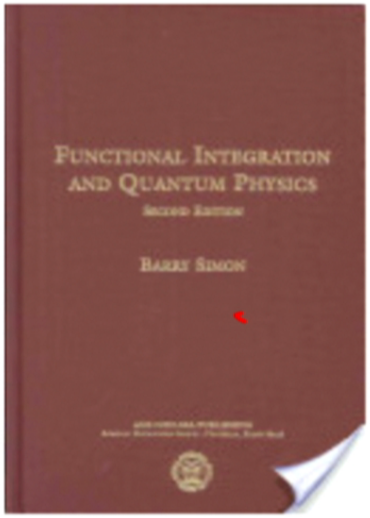
Title: Quantum Theory - Lecture 10

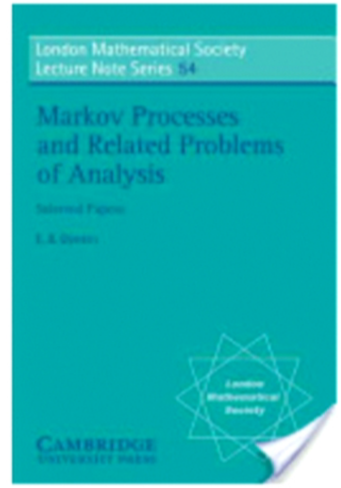
Date: Sep 26, 2011 09:00 AM

URL: http://pirsa.org/11090079

Abstract:




Advanced further reading (<u>not</u> needed for this course)




Barry Simon, "Functional Integration and Quantum Physics"

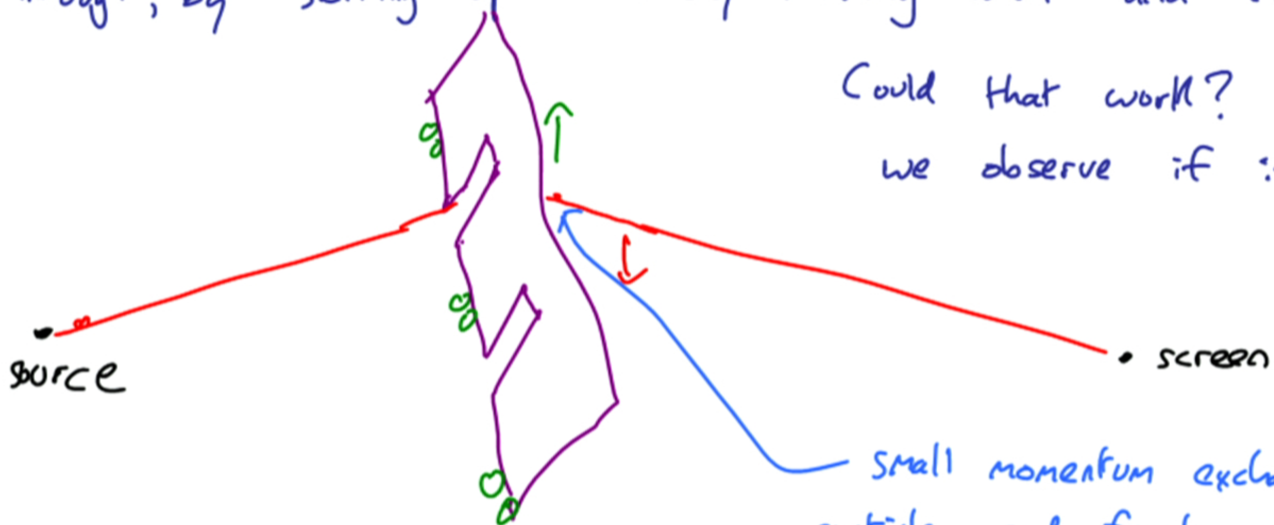
E.B. Dynkin, "Markov Processes and Related Problems of Analysis"

References on path integrals for diffusion, stochastic analysis, Wiener integrals – rigorous version of path integral with (specific) real-valued functionals.

# Things to think (or read) about re the double slit experiment

① We could try to infer indirectly which slit the particle went through, by setting up a freely moving wall and measuring recoil.

Could that work? What would we observe if it did?




small momentum exchange between particle and freely moving wall.

## Things to think (or read) about re the double slit experiment

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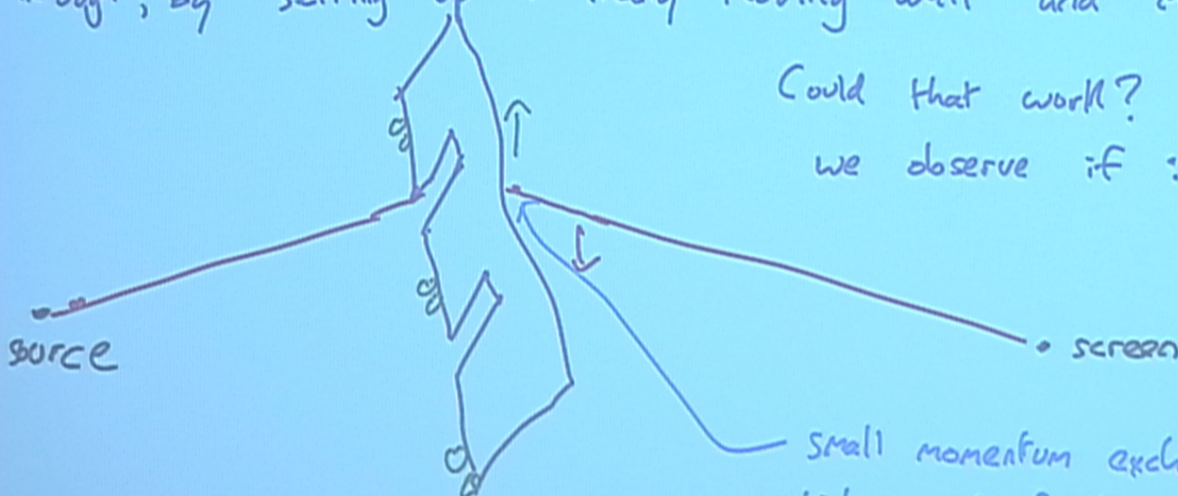


small momentum exchange between particle and freely moving wall.

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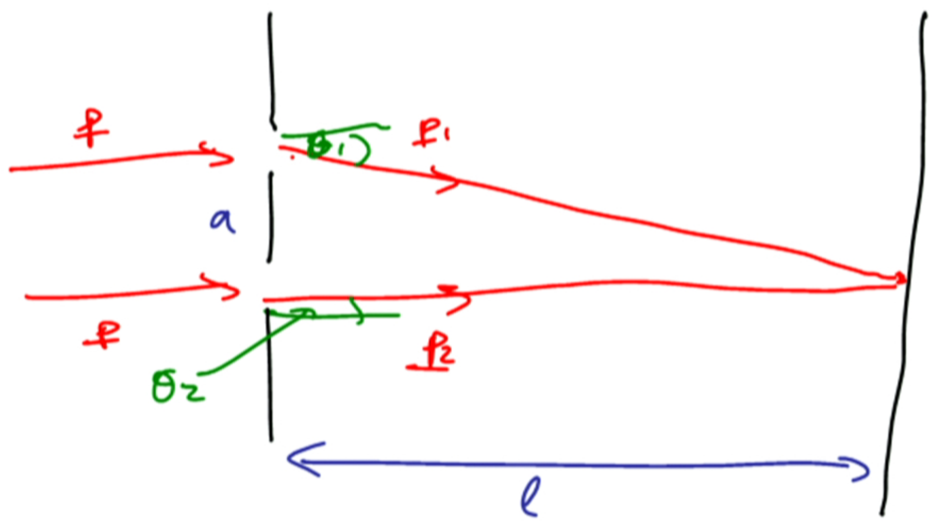


$$|p_1| \doteq |p| \doteq |p_2|$$

$$\theta_1 \doteq \frac{|\delta p_1|}{|p|}$$

$$\theta_2 \doteq \frac{|\delta p_2|}{|p|}$$

$$\theta_1 - \theta_2 \doteq \frac{a}{\ell}$$

$$\therefore \quad \frac{|p_1 - p_2|}{|p|} = \boxed{\frac{|\delta p|}{|p|} \doteq \frac{a}{\ell}}$$

where $\delta p$ is the difference in momentum "kick" given from the two slits.

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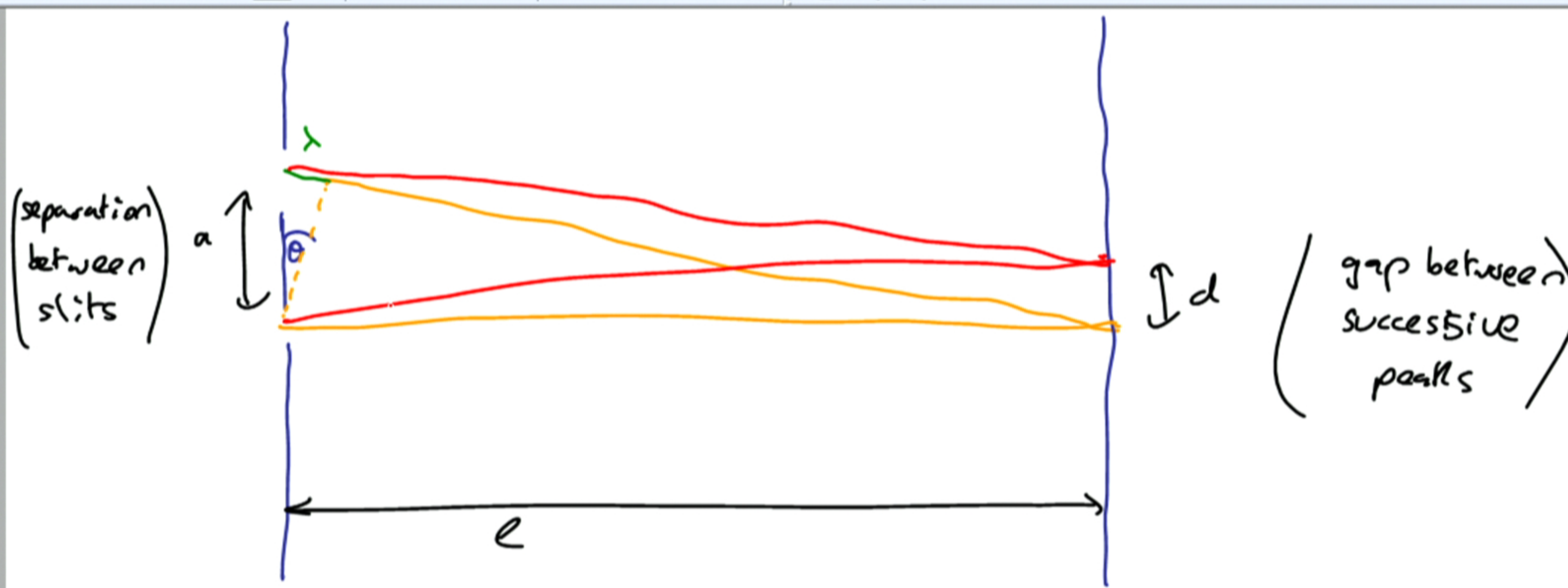


Orange triangle is essentially red triangle rotated by angle θ.

$$\lambda \doteqdot a\theta \,, \quad d \doteqdot l\theta \quad \Rightarrow \quad \boxed{\frac{a}{l} = \frac{\lambda}{d}}$$

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$$\boxed{\frac{a}{\ell} = \frac{\lambda}{d}} \qquad \boxed{\frac{|\delta p|}{|p|} \doteq \frac{a}{\ell}}$$

Now $|p| = \frac{h}{\lambda} \implies |\delta p| \doteq \frac{a|p|}{\ell} \doteq \left(\frac{\lambda}{d}\right)\left(\frac{h}{\lambda}\right) = \frac{h}{d}$

i.e. $d = \frac{h}{\delta p}$ — variation in screen momentum depending on slit path.

peak separation

But applying the uncertainty relation $\delta x \, \delta p \geq \hbar/2$ to the screen

if we can measure screen momentum to within uncertainty $\delta p$, then its position uncertainty $\delta x \geq \frac{\hbar}{2\delta p}$ – i.e. position of the peaks (and troughs) is unpredictable to within error $\geq \frac{\hbar}{2\delta p} \doteq^{*} d$ = (peak-to-peak separation).

*within order of magnitude

No interference detectable! Consistent with being able to infer path .....

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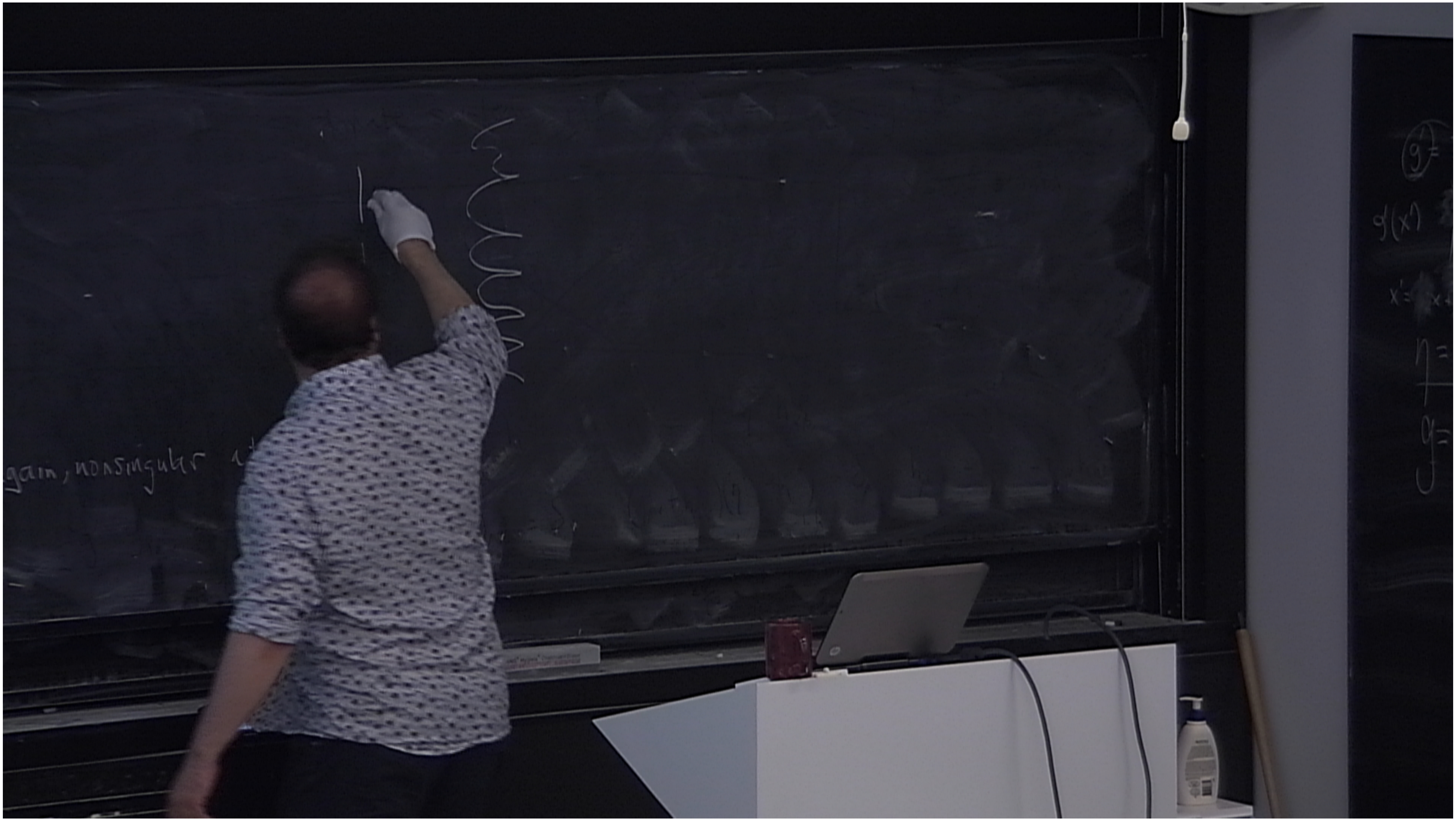
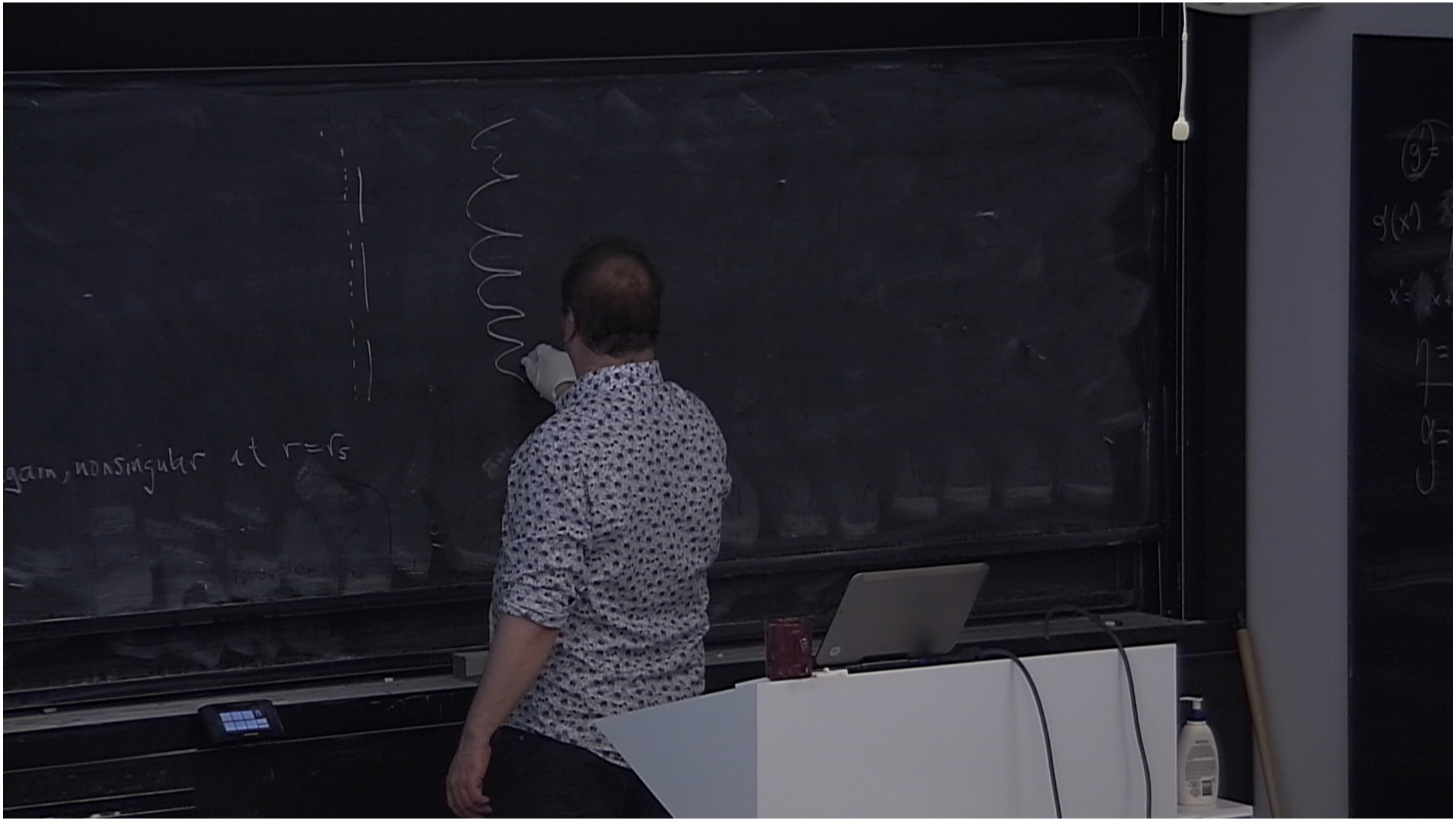
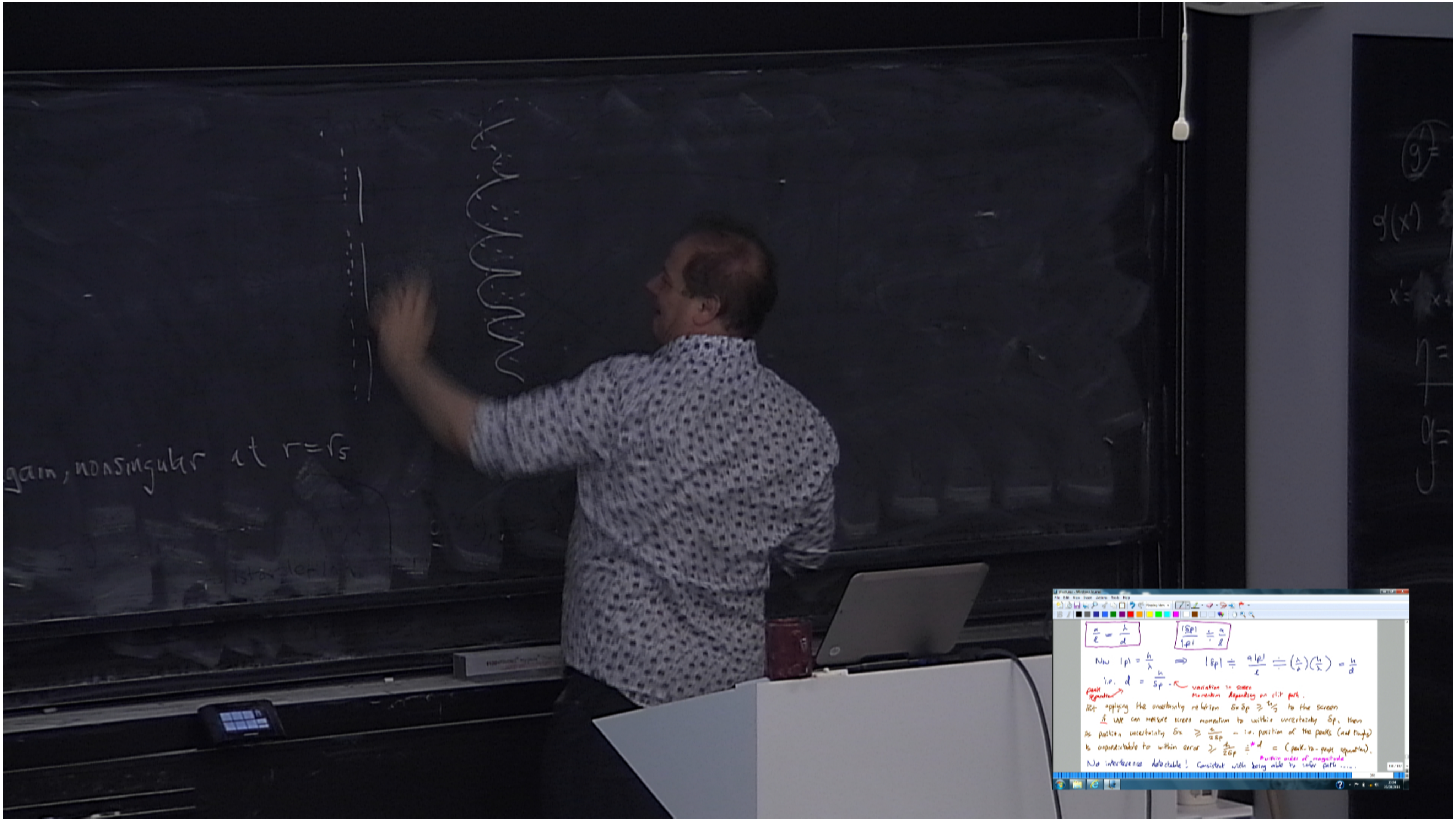

gain, nonsingular


gain, nonsingular at r=rs


gain, nonsingular at r=rs
$\frac{a}{\ell} = \frac{\lambda}{d}$
$\frac{|\delta p|}{|p|} \doteq \frac{a}{\ell}$
Now $|p| = \frac{h}{\lambda}$ $\Rightarrow$ $|\delta p| \doteq \frac{a|p|}{\ell} \doteq \left(\frac{\lambda}{d}\right)\left(\frac{h}{\lambda}\right) = \frac{h}{d}$
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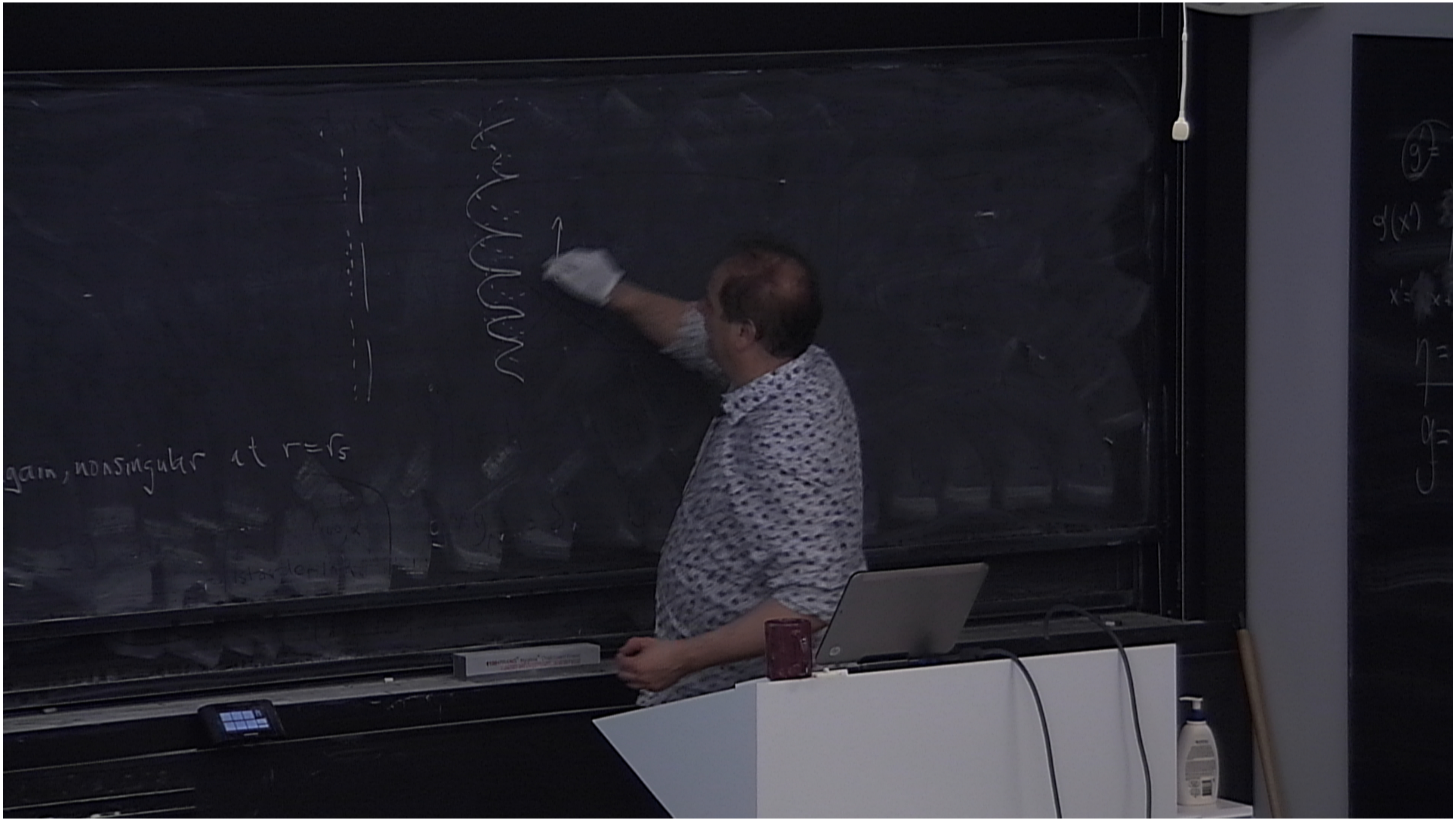
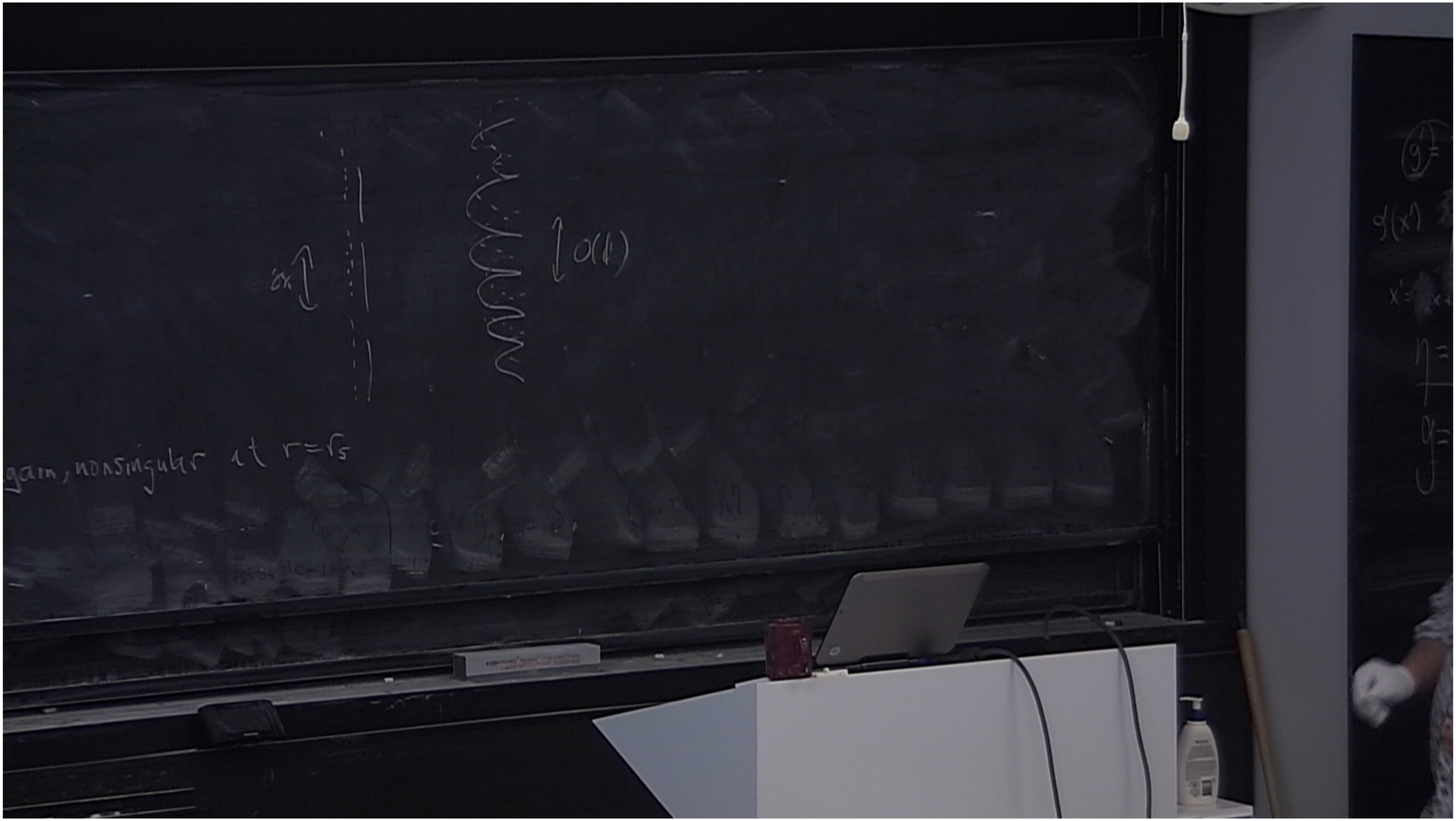
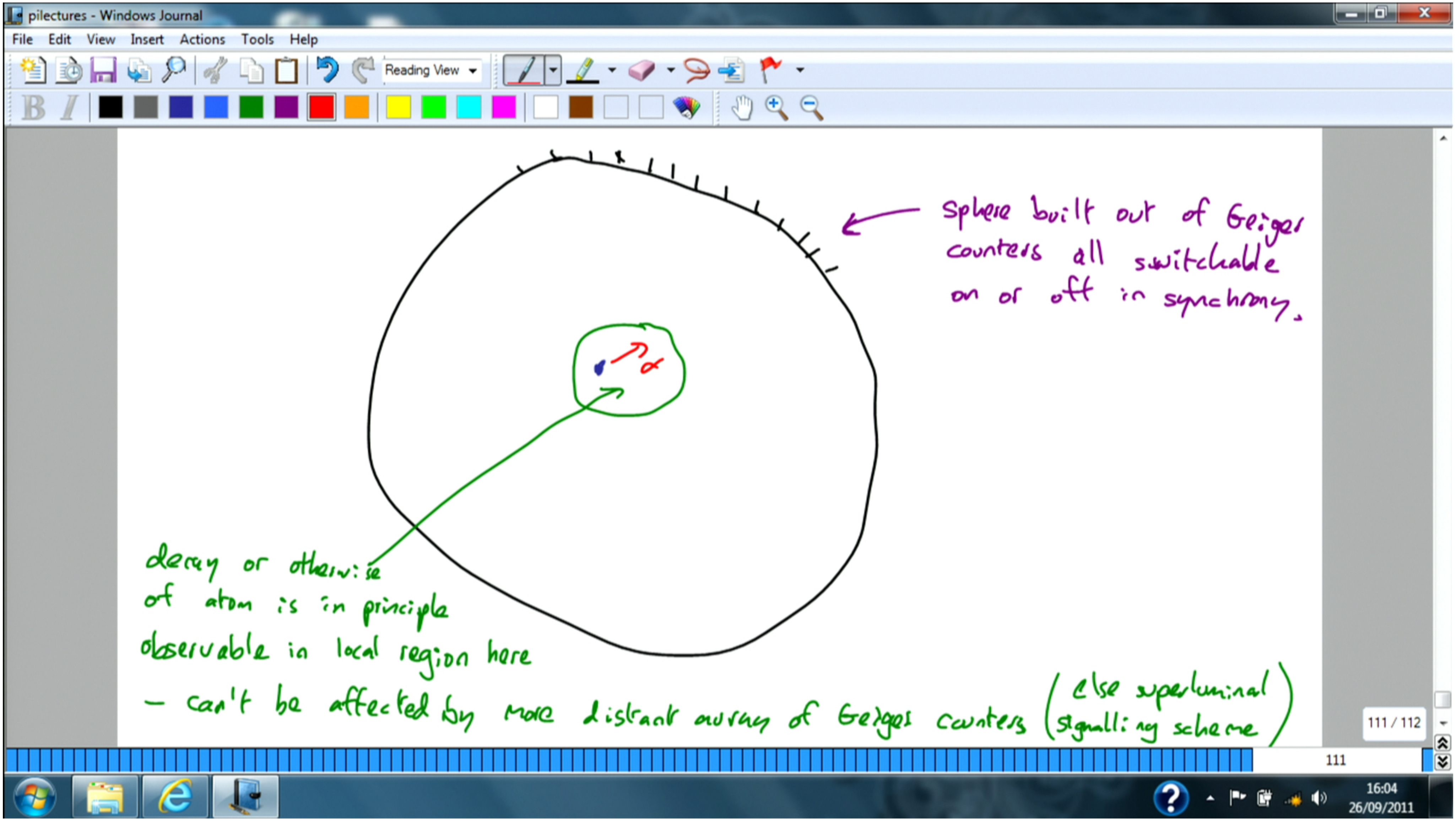

gain, nonsingular at r=rs


gain, nonsingular at r=rs
O(1)

## Things to think (or read) about re the double slit experiment

② We've obtained the wave function of a particle passing through the slits by thinking of the wall as a big detector / measuring device. It could be – it could e.g. flash whenever a particle hits – but usually it isn't. Suppose e.g. it reflects particles. Could that affect the predicted interference pattern?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Sphere built out of Geiger counters all switchable on or off in synchrony.
α
decay or otherwise of atom is in principle observable in local region here
– can't be affected by more distant array of Geiger counters
(else superluminal signalling scheme)
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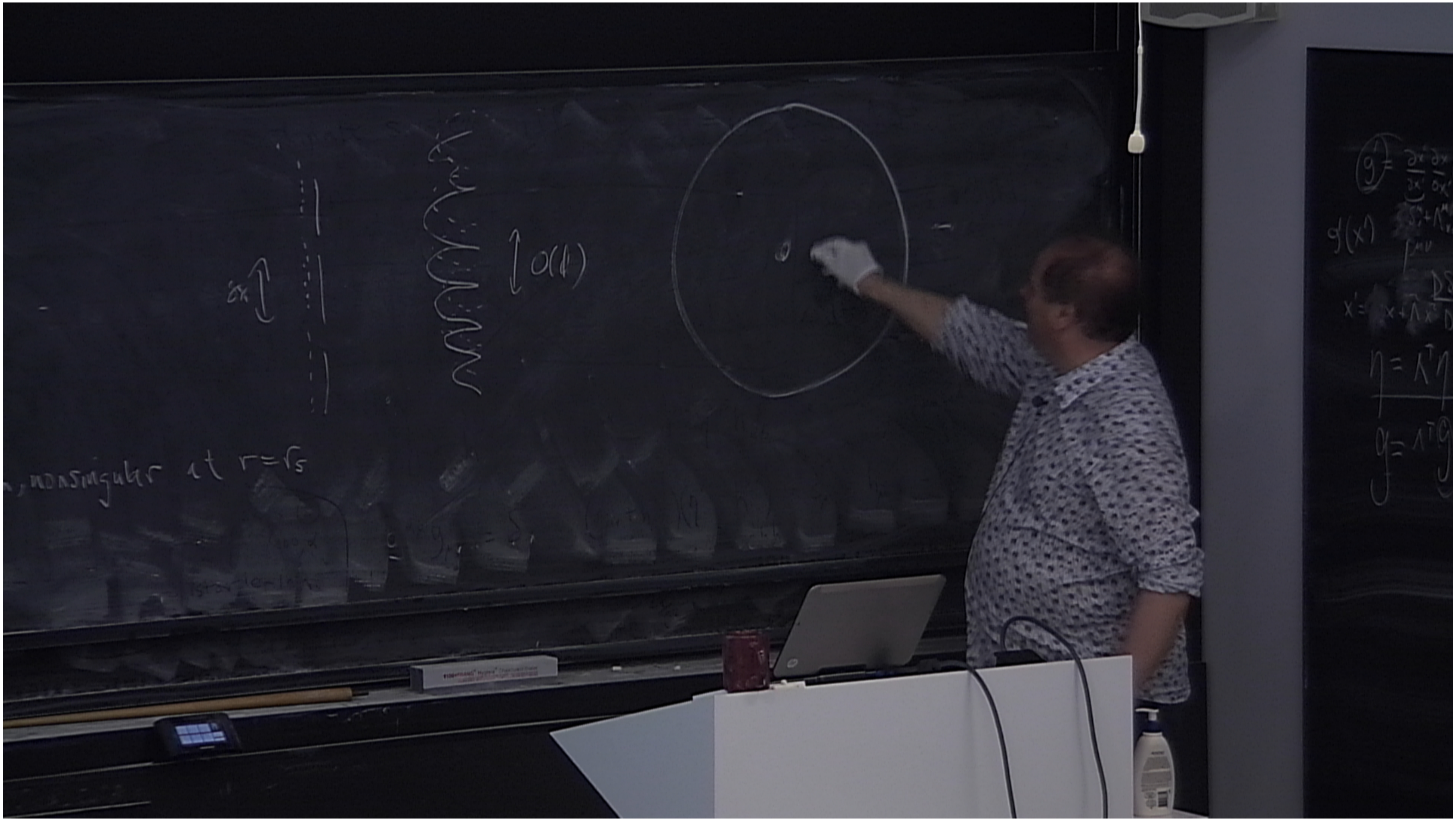
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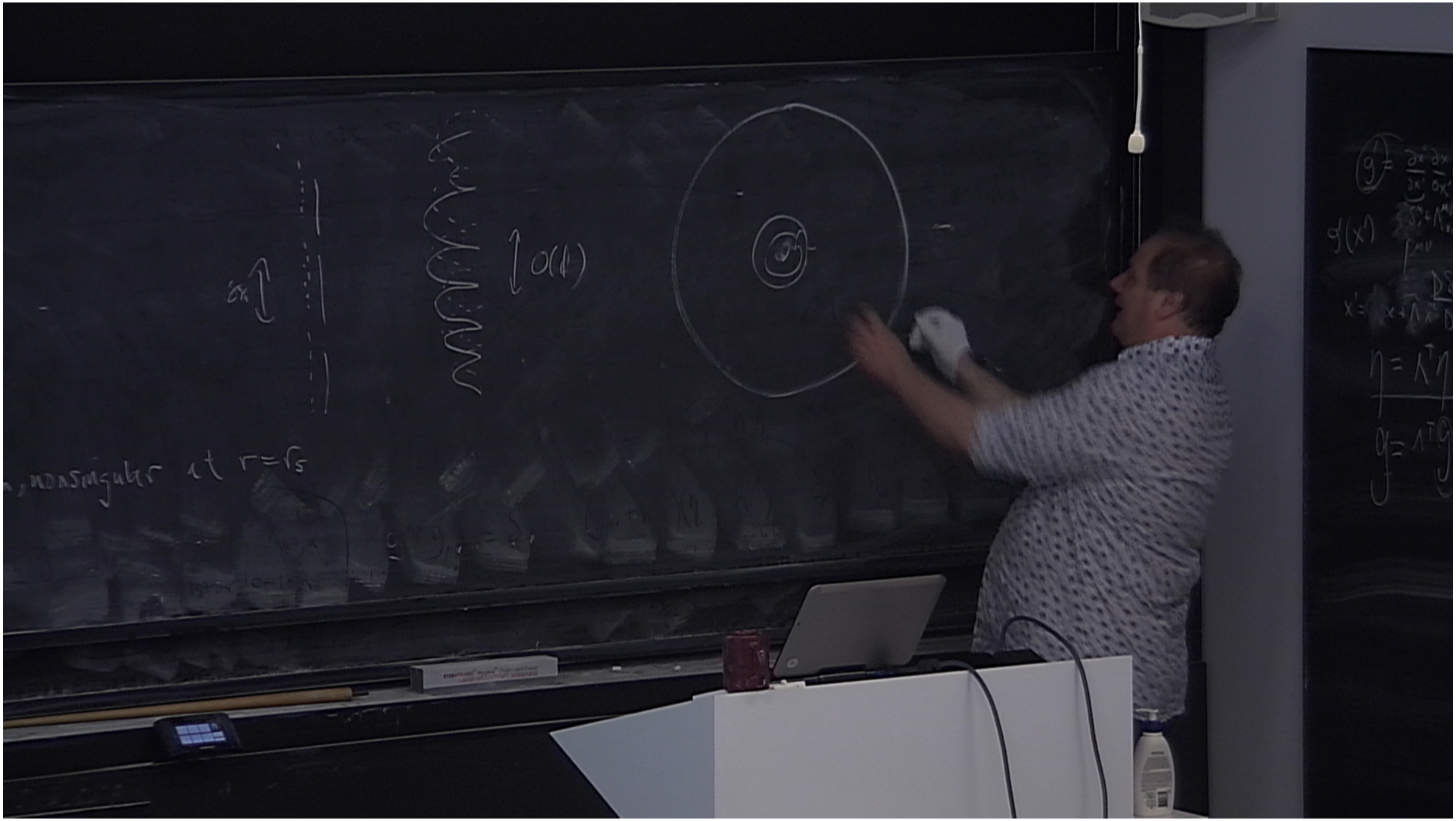
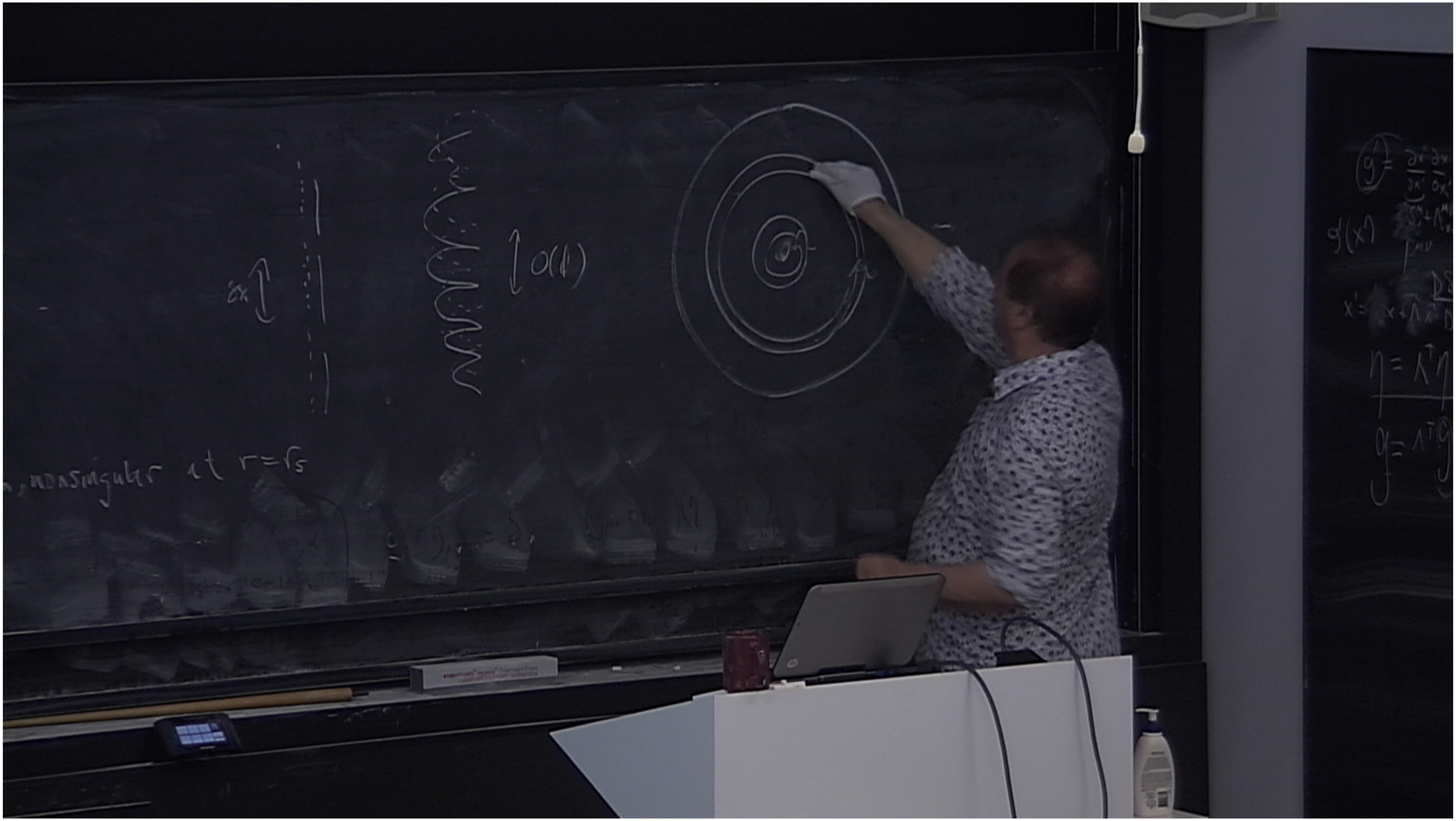
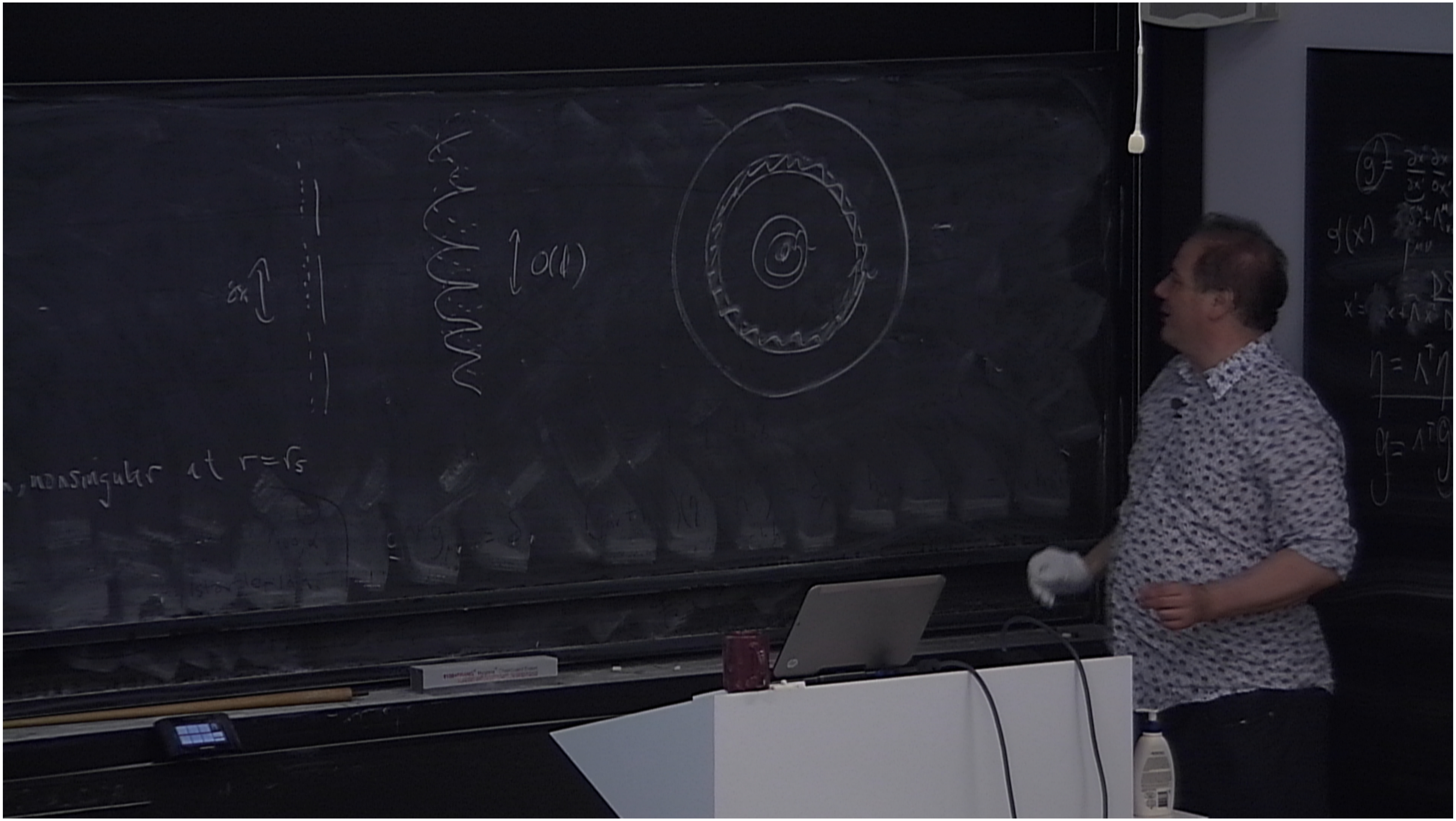

nonsingular at $r=r_s$


O(1)
nonsingular at r=r_s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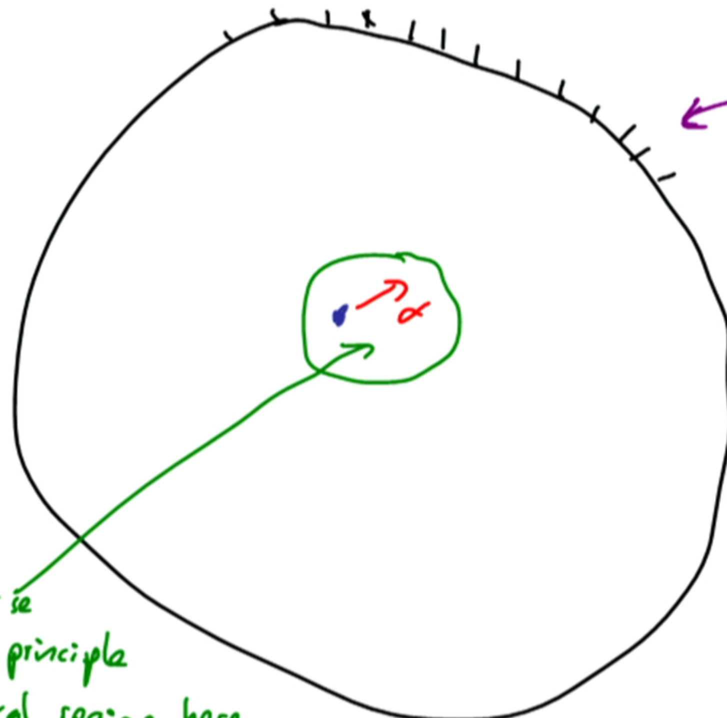
Sphere built out of Geiger counters all switchable on or off in synchrony.




decay or otherwise of atom is in principle observable in local region here

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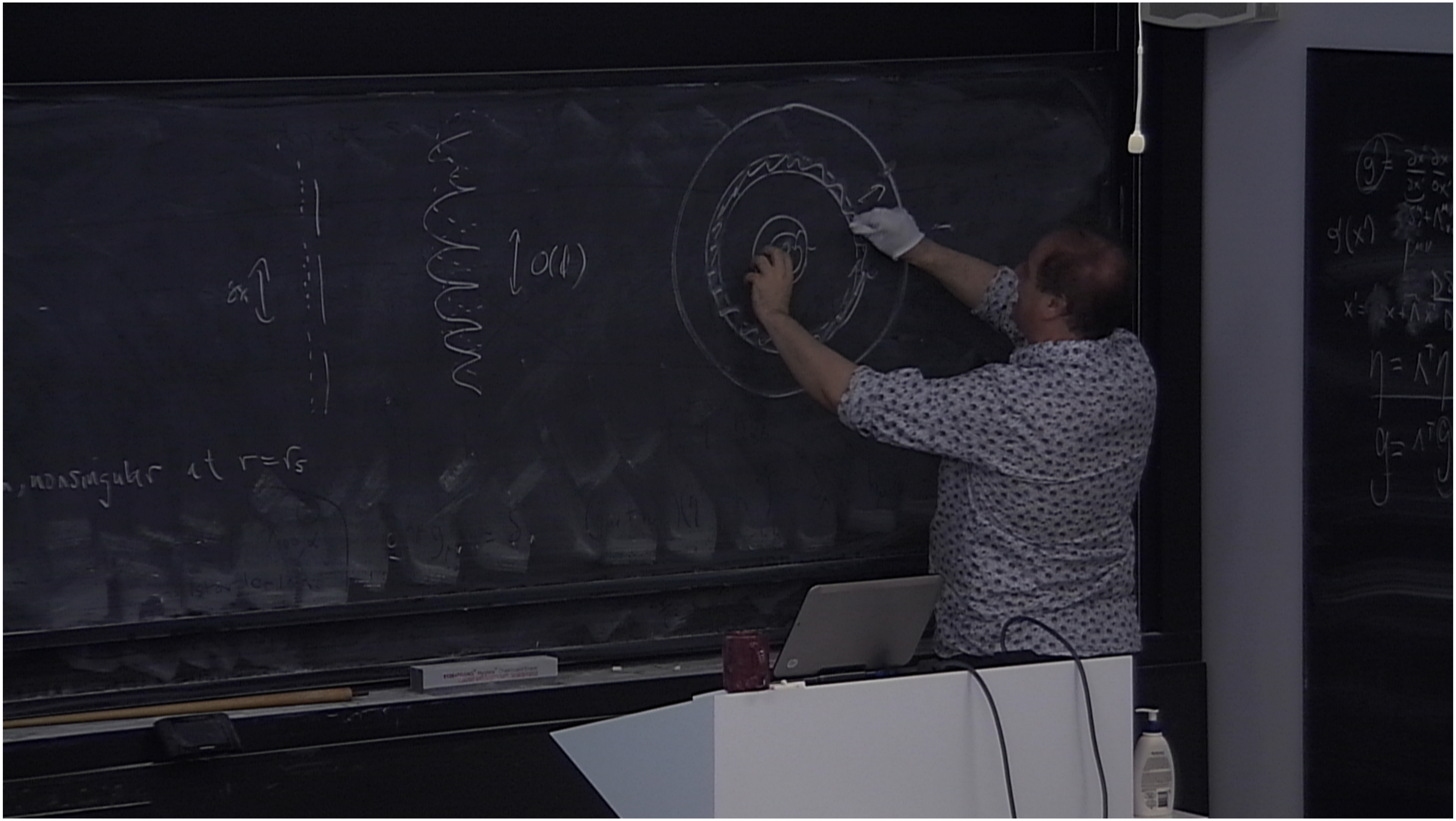
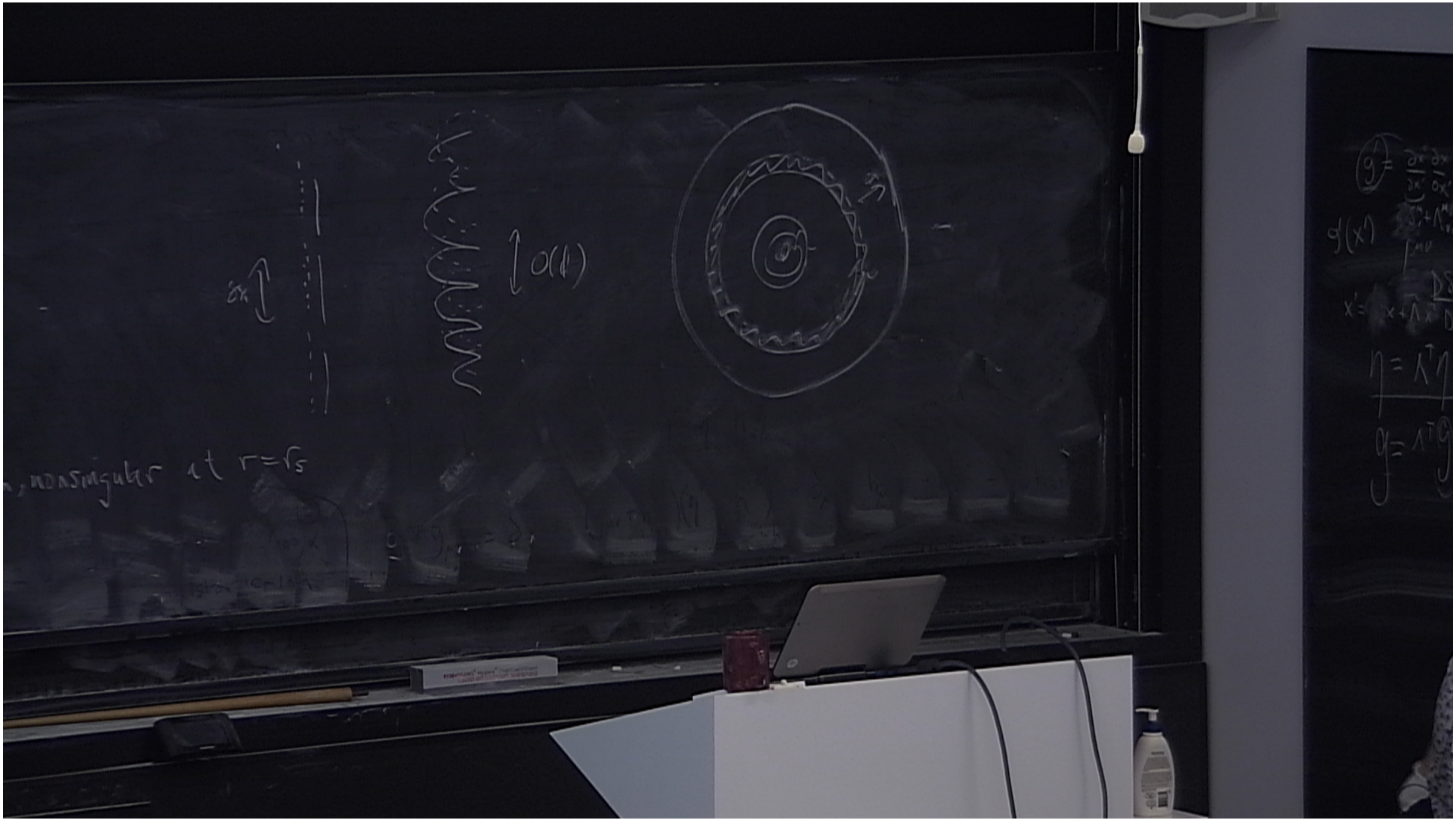

nonsingular at r=rs
δx
O(1)


$O(1)$
, nonsingular at $r=r_s$

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# Continuous position measurements and the quantum Zeno effect

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*Department of Physics, The University of Queensland, Queensland 4072, Australia*

(Dated: July 1993)

We present a new model of continuous (in time) position measurements on a quantum system using a single pseudo-classical meter. The non-selective evolution of the system is described by a master equation which is identical to that obtained from previous models. The selective evolution is described by a stochastic nonlinear Schrödinger equation. The significance of this equation is that the stochastic term has a physical interpretation. By carefully choosing the parameters which define the meter and the system-meter coupling, we obtain a meter pointer with well-defined position which undergoes fluctuations. This 'jitter' in the pointer position gives rise to the stochastic dynamical collapse of the system wavefunction. By the inclusion of feedback on the meter, the pointer is made to relax towards an appropriate readout. We apply this model to the selective measurement of the position of a particle in a double well potential. In contrast to a recent claim [Phys. Rev. A, **46**, 1199 (1992)] we show that truly continuous position measurements lead to a quantum Zeno effect in certain parameter regimes. This is manifest by the changing of the particle dynamics from coherent tunnelling between the well minima to incoherent flipping, as in a random telegraph. As the measurement strength increases, the average length of time the particle remains stuck in one well increases proportionally.

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# Mixed States and Density Matrices

Suppose you're given a state $|\psi\rangle$

and know it's one of $|\psi_1\rangle, \; \ldots\ldots, \; |\psi_n\rangle$

with respective probabilities $p_1, \; \ldots\ldots, \; p_n$

(And this is a complete list of possibilities : $\sum_{i=1}^{n} p_i = 1, \quad p_i \geq 0.$

And you can't learn anything more about the preparation of $|\psi\rangle$.)

How could this happen? Secretive colleague with a random number generator;

Imperfect preparation device with known error statistics, ......

What can you do with this information?

$|\psi\rangle$ is one of $|\psi_1\rangle, \ldots\ldots, |\psi_n\rangle$

with respective probabilities $p_1, \ldots\ldots, p_n$

What can you do with this information?

One option: Keep this list of states and probabilities, and keep track of how it changes as you apply Hamiltonian evolution or make measurements.

Apply $e^{-iHt/\hbar}$: $|\psi\rangle$ is one of $e^{-iHt/\hbar}|\psi_1\rangle, \ldots\ldots, e^{-iHt/\hbar}|\psi_n\rangle$

with respective probabilities $p_1, \ldots\ldots\ p_n$

Maybe not too cumbersome? – though it means solving the Schrödinger equation $n$ times (and $n$ may be big and the solutions may be hard to calculate).

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$|\psi\rangle$ is one of $|\psi_1\rangle, \ldots\ldots, |\psi_n\rangle$

with respective probabilities $p_1, \ldots\ldots p_n$

What if we apply a projective measurement $\{P_j\}$ ? $\left( P_j P_{j'} = \delta_{jj'} P_j , \; \sum_j P_j = I \right)$

$$\text{Prob}(\text{outcome } j) = \sum_i \text{Prob}(\text{outcome } j \mid \text{state } i)\, \text{Prob}(\text{state } i)$$

$$= \sum_i \langle \psi_i | P_j | \psi_i \rangle \, p_i .$$

State after outcome $j$ ? It's one of $\dfrac{P_j|\psi_1\rangle}{|P_j|\psi_1\rangle|}, \ldots\ldots, \dfrac{P_j|\psi_n\rangle}{|P_j|\psi_n\rangle|}$

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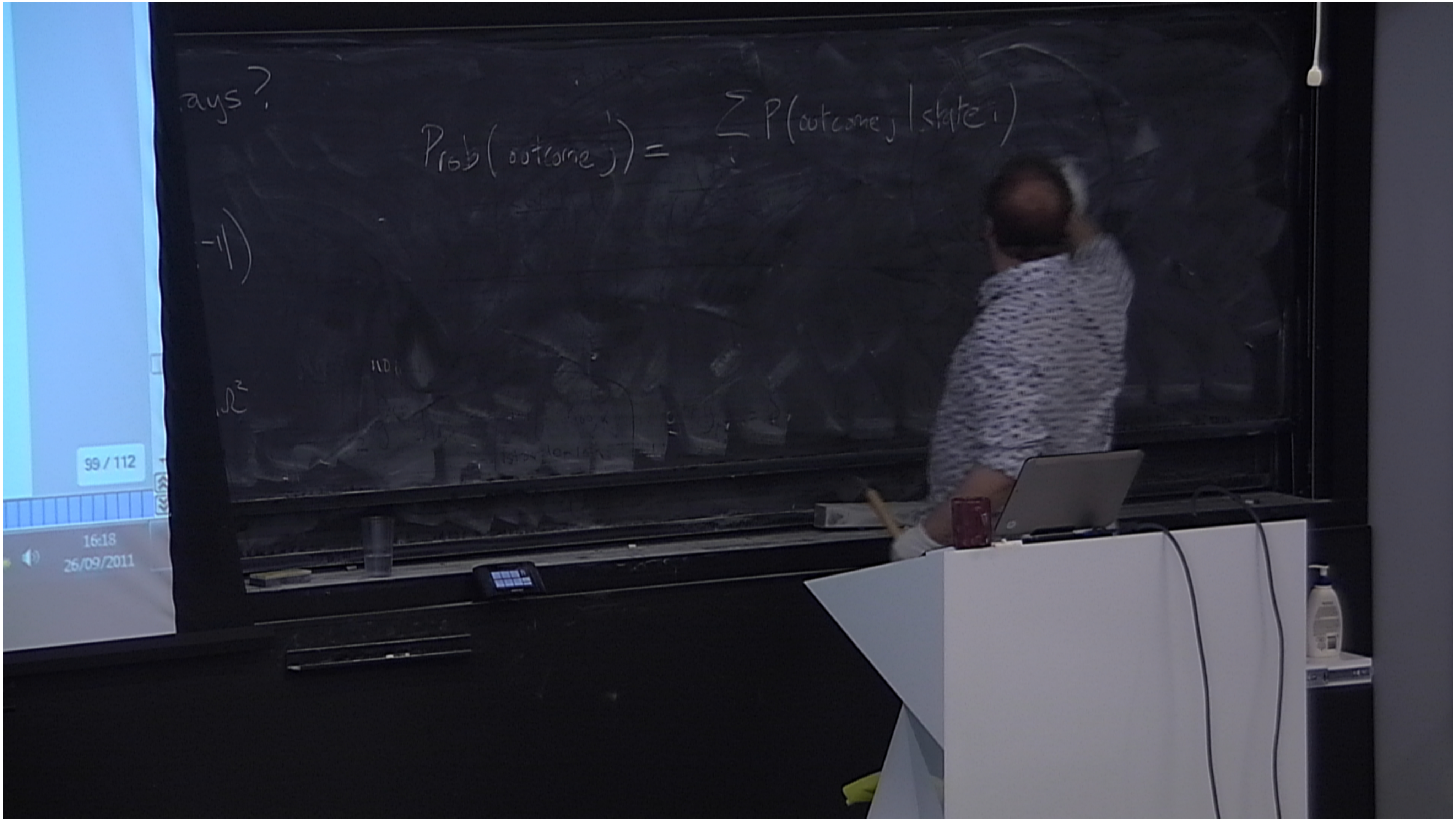

ays?
Prob(outcome j) = Σ P(outcome j | state i)
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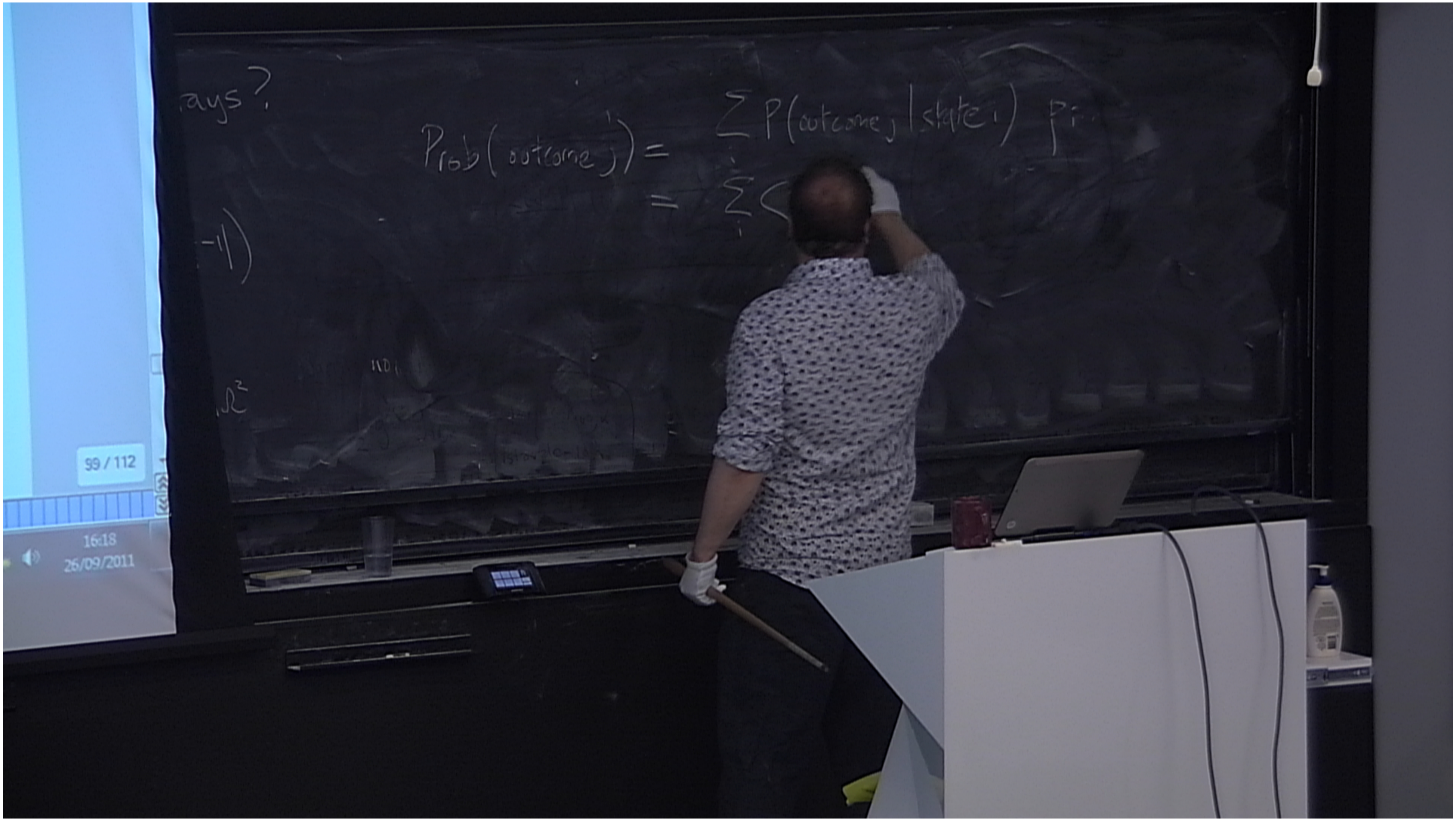
Prob( outcome j ) = Σ P(outcome j | state i) pi
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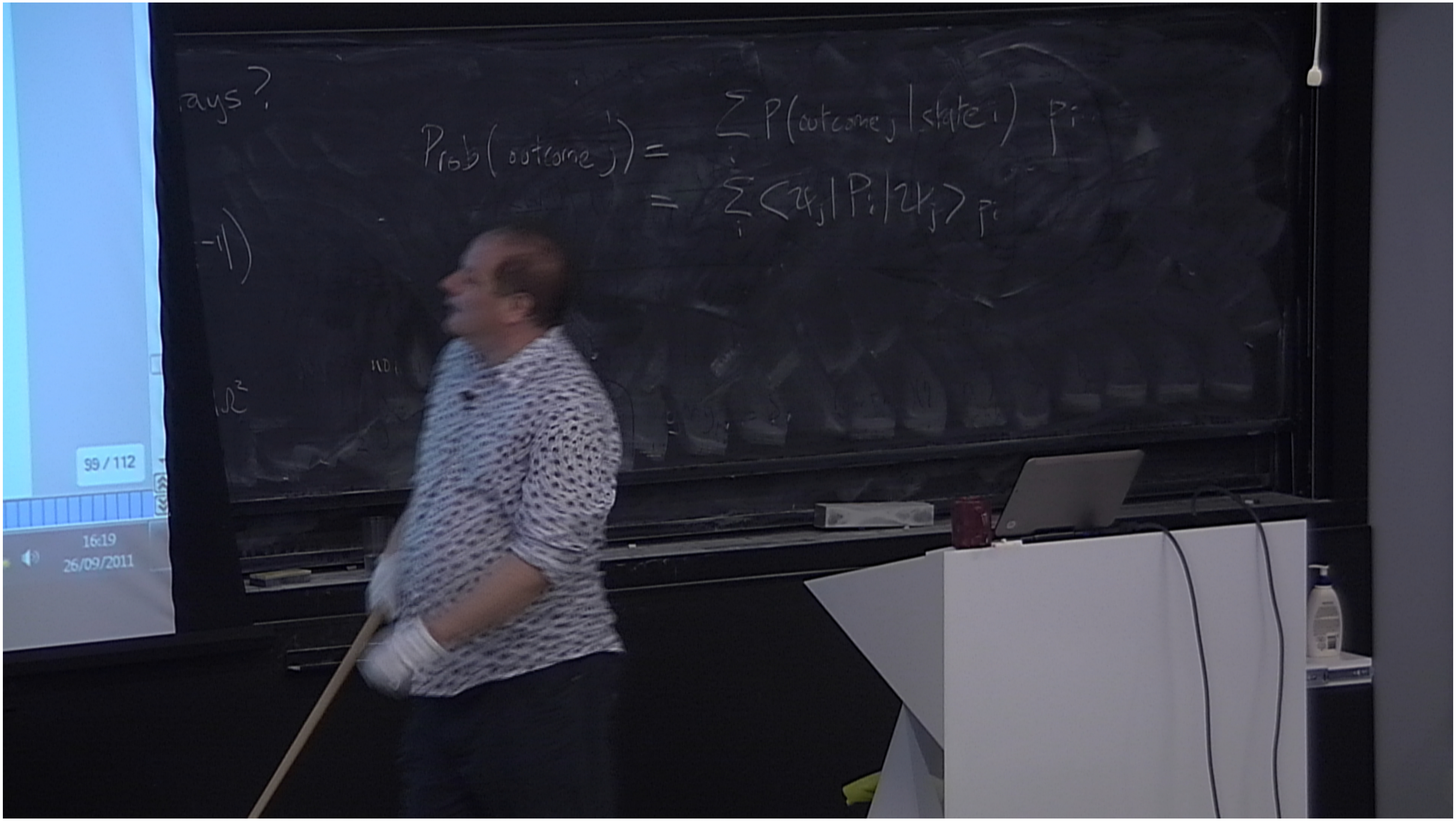
ays?
Prob(outcome j) = Σ P(outcome j | state i) p_i
= Σ ⟨ψ_j | P_i | ψ_j⟩ p_i
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$$\text{Prob}(\text{outcome } j) = \sum_i \text{Prob}(\text{outcome } j \mid \text{state } i)\ \text{Prob}(\text{state } i)$$

$$= \sum_i \langle \psi_i | P_j | \psi_i \rangle\, p_i \, .$$

State after outcome $j$? It's one of $\dfrac{P_j|\psi_1\rangle}{|P_j|\psi_1\rangle|}, \ldots\ldots, \dfrac{P_j|\psi_j\rangle}{|P_j|\psi_j\rangle|}$.

Which one? We don't know, but we can work out probabilities:

$$\text{Prob}(\text{initial state } i \mid \text{outcome } j) = \frac{\text{Prob}(\text{state } i \ \underline{\text{and}}\ \text{outcome } j)}{\text{Prob}(\text{outcome } j)}$$

$$= \frac{p_i \langle \psi_i | P_j | \psi_i \rangle}{\sum_i p_i \langle \psi_i | P_j | \psi_i \rangle}$$

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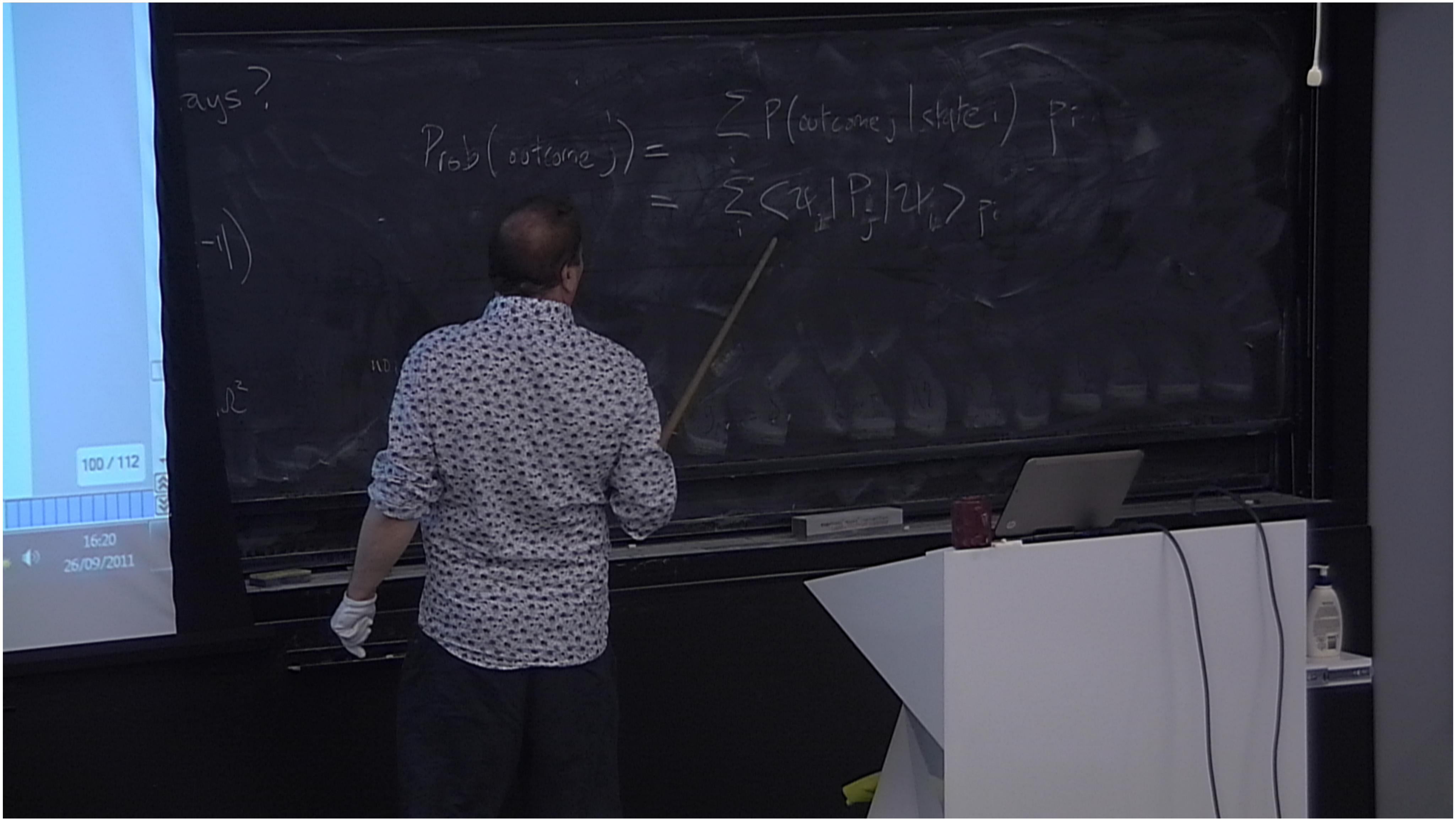
ays?
Prob(outcome_j) = Σ_i P(outcome_j | state_i) p_i
= Σ_i ⟨ψ_i | P_j | ψ_i⟩ p_i
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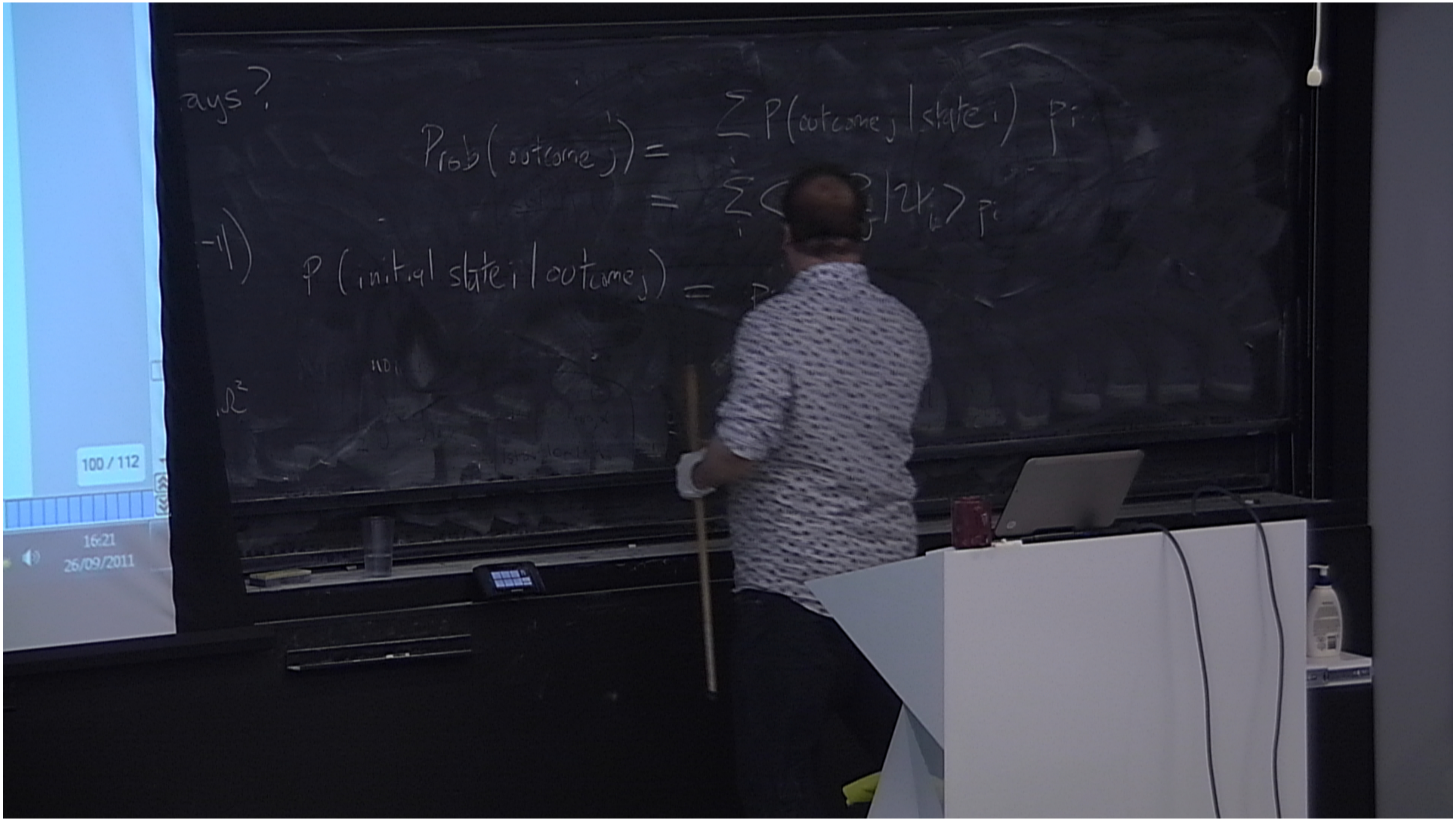
ays?
Prob(outcome_j) = Σ P(outcome_j | state_i) p_i
P(initial state_i | outcome_j) =
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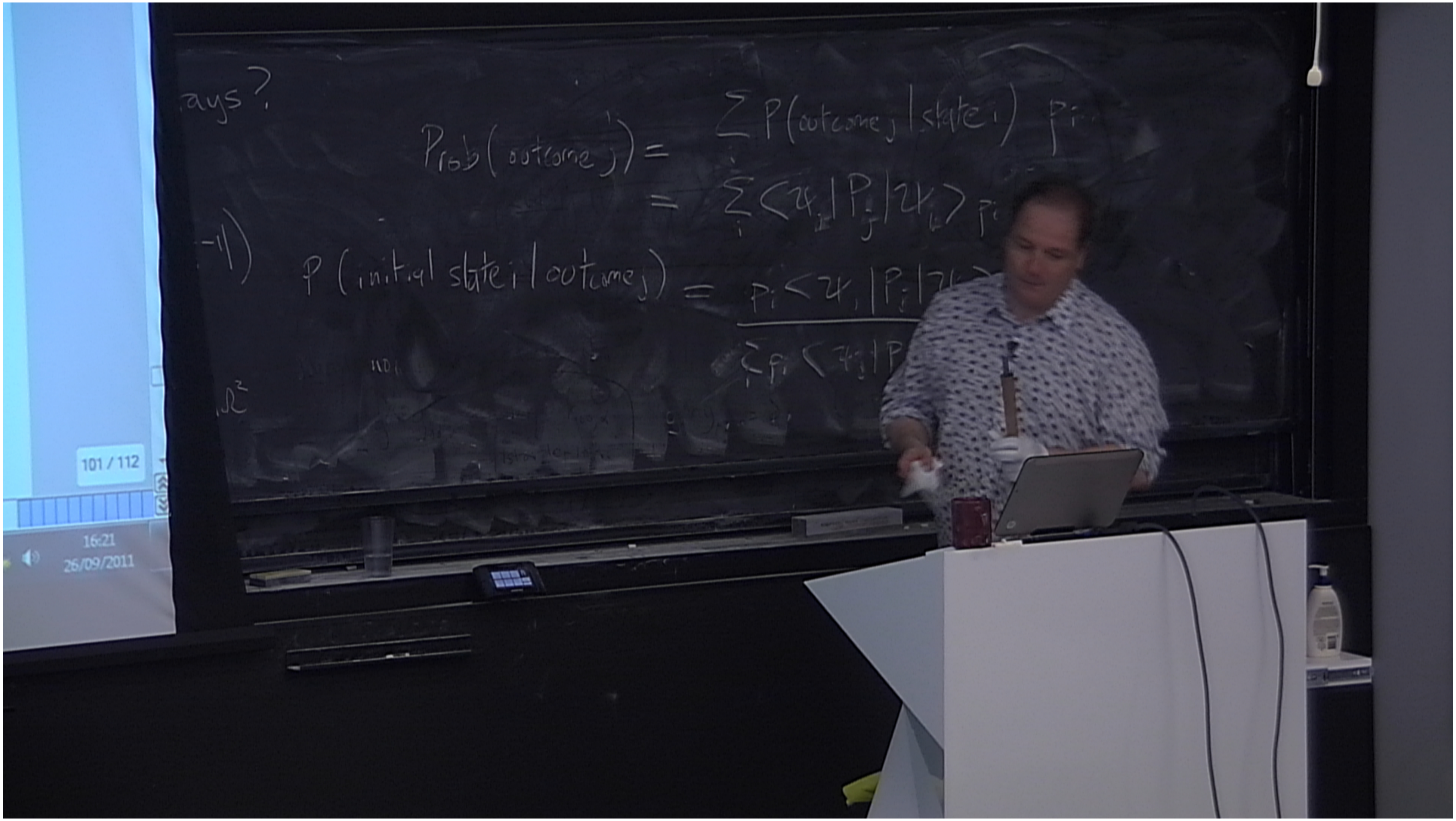

ays?
Prob ( outcome j ) =
P ( initial state i | outcome j ) =
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Our project: (1) define an operator on $H$, the density matrix

$$\rho(0) = \sum_{i=1}^{n} p_i \, |\psi_i(0)\rangle\langle\psi_i(0)| \quad \text{at time } t=0.$$

(2) Introduce the evolution law $i\hbar \frac{d}{dt}\rho(t) = [H, \rho(t)]$

which has solution

$$\rho(t) = \exp\left(\frac{-iHt}{\hbar}\right)\rho(0)\exp\left(\frac{iHt}{\hbar}\right)$$

(3) Introduce the measurement postulate: a measurement defined by projectors $\{P_j\}$ on $\rho$ produces outcome $j$ with probability

$$p_j = \mathrm{Tr}(P_j \rho P_j) = \mathrm{Tr}(P_j \rho)$$

and post-measurement state

$$\frac{P_j \rho P_j}{\mathrm{Tr}(P_j \rho P_j)} = \frac{P_j \rho P_j}{\mathrm{Tr}(P_j \rho)}$$

(4) Check this is all consistent!

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## A (nonquantum) puzzle

Full plane.

First passenger sits in random seat

Succeeding passengers sit in their allocated seat if they can: otherwise random

What's the probability the last passenger sits in their own seat?

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To check the consistency of our definition of the density matrix $\rho = \sum p_i |\psi_i\rangle\langle\psi_i|$

and the evolution law

$$\rho(t) = e^{-iHt/\hbar}\rho(0)e^{iHt/\hbar},$$

note that this diagram commutes

## Density Matrix – Time Evolution

$$\begin{tikzcd}
\{p_i, |\psi_i\rangle\} \arrow[r, dashed, "\text{Time Evolution}"] \arrow[d, dashed, "\text{Density Matrix}"'] & \{p_i, e^{-iHt/\hbar}|\psi_i\rangle\} \arrow[d, dashed, "\text{Density Matrix}"] \\
\rho(0) = \sum_{i=1}^{n} p_i |\psi_i\rangle\langle\psi_i| \arrow[r, dashed, "\text{Time Evolution}"'] & \rho(t) = e^{-iHt/\hbar}\rho(0)e^{iHt/\hbar}
\end{tikzcd}$$

Pretty obvious!

Good ....

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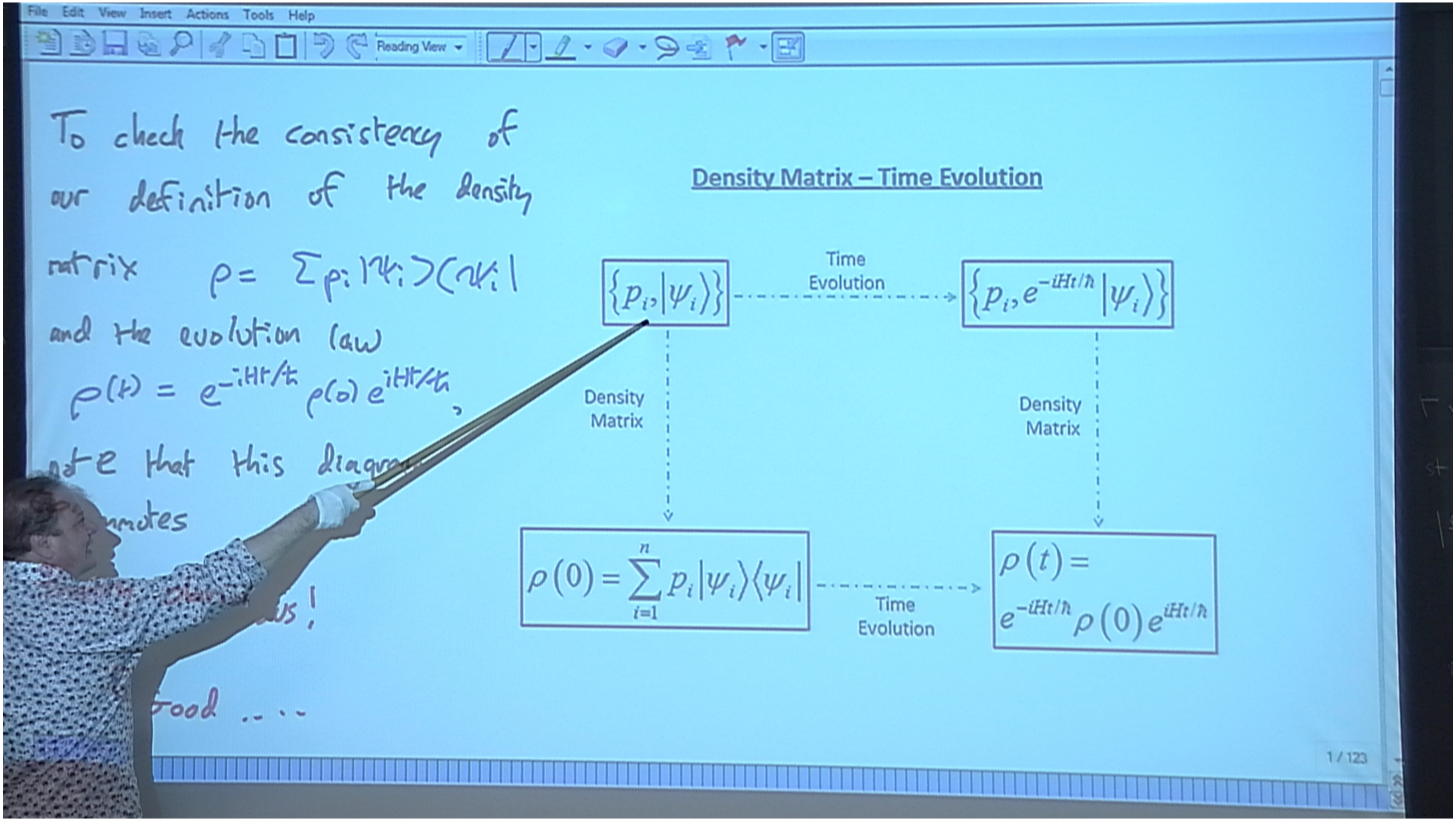
1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To check the consistency of
our definition of the density
matrix $\rho = \sum p_i |\psi_i\rangle\langle\psi_i|$
and the evolution law
$\rho(t) = e^{-iHt/\hbar}\rho(0)e^{iHt/\hbar}$,
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Good ....
Density Matrix – Time Evolution
$\{p_i, |\psi_i\rangle\}$
Time Evolution
$\{p_i, e^{-iHt/\hbar}|\psi_i\rangle\}$
Density Matrix
Density Matrix
$\rho(0) = \sum_{i=1}^{n} p_i |\psi_i\rangle\langle\psi_i|$
Time Evolution
$\rho(t) = e^{-iHt/\hbar}\rho(0)e^{iHt/\hbar}$
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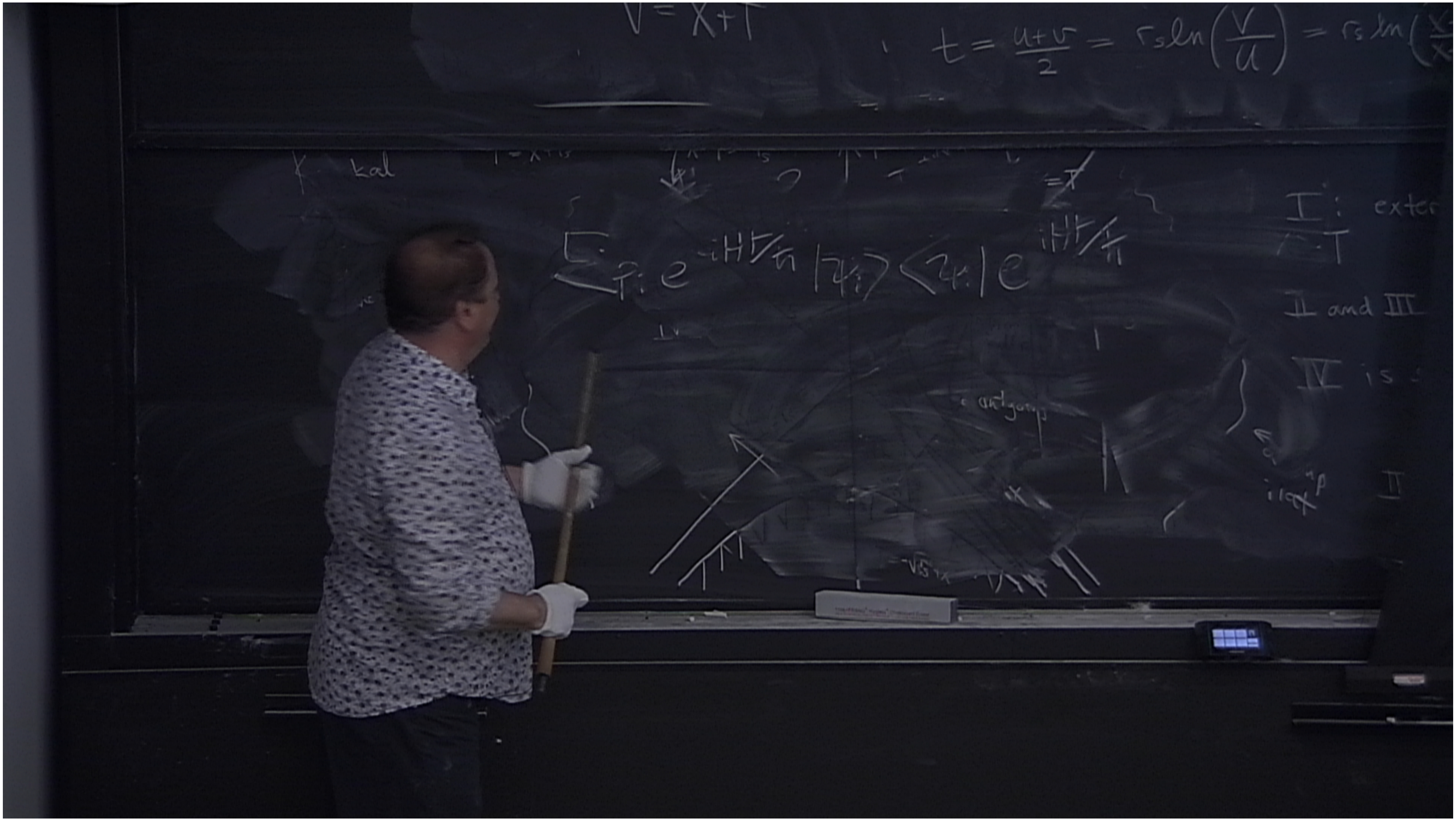
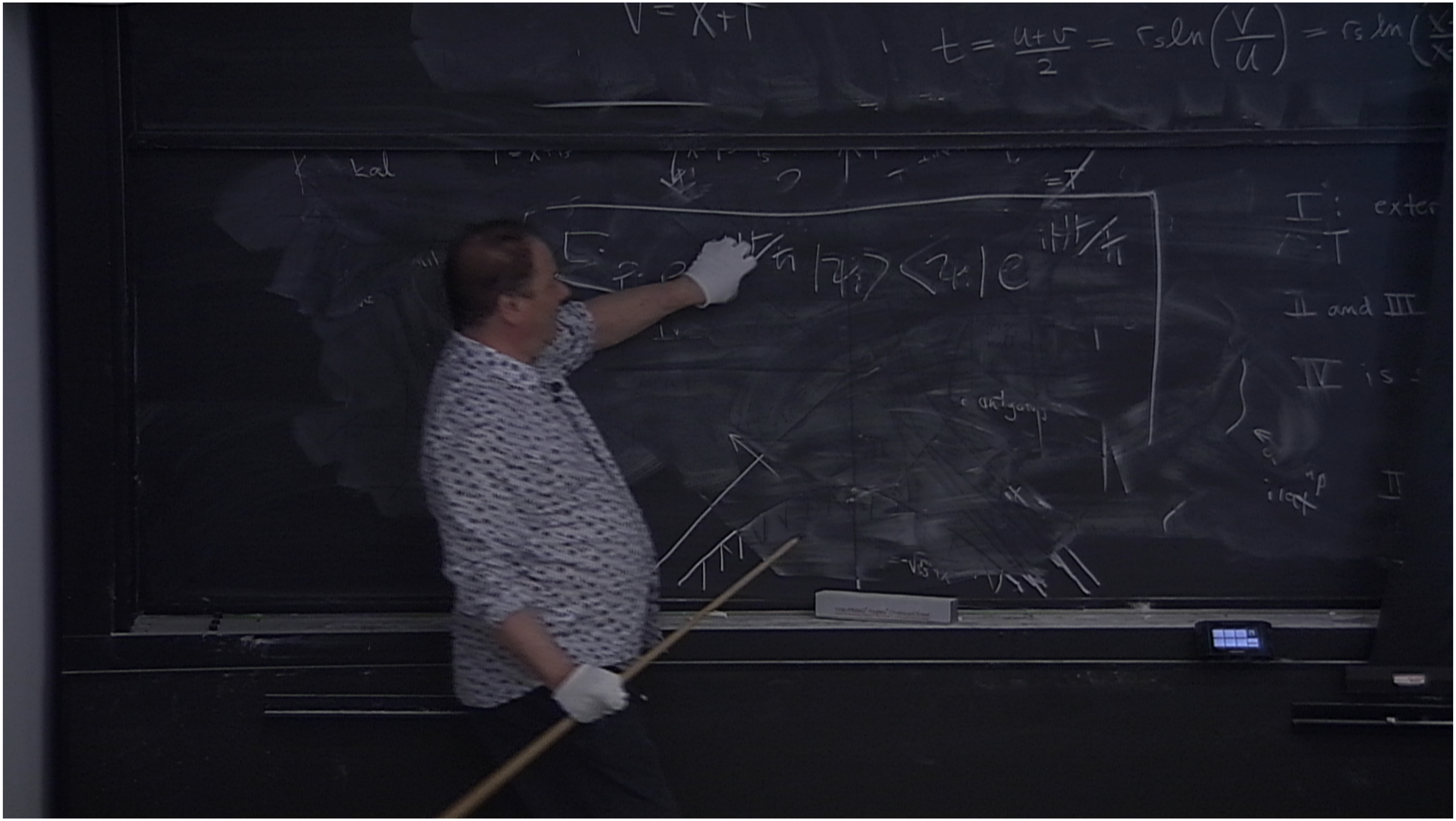
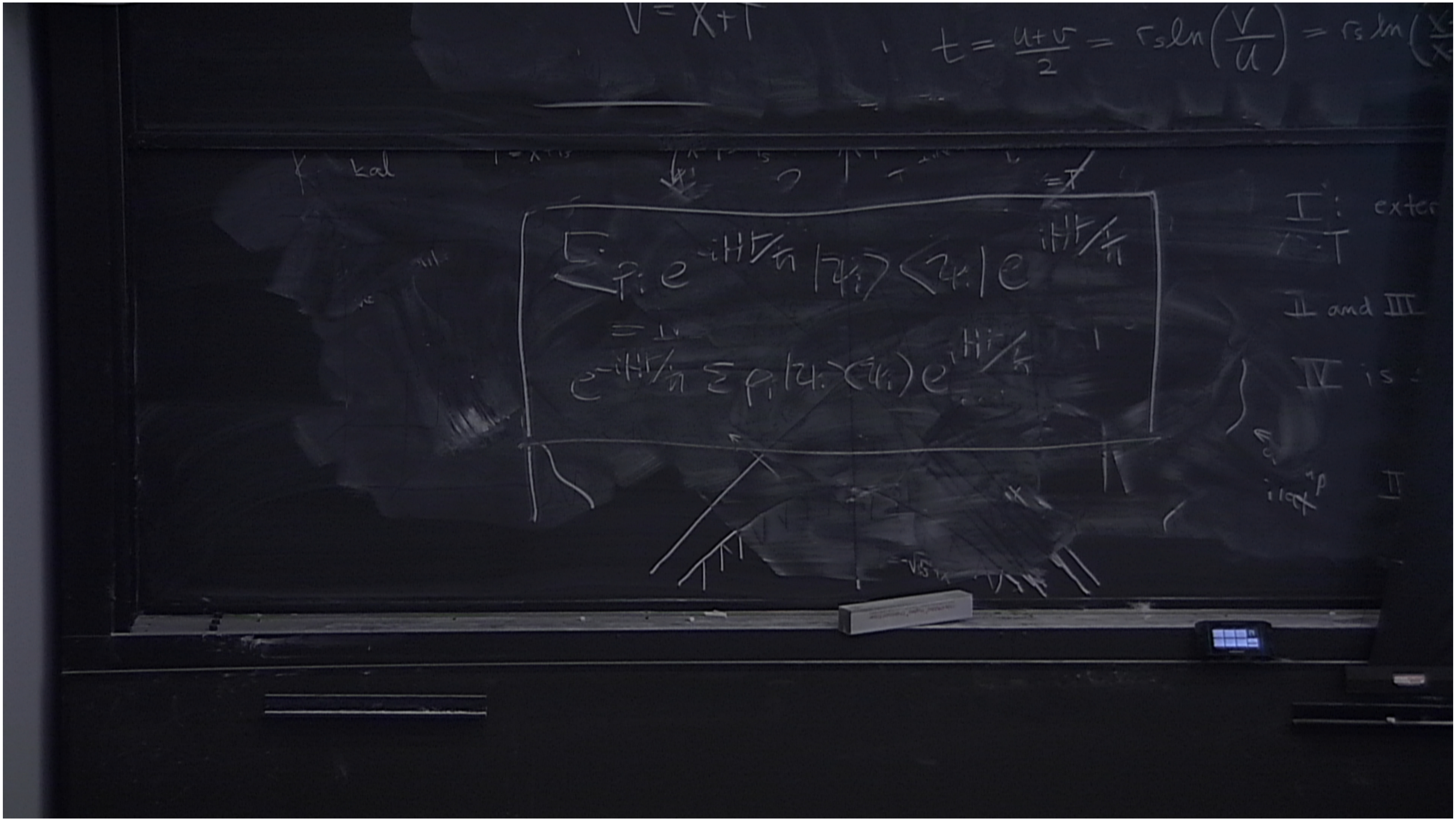

II and III


$t = \frac{u+v}{2} = r_s \ln\left(\frac{v}{u}\right) = r_s \ln$
I : exter
II and III
IV is


t = (u+v)/2 = r_s ln(V/U) = r_s ln(
Σ p_i e^{-iHt/ħ} |ψ_i⟩⟨ψ_i| e^{iHt/ħ}
e^{-iHt/ħ} Σ p_i |ψ_i⟩⟨ψ_i| e^{iHt/ħ}
I : exter
II and III
IV is

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## Density Matrix - Measurement

To check consistency of our definition of the density matrix and the measurement postulate, we need to check this diagram also commutes

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\boxed{\{p_i, |\psi_i\rangle\}} \arrow[r, dashed, "\text{measurement}", "\text{outcome } j"'] \arrow[d, dashed, "\text{density matrix}"] & \boxed{\left\{ p_i', \frac{P_j|\psi_i\rangle}{|P_j|\psi_i\rangle|} \right\}} \arrow[d, dashed, "\text{density matrix}"] \\
\boxed{\rho = \sum p_i |\psi_i\rangle\langle\psi_i|} \arrow[r, dashed, "\text{measurement}", "\text{outcome } j"'] & \boxed{\frac{P_j \rho P_j}{\mathrm{Tr}(P_j \rho P_j)}}
\end{tikzcd}$$

Not quite so obvious!

The a priori probabilities $\{p_i\}$ changed to $\{p_i'\}$ after a measurement with outcome $j$

(some states are likelier to produce $j$ than others)

We need to use our earlier calculation of the $\{p_i'\}$ to verify this works

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2

OK. So, explicitly:

$$\{p_i, |\psi_i\rangle\} \xrightarrow[\text{outcome } j]{\text{measurement}} \left\{ \frac{p_i \langle\psi_i|P_j|\psi_i\rangle}{\sum_i p_i\langle\psi_i|P_j|\psi_i\rangle}\,,\ \frac{P_j|\psi_i\rangle}{\left|P_j|\psi_i\rangle\right|} \right\}$$

density matrix

$$\rho = \sum p_i |\psi_i\rangle\langle\psi_i|$$

density matrix

$$\sum_i \left( \left( \frac{p_i \cancel{\langle\psi_i|P_j|\psi_i\rangle}}{\sum_i p_i \langle\psi_i|P_j|\psi_i\rangle} \right) \frac{P_j|\psi_i\rangle\langle\psi_i|P_j}{\cancel{\left|P_j|\psi_i\rangle\right|}^2} \right)$$

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3

OK. So, explicitly:

$$\{p_i, |\psi_i\rangle\} \xrightarrow[\text{outcome } j]{\text{measurement}} \left\{ \frac{p_i \langle\psi_i|P_j|\psi_i\rangle}{\sum_i p_i \langle\psi_i|P_j|\psi_i\rangle} \, , \, \frac{P_j|\psi_i\rangle}{|P_j|\psi_i\rangle|} \right\}$$

density matrix

$$\rho = \sum p_i |\psi_i\rangle\langle\psi_i|$$

density matrix

$$\sum_i \left( \left( \frac{p_i \cancel{\langle\psi_i|P_j|\psi_i\rangle}}{\sum_i p_i \langle\psi_i|P_j|\psi_i\rangle} \right) \frac{P_j|\psi_i\rangle\langle\psi_i|P_j}{\cancel{|P_j|\psi_i\rangle|}} \right)$$

$$= \frac{P_j \left(\sum_i p_i |\psi_i\rangle\langle\psi_i|\right) P_j}{\sum_i p_i \langle\psi_i|P_j|\psi_i\rangle}$$

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OK. So, explicitly:

$$\{p_i, |\psi_i\rangle\} \xrightarrow[\text{outcome } j]{\text{measurement}} \left\{ \frac{p_i\langle\psi_i|P_j|\psi_i\rangle}{\sum_i p_i \langle\psi_i|P_j|\psi_i\rangle} , \frac{P_j|\psi_i\rangle}{|P_j|\psi_i\rangle|} \right\}$$

$\downarrow$ density matrix

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$$\rho = \sum p_i |\psi_i\rangle\langle\psi_i|$$

$$\sum_i \left( \left( \frac{p_i \cancel{\langle\psi_i|P_j|\psi_i\rangle}}{\sum_i p_i \langle\psi_i|P_j|\psi_i\rangle} \right) \frac{P_j|\psi_i\rangle\langle\psi_i|P_j}{\cancel{|P_j|\psi_i\rangle|^2}} \right)$$

$$\|$$

$$\frac{P_j \rho P_j}{Tr(\rho P_j)} = \frac{P_j \left(\sum_i p_i |\psi_i\rangle\langle\psi_i|\right) P_j}{\sum_i p_i \langle\psi_i|P_j|\psi_i\rangle}$$

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$$\Big\downarrow \text{density matrix} \qquad\qquad \Big\downarrow \text{density matrix}$$

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So it works! Our definitions give the right answers for time evolution and for measurements, and so for any sequence of either or both of these.

But that's all there is in quantum theory! If we get the same predictions from $\rho$ as from our list of states and probabilities for any possible experiment, $\rho$ carries all the physical information we have available.

Mathematically, $\{ p_i, |\psi_i\rangle \}$ carries more info. than $\rho$.

But the extra information makes no practical difference to us.

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7

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Ensemble 1: probabilities $\{\frac{1}{2}, \frac{1}{2}\}$ states $\{|\uparrow\rangle, |\downarrow\rangle\}$

Ensemble 2: probabilities $\{\frac{1}{2}, \frac{1}{2}\}$ states $\{|\rightarrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle),$
$|\leftarrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle)\}$

These describe mathematically distinct ensembles, physically distinct situations.

But we can't distinguish them experimentally:

$$\rho_1 = \frac{1}{2}\left(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|\right)$$

$$\rho_2 = \frac{1}{4}\left((|\uparrow\rangle + |\downarrow\rangle)(\langle\uparrow| + \langle\downarrow|) + (|\uparrow\rangle - |\downarrow\rangle)(\langle\uparrow| - \langle\downarrow|)\right) = \rho$$

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$$\rho' = \frac{1}{4}\left((|\uparrow\rangle + |\downarrow\rangle)(\langle\uparrow| + \langle\downarrow|) + (|\uparrow\rangle - |\downarrow\rangle)(\langle\uparrow| - \langle\downarrow|)\right) = \rho$$

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Comment The states $\rho = \sum p_i |\psi_i\rangle\langle\psi_i|$ defined by a probabilistic mixture $\{p_i, |\psi_i\rangle\}$ are called proper mixed states, to distinguish them from improper mixed states (which we'll consider next).

(Horrible terminology, but it's standard.)

Pure and mixed states It's useful to think of density matrices as mathematical objects — operators on $\mathcal{H}$ of a certain form — and characterize some of their properties. For example:

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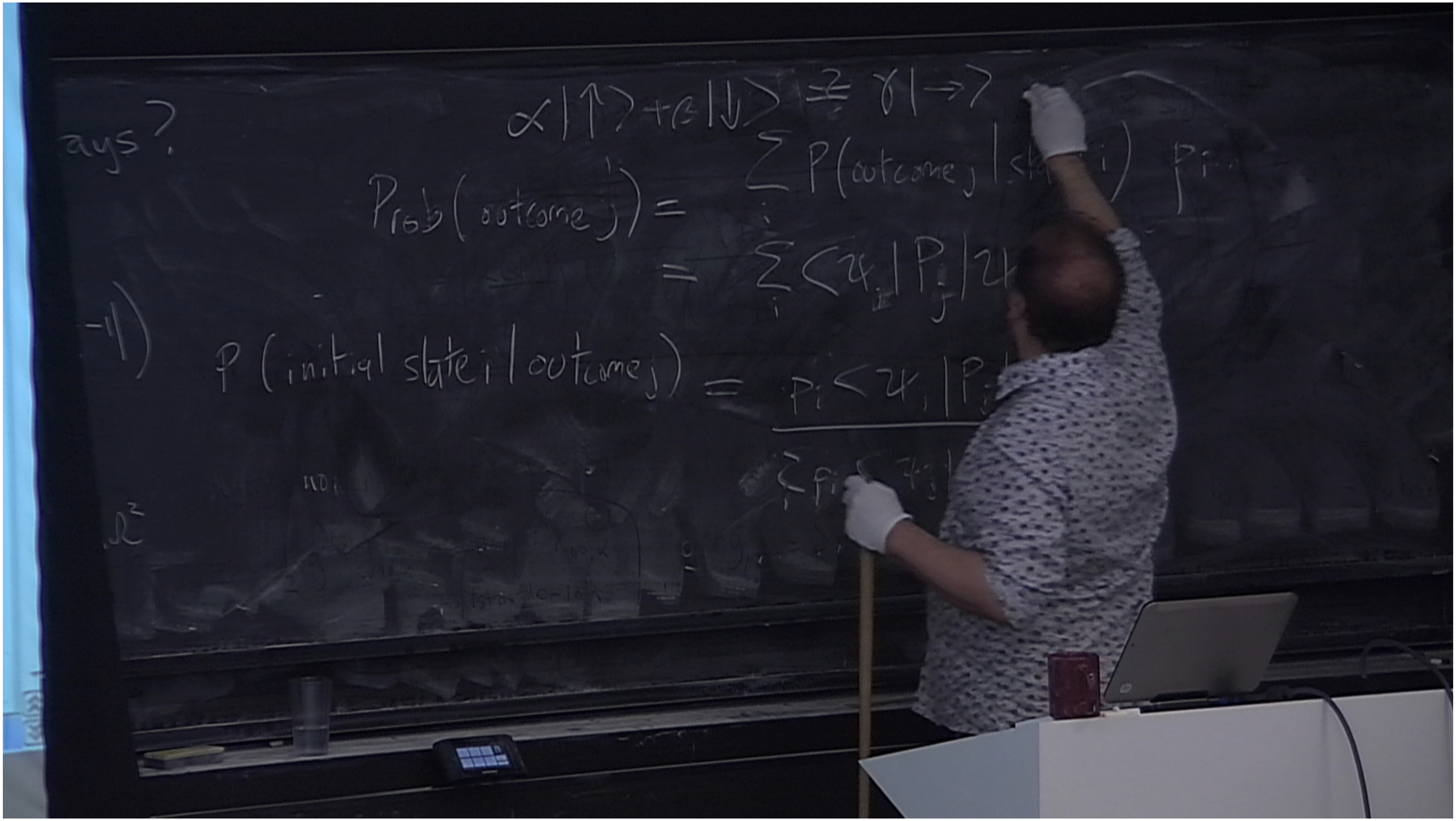
ays?
α|↑⟩ + β|↓⟩ ≟ γ|→⟩
Prob(outcome_j) =
P(initial state_i | outcome_j) =

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<u>Comment</u> The states $\rho = \sum p_i |\psi_i\rangle\langle\psi_i|$ defined by a probabilistic mixture $\{p_i, |\psi_i\rangle\}$ are called <u>proper mixed states</u>, to distinguish them from <u>improper mixed states</u> (which we'll consider next).

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<u>Pure and mixed states</u> It's useful to think of density matrices as mathematical objects — operators on $\mathcal{H}$ of a certain form — and characterize some of their properties. For example:

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**Lemma** Every density matrix is (i) self adjoint, $\rho = \rho^\dagger$,

(ii) positive semi-definite : $\langle\psi|\rho|\psi\rangle \geq 0$ for all $|\psi\rangle$.

(iii) normalised, $\mathrm{Tr}(\rho) = 1$.

And every operator on $\mathcal{H}$ obeying (i)-(iii) is a density matrix of some mixture $\{p_i, |\psi_i\rangle\}$.

**Proof** If $\rho = \sum p_i |\psi_i\rangle\langle\psi_i|$, then (i) $\rho^\dagger = \sum p_i |\psi_i\rangle\langle\psi_i| = \rho$,

(ii) $\langle\psi|\rho|\psi\rangle = \sum p_i \langle\psi|\psi_i\rangle\langle\psi_i|\psi\rangle = \sum p_i |\langle\psi|\psi_i\rangle|^2$,

since the probabilities $p_i \geq 0$

(iii) $\mathrm{Tr}(\rho) = \sum p_i \langle\psi_i|\psi_i\rangle = \sum p_i = 1$.

And if $\rho$ obeys (i)-(iii), then (i) tells us $\rho$ is diagonalizable: $\rho = \sum \lambda_i |e_i\rangle\langle e_i|$

(ii) tells us $\lambda_i \geq 0$, (iii) tells us $\sum \lambda_i = 1$. So $\rho$ is a density matrix for the mixture $\{\lambda_i, |e_i\rangle\}$

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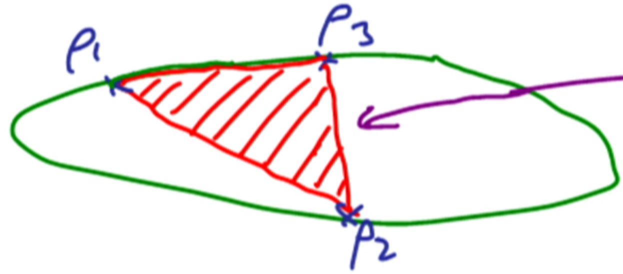
**Pure and mixed states** The set of density matrices is a convex subset of the space of operators on $\mathcal{H}$. That is, if $a_i \geq 0$ and $\sum a_i = 1$, and $\rho_i$ are density matrices, so is the convex sum $\rho = \sum a_i \rho_i$.

E.g.



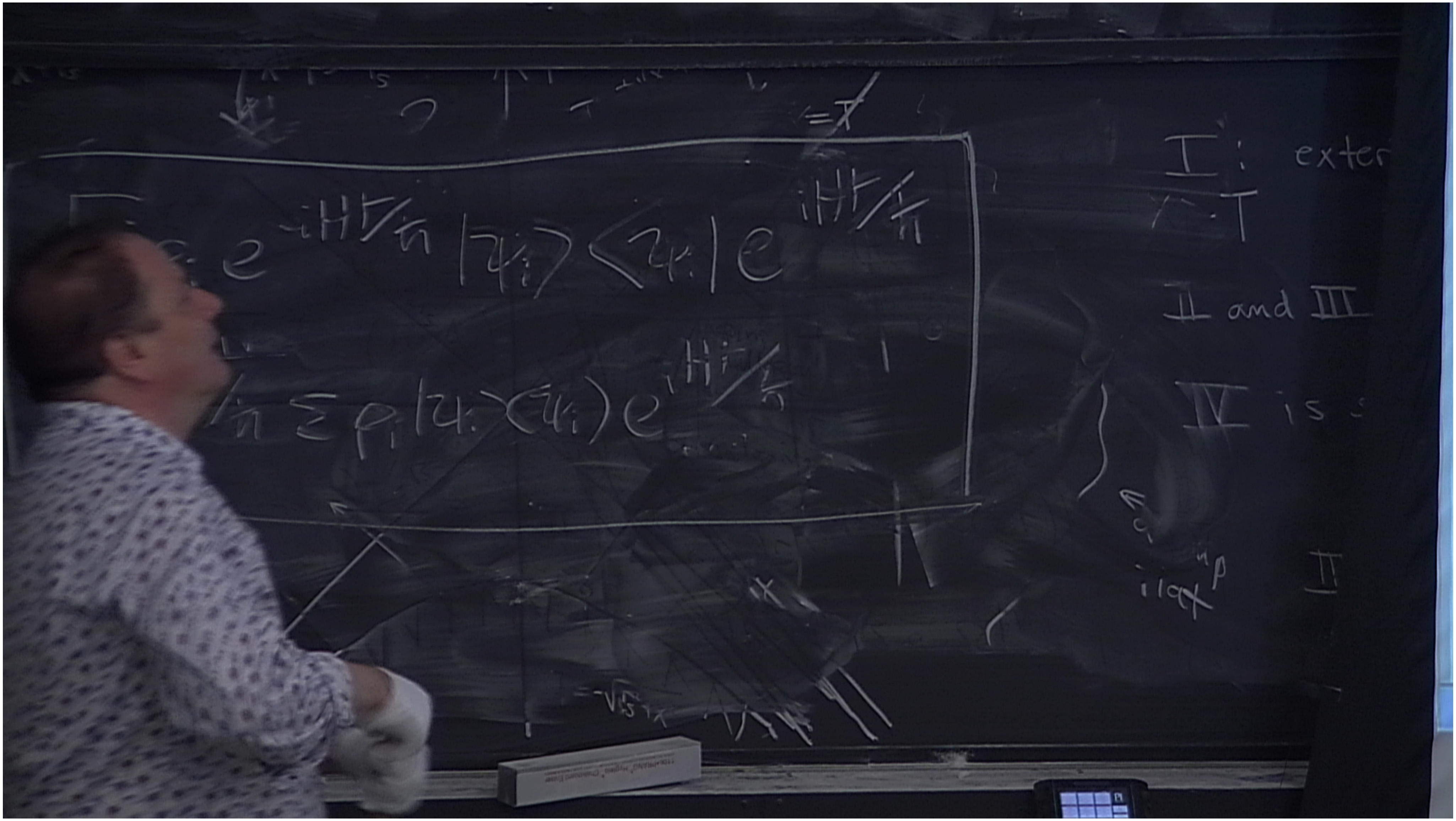

convex combinations of $\rho_1, \rho_2, \rho_3$ lie in the set.

To see this, use the last lemma. $\rho = \sum a_i \rho_i$ obeys

(i) $\rho^\dagger = \sum a_i^* \rho_i^\dagger = \sum a_i \rho_i = \rho$, (ii) $\langle\psi|\rho|\psi\rangle = \underbrace{\sum a_i \langle\psi|\rho_i|\psi\rangle}_{\geq 0}$

(iii) $\mathrm{Tr}(\rho) = \sum a_i \mathrm{Tr}(\rho_i) = \sum a_i = 1$.

So $\rho$ is also a density matrix. QED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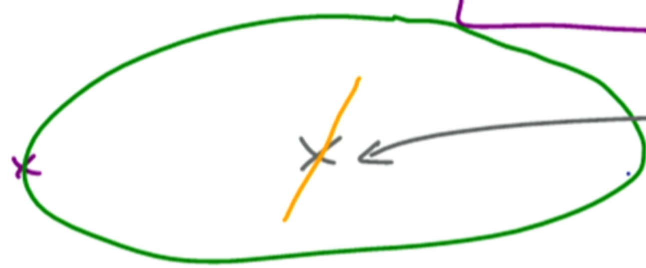
We say a density matrix $\rho$ is pure if all its convex decompositions are trivial: if $\rho = \sum a_i \rho_i$ with $a_i \geq 0$, $\sum a_i = 1$, $\rho_i$ density matrices then each $\rho_i = \rho$ (so $\rho = \sum a_i \rho_i$ just says $\rho = (\sum a_i)\rho = \rho$),

It's mixed if it's not pure.

$\boxed{\text{You can check: } \rho \text{ is pure} \Longleftrightarrow \rho = |\psi\rangle\langle\psi|}$

pure: an extremal point of the convex set – no line segment through it lies in the set.



mixed: lies on line segments that are in the set. (In principle boundary lines are allowed but won't matter for us.)

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