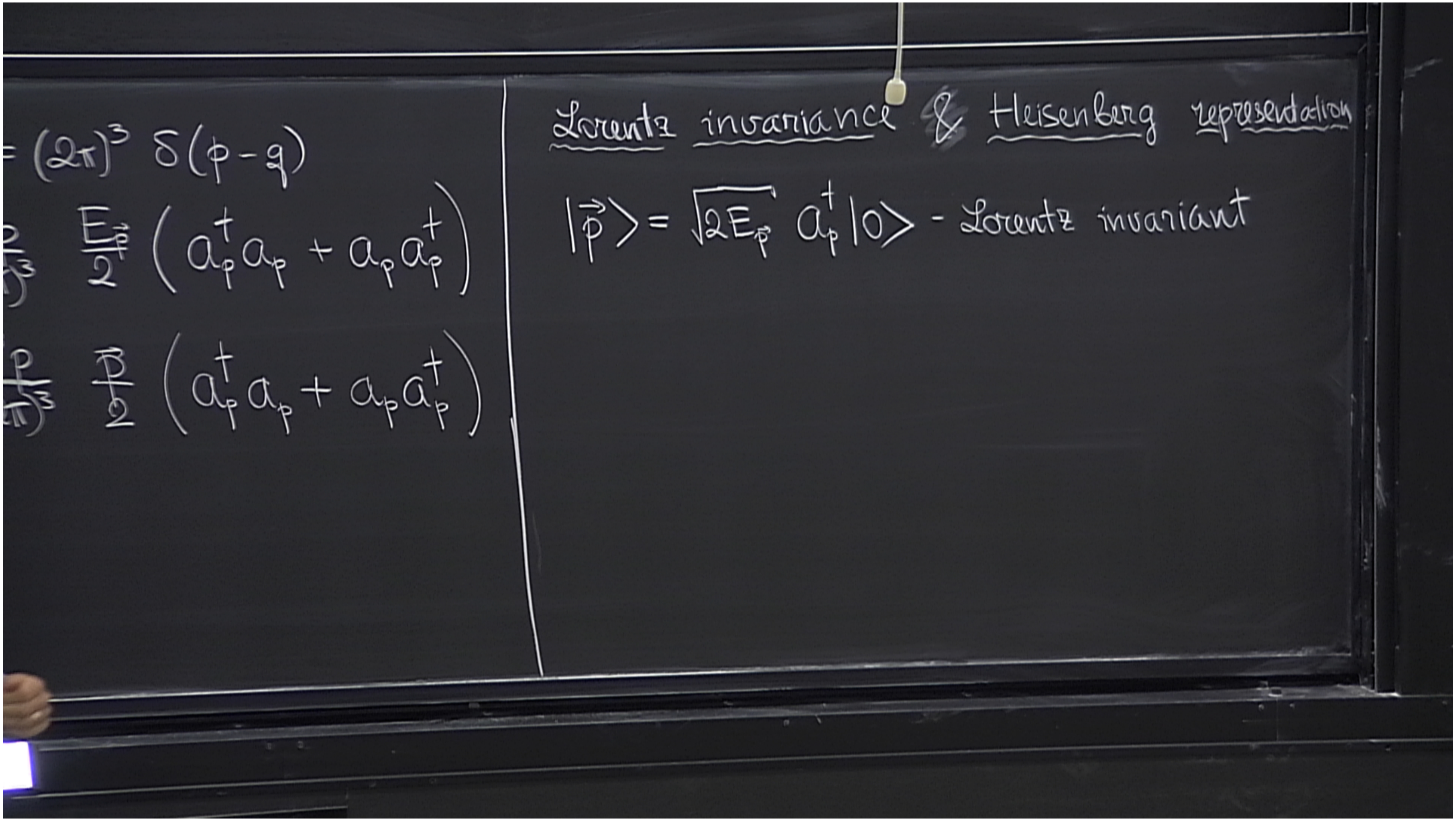
$$= (2\pi)^3\, \delta(p - q)$$

$$\frac{\;}{(2\pi)^3}\ \frac{E_{\vec p}}{2} \left(a_p^\dagger a_p + a_p a_p^\dagger \right)$$

$$\frac{p}{(2\pi)^3}\ \frac{\vec p}{2} \left(a_p^\dagger a_p + a_p a_p^\dagger \right)$$

Lorentz invariance & Heisenberg representation

$|\vec p\rangle = \sqrt{2E_{\vec p}}\; a_p^\dagger |0\rangle$ - Lorentz invariant


Lorentz invariance & Heisenberg representation
$|\vec{p}\rangle = \sqrt{2E_{\vec{p}}}\, a_p^\dagger |0\rangle$ - Lorentz invariant

$$= (2\pi)^3\, \delta(p-q)$$

$$\ldots \frac{\ldots}{(2\pi)^3}\, \frac{E_{\vec{p}}}{2} \left(a_p^\dagger a_p + a_p a_p^\dagger \right)$$

$$\ldots \frac{\ldots p}{(2\pi)^3}\, \frac{\vec{p}}{2} \left(a_p^\dagger a_p + a_p a_p^\dagger \right)$$

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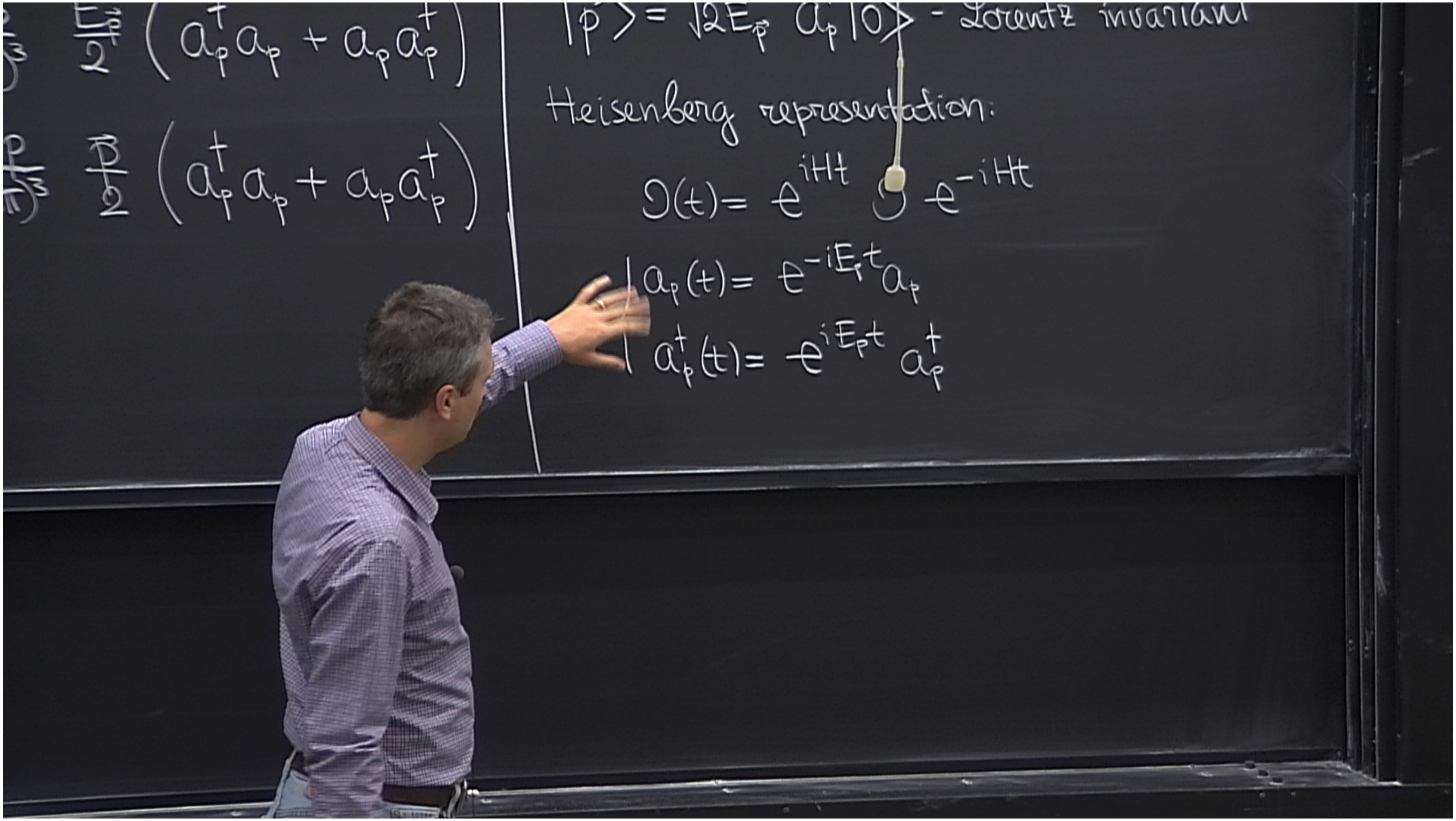
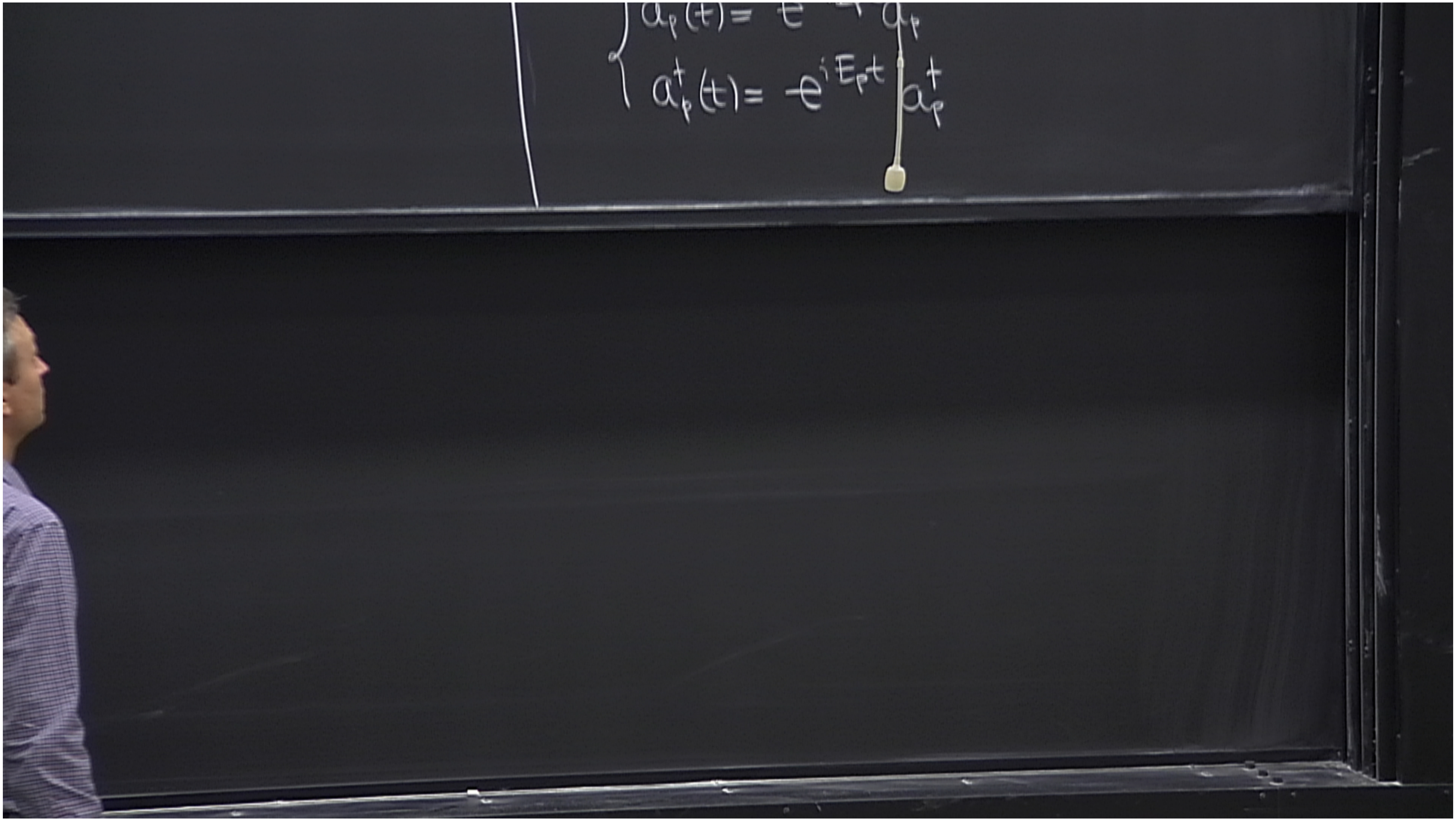
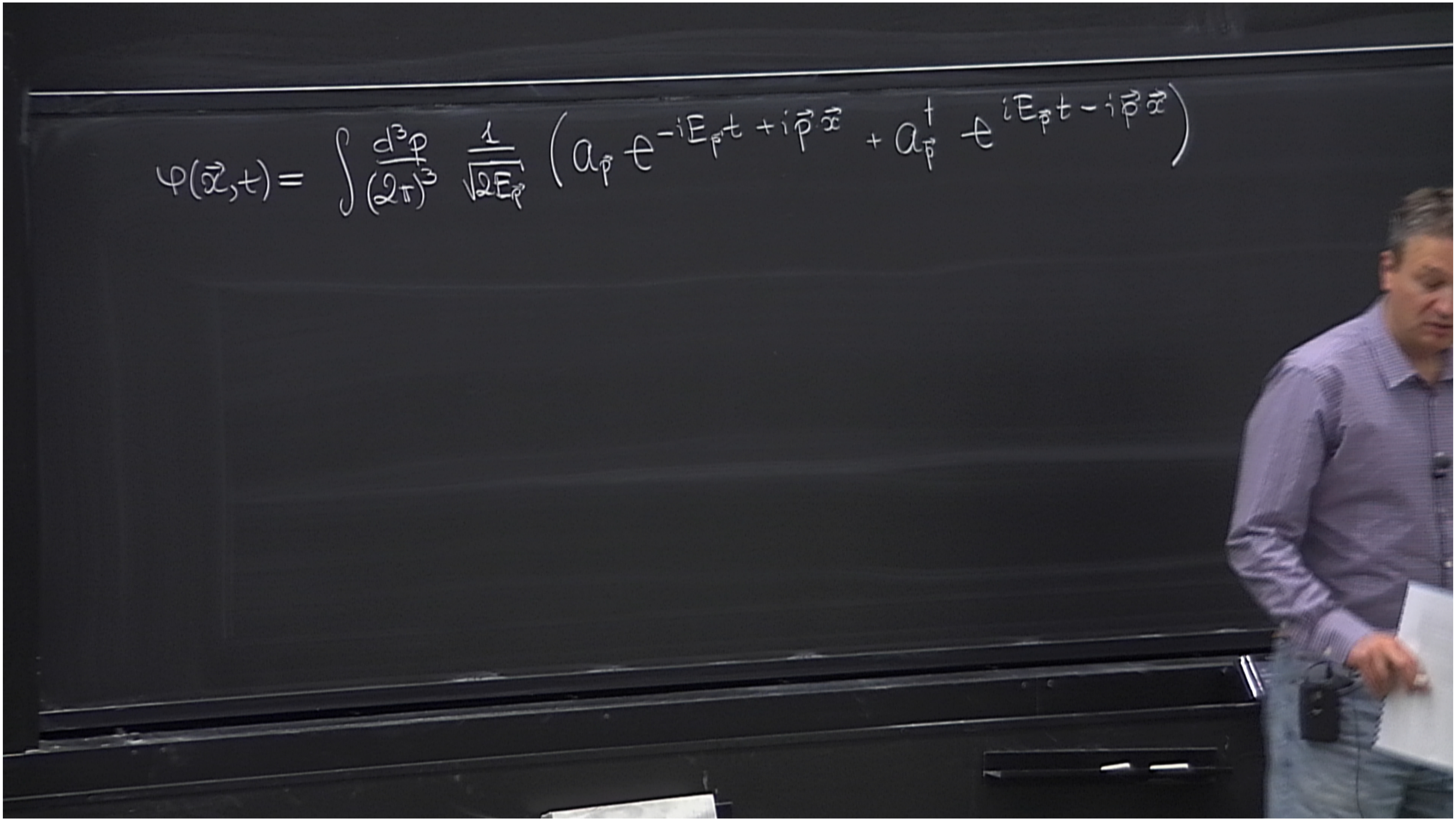
Heisenberg representation:


Heisenberg representation:
$O(t) = e^{iHt} O e^{-iHt}$
$a_p(t) = e^{-iE_p t} a_p$
$a_p^\dagger(t) = e^{iE_p t} a_p^\dagger$


$a_p^\dagger(t) = e^{iE_p t} a_p^\dagger$

$$\varphi(\vec{x},t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}}\, e^{-iE_{\vec{p}}t + i\vec{p}\cdot\vec{x}} + a_{\vec{p}}^{\dagger}\, e^{iE_{\vec{p}}t - i\vec{p}\vec{x}} \right)$$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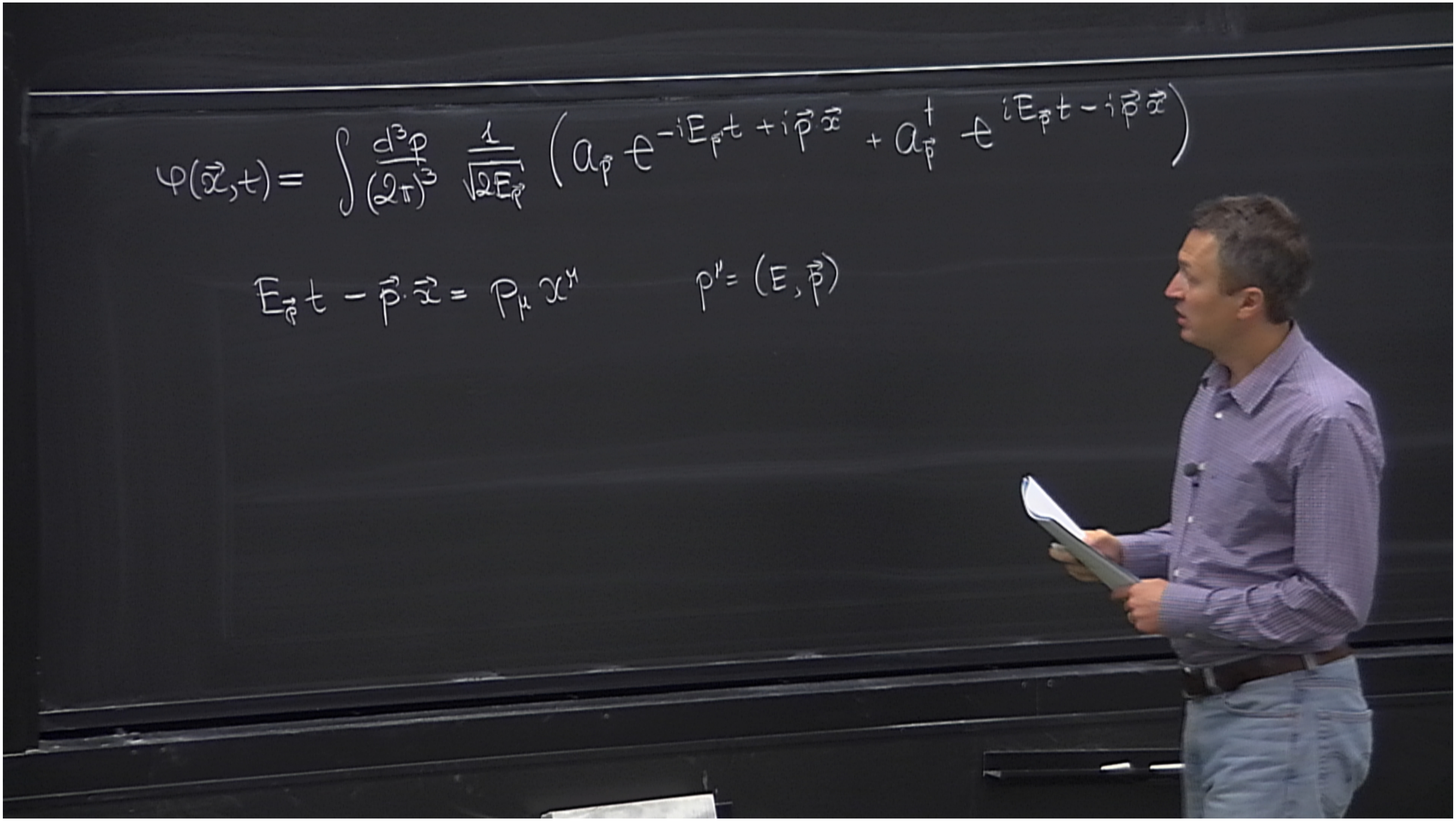
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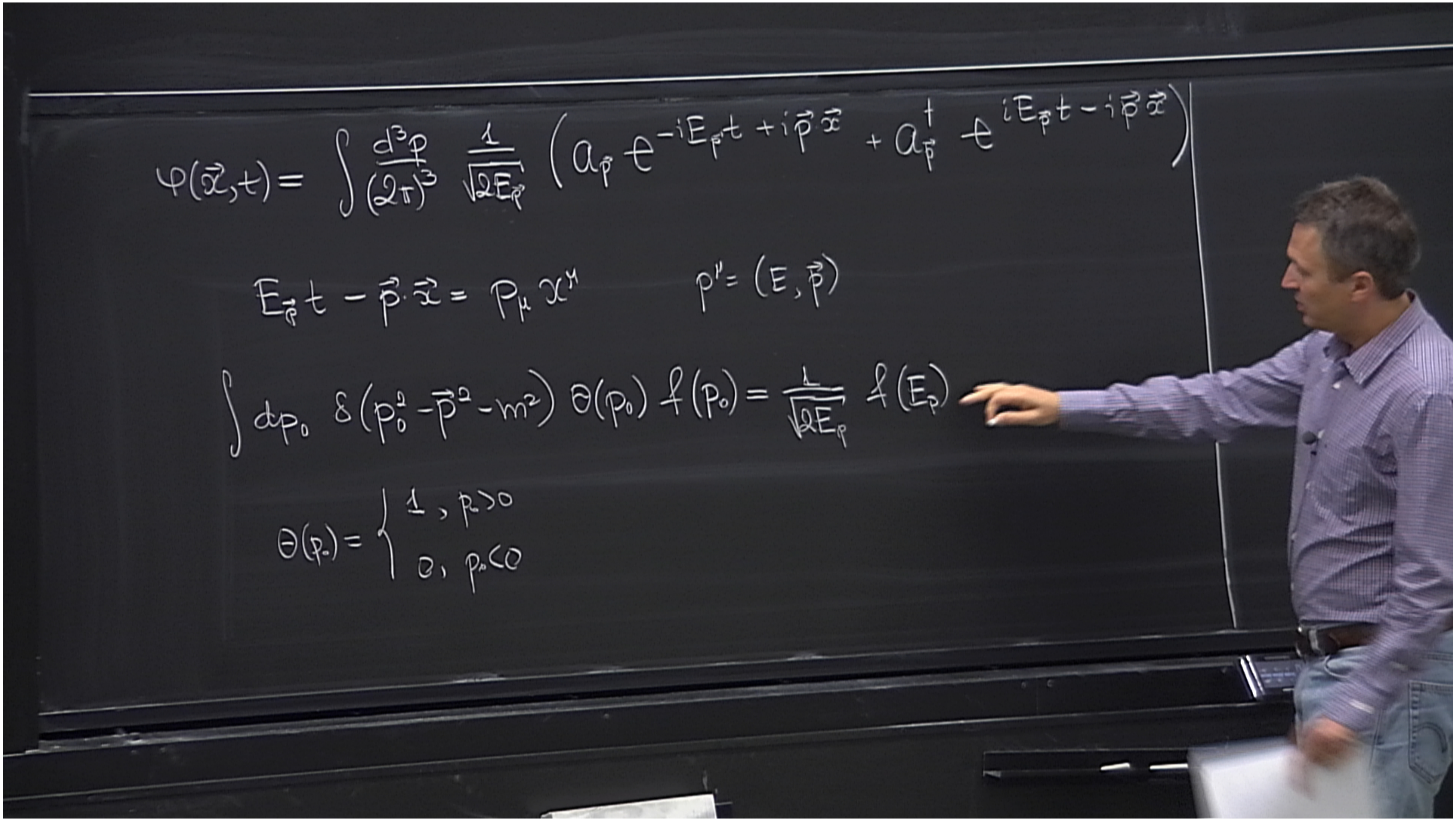
$E_{\vec{p}}t - \vec{p}\cdot\vec{x} = p_\mu x^\mu$
$p^\mu = (E, \vec{p})$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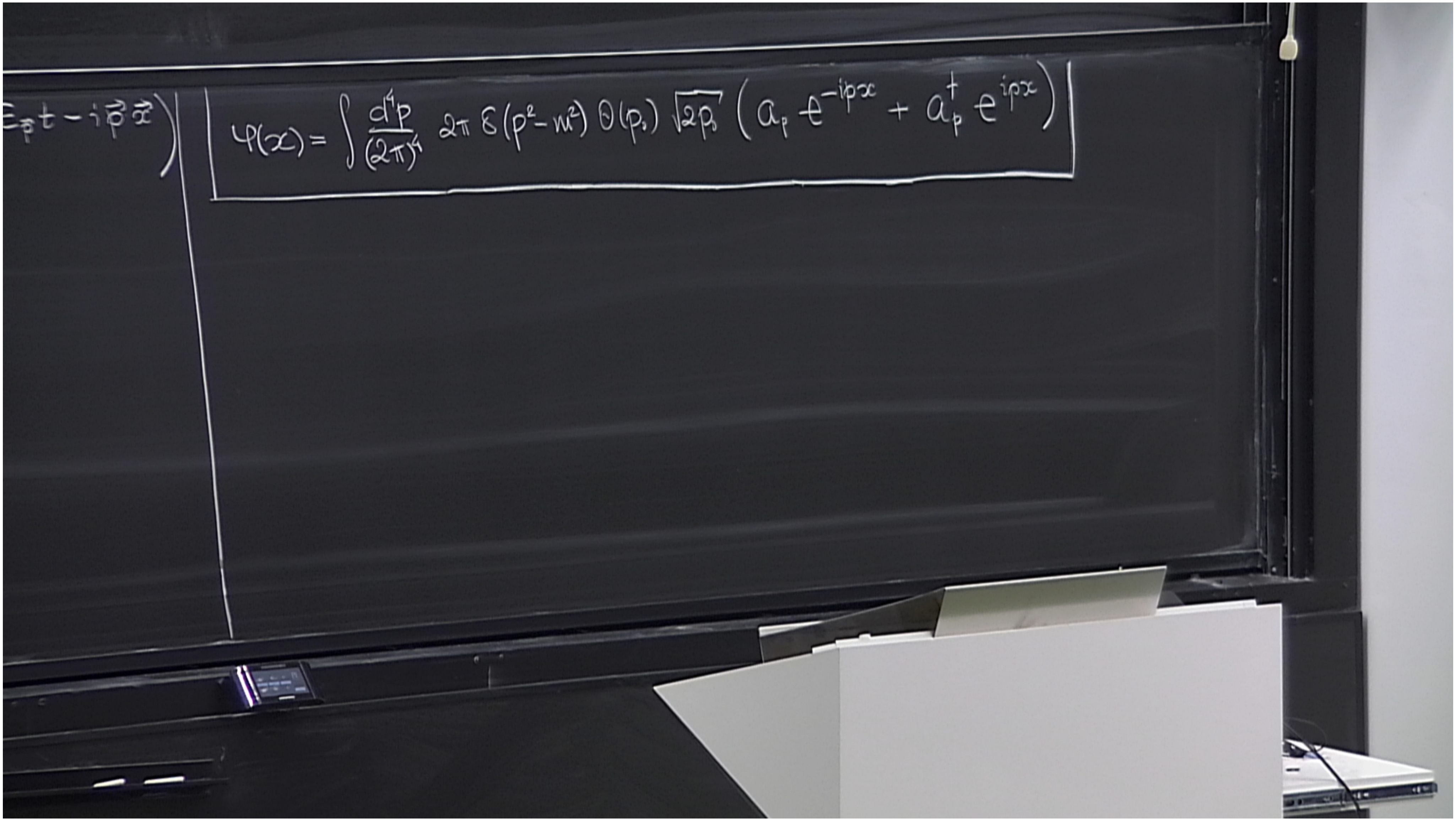
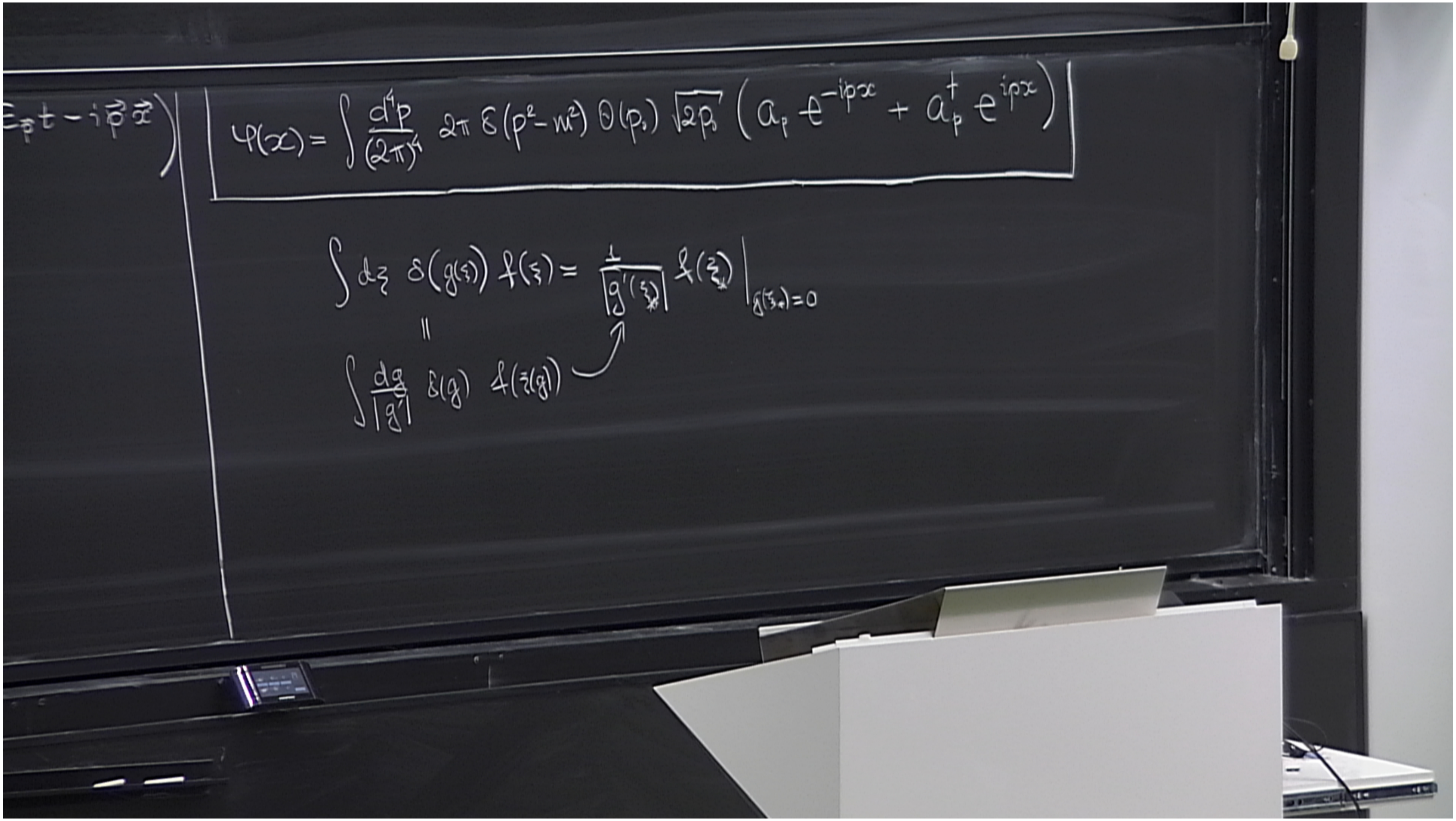
$E_{\vec{p}}t - \vec{p}\cdot\vec{x} = p_\mu x^\mu$
$p^\mu = (E, \vec{p})$
$\int dp_0 \, \delta(p_0^2 - \vec{p}^2 - m^2)\, \Theta(p_0)\, f(p_0) = \frac{1}{\sqrt{2E_p}} f(E_p)$
$\Theta(p_0) = \begin{cases} 1, & p_0 > 0 \\ 0, & p_0 < 0 \end{cases}$


$E_{\vec{p}} t - i\vec{p}\vec{x})$
$\varphi(x) = \int \frac{d^4p}{(2\pi)^4} 2\pi\, \delta(p^2 - m^2)\, \Theta(p_0) \sqrt{2p_0} \left(a_p e^{-ipx} + a_p^\dagger e^{ipx}\right)$


$E_{\vec{p}} t - i\vec{p}\vec{x})$
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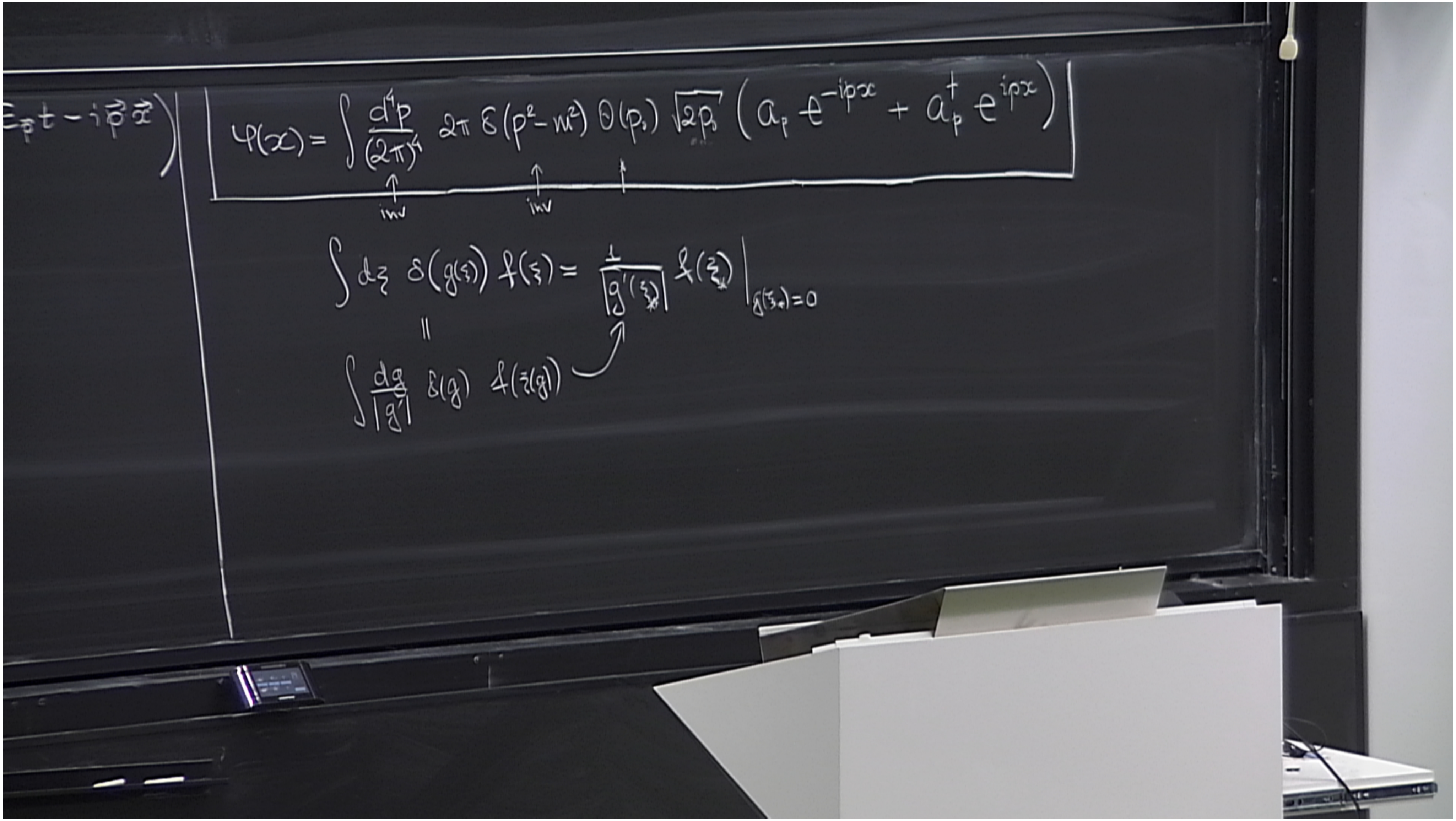
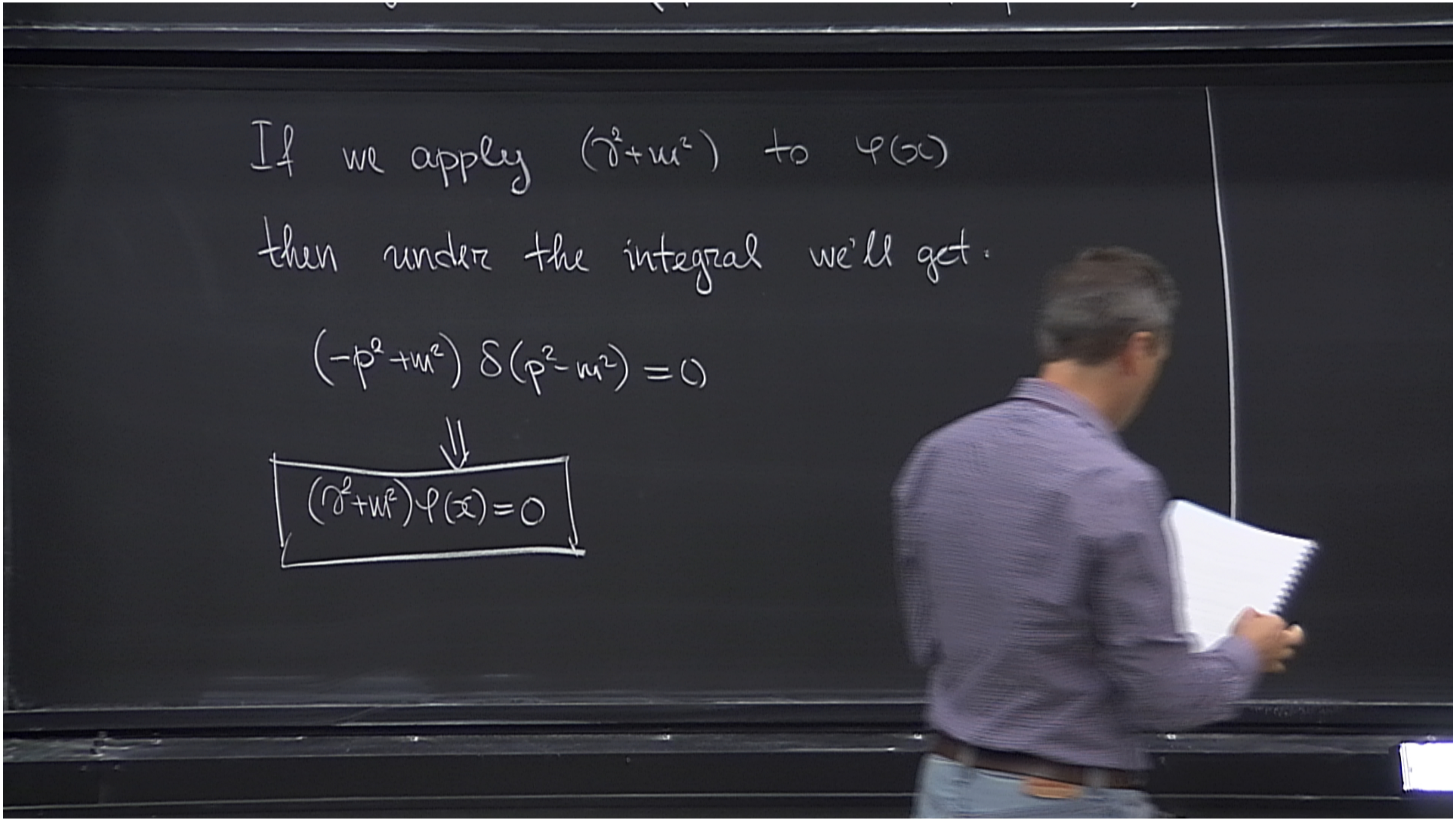
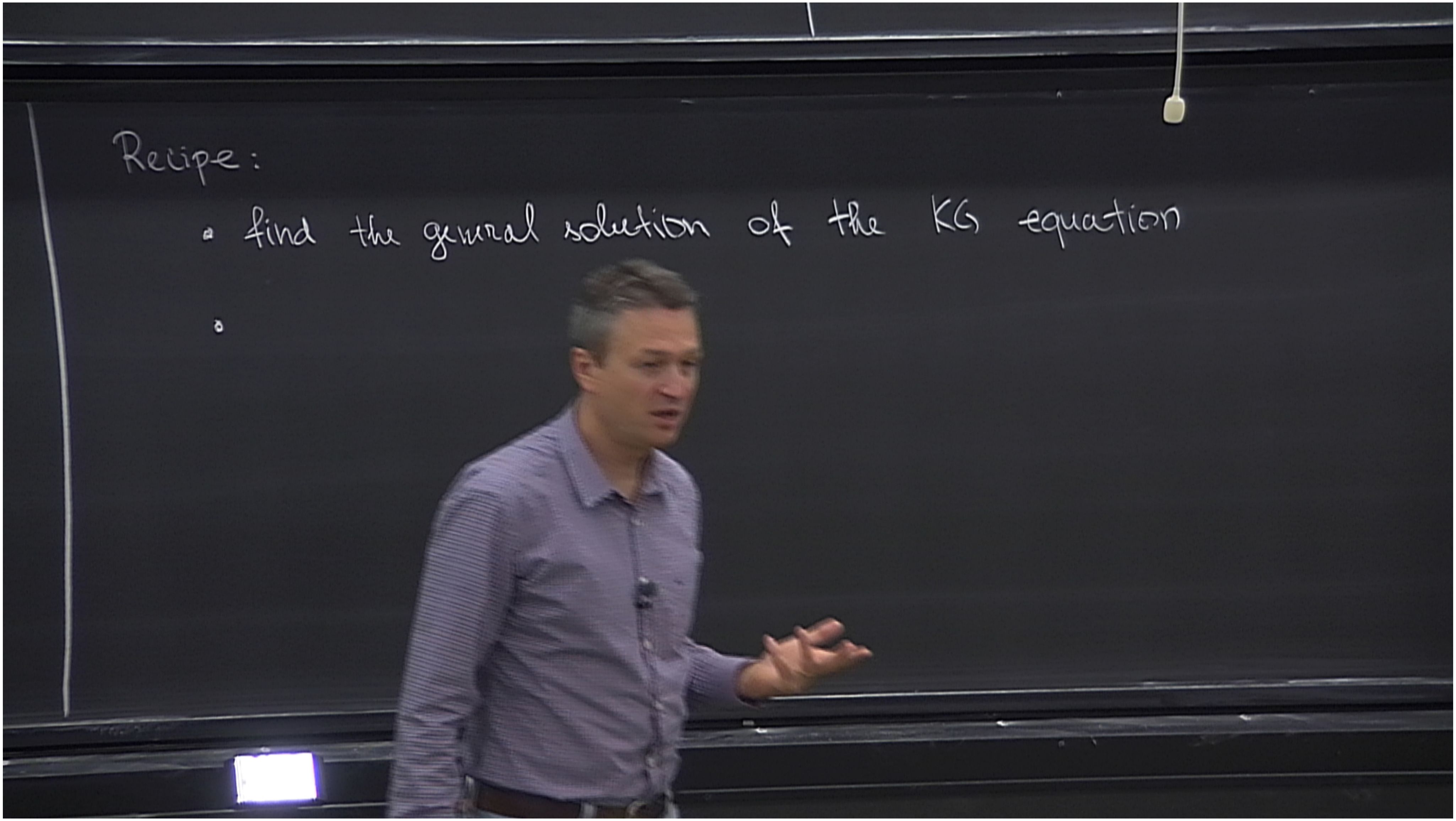
$\int d\xi \, \delta(g(\xi)) \, f(\xi) = \frac{1}{|g'(\xi_*)|} f(\xi_*) \Big|_{g(\xi_*)=0}$
$\int \frac{dg}{|g'|} \delta(g) \, f(\xi(g))$

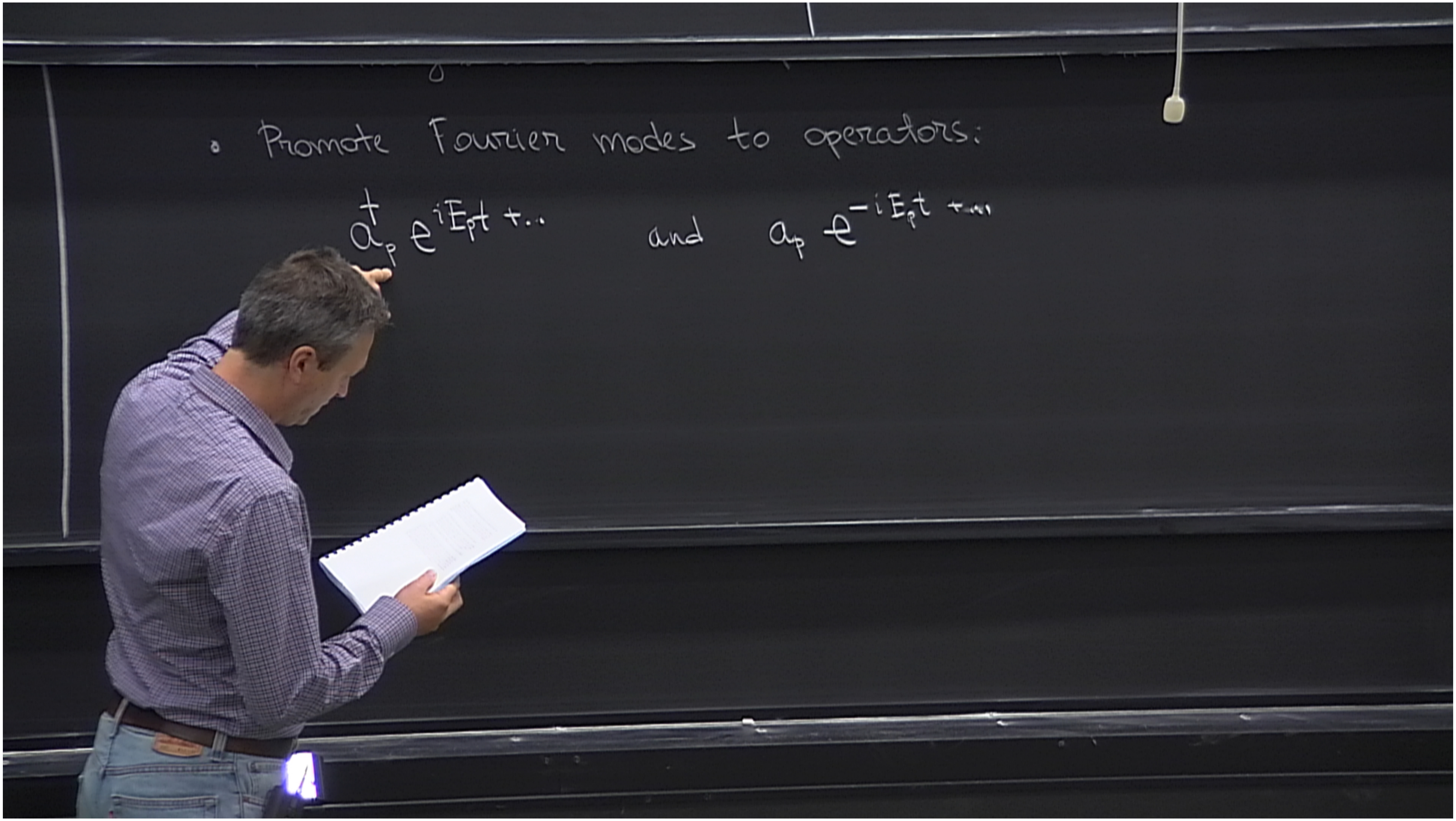

$E_p t - i\vec{p}\vec{x})$
$\varphi(x) = \int \frac{d^4p}{(2\pi)^4} \, 2\pi \, \delta(p^2 - m^2) \, \Theta(p_0) \, \sqrt{2p_0} \left(a_p e^{-ipx} + a_p^\dagger e^{ipx} \right)$
inv
inv
$\int d\xi \, \delta(g(\xi)) \, f(\xi) = \frac{1}{|g'(\xi_*)|} f(\xi_*) \Big|_{g(\xi_*)=0}$
$\int \frac{dg}{|g'|} \, \delta(g) \, f(\xi(g))$

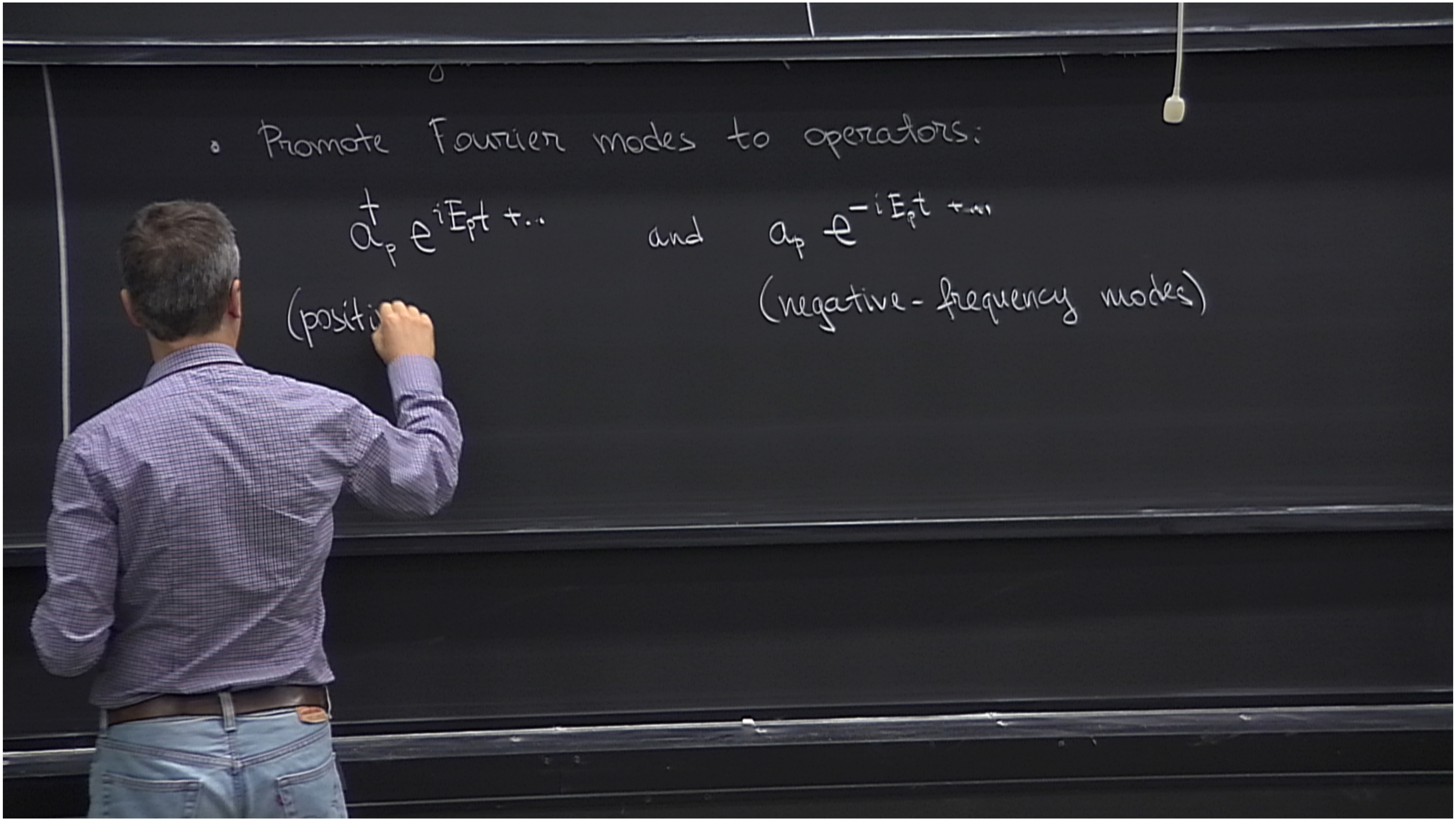

If we apply (∂^2+m^2) to $\varphi(x)$
then under the integral we'll get.
$(-p^2+m^2)\,\delta(p^2-m^2)=0$
$\Downarrow$
$(\partial^2+m^2)\varphi(x)=0$


Recipe:
• find the general solution of the KG equation
•

Recipe:

- find the general solution of the KG equation
- Promote Fourier modes to operators:


• Promote Fourier modes to operators:
$a_p^\dagger e^{iE_pt} + ..$ and $a_p e^{-iE_pt} + ...$


• Promote Fourier modes to operators:
$a_p^\dagger e^{iE_pt} + \dots$ and $a_p e^{-iE_pt} + \dots$
(positi
(negative-frequency modes)

Recipe:

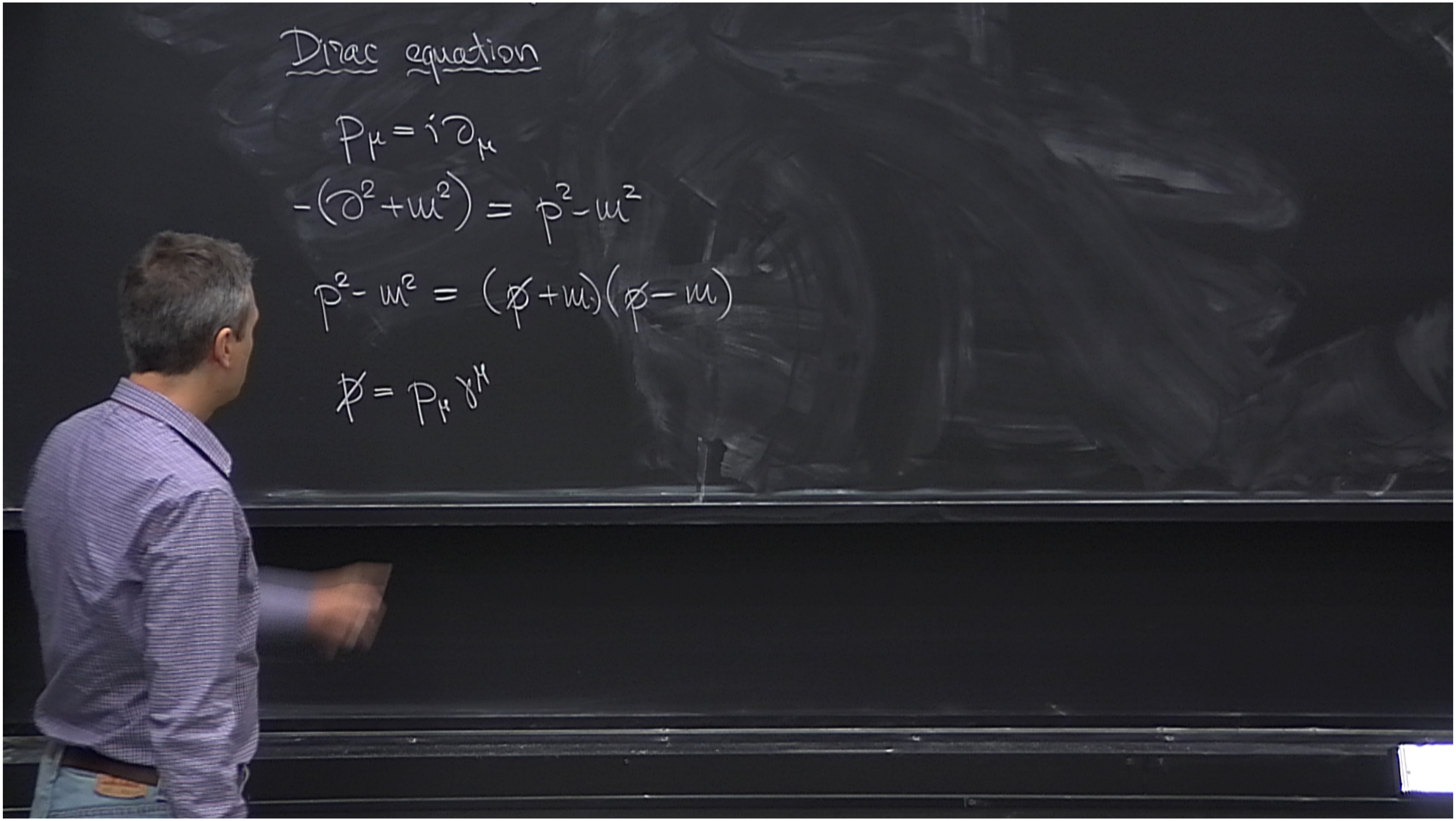
- find the general solution of the KG equation
- Promote Fourier modes to operators:

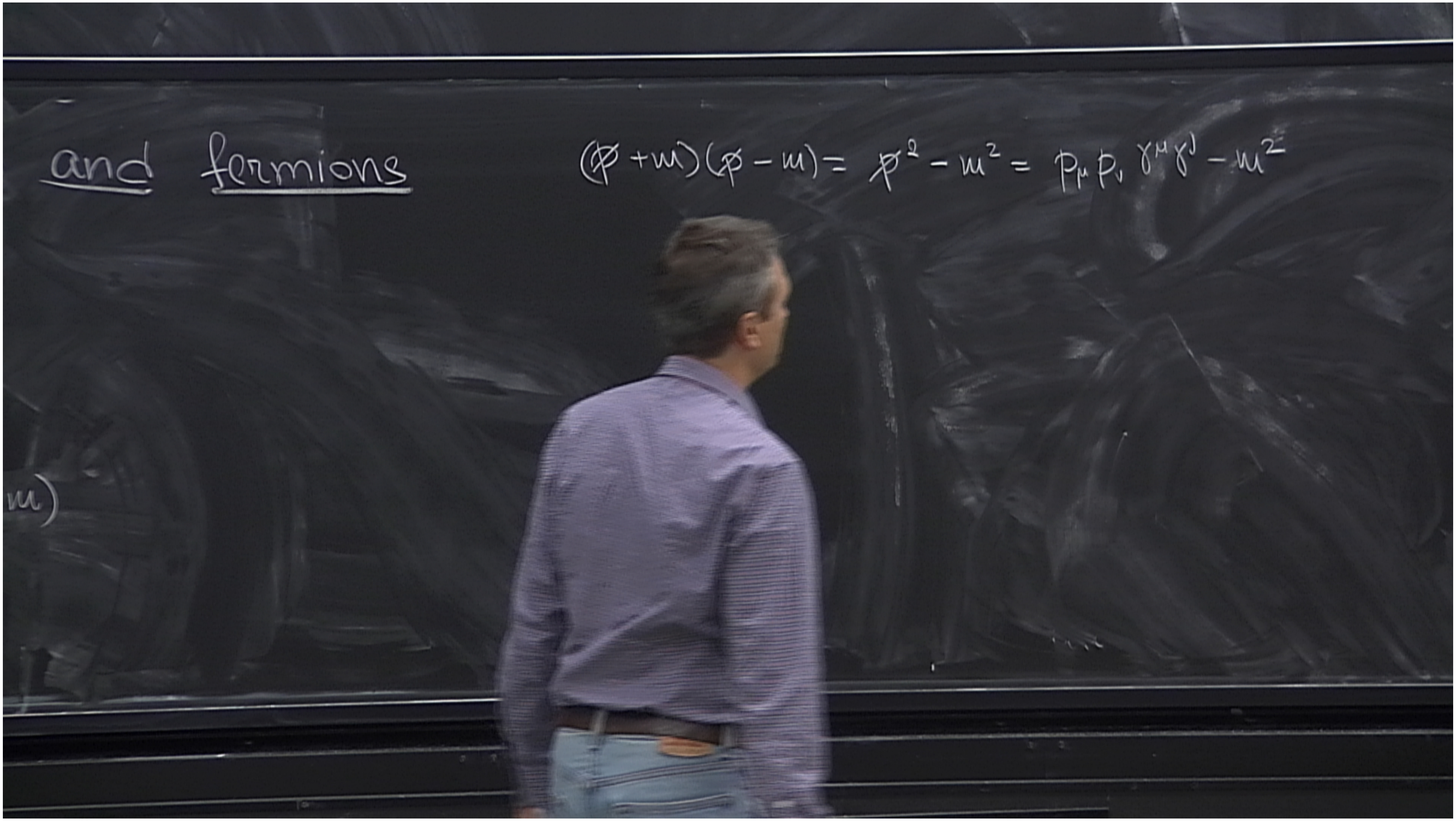
$$a_p^\dagger e^{iE_pt} + \ldots \quad \text{and} \quad a_p e^{-iE_pt} + \ldots \qquad \times \sqrt{2E_p}$$

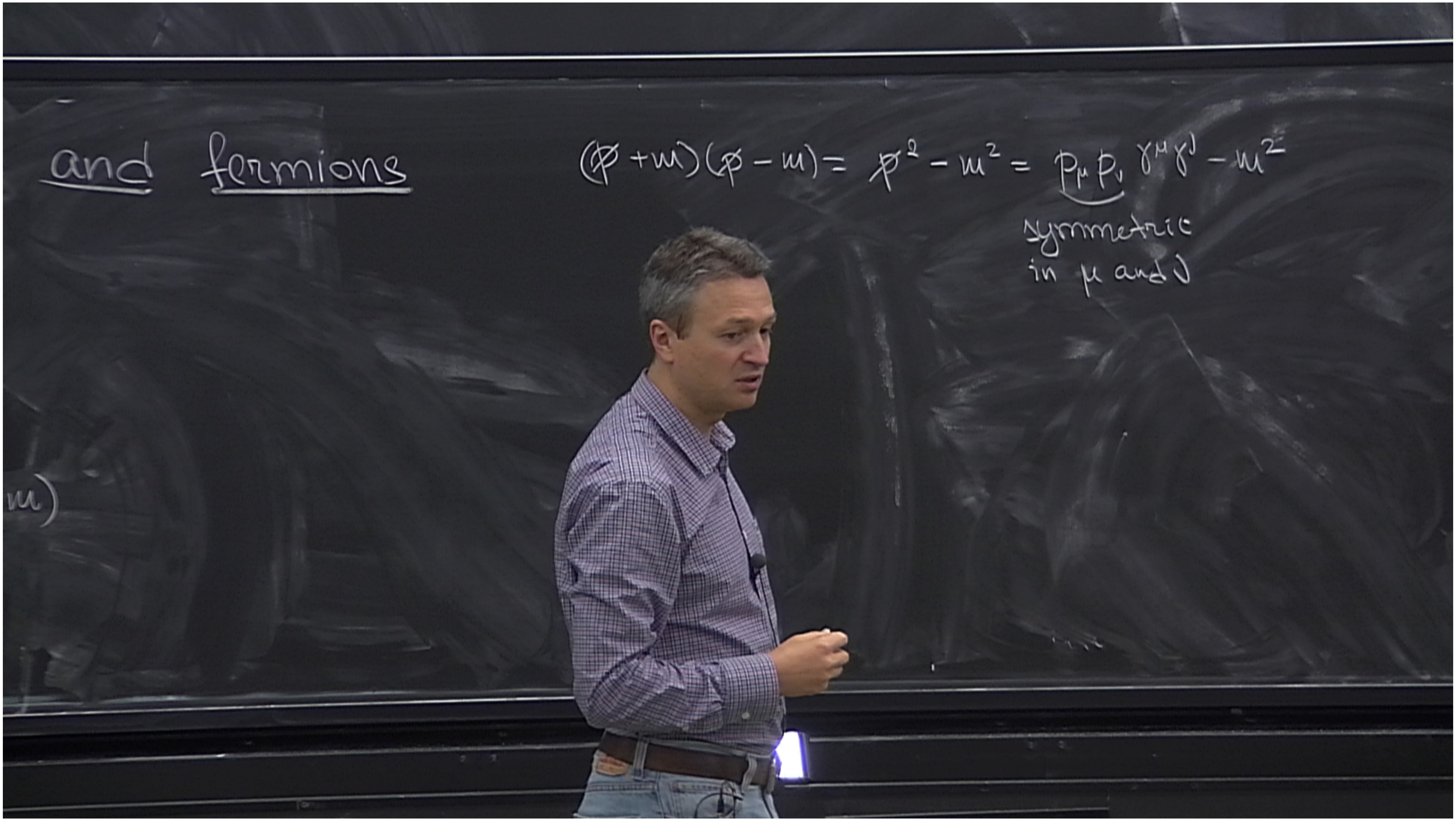
(positive-frequency modes) (negative-frequency modes)

$$[a_p, a_q^\dagger] = (2\pi)^3 \delta(p-q)$$

- L.I. states: $|\vec{p}\rangle = \sqrt{2E_p}\, a_p^\dagger |0\rangle$


Dirac equation
$p_\mu = i\partial_\mu$
$-(\partial^2 + m^2) = p^2 - m^2$
$p^2 - m^2 = (\not{p} + m)(\not{p} - m)$
$\not{p} = p_\mu \gamma^\mu$


and fermions
$(\not{p}+m)(\not{p}-m) = \not{p}^2 - m^2 = p_\mu p_\nu \gamma^\mu \gamma^\nu - m^2$
m)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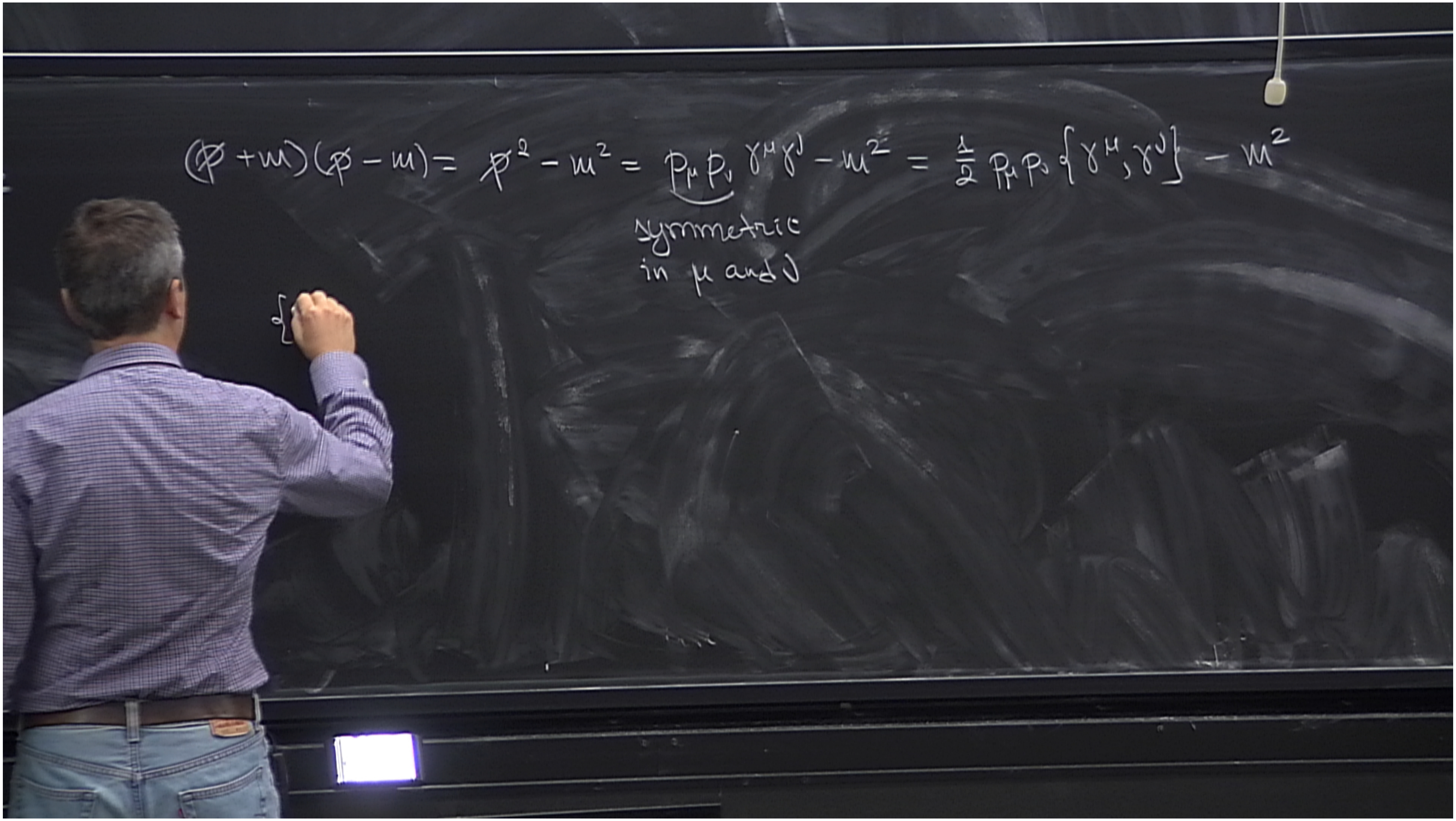
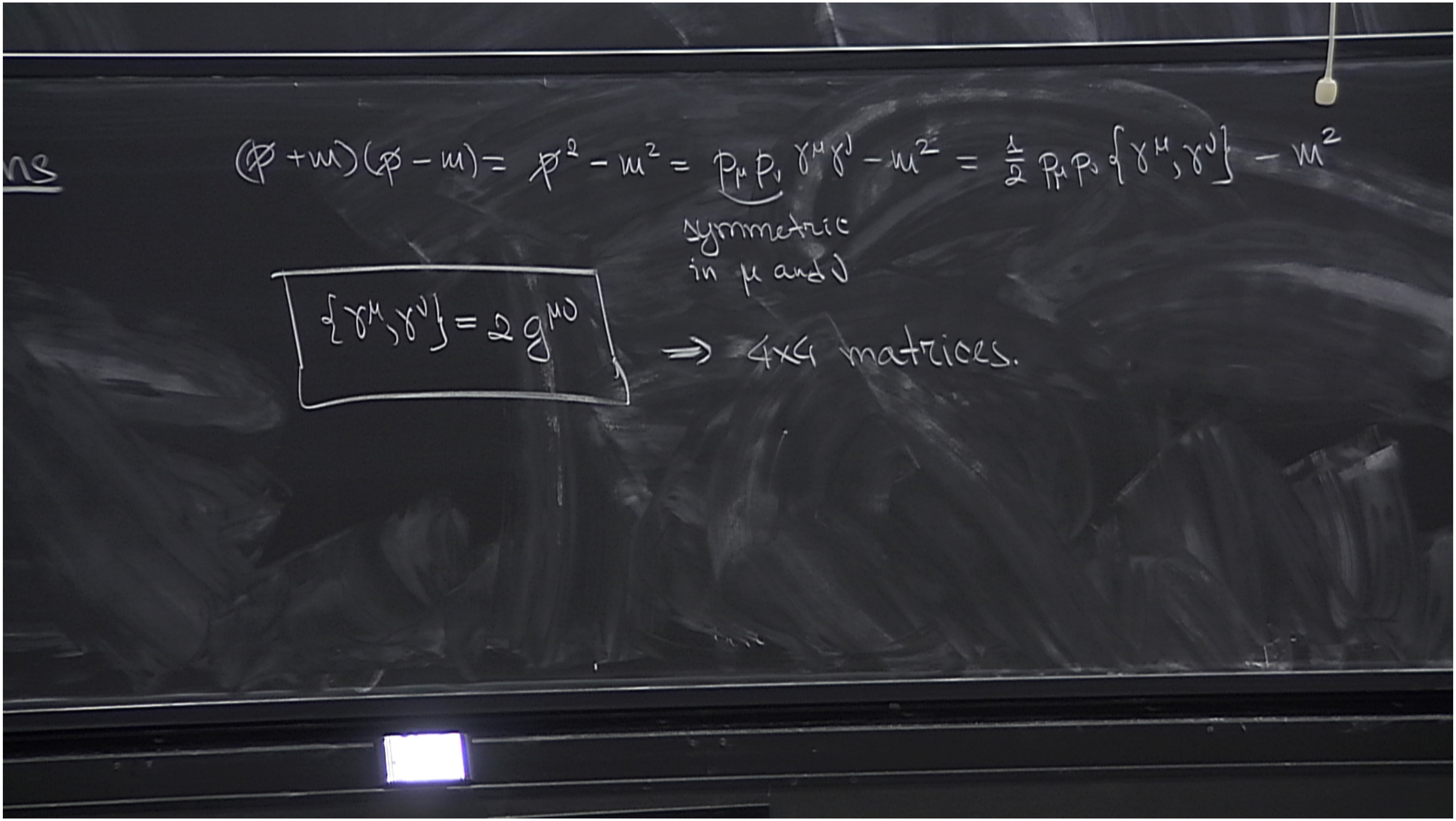
symmetric
in μ and ν


$(\not{p} + m)(\not{p} - m) = \not{p}^2 - m^2 = p_\mu p_\nu \gamma^\mu \gamma^\nu - m^2 = \frac{1}{2} p_\mu p_\nu \{\gamma^\mu, \gamma^\nu\} - m^2$
symmetric
in μ and ν

$$(\not{p} + m)(\not{p} - m) = \not{p}^2 - m^2 = \underbrace{p_\mu p_\nu}_{\text{symmetric in } \mu \text{ and } \nu} \gamma^\mu \gamma^\nu - m^2 = \frac{1}{2} p_\mu p_\nu \{\gamma^\mu, \gamma^\nu\} - m^2$$

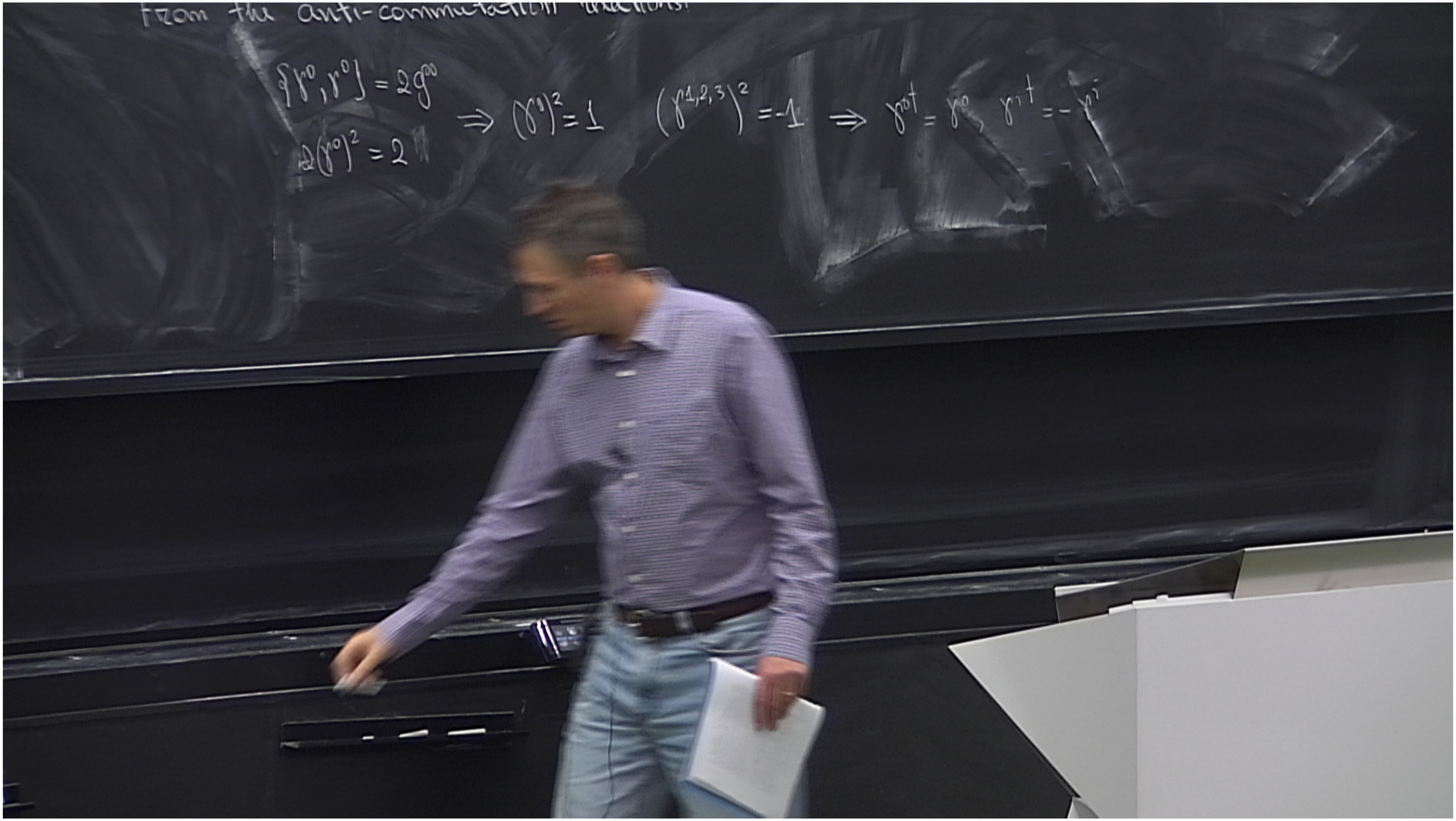
$$\boxed{\{\gamma^\mu, \gamma^\nu\} = 2 g^{\mu\nu}}$$


ns
$(\not{p}+m)(\not{p}-m) = \not{p}^2 - m^2 = p_\mu p_\nu \gamma^\mu \gamma^\nu - m^2 = \frac{1}{2} p_\mu p_\nu \{\gamma^\mu, \gamma^\nu\} - m^2$
symmetric in μ and ν
$\{\gamma^\mu, \gamma^\nu\} = 2 g^{\mu\nu}$
⟹ 4×4 matrices.

symmetric
in μ and ν

$$\{\gamma^\mu, \gamma^\nu\} = 2 g^{\mu\nu}$$

$\Longrightarrow$ 4x4 matrices.


from the anti-commutation
$\{\gamma^0,\gamma^0\} = 2g^{00}$
$2(\gamma^0)^2 = 2$
$\Rightarrow (\gamma^0)^2 = 1$
$(\gamma^{1,2,3})^2 = -1$
$\Rightarrow \gamma^{0\dagger} = \gamma^0, \gamma^{i\dagger} = -\gamma^i$