

Title: Relativity - Lecture 14

Date: Sep 30, 2011 09:00 AM

URL: <http://pirsa.org/11090078>

Abstract:

$\gamma = (1, 0, 0, 0)$ in t, r, θ, ϕ coords

$$\text{Kerr } ds^2 = -\left(1 - \frac{r_s r}{\rho^2}\right) dt^2 - \frac{r_s a r \sin^2 \theta}{\rho^2} (dt d\phi + d\phi dt) + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \frac{\sin^2 \theta}{\rho^2} [(r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta] d\phi^2$$

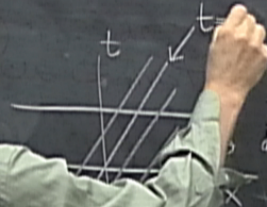
$$g^{\mu\nu} \partial_\mu r \partial_\nu r = 0$$

$$g_{rr} = g^{\mu\nu} \partial_\mu r \partial_\nu r = 0$$

$$\Delta = r^2 - r_s r + a^2$$

$$\rho^2 = r^2 + a^2 \cos^2 \theta$$

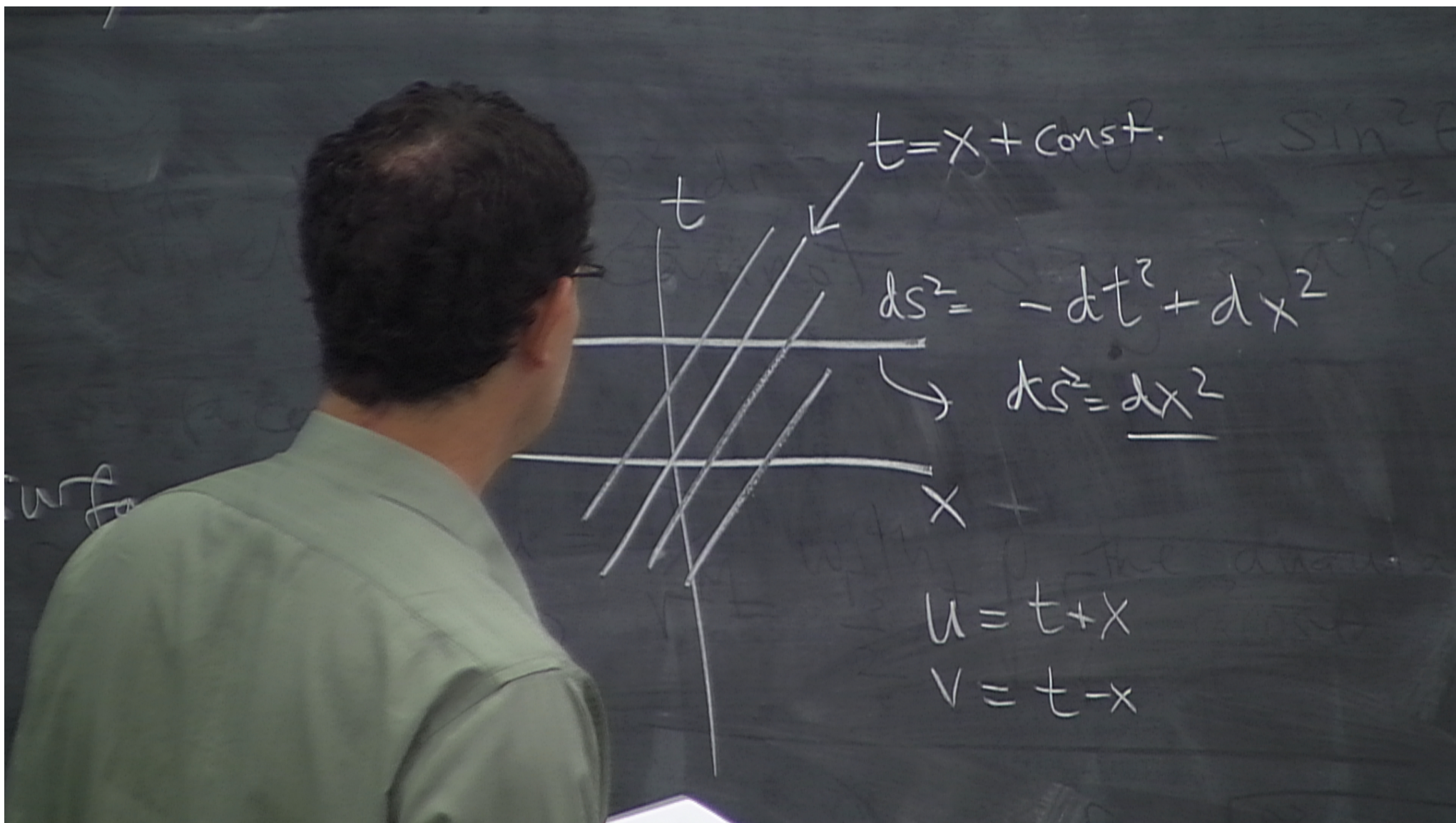
induced metric on an embedded



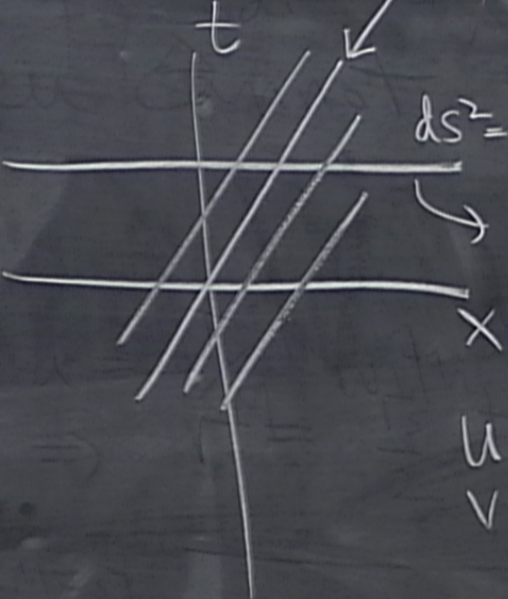
$$-dt^2 + dx^2$$

$$ds^2 = dx^2$$

induced metric on surface $t = \text{const.}$



$$\frac{a r \sin^2 \theta}{\rho^2} (dt d\phi + d\phi dt) + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta + \frac{\sin^2 \theta}{\rho^2} L(r+a)$$

$t = x + \text{const.}$

 $ds^2 = -dt^2 + dx^2$
 $\rightarrow ds^2 = \underline{dx^2}$
 induced metric on surface
 $u = x+t$
 $v = x-t$
 $ds^2 = du dv$

$$+ d\phi dt) + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \frac{\sin^2 \theta}{\rho^2} [(r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta] d\phi^2$$

$$t = x + \text{const.}$$

$$ds^2 = -dt^2 + dx^2$$

$$\rightarrow ds^2 = \underline{dx^2}$$

x

$$u = x + t : ds^2 = du dv$$

$$v = x - t$$

induced
metric

on surface $t = \text{const.}$

induced metric on surface $r = \text{const.}$

$$(r^2 + a^2) \sin^2 \theta d\phi^2$$

to the coordinate t .

$$P_t = m g_{\mu\nu} \dot{x}^\mu \quad \text{conserved.}$$

$$\theta=0,$$

$g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = g$ goes from -ve to positive as fall inside event horizon

$$\text{Kerr } ds^2 = -\left(1 - \frac{r_s r}{\rho^2}\right) dt^2 - \frac{r_s a \sin^2 \theta}{\rho^2} (dt d\phi + d\phi dt) + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \frac{\sin^2 \theta}{\rho^2} [(r^2 + a^2) d\phi - a \Delta \sin^2 \theta] d\phi^2$$

$$g^{\mu\nu} \partial_\mu r \partial_\nu r = 0$$

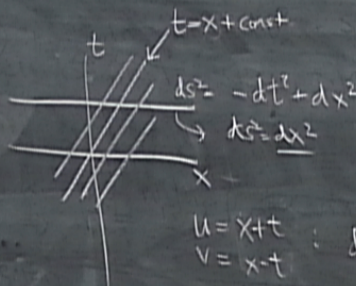
$$\Delta = r^2 - r_s r + a^2$$

$$\rho^2 = r^2 + a^2 \cos^2 \theta$$

$$g_{rr} = g^{\mu\nu} \partial_\mu r \partial_\nu r$$

induced metric on an embedded surface

Kerr: surface $\Delta=0$



induced metric on surface $t = \text{const.}$

$$u = x+t, \quad v = x-t, \quad ds^2 = du dv$$

induced metric on surface $v = \text{const.}$

$$g_{\mu\mu} = 0$$

$\Rightarrow$ metric components are zero in a null direction

to the coordinate t .

$$P_t = m g_{\mu\nu} \dot{x}^\mu \text{ conserved.}$$

$$\theta=0,$$

$g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = g$ goes from -ve to positive as fall inside event horizon

$$\text{Kerr } ds^2 = -\left(1 - \frac{r_s r}{\rho^2}\right) dt^2 - \frac{r_s a \sin^2 \theta}{\rho^2} (dt d\phi + d\phi dt) + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \frac{\sin^2 \theta}{\rho^2} [(r^2 + a^2) d\phi - a \Delta \sin^2 \theta] d\phi^2$$

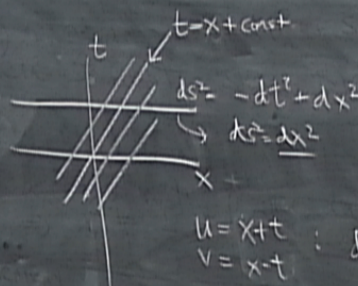
$$g^{\mu\nu} \partial_\mu \partial_\nu, r=0$$

$$\Delta = r^2 - r_s r + a^2$$

$$\rho^2 = r^2 + a^2 \cos^2 \theta$$

induced metric on an embedded surface

Kerr: Surface $\Delta=0$ $r=r_+$ or r_-



induced metric on surface $t = \text{const.}$

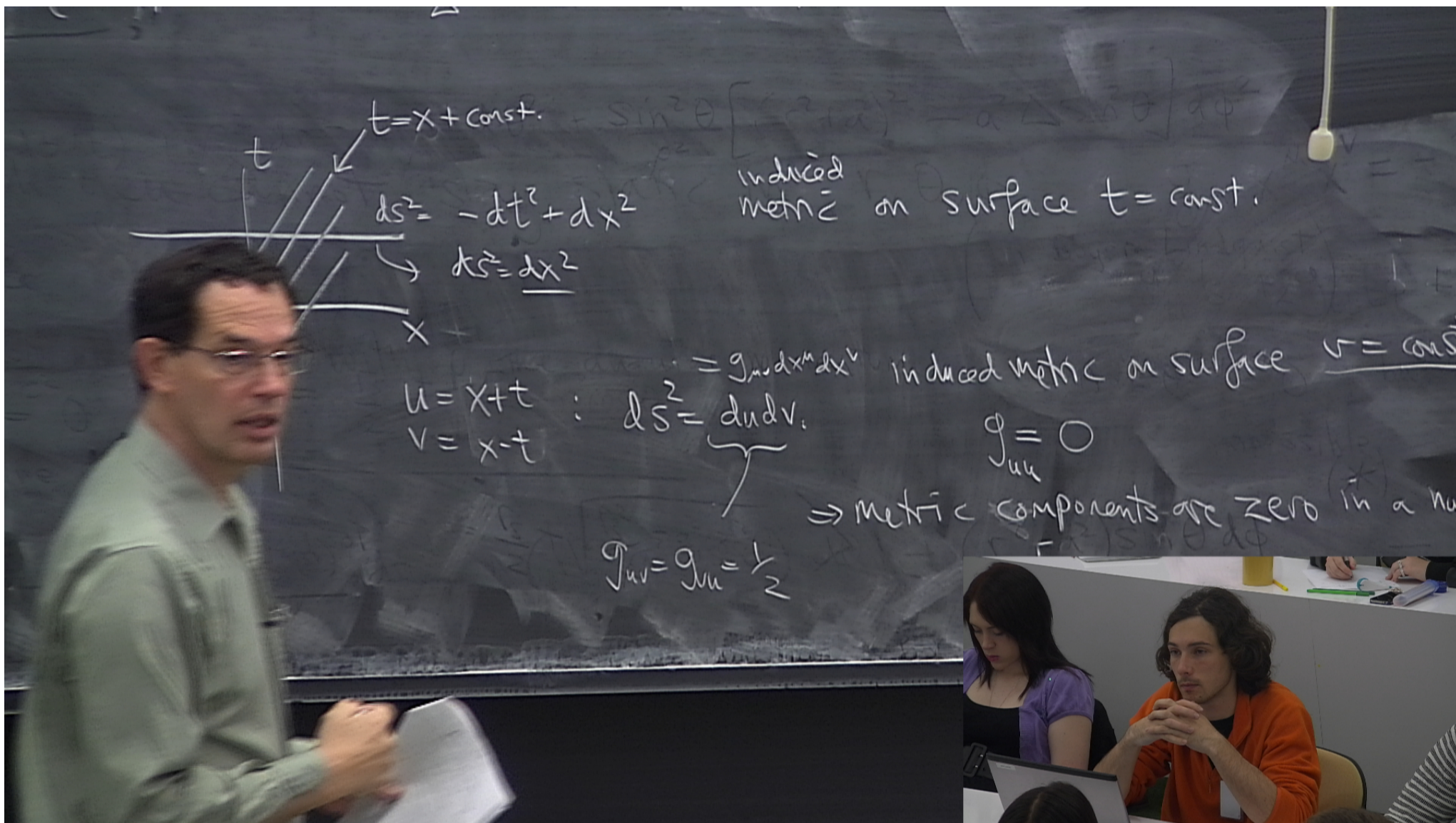
$$u = x+t, \quad v = x-t$$

$$ds^2 = du dv$$

induced metric on surface $r = \text{const}$

$$g_{\mu\nu} = 0$$

$\Rightarrow$ metric components are zero in a null direction



$$-\frac{r_s a r \sin^2 \theta}{\rho^2} (dt d\phi + d\phi dt) + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \frac{\sin^2 \theta}{\rho^2} [(r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta] d\phi^2$$

$t = x + \text{const.}$
 $ds^2 = -dt^2 + dx^2$
 $\rightarrow ds^2 = dx^2$

induced metric on surface $t = \text{const.}$

surface

$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$u = x+t$
 $v = x-t$

$ds^2 = du dv$
 $= g_{uv} dx^u dx^v$

$g_{uu} = 0$
 $\Rightarrow$ metric components are zero in a null direction.

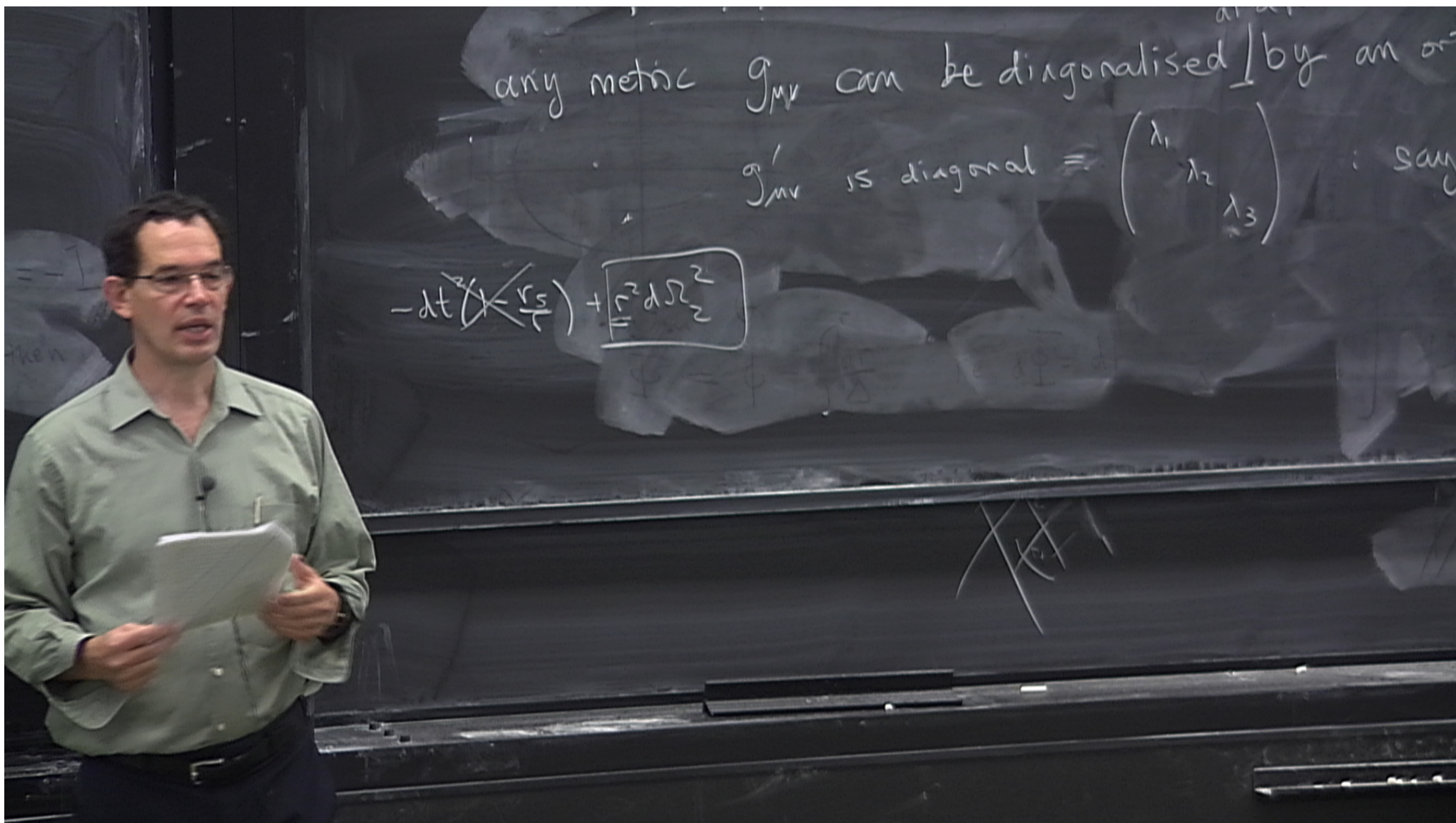
$\det = -\frac{1}{4}$

$\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$

$g_{uv} = g_{vu} = \frac{1}{2}$

any metric $g_{\mu\nu}$ can be diagonalised by an orthogonal transformation $x'^{\mu} = O^{\mu}_{\nu} x^{\nu}$

any metric $g_{\mu\nu}$ can be diagonalised ^{at a point} by an orthogonal transformation $x'^{\mu} = O^{\mu}_{\nu} x^{\nu}$
 $g'_{\mu\nu}$ is diagonal



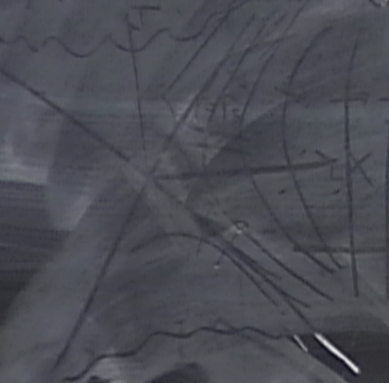
Kerr $\Delta=0$ Event horizons
 eg. $r=r_+$

$$g_{tt} \quad g_{t\phi}$$

$$g_{\phi t} \quad g_{\phi\phi}$$

$$= \begin{pmatrix} -\left(1 - \frac{r_s r}{\rho^2}\right) & -\frac{r_s a r \sin^2 \theta}{\rho^2} \\ -\frac{r_s a r \sin^2 \theta}{\rho^2} & (r^2 + a^2) \frac{\sin^2 \theta}{\rho^2} \end{pmatrix}$$

Kruskal



$$-dT^2 + dX^2$$

event horizons have an area

is you take any spacelike surface lying within the
 the same, whichever spacelike surface you choose.

eg. Schwarzschild
 $t = \text{const}$

$$-\left(1 - \frac{r_s}{r}\right) dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

Kerr $\Delta=0$ Event horizons
 eg. $r=r_+$ surface g_{tt} $g_{t\phi}$ $g_{\phi\phi}$

$$= \frac{1}{\rho^2} \begin{pmatrix} -(\rho^2 - \frac{r_s r}{r}) & -\frac{r_s a r \sin^2 \theta}{\rho^2} \\ -\frac{r_s a r \sin^2 \theta}{\rho^2} & \frac{(r^2 + a^2) \sin^2 \theta}{\rho^2} \end{pmatrix}$$

Claim: if $r^2 + a^2 = r_s r$ have an area (associated w

is you take $\rho^2 = r^2 + a^2(1 - \sin^2 \theta)$ surface lying within the same, whichever spacelike surface you choose.

eg. Schwarzschild $- (1 - \frac{r_s}{r}) dt^2 + (1 - \frac{r_s}{r})^{-1} dr^2 + r^2 d\Omega^2$
 $t = \text{const}$

these Δ are called

$$\begin{pmatrix} -\left(\frac{p^2}{r} - \frac{r_s r}{r^2}\right) & -\frac{r_s a r \sin^2 \theta}{p^2} \\ -\frac{r_s a r \sin^2 \theta}{p^2} & \frac{(r^2 + a^2)^2 \sin^2 \theta}{p^2} \end{pmatrix} \Delta = 0 = \frac{1}{p^2} \begin{pmatrix} a^2 \sin^2 \theta & -r_s a r \sin^2 \theta \\ -r_s a r \sin^2 \theta & (r_s r)^2 \sin^2 \theta \end{pmatrix}$$

observer at infinity
(associated with entropy of black hole)
area
face lying within the event horizon and fully enclosing the interior

space-like surface you choose.
 $r_s^2 d\Omega^2 = \text{area is just } 4\pi r_s^2$

$\gamma'_{\mu\nu}$ is diagonal $= \begin{pmatrix} 1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix}$: say surface is null if one of values vanishes

$$-dt^2 \left(\frac{r_s}{r} \right) + \boxed{r^2 d\Omega^2}$$

Kerr $\Delta=0$ event horizons
eg $r=r_+$

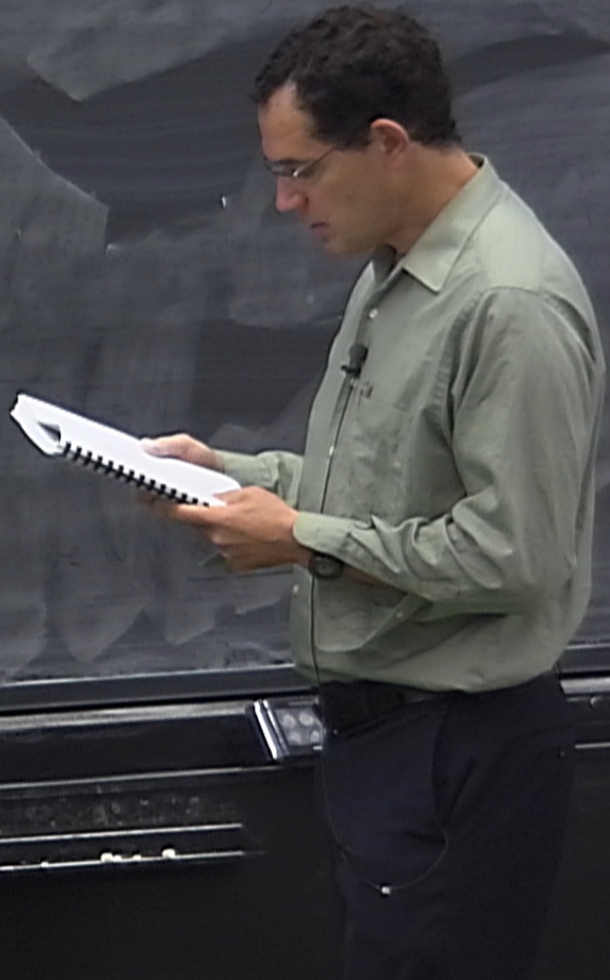
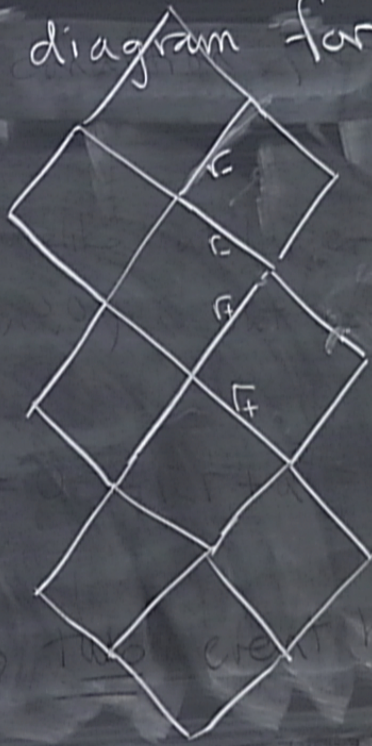
$$g_{tt} \quad g_{t\phi} \quad g_{\phi\phi} = \frac{1}{\rho^2} \begin{pmatrix} -\left(\rho^2 - \frac{r_s r}{r}\right) & -\frac{r_s a \sin^2 \theta}{\rho^2} \\ -\frac{r_s a \sin^2 \theta}{\rho^2} & \frac{(r^2 + a^2) \sin^2 \theta}{\rho^2} \end{pmatrix} \quad \Delta=0 = \frac{1}{\rho^2} \begin{pmatrix} a^2 \sin^2 \theta & -r_s a \sin^2 \theta \\ -r_s a \sin^2 \theta & (r_s r)^2 \sin^2 \theta \end{pmatrix}$$

Claim: if $r^2 a^2 = r_s r$ (associated with $\Delta=0$) $\Rightarrow \det=0 \Rightarrow \exists$ a null direction

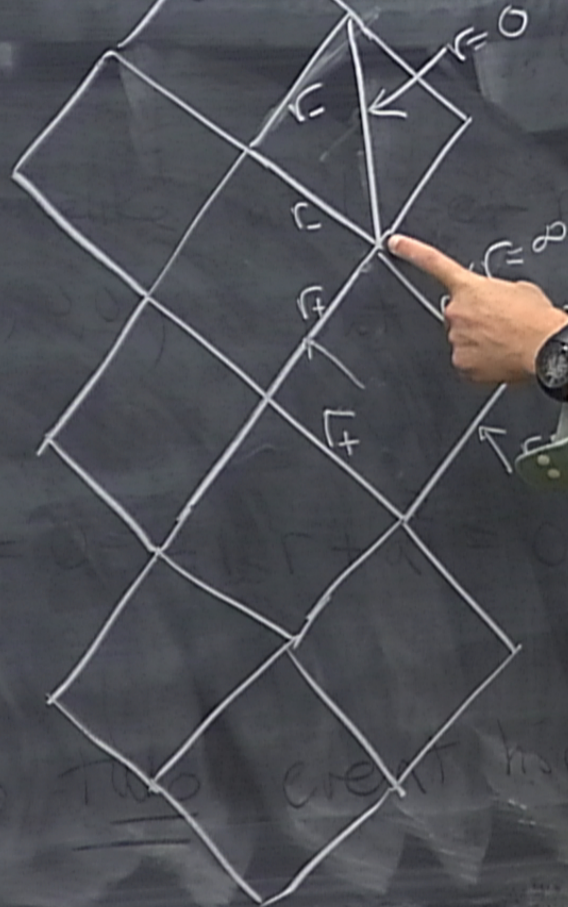
eg take $\rho^2 = r^2 + a^2(1 - \sin^2 \theta)$ place lying within the event horizon and enclosing the same whichever spacelike surface you choose.

eg Schwarzschild $-(1 - \frac{r_s}{r}) dt^2 + \frac{r^2}{1 - \frac{r_s}{r}} d\Omega^2$ $r_s \Delta = 0 = dt^2 + r^2 d\Omega^2$

Causal diagram for Kerr



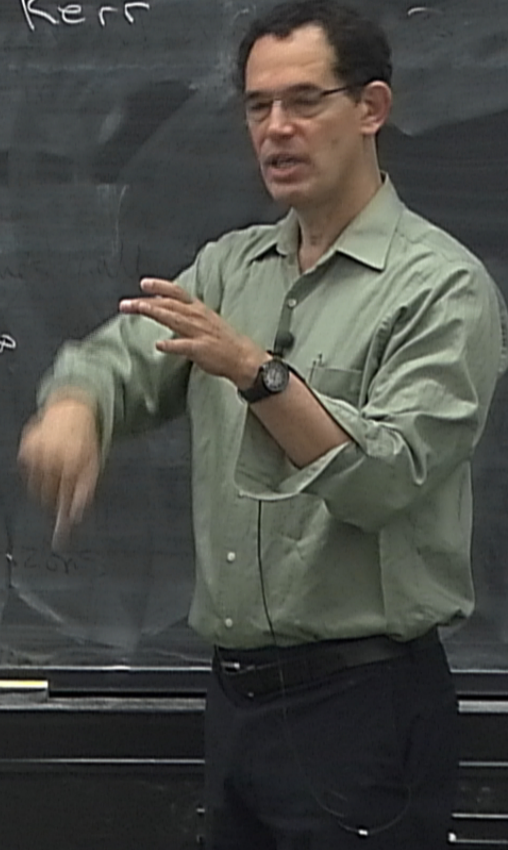
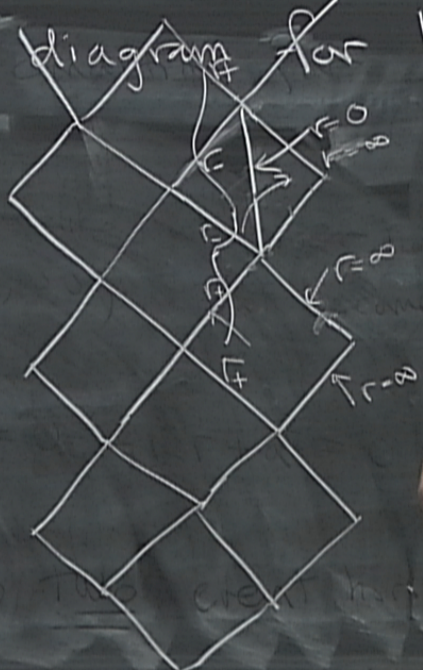
Causal diagram for Kerr

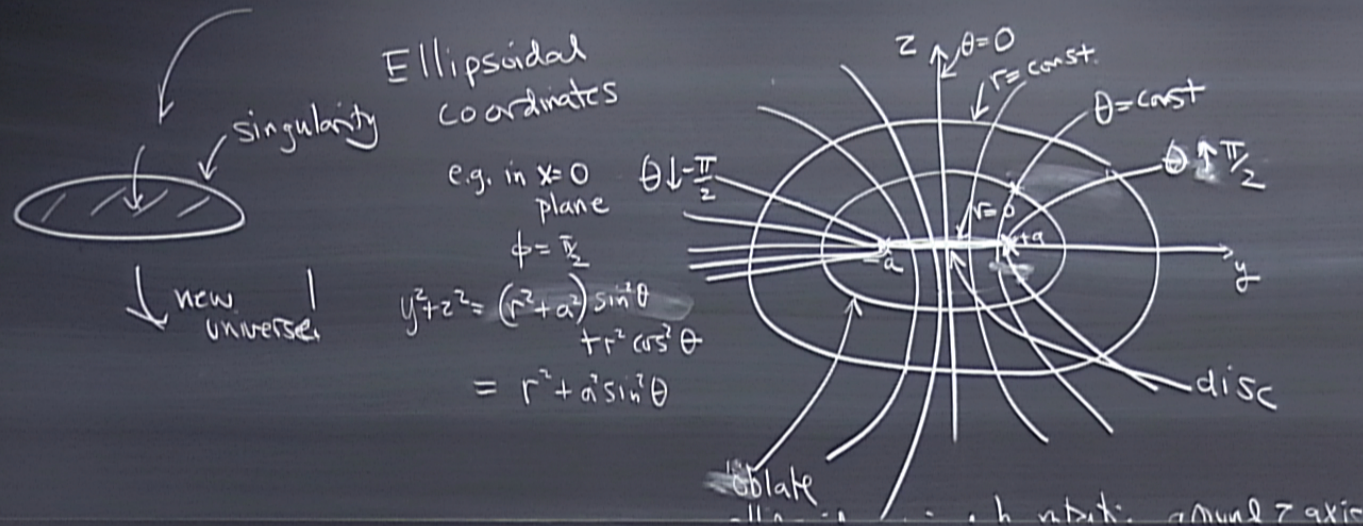


$$-dt \left(\frac{r_s}{r} \right) + \frac{r^2 d\Omega^2}{2}$$

$$\Phi = \phi - \int \frac{dr}{\Delta} \quad \text{ie } d\Phi = d\phi - \frac{dr}{\Delta}$$

Causal diagram for Kerr

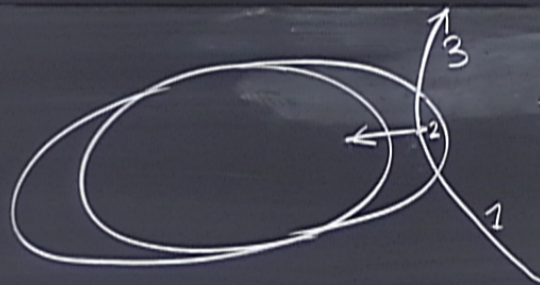




$$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

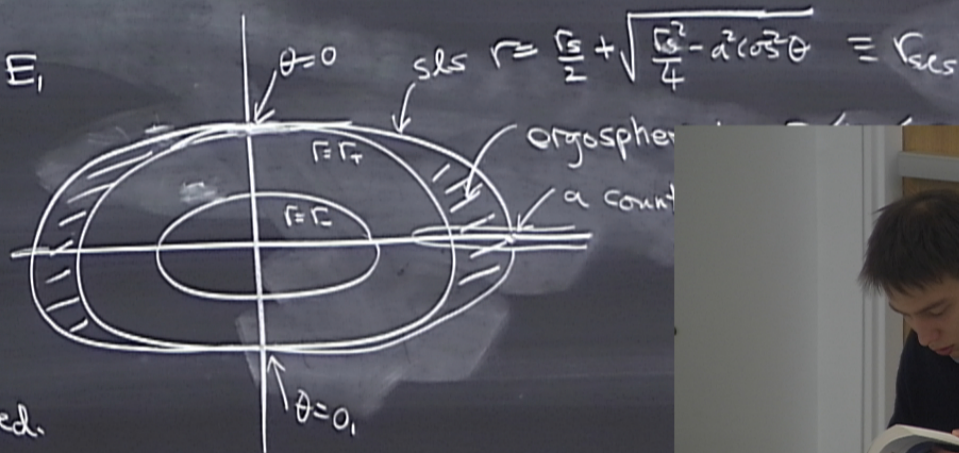
$$0 < r < \infty$$

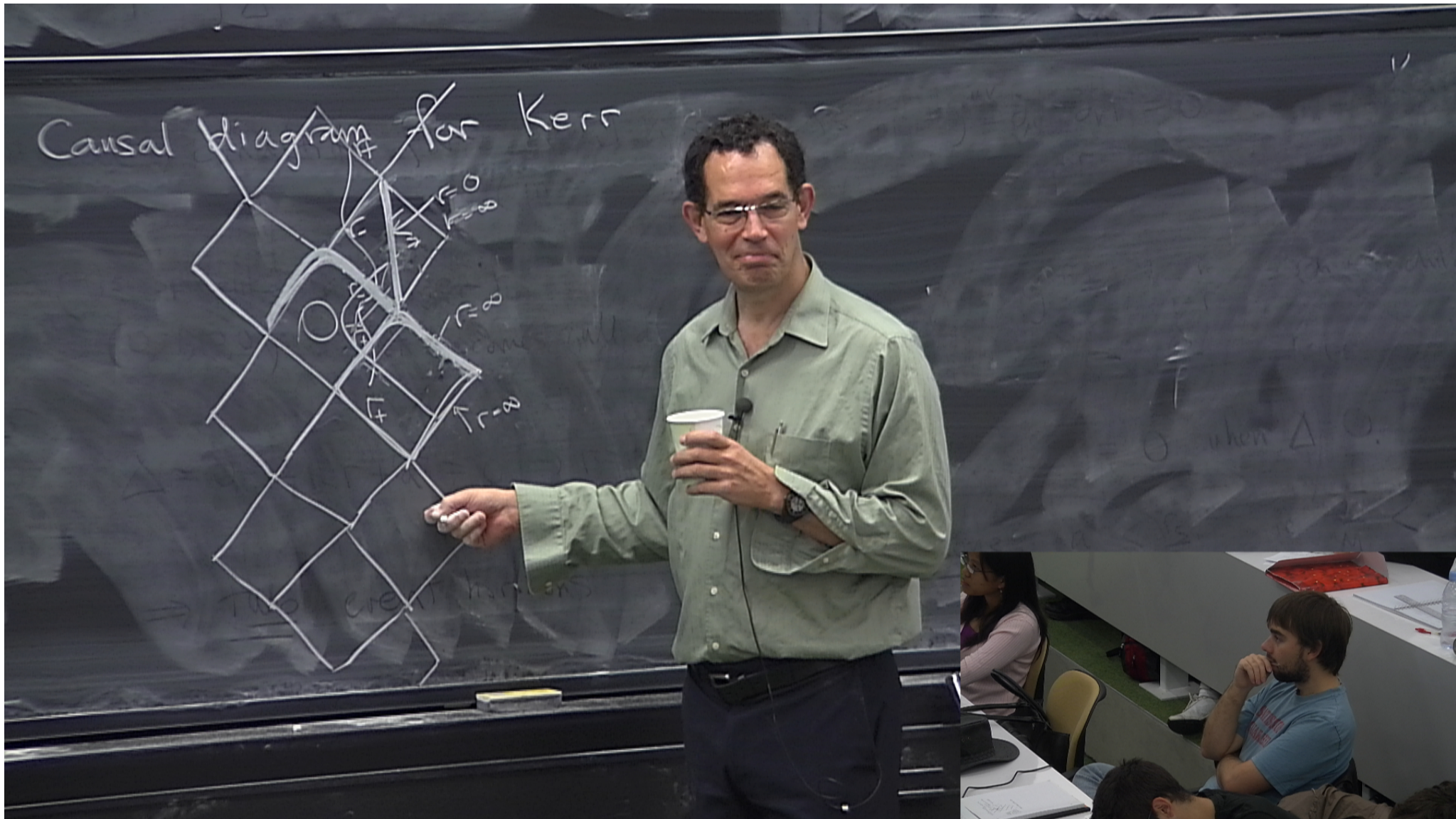
If calculate R^{μ}_{ν} (for Kerr)
 find singularity at $\rho^2 = 0$
 i.e. $r = 0$ a
 Kerr has a ring singularity!



Conserved energy is related
 to the coordinate t .

$$P_t = m g_{t\mu} \dot{x}^\mu \text{ conserved.}$$





Cosmology

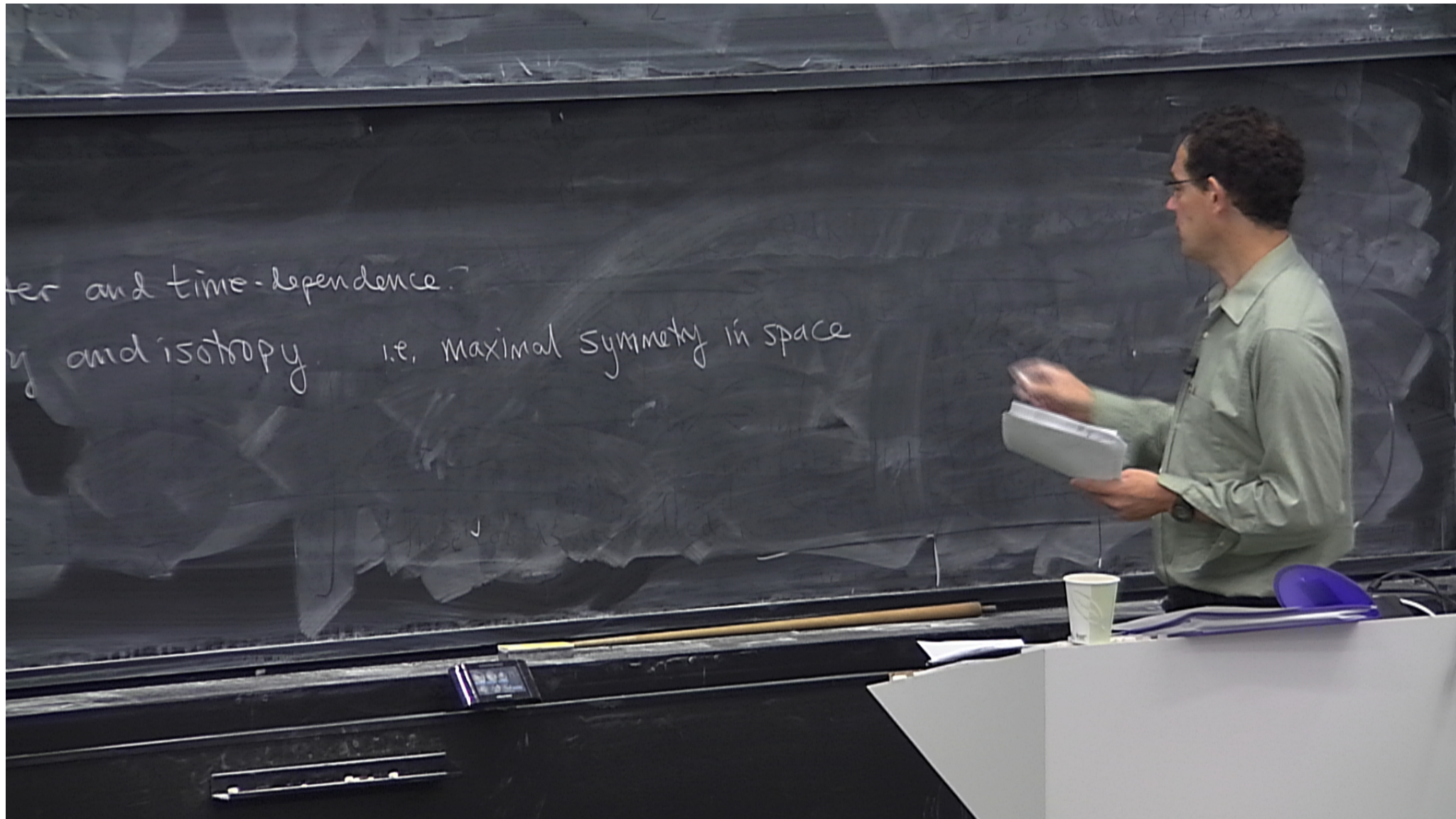
Want general solutions with matter and time-dependence.

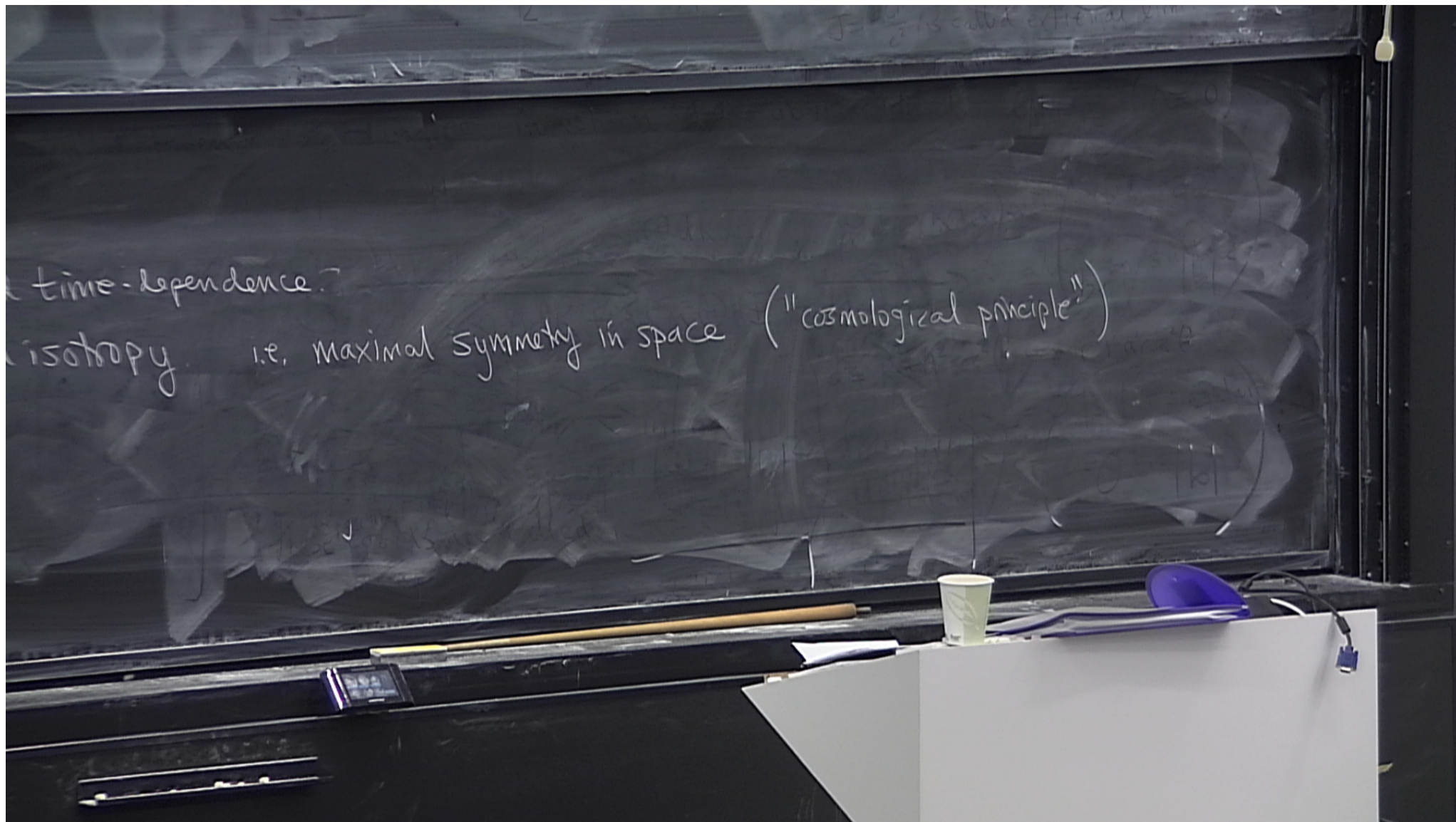
$$\Phi = \psi \int \frac{dr}{\Delta} \quad \text{ie. } d\Phi = d\psi \int \frac{dr}{\Delta}$$

Cosmology

Want general solutions with matter and time-dependence
to simplify, assume homogeneity and isotropy

$$\Phi = \phi \int \frac{dr}{\Delta} \quad \text{e.g. } d\Phi = d\left(\frac{r}{\Delta}\right)$$





Cosmology

Want general solutions with matter and time-dependence.

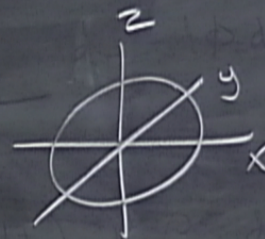
to simplify, assume homogeneity and isotropy i.e. maximal symmetry in space ("cosmological principle")

In d dimensions, greatest number of symmetries possible is $\frac{d(d+1)}{2}$ (Weinberg Ch. 13)

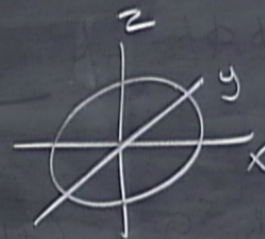
$d=3$:

S^3

$$\int_3: \quad x^2 + y^2 + z^2 + v^2 = 1 \quad \text{in } E^4$$



$$\int^3: \quad x^2 + y^2 + z^2 + v^2 = 1 \quad \text{in } E^4$$



$$ds^2 = dx^2 + dy^2 + dz^2 + dv^2$$

$$x = \sin \chi \sin \theta \cos \phi$$

$$y = \sin \chi \sin \theta \sin \phi$$

$$z = \sin \chi \cos \theta$$

$$v = \cos \chi$$



$$ds^2 = d\chi^2 + \sin^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2)$$

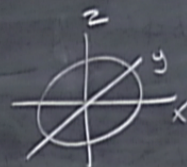
the coordinate t .

$$P_t = m g_{\mu\nu} \dot{x}^\mu \text{ conserved.}$$

$$\theta=0,$$

$$g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \text{ goes from -ve to positive}$$

$$S^3: x^2 + y^2 + z^2 + v^2 = 1 \text{ in } E^4$$



$$ds^2 = dx^2 + dy^2 + dz^2 + dv^2$$

$$x = \sin\chi \sin\theta \cos\phi$$

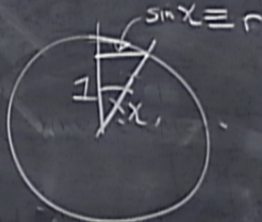
$$y = \sin\chi \sin\theta \sin\phi$$

$$z = \sin\chi \cos\theta$$

$$v = \cos\chi$$

$$ds^2 = d\chi^2 + \sin^2\chi (d\theta^2 + \sin^2\theta d\phi^2)$$

$$= \frac{dr^2}{1-r^2} + r^2 d\Omega^2$$

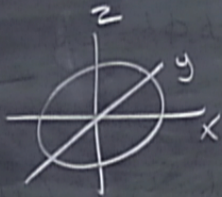


$$dr = \cos\chi d\chi$$

$$dr^2 = (1-r^2) d\chi^2$$

$$d\chi^2 = \frac{dr^2}{1-r^2}$$

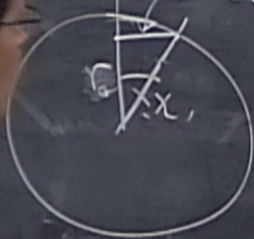
$\theta=0$
 $\rho_{\mu\nu} \sum^{\mu} \sum^{\nu} = q$ goes from -ve to positive as fall inside ergosphere



1 in E^4

$$ds^2 = d\chi^2 + \sin^2\chi (d\theta^2 + \sin^2\theta d\phi^2) = \frac{dr^2}{1 - \frac{r^2}{r_0^2}} + r^2 d\Omega^2$$

$r_0 \sin\chi \equiv r$ ← sphere of radius r_0



$$dr = r_0 \cos\chi d\chi$$

$$dr^2 = (1 - \frac{r^2}{r_0^2}) d\chi^2 r_0^2$$

$$d\chi^2 = \frac{dr^2}{1 - \frac{r^2}{r_0^2}}$$

$$= \frac{dr^2}{1 - k r^2} + r^2 d\Omega^2$$

$$k = \frac{1}{r_0^2}$$

$$+ \sin^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2)$$

$$= \frac{dr^2}{1 - \frac{r^2}{r_0^2}} + r^2 d\Omega^2$$

← sphere of radius r_0

$$dr = r_0 \cos \chi d\chi$$

$$dr^2 = (1 - \frac{r^2}{r_0^2}) d\chi^2 r_0^2$$

$$d\chi^2 = \frac{dr^2}{1 - \frac{r^2}{r_0^2}}$$

$$= \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2$$

$$k = \frac{1}{r_0^2}$$

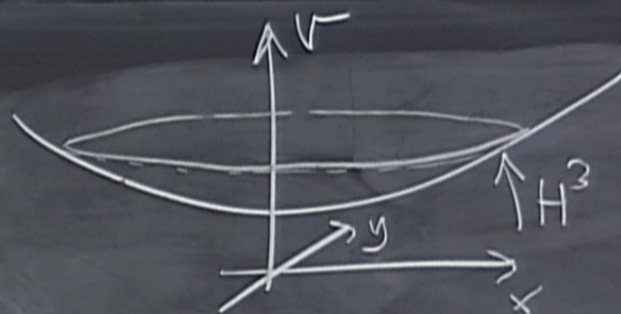
E^3 obtained by
let $r_0 \rightarrow \infty$

H^3

$$x^2 + y^2 + z^2 - v^2 = -1$$

in $d=4$ Minkowski

$$dx^2 + dy^2 + dz^2 - dv^2$$



H^3

$$x^2 + y^2 + z^2 - v^2 = -1$$

in $d=4$ Minkowski

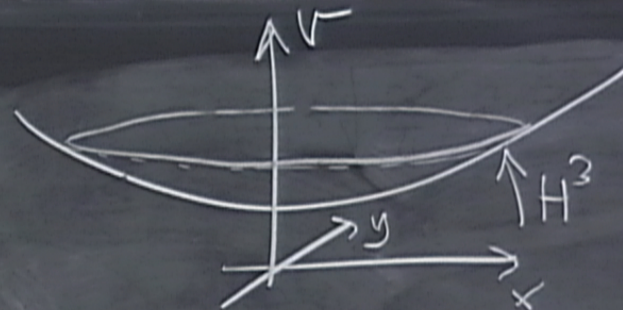
$$dx^2 + dy^2 + dz^2 - dv^2$$

$$x = \sinh \chi \sin \theta \cos \phi$$

⋮

$$z = \sinh \chi \cos \theta$$

$$v = \cosh \chi$$



$m_{\text{SUSY}}^{\text{grav}} = (1, 0, 0, 0)$ in (v, χ, θ, ϕ) coords.

H^3

$$x^2 + y^2 + z^2 - v^2 = -1$$

in $d=4$ Minkowski

$$dx^2 + dy^2 + dz^2 - dv^2$$

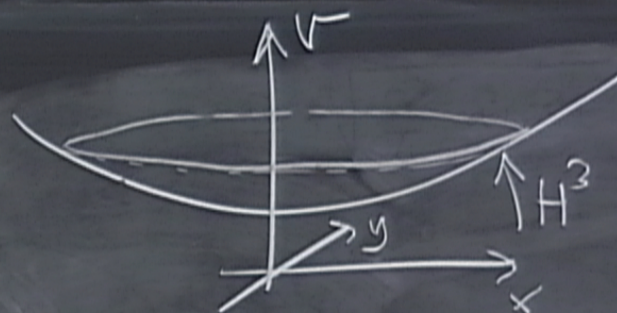
$$x = \sinh \chi \sin \theta \cos \phi$$

!

$$z = \sinh \chi \cos \theta$$

$$\cosh \chi$$

$$ds_{H^3}^2 = dx^2 + \sinh^2 \chi d\Omega^2$$



H^3

$$x^2 + y^2 + z^2 - v^2 = -1$$

in $d=4$ Minkowski

$$dx^2 + dy^2 + dz^2 - dv^2$$

$$x = \sinh \chi \sin \theta \cos \varphi$$

⋮

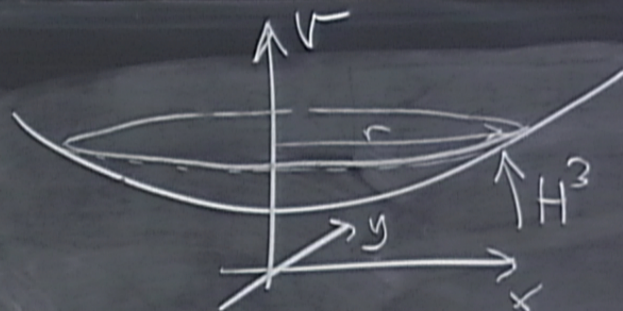
$$z = \sinh \chi \cos \theta$$

$$v = \cosh \chi$$

$$ds_{H^3}^2 = dx^2 + \sinh^2 \chi d\Omega^2$$

$$r = r_0 \sinh \chi \quad (H^3 \text{ of radius } r_0)$$

$$= \frac{dr^2}{1 + \frac{r^2}{r_0^2}} + r^2 d\Omega^2$$



H^3

$$x^2 + y^2 + z^2 - v^2 = -1$$

in $d=4$ Minkowski

$$dx^2 + dy^2 + dz^2 - dv^2$$

$$x = \sinh \chi \sin \theta \cos \varphi$$

$\vdots$

$$z = \sinh \chi \cos \theta$$

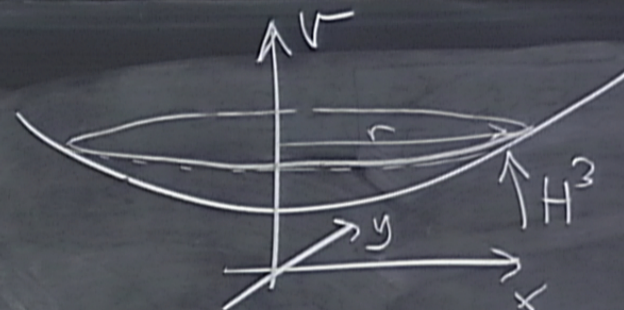
$$v = \cosh \chi$$

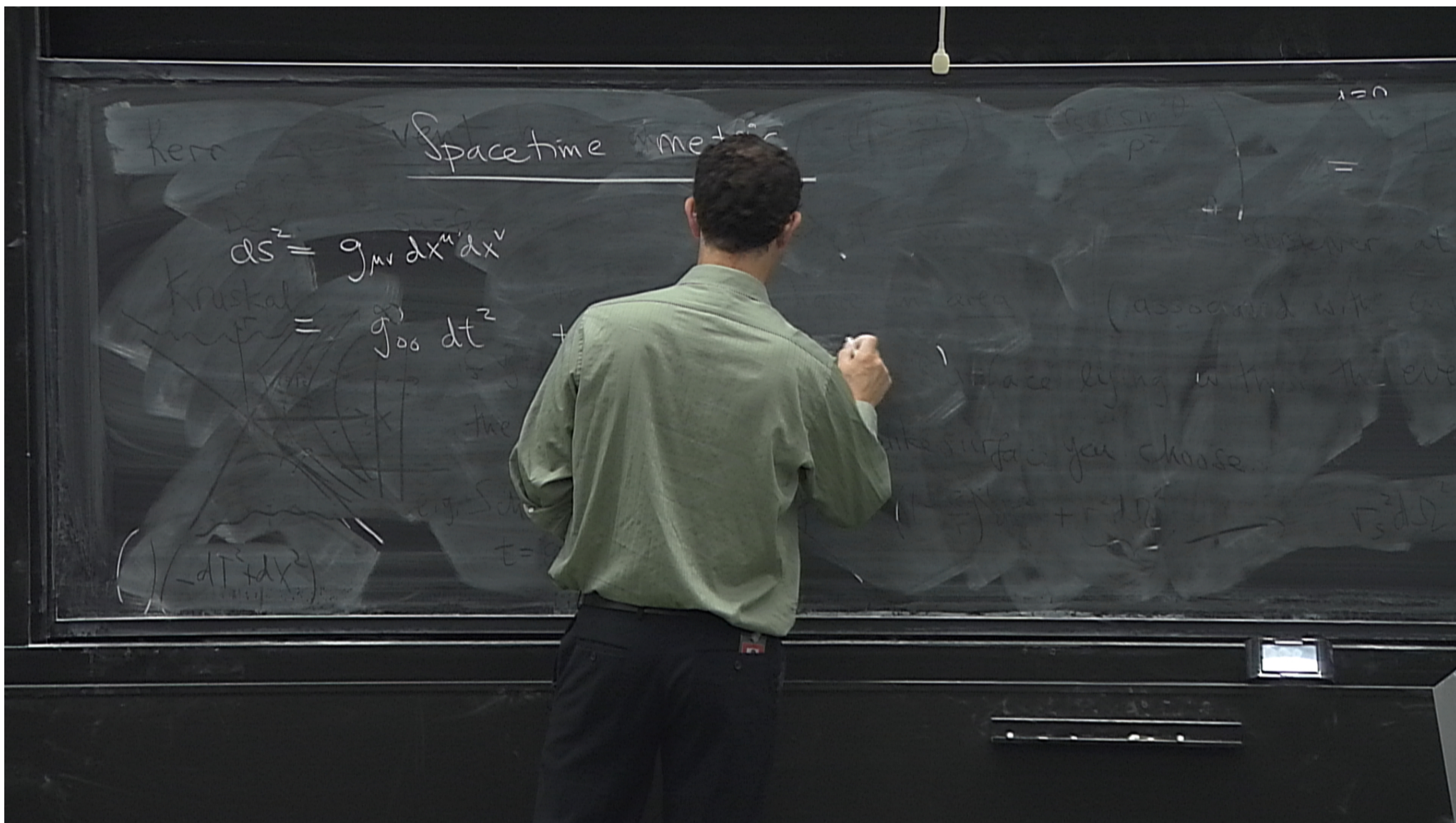
$$ds_{H^3}^2 = dx^2 + \sinh^2 \chi d\Omega^2$$

$$r = r_0 \sinh \chi \quad (H^3 \text{ of radius } r_0)$$

$$= \frac{dr^2}{1 + \frac{r^2}{r_0^2}} + r^2 d\Omega^2$$

$m_{\text{cov}}^{\mu\nu} = \eta^{\mu\nu} = (1, 0, 0, 0)$ in $(t, \theta, \varphi, \chi)$ coords.





Spacetime metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

Kruskal

$$= g_{00} dt^2 + 2g_{0i} dt dx^i + g_{ij} dx^i dx^j$$

the same, whichever spacelike surface you choose

e.g. Schwarzschild
 $t = \text{const}$

$$(-dt^2 + dx^2)$$

$$\begin{aligned}x &= \sin\chi \sin\theta \cos\phi \\y &= \sin\chi \sin\theta \sin\phi \\z &= \sin\chi \cos\theta \\r &= \cos\chi\end{aligned}$$

$$ds^2 = d\chi^2 + \sin^2\chi (d\theta^2 + \sin^2\theta d\phi^2)$$

$r \sin\chi \equiv r$



$$dr = r_0 \cos\chi d\chi$$

$$dr^2 = (1 - r_0^2 \sin^2\chi) d\chi^2$$

$$d\chi^2 = \frac{dr^2}{1 - r_0^2 \sin^2\chi}$$

System: $x^i = (x, y, z, v)$

$$x^i = O^{ij} x^j \quad O^T O = 1$$

$O(4)$

H

$$x + y + z + v = -1$$

In $d=4$ Minkowski

$$dx^2 + dy^2 + dz^2 - dv^2$$

$$x = \sinh\chi \sin\theta \cos\phi$$

$$y = \sinh\chi \sin\theta \sin\phi$$

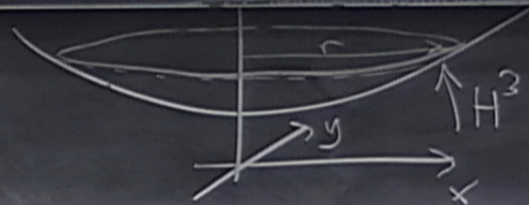
$$z = \sinh\chi \cos\theta$$

$$v = \cosh\chi$$

$$ds_{H_3}^2 = d\chi^2 + \sinh^2\chi d\Omega^2$$

$$r = r_0 \sinh\chi \quad (H^3 \text{ of radius } r_0)$$

$$= \frac{dr^2}{1 + r_0^2 \frac{r^2}{r_0^4}} + r^2 d\Omega^2$$



Spacetime metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

$$= \underbrace{g_{00}}_{\text{Scalar, invariant}} dt^2 + 2g_{0i} dt dx^i + g_{ij} dx^i dx^j$$

Symmetry: $\Rightarrow g_{00}(t)$ same whichever spacelike surface you choose

e.g. Schwarzschild $- (1 - \frac{r_s}{r}) dt^2 + \dots$
 $t = \text{const}$

$$(-dt^2 + dx^2)$$

Kerr

Spacetime metric

$$\gamma_{ij} dx^i dx^j = \text{line element on } \begin{cases} S^3 \\ E^3 \\ H^2 \end{cases}$$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

$$= \underbrace{g_{00}}_{\text{scalar, invariant}} dt^2 + 2 \underbrace{g_{0i} dt dx^i}_{\text{vector, invariant under spatial symm}} + g_{ij} dx^i dx^j$$

Symmetry:

$$\Rightarrow g_{00}(t)$$

same which spatial symm doesn't change for you choose.

$$= 0$$

e.g. Schwarzschild

$t = \text{const}$

$$= g_{00} dt^2 + a^2(t) \gamma_{ij} dx^i dx^j$$

$$\gamma_{ij} dx^i dx^j = dr^2 + r^2 d\Omega^2$$

here

Spacetime metric

$\gamma_{ij} dx^i dx^j =$ line element on $\begin{cases} S^3 \\ E^3 \\ H^2 \end{cases}$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

Kruskal

$$= \underbrace{g_{00}}_{\text{scalar, invariant}} dt^2 + 2 \underbrace{g_{0i} dt dx^i}_{\text{vector, invariant under spatial symm}} + \underbrace{g_{ij} dx^i dx^j}_{\text{space lying on } dt^2}$$

Symmetry: $\Rightarrow g_{00}(t)$ same which spatial symm doesn't change for you choose.

$\hat{a}(t) \gamma_{ij}(x)$

$= 0$

e.g. Schwarzschild
 $t = \text{const}$

Kerr

Spacetime metric

$\gamma_{ij} dx^i dx^j =$ line element on $\begin{cases} S^3 \\ E^3 \\ H^2 \end{cases}$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

Kruskal

$$= \underbrace{g_{00}}_{\text{Scalar, invariant}} dt^2 + 2 \underbrace{g_{0i}}_{\text{vector, invariant under spatial symm}} dt dx^i + \underbrace{g_{ij}}_{\text{metric on space lying on } t=\text{const}} dx^i dx^j$$

Symmetry: $\Rightarrow g_{00}(t)$ same which spatial symm does not change for you choose.

$$= 0$$

(associated with a particular spacetime)

$$= g_{00} dt^2 + a^2(t) \gamma_{ij} dx^i dx^j$$

$-dt^2$ $dt = \sqrt{-g_{00}} dt$

e.g. Schwarzschild $t = \text{const}$

$$r_s^2 d\Omega^2 = a^2(t) \gamma_{ij} dx^i dx^j$$

translations

metric

$$\gamma_{ij} dx^i dx^j = \text{line element on } \begin{cases} S^3 \\ E^3 \\ H^2 \end{cases} \quad \begin{aligned} & \frac{dr^2}{1-(r_0)^2} + r^2 d\Omega^2 \\ & dr^2 + r^2 d\Omega^2 \\ & \frac{dr^2}{1+(r_0)^2} + r^2 d\Omega^2 \end{aligned}$$

$$g_{0i} dt dx^i + g_{ij} dx^i dx^j = g_{00} dt^2 + a^2(t) \gamma_{ij} dx^i dx^j$$

factor, invariant under spatial symms

$a^2(t) \gamma_{ij}(x)$

space-time surface lying within the event horizon and enclosing the interior

face you choose.

$$-d\tilde{t}^2 \quad d\tilde{t} = \sqrt{-g_{00}} dt$$

$$-dt^2 + a^2(t) \gamma_{ij} dx^i dx^j$$

Matter: assume perfect fluid

$$T^{\mu\nu} = (p + \rho) u^\mu u^\nu + p g^{\mu\nu}$$

u^μ

Matter: assume perfect fluid

$$T^{\mu\nu} = (p + \rho) u^\mu u^\nu + p g^{\mu\nu}$$

$$u^\mu = (u^0, u^i)$$

$$= (u^0, \underline{0}) \text{ by isotropy}$$

$$g_{\mu\nu} u^\mu u^\nu = -1$$

$$= -u^0{}^2 \Rightarrow u^0 = +1 \text{ (future pointing)}$$

Matter: assume perfect fluid

$$g^{\mu\nu} = \begin{pmatrix} -1 & \\ & \frac{1}{a^2} \gamma_{ij} \end{pmatrix}$$

$$T^{\mu\nu} = (p + \rho) u^\mu u^\nu + p g^{\mu\nu}$$

$$u^\mu = (u^0, u^i)$$

$$= (u^0, 0) \text{ by isotropy}$$

$$\Rightarrow u^\mu = (1, 0) \Rightarrow T^{\mu\nu} = \begin{pmatrix} \end{pmatrix}$$

$$g_{\mu\nu} u^\mu u^\nu = -1$$

$$= -u^0{}^2 \Rightarrow u^0 = +1 \text{ (future pointing)}$$

Matter: assume perfect fluid

$$g^{\mu\nu} = \begin{pmatrix} -1 & \\ & \frac{1}{a^2} \delta^{ij} \end{pmatrix}$$

$$T^{\mu\nu} = (p + \rho) u^\mu u^\nu + p g^{\mu\nu}$$

$$u^\mu = (u^0, u^i)$$

$$= (u^0, 0) \text{ by isotropy}$$

$$\Rightarrow u^\mu = (1, 0) \Rightarrow T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & \boxed{p \frac{1}{a^2} \delta^{ij}} & & \\ 0 & & & \\ 0 & & & \end{pmatrix}$$

$$g_{\mu\nu} u^\mu u^\nu = -1$$

$$= -u^0{}^2 \Rightarrow u^0 = +1 \text{ (future pointing)}$$

Einstein equations $G_{\mu\nu} = 8\pi G T_{\mu\nu}$

$$g_{\mu\nu} = \begin{pmatrix} -1 & \\ & a^2 \gamma_{ij} \end{pmatrix}$$

$$g^{\mu\nu} = \begin{pmatrix} -1 & \\ & a^2 \gamma^{ij} \end{pmatrix}$$

$$\Gamma_{ij}^0 = a \dot{a} \gamma_{ij}$$

$$\Gamma_{0j}^i = \frac{\dot{a}}{a} \delta_j^i$$

$$\Gamma_{jk}^i = \Gamma_{jk}^i(\gamma)$$

ellipsoid, obtained by rotating around z axis.

Einstein equations $G_{\mu\nu} = 8\pi G T_{\mu\nu}$

$$g_{\mu\nu} = \begin{pmatrix} -1 & \\ & a^2 \gamma_{ij} \end{pmatrix}$$

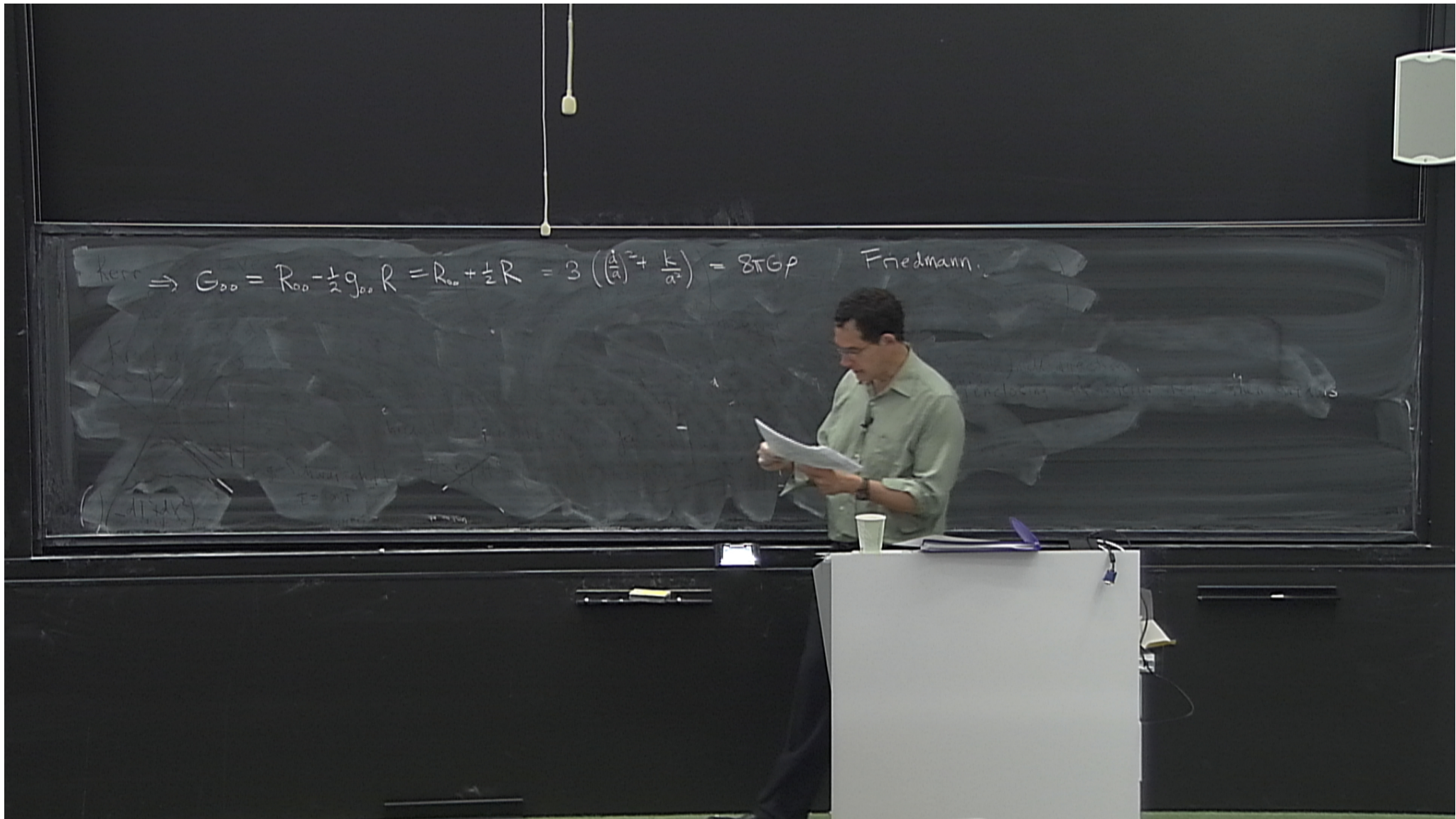
$$g^{\mu\nu} = \begin{pmatrix} -1 & \\ & a^2 \gamma^{ij} \end{pmatrix}$$

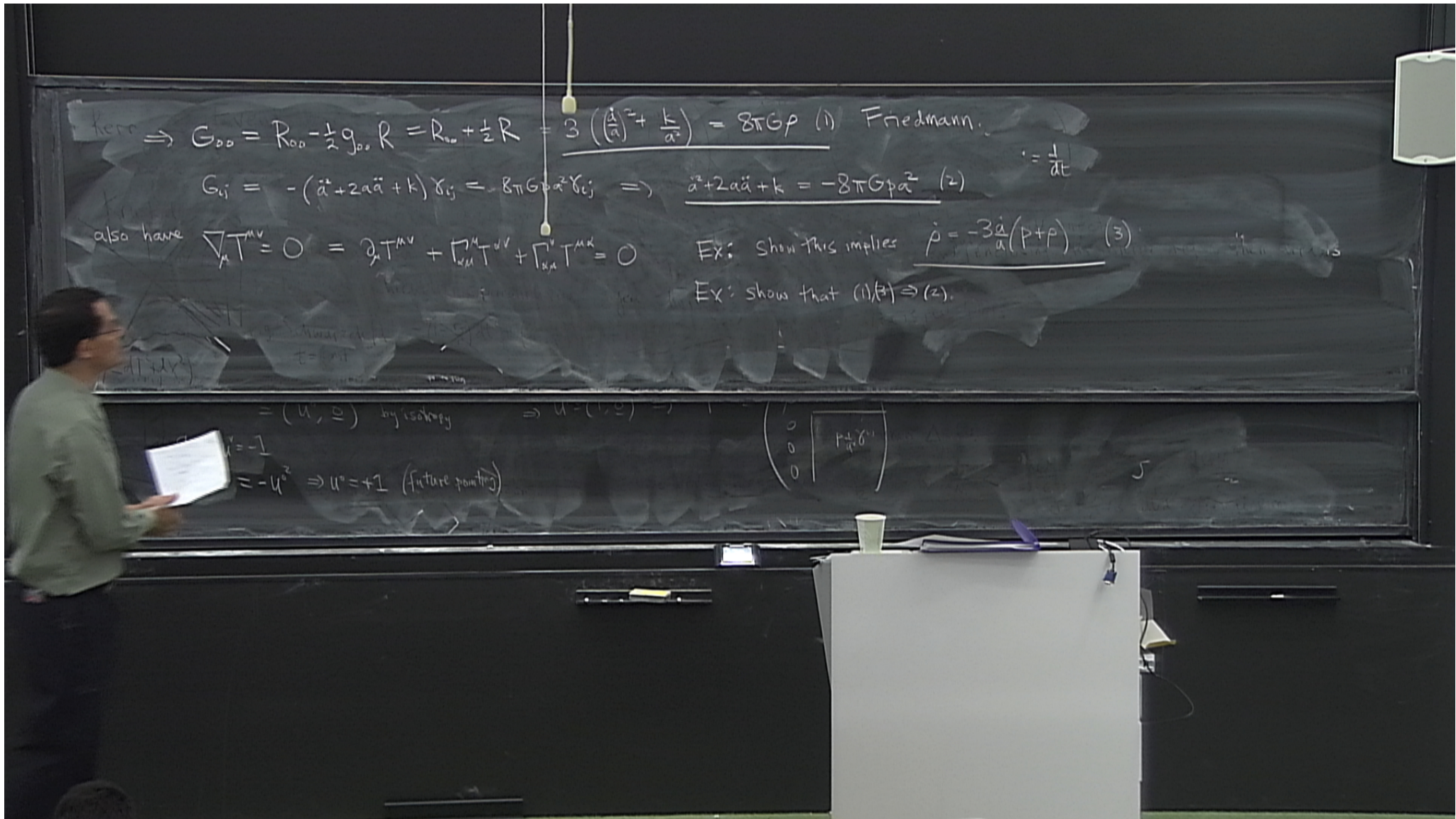
$$\Gamma_{ij}^0 = a \dot{a} \gamma_{ij}$$

$$\Gamma_{0j}^i = \frac{\dot{a}}{a} \delta_j^i$$

$$\Gamma_{jk}^i = \Gamma_{jk}^i(\gamma) \text{ ie. } \Gamma \text{ on } \begin{cases} S^3 \\ E^3 \\ H^3 \end{cases}$$

ellipsoids obtained by rotating around z -axis





$\Rightarrow G_{00} = R_{00} - \frac{1}{2}g_{00}R = R_{00} + \frac{1}{2}R = 3\left(\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right) = 8\pi G\rho$ (1) Friedmann.

$G_{ij} = -(\ddot{a} + 2a\ddot{a} + k)\gamma_{ij} = -8\pi G p a^2 \gamma_{ij} \Rightarrow \ddot{a} + 2a\ddot{a} + k = -8\pi G p a^2$ (2)

also have $\nabla_\mu T^{\mu\nu} = 0 = \partial_\mu T^{\mu\nu} + \Gamma_{\mu\lambda}^\mu T^{\lambda\nu} + \Gamma_{\mu\lambda}^\nu T^{\mu\lambda} = 0$ Ex: show this implies $\dot{\rho} = -\frac{3\dot{a}}{a}(\rho + p)$ (3)

Ex: show that (1)/(2) $\Rightarrow$ (3).

$= (u^\mu, 0)$ by isotropy $\Rightarrow u = (1, 0)$
 $u^\mu = -1$
 $= -u^2 \Rightarrow u^0 = +1$ (future pointing)

$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix} \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix}$

$$\underline{R = 3 \left(\left(\frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} \right) = 8\pi G \rho} \quad (1) \quad \text{Friedmann.}$$

$$8\pi G \rho a^2 \chi_{ij} \Rightarrow \underline{\ddot{a}^2 + 2a\ddot{a} + k = -8\pi G p a^2} \quad (2)$$

$$\nabla^\mu + \Gamma_{\alpha\mu}^\nu T^{\mu\alpha} = 0 \quad \text{Ex: Show this implies} \quad \underline{\dot{\rho} = -3 \frac{\dot{a}}{a} (p + \rho)} \quad (3)$$

Ex: that (1)/(3) $\Rightarrow$ (2).