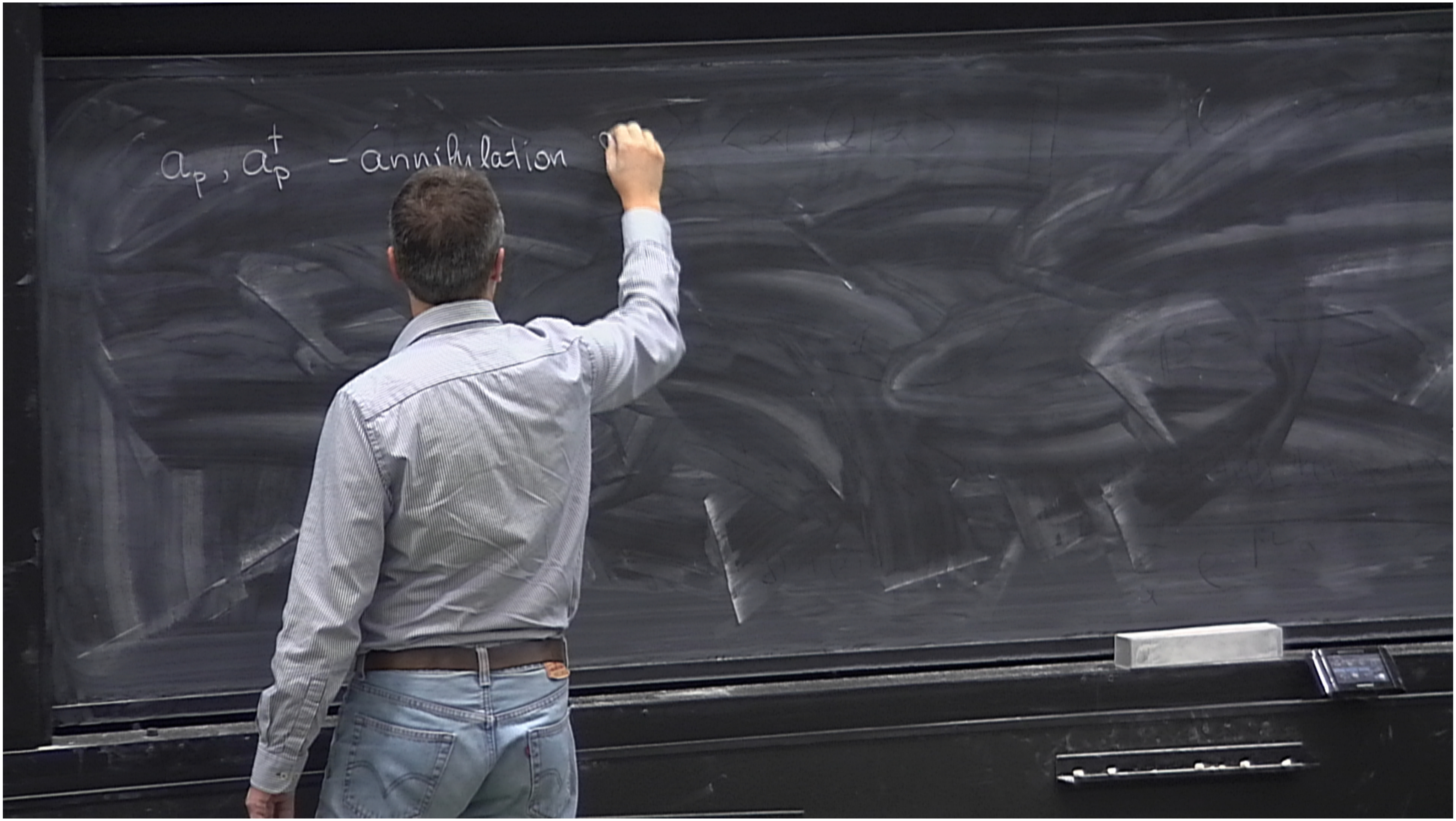
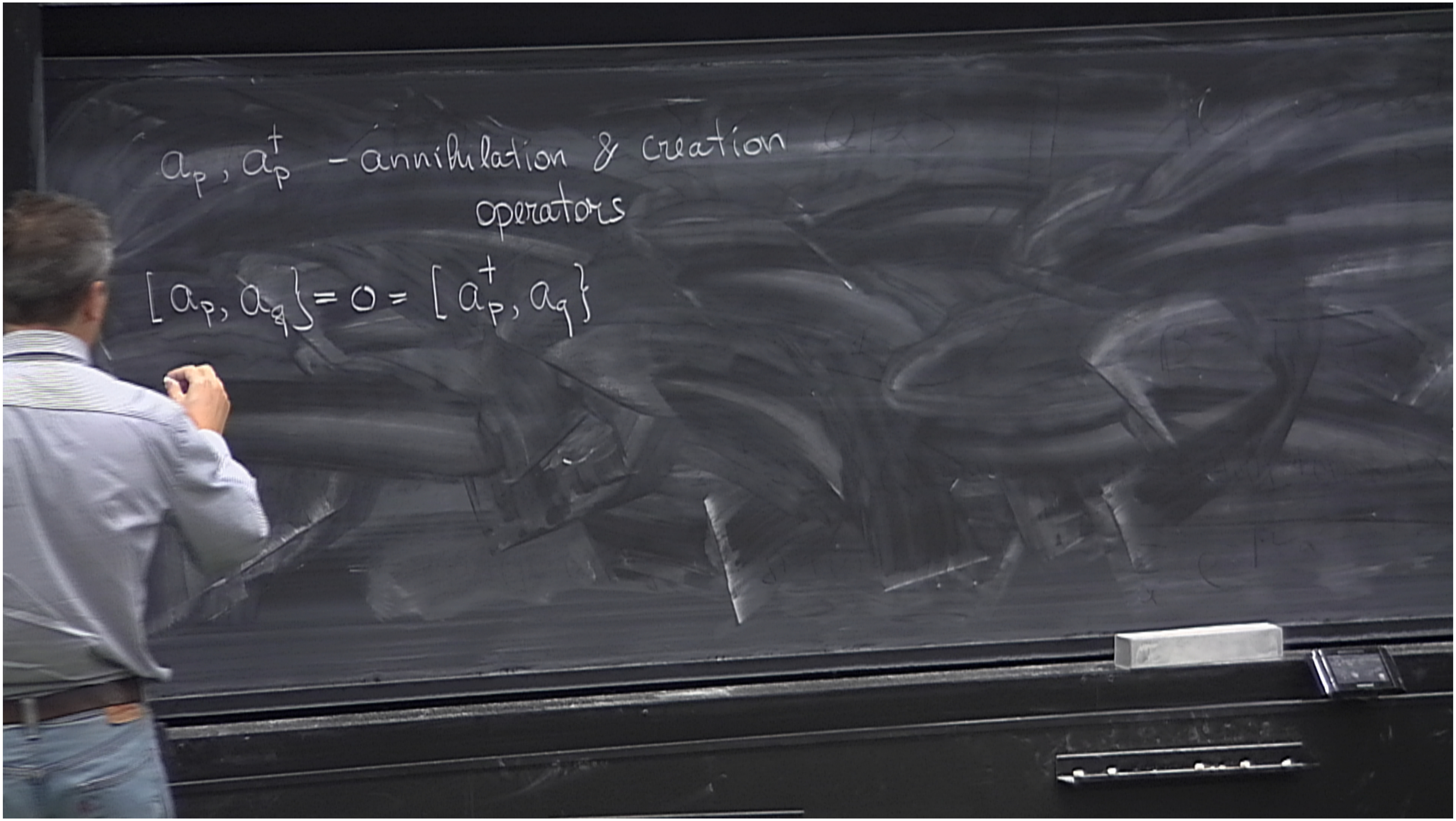
Title: Quantum Field Theory I - Lecture 2

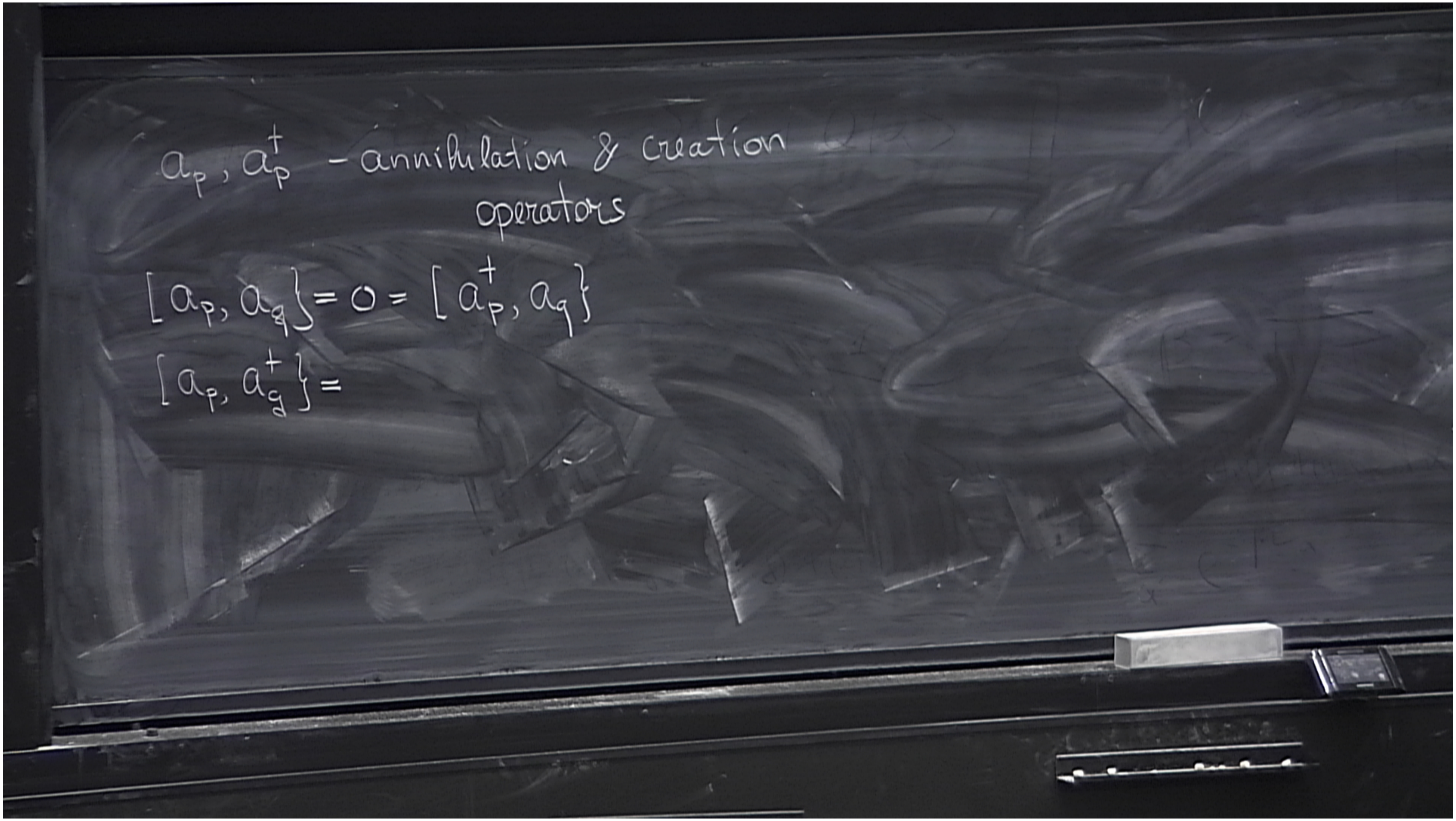
Date: Oct 04, 2011 09:00 AM

URL: http://pirsa.org/11100007

Abstract:


a_p , a_p^† - annihilation


a_p, $a_p^\dagger$ - annihilation & creation operators
$[a_p, a_q\} = 0 = [a_p^\dagger, a_q\}$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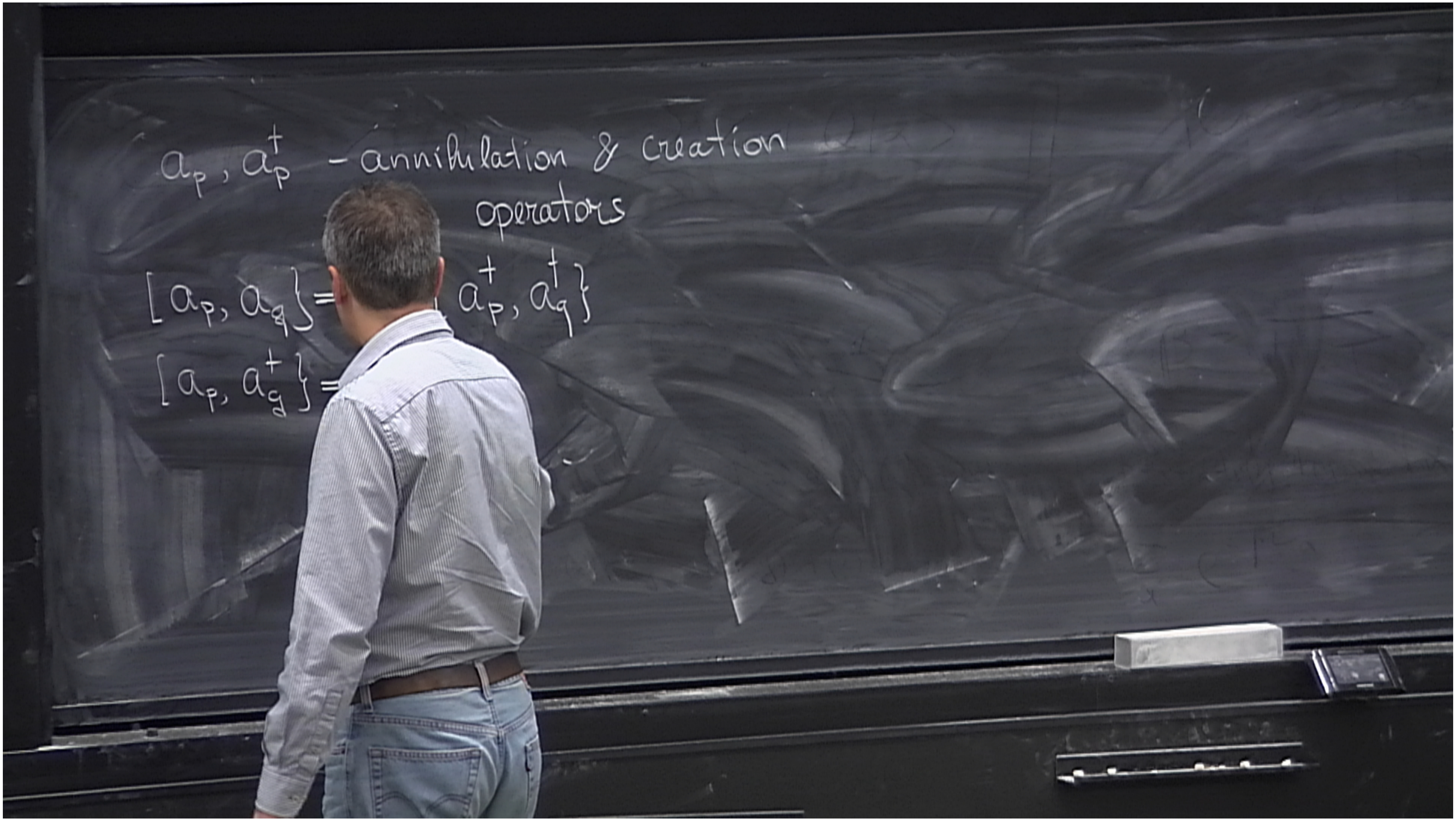
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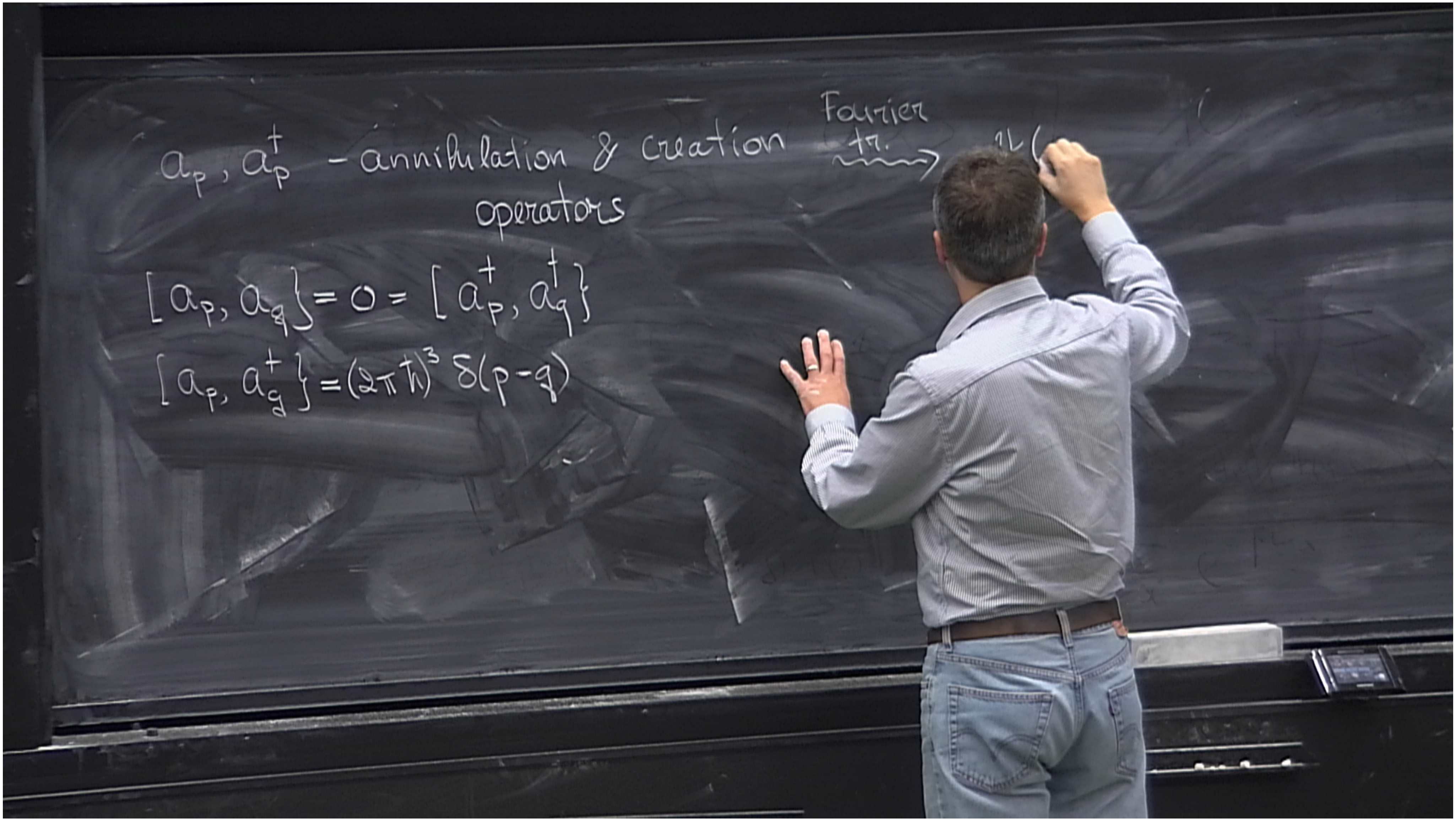
$a_p, a_p^\dagger$ - annihilation & creation
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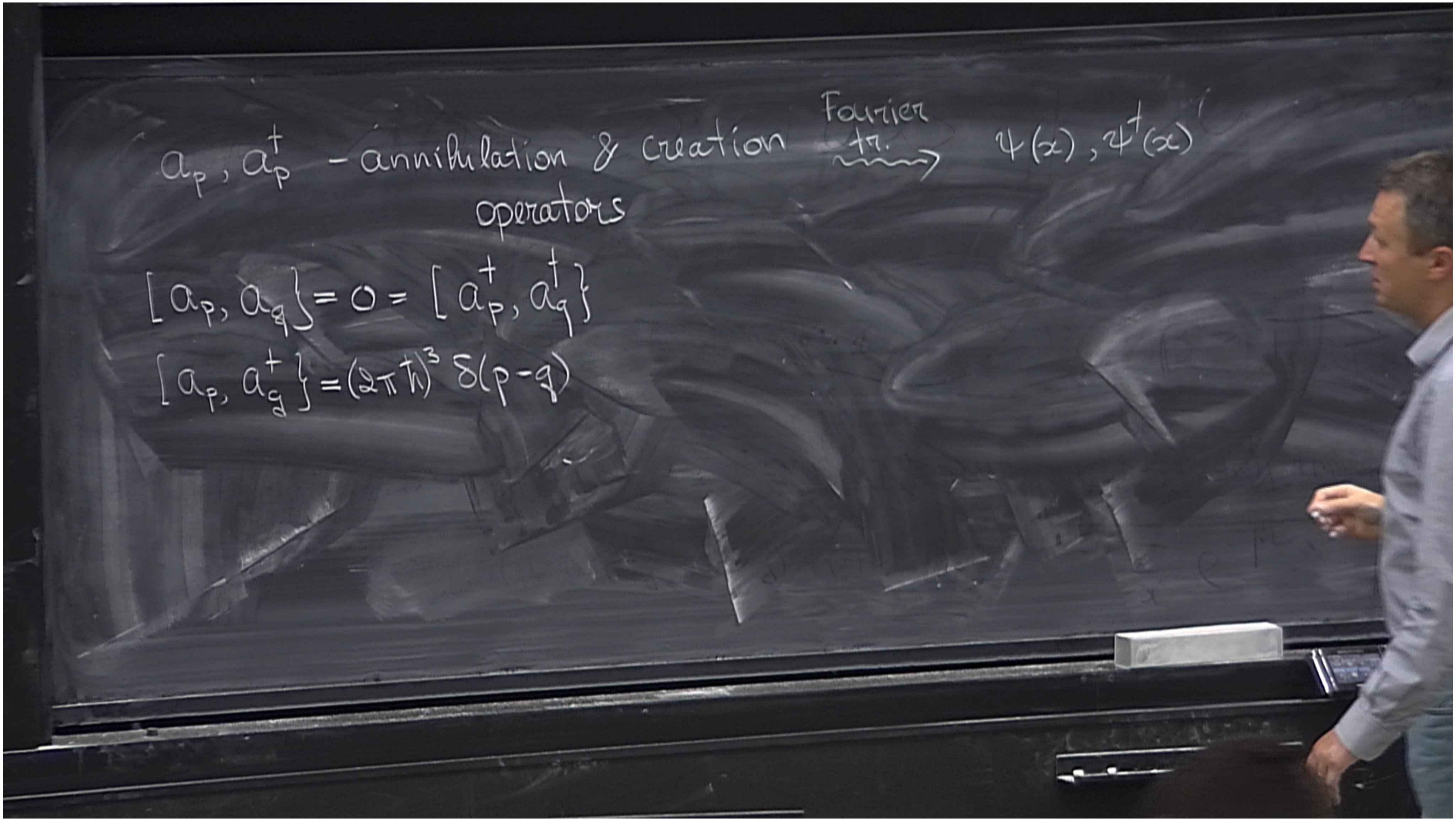
Fourier
tr.
$[a_p, a_q\} = 0 = [a_p^\dagger, a_q^\dagger\}$
$[a_p, a_q^\dagger\} = (2\pi\hbar)^3 \delta(p-q)$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$\psi(x), \psi^\dagger(x)$
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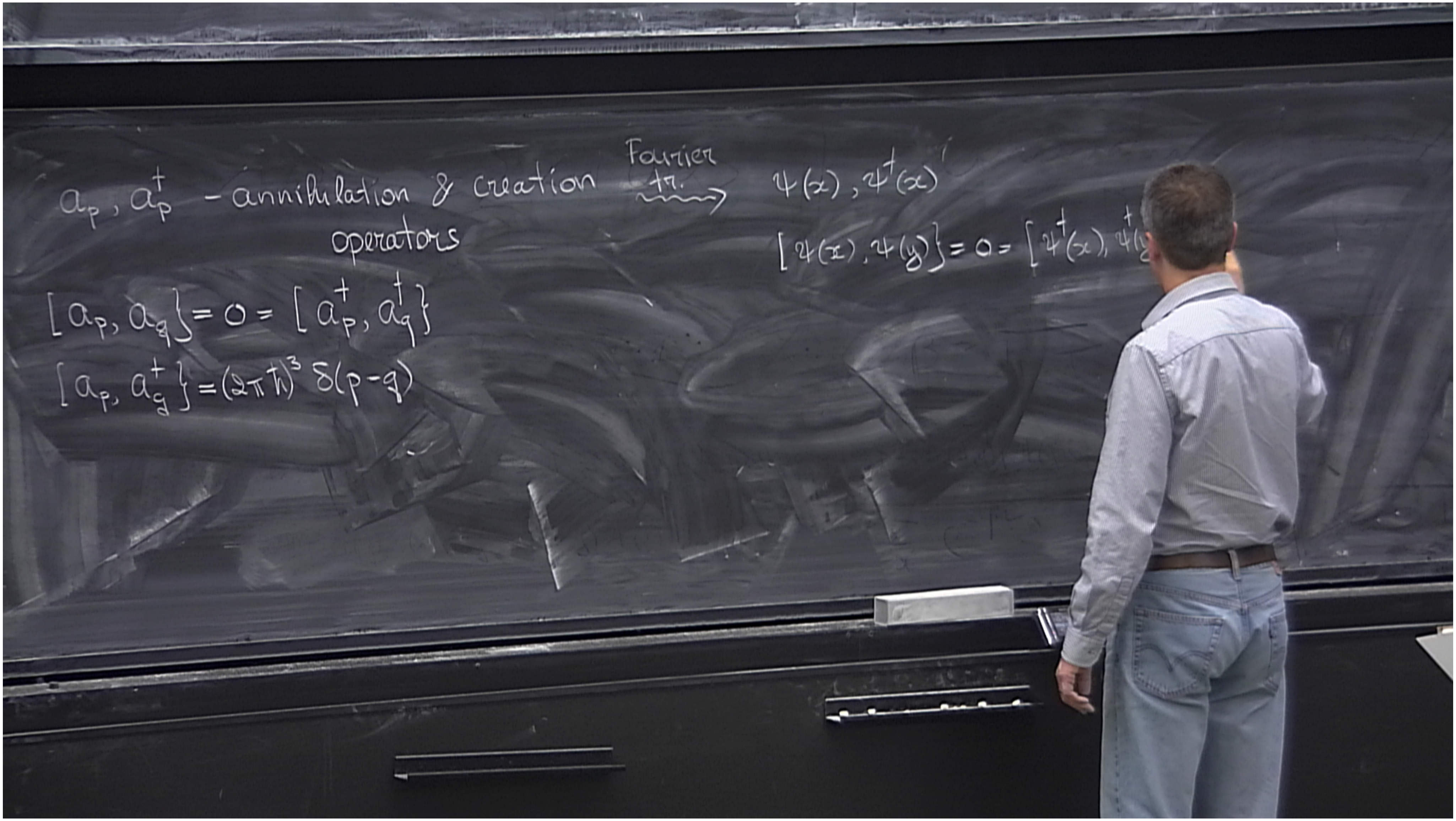
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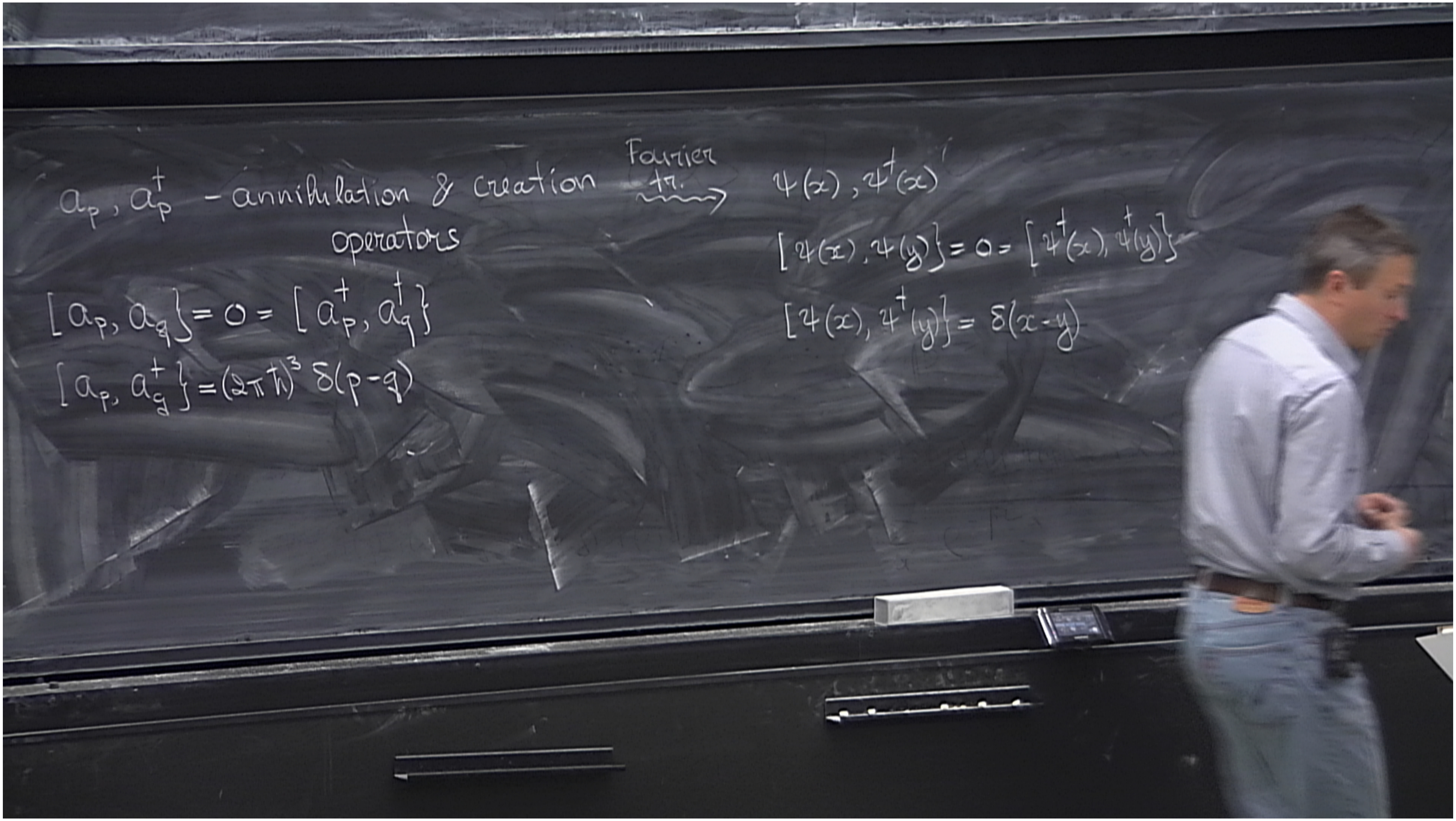
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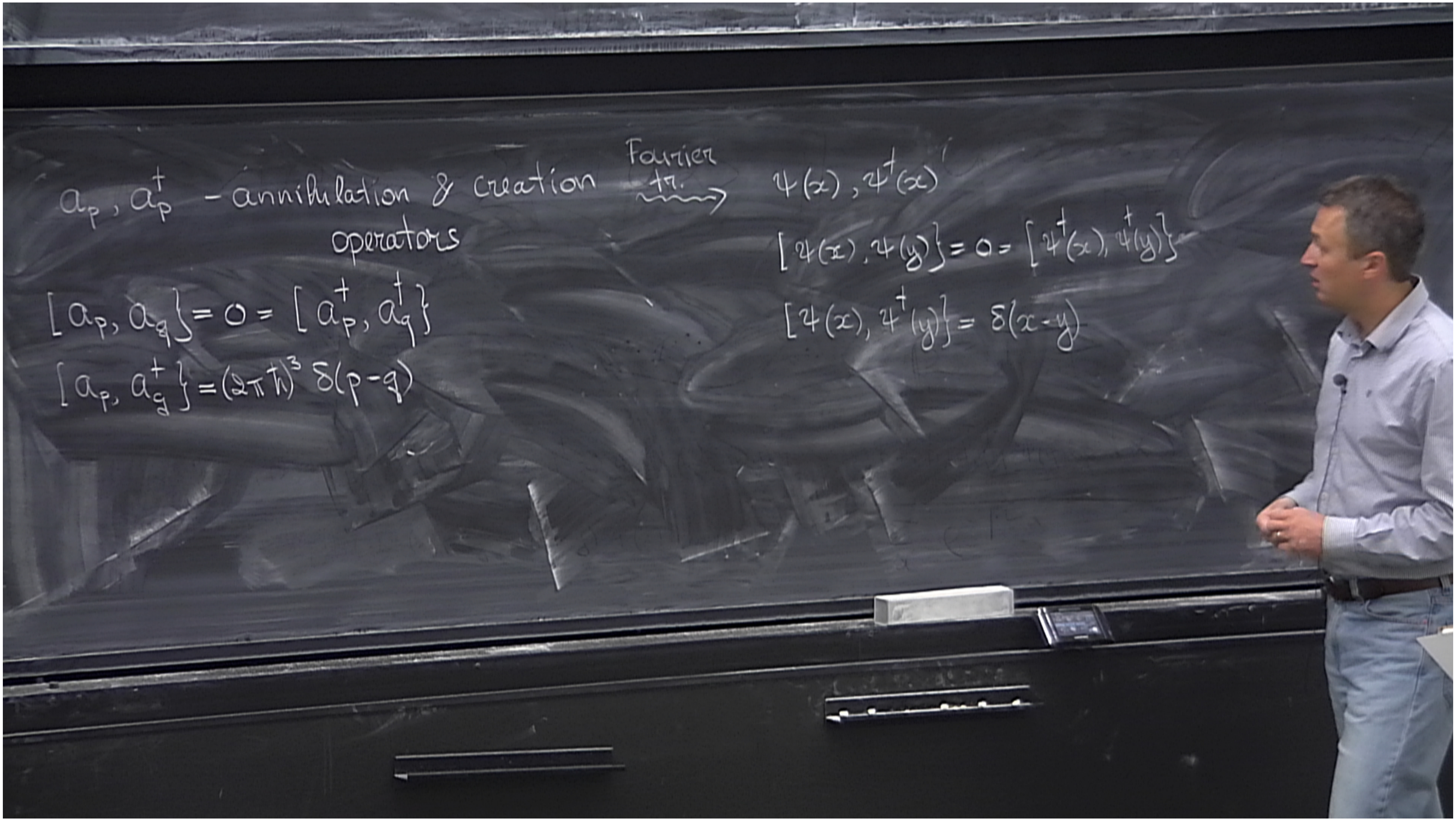
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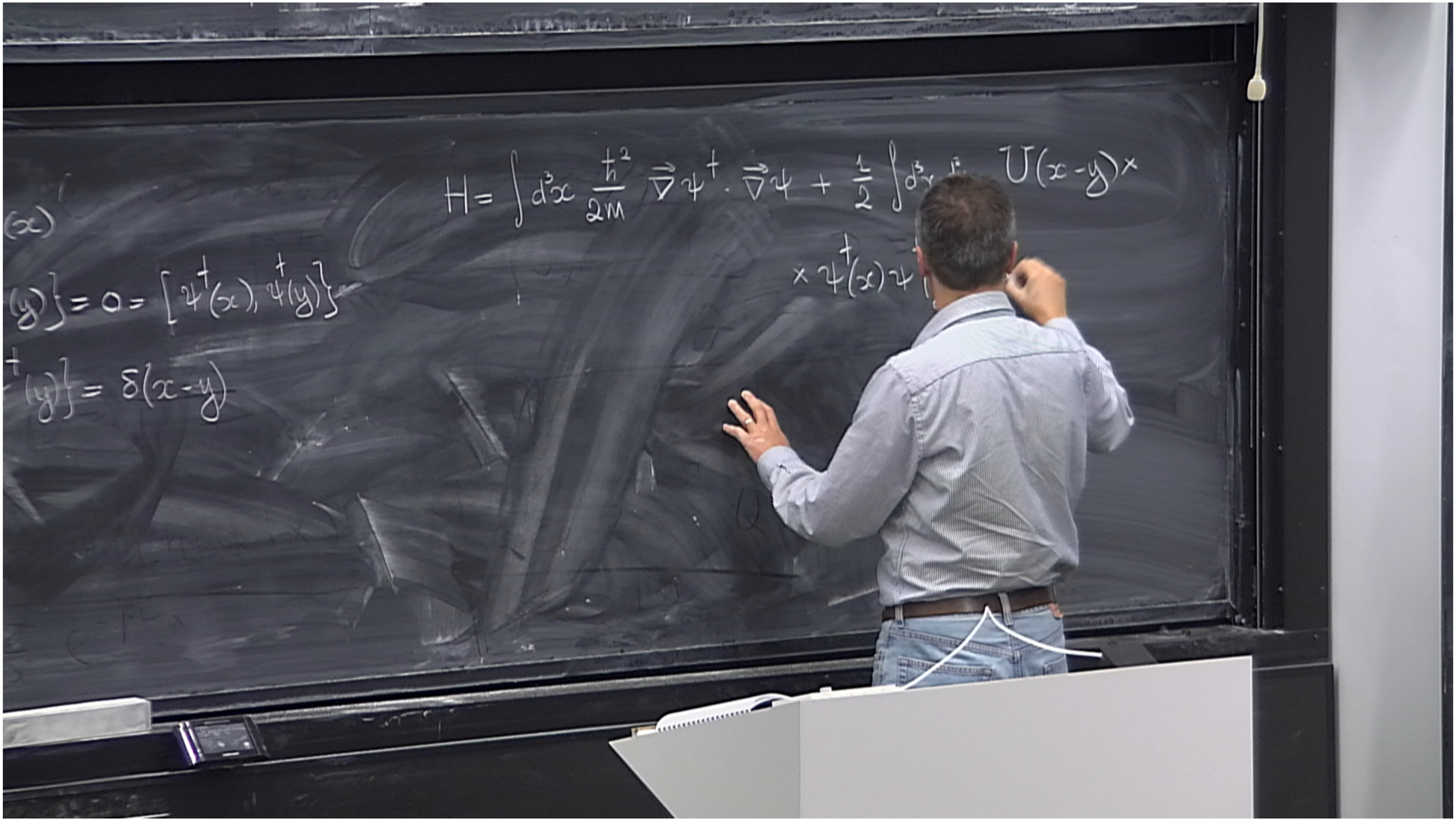
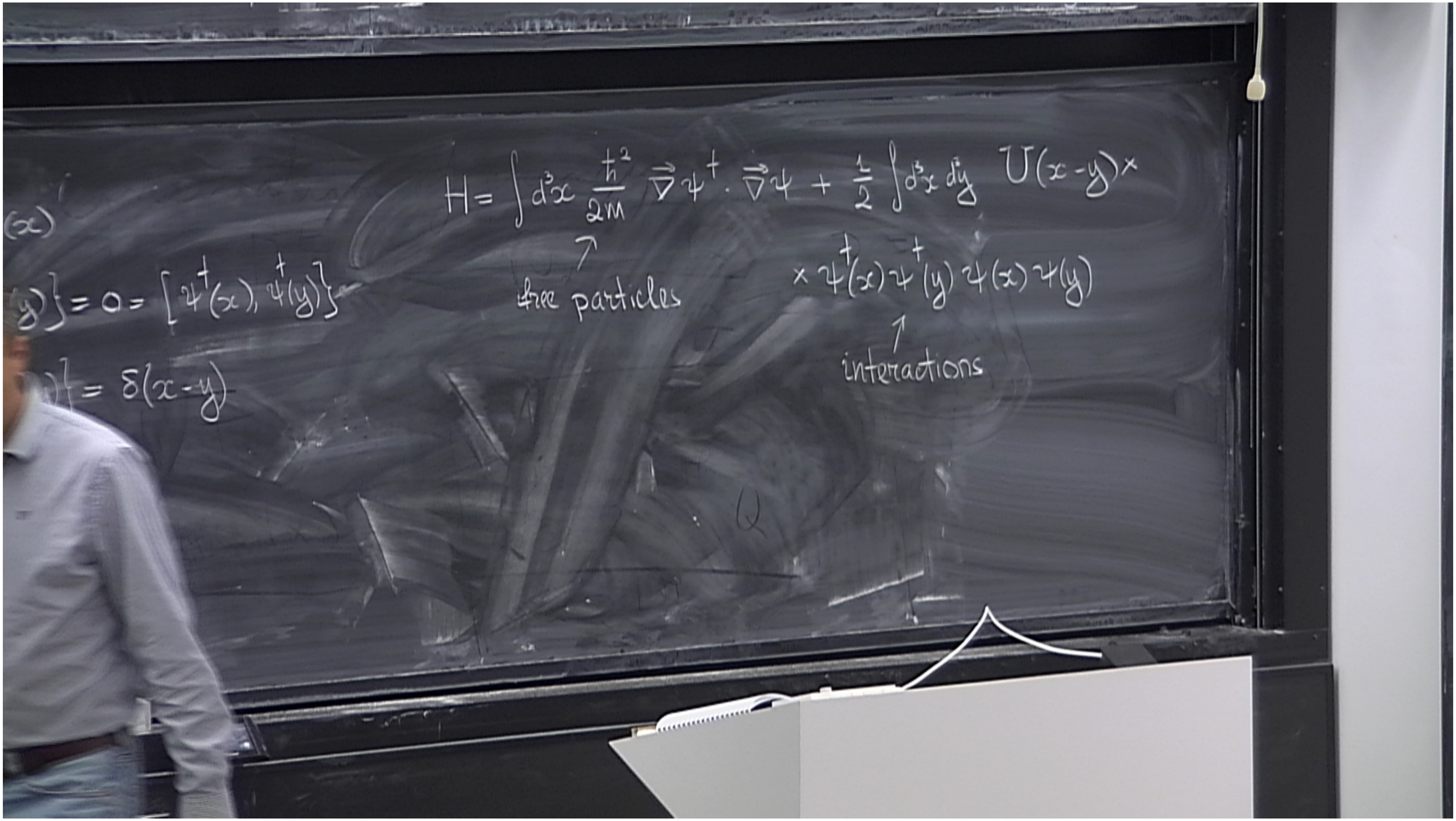

H = ∫d³x ħ²/2m ∇ψ† · ∇ψ + 1/2 ∫d³x ... U(x-y)×
× ψ†(x)ψ†
(x)
(y)} = 0 = [ψ†(x), ψ†(y)]
†(y)} = δ(x-y)


H = ∫d³x ħ²/2m ∇ψ† · ∇ψ + 1/2 ∫d³x d³y U(x − y) ×
× ψ†(x) ψ†(y) ψ(x) ψ(y)
free particles
interactions
(x)
y)} = 0 = [ψ†(x), ψ†(y)}
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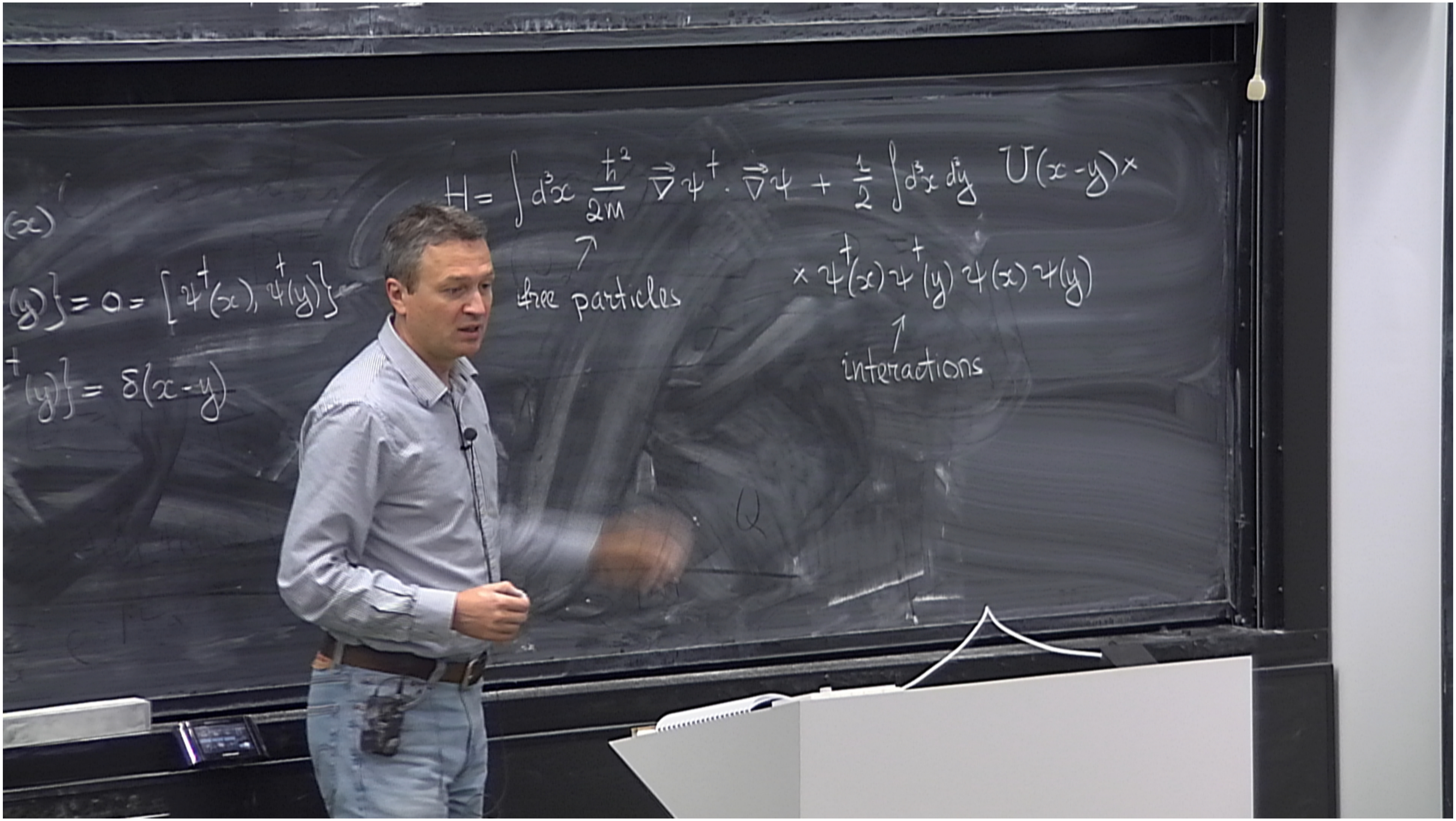

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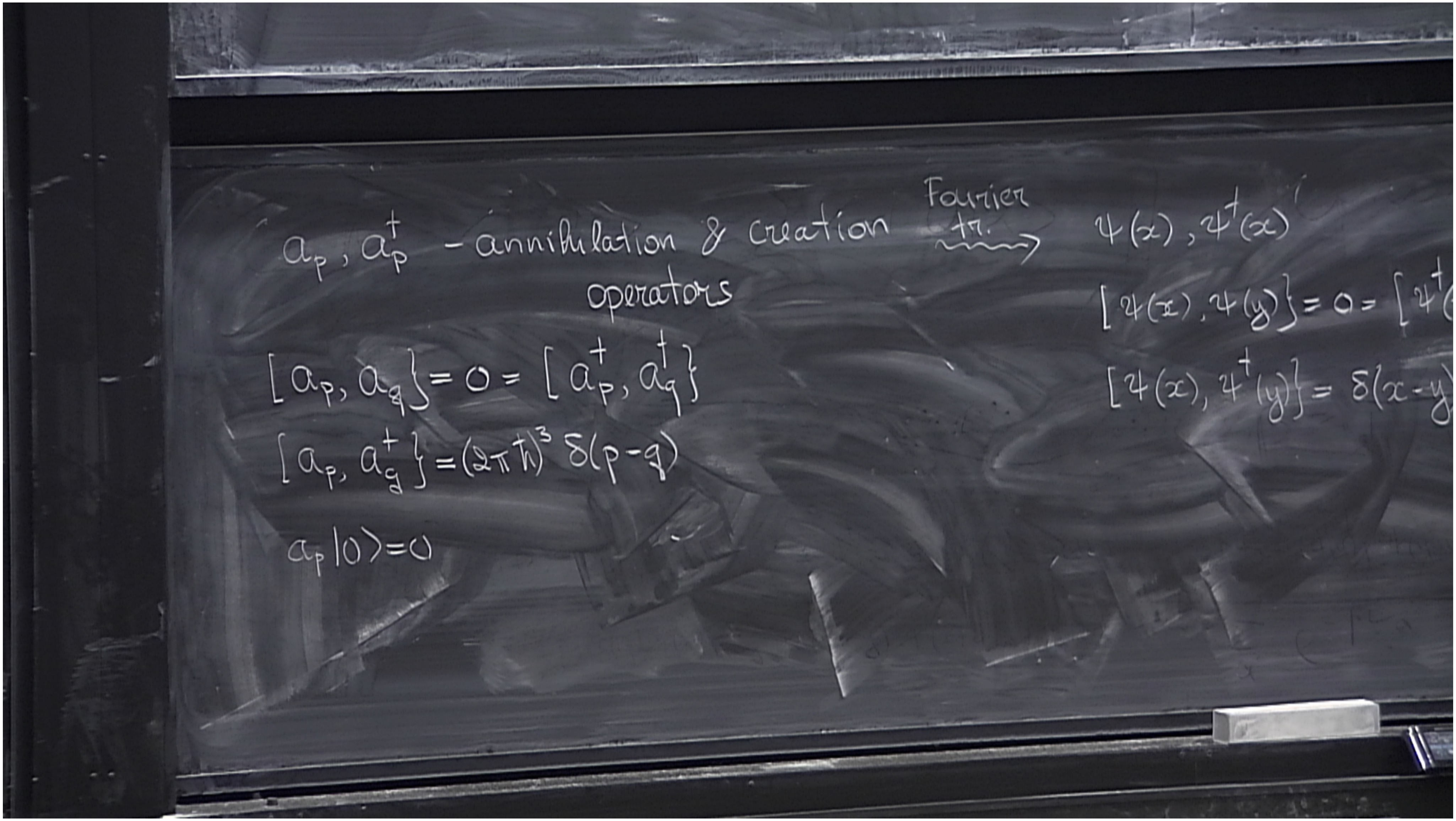
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[a_p, a_q^†} = (2πħ)^3 δ(p-q)
a_p|0⟩=0
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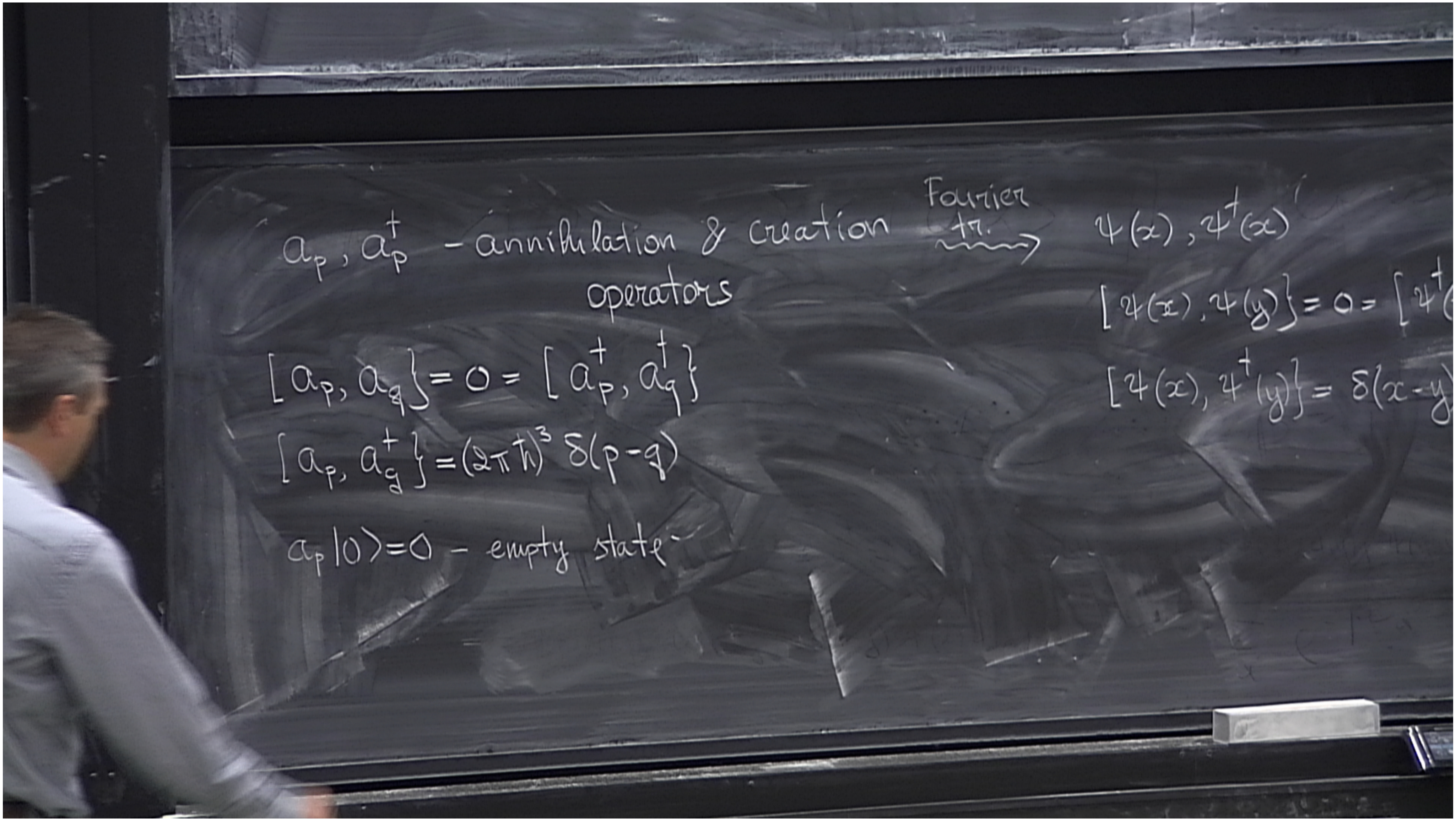
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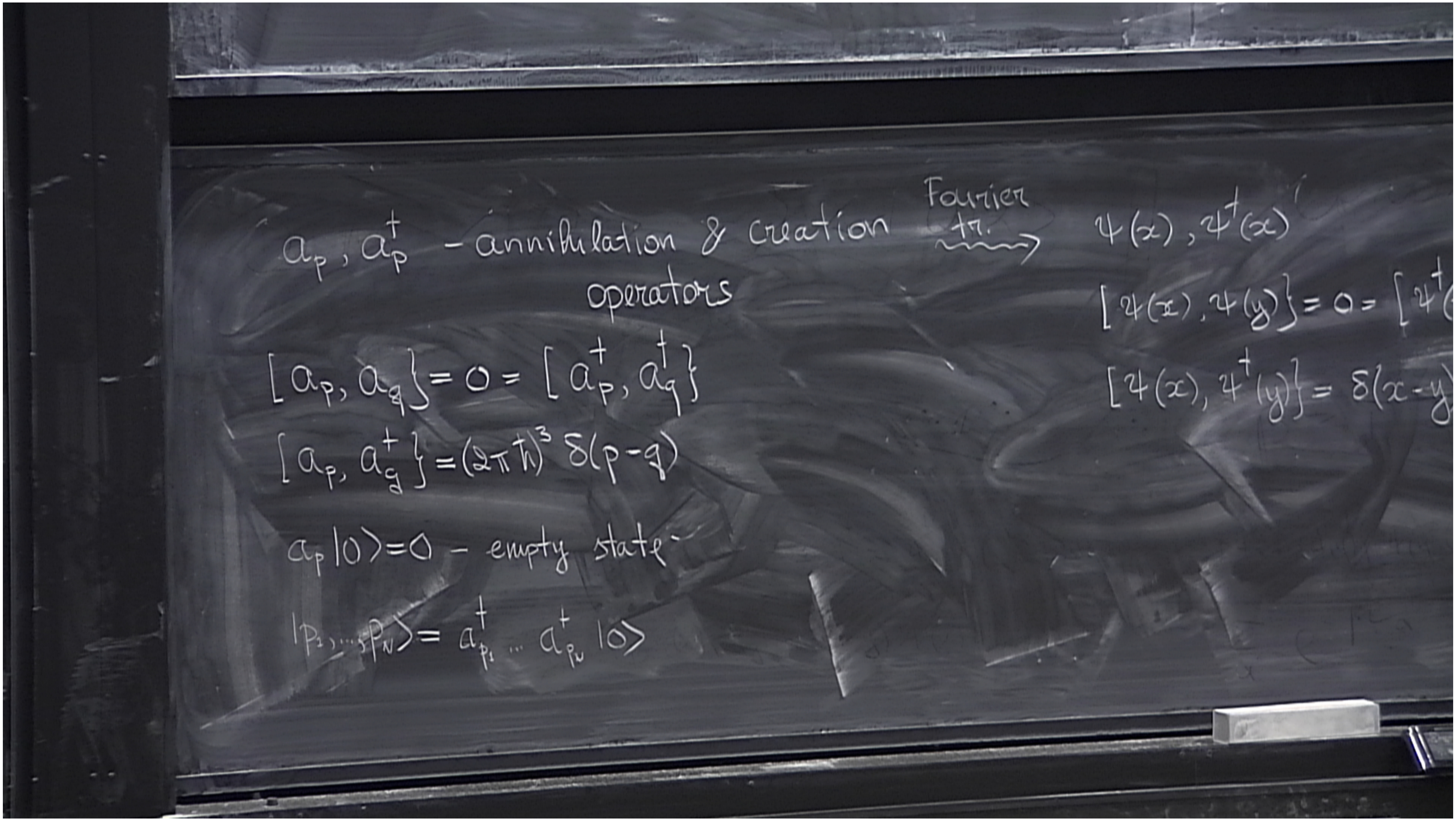
$a_p|0\rangle = 0$ - empty state.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$|p_1, \ldots, p_N\rangle = a_{p_1}^\dagger \ldots a_{p_N}^\dagger |0\rangle$
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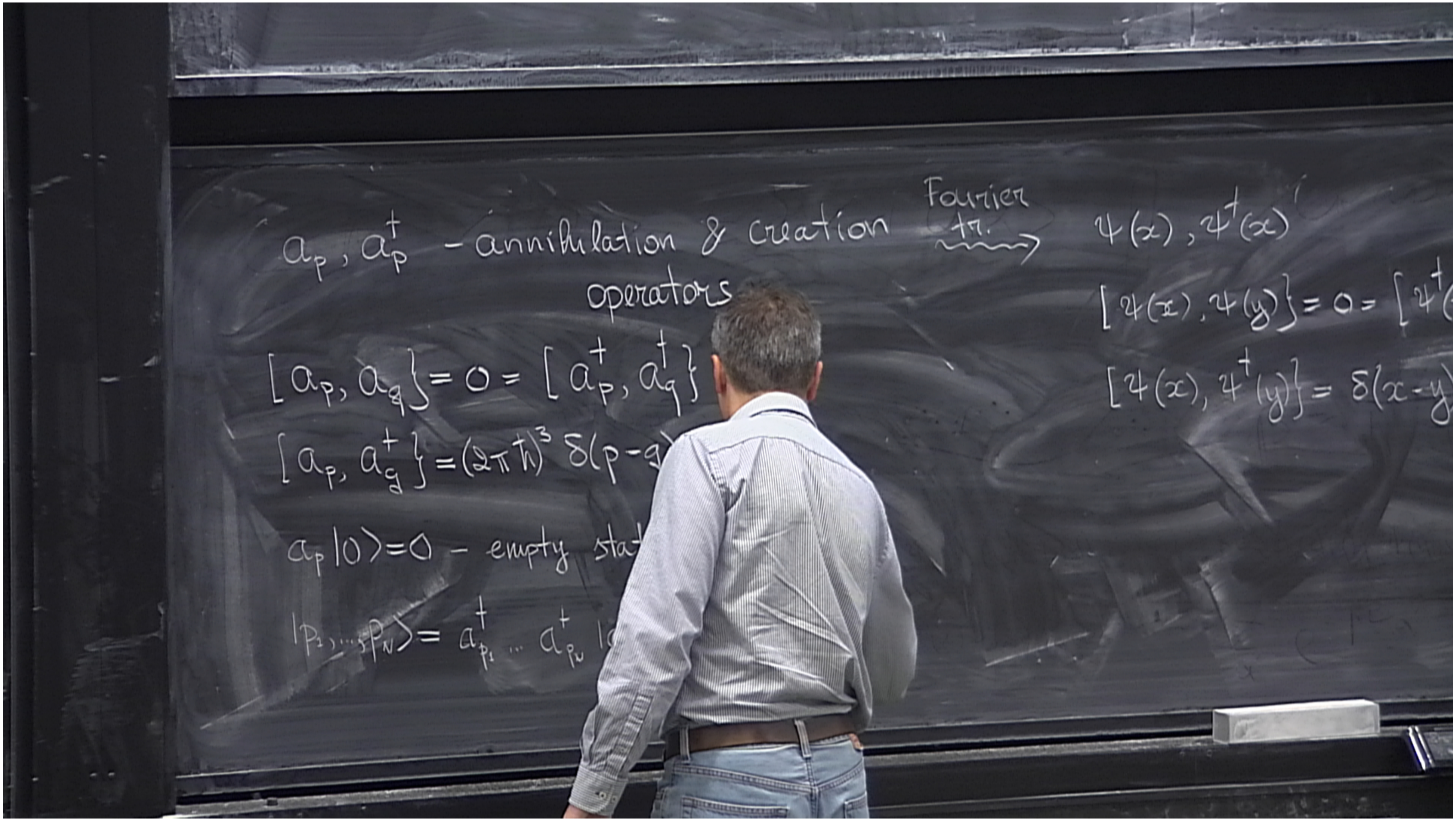

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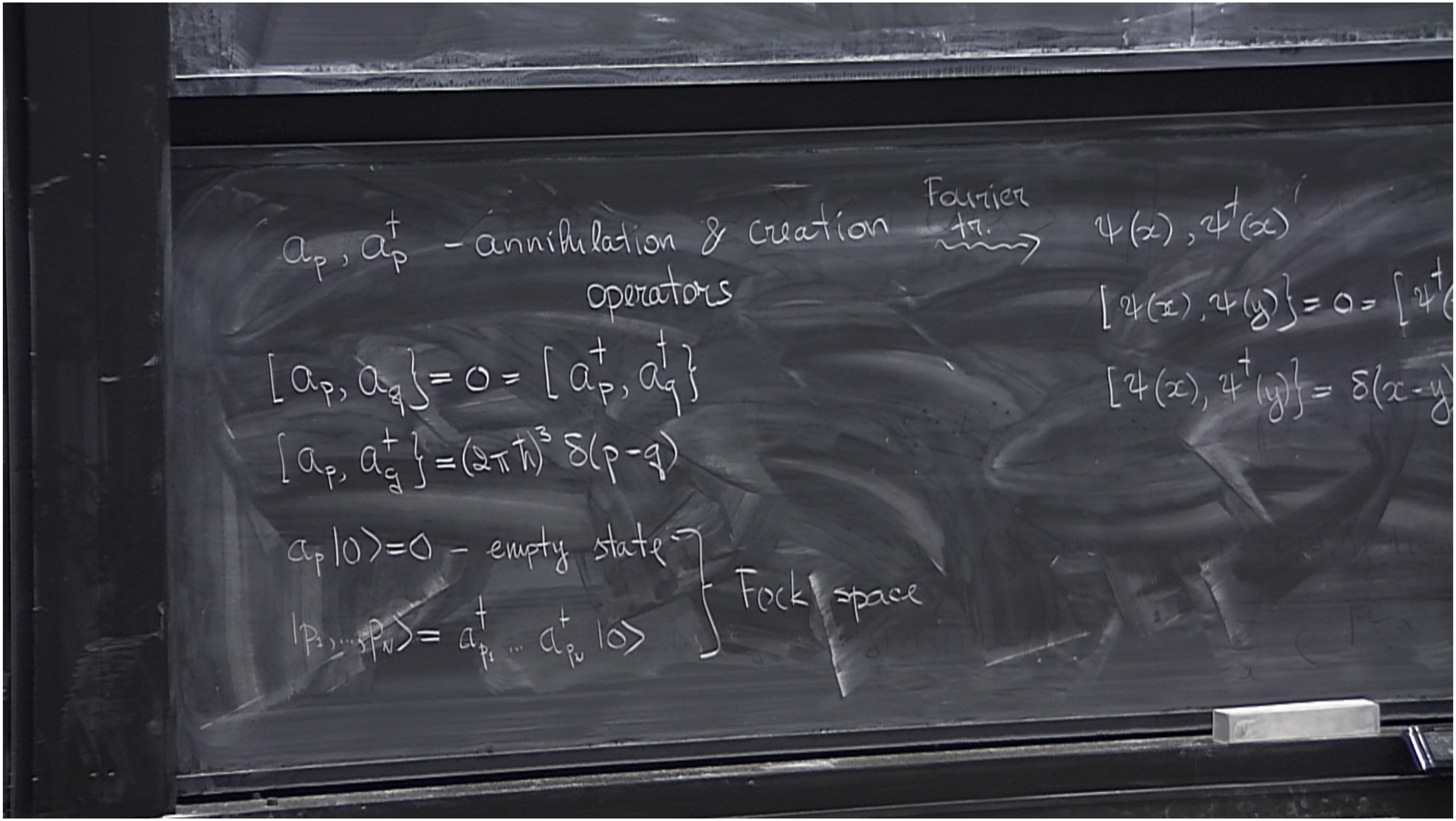
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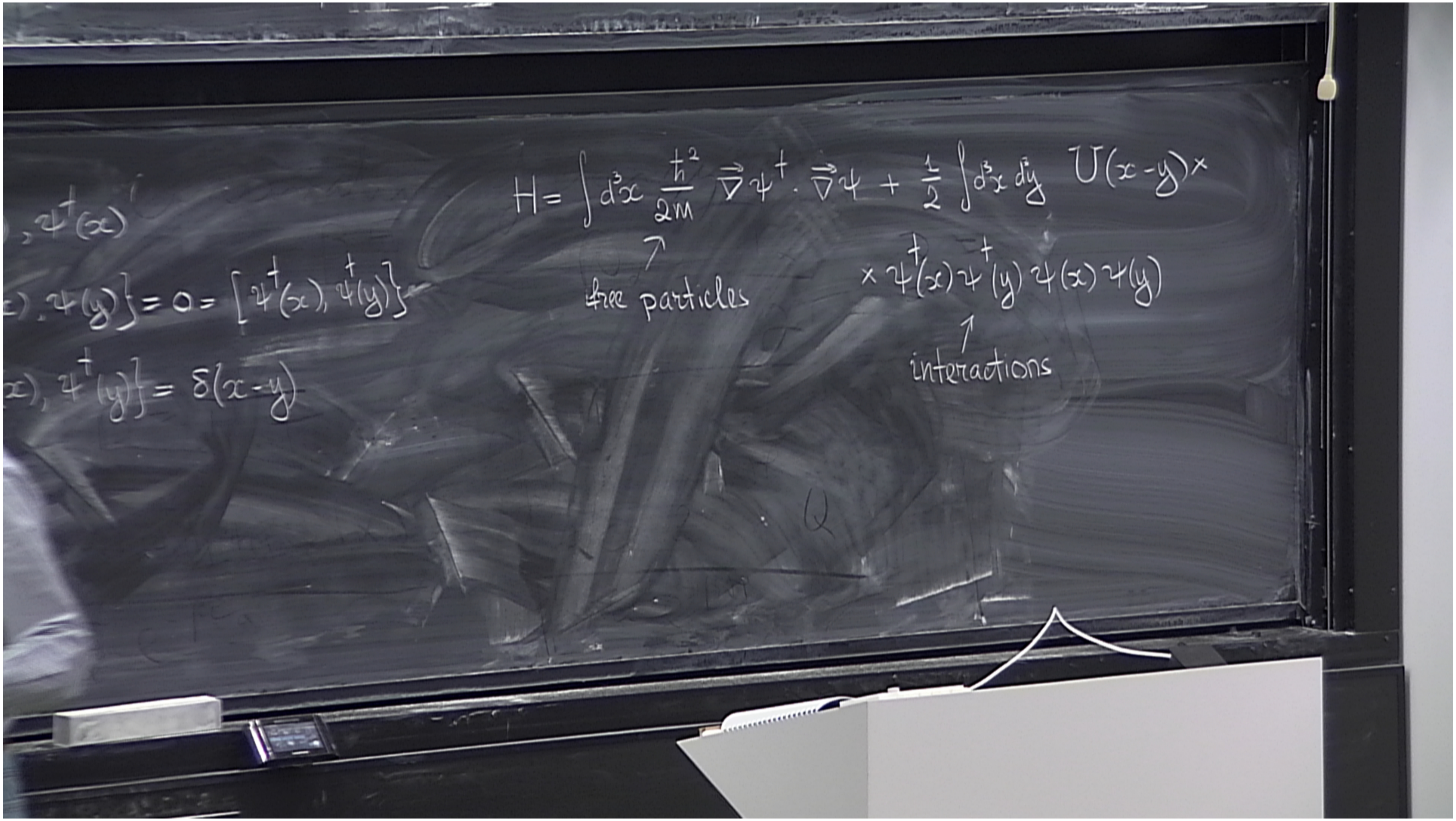
$|p_1, \dots, p_N\rangle = a_{p_1}^\dagger \dots a_{p_N}^\dagger |0\rangle$
Fock space


$H = \int d^3x \frac{\hbar^2}{2m} \vec{\nabla}\psi^\dagger \cdot \vec{\nabla}\psi + \frac{1}{2}\int d^3x\, d^3y\, U(x-y) \times$
$\times \psi^\dagger(x)\psi^\dagger(y)\psi(x)\psi(y)$
free particles
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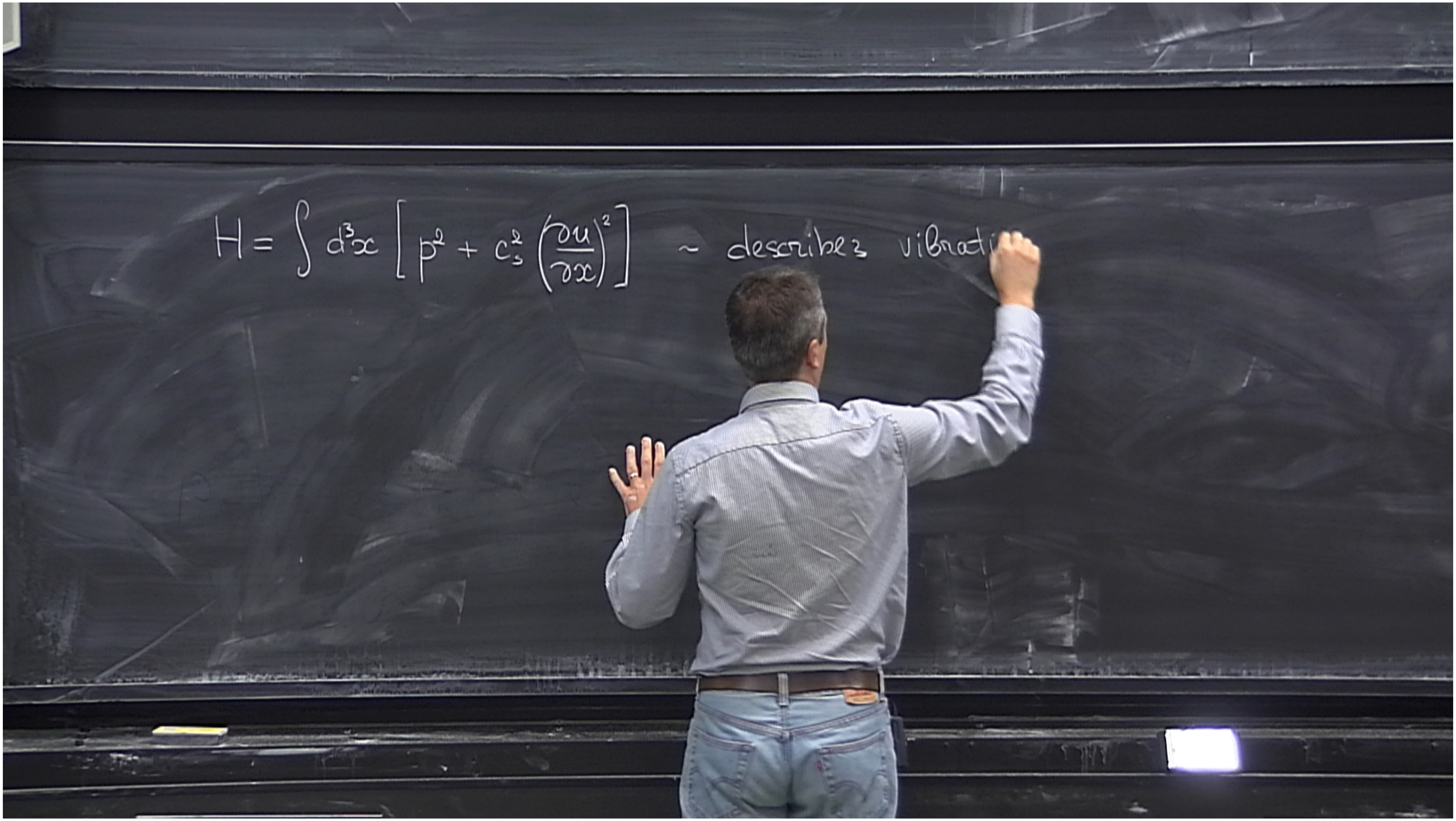
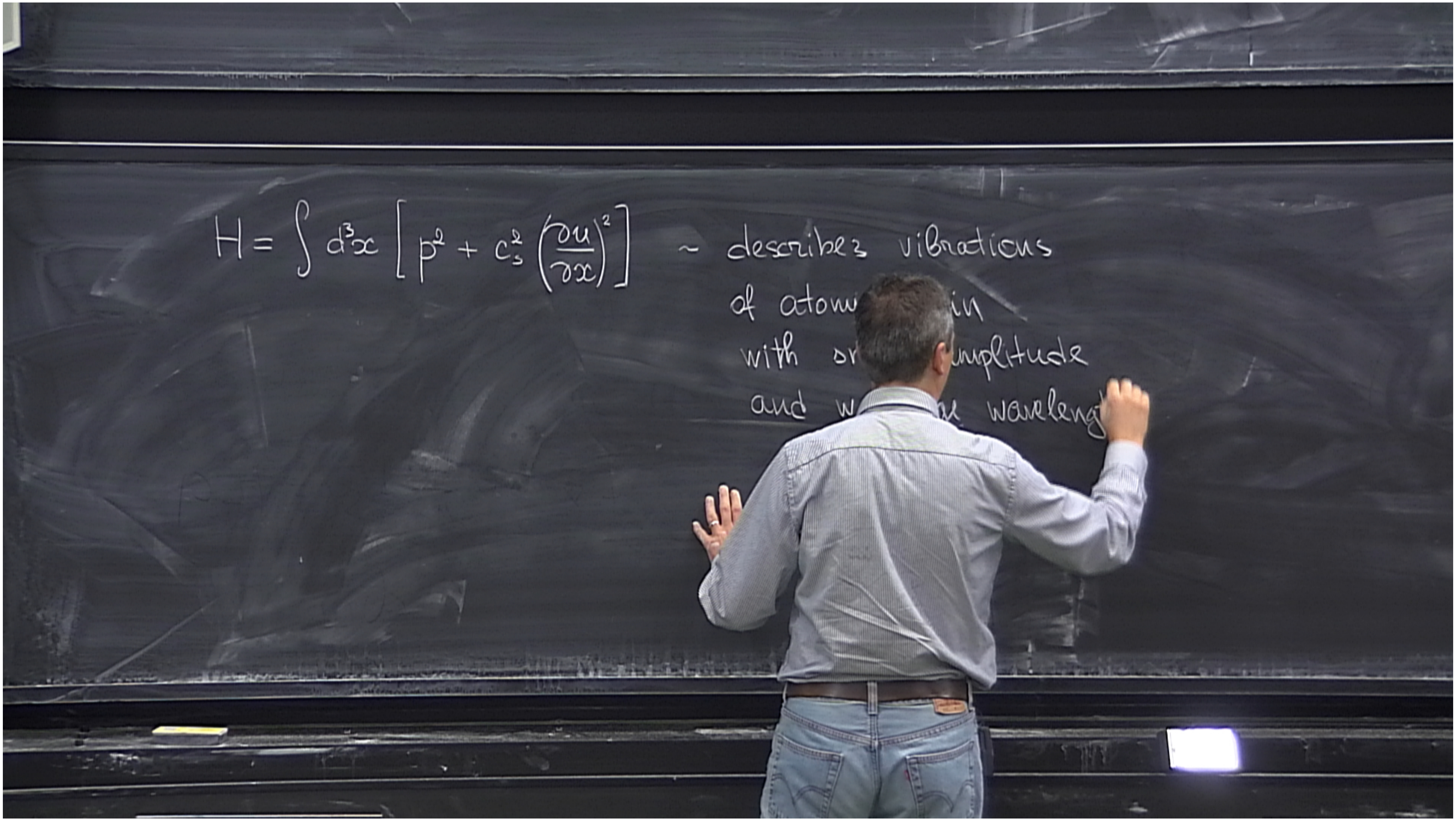

$H = \int d^3x \left[p^2 + c_s^2 \left(\frac{\partial u}{\partial x}\right)^2 \right]$ ~ describes vibrati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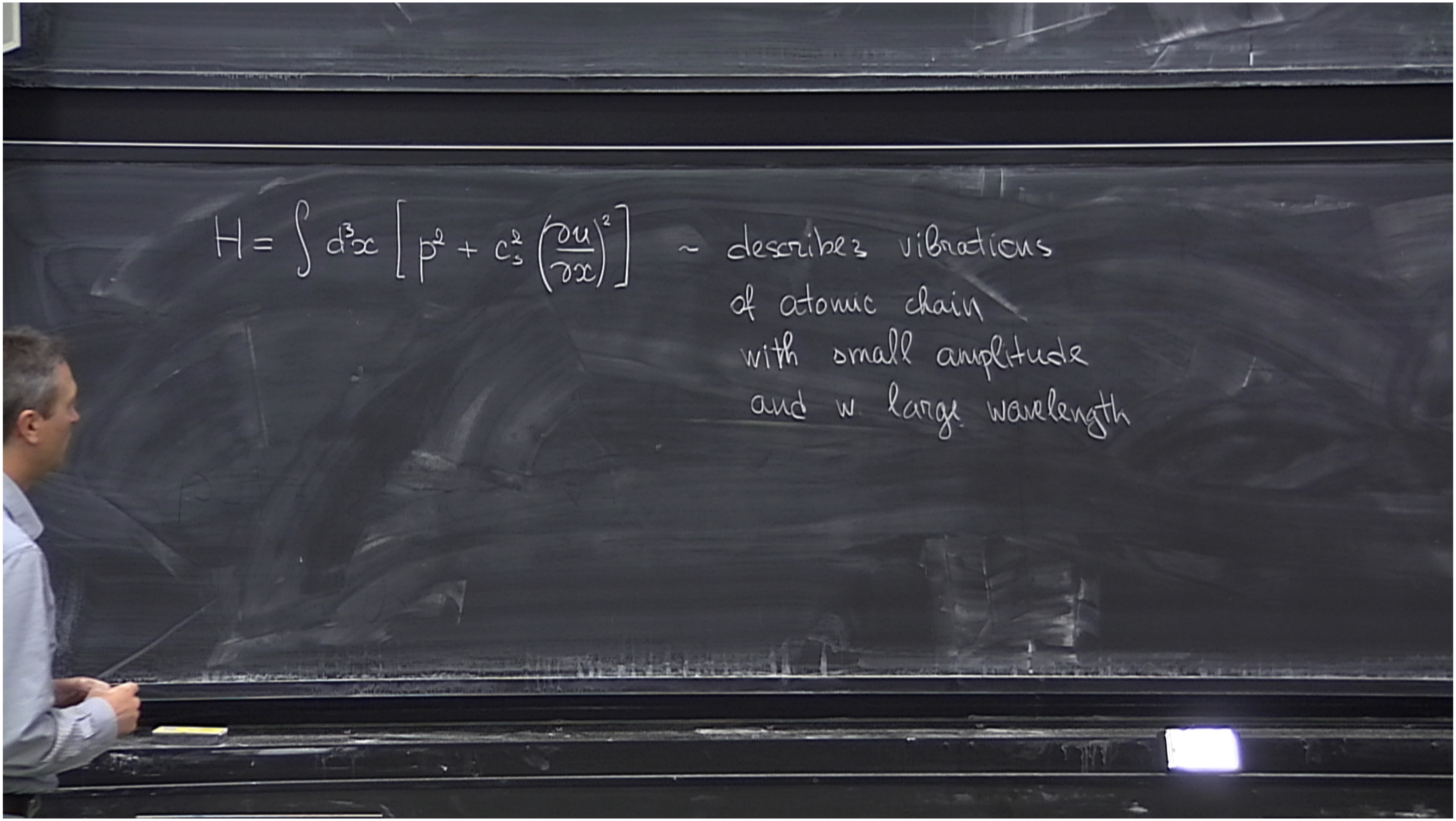
~ describes vibrations
with
and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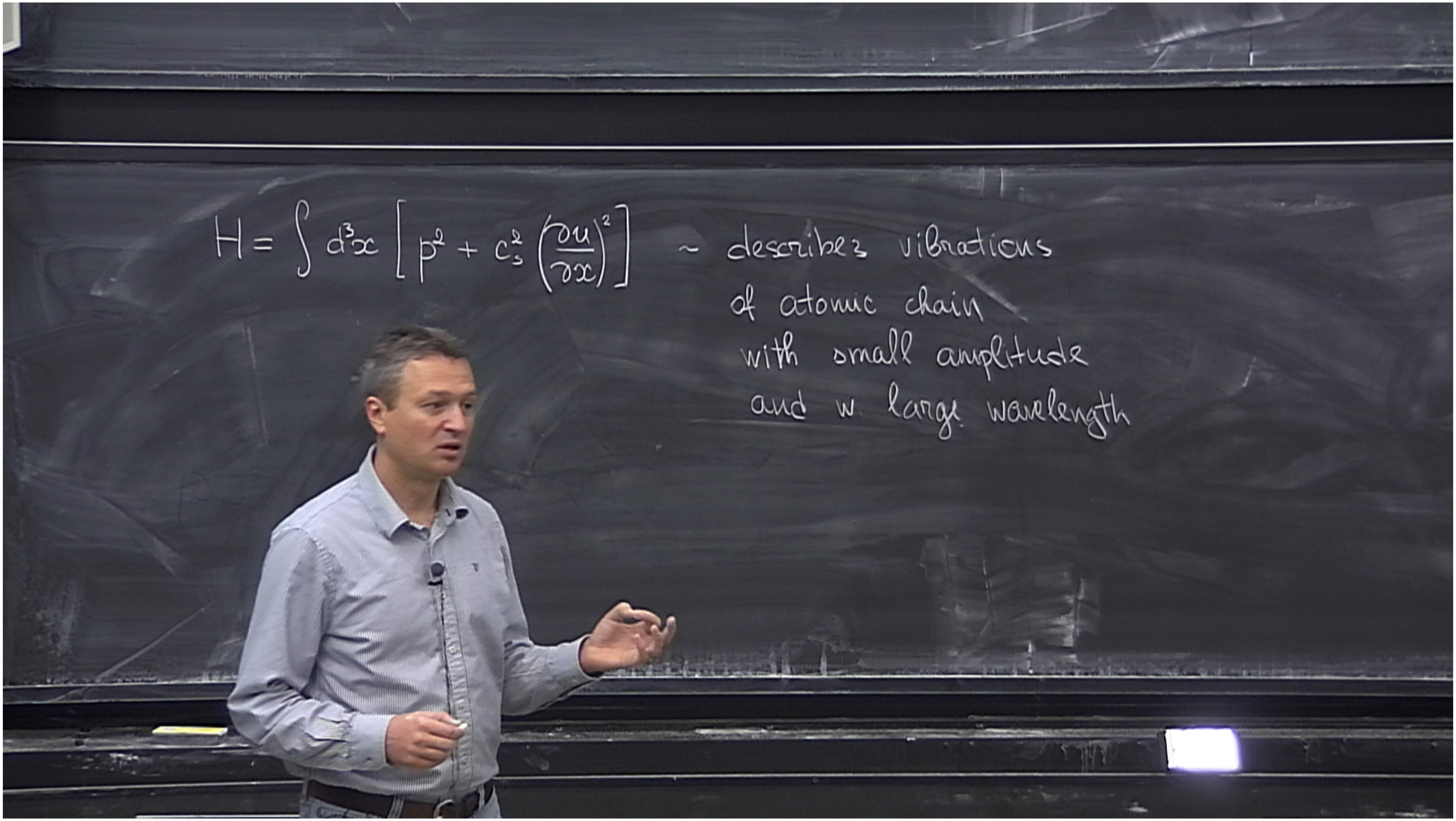
of atomic chain
with small amplitude
and w. large wavelength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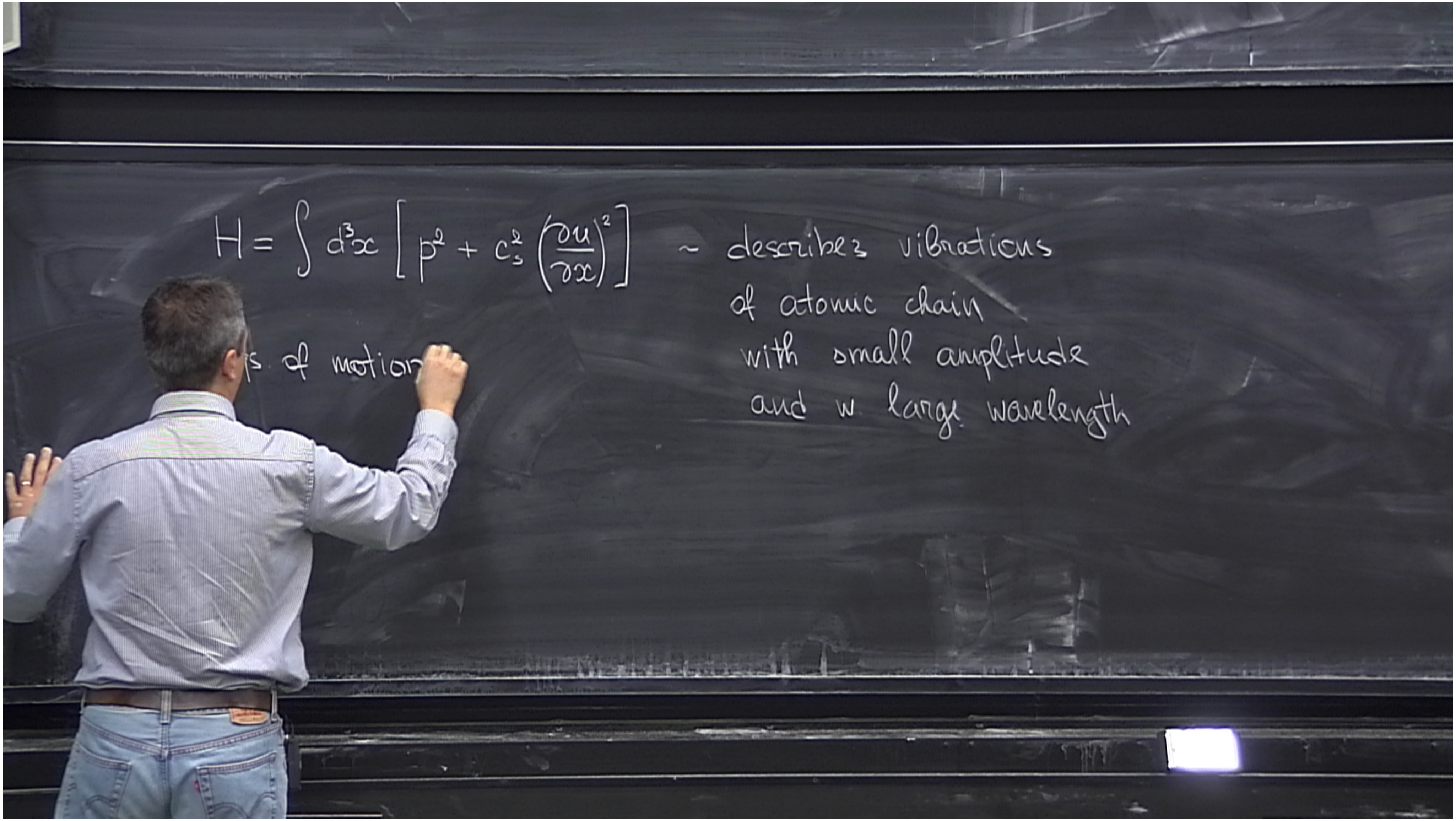
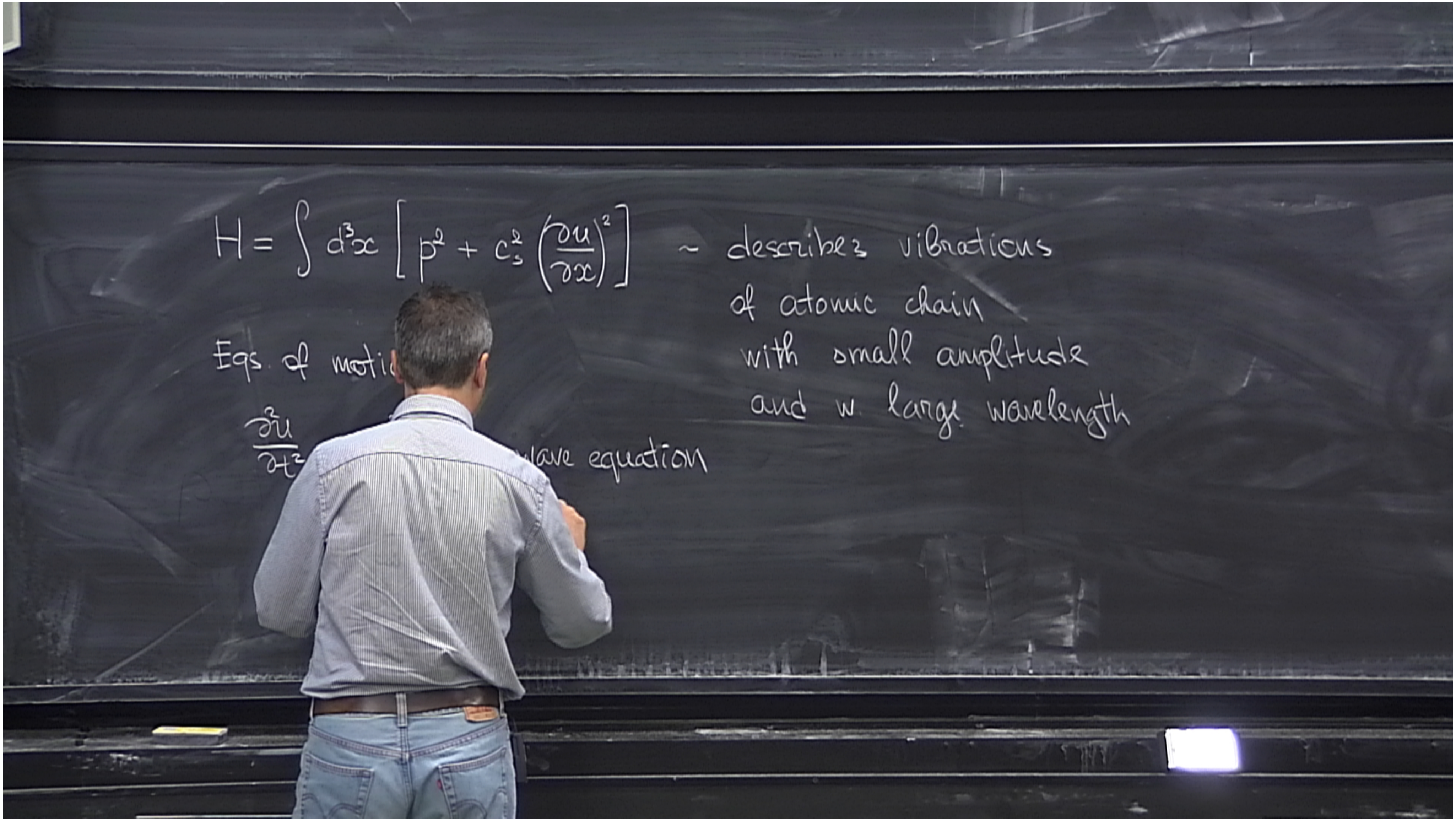

H = ∫ d³x [p² + c_s² (∂u/∂x)²]
~ describes vibrations
of atomic chain
with small amplitude
and w. large wavelength
s. of motion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Eqs. of moti
wave equation

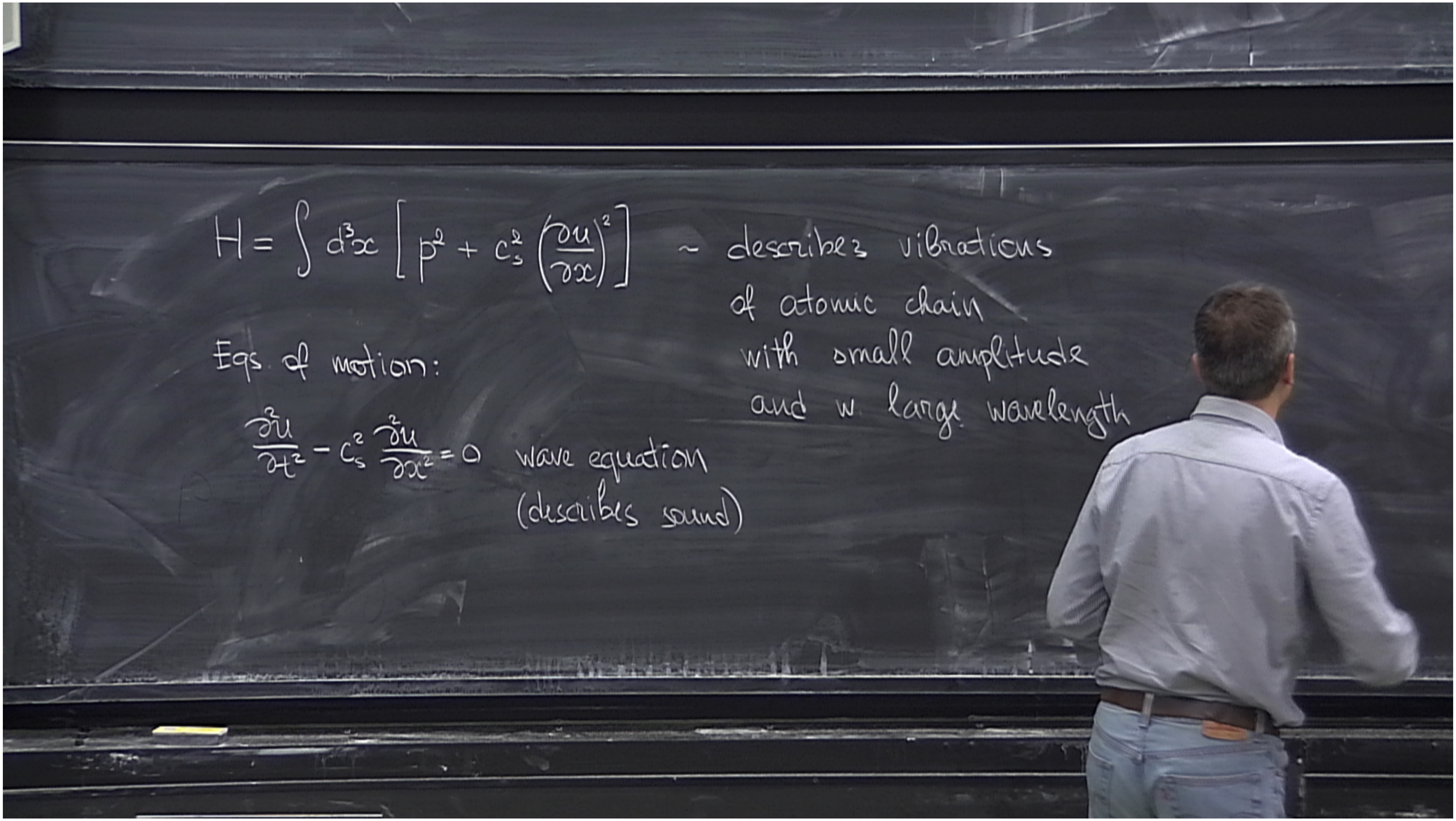
$$H = \int d^3x \left[p^2 + c_s^2 \left(\frac{\partial u}{\partial x} \right)^2 \right]$$

~ describes vibrations of atomic chain with small amplitude and w. large wavelength

Eqs. of motion:

$$\frac{\partial^2 u}{\partial t^2} - c_s^2 \frac{\partial^2 u}{\partial x^2} = 0$$

wave equation (describes sound)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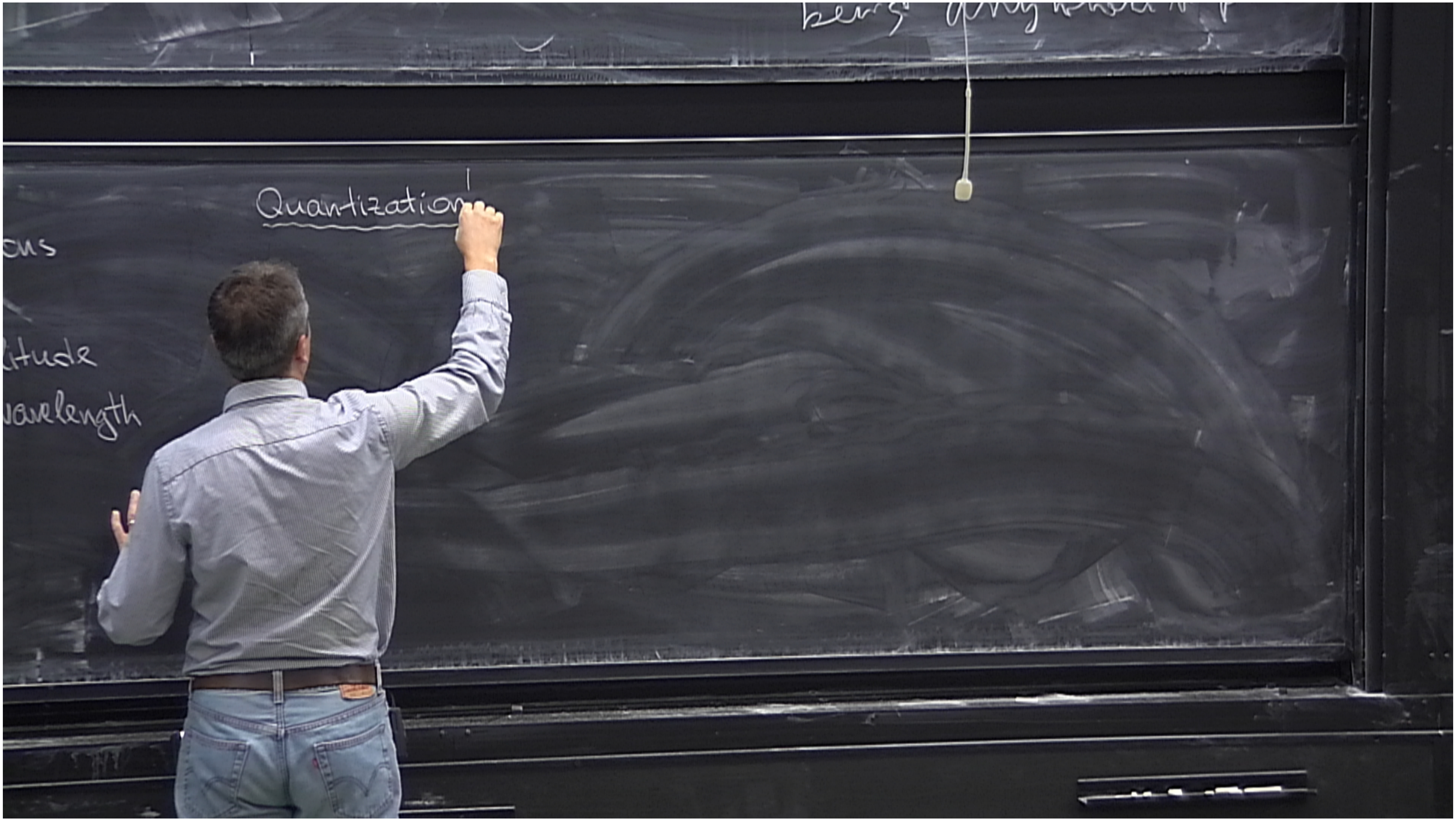
Quantization
ons
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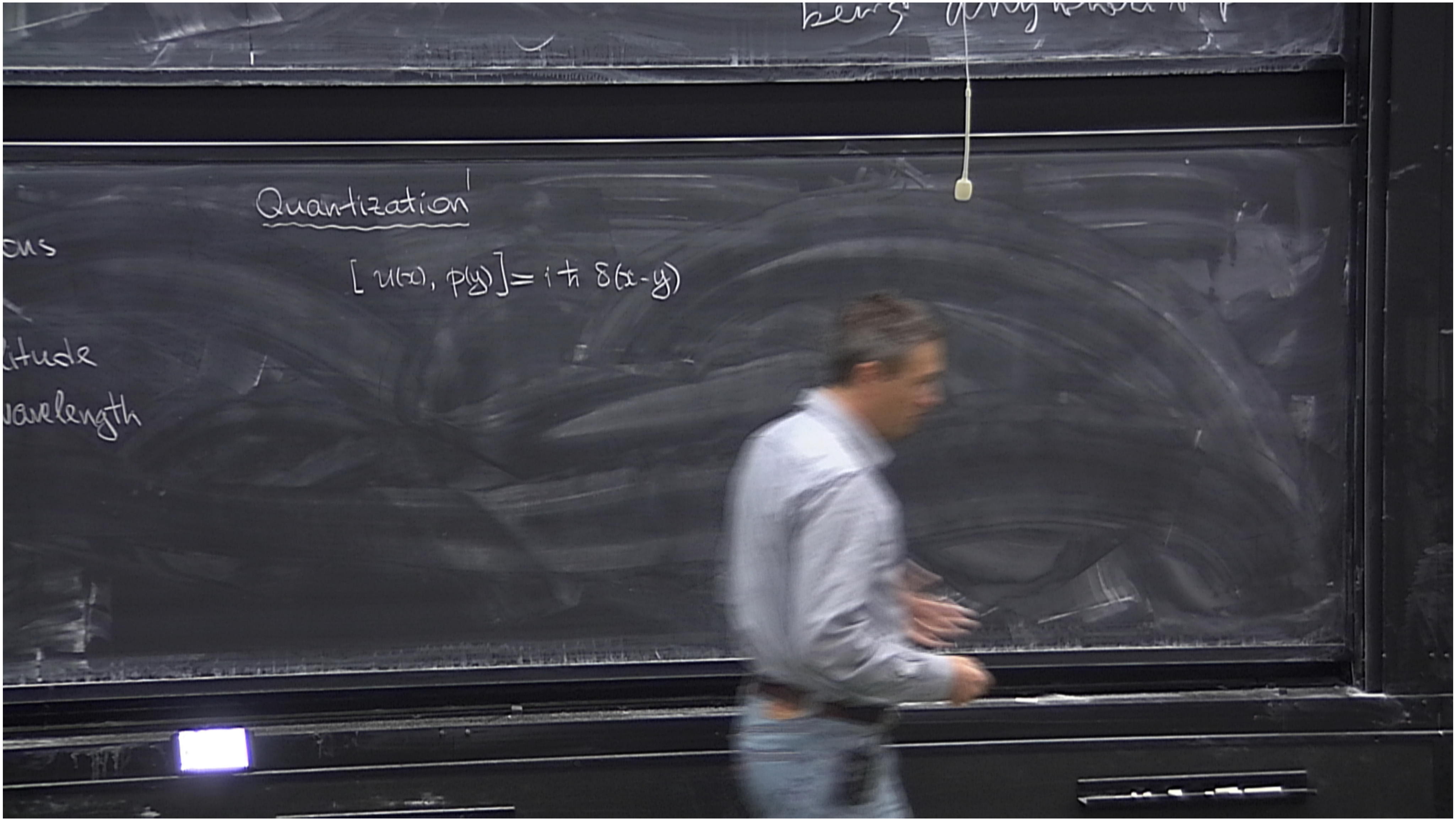
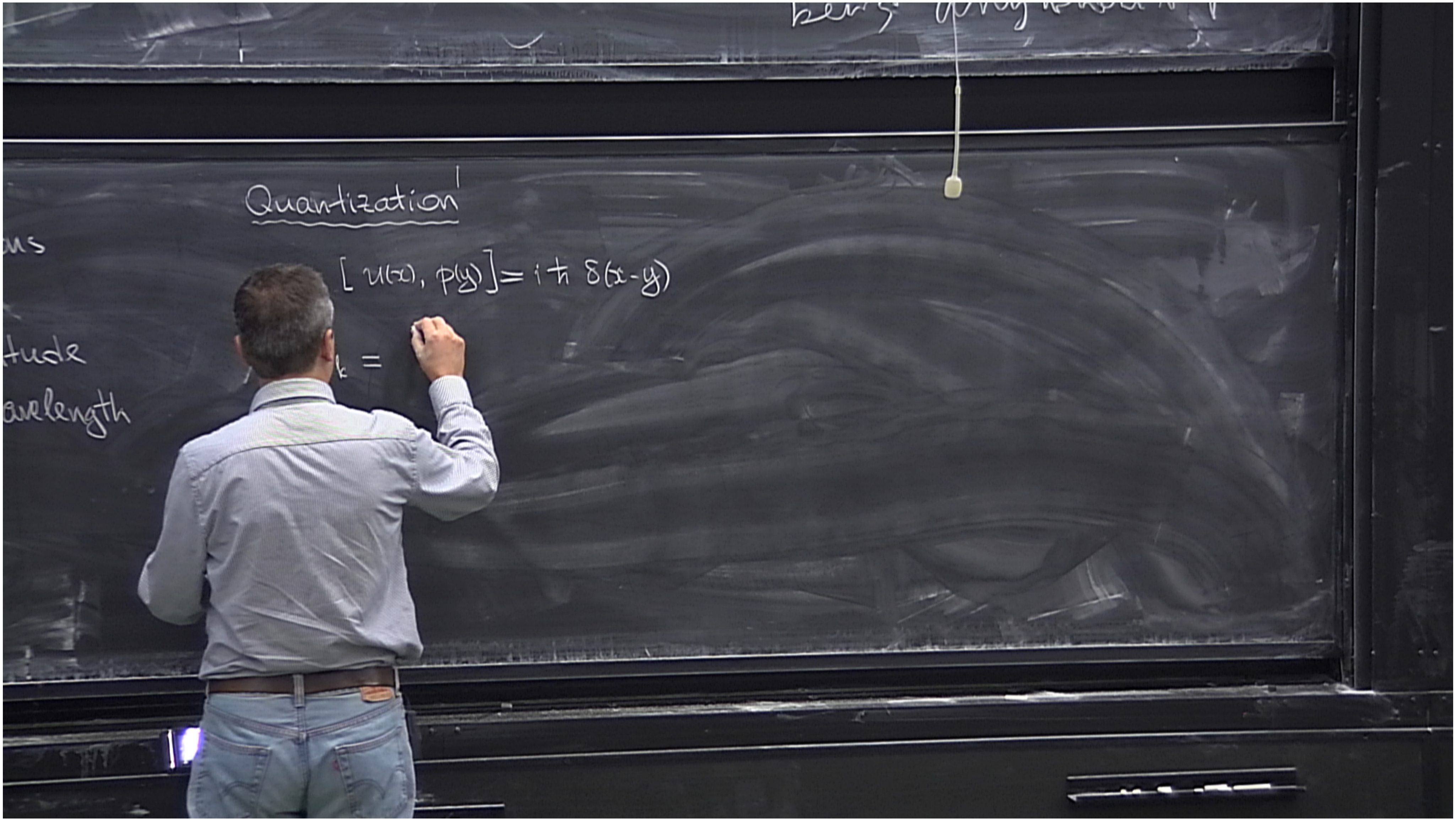
wavelength


Quantization
$[u(x), p(y)] = i\hbar\, \delta(x-y)$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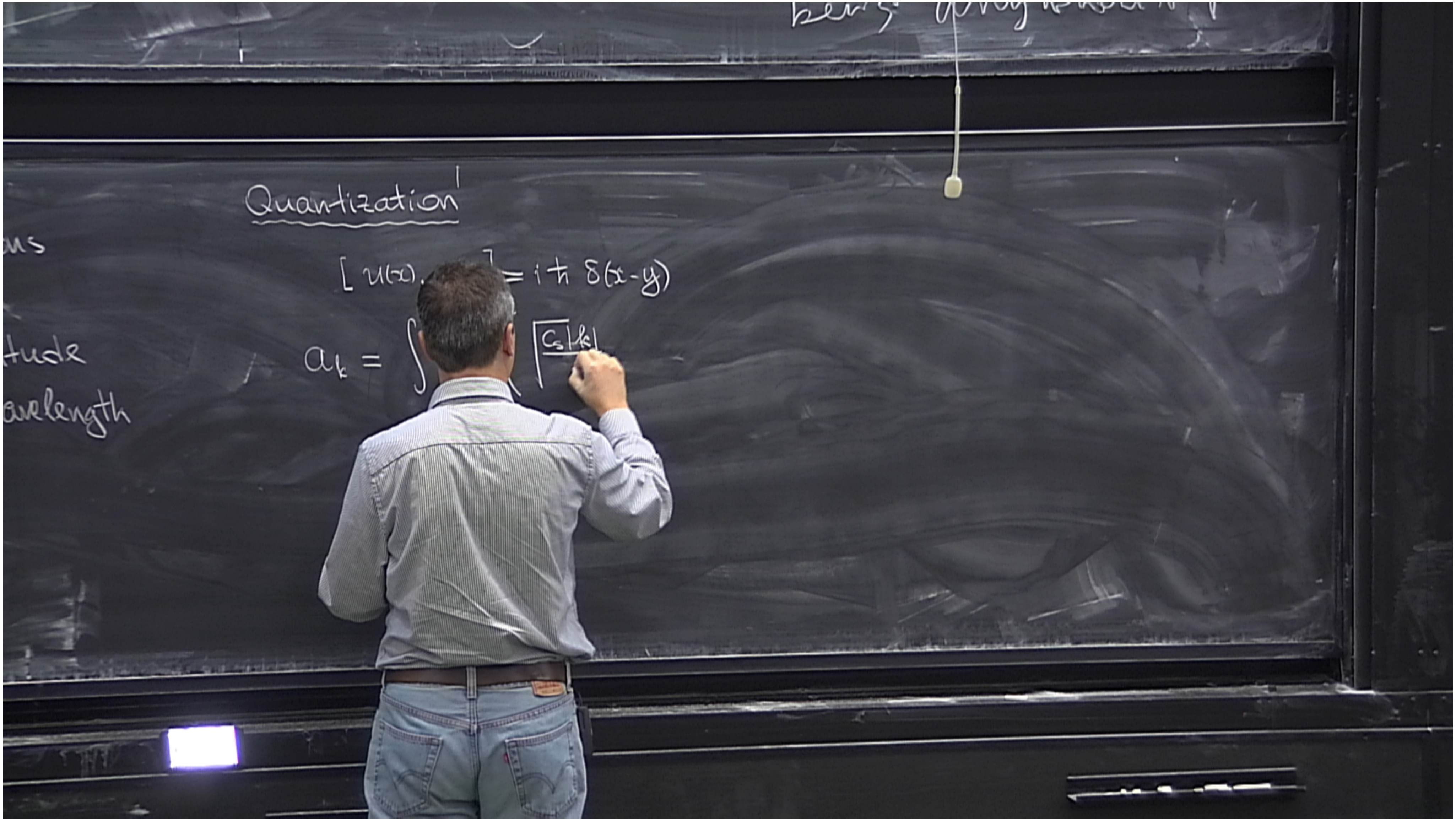

Quantization
[u(x), ...] = iℏ δ(x-y)
a_k = ∫ ... √(c_s|k| ...
...ns
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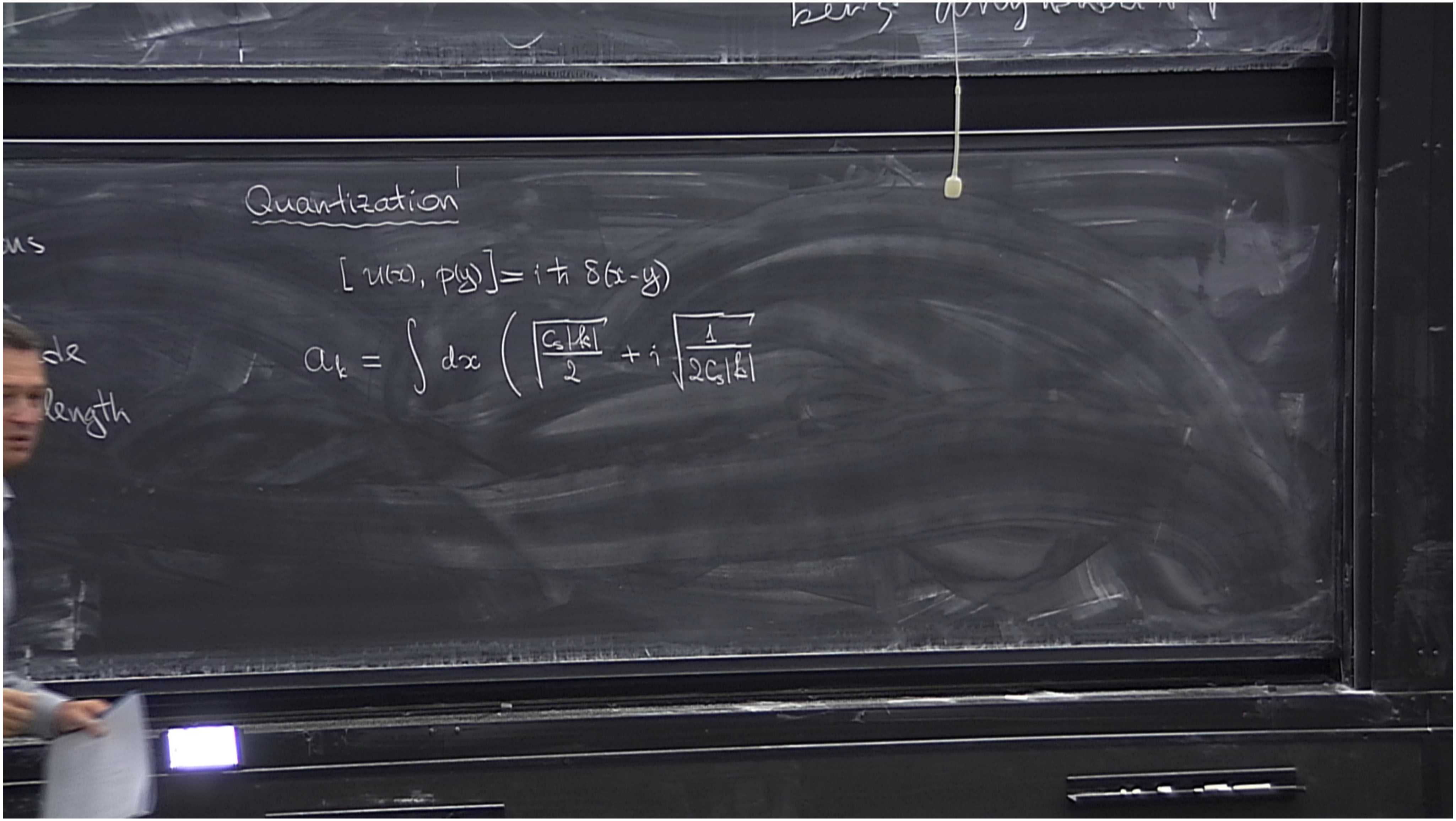
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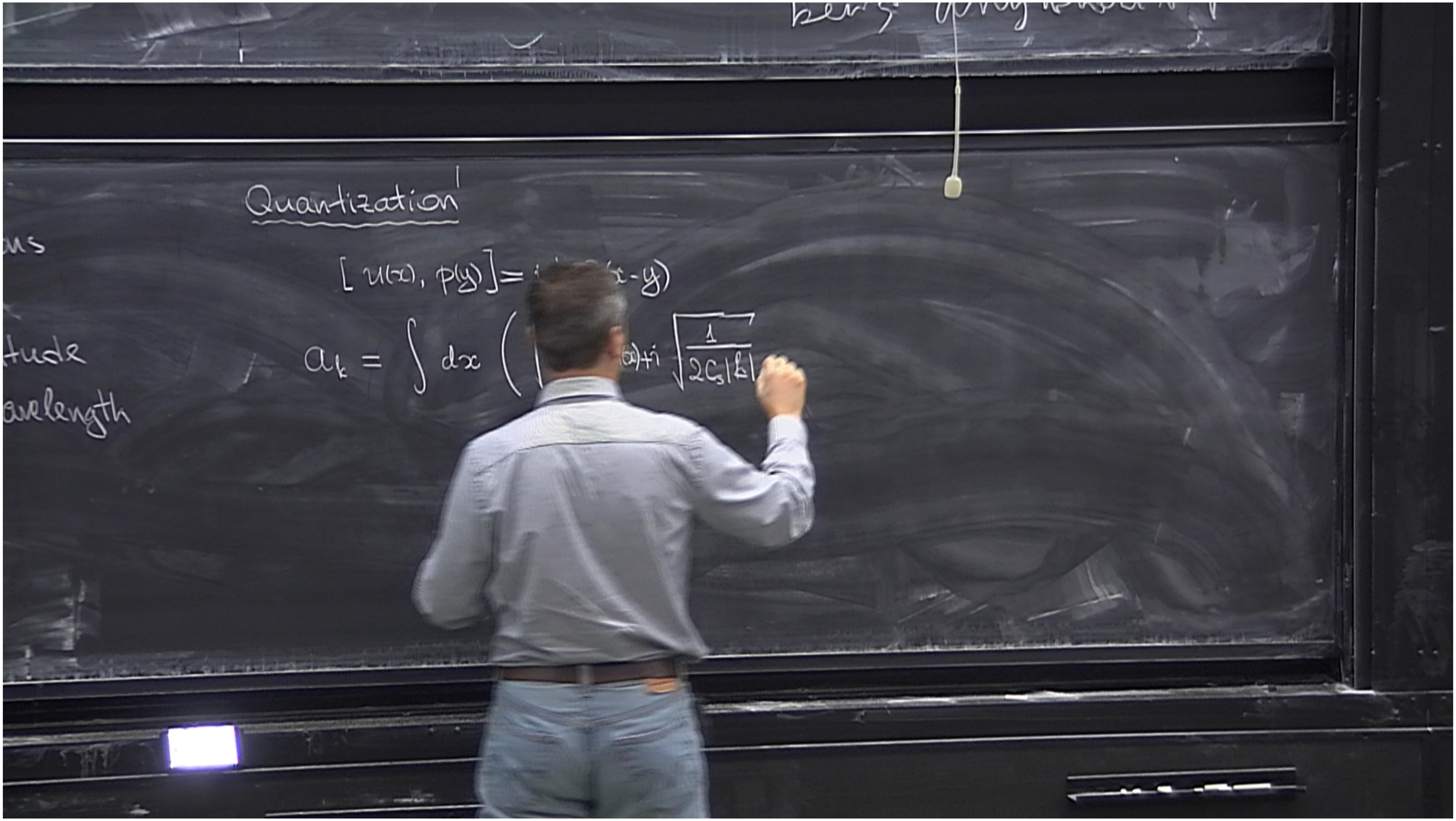
[u(x), p(y)] = i ħ δ(x - y)
a_k = ∫ dx (√(c_s|k| / 2) + i √(1 / (2c_s|k|))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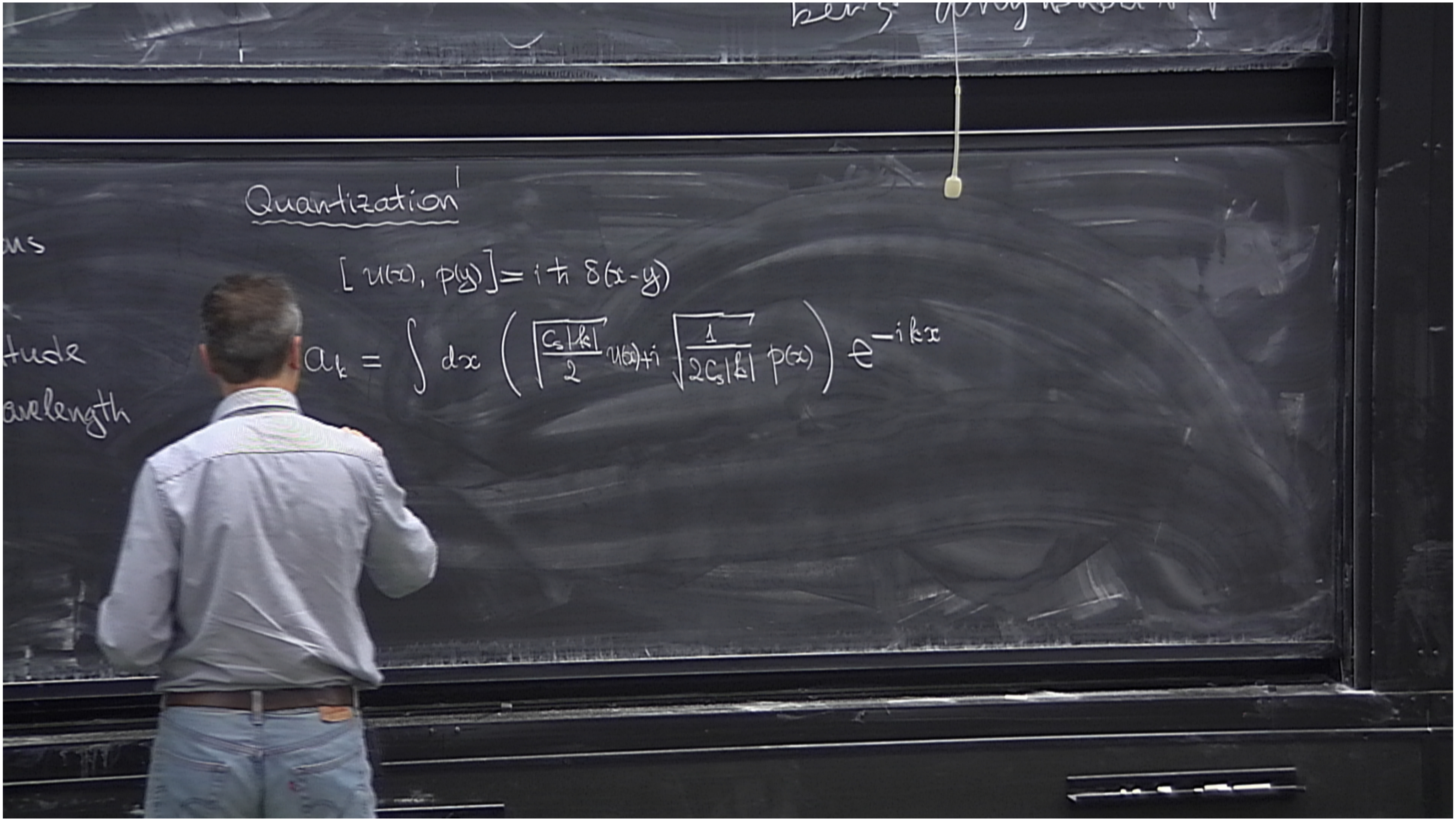
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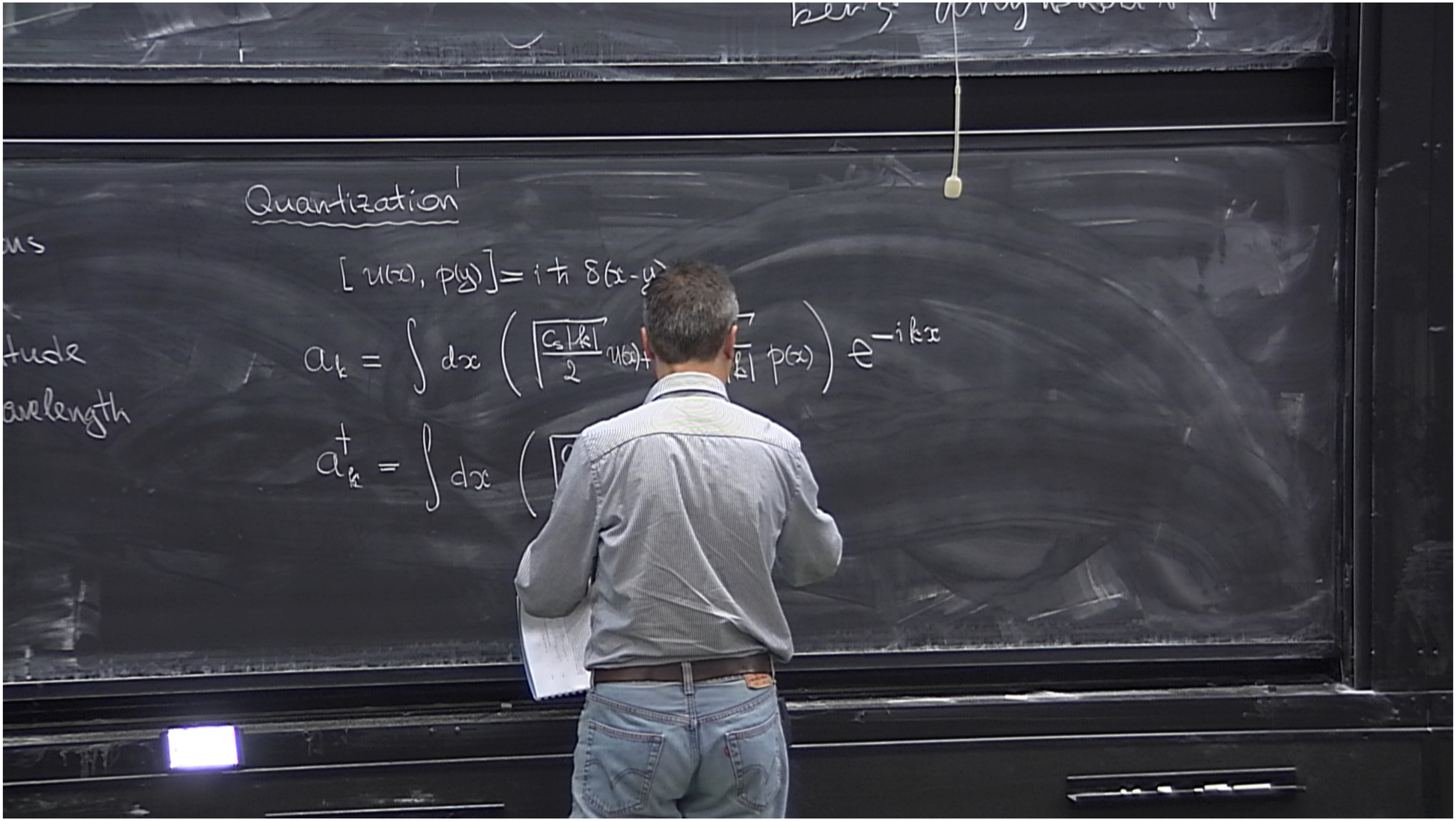
[u(x), p(y)] = i ħ δ(x - y)
a_k = ∫ dx (√(c_s|k|/2) u(x) + i √(1/(2c_s|k|)) p(x)) e^{-ikx}
ns
tude
avelength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a_k^† = ∫ dx (
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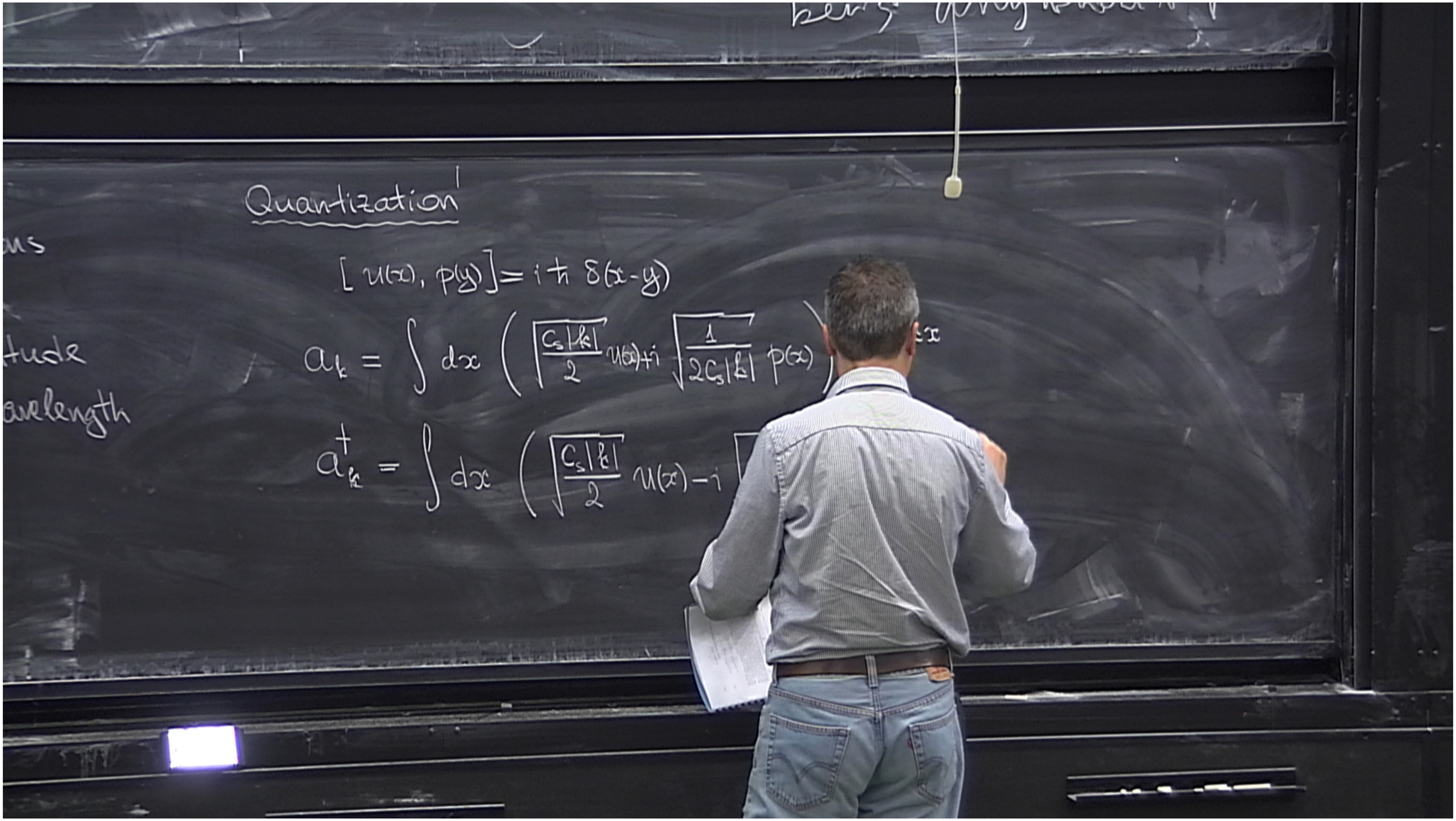
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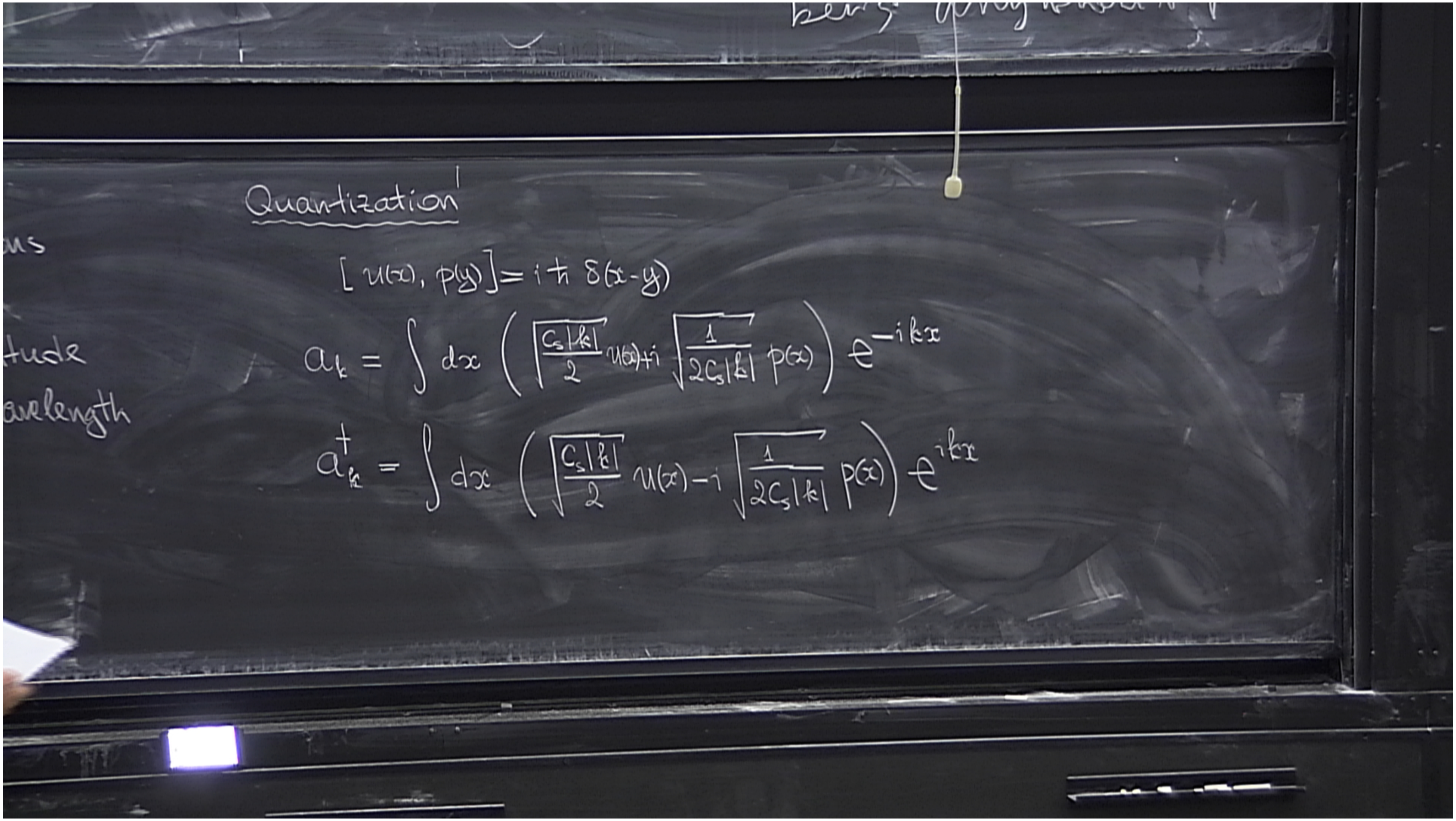
a†_k = ∫ dx (√(c_s|k|/2) u(x) − i
wavelength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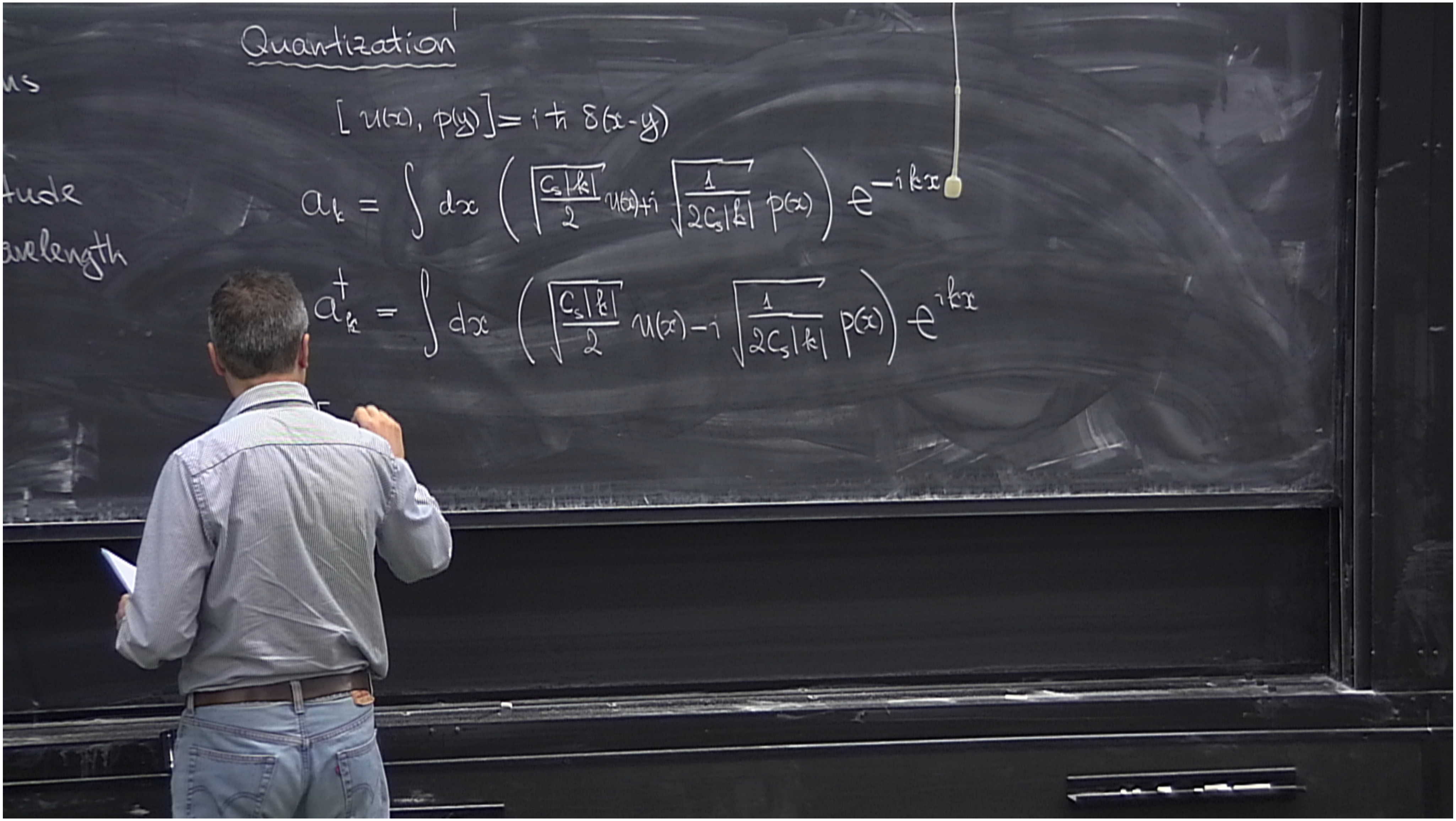
$a_k = \int dx \left(\sqrt{\frac{c_s|k|}{2}}\, u(x) + i \sqrt{\frac{1}{2c_s|k|}}\, p(x) \right) e^{-ikx}$
$a_k^\dagger = \int dx \left(\sqrt{\frac{c_s|k|}{2}}\, u(x) - i \sqrt{\frac{1}{2c_s|k|}}\, p(x) \right) e^{ikx}$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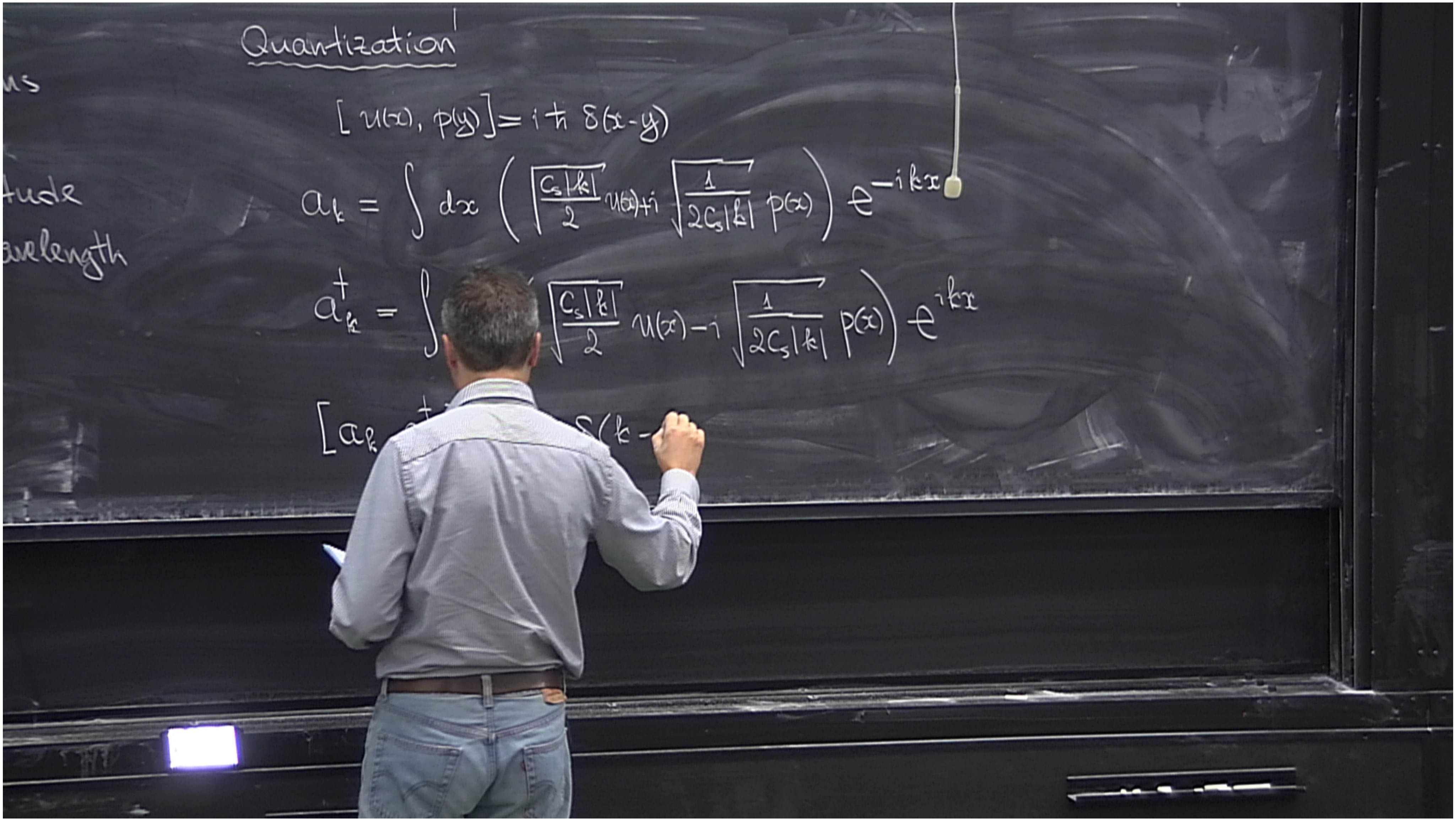

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$a_k^\dagger = \int \ldots \sqrt{\frac{c_s|k|}{2}}\, u(x) - i \sqrt{\frac{1}{2c_s|k|}}\, p(x) \Big) e^{ikx}$
$[a_k, a^\dagger \ldots \delta(k - \ldots$

us

tude

avelength

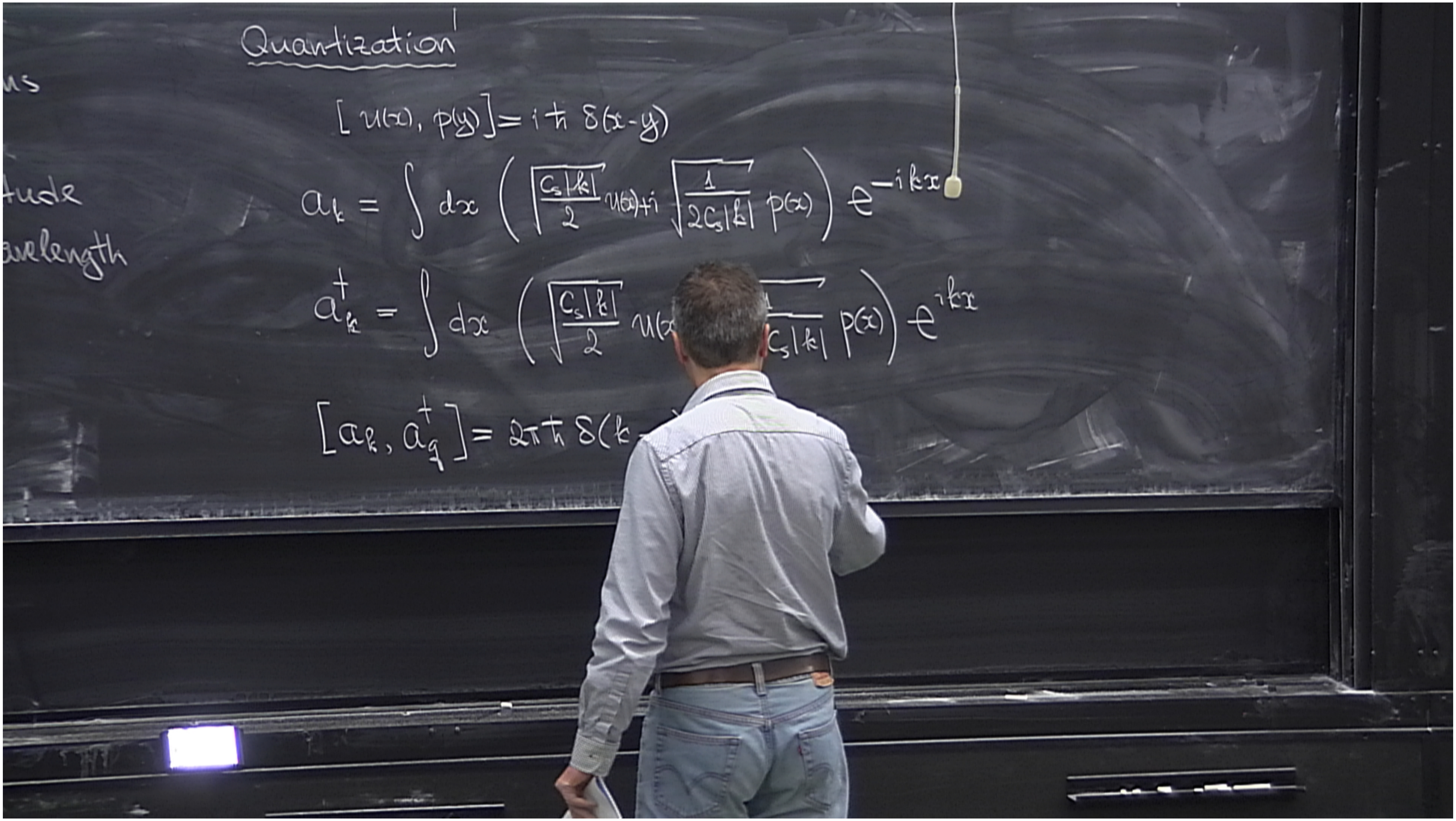
Quantization

$$[u(x), p(y)] = i\hbar\,\delta(x-y)$$

$$a_k = \int dx \left(\sqrt{\frac{c_s|k|}{2}}\, u(x) + i \sqrt{\frac{1}{2c_s|k|}}\, p(x) \right) e^{-ikx}$$

$$a_k^\dagger = \int dx \left(\sqrt{\frac{c_s|k|}{2}}\, u(x) - i \sqrt{\frac{1}{2c_s|k|}}\, p(x) \right) e^{ikx}$$

$$[a_k, a_q^\dagger] = 2\pi\hbar\,\delta(k-q)$$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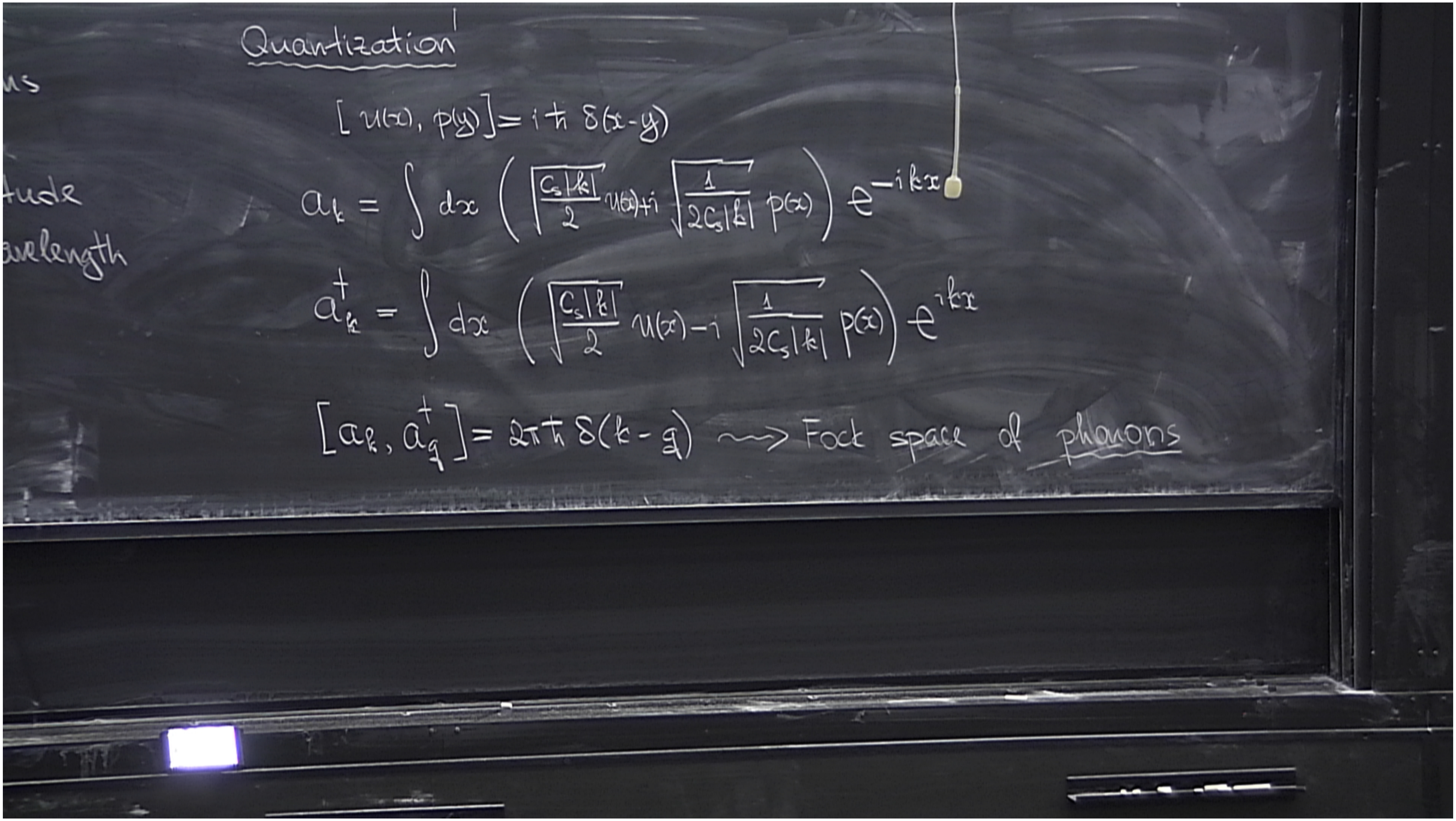
$a_k^\dagger = \int dx$
$[a_k, a_q^\dagger] = 2\pi\hbar\, \delta(k$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$[a_k, a_q^\dagger] = 2\pi\hbar\, \delta(k-q)$ ⇝ Fock space of phonons

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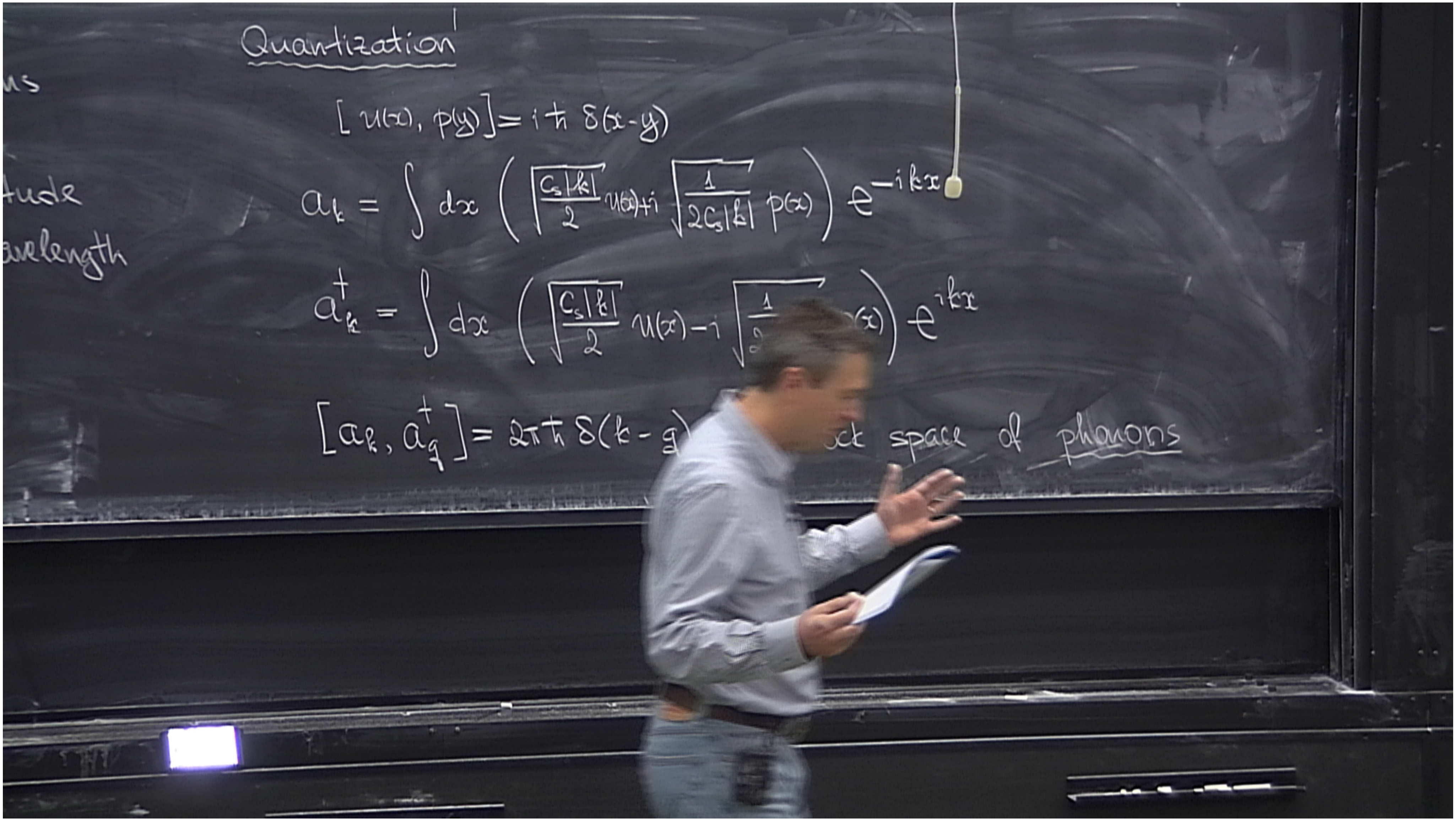
Quantization

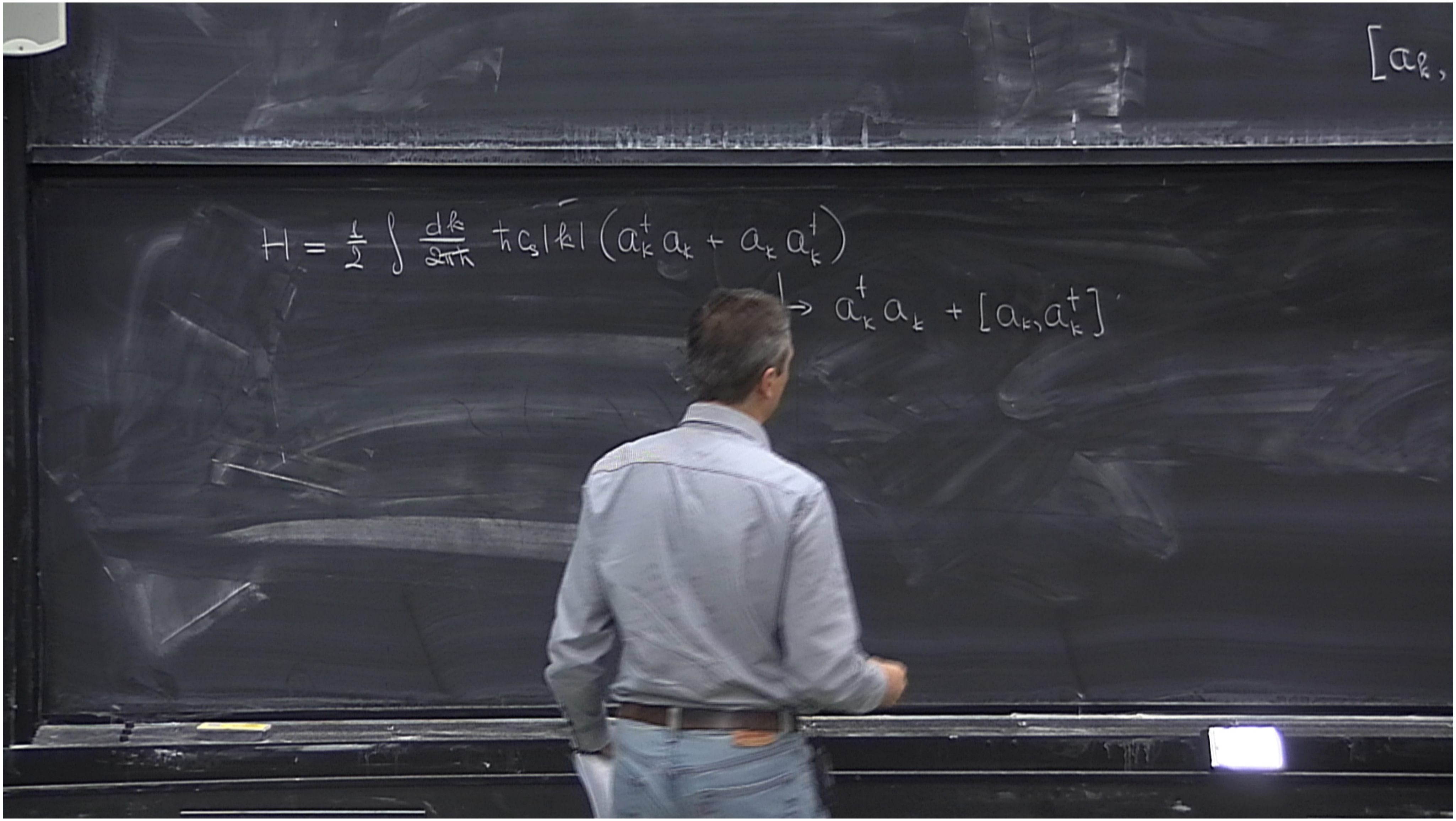
$$[u(x), p(y)] = i\hbar\,\delta(x-y)$$

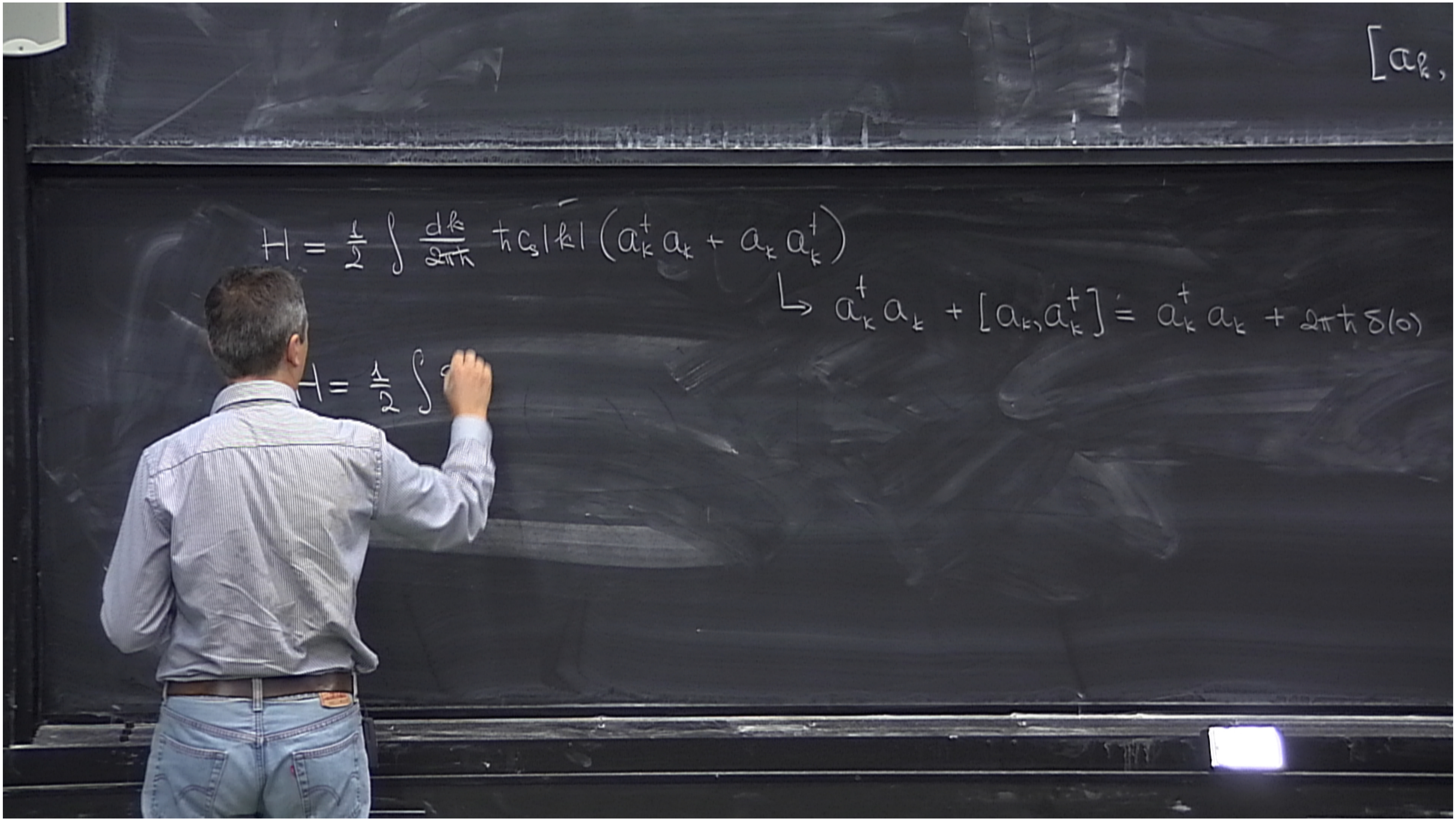
$$a_k = \int dx \left(\sqrt{\frac{c_s|k|}{2}}\, u(x) + i \sqrt{\frac{1}{2c_s|k|}}\, p(x) \right) e^{-ikx}$$

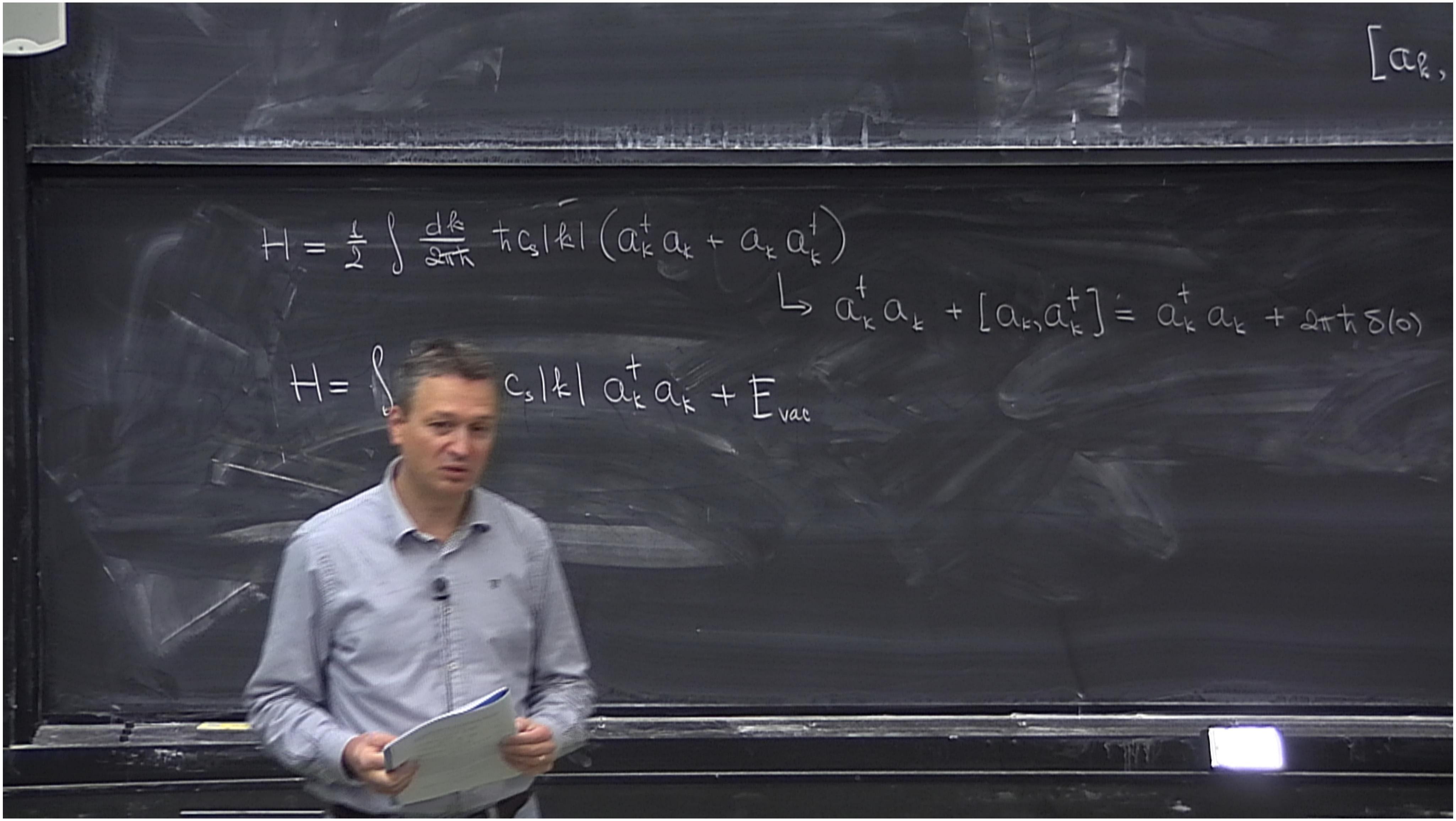
$$a_k^\dagger = \int dx \left(\sqrt{\frac{c_s|k|}{2}}\, u(x) - i \sqrt{\frac{1}{2c_s|k|}}\, p(x) \right) e^{ikx}$$

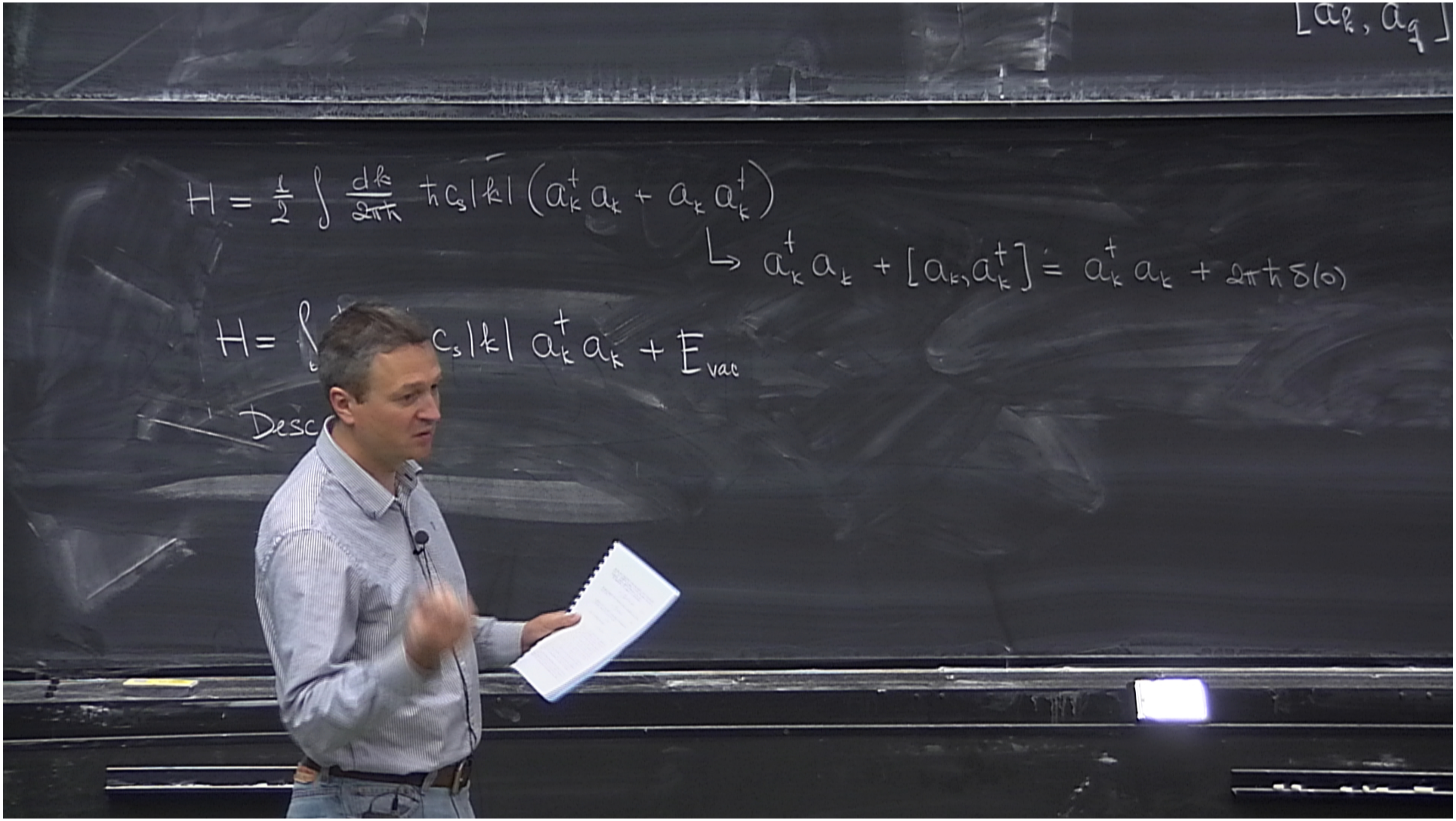
$$[a_k, a_q^\dagger] = 2\pi\hbar\,\delta(k-q)$$ $\rightsquigarrow$ Fock space of phonons

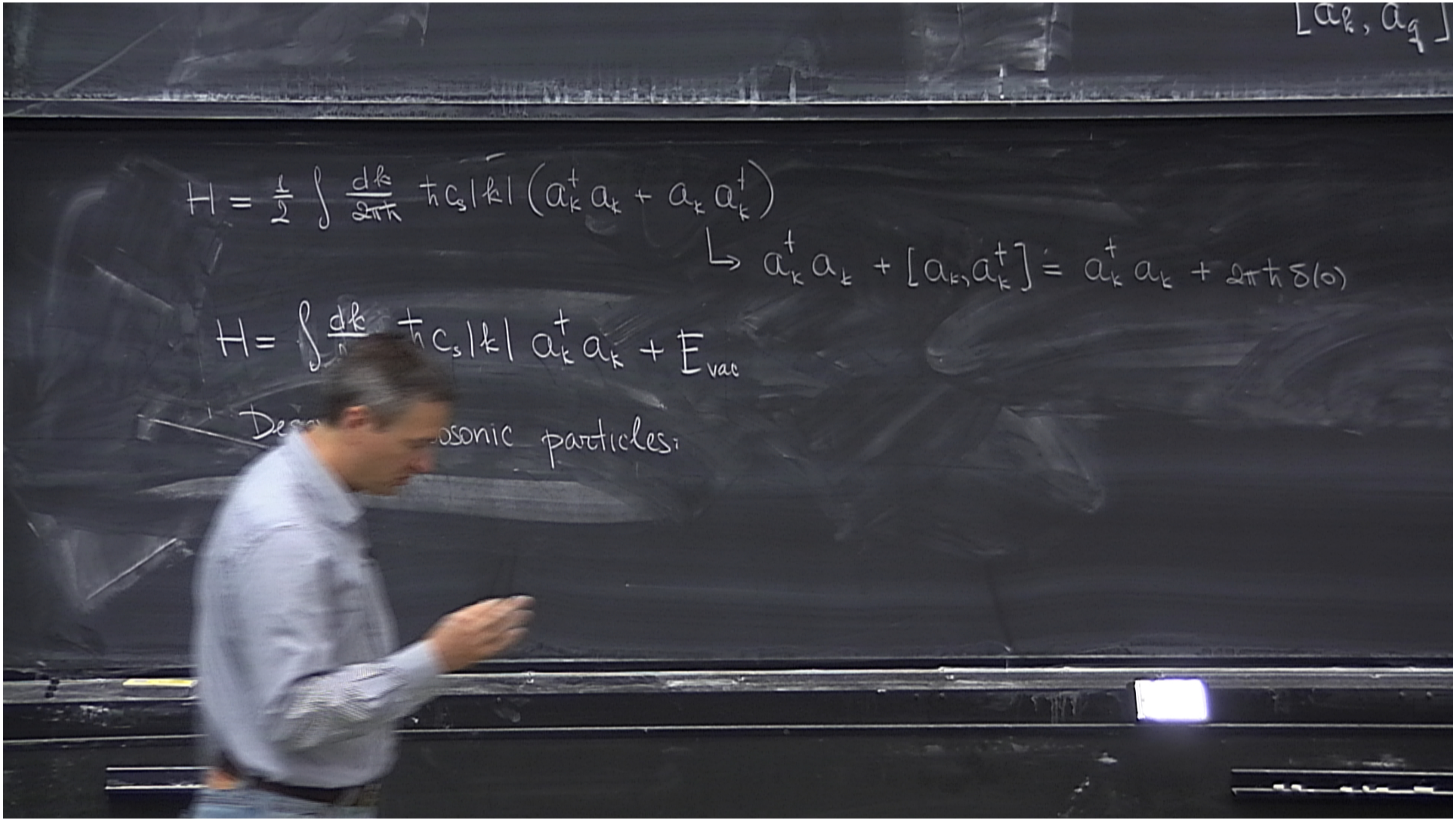

Quantization
[u(x), p(y)] = iħ δ(x-y)
a_k = ∫ dx (√(c_s|k|/2) u(x) + i √(1/(2c_s|k|)) p(x)) e^{-ikx}
a_k^† = ∫ dx (√(c_s|k|/2) u(x) − i
) e^{ikx}
[a_k, a_q^†] = 2πħ δ(k - q)
space of phonons

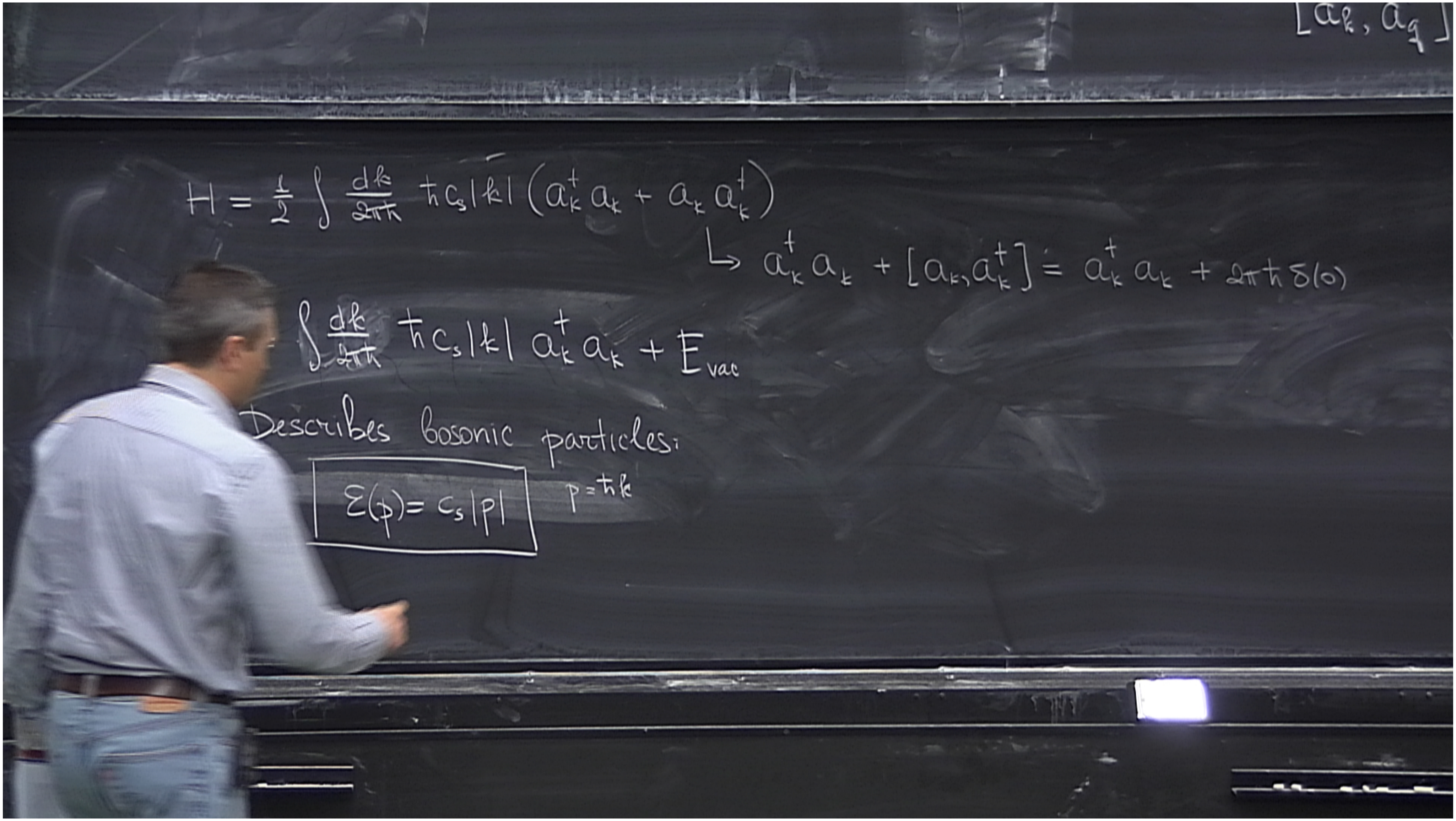

$H = \frac{1}{2} \int \frac{dk}{2\pi\hbar} \hbar c_s |k| (a_k^\dagger a_k + a_k a_k^\dagger)$
$\hookrightarrow a_k^\dagger a_k + [a_k, a_k^\dagger]$
$[a_k,$

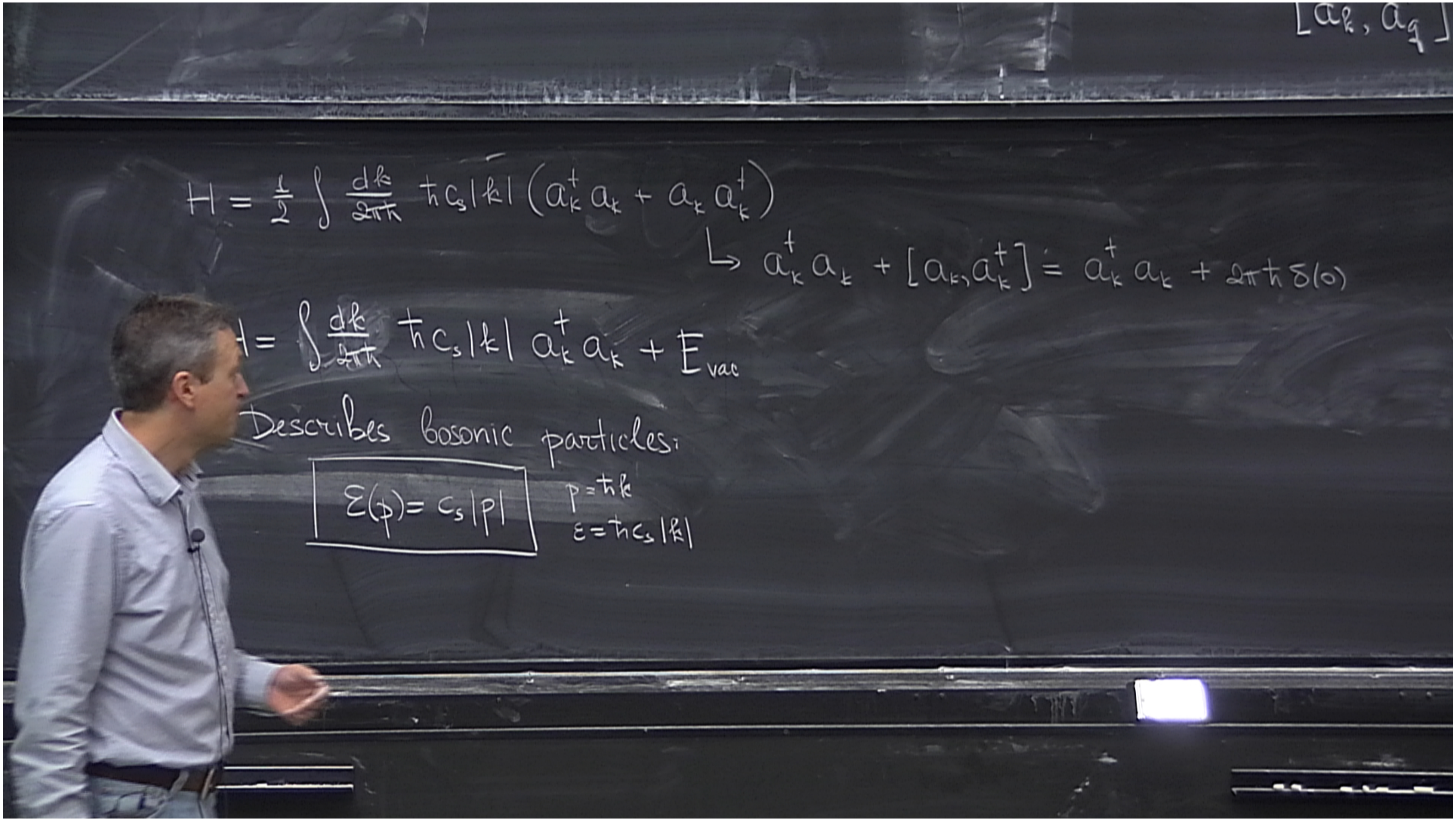

$[a_k,$
$H = \frac{1}{2} \int \frac{dk}{2\pi\hbar} \hbar c_s |k| (a_k^\dagger a_k + a_k a_k^\dagger)$
$\hookrightarrow a_k^\dagger a_k + [a_k, a_k^\dagger] = a_k^\dagger a_k + 2\pi\hbar\,\delta(0)$
$H = \frac{1}{2} \int$

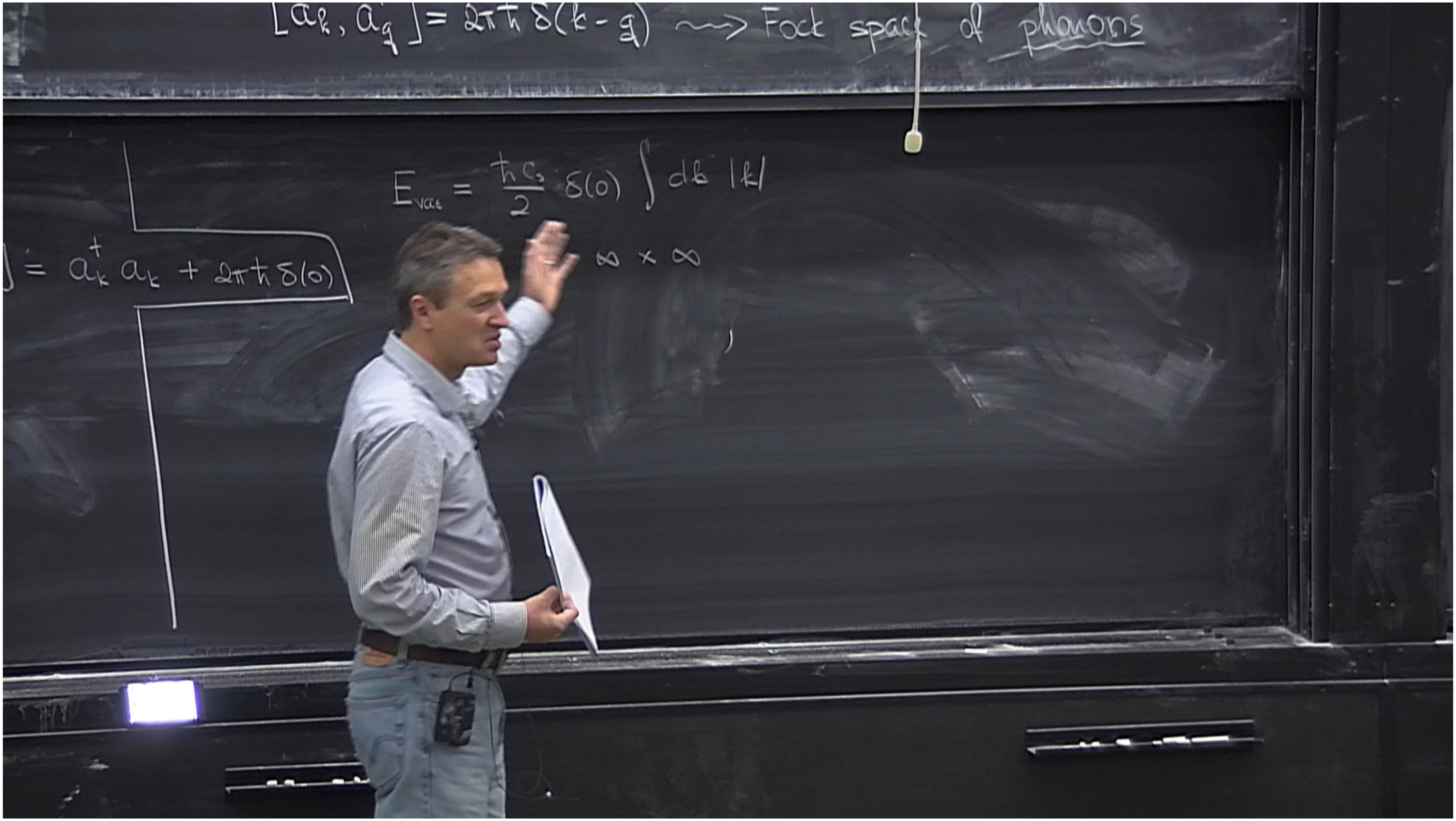

[a_k,
$H = \frac{1}{2} \int \frac{dk}{2\pi\hbar} \hbar c_s |k| (a_k^\dagger a_k + a_k a_k^\dagger)$
$\hookrightarrow a_k^\dagger a_k + [a_k, a_k^\dagger] = a_k^\dagger a_k + 2\pi\hbar\,\delta(0)$
$H = \int$ $c_s |k|\, a_k^\dagger a_k + E_{vac}$

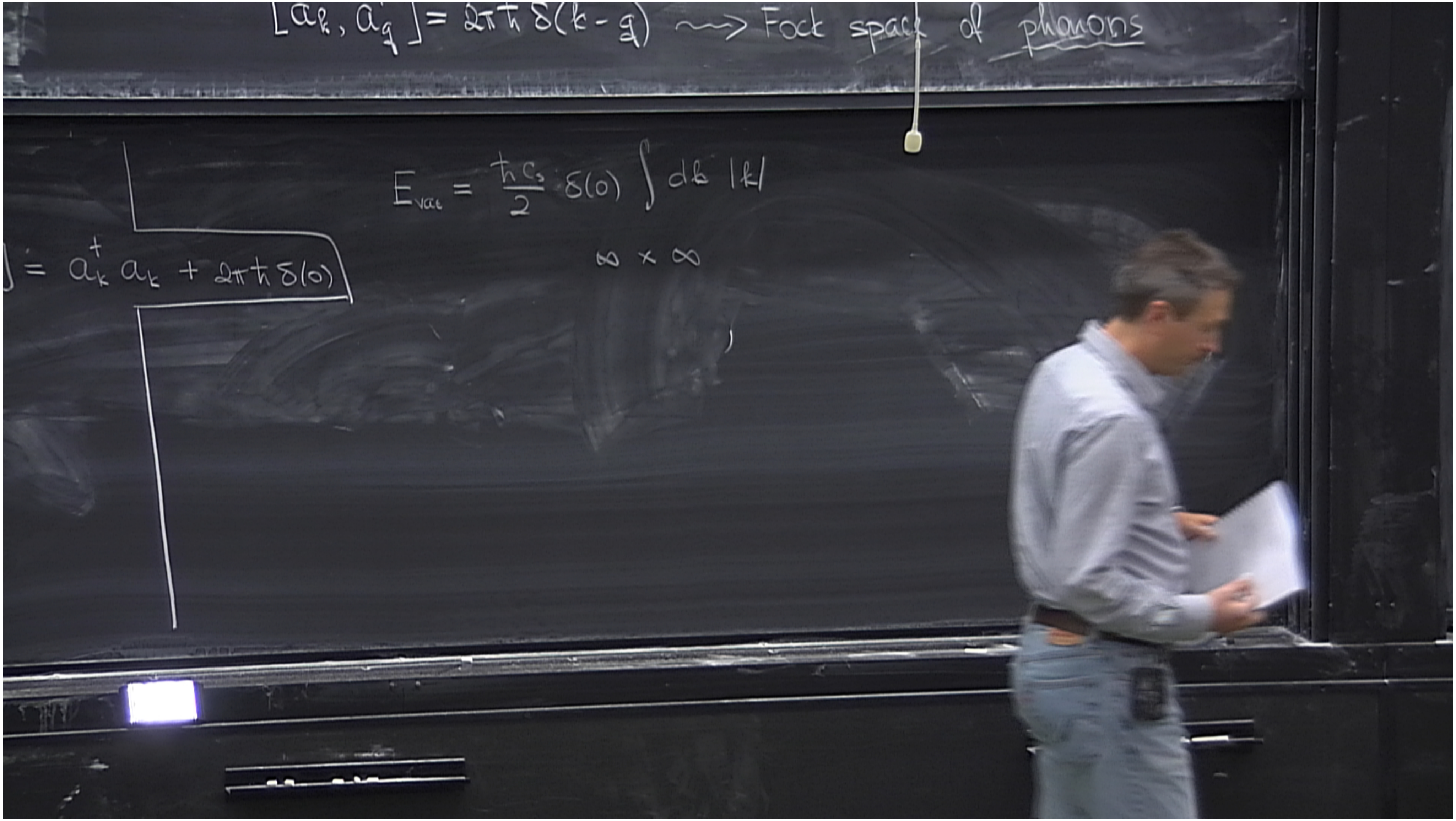

[ak, aq]
H = 1/2 ∫ dk/2πħ ħ cs |k| (a†k ak + ak a†k)
↳ a†k ak + [ak, a†k] = a†k ak + 2πħ δ(0)
H = ∫ cs |k| a†k ak + Evac
Desc

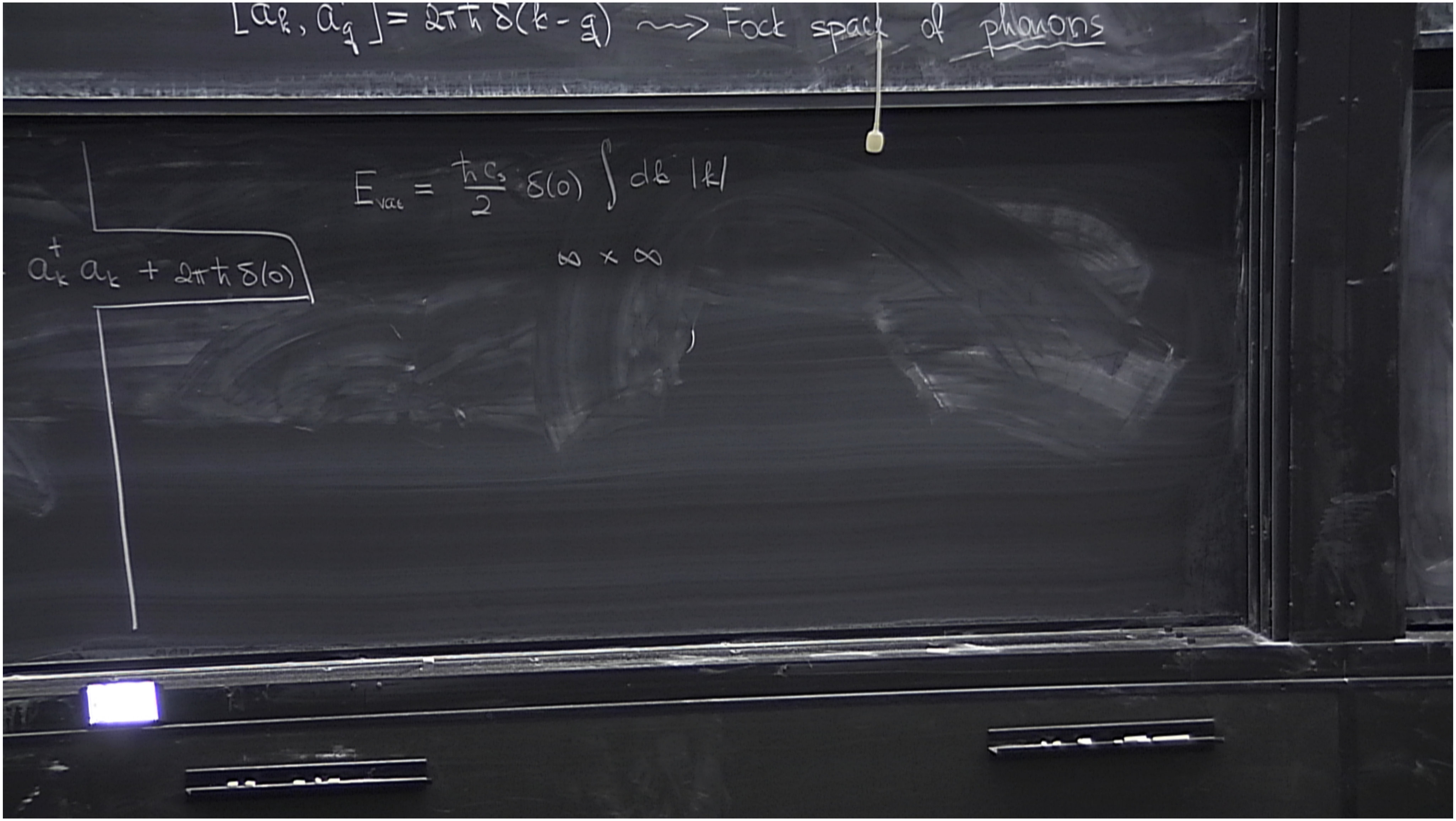

$[a_k, a_q]$
$H = \frac{1}{2} \int \frac{dk}{2\pi\hbar} \hbar c_s |k| (a_k^\dagger a_k + a_k a_k^\dagger)$
$\hookrightarrow a_k^\dagger a_k + [a_k, a_k^\dagger] = a_k^\dagger a_k + 2\pi\hbar\delta(0)$
$H = \int \frac{dk}{} \hbar c_s |k| a_k^\dagger a_k + E_{vac}$
osonic particles:

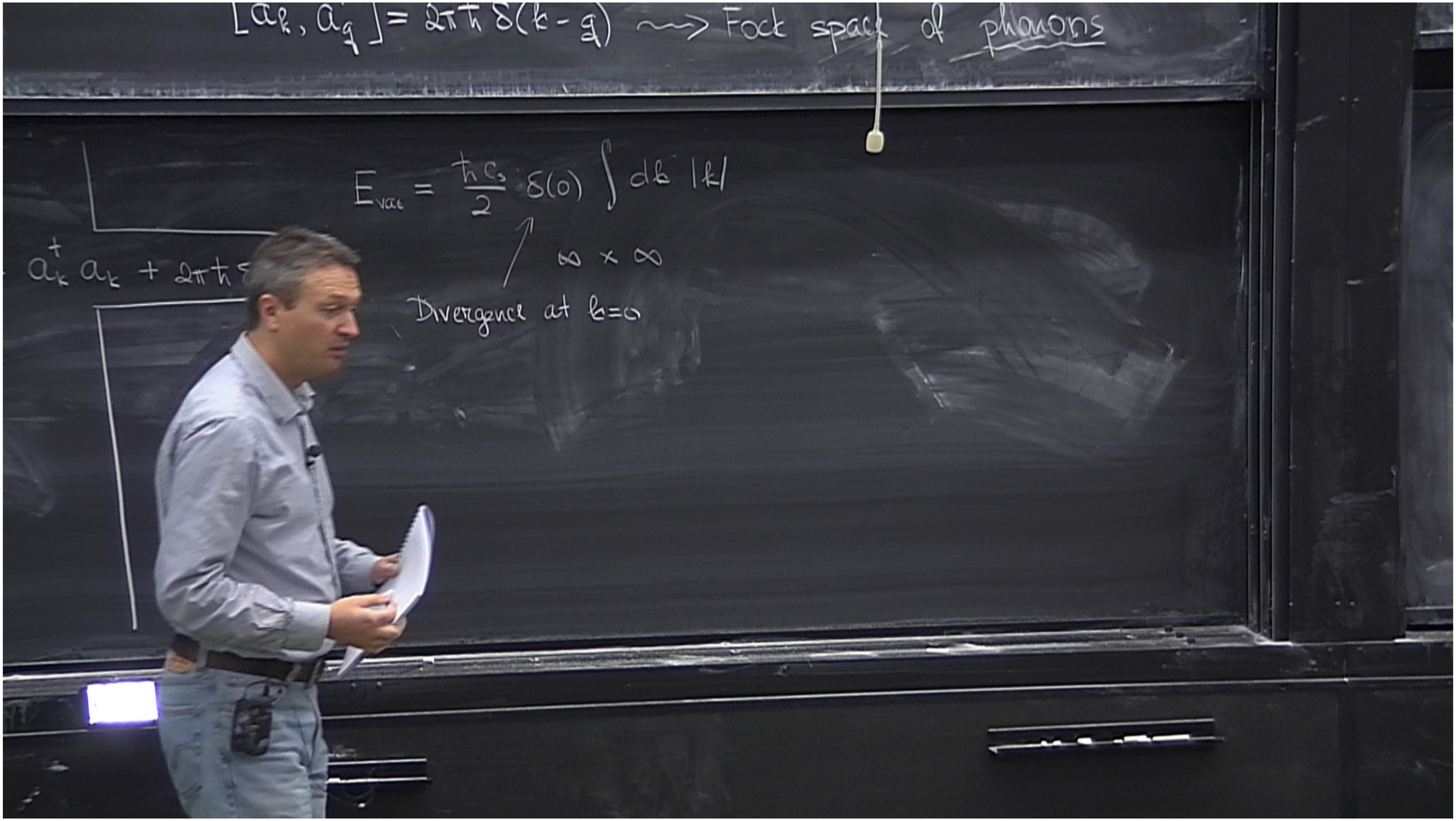

$[a_k, a_q]$
$H = \frac{1}{2} \int \frac{dk}{2\pi\hbar} \hbar c_s |k| (a_k^\dagger a_k + a_k a_k^\dagger)$
$\hookrightarrow a_k^\dagger a_k + [a_k, a_k^\dagger] = a_k^\dagger a_k + 2\pi\hbar\delta(0)$
$\int \frac{dk}{2\pi\hbar} \hbar c_s |k| a_k^\dagger a_k + E_{vac}$
Describes bosonic particles:
$\mathcal{E}(p) = c_s |p|$
$p = \hbar k$

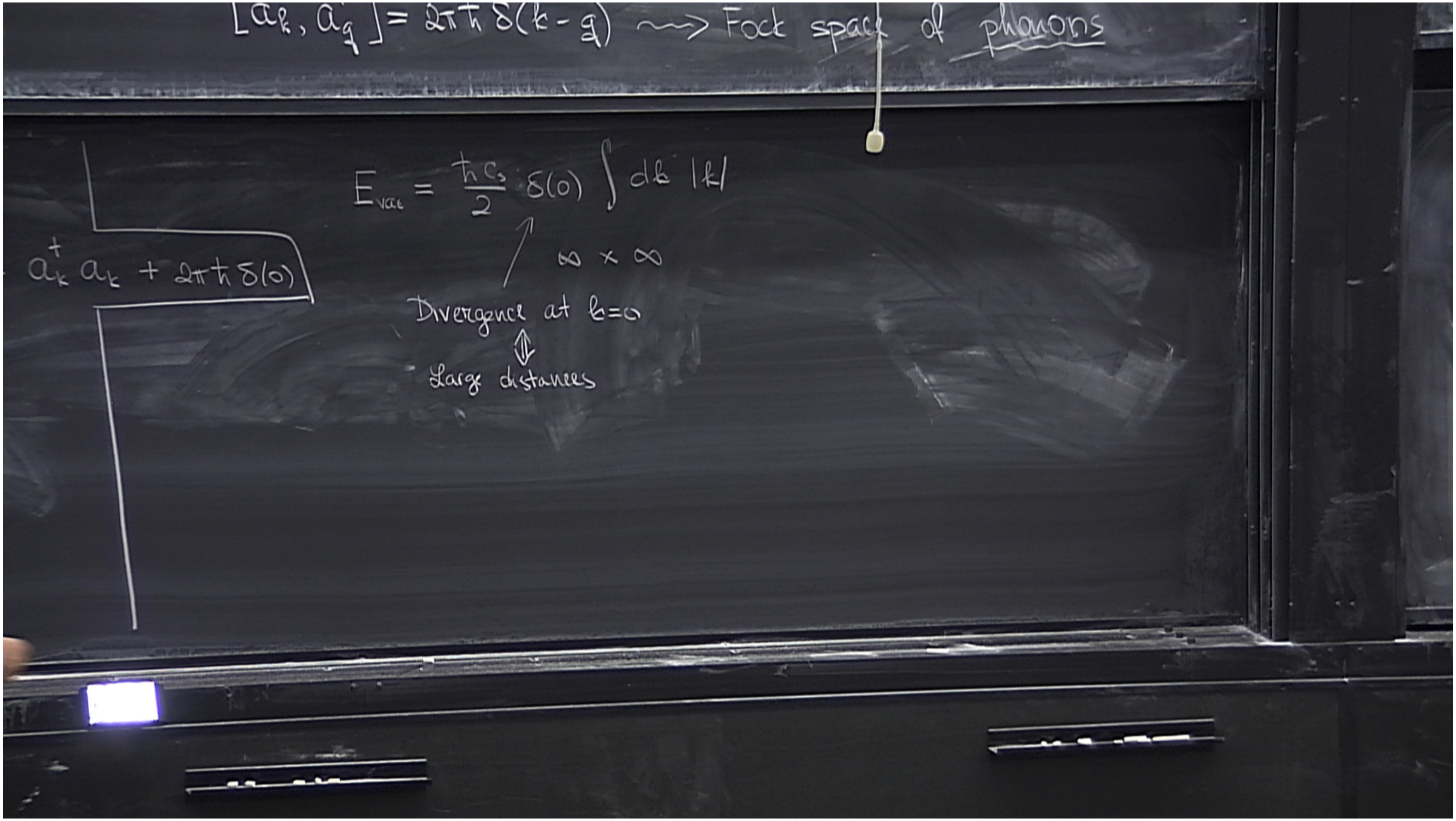

$[a_k, a_q]$
$H = \frac{1}{2} \int \frac{dk}{2\pi\hbar} \hbar c_s |k| (a_k^\dagger a_k + a_k a_k^\dagger)$
$\hookrightarrow a_k^\dagger a_k + [a_k, a_k^\dagger] = a_k^\dagger a_k + 2\pi\hbar\,\delta(0)$
$H = \int \frac{dk}{2\pi\hbar} \hbar c_s |k|\, a_k^\dagger a_k + E_{vac}$
Describes bosonic particles:
$\varepsilon(p) = c_s |p|$
$p = \hbar k$
$\varepsilon = \hbar c_s |k|$


[a_k , a_q†] = 2πħ δ(k - q) ~> Fock space of phonons
E_vac = ħc_s/2 δ(0) ∫ dk |k|
∞ × ∞
= a_k† a_k + 2πħ δ(0)

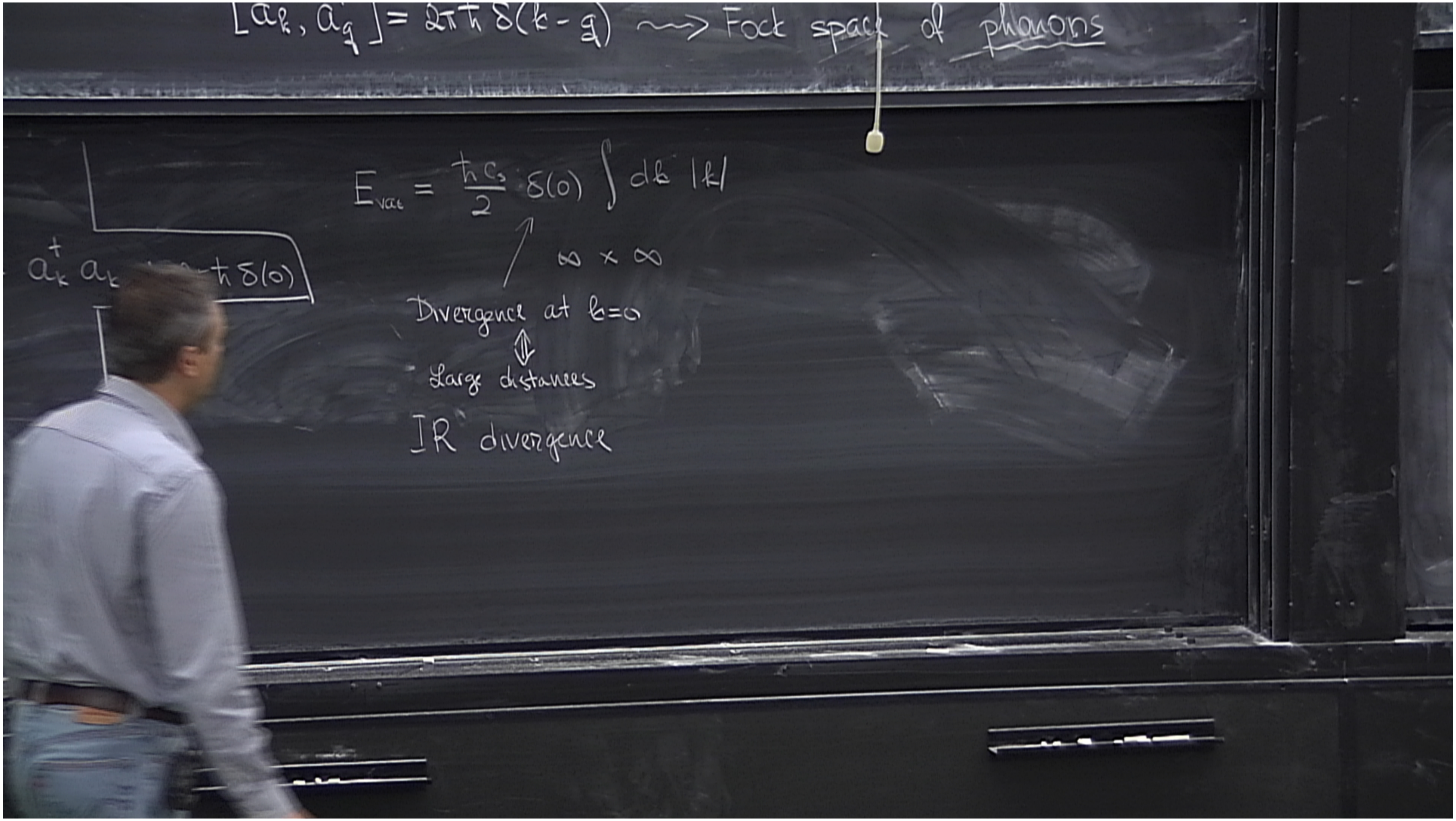

$[a_k, a_q^\dagger] = 2\pi\hbar\,\delta(k-q)$ ⟶ Fock space of phonons
$E_{vac} = \frac{\hbar c_s}{2}\,\delta(0)\int dk\,|k|$
$\infty \times \infty$
$= a_k^\dagger a_k + 2\pi\hbar\,\delta(0)$

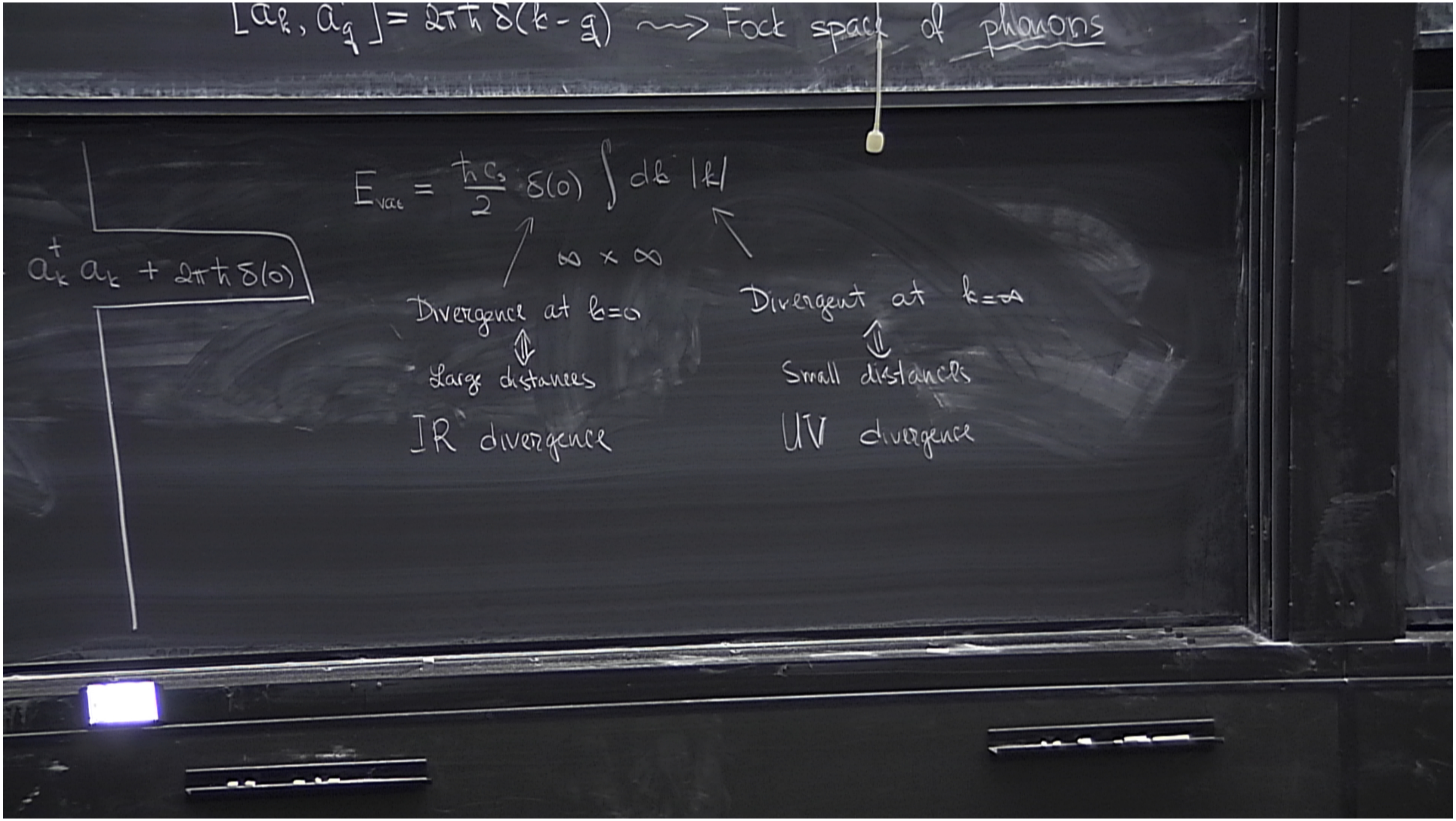

[a_k, a_q†] = 2πħ δ(k - q) ~~> Fock space of phonons
E_vac = ħc_s/2 δ(0) ∫ dk |k|
∞ × ∞
a_k† a_k + 2πħ δ(0)


Fock space of phonons
Divergence at k=0

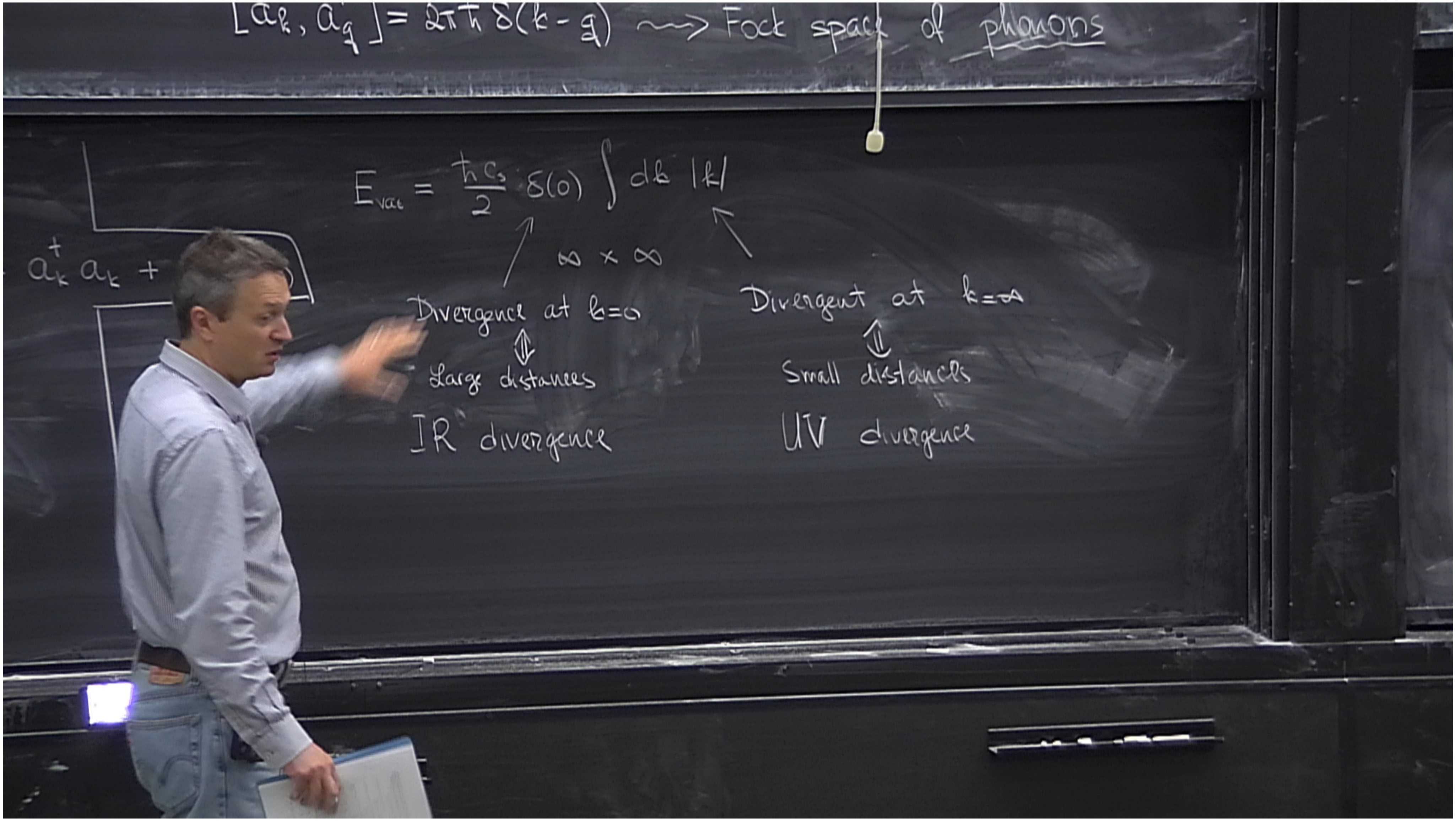


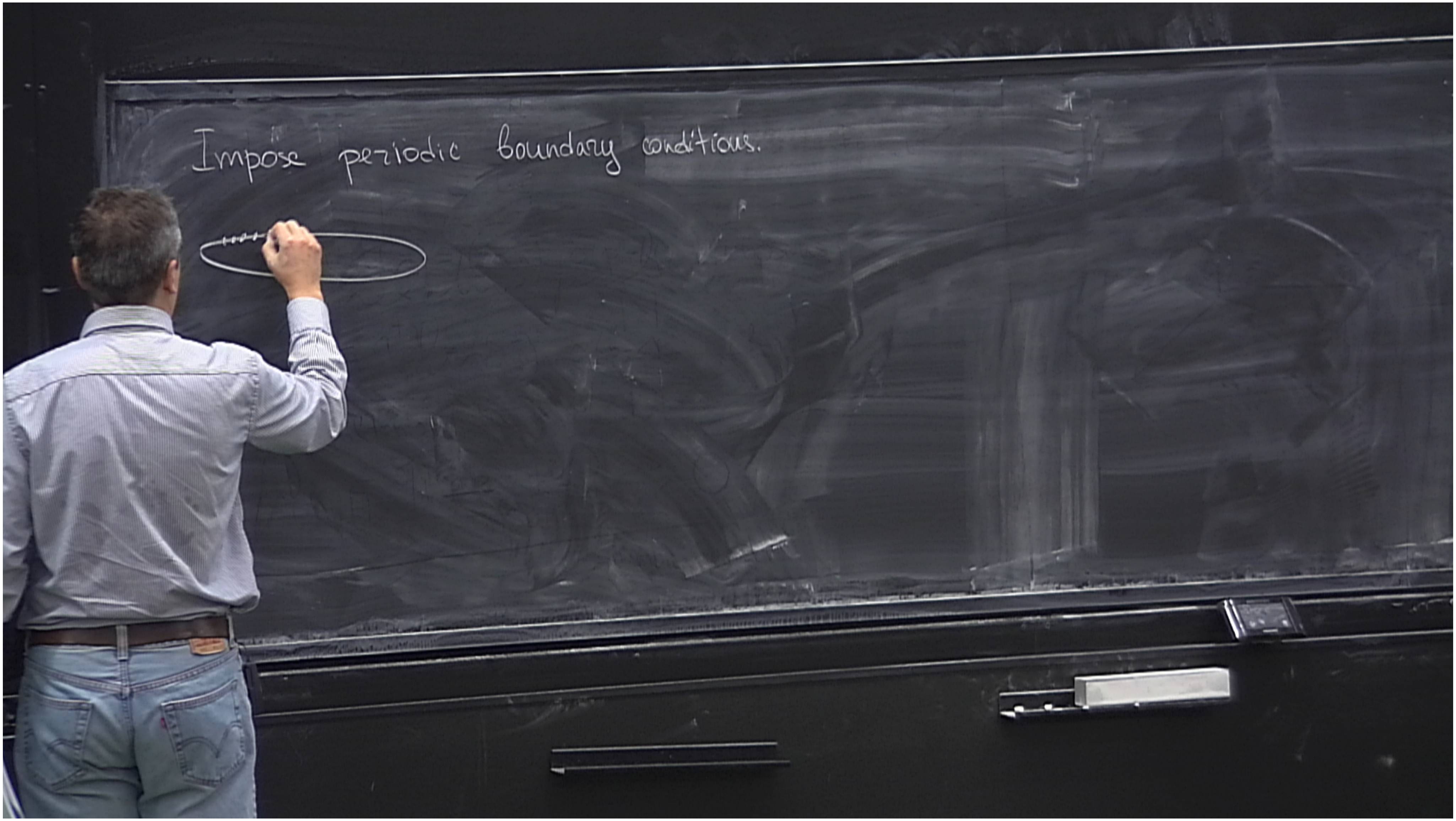
$[a_k, a_q^\dagger] = 2\pi\hbar\,\delta(k-q)$ ⇝ Fock space of phonons
$E_{vac} = \frac{\hbar c_s}{2}\,\delta(0)\int dk\,|k|$
$\infty \times \infty$
Divergence at $k=0$
Large distances
$a_k^\dagger a_k + 2\pi\hbar\,\delta(0)$

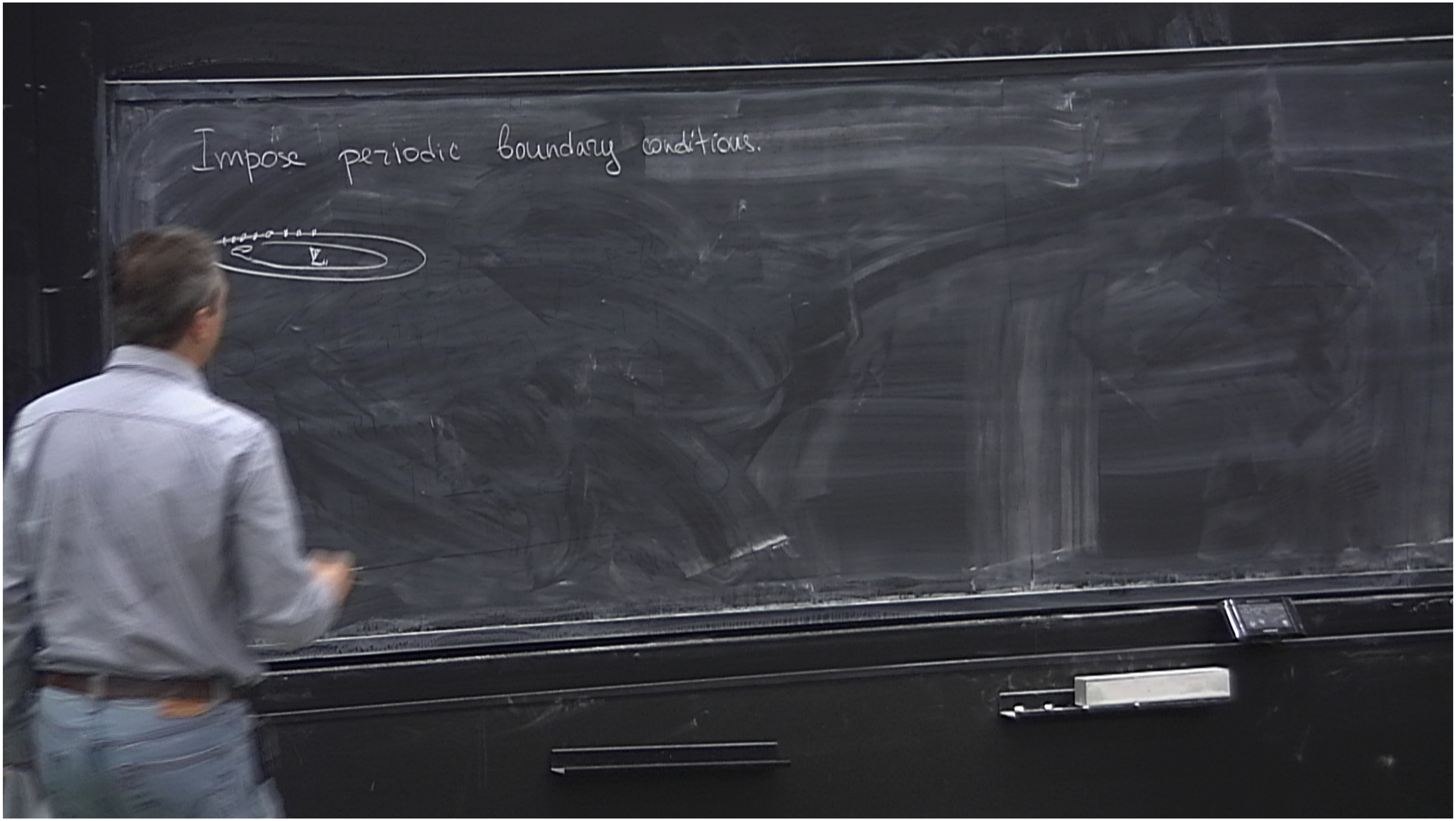

Fock space of phonons
Divergence at k=0
Large distances
IR divergence

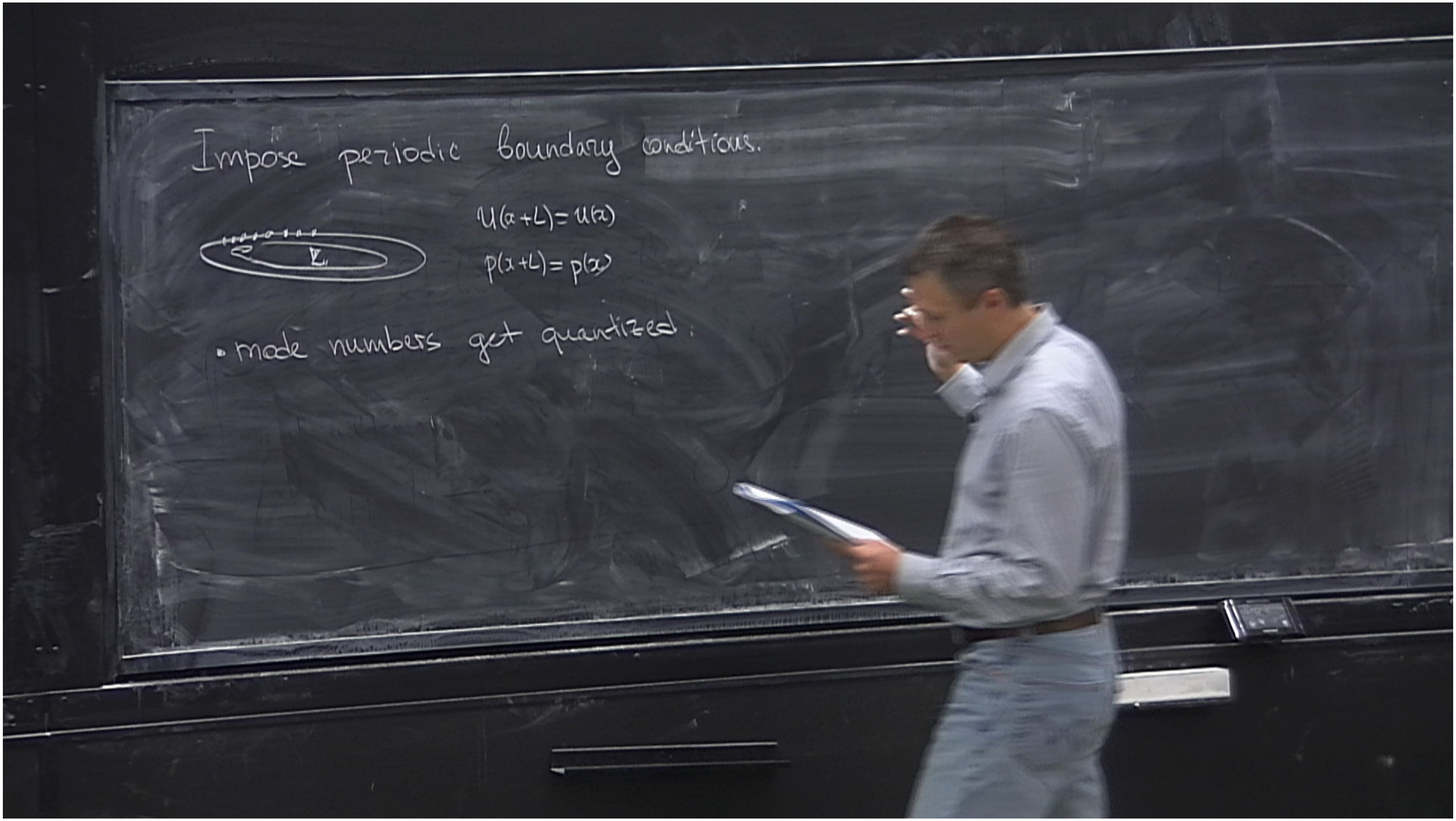


$[a_k, a_q^\dagger] = 2\pi\hbar\,\delta(k-q)$ ⇝ Fock space of phonons
$a_k^\dagger a_k + 2\pi\hbar\,\delta(0)$
$E_{vac} = \frac{\hbar c_s}{2}\,\delta(0) \int dk\, |k|$
$\infty \times \infty$
Divergence at $k=0$
⇕
Large distances
IR divergence
Divergent at $k=\infty$
⇕
Small distances
UV divergence

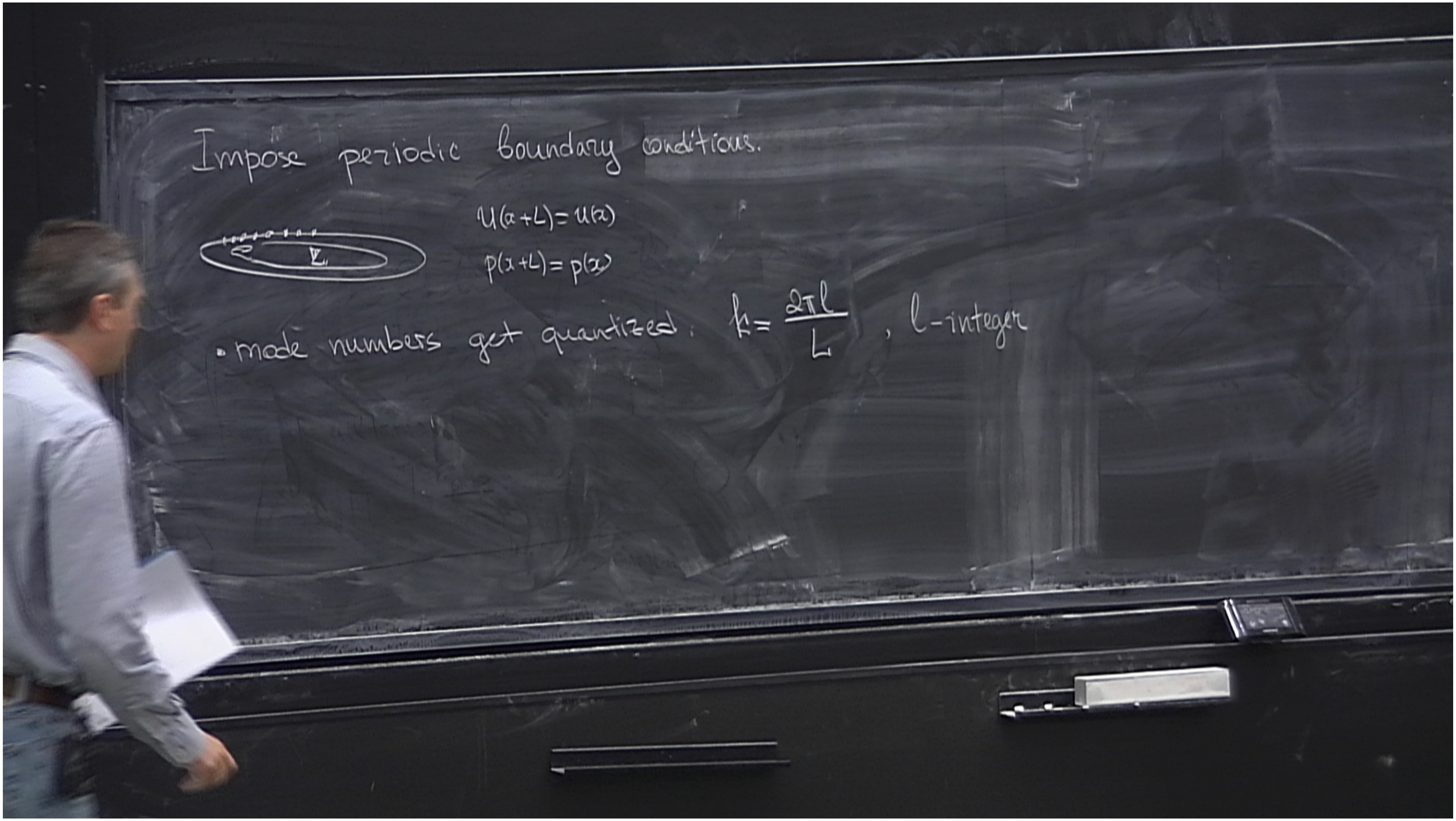

Fock space of phonons
$E_{vac} = \frac{\hbar c_s}{2} \delta(0) \int dk \, |k|$
$\infty \times \infty$
Divergence at k=0
Large distances
IR divergence
Divergent at k=∞
Small distances
UV divergence

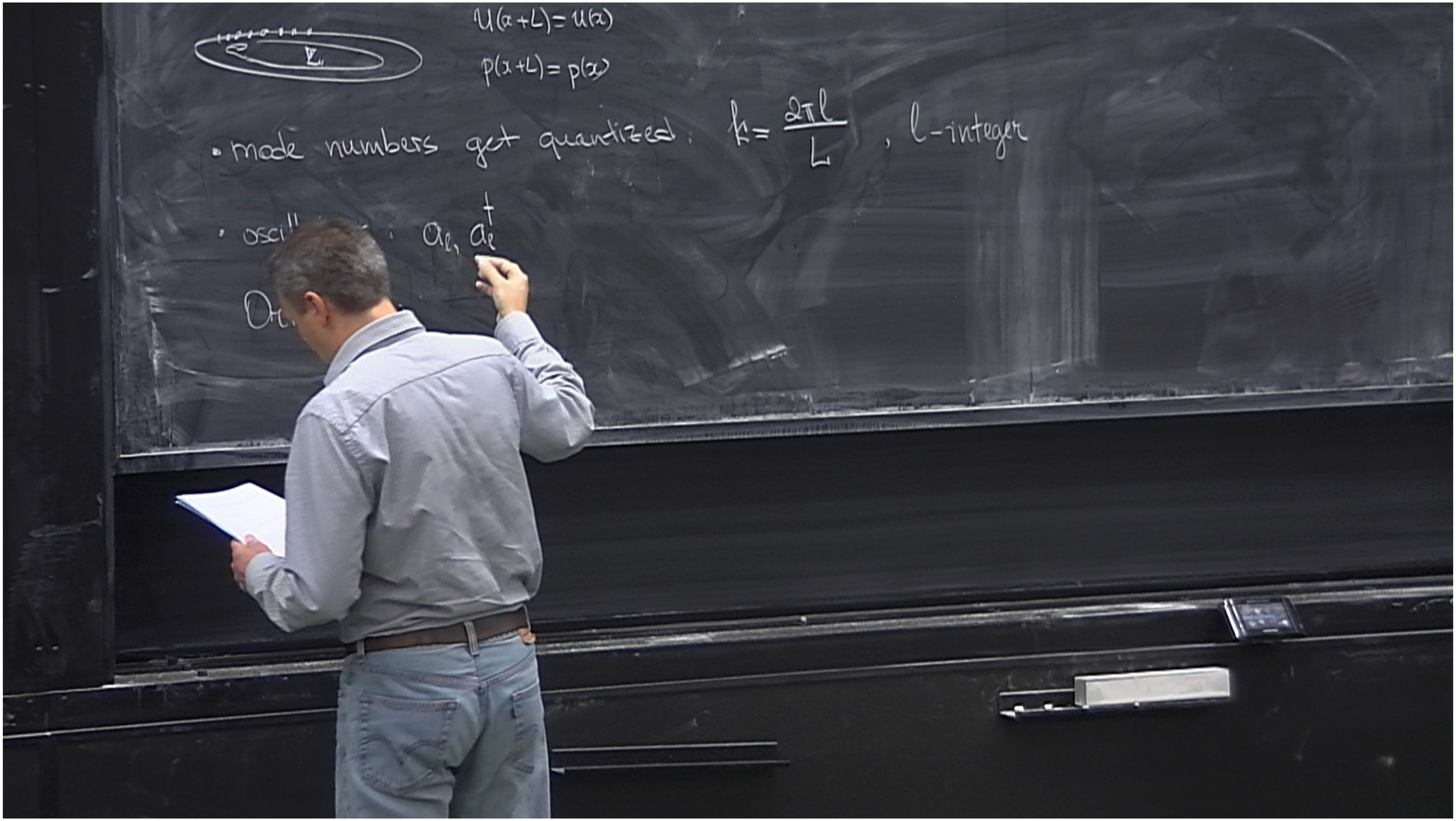

Impose periodic boundary conditions.

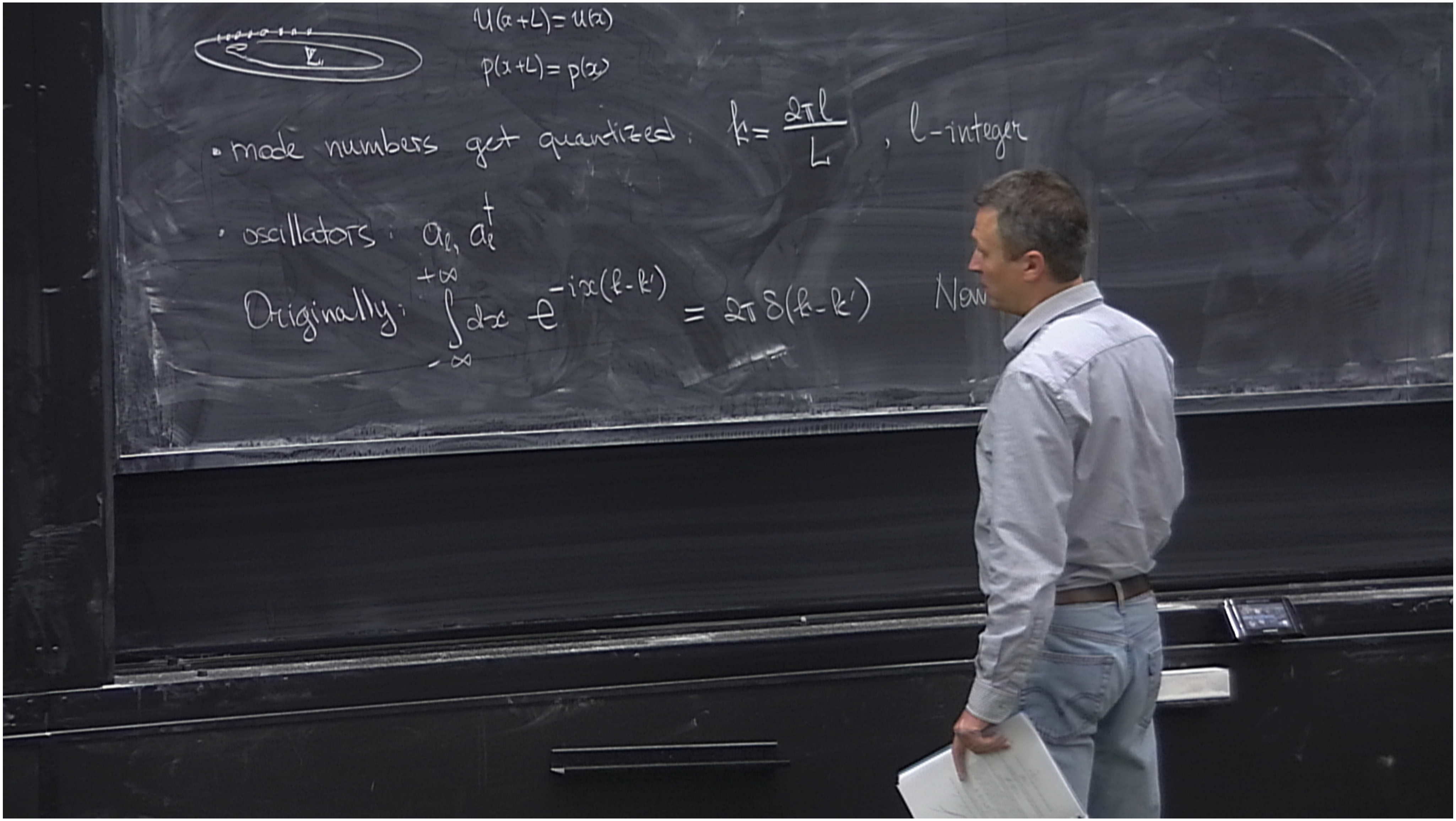

Impose periodic boundary conditions.

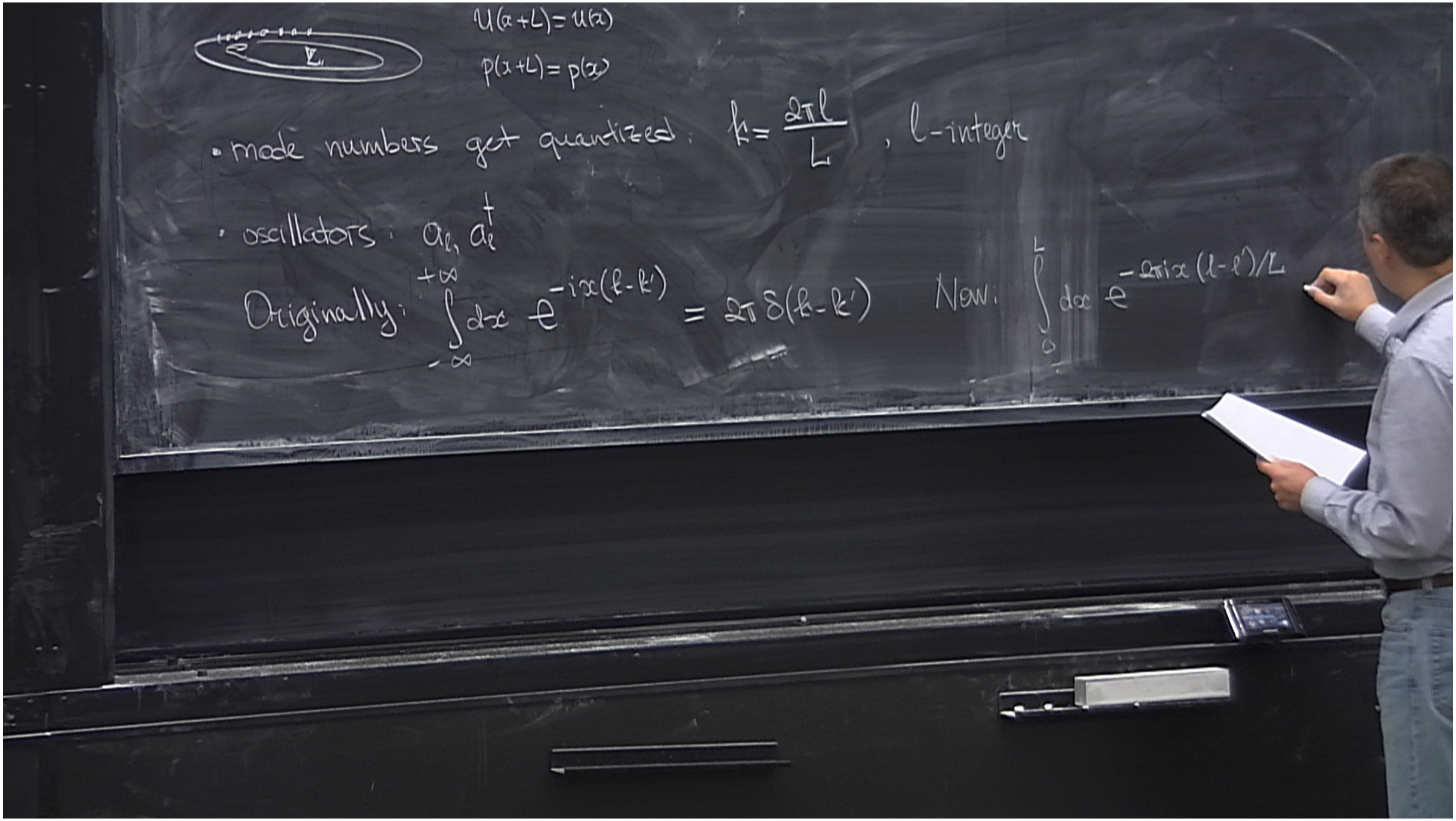


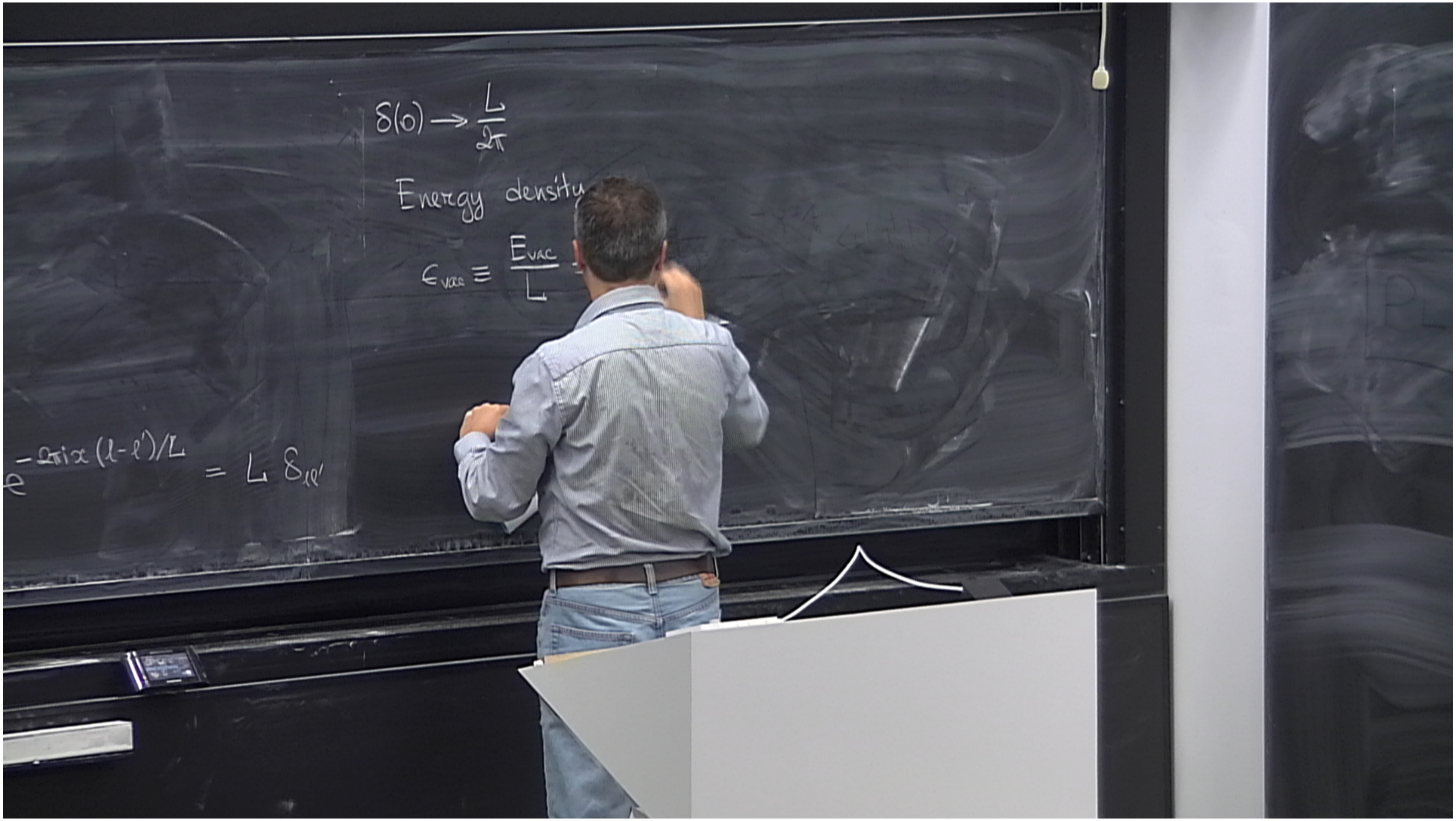
Impose periodic boundary conditions.
$u(x+L) = u(x)$
$p(x+L) = p(x)$
L
• mode numbers get quantized:

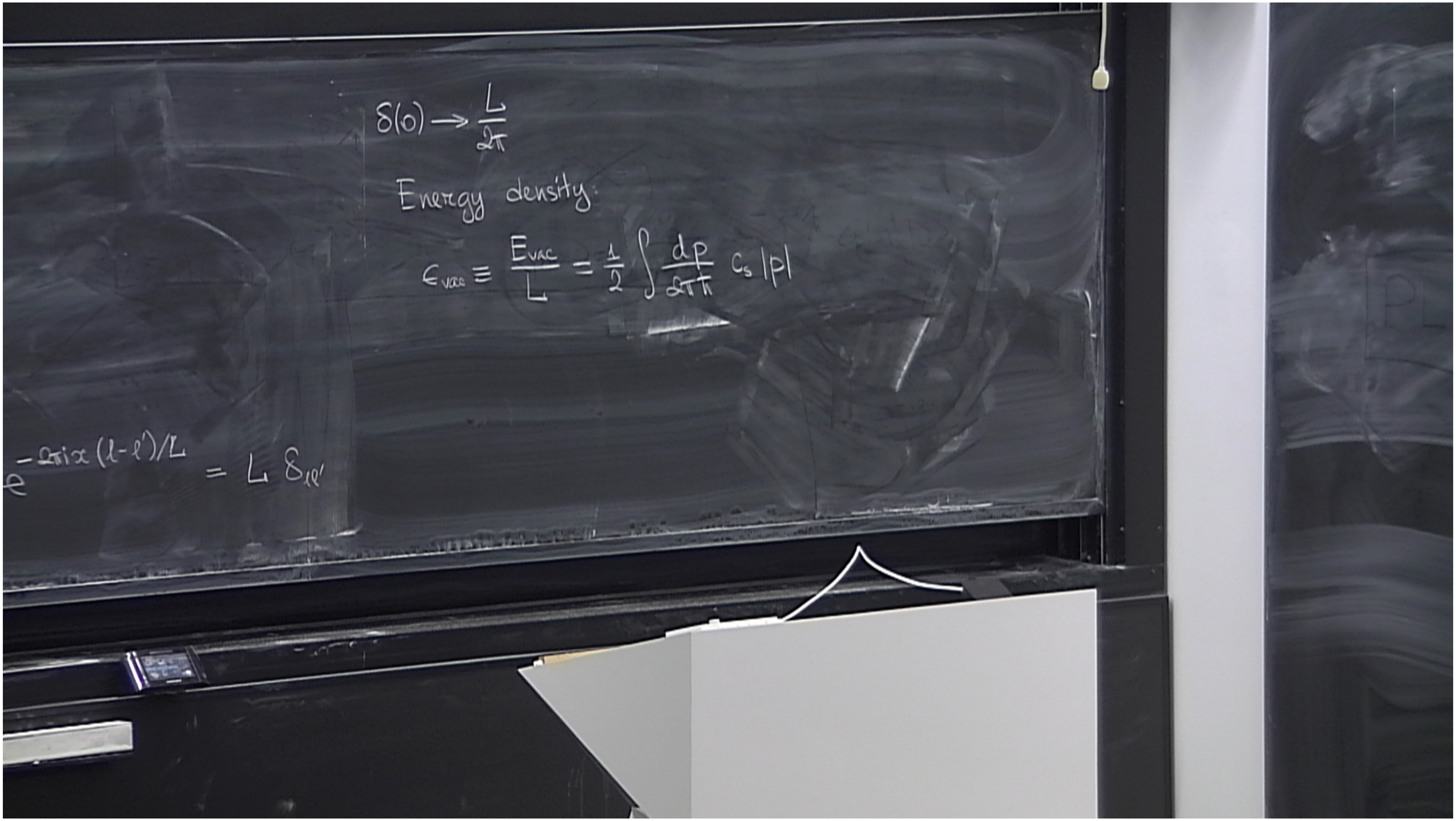

Impose periodic boundary conditions.
u(x+L)=u(x)
p(x+L)=p(x)
L
• mode numbers get quantized: k = 2πl/L , l-integer


U(x+L) = U(x)
p(x+L) = p(x)
• mode numbers get quantized: k = 2πl / L , l-integer
a_l, a_l^†

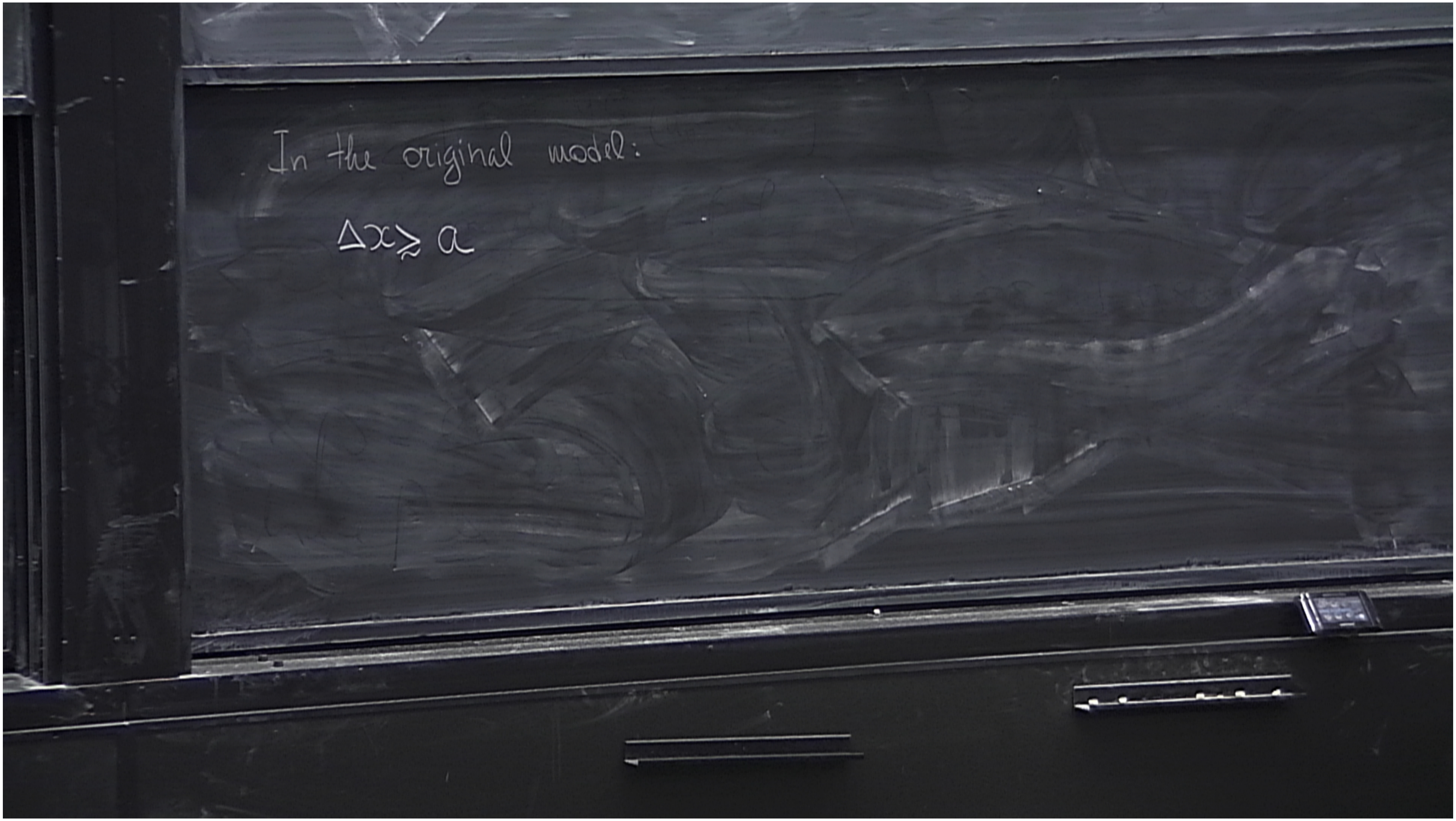

u(x+L) = u(x)
p(x+L) = p(x)
L
• mode numbers get quantized: k = 2πl / L , l-integer
• oscillators: a_l, a_l^†
Originally: ∫_{-∞}^{+∞} dx e^{-ix(k-k')} = 2π δ(k-k')
Now


$u(x+L)=u(x)$
$p(x+L)=p(x)$
L
• mode numbers get quantized: $k=\frac{2\pi l}{L}$, l-integer
• oscillators: $a_l, a_l^\dagger$
Originally: $\int_{-\infty}^{+\infty} dx\, e^{-ix(k-k')} = 2\pi\delta(k-k')$
Now: $\int_0^L dx\, e^{-2\pi i x(l-l')/L}$


$\delta(0) \to \frac{L}{2\pi}$
Energy density
$\epsilon_{vac} \equiv \frac{E_{vac}}{L}$
$e^{-2\pi i x (l-l')/L} = L\, \delta_{ll'}$

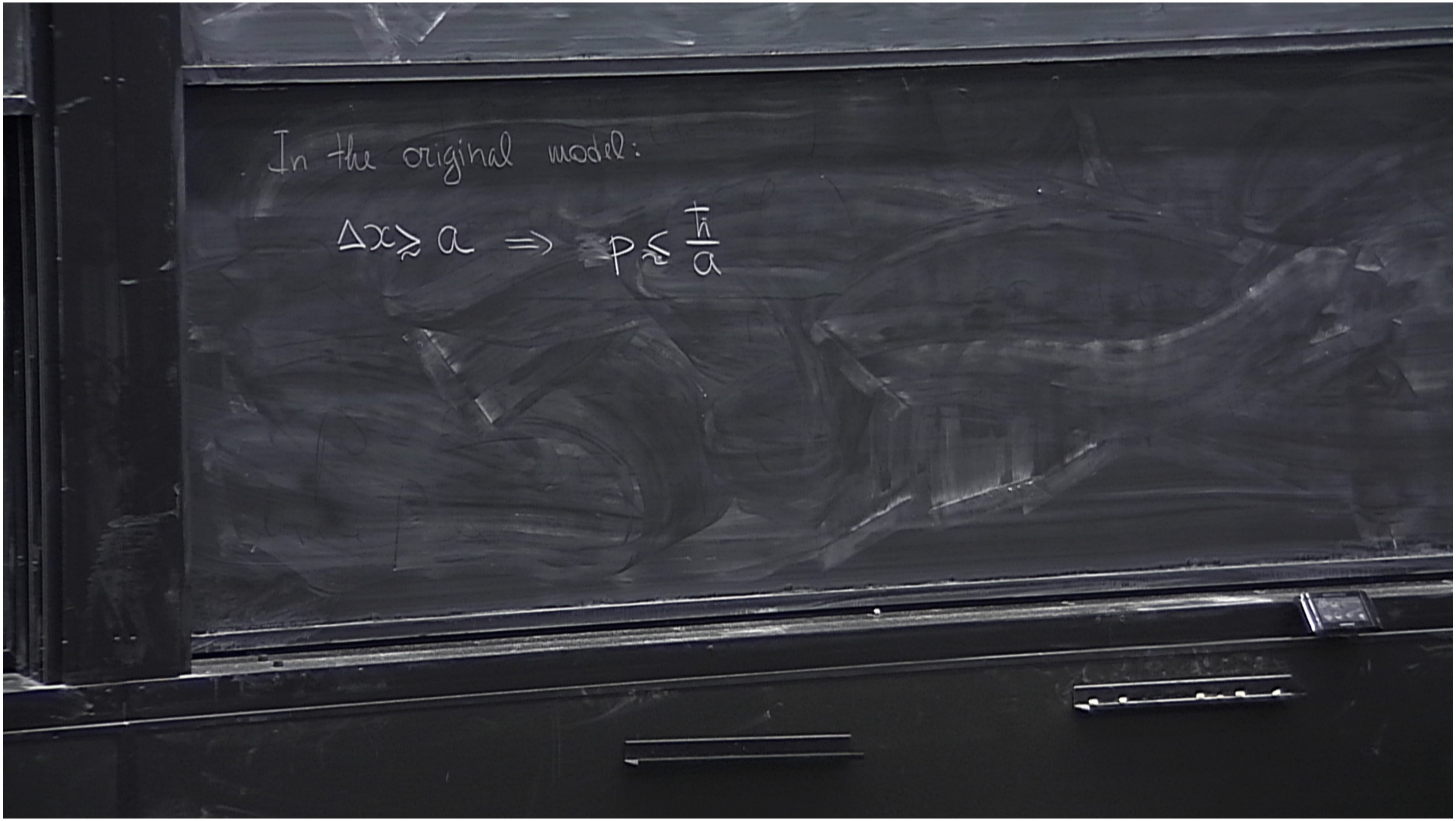

$\delta(0) \to \frac{L}{2\pi}$
Energy density:
$\epsilon_{vac} \equiv \frac{E_{vac}}{L} = \frac{1}{2}\int \frac{dp}{2\pi\hbar}\, c_s |p|$
$e^{-2\pi i x(l-l')/L} = L\,\delta_{ll'}$

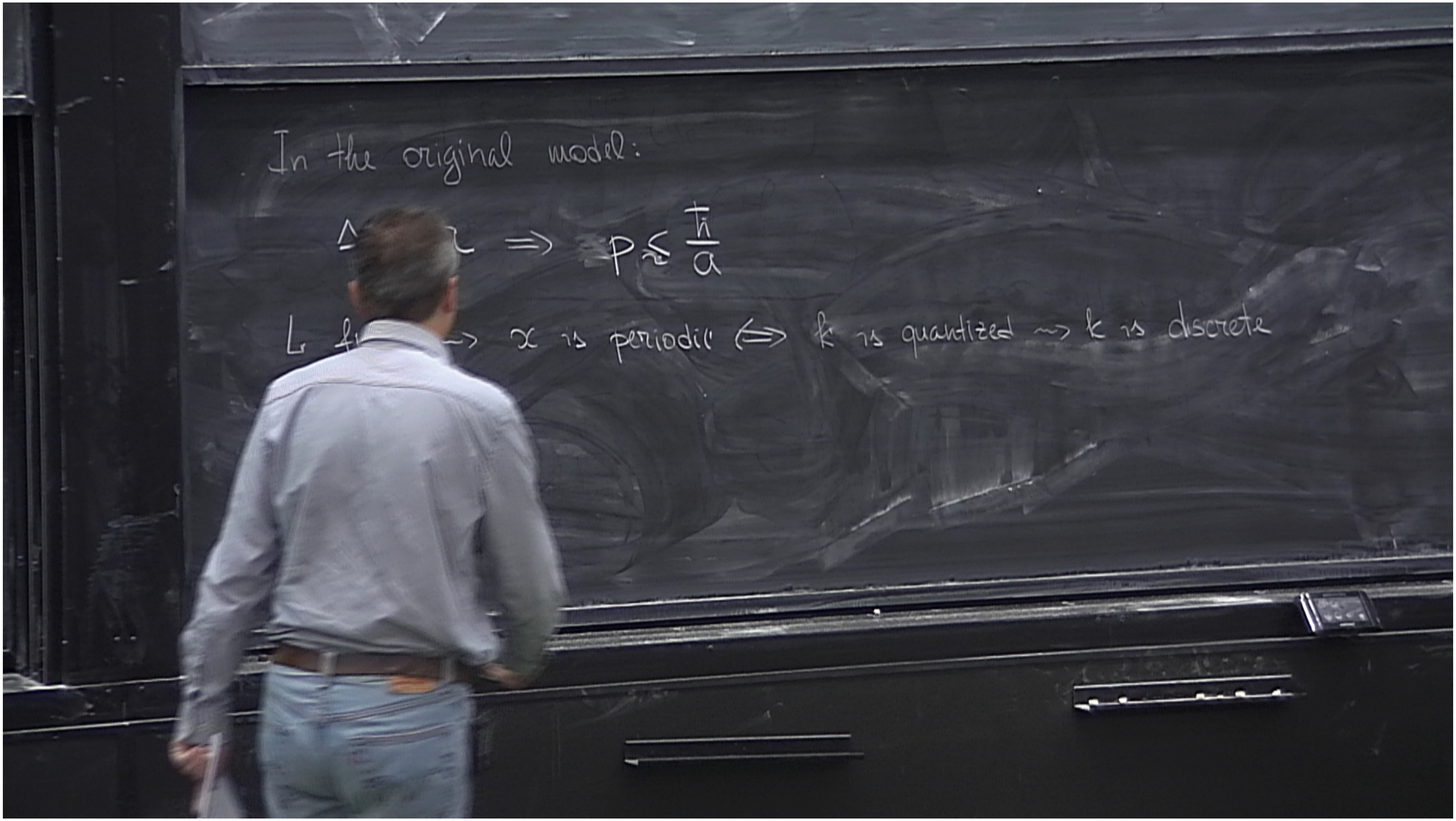

δ(0) → L/2π
Energy density:
ε_vac ≡ E_vac/L = 1/2 ∫ dp/2πħ c_s |p|
e^{-2πix(l-l')/L} = L δ_{ll'}

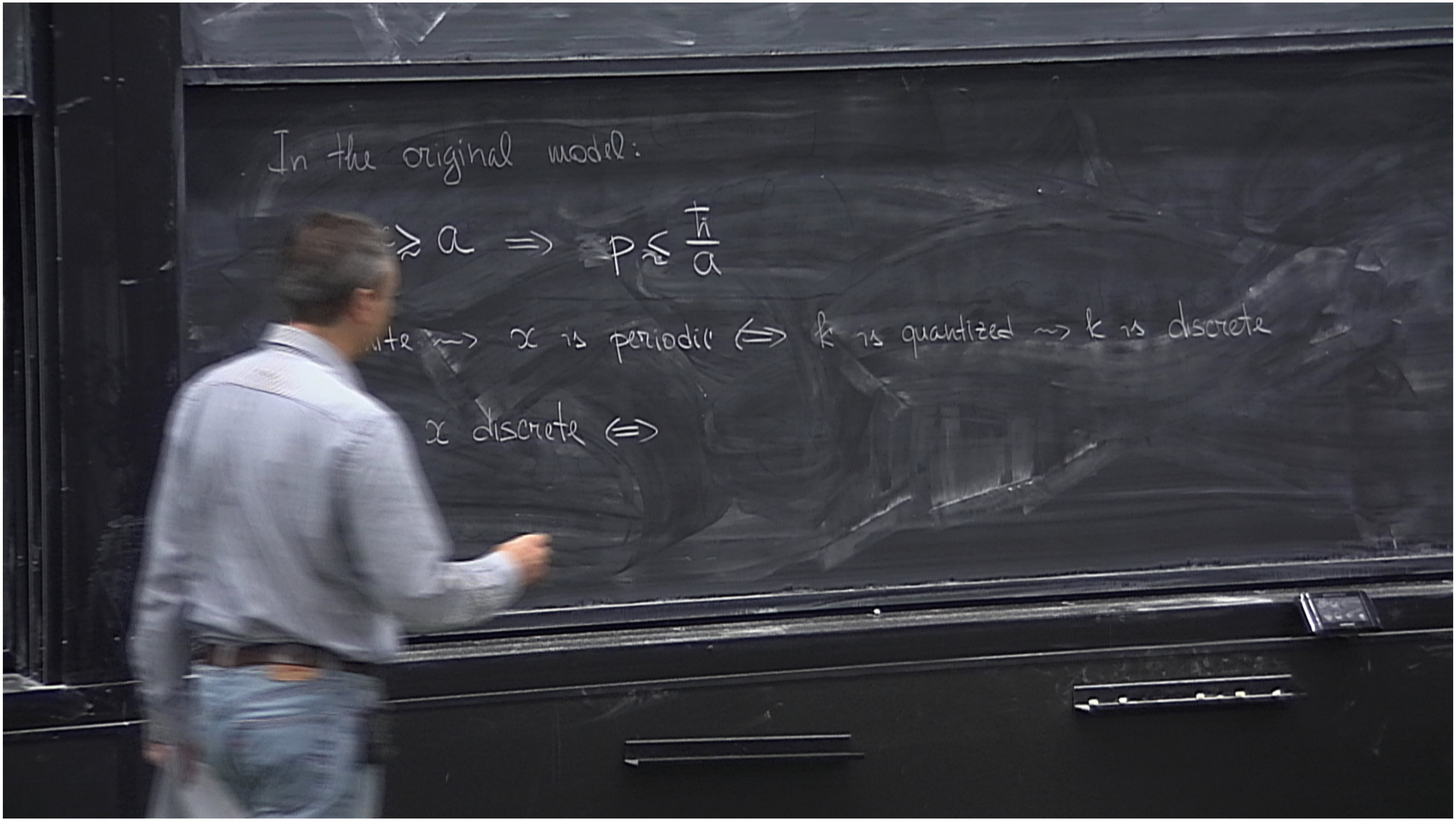

In the original model:
$\Delta x \gtrsim a$

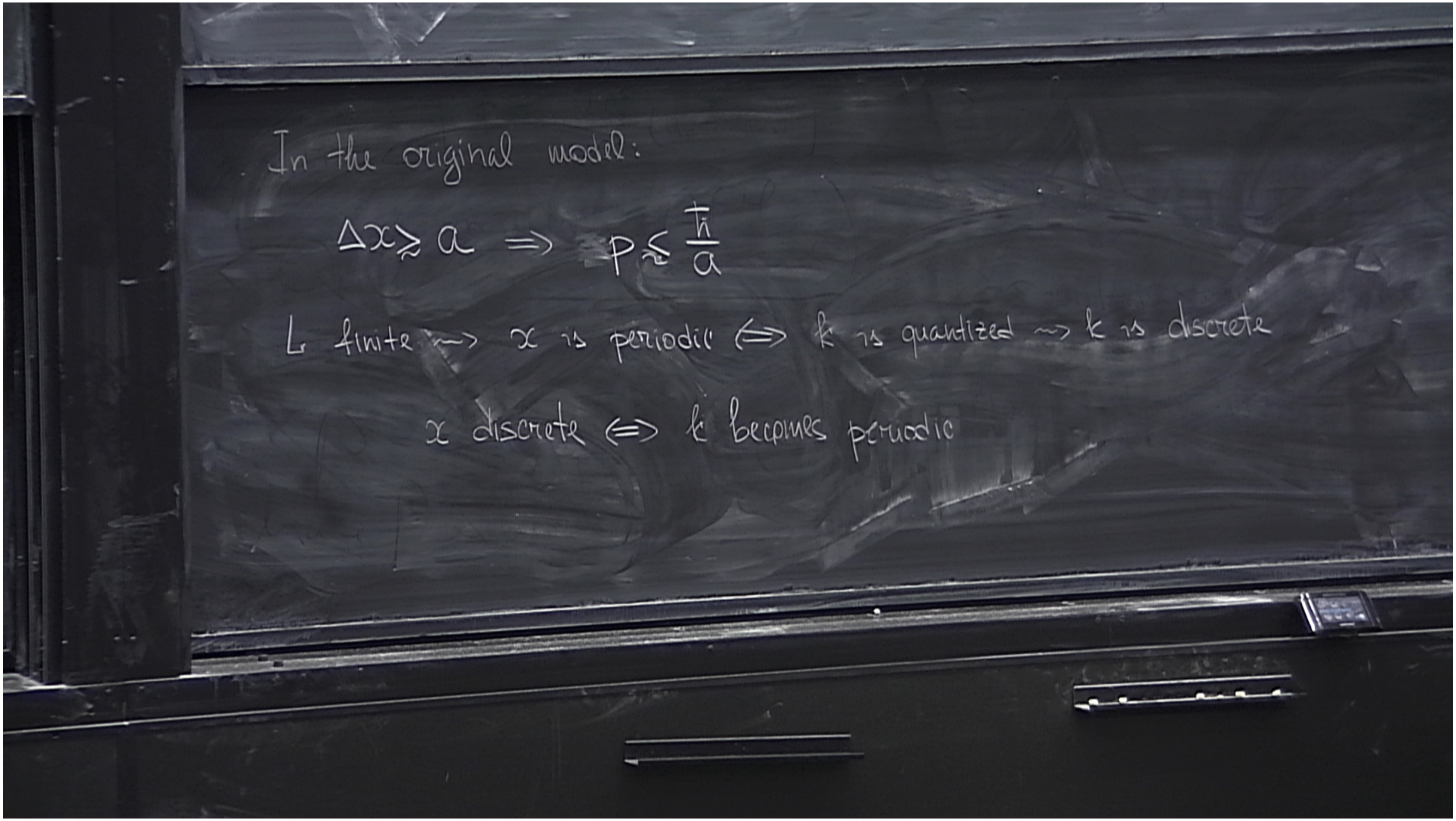
In the original model:

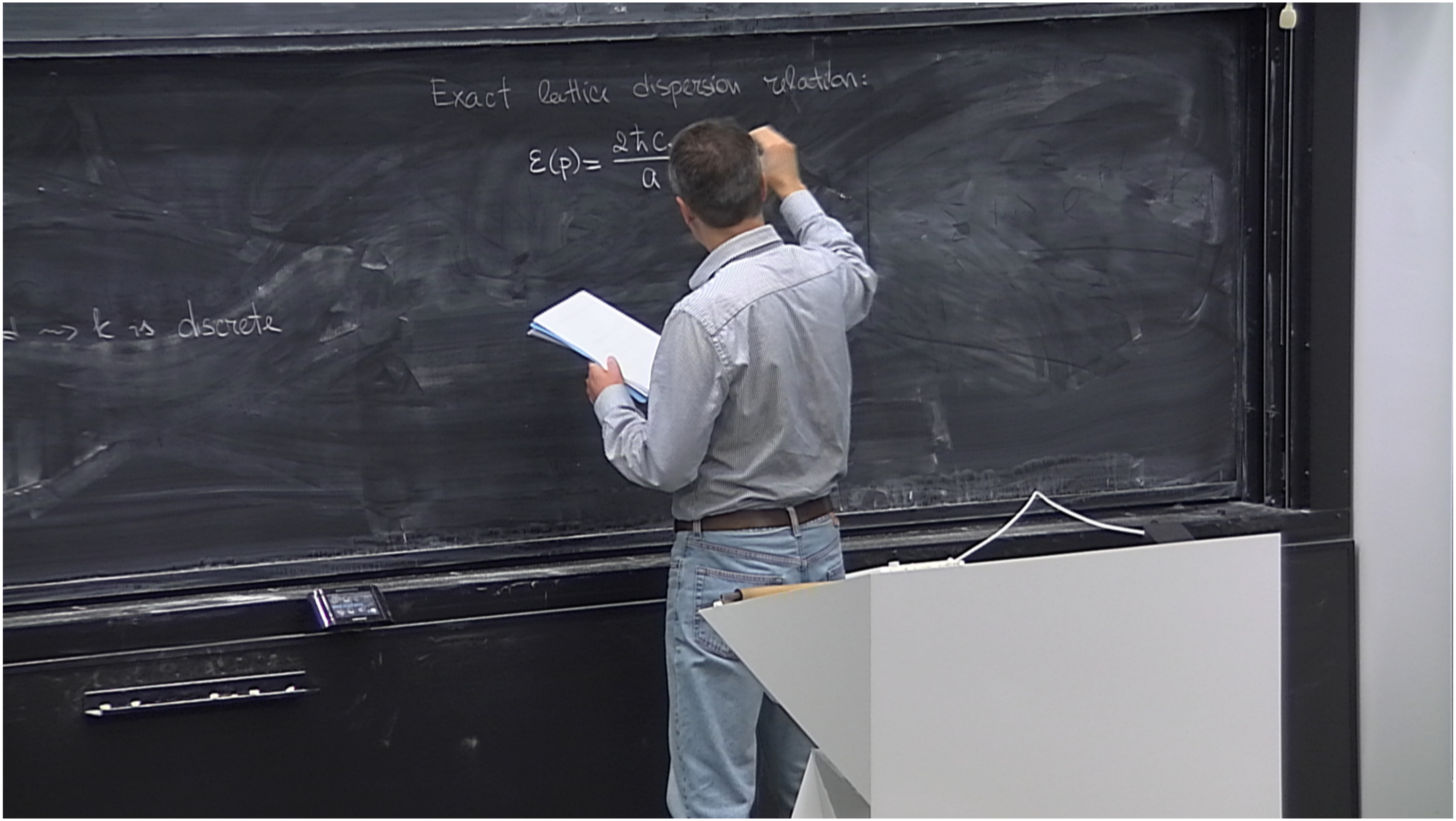
$$\Delta x \gtrsim a \implies \Delta p \lesssim \frac{\hbar}{a}$$


In the original model:
$\Delta x \gtrsim a \Rightarrow p \lesssim \frac{\hbar}{a}$

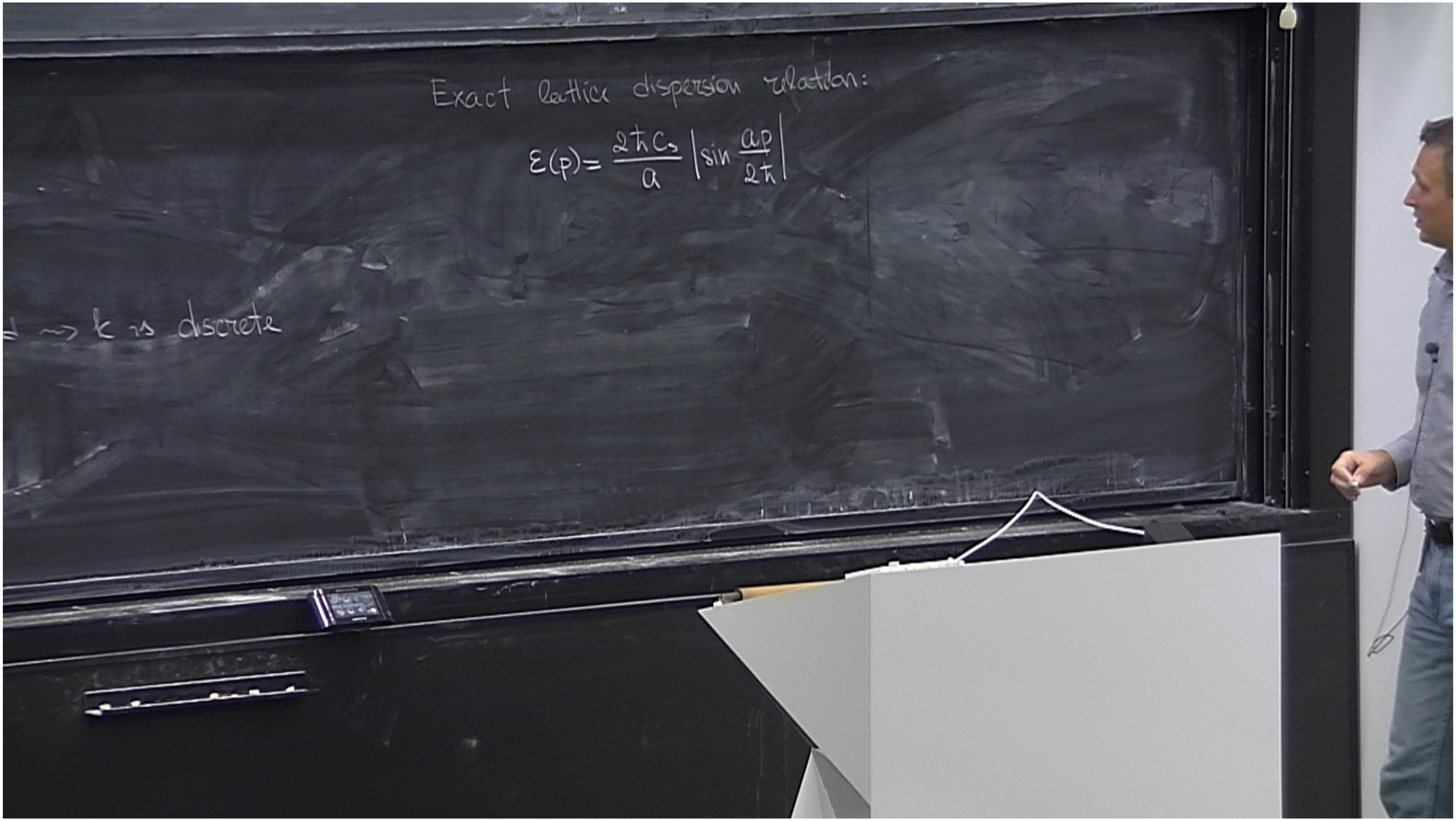

In the original model:
⇒ p ≲ ħ/a
x is periodic ⟺ k is quantized ⇝ k is discrete

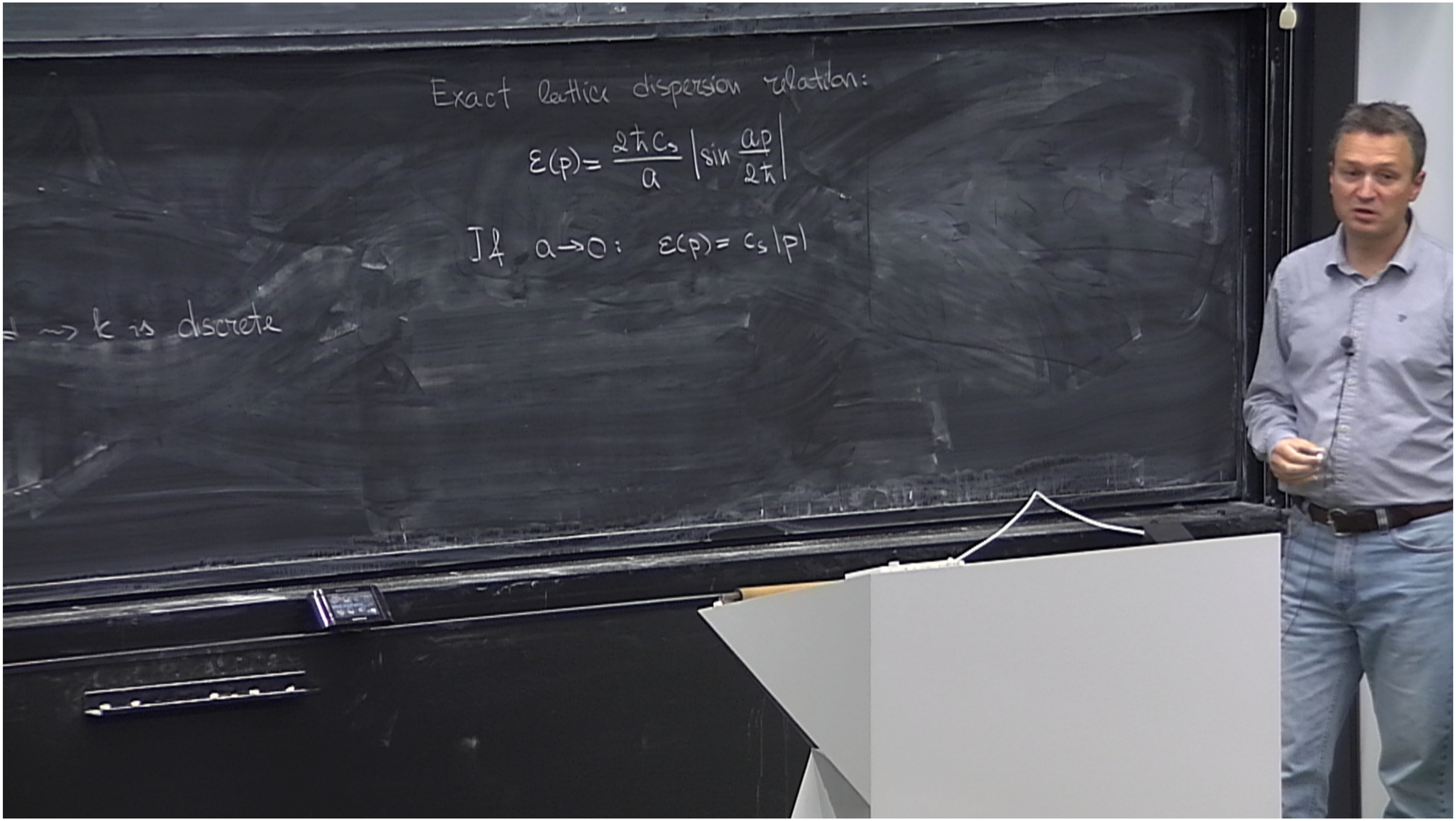

In the original model:
≳ a ⟹ p ≲ ħ/a
x is periodic ⟺ k is quantized ⇝ k is discrete
x discrete ⟺

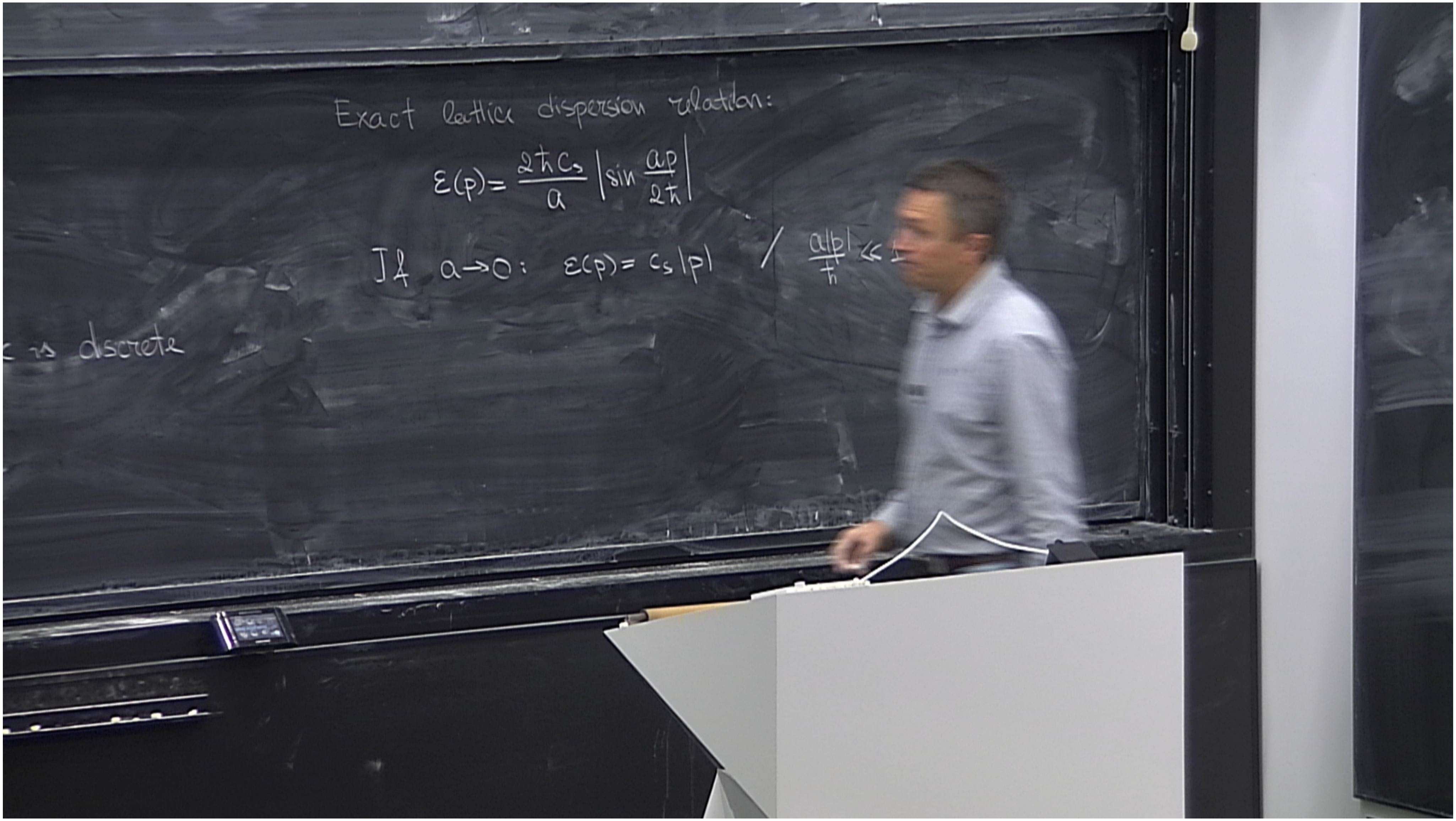

In the original model:
Δx ≳ a ⇒ p ≲ ħ/a
L finite ⇝ x is periodic ⟺ k is quantized ⇝ k is discrete
x discrete ⟺ k becomes periodic

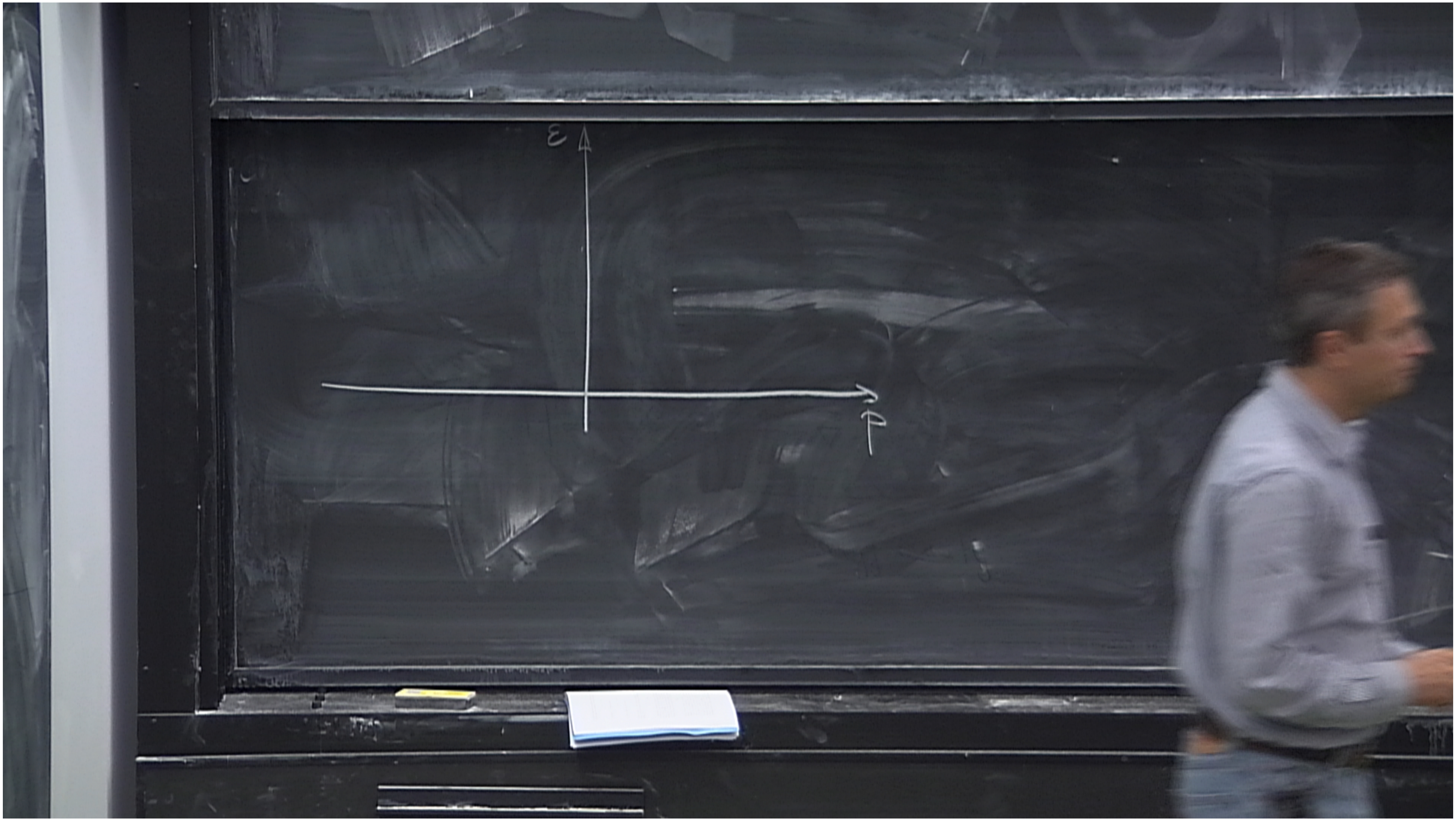


Exact lattice dispersion relation:
ε(p) = 2ħc / a
k is discrete

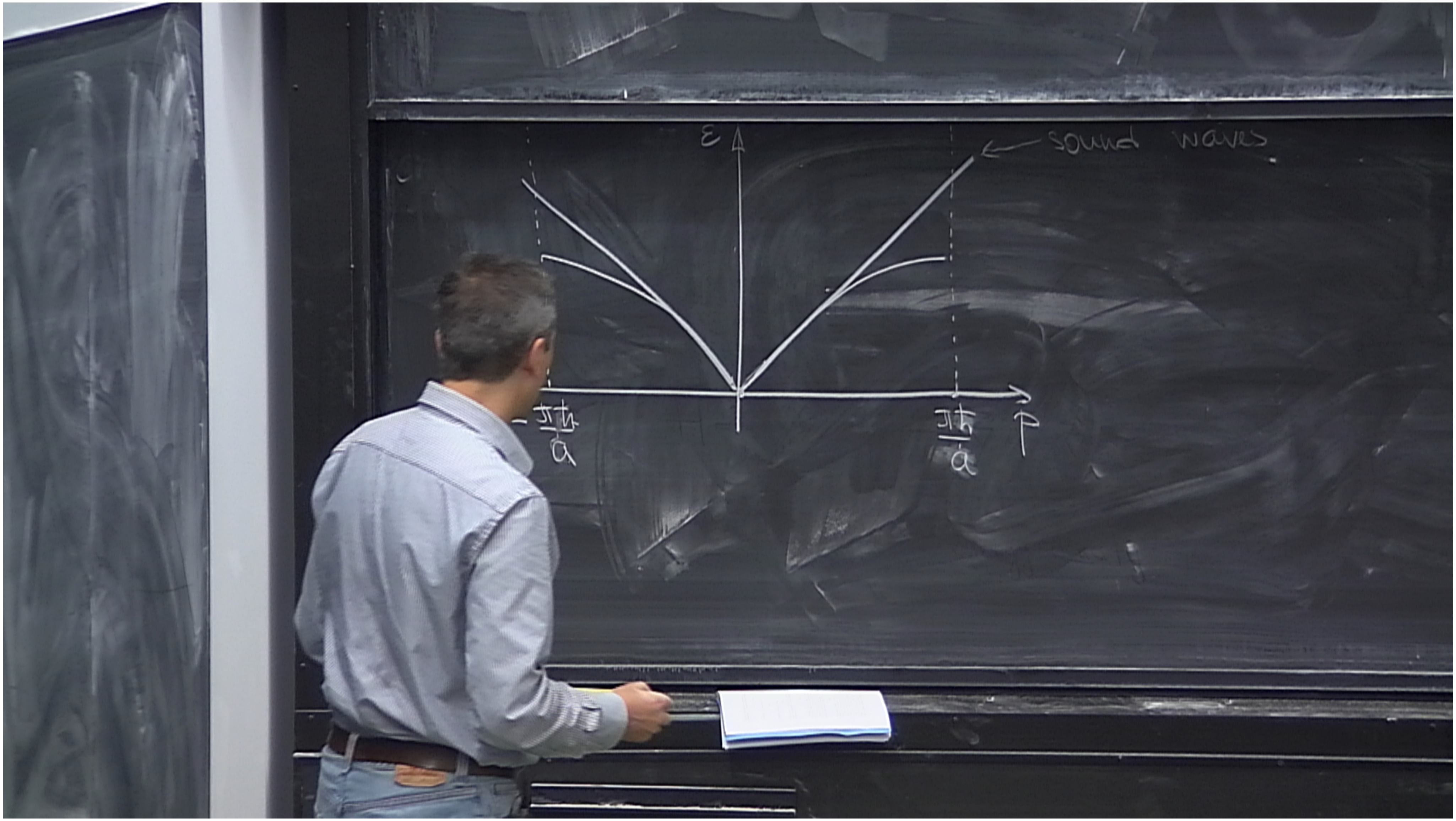

Exact lattice dispersion relation:
$\varepsilon(p) = \frac{2\hbar c_s}{a}\left|\sin\frac{ap}{2\hbar}\right|$
⇝ k is discrete


Exact lattice dispersion relation:
$\varepsilon(p) = \frac{2\hbar c_s}{a} \left| \sin \frac{ap}{2\hbar} \right|$
If $a \to 0$: $\varepsilon(p) = c_s |p|$
$\rightsquigarrow$ k is discrete

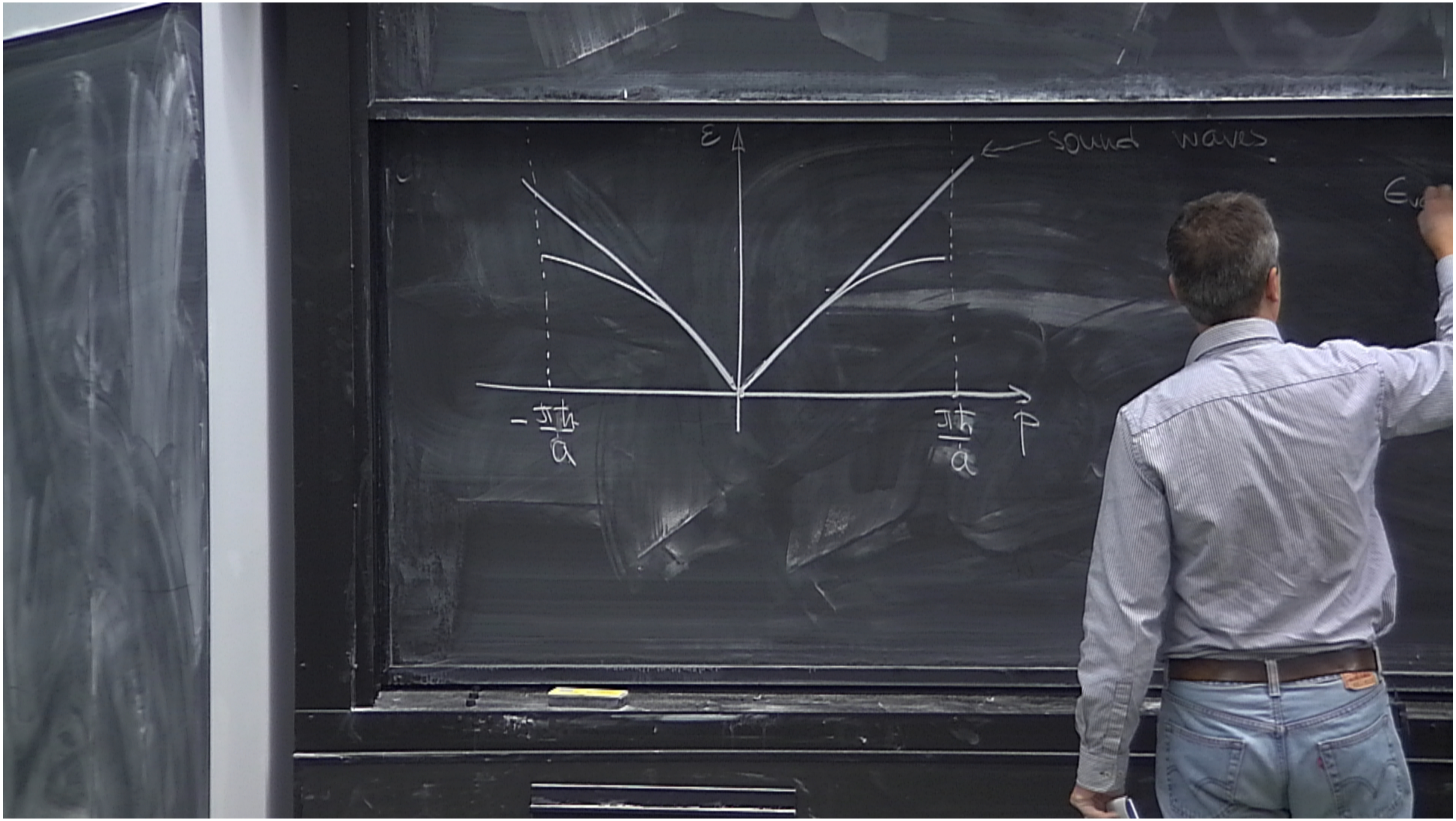

Exact lattice dispersion relation:
ε(p) = (2ħ c_s / a) |sin (ap / 2ħ)|
If a→0: ε(p) = c_s |p| / a|p|/ħ ≪
is discrete


ε
p

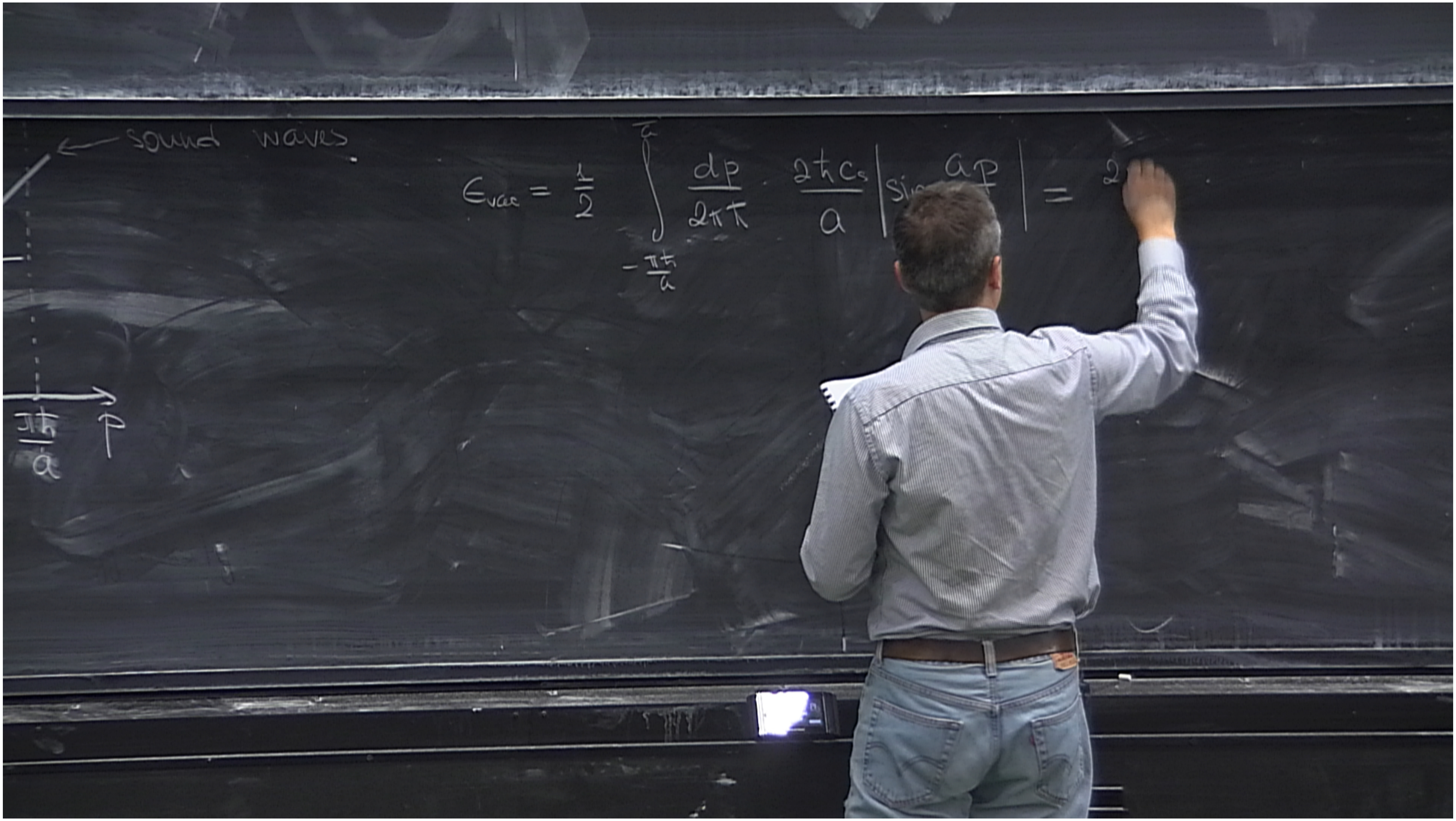

E
P



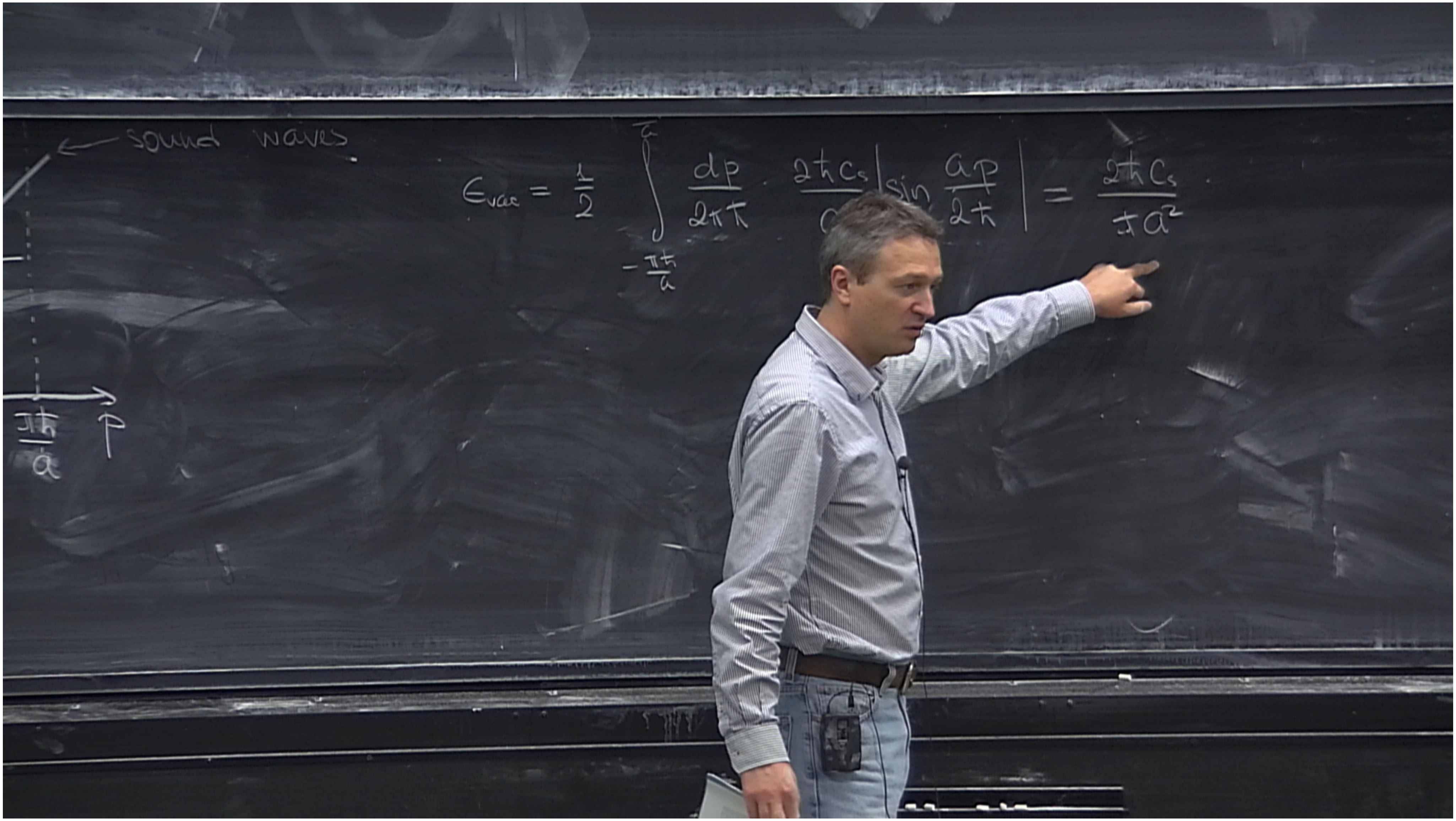
ε
sound waves
$-\frac{\pi\hbar}{a}$
$\frac{\pi\hbar}{a}$
p

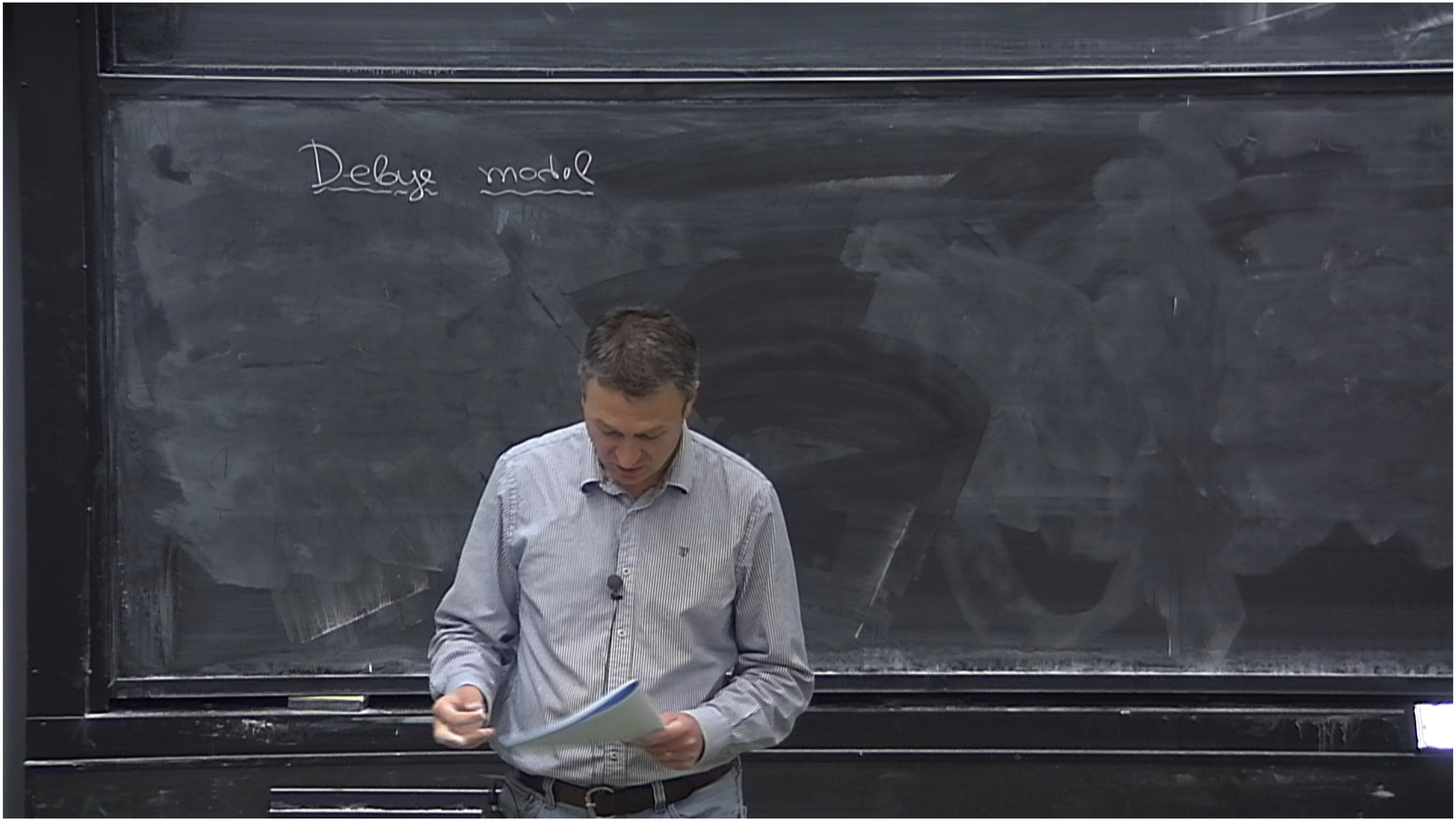


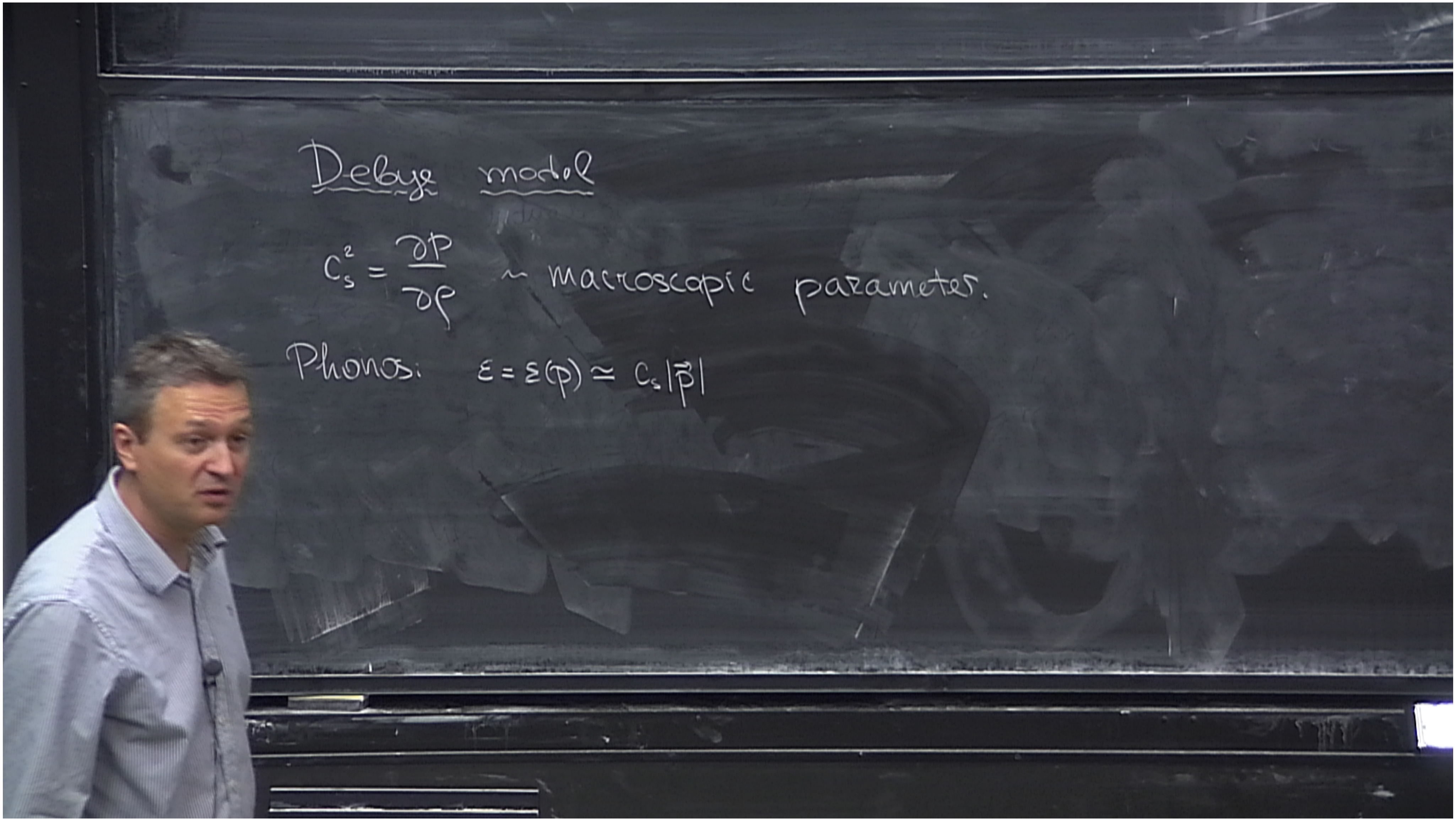
ε
sound waves
$-\frac{\pi\hbar}{a}$
$\frac{\pi\hbar}{a}$
p

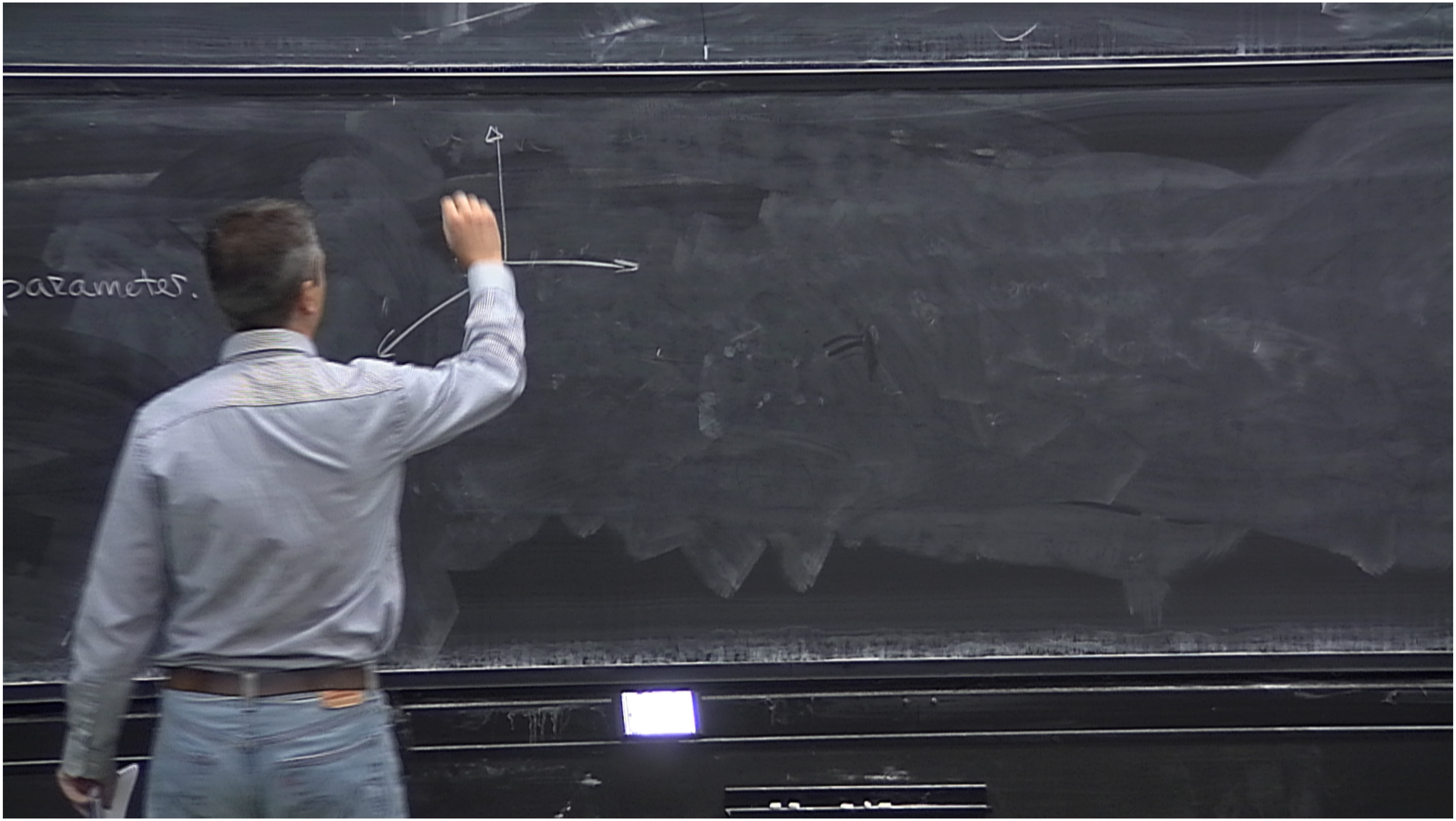


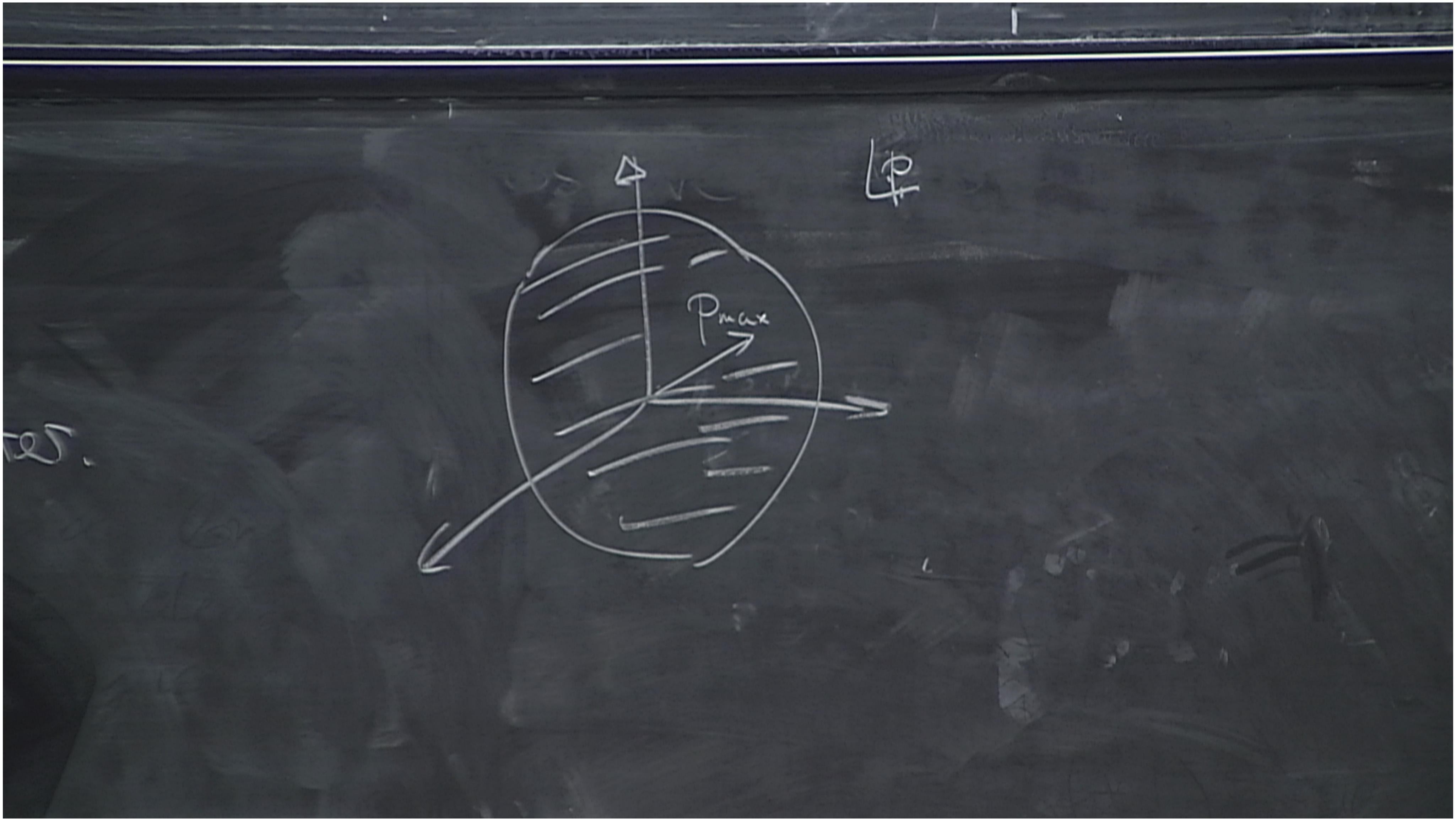
sound waves


sound waves
$\epsilon_{vac} = \frac{1}{2} \int_{-\pi\hbar/a}^{\pi\hbar/a} \frac{dp}{2\pi\hbar} \frac{2\hbar c_s}{a} \left| \sin \frac{ap}{2\hbar} \right| = \frac{2\hbar c_s}{\pi a^2}$
p

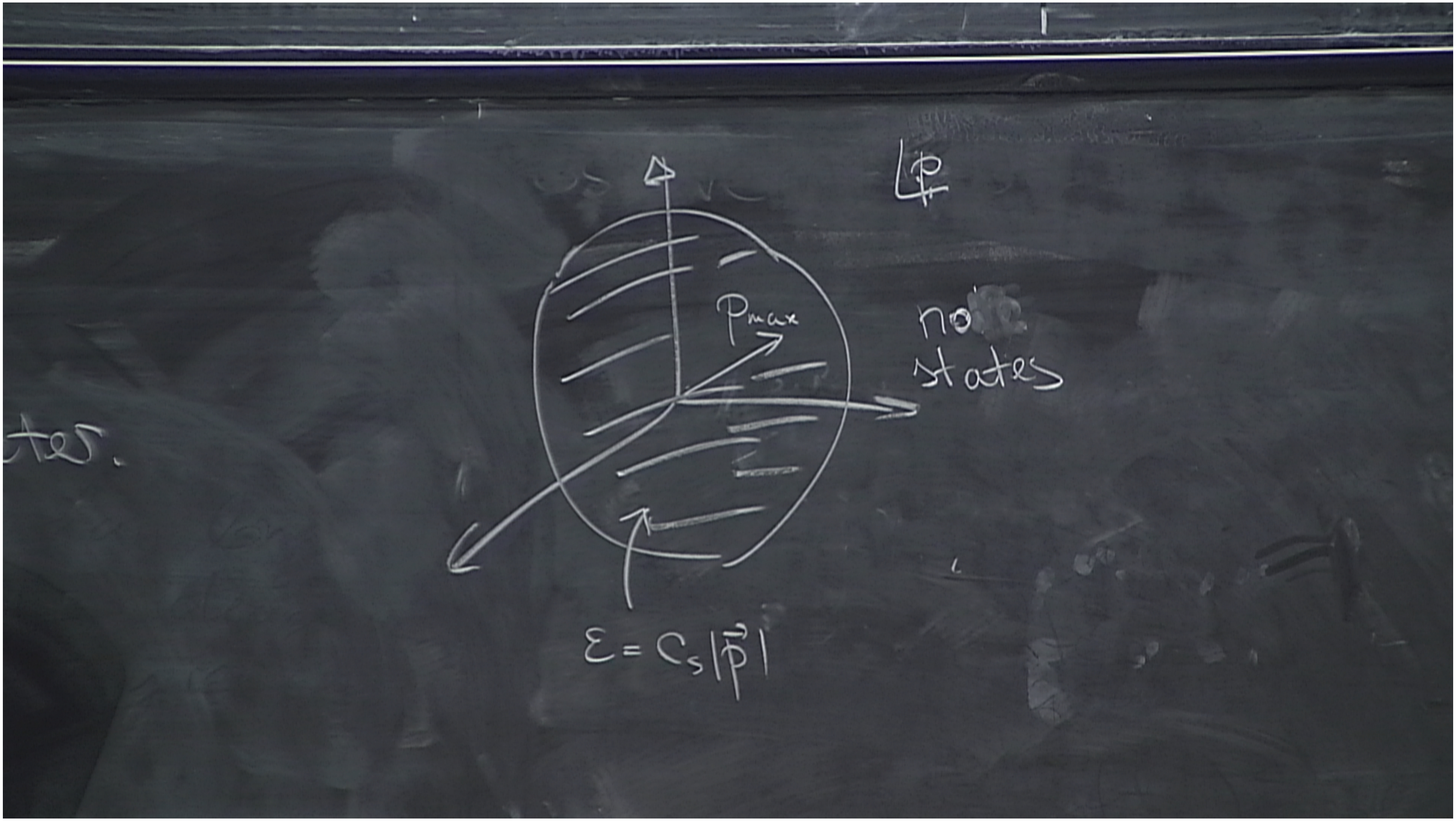

Debye model


Debye model
$c_s^2 = \frac{\partial p}{\partial \rho}$ ~ macroscopic parameter.
Phonos: $\varepsilon = \varepsilon(p) \simeq c_s|\vec{p}|$

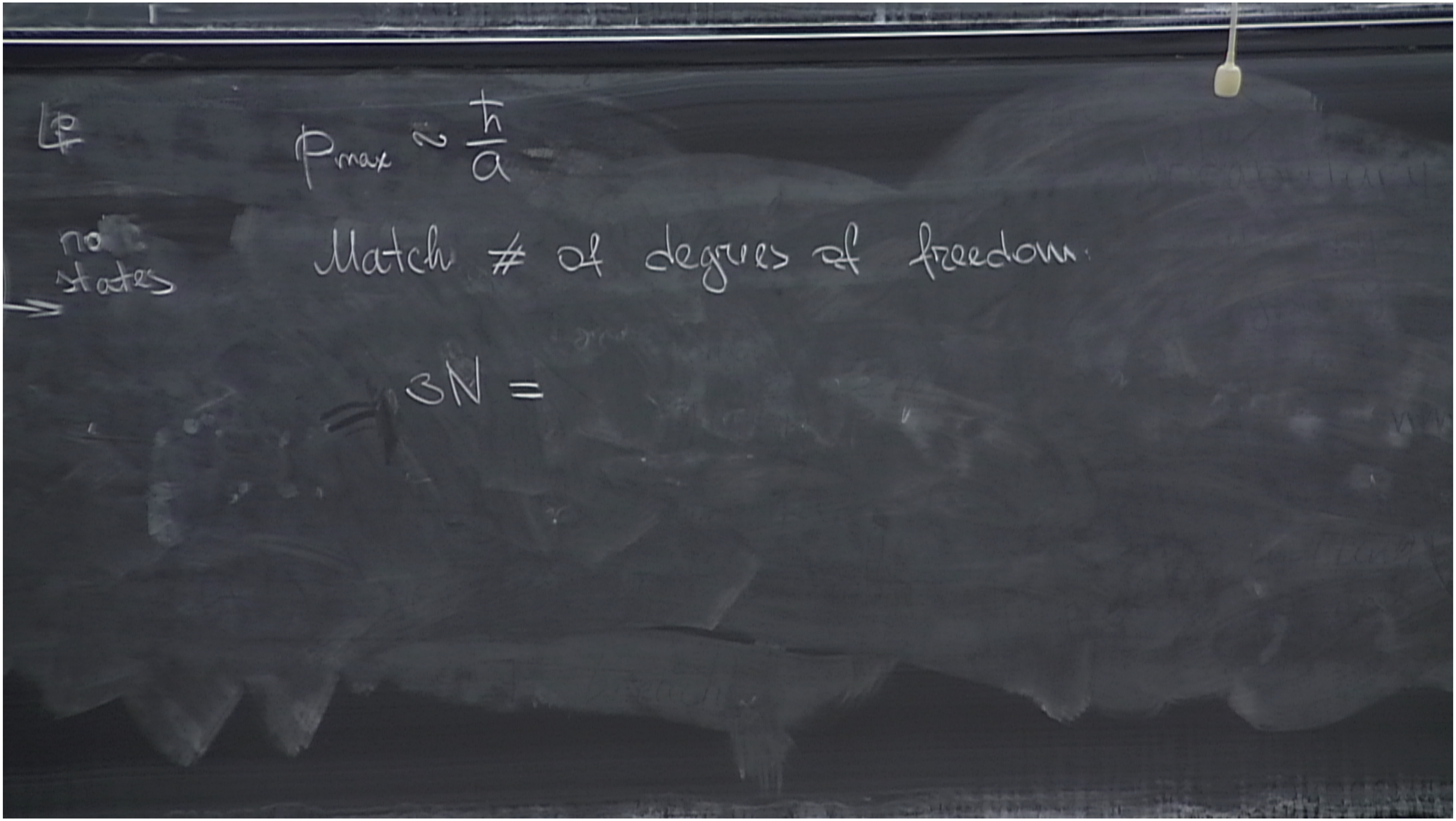

parameter.

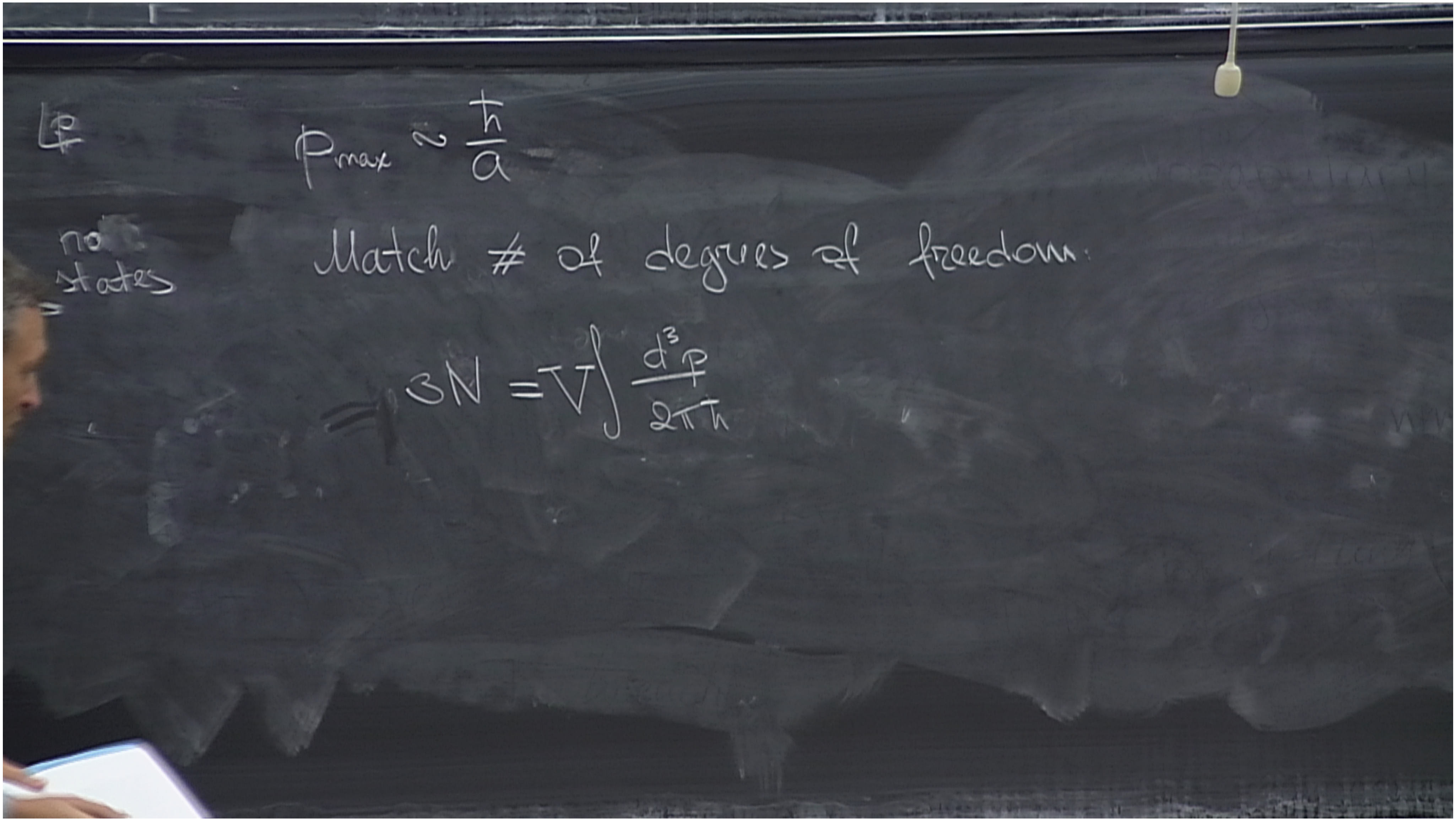


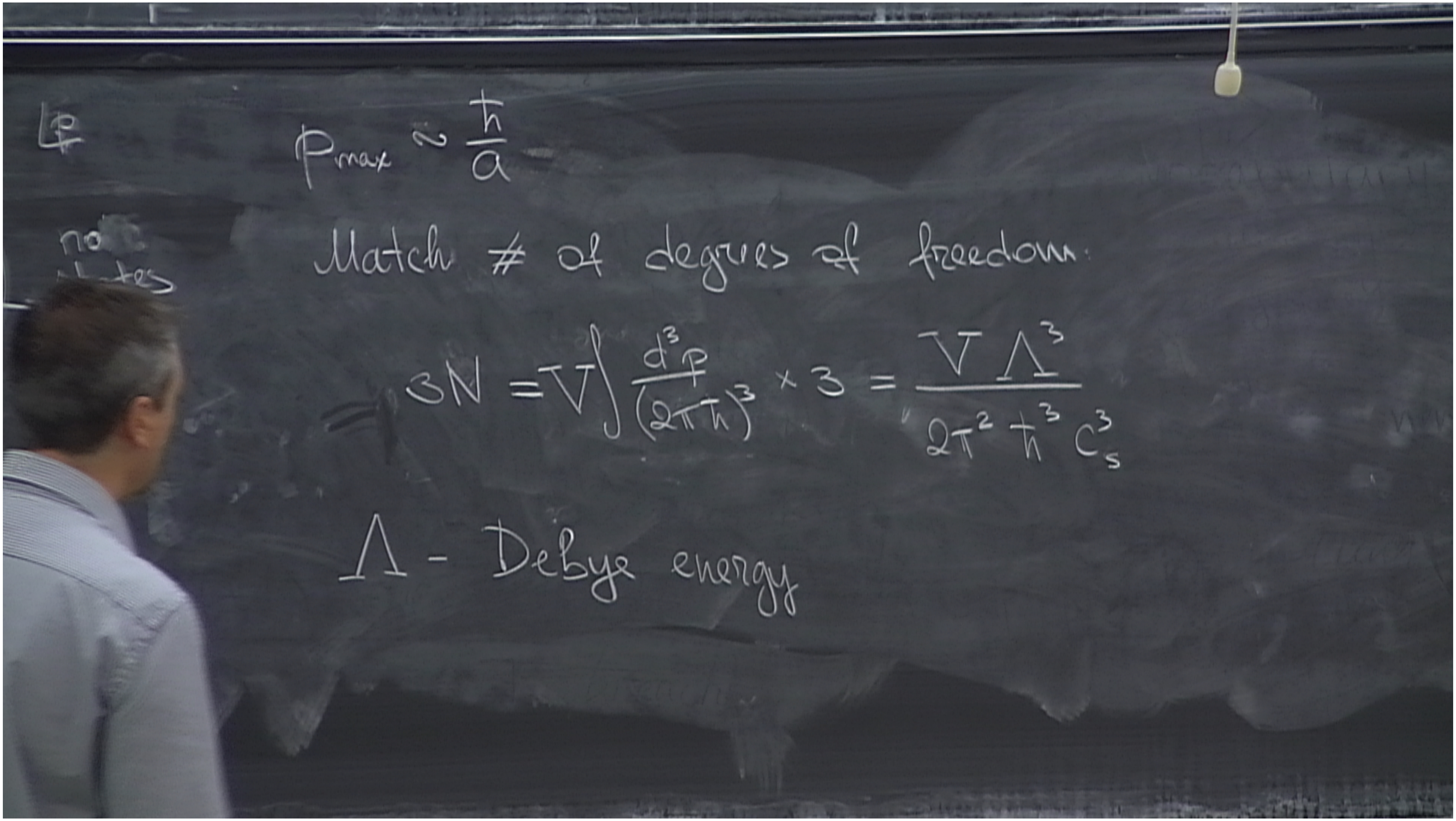
p_{max}

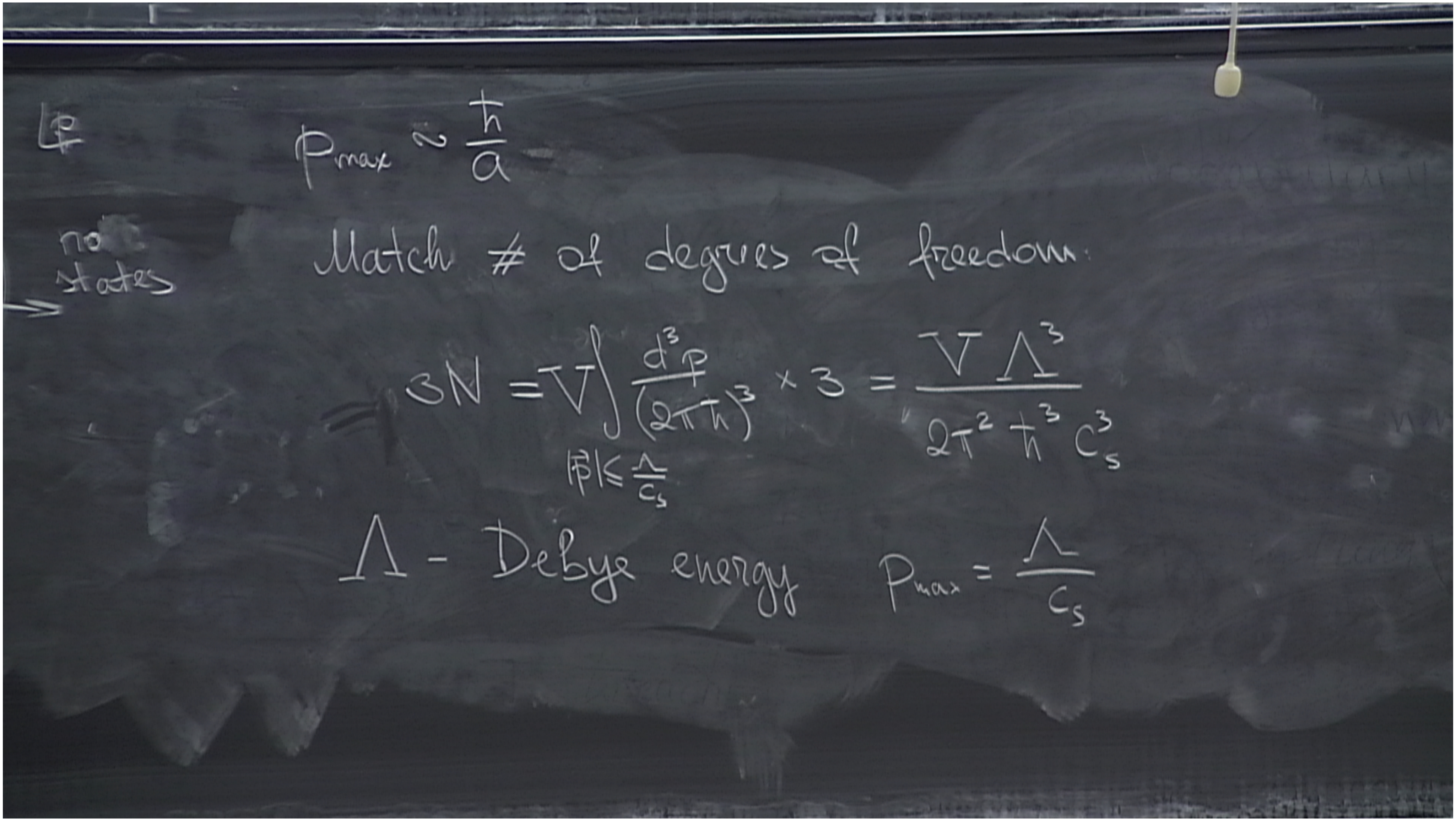


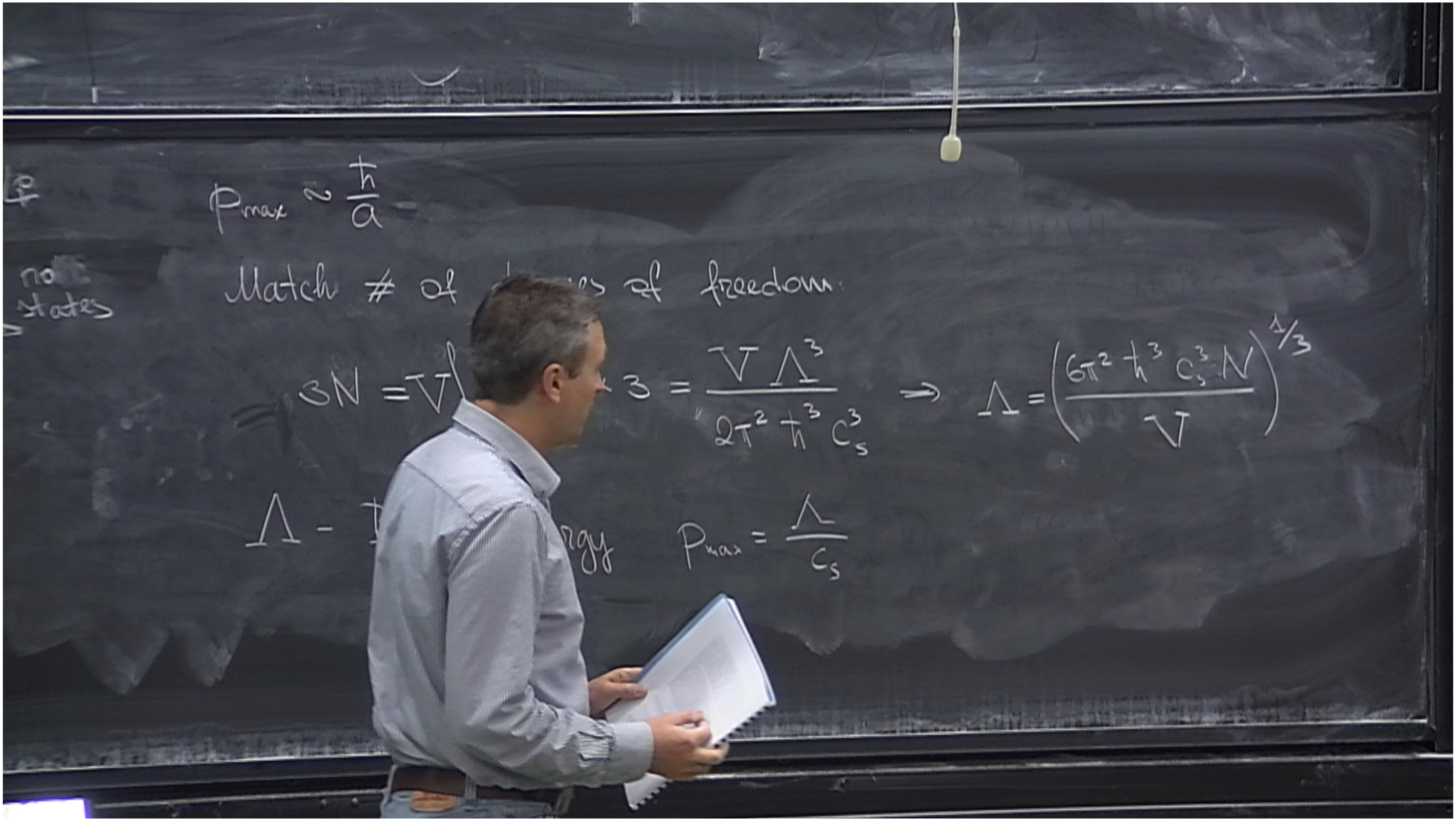
ates.
p_{max}
no
states
$\varepsilon = c_s|\vec{p}|$


$p_{max} \approx \frac{\hbar}{a}$
Match # of degrees of freedom
$3N =$


p_max ~ ħ/a
Match # of degrees of freedom:
3N = V ∫ d³p / 2πħ
states


$p_{max} \approx \frac{\hbar}{a}$
Match # of degrees of freedom:
$3N = V\int \frac{d^3p}{(2\pi\hbar)^3} \times 3 = \frac{V\Lambda^3}{2\pi^2 \hbar^3 c_s^3}$
Λ - Debye energy


$p_{max} \sim \frac{\hbar}{a}$
Match # of degrees of freedom:
$\Rightarrow 3N = V \int_{|\vec{p}| \le \frac{\Lambda}{c_s}} \frac{d^3p}{(2\pi\hbar)^3} \times 3 = \frac{V \Lambda^3}{2\pi^2 \hbar^3 c_s^3}$
Λ - Debye energy $p_{max} = \frac{\Lambda}{c_s}$



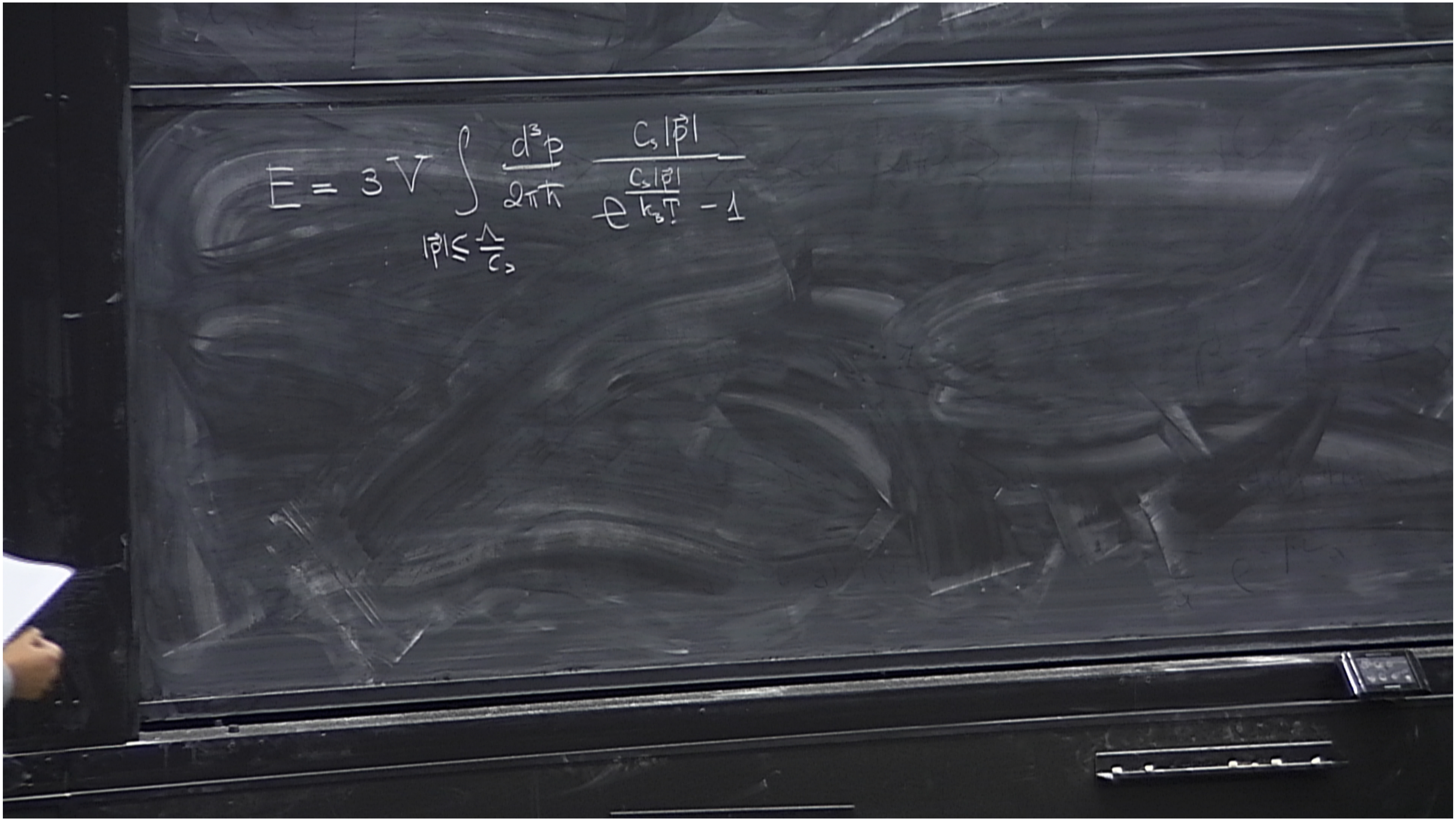
Pmax ≈ ħ/a
Match # of
of freedom
states
3N =
3 = VΛ³ / 2π² ħ³ cs³
Λ = (6π² ħ³ cs³ N / V)^1/3
Λ -
Pmax = Λ / cs

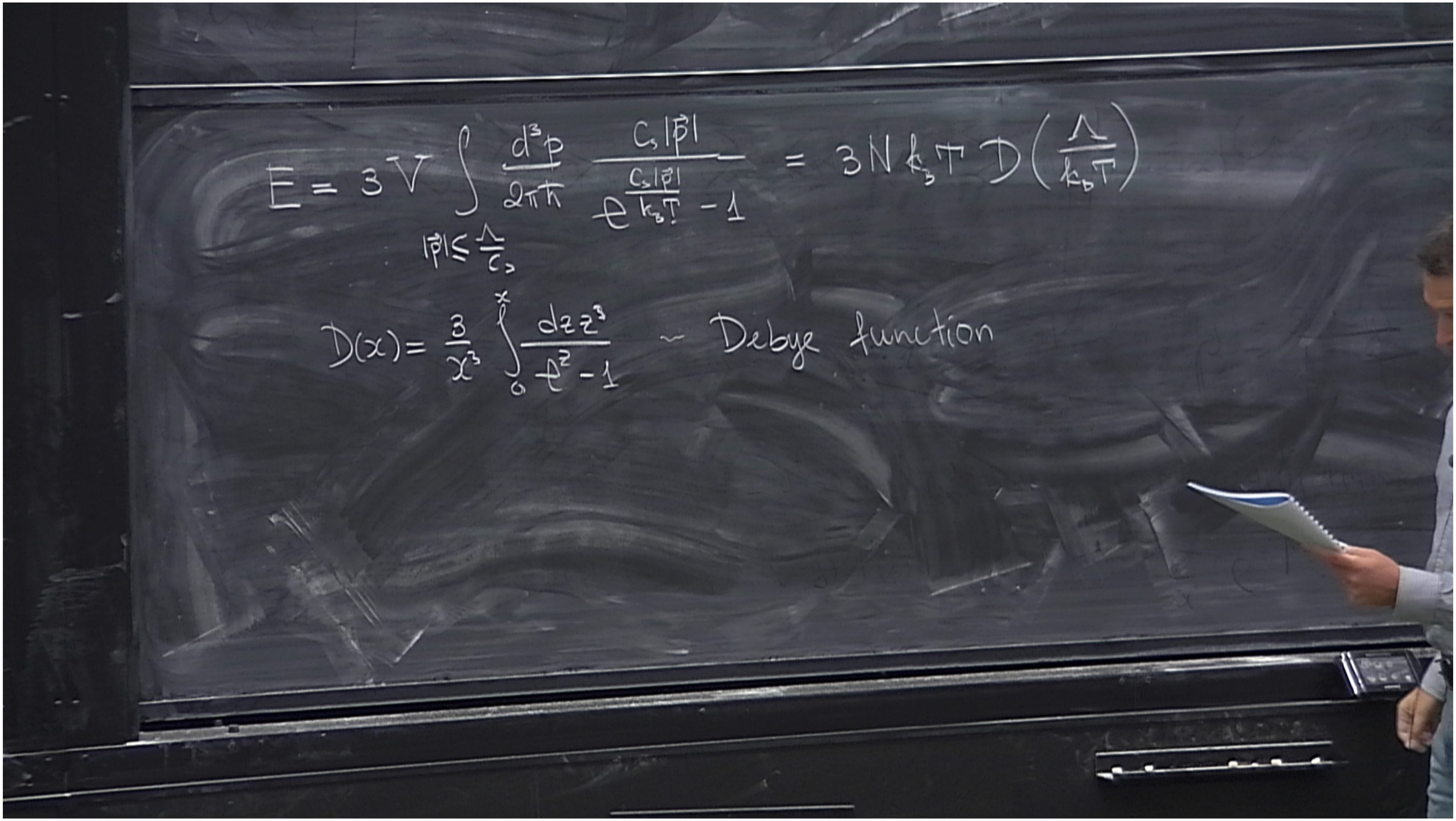
$$p_{max} \sim \frac{\hbar}{a}$$

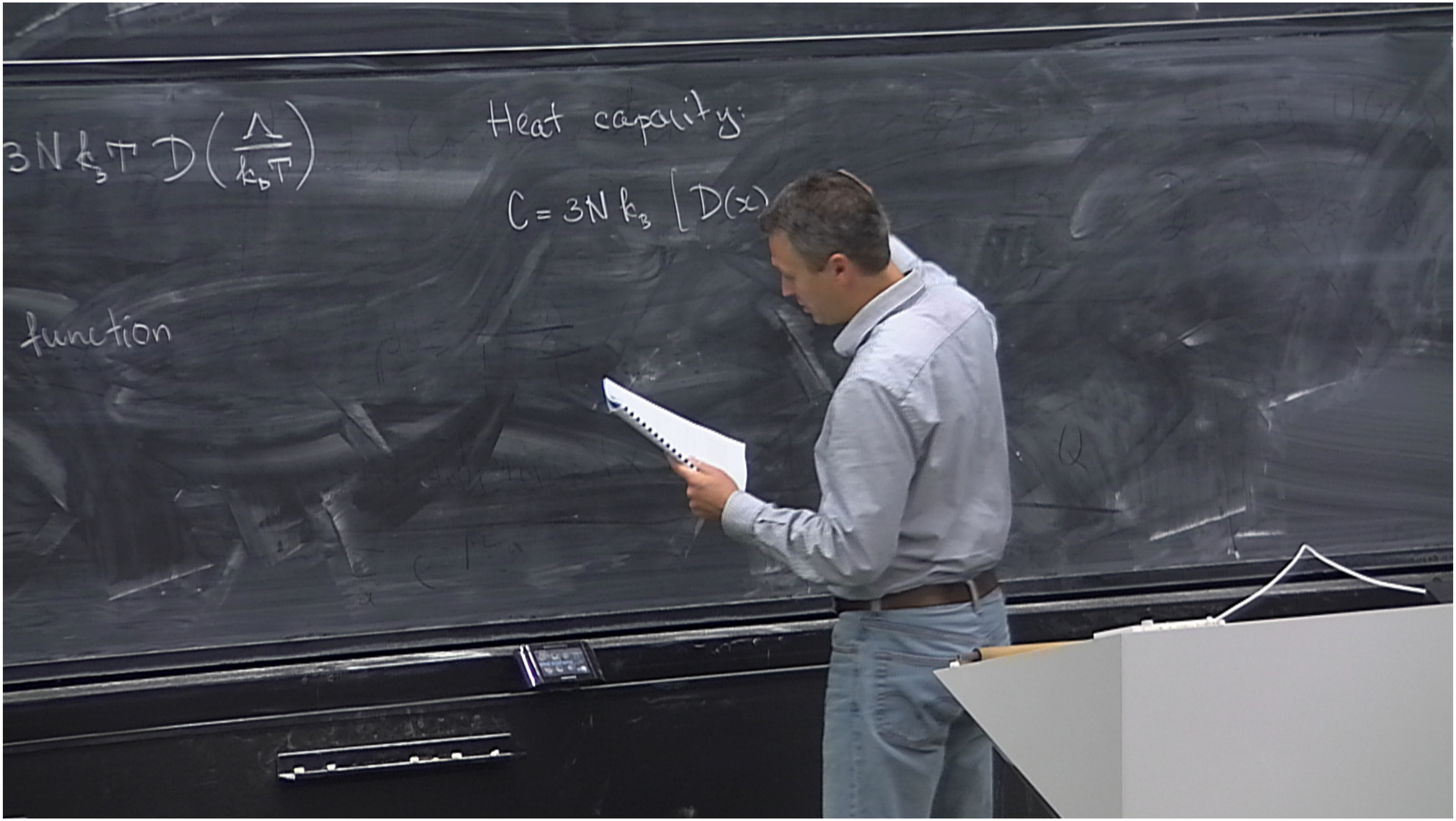
Match # of degrees of freedom:

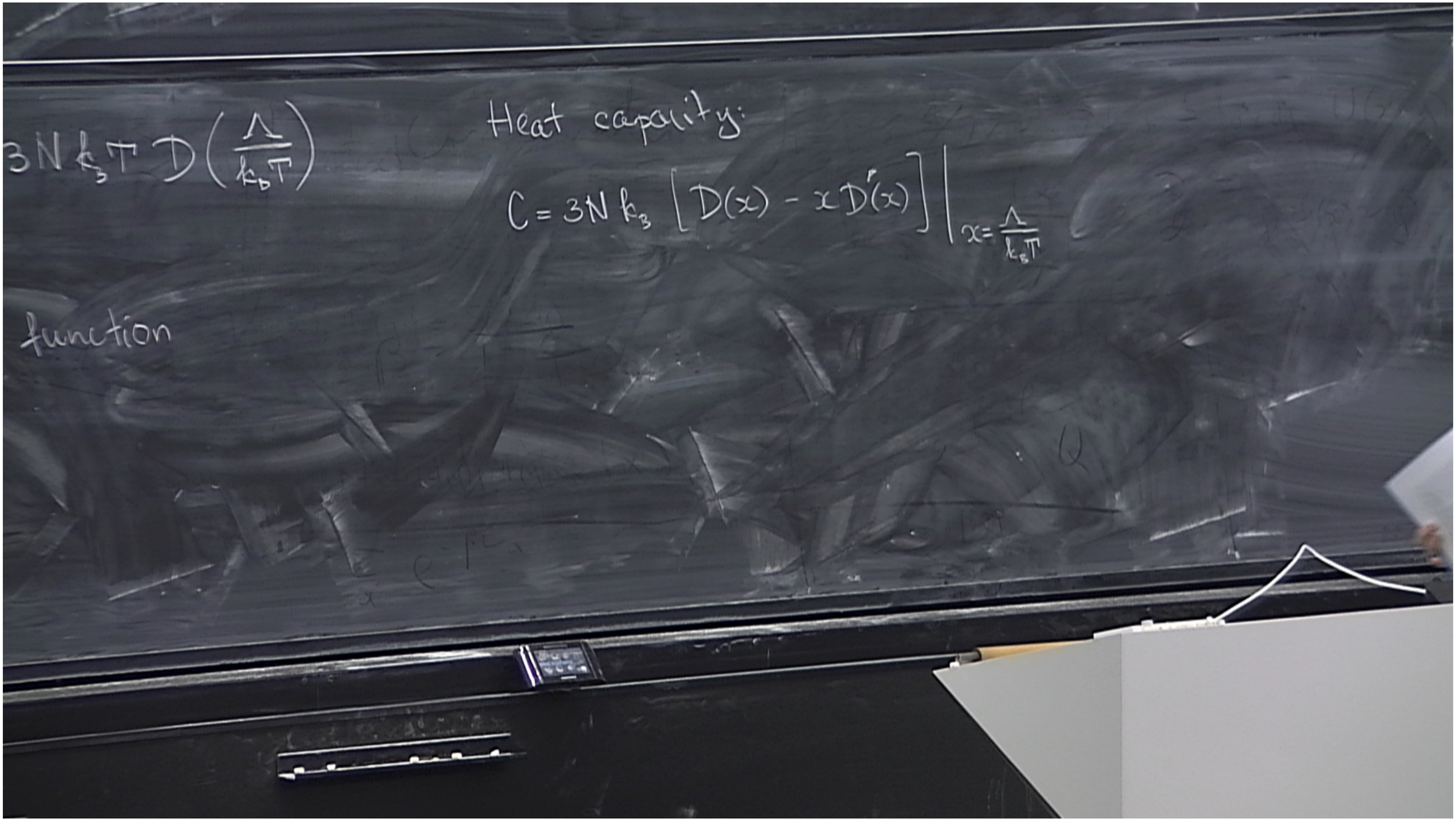
$$\Rightarrow 3N = V \int_{|\vec{p}| \le \frac{\Lambda}{c_s}} \frac{d^3p}{(2\pi\hbar)^3} \times 3 = \frac{V \Lambda^3}{2\pi^2 \hbar^3 c_s^3} \Rightarrow \Lambda = \left(\frac{6\pi^2 \hbar^3 c_s^3 N}{V} \right)^{1/3}$$

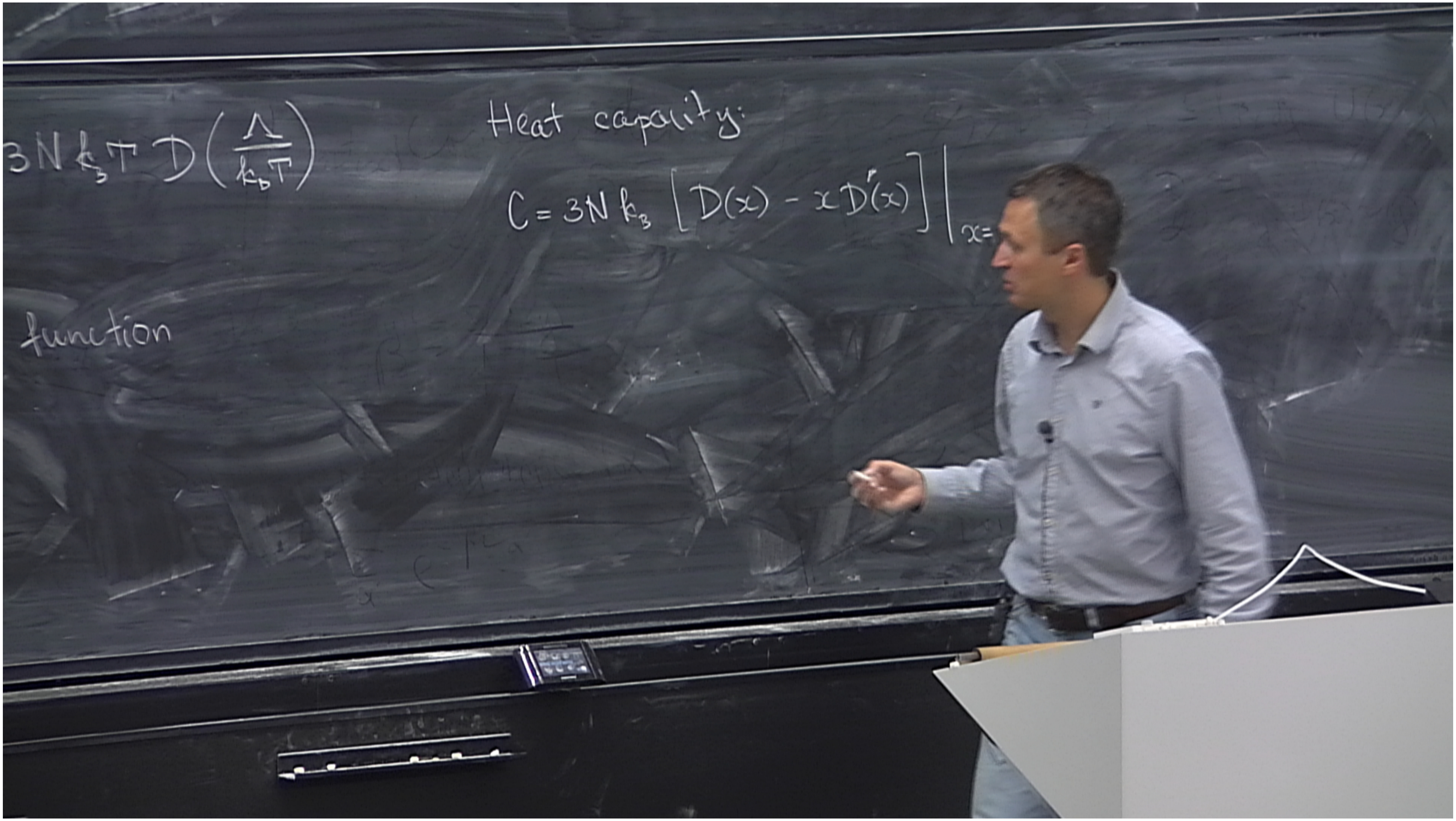
Λ - Debye energy $\quad p_{max} = \frac{\Lambda}{c_s}$

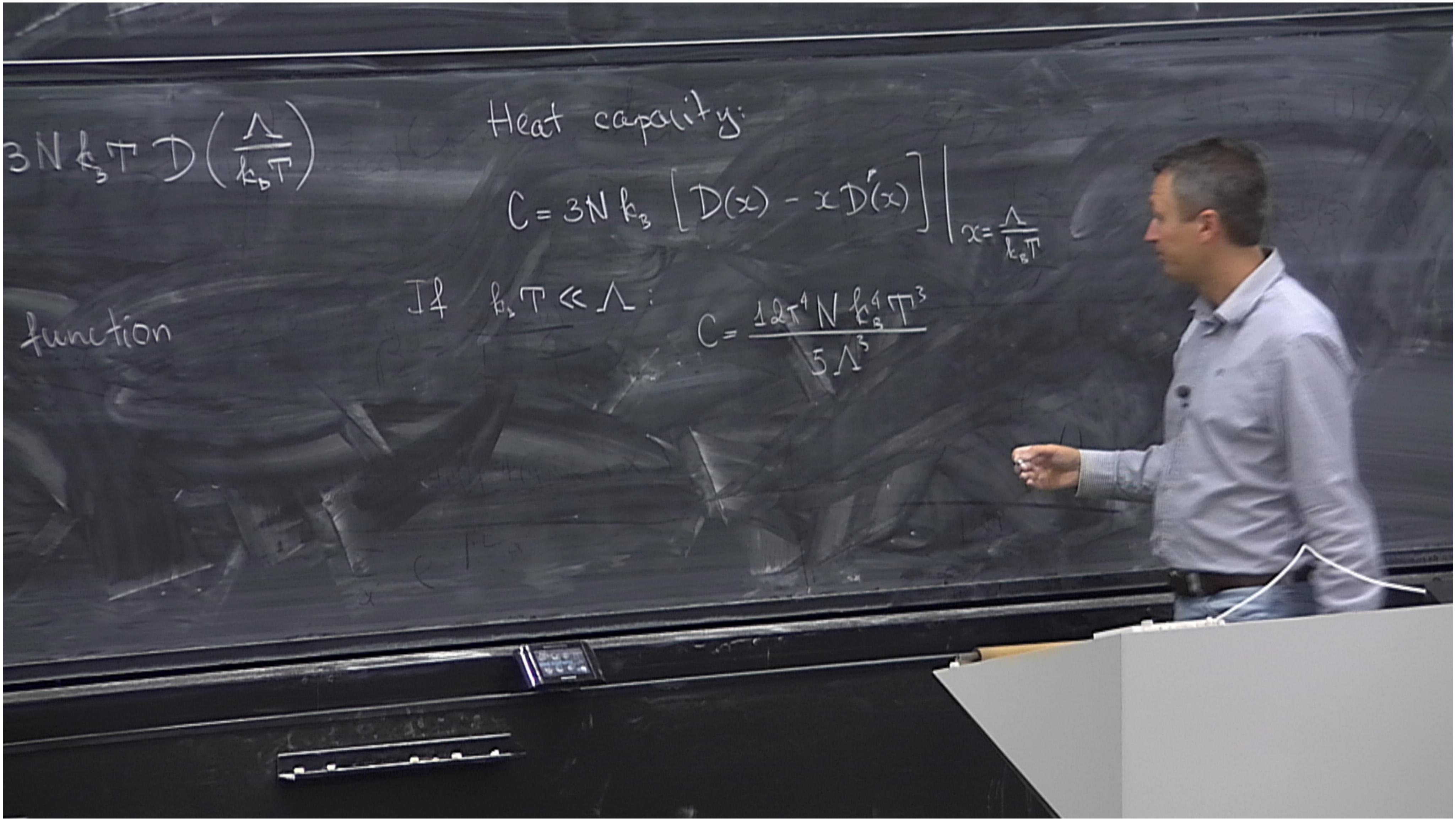

$E = 3V \int_{|\vec{p}| \leq \frac{\Lambda}{c_s}} \frac{d^3p}{2\pi\hbar} \frac{c_s|\vec{p}|}{e^{\frac{c_s|\vec{p}|}{k_B T}} - 1}$

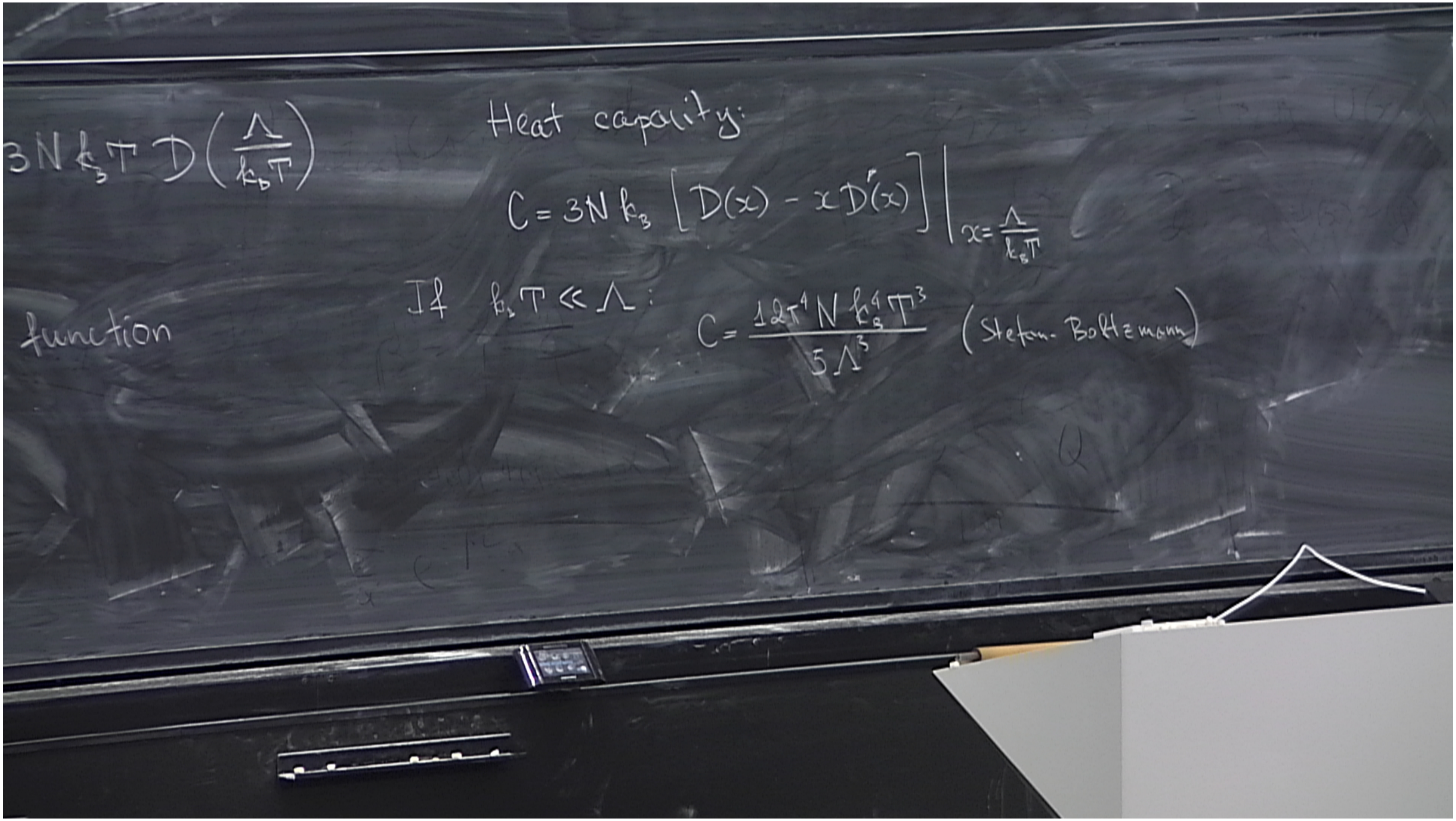

$E = 3V \int_{|\vec{p}| \leq \frac{\Lambda}{c_s}} \frac{d^3p}{2\pi\hbar} \frac{c_s|\vec{p}|}{e^{\frac{c_s|\vec{p}|}{k_B T}} - 1} = 3N k_B T \, D\left(\frac{\Lambda}{k_B T}\right)$
$D(x) = \frac{3}{x^3} \int_0^x \frac{dz\, z^3}{e^z - 1}$ ~ Debye function

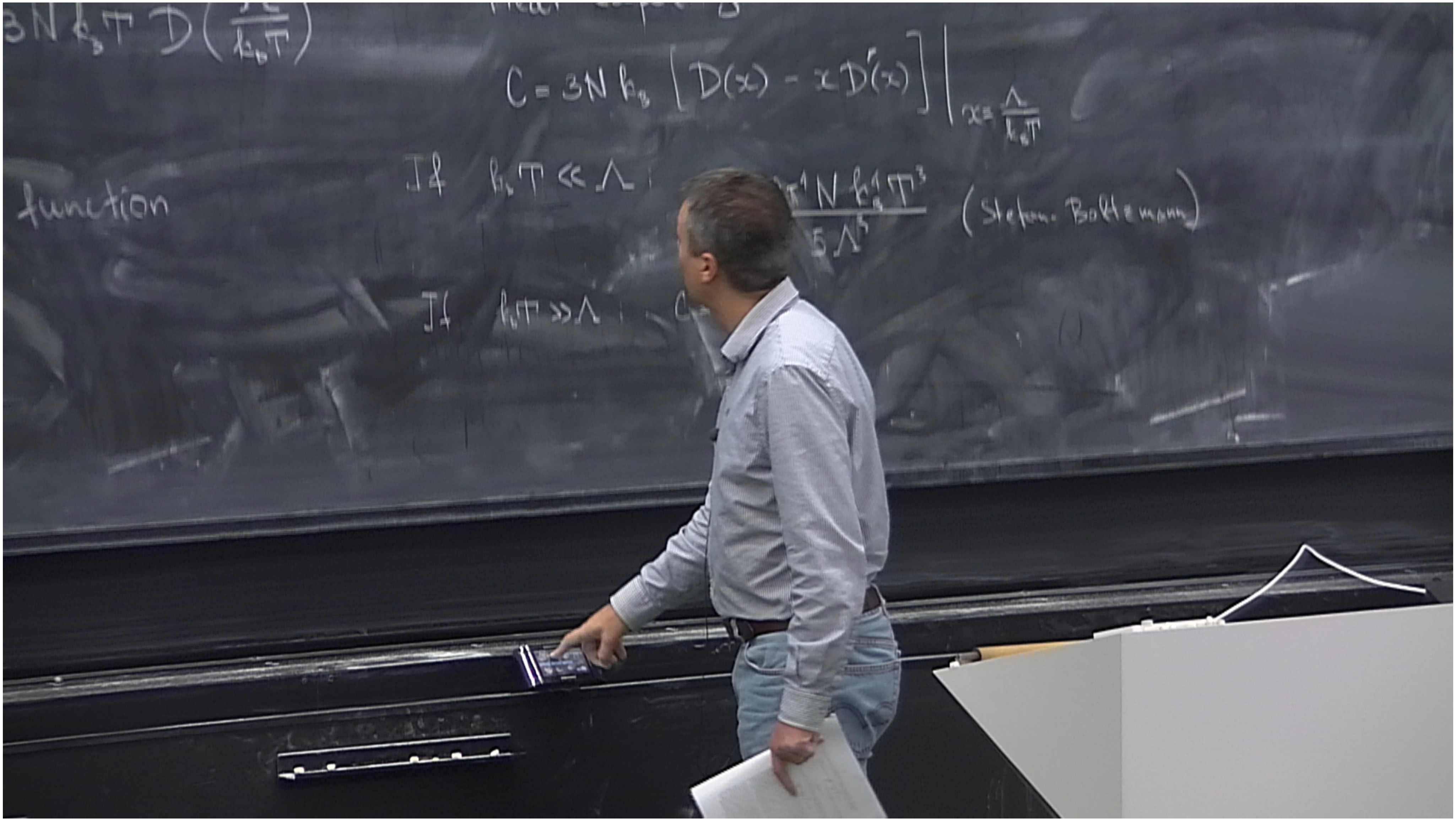

Heat capacity:
function

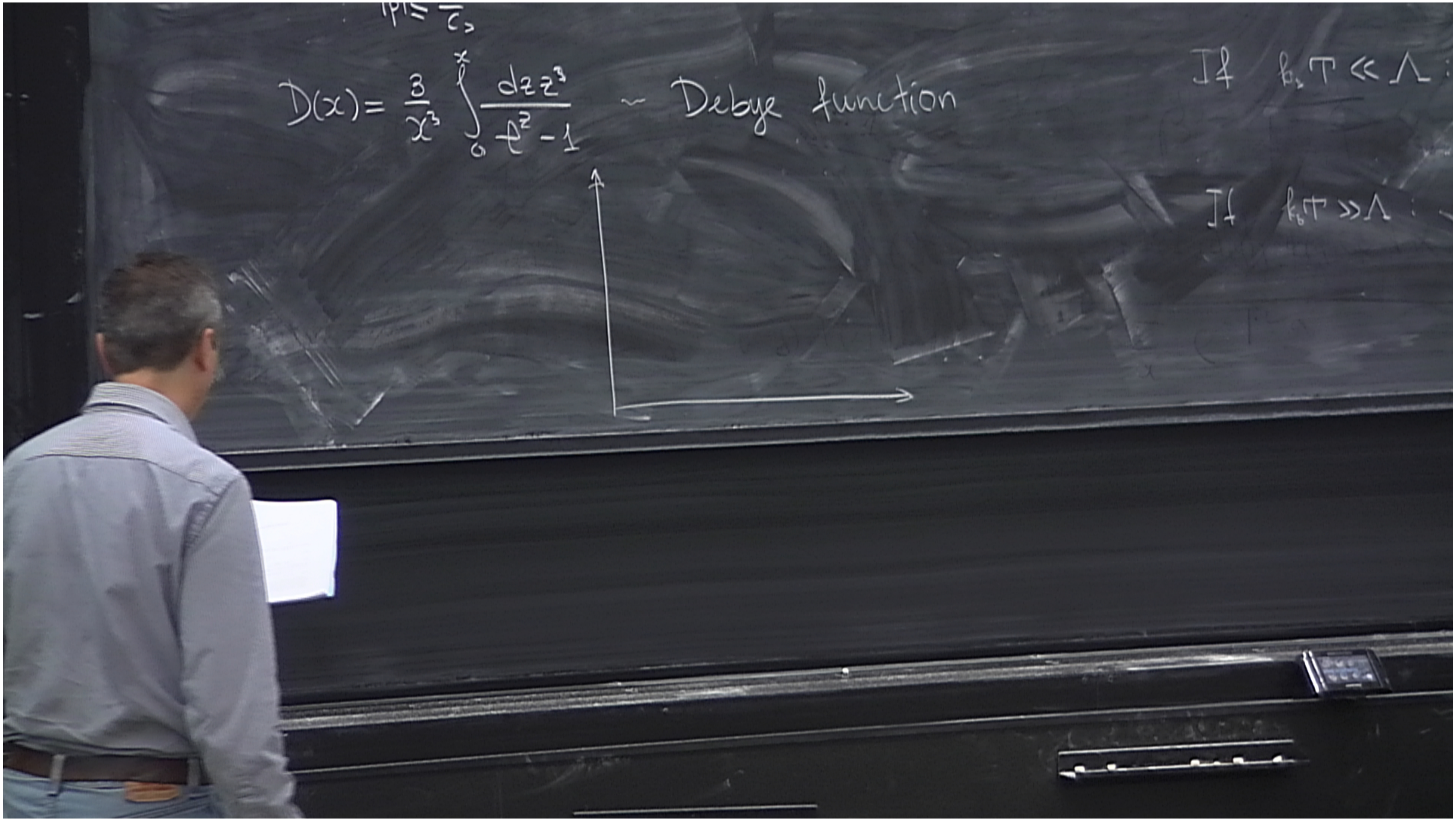

3 N k_B T D(Λ / k_B T)
Heat capacity:
C = 3N k_B [D(x) − x D'(x)] |_{x = Λ / k_B T}
function

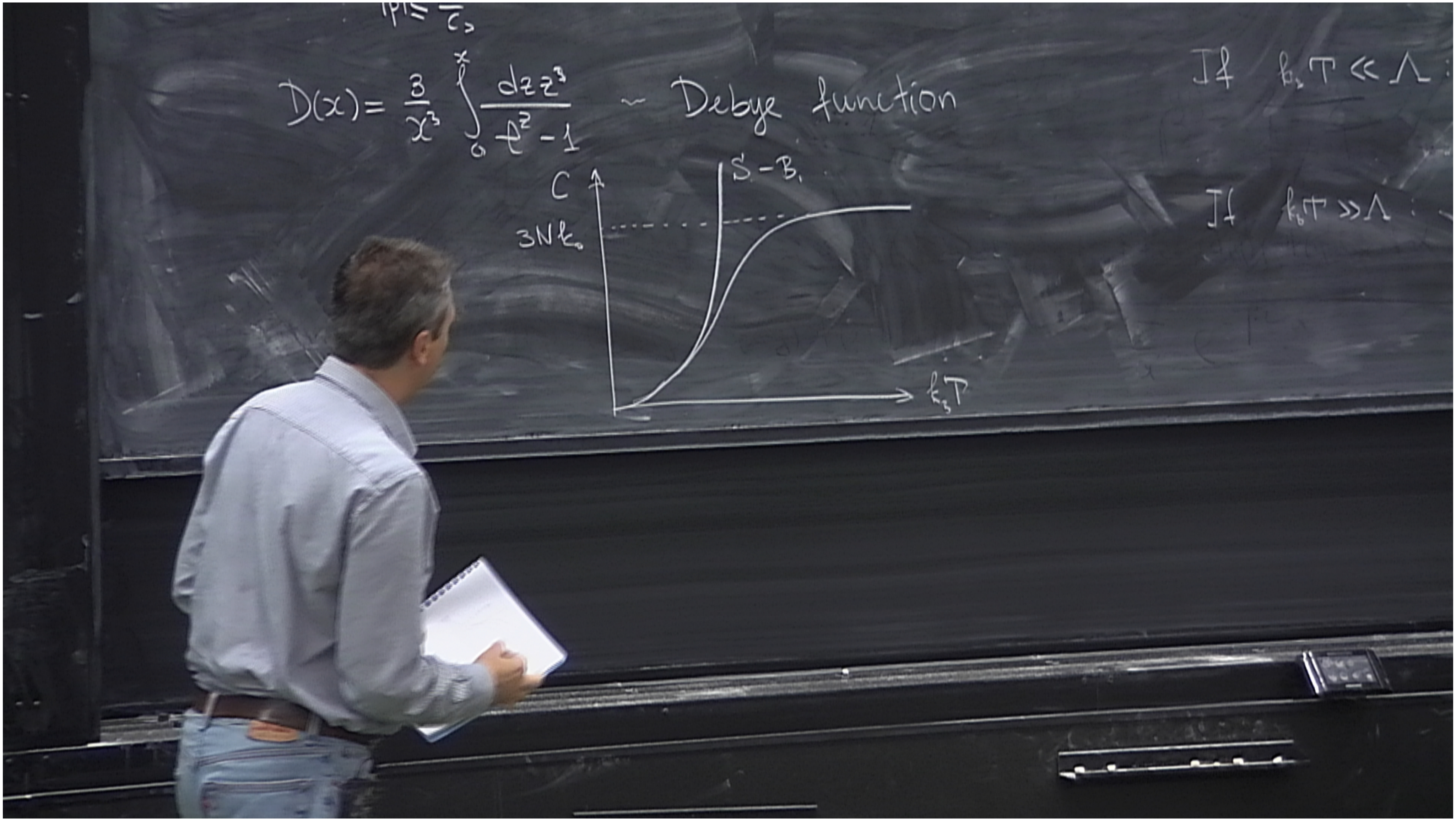

3N k_B T D(Λ/k_B T)
Heat capacity:
C = 3N k_B [D(x) - x D'(x)] |_{x=
function


3N k_B T D(Λ / k_B T)
Heat capacity:
C = 3N k_B [D(x) - x D'(x)] | x = Λ / k_B T
function
If k_B T ≪ Λ:
C = 12π^4 N k_B^4 T^3 / 5Λ^3

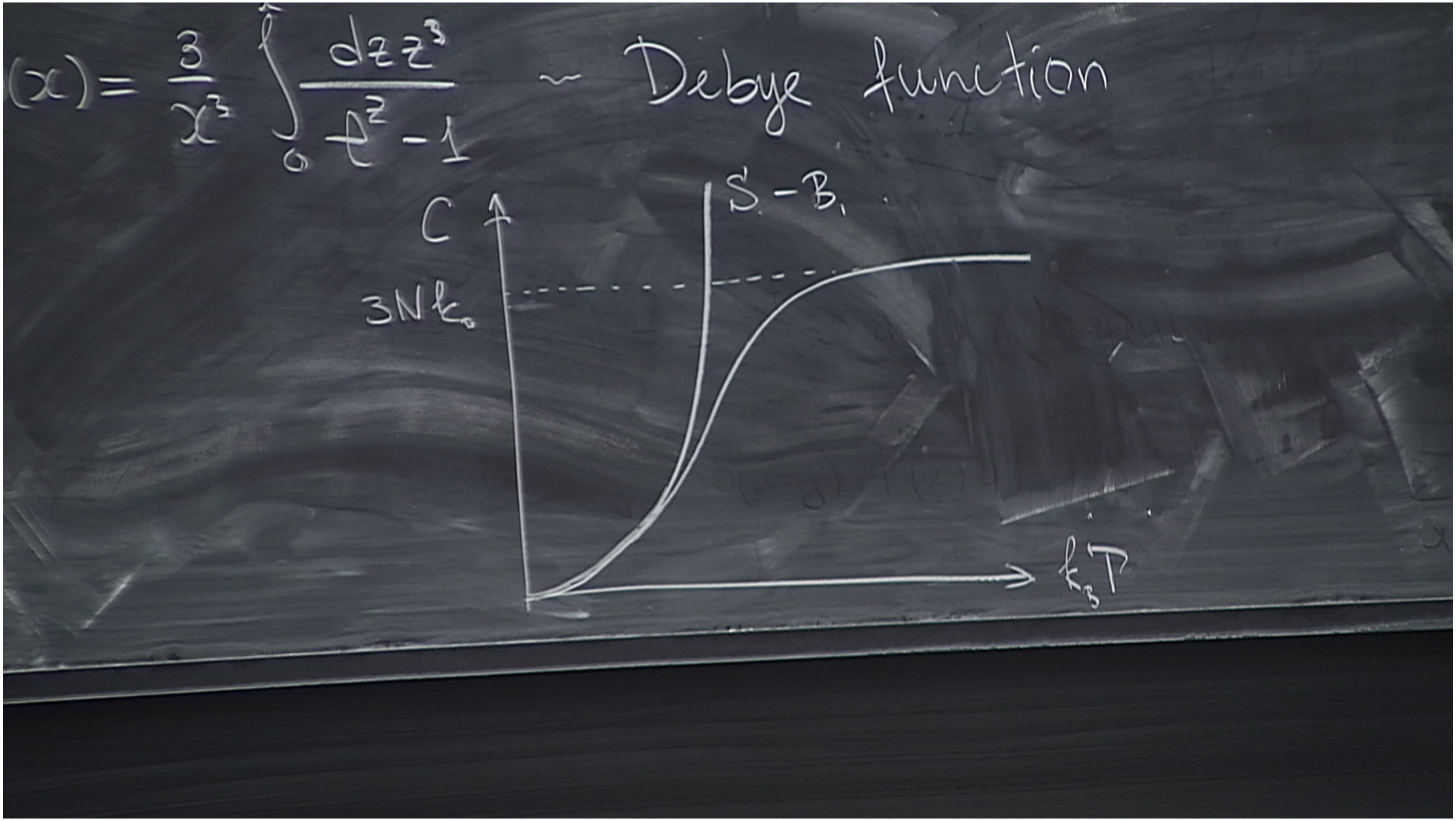

$3Nk_BT\,D\left(\frac{\Lambda}{k_BT}\right)$
function
Heat capacity:
$C = 3Nk_B\left[D(x) - xD'(x)\right]\Big|_{x=\frac{\Lambda}{k_BT}}$
If $k_BT \ll \Lambda$:
$C = \frac{12\pi^4 N k_B^4 T^3}{5\Lambda^3}$
(Stefan-Boltzmann)


function
If
If


D(x) = 3/x³ ∫₀ˣ dz z³/(e^z − 1) ~ Debye function
If k_B T ≪ Λ :
If k_B T ≫ Λ :



$D(x) = \frac{3}{x^3}\int_0^x \frac{dz\, z^3}{e^z - 1}$ ~ Debye function
If $k_B T \ll \Lambda$:
If $k_B T \gg \Lambda$:
C
$3Nk_B$
S-B.
$k_B T$


(x) = 3/x^3 ∫ dz z^3 / (e^z − 1) ~ Debye function
C
S.-B.
3Nk
k_B T