

Title: Relativity - Lecture 10

Date: Sep 23, 2011 09:00 AM

URL: <http://pirsa.pi.local/11090077>

Abstract:

Spherical symmetric solution of $R_{\mu\nu} = 0$

possible to choose coordinates such that

$$ds^2 = -B(r,t)dt^2 + A(r,t)dr^2 + r^2 d\Omega^2$$

$$(d\theta^2 + \sin^2\theta d\phi^2)$$



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$$\Gamma_{\nu\lambda}^{\mu} = \frac{1}{2} g^{\mu\alpha} (g_{\nu\alpha,\lambda} + g_{\lambda\alpha,\nu} - g_{\nu\lambda,\alpha})$$

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$$\Gamma_{\nu\lambda}^{\mu} = \frac{1}{2} g^{\mu\alpha} (g_{\nu\alpha,\lambda} + g_{\lambda\alpha,\nu} - g_{\nu\lambda,\alpha})$$

$$(x^0, x^1, x^2, x^3) = (t, r, \theta, \varphi)$$

$$\Gamma_{00}^0 = \frac{1}{2} g^{00}$$

$$\rightarrow \frac{1}{2} (u^{\mu} + v^{\mu}) (a_{\mu}) \rightarrow$$

$$\text{sf } R_{\mu\nu} = 0$$

$$(d\theta^2 + \sin^2\theta d\varphi^2)$$

$$t^2 + A(r, t) dr^2 + r^2 d\Omega^2$$

$$g_{\mu\nu} = \begin{pmatrix} -B & & & \\ & A & & \\ & & r^2 & \\ & & & r^2 \sin^2\theta \end{pmatrix}$$

$$g^{\mu\nu} = \begin{pmatrix} -B^{-1} & & & \\ & A^{-1} & & \\ & & r^{-2} & \\ & & & \frac{1}{r^2 \sin^2\theta} \end{pmatrix}$$

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$$(x^0, x^1, x^2, x^3) = (t, r, \theta, \varphi)$$

$$\Gamma_{\alpha\beta}^{\alpha} = \frac{1}{2} g^{\alpha\gamma} g_{\alpha\gamma,\beta}$$

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$$(x^0, x^1, x^2, x^3) = (t, r, \theta, \varphi)$$

$$\Gamma_{00}^0 = \frac{1}{2} g^{00} g_{00,0} = \frac{1}{2} \frac{\dot{B}}{B}$$

$$\frac{\partial}{\partial t} = \cdot, \quad \frac{\partial}{\partial r} = ' , \quad \frac{\partial}{\partial \theta} = \cdot, \quad \frac{\partial}{\partial \varphi} = \cdot$$

$$\Rightarrow \boxed{R_{\alpha\beta} = 0}$$

Spherical symmetric solution of $R_{\mu\nu} = 0$

possible to choose coordinates such that

$$ds^2 = -B(r,t)dt^2 + A(r,t)dr^2 + r^2 d\Omega^2$$

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$$\Gamma_{\nu\lambda}^{\mu} = \frac{1}{2} g^{\mu\alpha} (g_{\alpha\nu,\lambda} + g_{\alpha\lambda,\nu} - g_{\nu\lambda,\alpha})$$

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$$(x^0, x^1, x^2, x^3) = (t, r, \theta, \varphi)$$

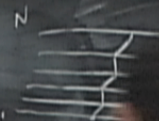
$$\frac{\partial}{\partial t} = \cdot$$

$$\frac{\partial}{\partial r} = \cdot$$

$$\Gamma_{00}^0 = \frac{1}{2} g^{00} g_{00,0} = \frac{1}{2} \frac{\dot{B}}{B}$$

$$\Gamma_{10}^0 = \Gamma_{01}^0 = \frac{1}{2} g^{00} (g_{00,1} + g_{01,0} - g_{10,0}) = \frac{1}{2} \frac{B'}{B}$$

sufficiently great



All experimental predictions

$$\Gamma_{rr}^t = \frac{1}{2} \frac{\dot{A}}{B}$$

$$\Gamma_{rt}^r = \Gamma_{tr}^r = \frac{\dot{A}}{2A}$$

$$\Gamma_{tt}^r = \frac{B'}{2A}$$

$$\Gamma_{rr}^r = \frac{A'}{2A}$$

$$\Gamma_{\theta\theta}^r = -\frac{r}{A}$$

$$\Gamma_{\varphi\varphi}^r = -\frac{r \sin^2 \theta}{A}$$

$$\Gamma_{\theta r}^\theta = \Gamma_{r\theta}^\theta = \frac{1}{r}$$

$$\Gamma_{\varphi\varphi}^\theta = -\sin \theta \cos \theta$$

$$\Gamma_{\varphi r}^\varphi = \Gamma_{r\varphi}^\varphi = \frac{1}{r}$$

$$\Gamma_{\varphi\theta}^\varphi = \Gamma_{\theta\varphi}^\varphi = \cot \theta$$

$$R_{\mu\kappa} = R^{\lambda}_{\mu\lambda\kappa} = - \left(\Gamma^{\lambda}_{\mu\lambda,\kappa} - \Gamma^{\lambda}_{\mu\kappa,\lambda} + \Gamma^{\lambda}_{\mu\lambda} \Gamma^{\lambda}_{\kappa\lambda} - \Gamma^{\lambda}_{\mu\kappa} \Gamma^{\lambda}_{\lambda\lambda} \right)$$

$$R_{\mu\kappa} = R^{\lambda}_{\mu\lambda\kappa} = - \left(\Gamma^{\lambda}_{\mu\lambda,\kappa} - \Gamma^{\lambda}_{\mu\kappa,\lambda} + \Gamma^{\lambda\lambda}_{\mu\lambda} \Gamma^{\lambda}_{\kappa\lambda} - \Gamma^{\lambda\lambda}_{\mu\kappa} \Gamma^{\lambda}_{\lambda\lambda} \right)$$

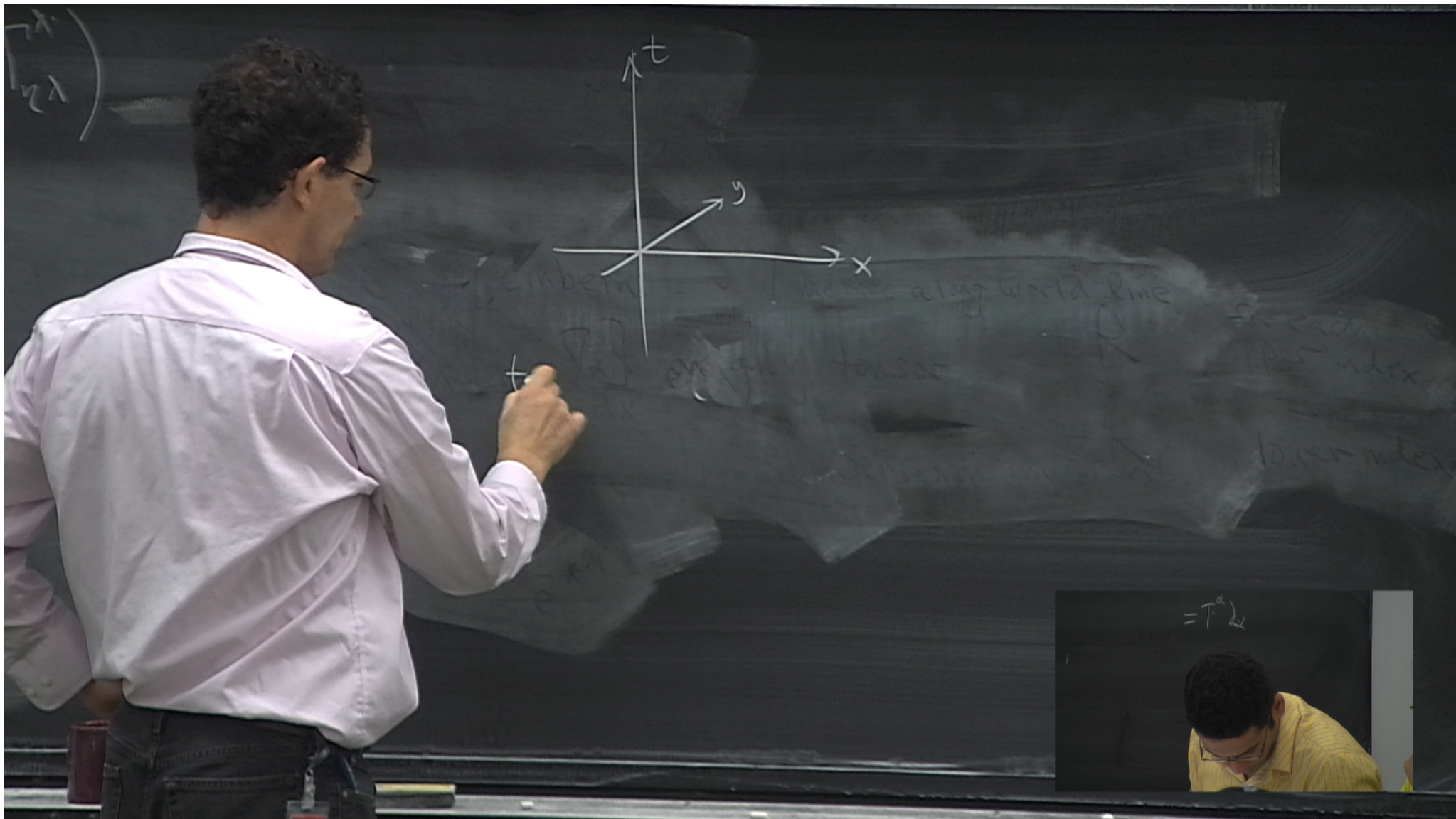
$$\rightarrow R_{rr} = - \frac{B''}{2B} + \frac{B'^2}{4B^2} + \frac{A'B'}{4AB} + \frac{A'}{Ar} - \frac{\ddot{A}}{2B} + \frac{\dot{A}\dot{B}}{4B^2} + \frac{\dot{A}^2}{4AB}$$

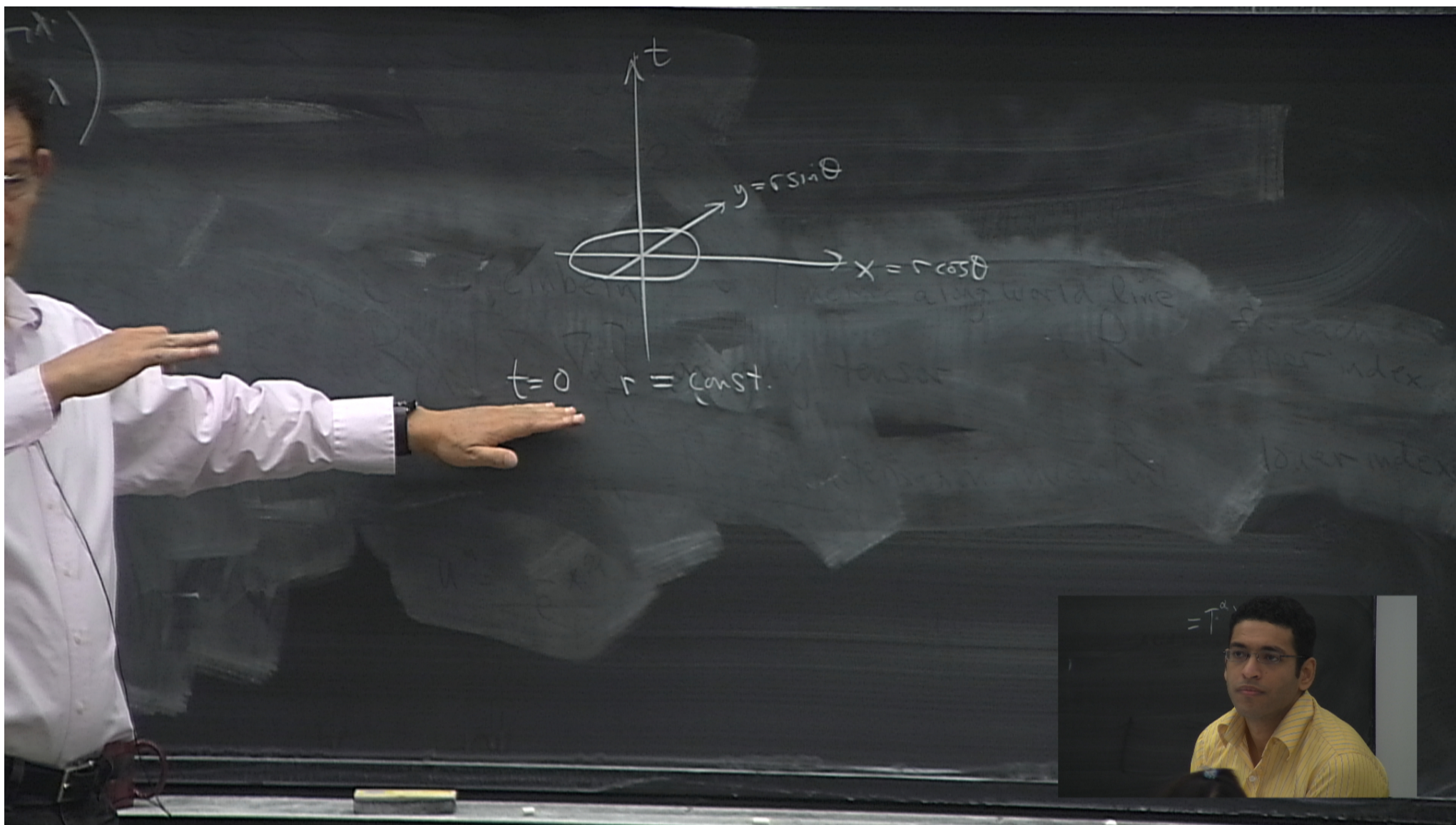
$$R_{\theta\theta} = 1 - \frac{1}{A} + \frac{rA'}{2A^2} - \frac{rB'}{2AB}$$

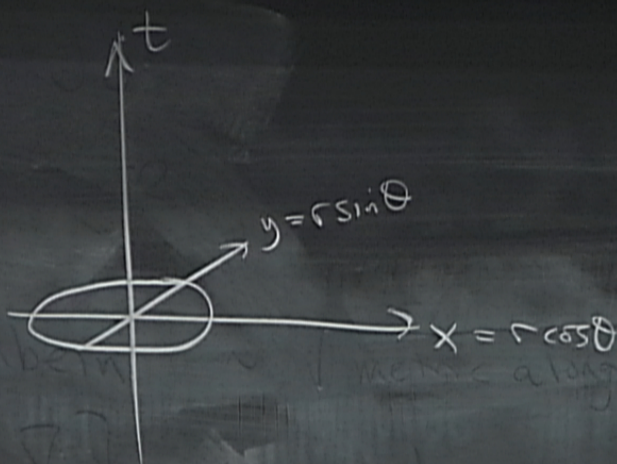
$$R_{tt} = \frac{B''}{2A} - \frac{B'A'}{4A^2} + \frac{B'}{Ar} - \frac{B'^2}{4AB} - \frac{\ddot{A}}{2A} + \frac{\dot{A}^2}{4A^2} + \frac{\dot{B}\dot{A}}{4AB}$$

$$R_{tr} = \frac{\dot{A}}{Ar}$$





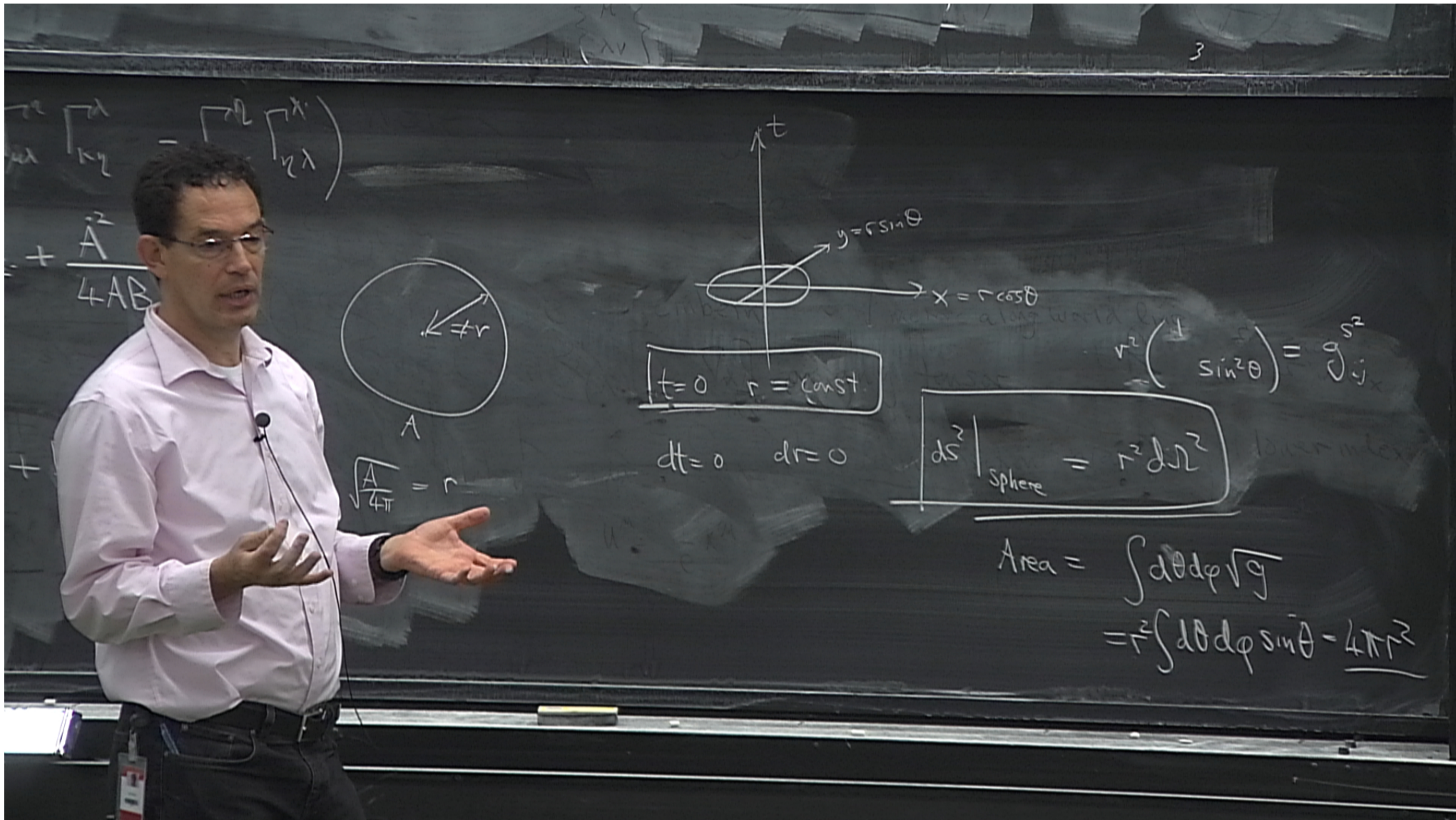


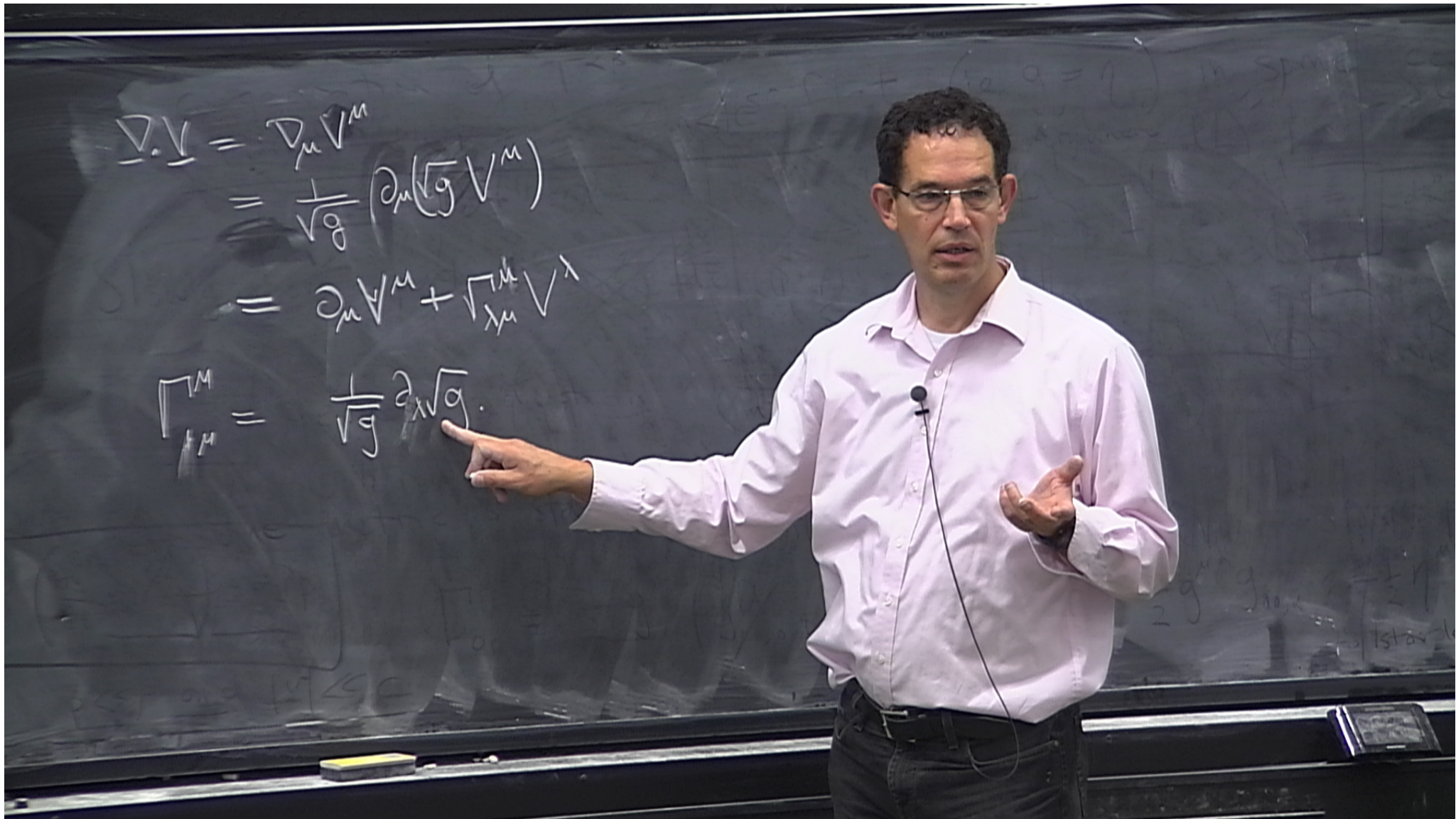


$$\underline{t=0} \quad r = \text{const.}$$

$$dt=0 \quad dr=0 \quad ds^2|_{\text{sphere}} = r^2 d\Omega^2$$

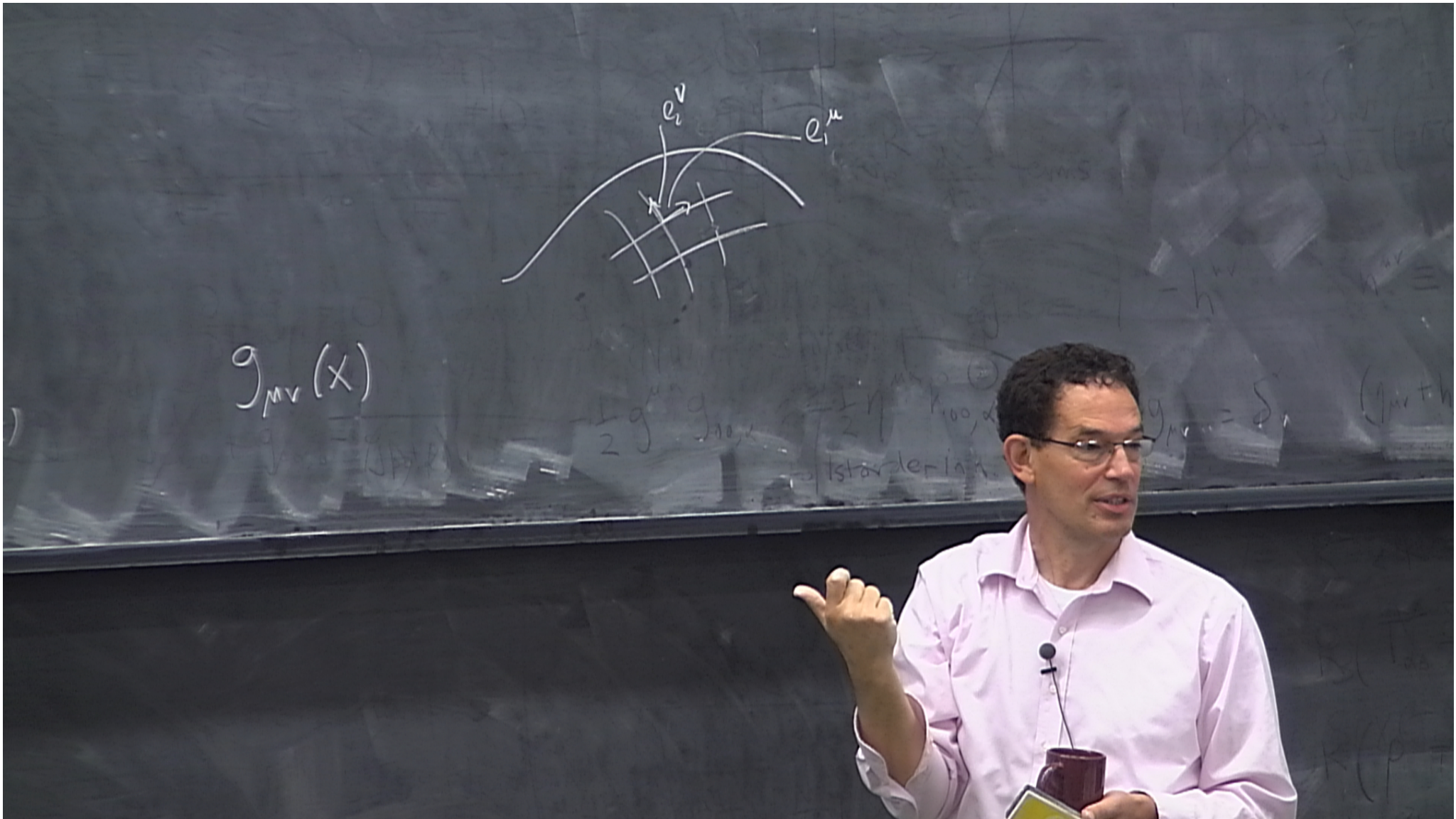
$$\begin{aligned} \text{Area} &= \int d\theta d\phi \sqrt{g} \\ &= r^2 \int d\theta d\phi \sin\theta = \underline{4\pi r^2} \end{aligned}$$



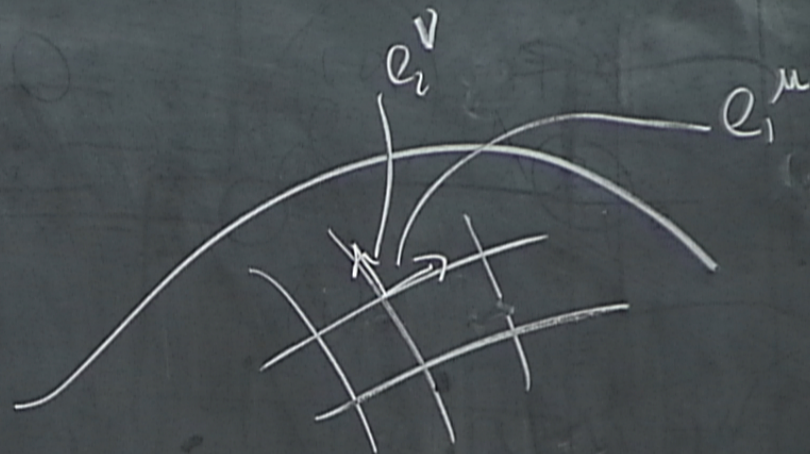


$$\begin{aligned}\underline{\nabla \cdot V} &= \nabla_\mu V^\mu \\ &= \frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} V^\mu) \\ &= \partial_\mu V^\mu + \Gamma^\mu_{\lambda\mu} V^\lambda\end{aligned}$$

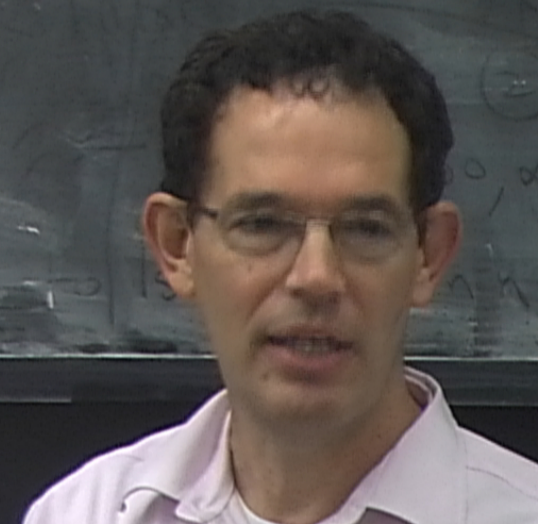
$$\Gamma^\mu_{\lambda\mu} = \frac{1}{\sqrt{g}} \partial_\lambda \sqrt{g}.$$

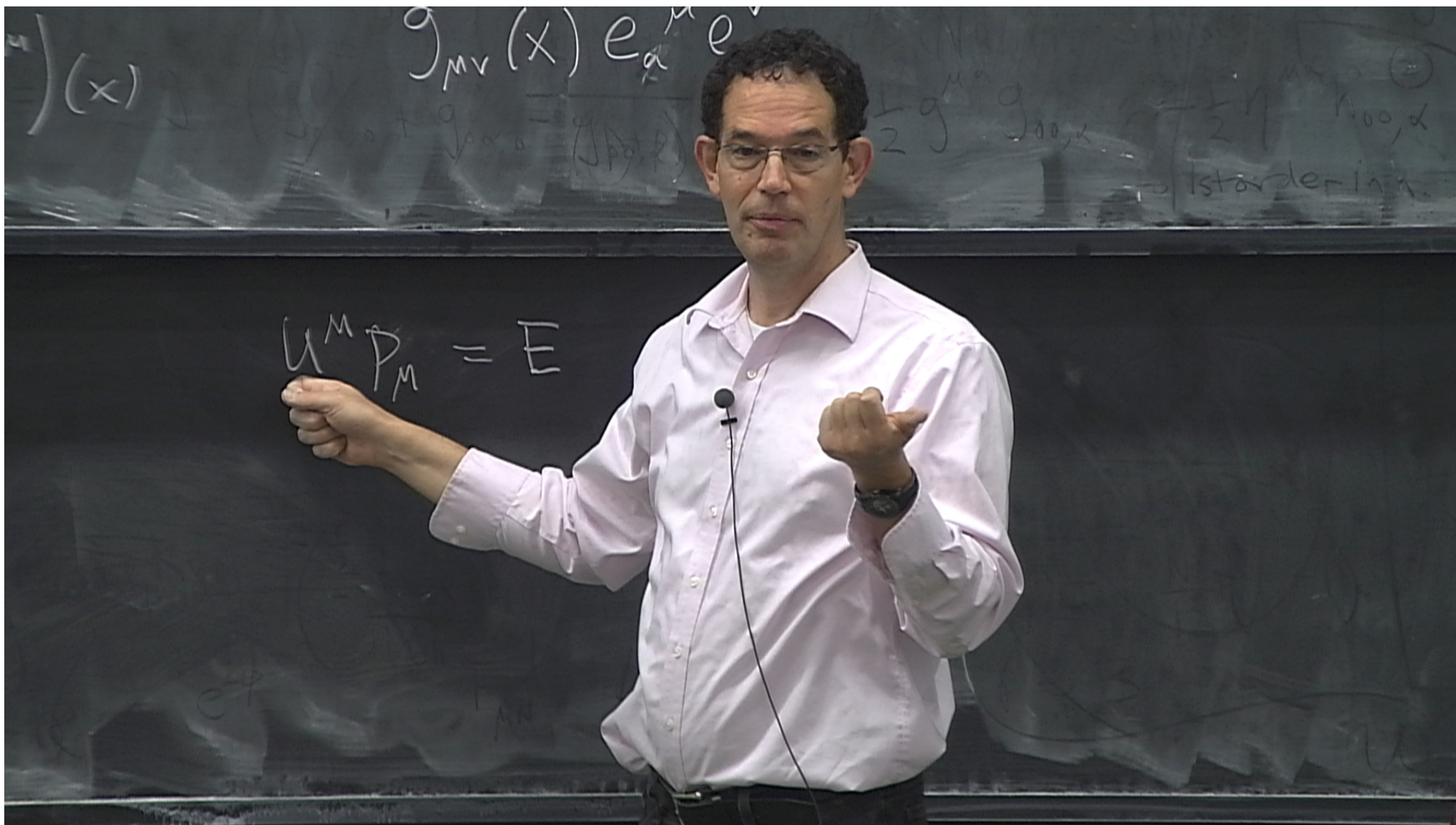


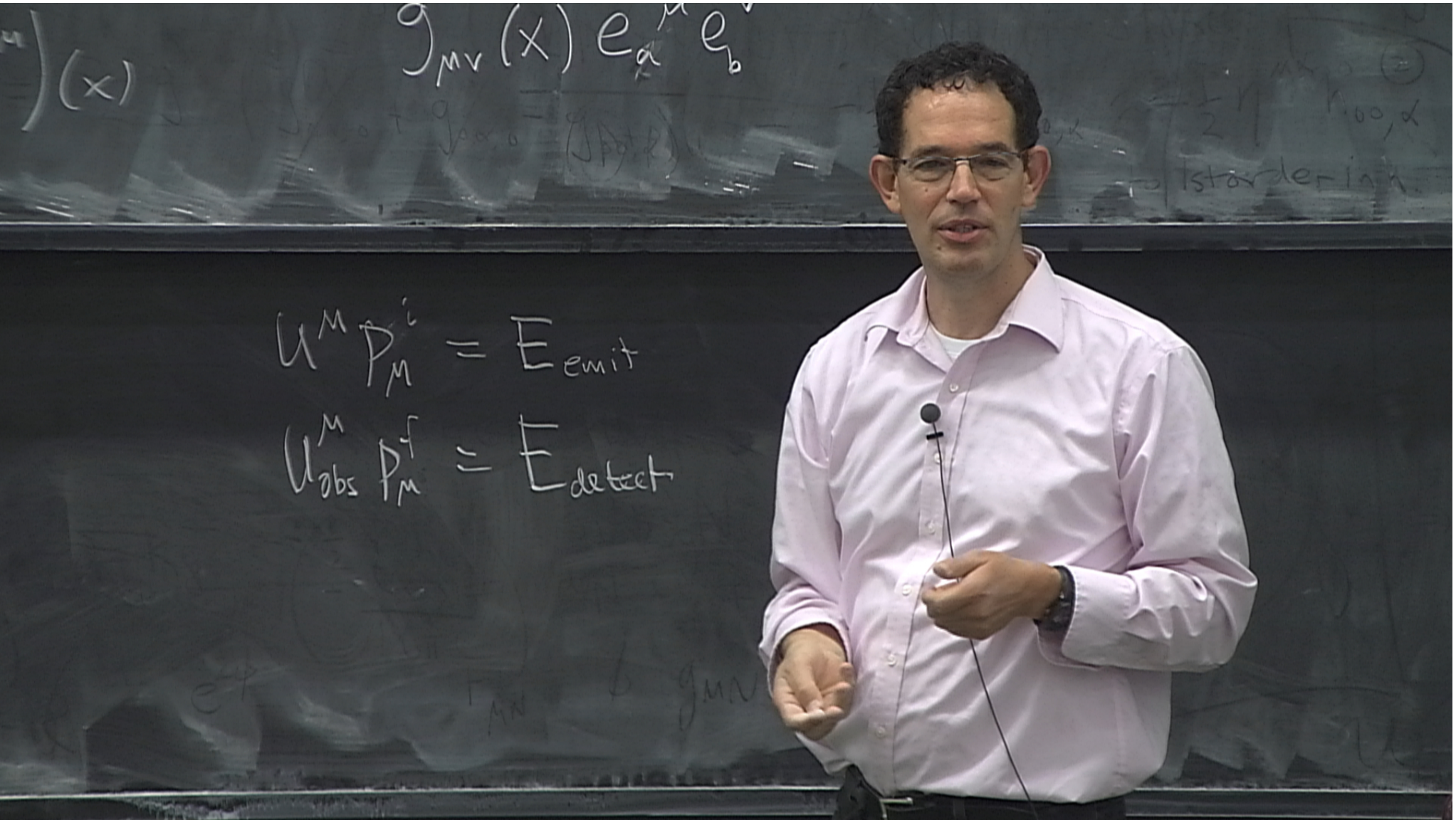
$$g_{\mu\nu}(x) e_a^\mu e_b^\nu$$



$$-\frac{1}{2} g^{\mu\alpha} g_{\alpha\beta, \lambda}$$







$$R_{tr} = 0 \Rightarrow \frac{\partial}{\partial t} A(r, t) = 0. \quad : A = A(r).$$

$$\sum_b T_b + \int R - \dots$$

e^{μ} Γ_{MN} $g_{\mu\nu}$

$$R_{tr} = 0 \Rightarrow \frac{\partial}{\partial t} A(r, t) = 0. \quad : A = A(r).$$

$\Rightarrow$ all time derivative terms drop out.

$$\sum_i T_i + \int R = \dots$$

e^{μ}

Γ_{MN}

g_{MN}

$$R_{\mu\nu} = R^{\lambda}_{\mu\nu\lambda} = -(\Gamma^{\lambda}_{\mu\alpha}\Gamma^{\alpha}_{\nu\lambda} - \Gamma^{\lambda}_{\nu\alpha}\Gamma^{\alpha}_{\mu\lambda} - \Gamma^{\lambda}_{\mu\lambda}\Gamma^{\alpha}_{\nu\alpha} + \Gamma^{\lambda}_{\nu\lambda}\Gamma^{\alpha}_{\mu\alpha})$$

$$\rightarrow R_{rr} = -\frac{g''}{2g} + \frac{g'''}{4g^2} + \frac{A'}{4AB} + \frac{A'}{A^2} + \frac{A'}{2g} + \frac{A'}{4g} + \frac{A'}{4AB}$$

$$R_{\theta\theta} = 1 - \frac{A}{r^2} + \frac{A'}{r^2} - \frac{A''}{2AB}$$

$$R_{tt} = \frac{g'}{2g} - \frac{g''}{4g^2} + \frac{A'}{4AB} - \frac{A''}{4AB} - \frac{A}{2g} + \frac{A}{4g} + \frac{A}{4AB}$$

$$R_t = \frac{A}{Ar}$$

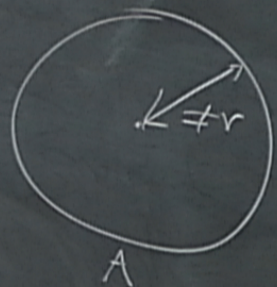
$$R_{\mu\kappa} = R^{\lambda}_{\mu\lambda\kappa} = - \left(\Gamma^{\lambda}_{\mu\lambda,\kappa} - \Gamma^{\lambda}_{\mu\kappa,\lambda} + \Gamma^{\lambda}_{\mu\lambda} \Gamma^{\lambda}_{\kappa\lambda} - \Gamma^{\lambda}_{\mu\kappa} \Gamma^{\lambda}_{\lambda\lambda} \right)$$

$$\rightarrow R_{rr} = -\frac{B''}{2B} + \frac{B'^2}{4B^2} + \frac{A'B'}{4AB} + \frac{A'}{Ar} - \frac{\ddot{A}}{2B} + \frac{\dot{A}\dot{B}}{4B^2} + \frac{\dot{A}^2}{4AB}$$

$$K_{\theta\theta} = 1 - \frac{1}{A} + \frac{rA'}{2A^2} - \frac{rB'}{2AB}$$

$$R_{tt} = \frac{B''}{2A} - \frac{B'A'}{4A^2} + \frac{B'}{Ar} - \frac{B'^2}{4AB} - \frac{\ddot{A}}{2A} + \frac{\dot{A}^2}{4A^2} + \frac{\dot{B}\dot{A}}{4AB}$$

$$R_{tr} = \frac{\dot{A}}{Ar}$$



$$\sqrt{\frac{A}{r}} = r$$

$$R_{tr} = 0 \Rightarrow \frac{\partial}{\partial t} A(r) = 0, \quad A = A(r),$$

$\Rightarrow$ all time derivative terms adding out.

$$R_{rr} = R_{tt} = 0$$

$$R_{tr} = 0 \Rightarrow \frac{\partial}{\partial t} A(r, t) = 0. \quad : A = A(r).$$

$\Rightarrow$ all time derivative terms drop out.

$$R_{rr} = R_{tt} = 0 \Rightarrow \frac{R_{rr}}{A} + \frac{R_{tt}}{B} = \frac{A'}{A^2 r} + \frac{B'}{ABr} = 0 \Rightarrow BA' + AB' = 0$$

$$R_{tr} = 0 \Rightarrow \frac{\partial}{\partial t} A(r, t) = 0. \quad : A = A(r).$$

$\Rightarrow$ all time derivative terms drop out.

$$R_{rr} = R_{tt} = 0 \Rightarrow \frac{R_{rr}}{A} + \frac{R_{tt}}{B} = \frac{A'}{A^2 r} + \frac{B'}{ABr} = 0 \Rightarrow BA' + AB' = 0$$

$$\Rightarrow (AB)' = 0$$

$$\Rightarrow A = \frac{\text{const}}{B}$$

Fix $f(t)$ with boundary conditions i.e. $g = 1$ in space $\mathbb{R}^3$

asymptotically Minkowski b.c.s.

$$R_{rr} - K_{tt} = 0 \Rightarrow \frac{R_{rr}}{A} + \frac{R_{tt}}{B} = \frac{A'}{A^2 r} + \frac{B'}{ABr} = 0 \Rightarrow BA' + AB' = 0$$

$$\Rightarrow (AB)' = 0$$

$$\Rightarrow A = \frac{f(t)}{B}$$

Fix $f(t)$ with boundary conditions

asymptotically Minkowski bcs : as $r \rightarrow \infty$ $ds^2 \rightarrow ds^2_{\text{MINKOWSKI}} = -dt^2 + dr^2 + r^2 d\Omega^2$.

Fix $f(t)$ with boundary conditions

asymptotically Minkowski bcs : as $r \rightarrow \infty$ $ds^2 \rightarrow ds^2_{\text{Minkowski}} = -dt^2 + dr^2 + r^2 d\Omega^2$.

ie $B \rightarrow 1$ $r \rightarrow \infty$
 $A \rightarrow 1$ "

Fix $f(t)$ with boundary conditions

asymptotically Minkowski bcs : as $r \rightarrow \infty$ $ds^2 \rightarrow ds^2_{\text{Minkowski}} = -dt^2 + dr^2 + r^2 d\Omega^2$.

$$\left. \begin{array}{l} \text{i.e. } B \rightarrow 1 \\ A \rightarrow 1 \end{array} \right\} \begin{array}{l} r \rightarrow \infty \\ " \end{array} \Rightarrow f(t) = 1$$

Fix $f(t)$ with boundary conditions

asymptotically Minkowski bcs : as $r \rightarrow \infty$ $ds^2 \rightarrow ds^2_{\text{Minkowski}} = -dt^2 + dr^2 + r^2 d\Omega^2$.

$$\left. \begin{array}{l} \text{i.e. } B \rightarrow 1 \\ A \rightarrow 1 \end{array} \right\} \begin{array}{l} r \rightarrow \infty \\ " \end{array} \Rightarrow f(t) = 1 \Rightarrow \underline{A(r) = \frac{1}{B(r)}}$$

$$\Rightarrow A = \frac{g_{tt}}{B}$$

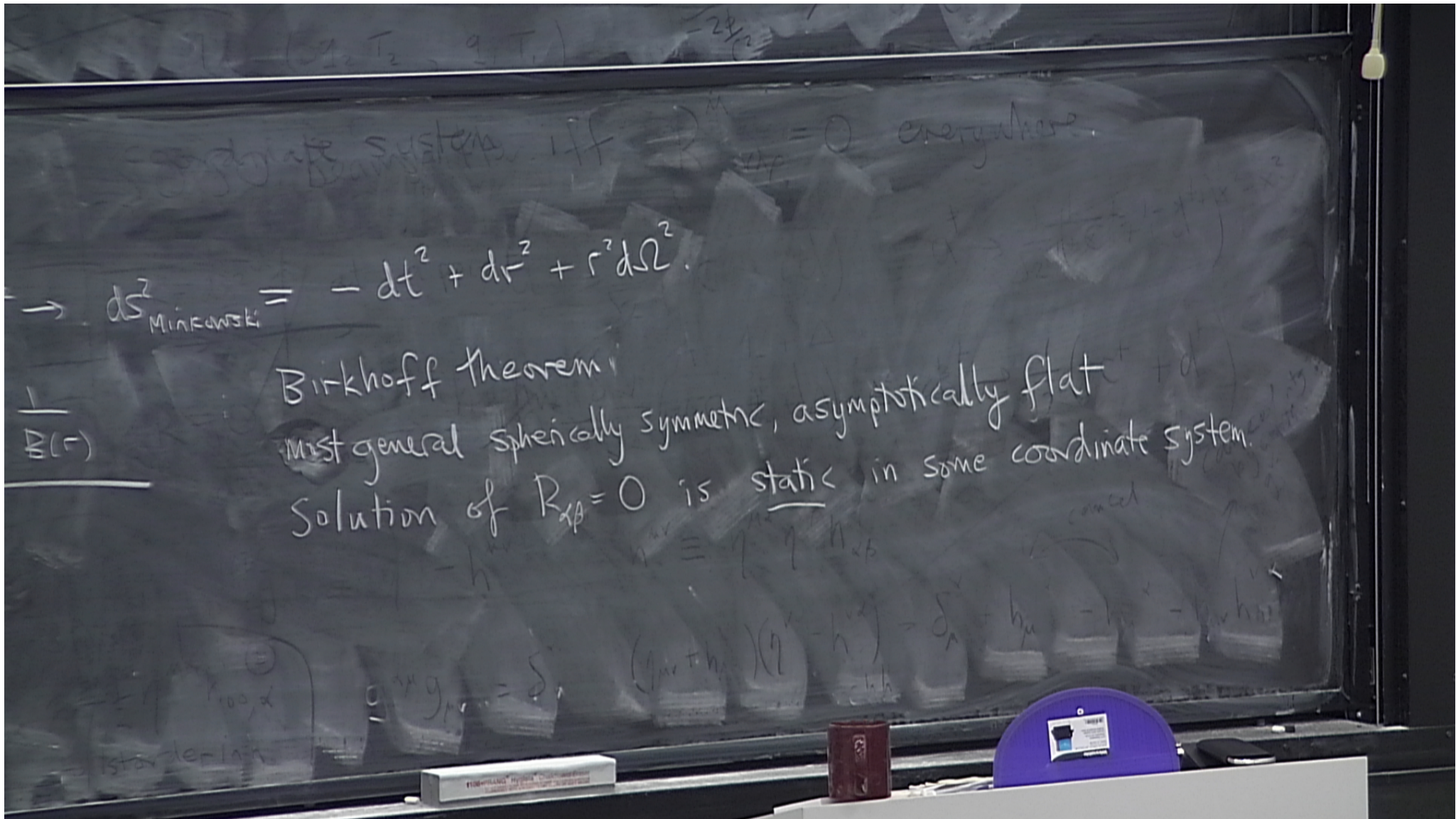
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asymptotically Minkowski bcs : as $r \rightarrow \infty$ $ds^2 \rightarrow ds^2_{\text{Minkowski}} = -dt^2 + dr^2 + r^2 d\Omega^2$

$$\left. \begin{array}{l} \text{ie } B \rightarrow 1 \\ A \rightarrow 1 \end{array} \right\} r \rightarrow \infty \Rightarrow f(t) = 1 \Rightarrow \underline{A(r) = \frac{1}{B(r)}}$$

Birkhoff theorem:

most general spherically symmetric, asymptotically flat
solution of $R_{\mu\nu} = 0$ is static in some coordinate system



$$A = \frac{1}{B} \Rightarrow R_{\text{tot}} = 1 - B - rB' = 0$$

$$\Rightarrow \frac{d}{dr}(rB) = 1$$

$$\Rightarrow rB = r + \text{const}$$

$$\underline{B = 1 + \frac{\text{const}}{r} = \frac{1}{A}}$$

$$\Rightarrow rB = r + \text{const}$$

$$B = 1 + \frac{\text{const}}{r} = \frac{1}{A}$$

But recall that when

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, |h_{\mu\nu}| \ll 1 \Rightarrow g_{00} = -1 + h_{00} = -\left(1 + \frac{2\phi}{c^2}\right)$$

$\phi = \text{gra}$

at lar

$$\frac{1}{i} \frac{\partial}{\partial t}$$

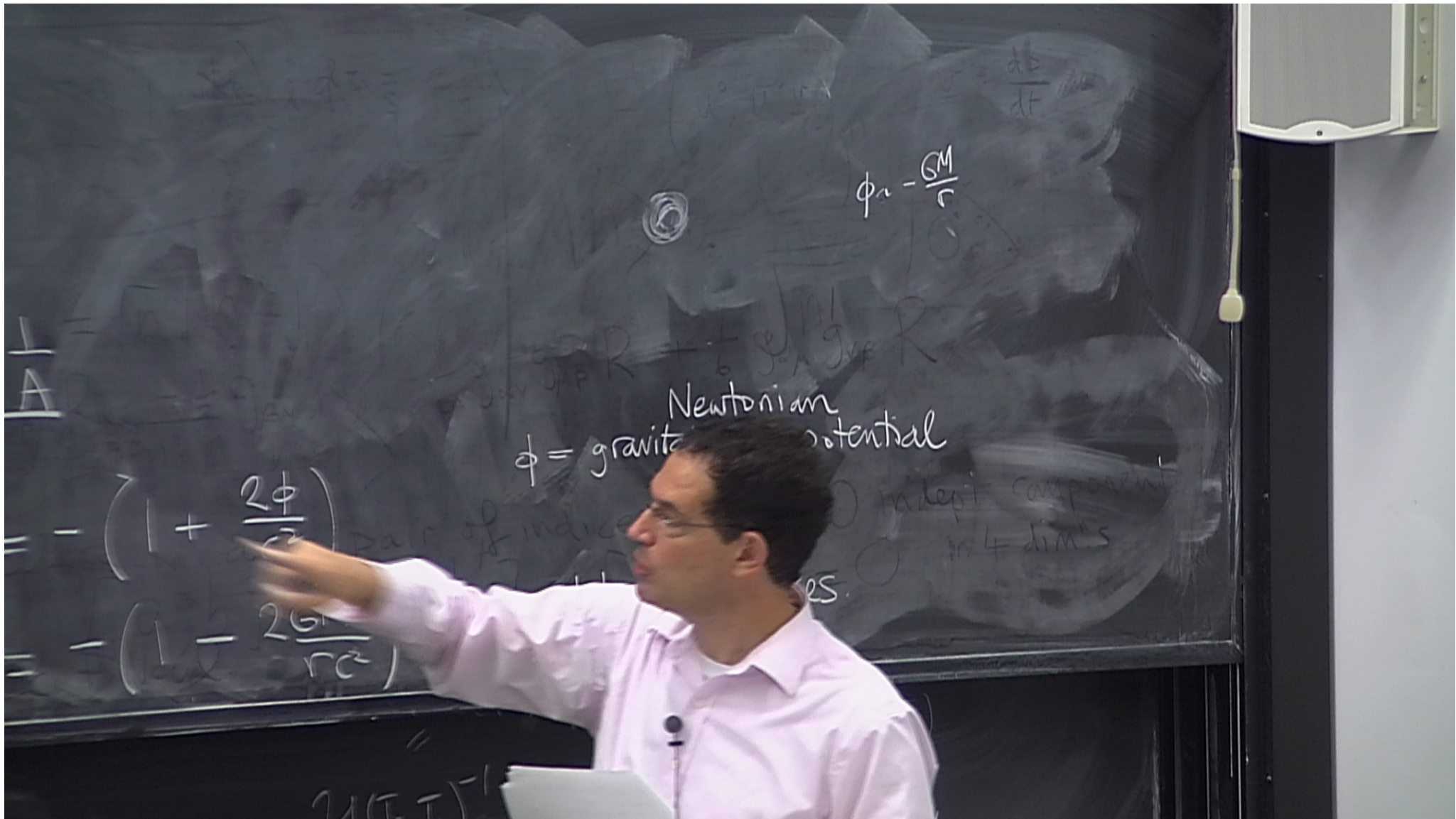
fields $\phi \rightarrow$ operator U_ϕ

$$U(T_2 T_1)^{-1}$$

$$\int \frac{\delta L}{\delta g} \delta g + r \delta g - H \delta g$$

natural product is 1-order of unity

Newtonian
 $\phi = \text{gravitational potential}$
at large distances



But recall that when

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ϕ = Newtonian gravitational potential

at large distances

So, we can write $B = 1 - \frac{2GM}{rc^2}$, where M is the "effective mass" of the solution at large r .

$$1 - \frac{2GM}{rc^2}$$

, where M is the "effective mass" of the solution at large r

$$\Rightarrow ds^2 = -\left(1 - \frac{2GM}{rc^2}\right) dt^2 + \frac{1}{\left(1 - \frac{2GM}{rc^2}\right)} + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

$$1 - \frac{2GM}{rc^2}$$

, where M is the "effective mass" of the solution at large

Schwarzschild
1920's

$$\Rightarrow ds^2 = -\left(1 - \frac{2GM}{rc^2}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2GM}{rc^2}\right)} + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

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1) singular at $r=0$: only solves $R_{\alpha\beta} = 0$ away from $r=0$.

$$2) r_s = \frac{2GM}{c^2}$$

$$1 - \frac{2GM}{rc^2}$$

, where M is the "effective mass" of the solution at large

Schwarzschild
1920's

$$\Rightarrow ds^2 = -\left(1 - \frac{2GM}{rc^2}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2GM}{rc^2}\right)} + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

1) singular at $r=0$: only solves $R_{\alpha\beta} = 0$ away from $r=0$.

2) $r_s = \frac{2GM}{c^2}$: $g_{\mu\nu}$ components look singular.