Title: The Holographic Fluid Dual to Vacuum Einstein Gravity

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Abstract: I'll discuss how to systematically construct a (d+2)-dimensional solution of the vacuum Einstein equations that is dual to a (d+1)-dimensional fluid satisfying the incompressible Navier-Stokes equations with specific higher-derivative corrections. The solution takes the form of a non-relativistic gradient expansion that is in direct correspondence with the hydrodynamic expansion of the dual fluid. The dual fluid has nevertheless an underlying description in terms of relativistic hydrodynamics, with the unusual property of having a vanishing equilibrium energy density. Using the gravitational results, as well as an interesting and exact constraint on its stress tensor, we identify the transport coefficients of the dual fluid. A simple Lagrangian model is sufficient to realise its key properties.

Holography

There is a general expectation, based on black hole physics, that any gravitational theory should be *holographic*, *i.e.*, it should admit a dual description in terms of a non-gravitational quantum theory in one dimension less.

- If gravitational theories are indeed holographic, one should be able to recover generic features of quantum theories through gravitational computations.

Hydrodynamics

- One such generic feature is the existence of a *hydrodynamic description* capturing the long-wavelength behaviour near to thermal equilibrium.

- One then expects to find the same feature on the gravitational side, *i.e.*, there should exist a bulk solution corresponding to the thermal state, and nearby solutions corresponding to the hydrodynamic regime.

In AdS/CFT

These expectations are beautifully realised in AdS/CFT:

Thermal state	⇔	AdS black hole
Relativistic hydrodynamics	⇔	Relativistic gradient expansion solution of bulk

➢ Solutions describing non-equilibrium configurations are well approximated by hydrodynamics at late times.

e.g., [Witten (1998)], [Policastro, Son & Starinets (2001)], [Janik & Peschanski (2005)], [Bhattacharyya, Hubeny, Minwalla & Rangamani (2007)], [Chesler & Yaffe (2010)]...

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For vacuum Einstein gravity

For the case of vacuum Einstein gravity, we will see that:

Thermal state	$\Leftrightarrow$	Rindler space
Incompressible Navier-Stokes + corrections	$\Leftrightarrow$	Non-relativistic gradient expansion solution of bulk

One may then use the properties of these solutions in order to obtain clues to the nature of the dual theory.

Plan

1. Equilibrium configurations
2. From equilibrium configurations to hydrodynamics
3. Solving for the bulk geometry
4. The underlying *relativistic* fluid
5. A model for the dual fluid
6. Conclusions and open questions

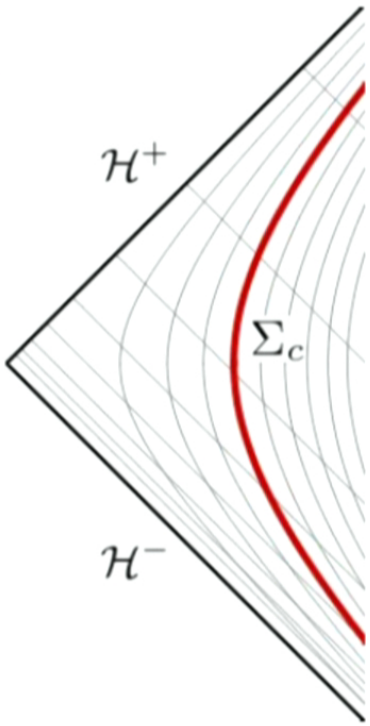
Rindler spacetime

Let's start with flat space written in ingoing Rindler coordinates:

$$ds^2 = -r d\tau^2 + 2 d\tau dr + dx_i dx^i$$

i.e. Minkowski space $ds^2 = -dT^2 + dX^2$ parametrised by timelike hyperbolae $X^2 - T^2 = 4r$ and ingoing null geodesics $X + T = e^{\tau/2}$.

Consider now the portion of spacetime bounded by the surface Σ_c defined by $r = r_c$ and the future horizon $\mathcal{H}^+$.



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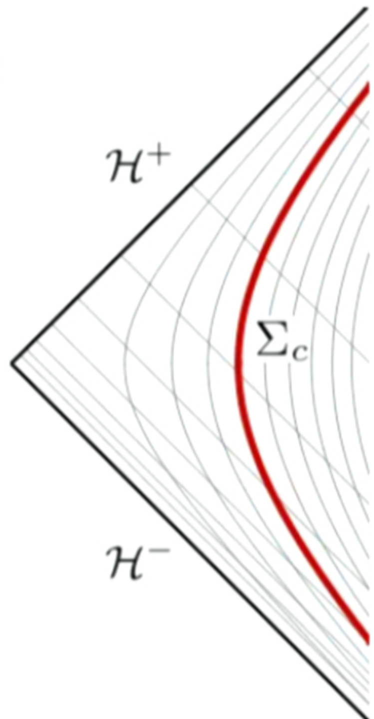

Equilibrium configurations

We can now generate general equilibrium configurations with arbitrary constant p and u^a by acting with diffeomorphisms.

We require that:

1. The induced metric on Σ_c is preserved.
2. The Brown-York stress tensor on Σ_c remains that of a perfect fluid.
3. The bulk metric remains stationary w.r.t. ∂_τ and homogeneous in the x^i directions.

One can prove that only two infinitesimal diffeomorphisms exist with all these properties.

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Equilibrium configurations

Exponentiating, the corresponding finite diffeomorphisms are:

- A constant boost

$$\sqrt{r_c}\tau \to \gamma\sqrt{r_c}\tau - \gamma\beta_i x^i, \qquad x^i \to x^i - \gamma\beta^i\sqrt{r_c}\tau + (\gamma - 1)\frac{\beta^i\beta_j}{\beta^2}x^j,$$

where $\gamma = (1-\beta^2)^{-1/2}$ and $\beta_i = v_i/\sqrt{r_c}$.

- A constant linear shift of r and re-scaling of τ,

$$r \to r - r_h, \qquad \tau \to (1 - r_h/r_c)^{-1/2}\tau.$$

This second transformation shifts the position of the horizon to $r = r_h < r_c$.

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Equilibrium configurations

Applying these two transformations, the resulting metric is

$$\mathrm{d}s^2 = -2pu_a \mathrm{d}x^a \mathrm{d}r + [\gamma_{ab} - p^2(r - r_c)u_a u_b]\mathrm{d}x^a \mathrm{d}x^b.$$

Clearly, the induced metric on Σ_c is still γ_{ab}. Moreover, the Brown-York stress tensor is that of a perfect fluid with

$$\rho = 0, \qquad p = \frac{1}{\sqrt{r_c - r_h}}, \qquad u^a = \frac{1}{\sqrt{r_c - v^2}}(1,\, v_i).$$

This solution therefore indeed describes an arbitrary equilibrium configuration with constant p and v_i. Note however the energy density is still zero.

We can also analyse the thermodynamics. The Rindler horizon is a Killing horizon, and normalising $\xi^2 = -1$ on Σ_c, the Unruh temperature satisfies $sT = p$ where the entropy density $s = 1/4G$.

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From equilibrium configurations to hydrodynamics

Having found equilibrium solutions with constant p and v_i, let's now allow these variables to *slowly* vary:

$$v_i \to \epsilon v_i(\epsilon^2 \tau, \epsilon \vec{x}), \qquad p \to \frac{1}{\sqrt{r_c}} + \epsilon^2 \frac{1}{r_c^{3/2}} P(\epsilon^2 \tau, \epsilon \vec{x}), \qquad \epsilon \ll 1.$$

Expanding in ϵ, we find

$$\begin{aligned} \mathrm{d}s^2 &= -r\mathrm{d}\tau^2 + 2\mathrm{d}\tau\mathrm{d}r + \mathrm{d}x_i\mathrm{d}x^i \\ &\quad - 2\left(1 - \frac{r}{r_c}\right) v_i \mathrm{d}x^i \mathrm{d}\tau - \frac{2v_i}{r_c}\mathrm{d}x^i\mathrm{d}r \\ &\quad + \left(1 - \frac{r}{r_c}\right)\left[(v^2 + 2P)\mathrm{d}\tau^2 + \frac{v_i v_j}{r_c}\mathrm{d}x^i\mathrm{d}x^j\right] + \left(\frac{v^2}{r_c} + \frac{2P}{r_c}\right)\mathrm{d}\tau\mathrm{d}r + O(\epsilon^3), \end{aligned}$$

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From Navier-Stokes to Einstein

Through independent considerations, Bredberg, Keeler, Lysov & Strominger [arXiv:1101.2451] proposed the metric:

$$
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\mathrm{d}s^2 = & -r\mathrm{d}\tau^2 + 2\mathrm{d}\tau\mathrm{d}r + \mathrm{d}x_i\mathrm{d}x^i \\
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& - \frac{(r^2 - r_c^2)}{r_c}\partial^2 v_i \mathrm{d}x^i\mathrm{d}\tau + O(\epsilon^4)
\end{aligned}
$$

This satisfies $R_{\mu\nu} = O(\epsilon^4)$ provided P and v_i obey the incompressible Navier-Stokes equations to $O(\epsilon^4)$!

Incompressible Navier-Stokes

The incompressible Navier-Stokes equations read

$$\partial_\tau v_i + v^j \partial_j v_i - \eta \partial^2 v_i + \partial_i P = 0, \qquad \partial_i v^i = 0,$$

and have the interesting non-relativistic scaling symmetry

$$v_i \to \epsilon v_i(\epsilon^2 \tau, \epsilon \vec{x}), \qquad P \to \epsilon^2 P(\epsilon^2 \tau, \epsilon \vec{x}).$$

Starting from relativistic fluid mechanics, one may recover the incompressible Navier-Stokes equations, along with specific higher-derivative corrections, by taking the hydrodynamic limit $\epsilon \to 0$.

- The incompressible Navier-Stokes equations capture the universal long-wavelength behaviour of essentially any fluid.
- Higher-derivative corrections are naturally organised according to their scaling with ϵ.

Solving to all orders

We can now construct the bulk metric to all orders using the gradient expansion

$$\partial_r \sim 1, \qquad \partial_i \sim \epsilon, \qquad \partial_\tau \sim \epsilon^2,$$

corresponding to the hydrodynamic limit.

➠ To do so, suppose first we have a solution at order ϵ^{n-1}. Let's now add a new term to the metric $g^{(n)}_{\mu\nu}$ at order ϵ^n. The Ricci tensor is

$$R^{(n)}_{\mu\nu} = \delta R^{(n)}_{\mu\nu} + \hat{R}^{(n)}_{\mu\nu}.$$

Here, $\delta R^{(n)}_{\mu\nu}$ is the contribution at order ϵ^n due to the new term $g^{(n)}_{\mu\nu}$, while $\hat{R}^{(n)}_{\mu\nu}$ is the nonlinear contribution at order ϵ^n from the metric at lower orders.

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- We know $\delta R^{(n)}_{\mu\nu}$ from the usual linearised formula. Since

$$\partial_r \sim 1, \qquad \partial_i \sim \epsilon, \qquad \partial_\tau \sim \epsilon^2,$$

we need only keep r-derivatives in this formula, since the rest are higher order. This gives:

$$\delta R^{(n)}_{rr} = -\frac{1}{2}\partial_r^2 g^{(n)}_{ii},$$
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- We can now set $R^{(n)}_{\mu\nu} = \delta R^{(n)}_{\mu\nu} + \hat{R}^{(n)}_{\mu\nu} = 0$ and solve for $g^{(n)}_{\mu\nu}$ in terms of the metric at lower orders.

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Integrability conditions

- For this to be possible, the following integrability conditions must be satisfied:

$$0 = \partial_r(\hat{R}^{(n)}_{ii} - r\hat{R}^{(n)}_{rr}) - \hat{R}^{(n)}_{rr}, \qquad 0 = \hat{R}^{(n)}_{\tau a} + r\hat{R}^{(n)}_{ra}.$$

- The Bianchi identities at order ϵ^n give

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- The vanishing of $f^{(n)}_a(\tau, \vec{x})$ follows from conservation of the Brown-York stress tensor on Σ_c. Using the Gauss-Codazzi identity,

$$\nabla^b T_{ab}\big|^{(n)}_{\Sigma_c} = [2\nabla^b(K\gamma_{ab} - K_{ab})]^{(n)} = [-2R_{a\mu}N^\mu]^{(n)} = -\frac{2}{\sqrt{r_c}} f^{(n)}_a(\tau, \vec{x}).$$

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- The vanishing of $f^{(n)}_a(\tau, \vec{x})$ follows from conservation of the Brown-York stress tensor on Σ_c. Using the Gauss-Codazzi identity,

$$\nabla^b T_{ab}\big|^{(n)}_{\Sigma_c} = [2\nabla^b(K\gamma_{ab} - K_{ab})]^{(n)} = [-2R_{a\mu}N^\mu]^{(n)} = -\frac{2}{\sqrt{r_c}} f^{(n)}_a(\tau, \vec{x}).$$

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- Thus, conservation of the Brown-York stress tensor on Σ_c is necessary for the bulk equations to be integrated.

- From the perspective of the dual fluid, conservation of the Brown-York stress tensor is equivalent to incompressibility (at ϵ^2 order) and the Navier-Stokes equation (at ϵ^3 order). At higher orders in ϵ we obtain corrections to these equations.

- To complete our integration scheme, we choose the gauge $g^{(n)}_{r\mu} = 0$ and impose boundary conditions such that the metric on Σ_c is preserved, and that the solution is regular on the future horizon $\mathcal{H}^+$. (This suppresses the appearance of divergent logarithmic terms in the metric).

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- Our final integration scheme is thus

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\begin{aligned}
g^{(n)}_{r\mu} &= 0,\\
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Fluid gauge conditions

This remaining freedom may be fixed by choosing appropriate gauge conditions for the dual fluid.

➢ $F_i^{(n)}$ may be fixed by imposing Landau gauge on the fluid:

$$0 = u^a T_{ab} h^b_c$$

i.e. the momentum density $T_{\tau i}$ vanishes in the local rest frame.
This is effectively a definition of the fluid velocity u^a.

➢ $F_\tau^{(n)}$ is fixed by imposing that there are no corrections to the isotropic part of the stress tensor:

$$T_{ij}^{\text{iso}} = \Big(\frac{1}{\sqrt{r_c}} + \frac{P}{r_c^{3/2}}\Big)\delta_{ij}.$$

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Bulk solution

We computed this bulk solution through to ϵ^5 order, for arbitrary spacetime dimension.

- For example, at ϵ^3 order, the only nonzero term is:

$$g^{(3)}_{\tau i} = \frac{(r - r_c)}{2r_c}\left[(v^2 + 2P)\frac{2v_i}{r_c} + 4\partial_i P - (r + r_c)\partial^2 v_i\right].$$

- At ϵ^4 order, the nonzero terms are $g^{(4)}_{\tau\tau}$ and $g^{(4)}_{ij}$.
- At ϵ^5 order, only $g^{(5)}_{\tau i}$ is nonzero. [See arXiv:1103.3022]

This behaviour makes sense since all scalars and tensors constructed from v_i, P and their derivatives are of even order in ϵ, while all vector quantities are odd.

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Recovering Navier-Stokes and incompressibility

From our unique bulk solution, we recover the Navier-Stokes and incompressibility equations, along with a specific set of corrections.

These arise from the momentum constraint on Σ_c:

$$0 = \nabla^b T_{ab}\Big|_{\Sigma_c} = 2\nabla^b(K\gamma_{ab} - K_{ab})$$

At even orders in ϵ we recover the incompressibility equation plus corrections,

$$\partial_i v_i = \frac{1}{r_c} v_i \partial_i P - v_i \partial^2 v_i + 2\partial_{(i} v_{j)} \partial_i v_j + O(\epsilon^6),$$

while at odd orders we recover Navier-Stokes plus corrections,

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2. From equilibrium configurations to hydrodynamics
3. Solving for the bulk geometry
4. The underlying *relativistic* fluid
5. A model for the dual fluid
6. Conclusions and open questions

The underlying *relativistic* fluid

As the ϵ-expansion is non-relativistic, T_{ab} appears to be non-relativistic. In fact, however, there is an underlying *relativistic* stress tensor which, when expanded out in ϵ, reproduces our above results.

- Agrees with expectation that the dual holographic theory should be relativistic.
- The relativistic stress tensor is much simpler: all information is encoded in only a few transport coefficients.

Characterising the dual fluid

In general,

$$T_{ab} = \rho u_a u_b + p h_{ab} + \Pi^{\perp}_{ab}, \qquad u^a \Pi^{\perp}_{ab} = 0,$$

where $\Pi^{\perp}_{ab}$ represents dissipative corrections and may be expanded in fluid gradients.

One unusual feature compared to standard relativistic hydrodynamics, however, is that the equilibrium energy density vanishes:

From our bulk solution, the energy density in the local rest frame is given by

$$\rho = T_{ab} u^a u^b = -\frac{1}{2\sqrt{r_c}} \sigma_{ij}\sigma_{ij} + O(\epsilon^6), \qquad \sigma_{ij} = 2\partial_{(i} v_{j)}.$$

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We may understand this curious property as follows. First, we note that the Hamiltonian constraint on Σ_c imposes

$$dT_{ab}T^{ab} = T^2.$$

Inserting our relativistic decomposition, we find

$$0 = \rho\big((d-1)\rho + 2dp + 2\Pi^{\perp}\big) + d\Pi^{\perp}_{ab}\Pi^{\perp ab} - (\Pi^{\perp})^2.$$

This determines ρ in terms of p and $\Pi^{\perp}_{ab}$. (Note there are two solution branches: here, we need the one corresponding to $\rho = 0 + O(\partial^2)$).

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First-order relativistic hydrodynamics

At first order in fluid gradients,

$$\Pi^{\perp}_{ab} = -2\eta\mathcal{K}_{ab} + O(\partial^2), \qquad \mathcal{K}_{ab} = h^c_a h^d_b \partial_{(c}u_{d)}.$$

Note there is no bulk viscosity term $\xi\mathcal{K}h_{ab}$, since $\mathcal{K} = \partial_a u^a$ and the fluid is incompressible: $0 = u^a\partial^b T_{ab} = -p\partial_a u^a + O(\partial^2)$.

Expanding T_{ab} in ϵ we find $\eta = 1$, hence $\eta/s = 1/4\pi$.

The 'equation of state' fixes

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Second-order relativistic hydrodynamics

The full expansion for $\Pi^{\perp}_{ab}$ to second order in gradients is

$$\Pi^{\perp}_{ab} = -2\eta\mathcal{K}_{ab} + c_1\mathcal{K}^c_a\mathcal{K}_{cb} + c_2\mathcal{K}^c_{(a}\Omega_{|c|b)} + c_3\Omega_a{}^c\Omega_{cb} + c_4 h^c_a h^d_b \partial_c\partial_d \ln p + c_5\mathcal{K}_{ab}\, D\ln p + c_6 D^{\perp}_a \ln p\, D^{\perp}_b \ln p + O(\partial^3),$$

where $D^{\perp}_a = h^b_a\partial_b$ and $D = u^a\partial_a$ and the vorticity $\Omega_{ab} = h^c_a h^d_b \partial_{[c}u_{d]}$.

- There are six second-order transport coefficients: c_1, c_2, etc.

Expanding this expression in ϵ and comparing with T_{ab} from our gravity calculation we find:

$$\eta = 1, \qquad 2c_1 = c_2 = c_3 = c_4 = -4\sqrt{r_c}.$$

These five simple terms encode our entire T_{ab} to ϵ^5 order!

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A model for the dual fluid

We now propose a simple Lagrangian model for the dual fluid.

We focus on the non-dissipative part of the stress tensor,

$$T_{ab} = ph_{ab} = p(\gamma_{ab} + u_a u_b),$$

describing a fluid with nonzero pressure but vanishing energy density in the local rest frame.

(To get the dissipative part would need to couple to a heat bath.)

A model for the dual fluid

Consider the action:

$$S = \int \mathrm{d}^{d+1}x \sqrt{-\gamma}\sqrt{-(\partial\phi)^2}.$$

cf. cuscuton model! [Afshordi, Chung, & Geshnizjani (06)]

The field equations describe *potential flow*

$$\nabla^a u_a = 0, \qquad u_a = \frac{\partial_a \phi}{\sqrt{X}}, \qquad X = -(\partial\phi)^2.$$

The stress tensor is of the required form

$$T_{ab} = \sqrt{X}\gamma_{ab} + \frac{1}{\sqrt{X}}\partial_a\phi\partial_b\phi = \sqrt{X}h_{ab}, \qquad \text{i.e.} \quad p = \sqrt{X}.$$

One way to obtain this sqrt action is to start with $\mathcal{L}(X,\phi)$ then impose

$$0 = \rho = T_{ab}u^a u^b = \left(-2\frac{\delta\mathcal{L}}{\delta\gamma^{ab}} + \gamma_{ab}\mathcal{L}\right)\frac{\partial^a\phi\partial^b\phi}{X} = 2X\frac{\delta\mathcal{L}}{\delta X} - \mathcal{L}$$

A model for the dual fluid

- The equilibrium configuration with $p = 1/\sqrt{r_c}$ in the rest frame corresponds to taking

$$\phi = \tau,$$

so $v_i \sim \partial_i \phi = 0$. This breaks Lorentz invariance, as does any choice of u_a.

- To model small fluctuations about this background we set

$$\phi = \tau + \delta\phi(\tau, \vec{x}).$$

One can then solve for the 3-velocity v_i and pressure fluctuation P:

$$v_i = -\frac{r_c \delta\phi_{,i}}{(1 + \delta\dot{\phi})}, \qquad P = r_c\Big[(1 + 2\delta\dot{\phi} + \delta\dot{\phi}^2 - r_c \delta\phi_{,i}\delta\phi_{,i})^{1/2} - 1\Big].$$

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Remarks

- The action is nonlocal: the expansion around the background solution involves an infinite number of derivatives.
- One can easily couple to other types of matter (Ψ, Φ, A_a), provided one expands around zero background values of these fields.
- Connection with brane action? e.g. $(d+1)$-dim brane embedded in $(d+2)$-dim Minkowski target space In static gauge this is

$$S = -T \int \mathrm{d}^{d+1}x \sqrt{1 + (\partial Y)^2},$$

where Y is the transverse coordinate to the brane. Taking the tensionless limit $T \to 0$ while keeping $\phi = TY$ fixed,

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- One can easily couple to other types of matter (Ψ, Φ, A_a), provided one expands around zero background values of these fields.
- Connection with brane action? e.g. $(d+1)$-dim brane embedded in $(d+2)$-dim Minkowski target space In static gauge this is

$$S = -T \int \mathrm{d}^{d+1}x \sqrt{1 + (\partial Y)^2},$$

where Y is the transverse coordinate to the brane. Taking the tensionless limit $T \to 0$ while keeping $\phi = TY$ fixed,

$$S = -\int \mathrm{d}^{d+1}x \sqrt{(\partial\phi)^2}.$$

Still missing minus sign inside sqrt $\to$ use target space signature $(d, 2)$?

Conclusions

- There is a direct relation between $(d+2)$-dimensional Ricci-flat metrics and $(d+1)$-dimensional fluids satisfying the incompressible Navier-Stokes equations, corrected by specific higher-derivative terms.

- The dual fluid has vanishing equilibrium energy density but nonzero pressure. There is an underlying relativistic hydrodynamic description. We computed the viscosity and four of the six second-order transport coefficients 'holographically'.

- A simple sqrt Lagrangian captures the non-dissipative properties of the fluid.

Open questions

Many questions remain:

➢ Is there a manifestly relativistic construction of the bulk metric? Does the solution exist globally? What if we add matter to the bulk?

➢ Does the correspondence extend beyond the hydrodynamic regime on the field theory side, and/or the classical gravitational description on the bulk side? Is there a string embedding? Can we get the sqrt action from branes?

➢ How far can flat space holography be developed? Is there a holographic dictionary relating bulk computations to quantities in the dual field theory on Σ_c?

➢ By the equivalence principle, our construction should hold locally in any small neighbourhood. Can one patch together such a 'local' holographic description of neighbourhoods to obtain a global holographic description of general spacetimes?