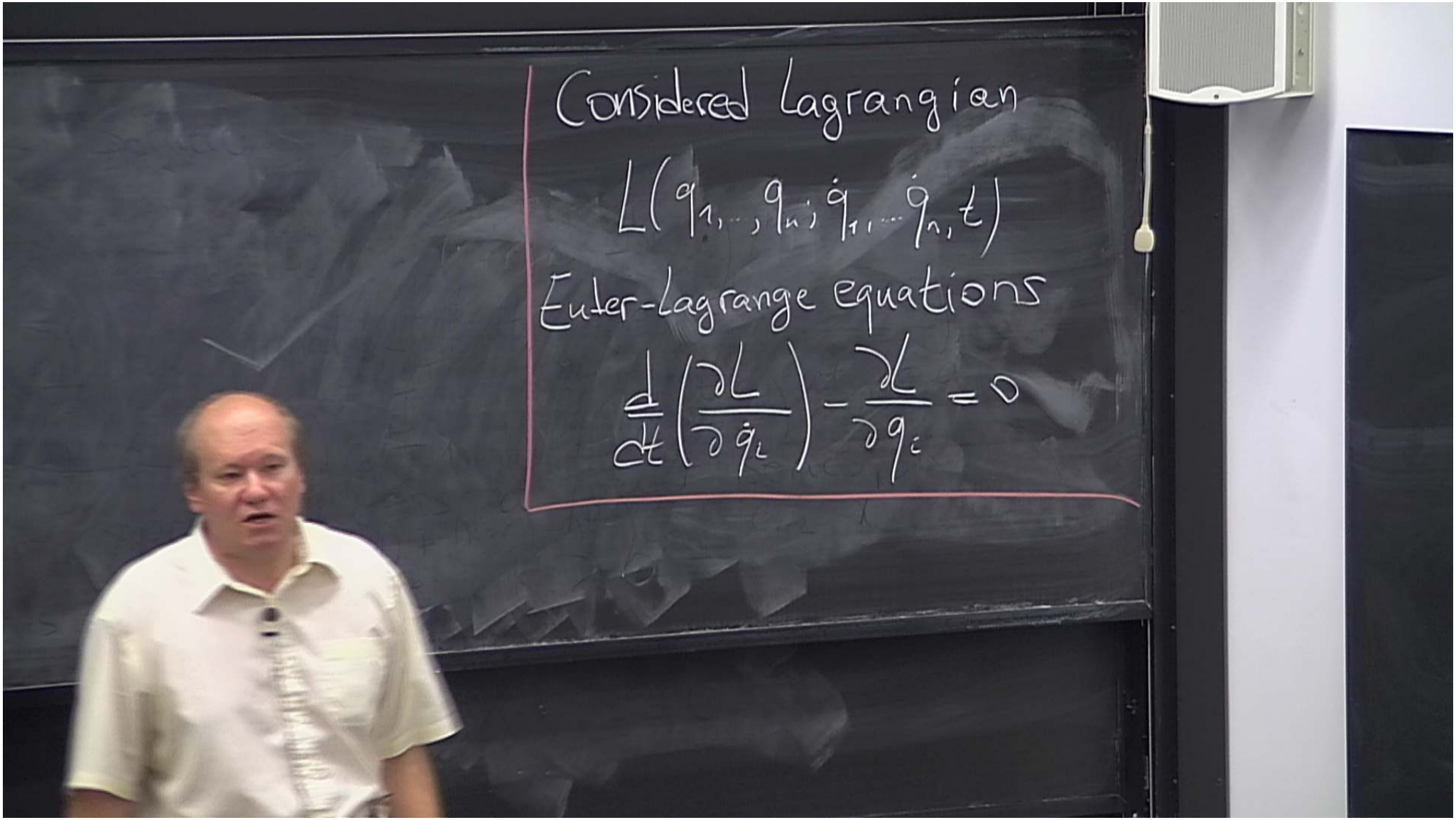
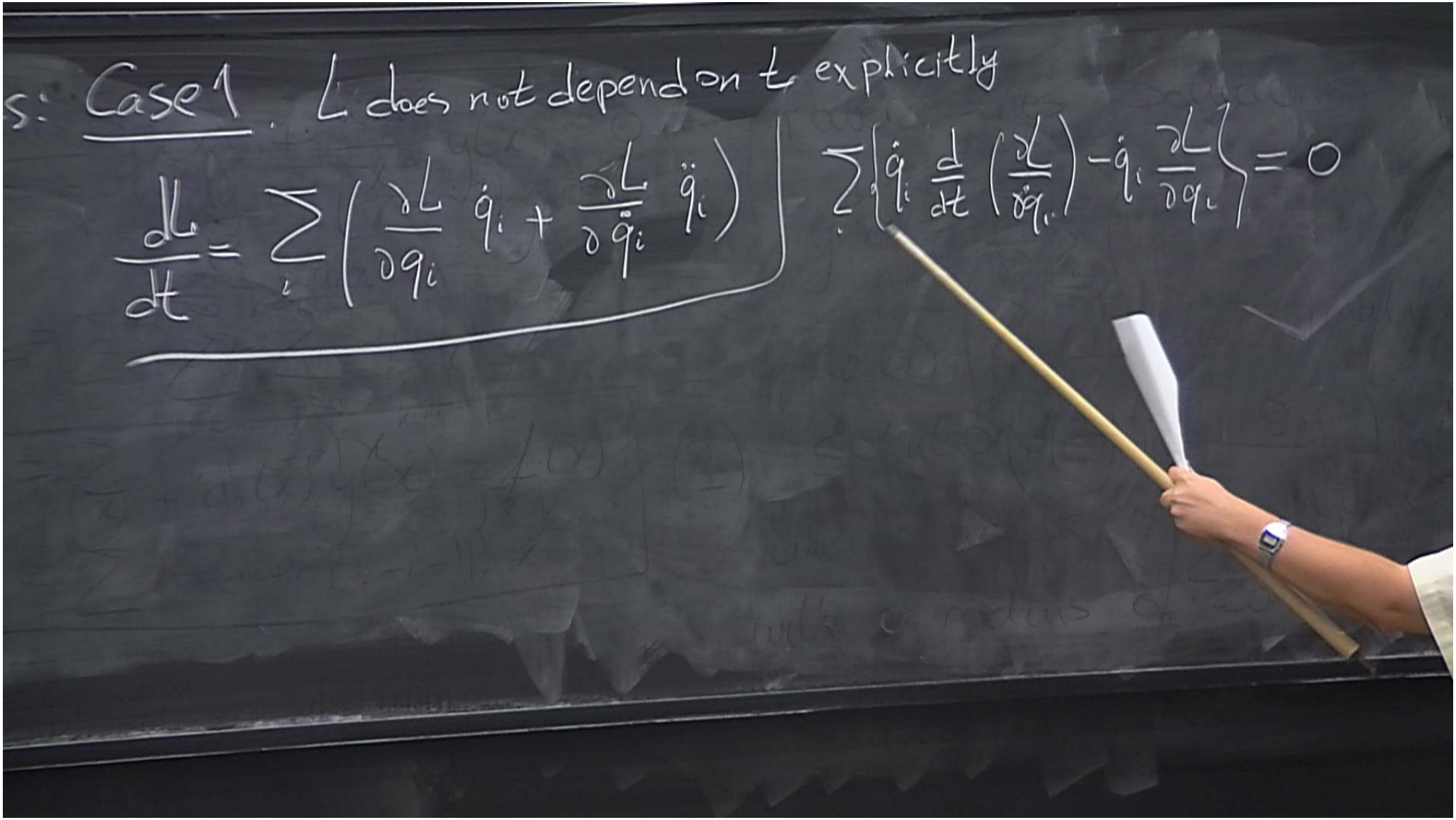
Title: Evaluation of Integrals and Calculus of Variations - Lecture 4

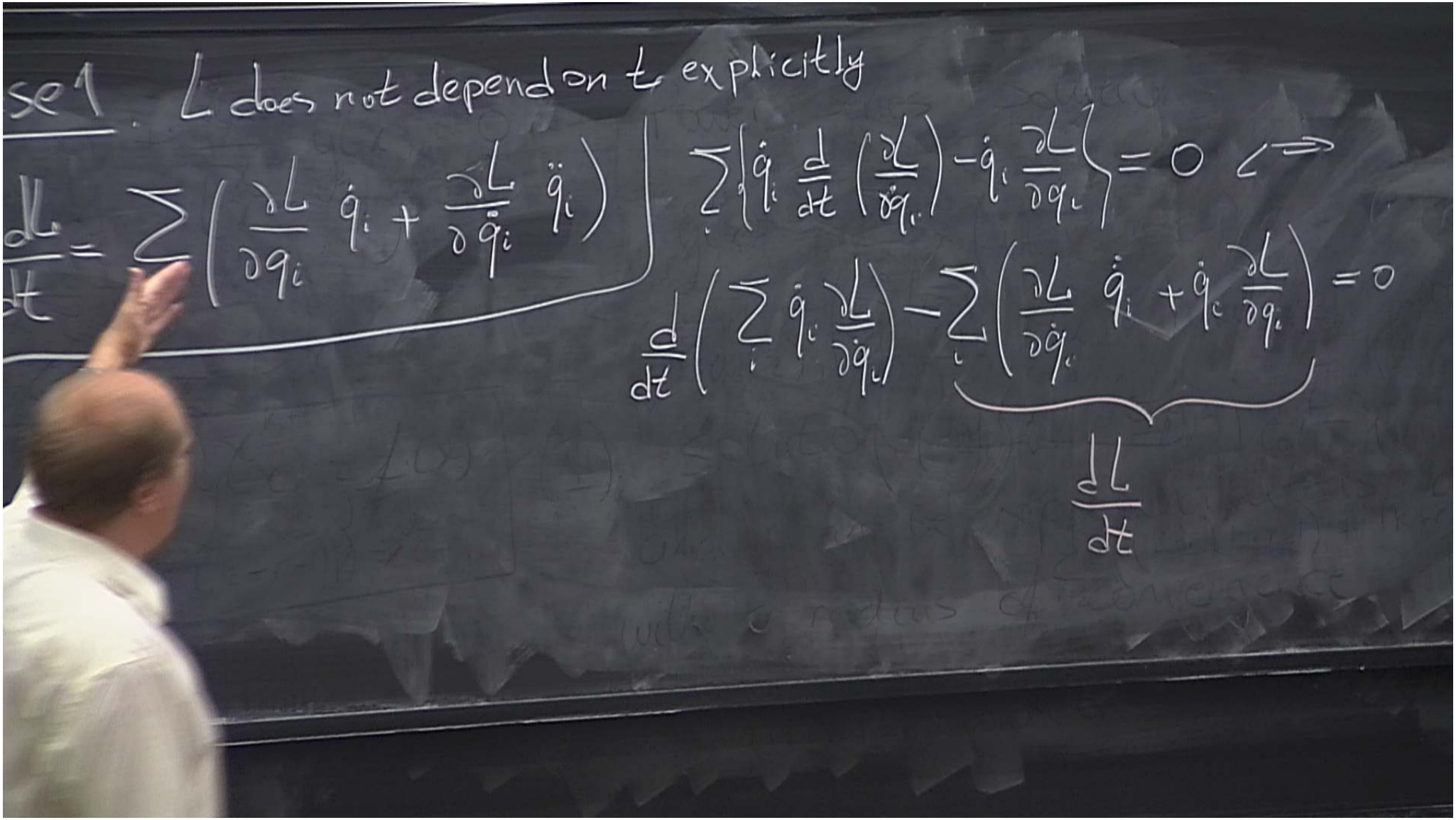
Date: Sep 01, 2011 09:00 AM

URL: http://pirsa.org/11090069

Abstract:


Considered Lagrangian
$L(q_1, \ldots, q_n; \dot{q}_1, \ldots \dot{q}_n, t)$
Euler-Lagrange equations
$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$


s: Case 1. L does not depend on t explicitly
$\frac{dL}{dt} = \sum_i \left(\frac{\partial L}{\partial q_i} \dot{q}_i + \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i \right)$
$\sum_i \left\{ \dot{q}_i \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \dot{q}_i \frac{\partial L}{\partial q_i} \right\} = 0$


se 1. L does not depend on t explicitly
$\frac{dL}{dt} = \sum \left(\frac{\partial L}{\partial q_i} \dot{q}_i + \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i \right)$
$\sum_i \left\{ \dot{q}_i \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \dot{q}_i \frac{\partial L}{\partial q_i} \right\} = 0 \Leftrightarrow$
$\frac{d}{dt} \left(\sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} \right) - \sum_i \left(\frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i + \dot{q}_i \frac{\partial L}{\partial q_i} \right) = 0$
$\frac{dL}{dt}$

Special cases: Case 1 . L does not depend on t explicitly

$\frac{\partial L}{\partial t} = 0$!

$$\frac{dL}{dt} = \sum_i \left(\frac{\partial L}{\partial q_i} \dot{q}_i + \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i \right)$$

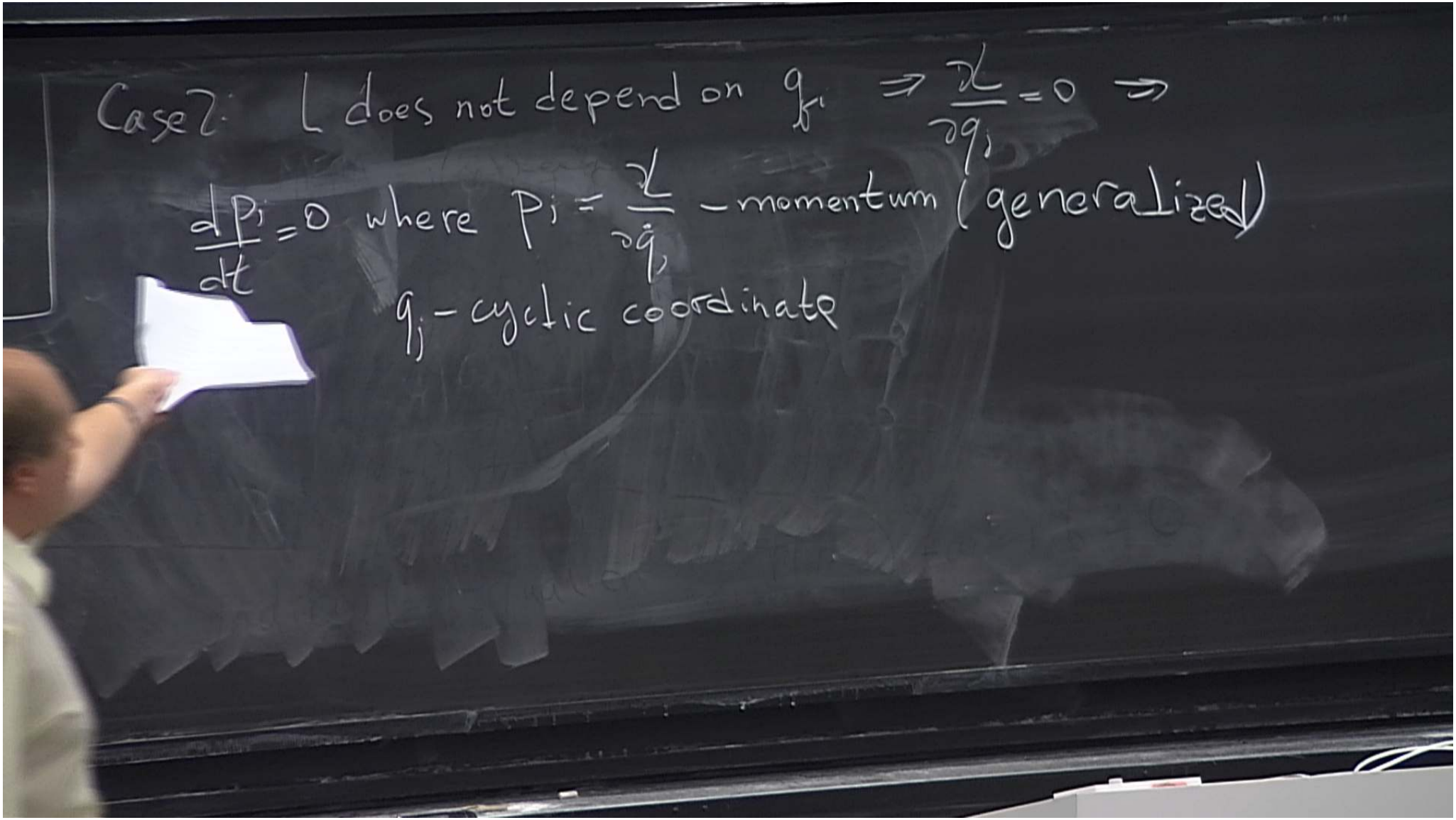
$$\sum_i \left\{ \dot{q}_i \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \right.$$

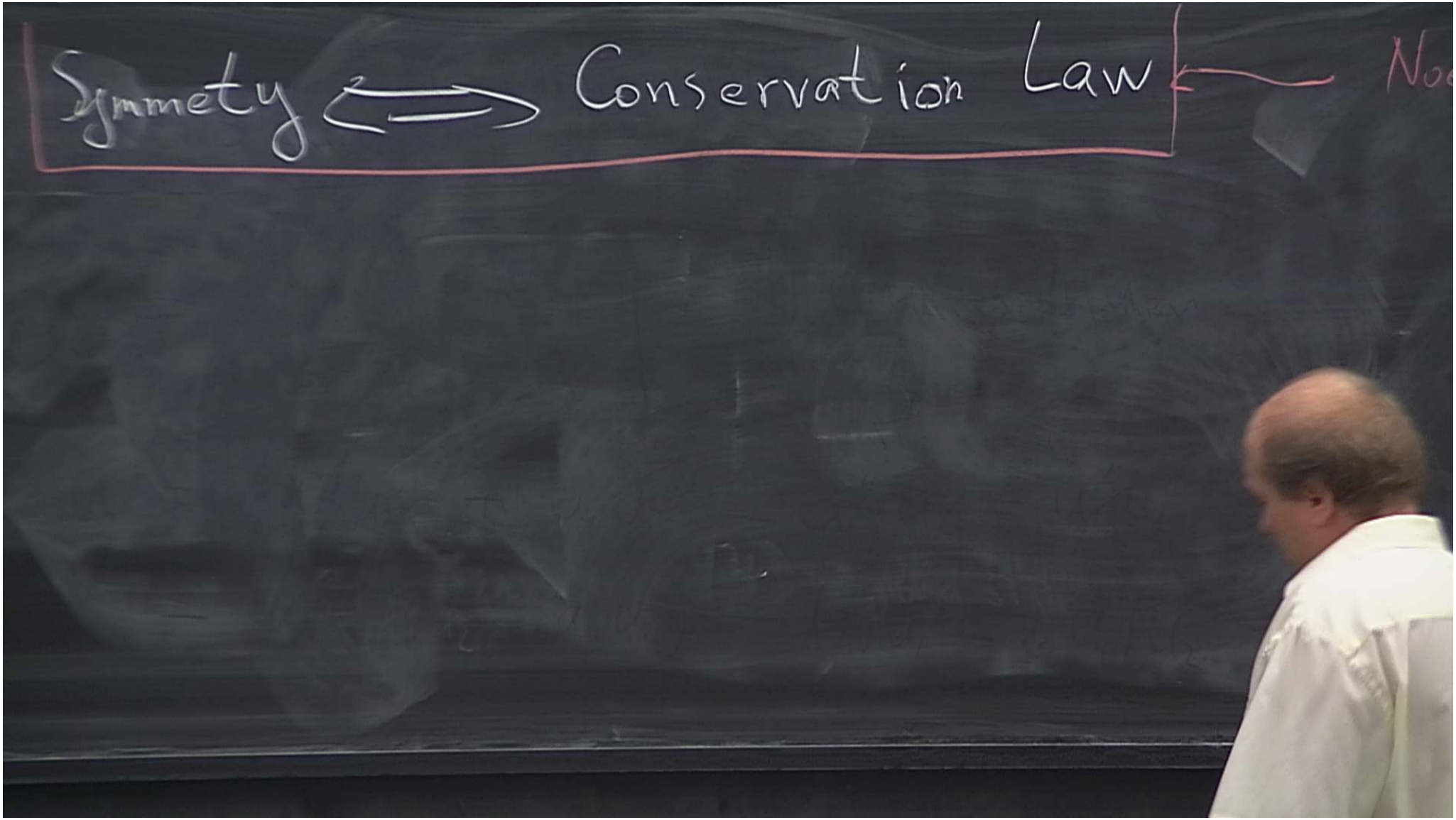
$$\frac{d}{dt} \left(\sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} \right) -$$

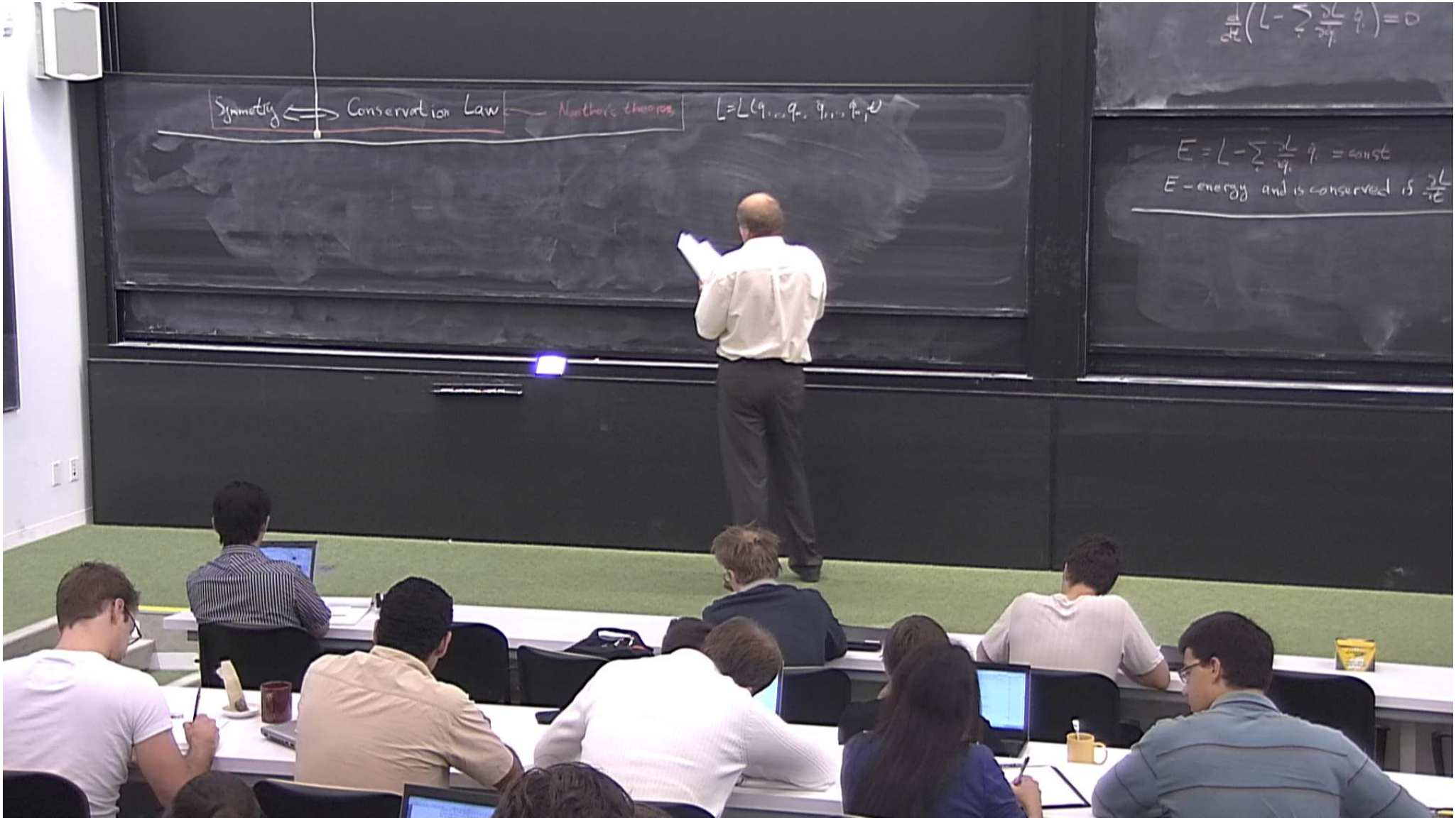
$$\frac{d}{dt} \left(L - \sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \right) = 0$$

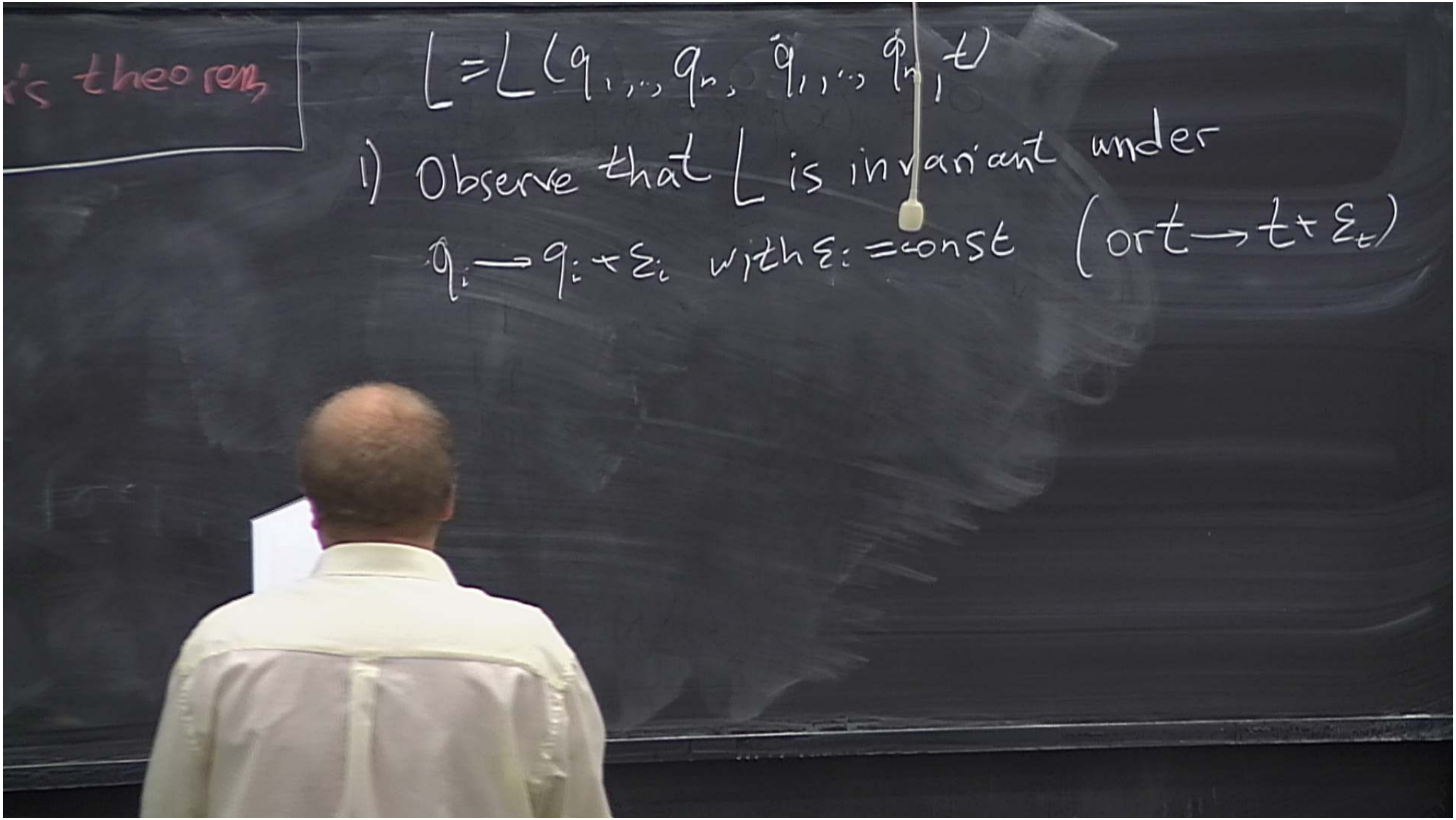
$$E = L - \sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i = \text{const}$$

E - energy and is conserved if $\frac{\partial L}{\partial t} = 0$


Case 2: L does not depend on $q_j \Rightarrow \frac{\partial L}{\partial q_j} = 0 \Rightarrow$
$\frac{dp_j}{dt} = 0$ where $p_j = \frac{\partial L}{\partial \dot{q}_j}$ - momentum (generalized)
q_j - cyclic coordinate


Symmetry ⟺ Conservation Law ⟵


Symmetry ⟺ Conservation Law
Noether's theorem
$L = L(q_1 \dots q_n, \dot{q}_1 \dots \dot{q}_n, t)$
$E = L - \sum \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i = \text{const}$
E - energy and is conserved if $\frac{\partial L}{\partial t}$


's theorem
$L = L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t)$
1) Observe that L is invariant under
$q_i \to q_i + \varepsilon_i$ with $\varepsilon_i = \text{const}$ (or $t \to t + \varepsilon_t$)

$L = L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t)$

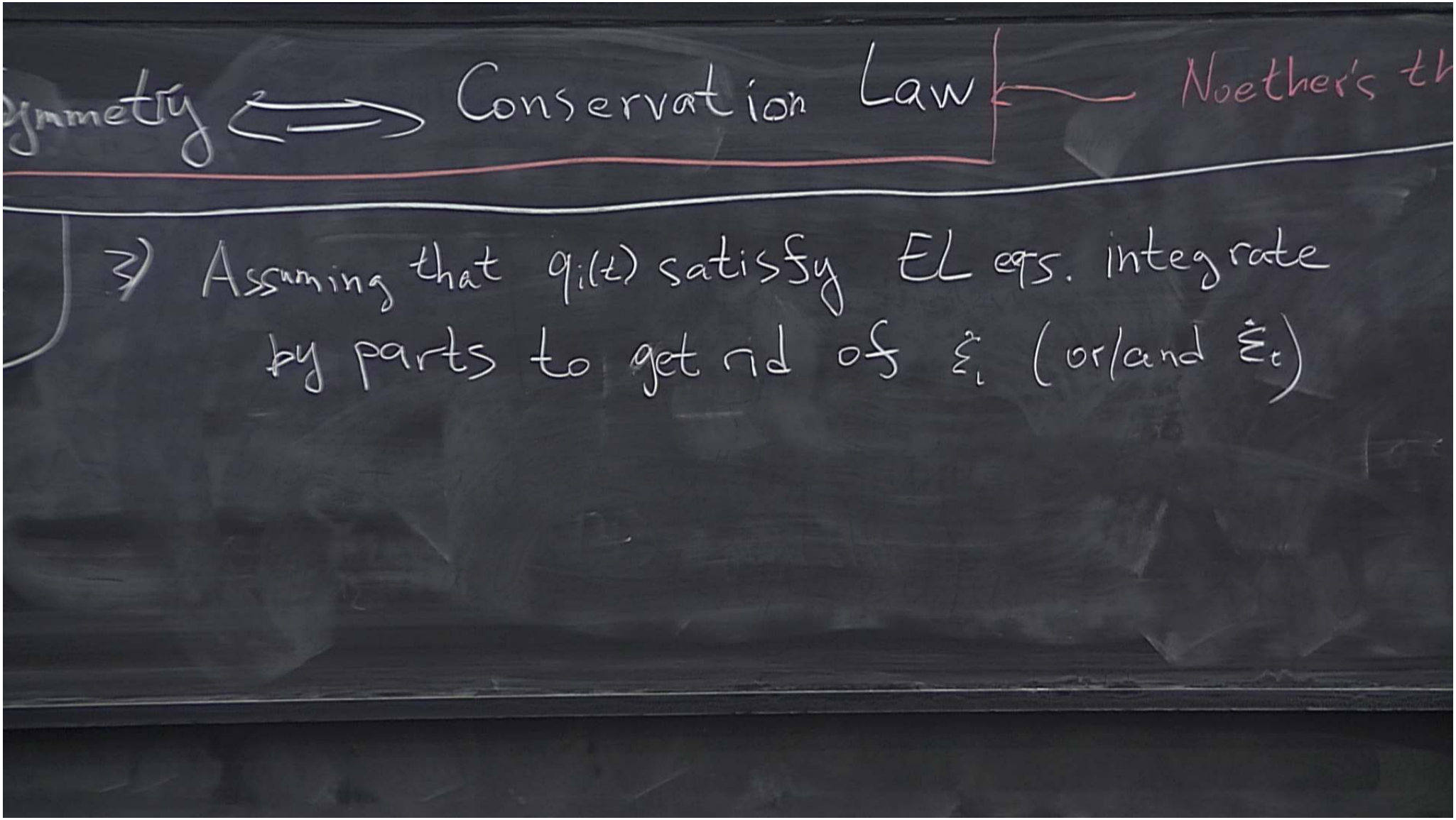
1) Observe that L is invariant under

$q_i \longrightarrow q_i + \varepsilon_i$ with $\varepsilon_i = \text{const}$ (or $t \longrightarrow t + \varepsilon_t$)

2) Obtain δS, under $q_i(t) \longrightarrow q_i(t) + \varepsilon_i(t)$

with $\varepsilon_i(t_i) = \varepsilon_i(t_f) = 0$

(or $q_i(t) \longrightarrow q_i(t) + \dot{q}_i\, \varepsilon_t(t)$)


ymmetry ⟺ Conservation Law ⟵ Noether's th
3) Assuming that $q_i(t)$ satisfy EL eqs. integrate by parts to get rid of ξ_i (or/and $\dot{\xi}_i$)

metry $\Longleftrightarrow$ Conservation Law $\longleftarrow$ Noether's theo

3) Assuming that $q_i(t)$ satisfy EL eqs. integrate by parts to get rid of $\dot{\xi}_i$ (or/and $\dot{\varepsilon}_t$)

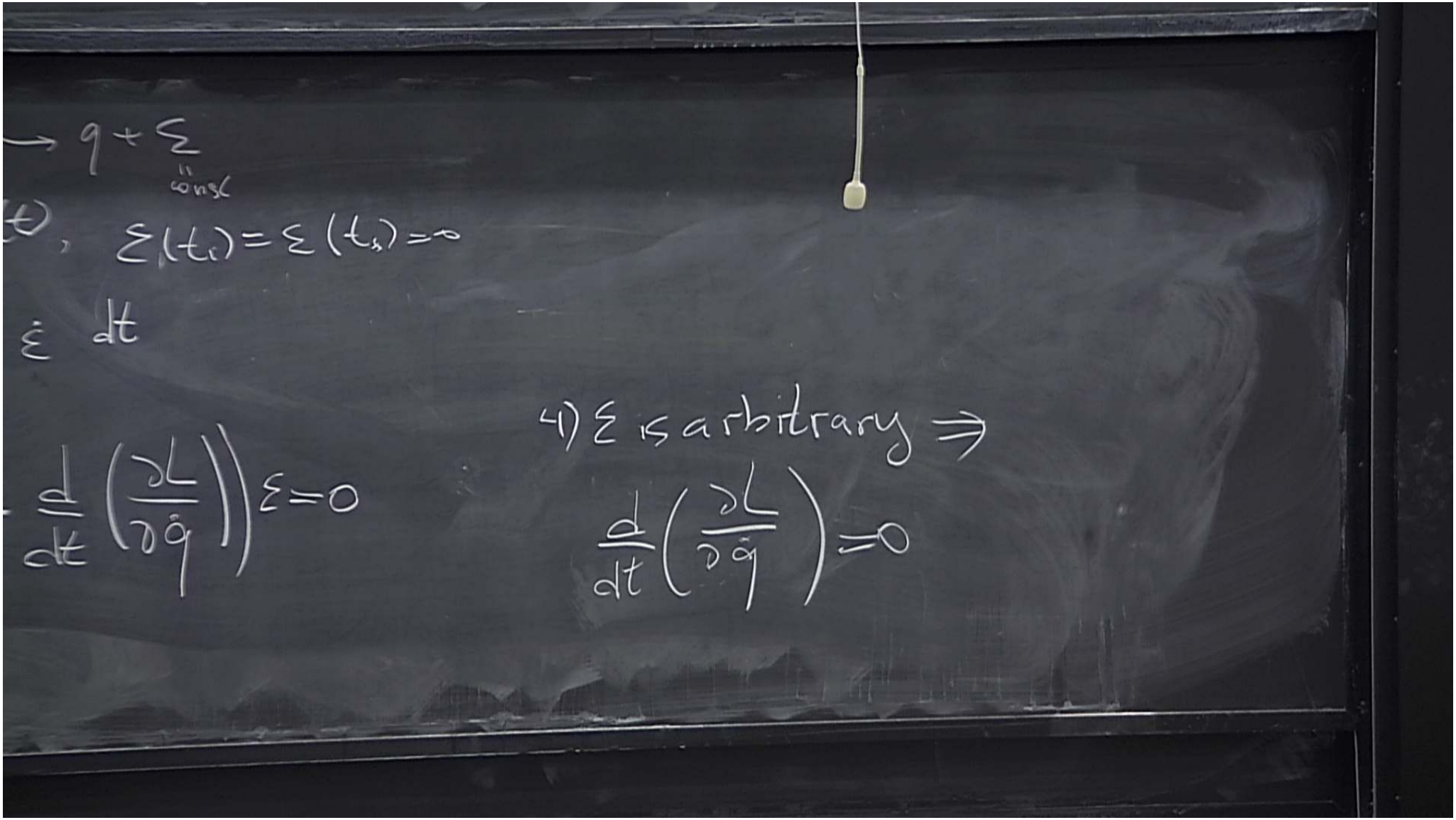
4) $\delta S = 0 \Rightarrow$ assuming that ξ_i (or ε_t) are arbitrary, equate to zero what stands before them.

$L = L(\dot{q})$ 1) Invariant if $q \to q + \varepsilon$ (const)

2) $q(t) \longrightarrow q(t) + \varepsilon(t)$, $\varepsilon(t_i) = \varepsilon(t_f) = 0$

$$\delta S = \int_{t_i}^{t_f} \frac{\partial L}{\partial \dot{q}} \dot{\varepsilon} \, dt$$

3) $$\delta S = \int_{t_i}^{t_f} dt \left(- \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \right) \varepsilon = 0$$


→ q + ε
const
t), ε(t₁) = ε(t₂) = 0
ε dt
d/dt (∂L/∂q̇)) ε = 0
4) ε is arbitrary ⇒
d/dt (∂L/∂q̇) = 0

$\rightarrow q + \varepsilon$ (const)

$t)$, $\varepsilon(t_i) = \varepsilon(t_s) = 0$

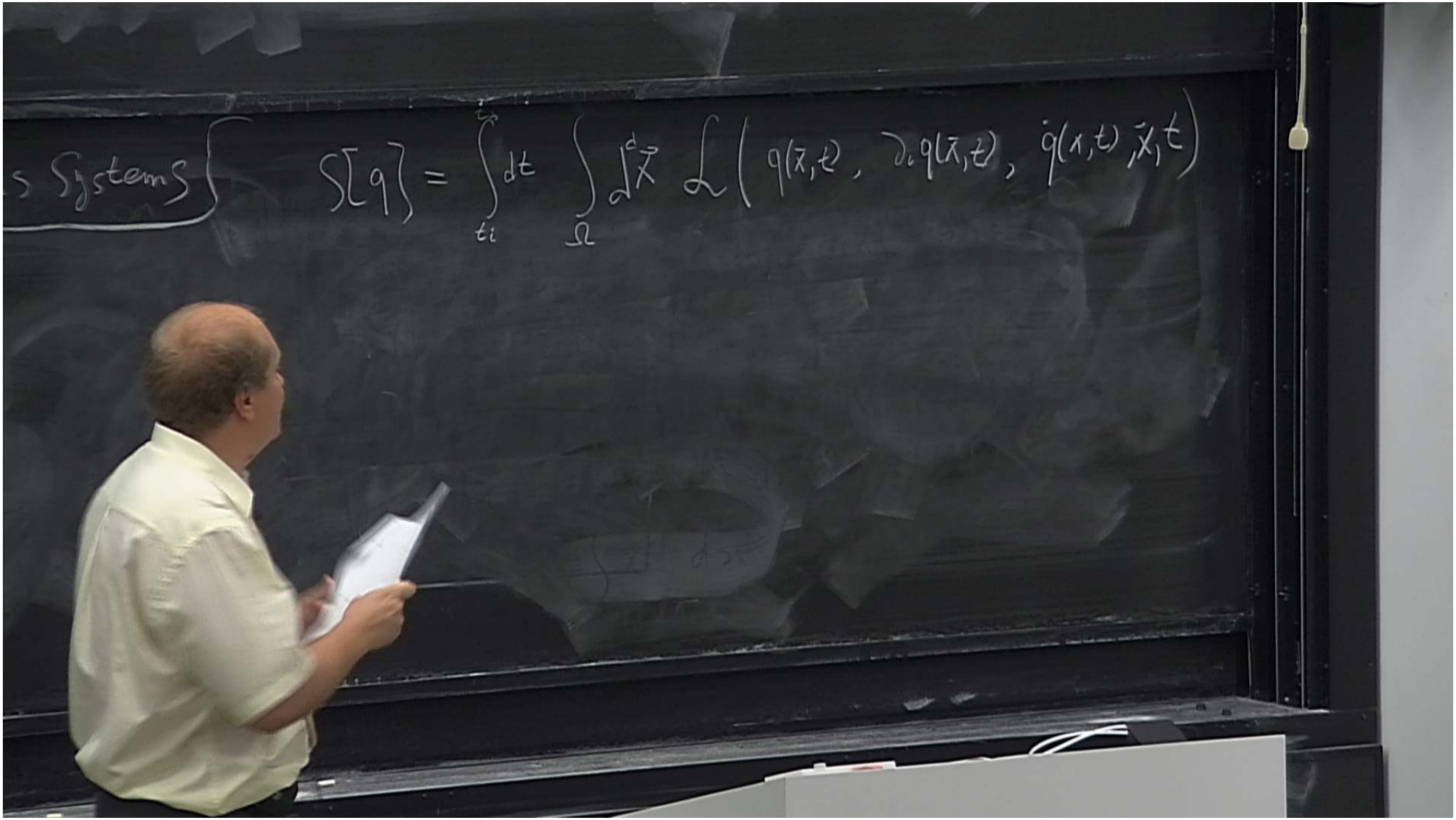
$\dot{\varepsilon}\ dt$

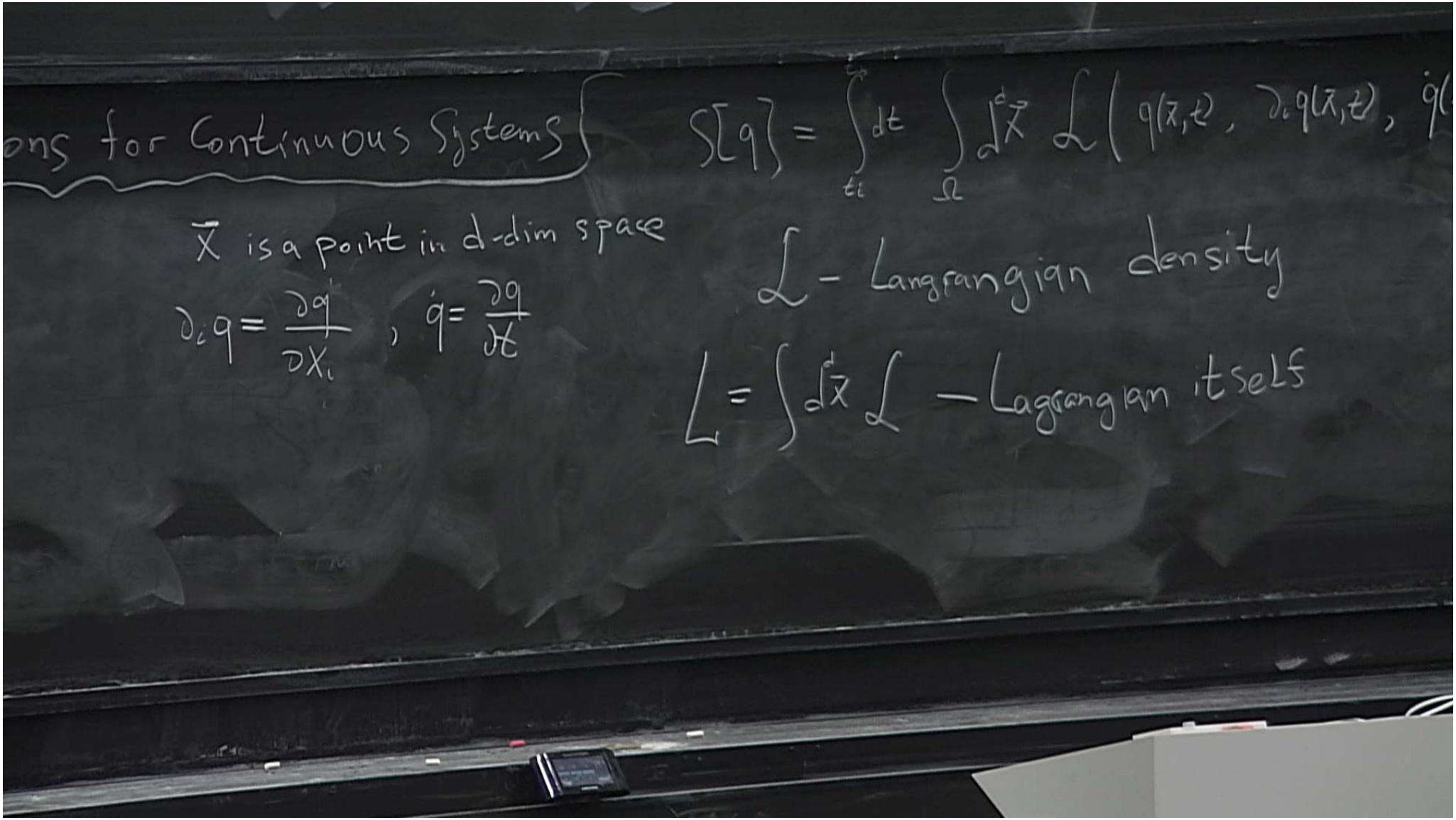
$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\varepsilon = 0$

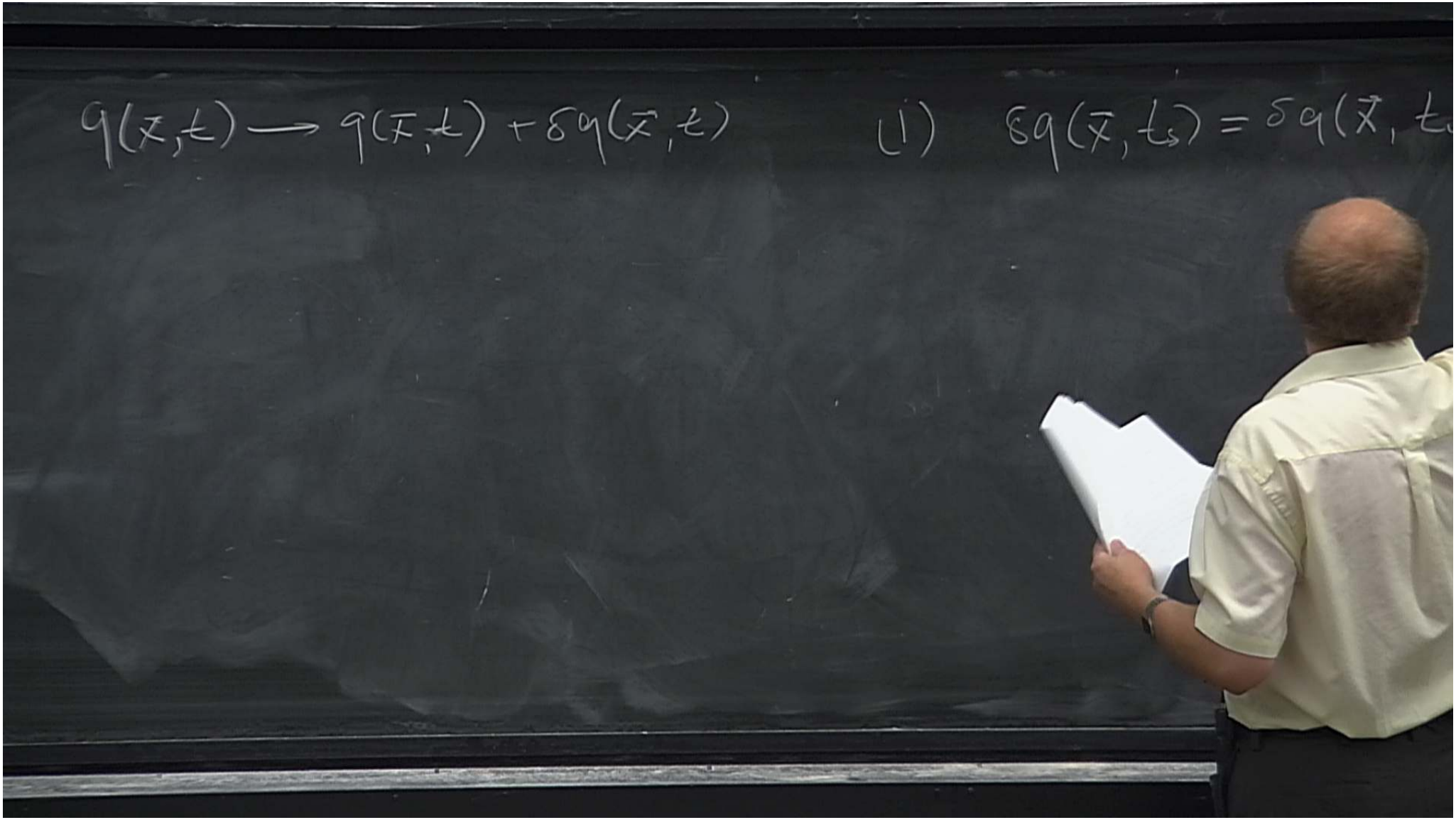
Ex: Apply the Noether's strategy to show that E is conserved for $L = L(q, \dot{q})$

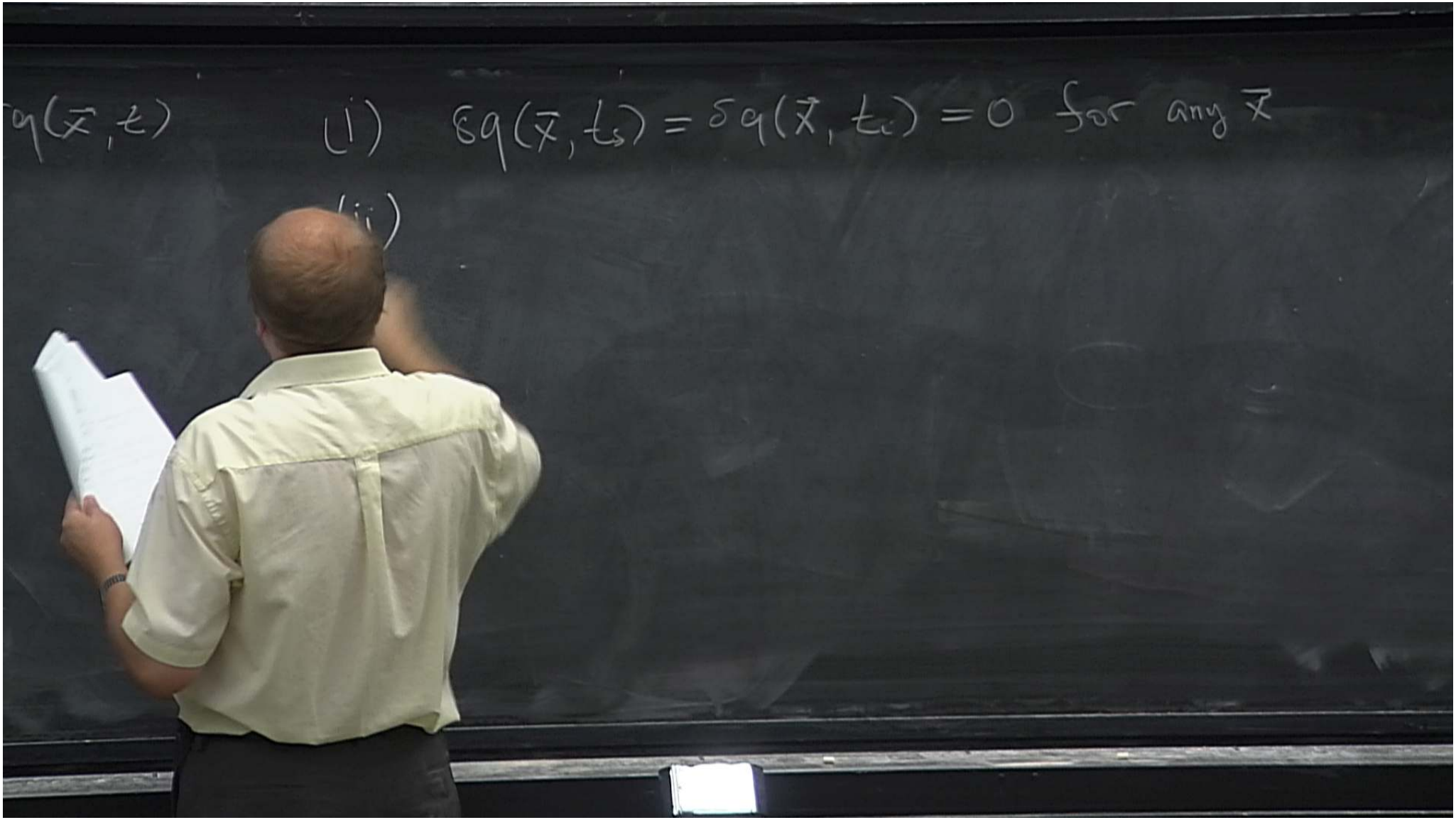
4) ε is arbitrary $\Rightarrow$

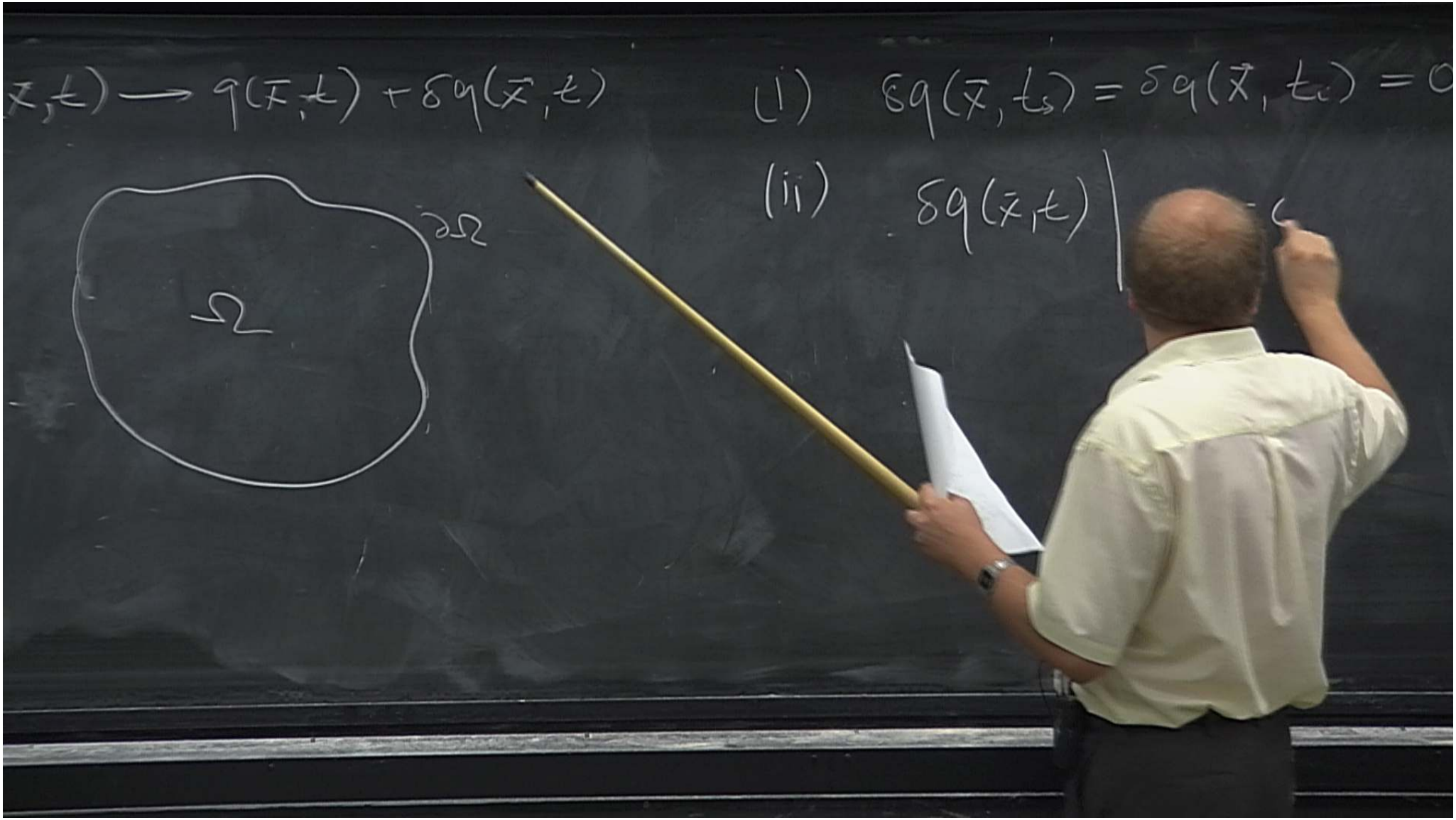
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0$$

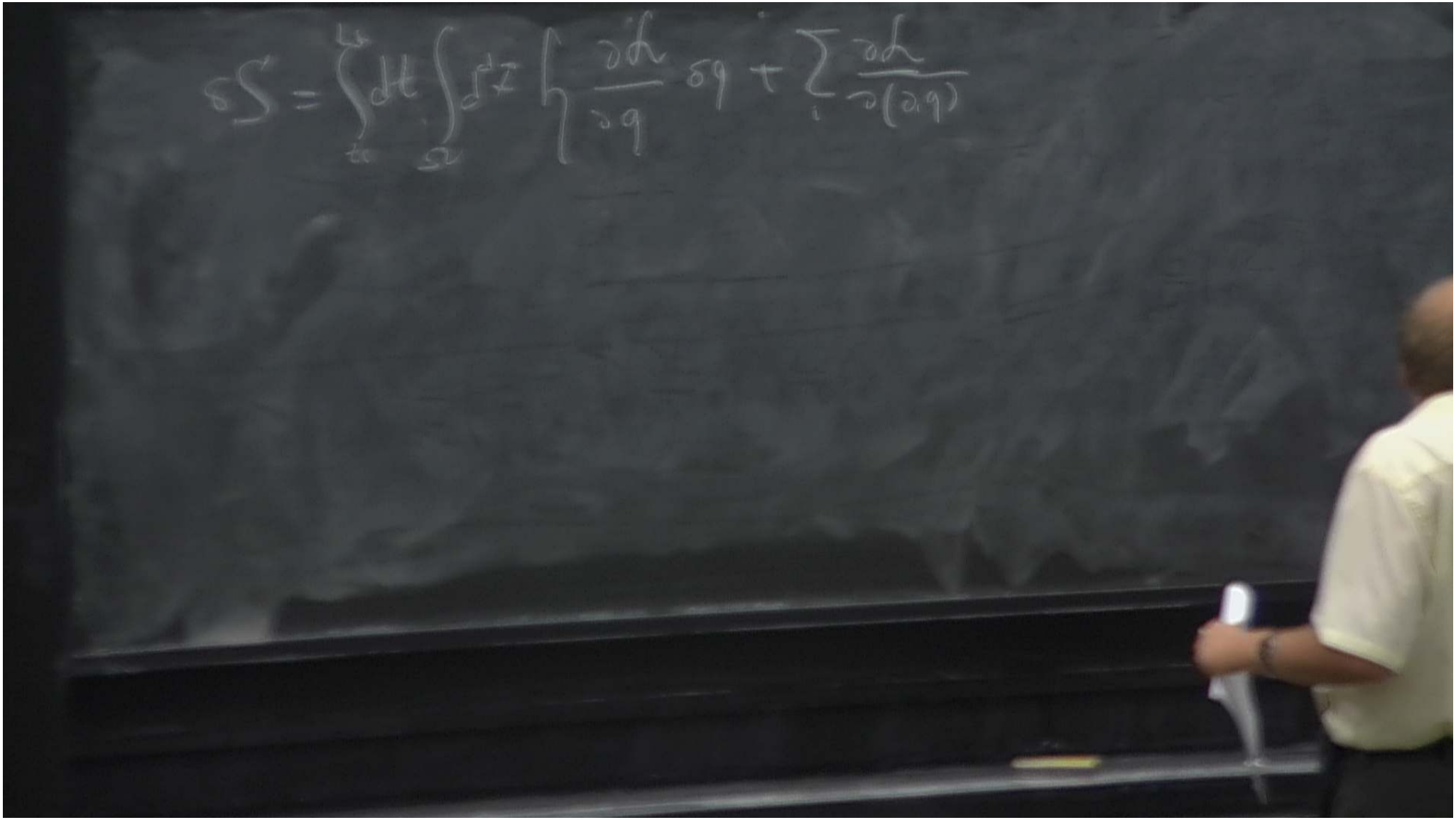

s Systems
$S[q] = \int_{t_i}^{t_f} dt \int_{\Omega} d^d\vec{x}\, \mathcal{L}\left(q(\vec{x},t),\ \partial_i q(\vec{x},t),\ \dot{q}(\vec{x},t),\ \vec{x},\ t\right)$

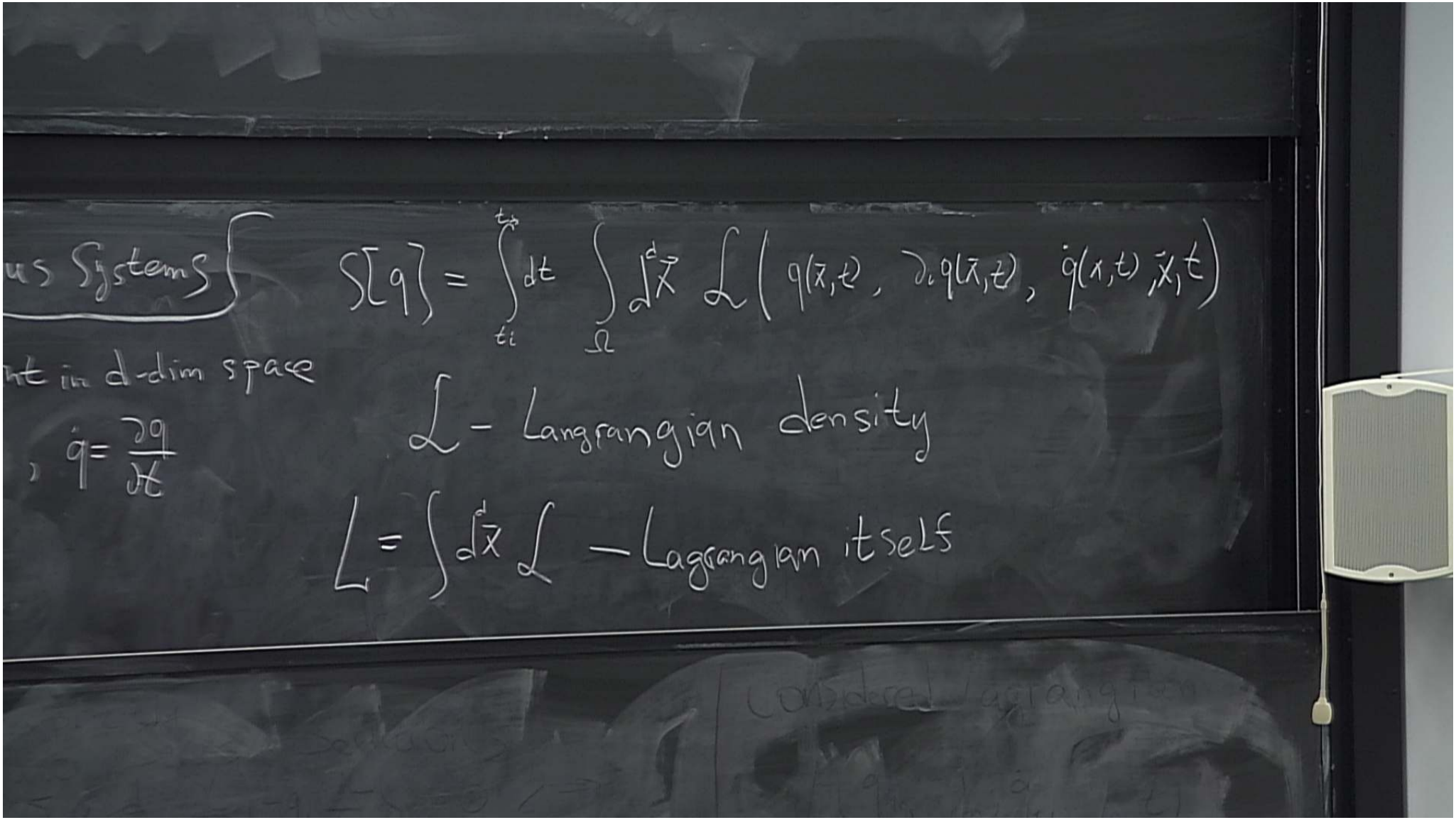

ons for Continuous Systems
$S[q] = \int_{t_i}^{t_f} dt \int_{\Omega} d^d\vec{x}\, \mathcal{L}(q(\vec{x},t), \partial_i q(\vec{x},t), \dot{q}($
$\vec{x}$ is a point in d-dim space
$\partial_i q = \frac{\partial q}{\partial x_i}$, $\dot{q} = \frac{\partial q}{\partial t}$
$\mathcal{L}$ - Lagrangian density
$L = \int d^d\vec{x}\, \mathcal{L}$ - Lagrangian itself


$q(x,t) \longrightarrow q(x,t) + \delta q(x,t)$
(1) $\delta q(x,t_1) = \delta q(x,t_2)$


q(x, t)
(i) δq(x, t_i) = δq(x, t_f) = 0 for any x
(ii)


$(x,t) \longrightarrow q(x,t) + \delta q(x,t)$
$\partial\Omega$
Ω
(i) $\delta q(x,t_0) = \delta q(x,t_1) = 0$
(ii) $\delta q(x,t)$


$\delta S = \int_{t_0}^{t_1} dt \int_{\Omega} d^3x \left(\frac{\partial \mathcal{L}}{\partial q} \delta q + \sum_i \frac{\partial \mathcal{L}}{\partial (\partial_i q)} \right)$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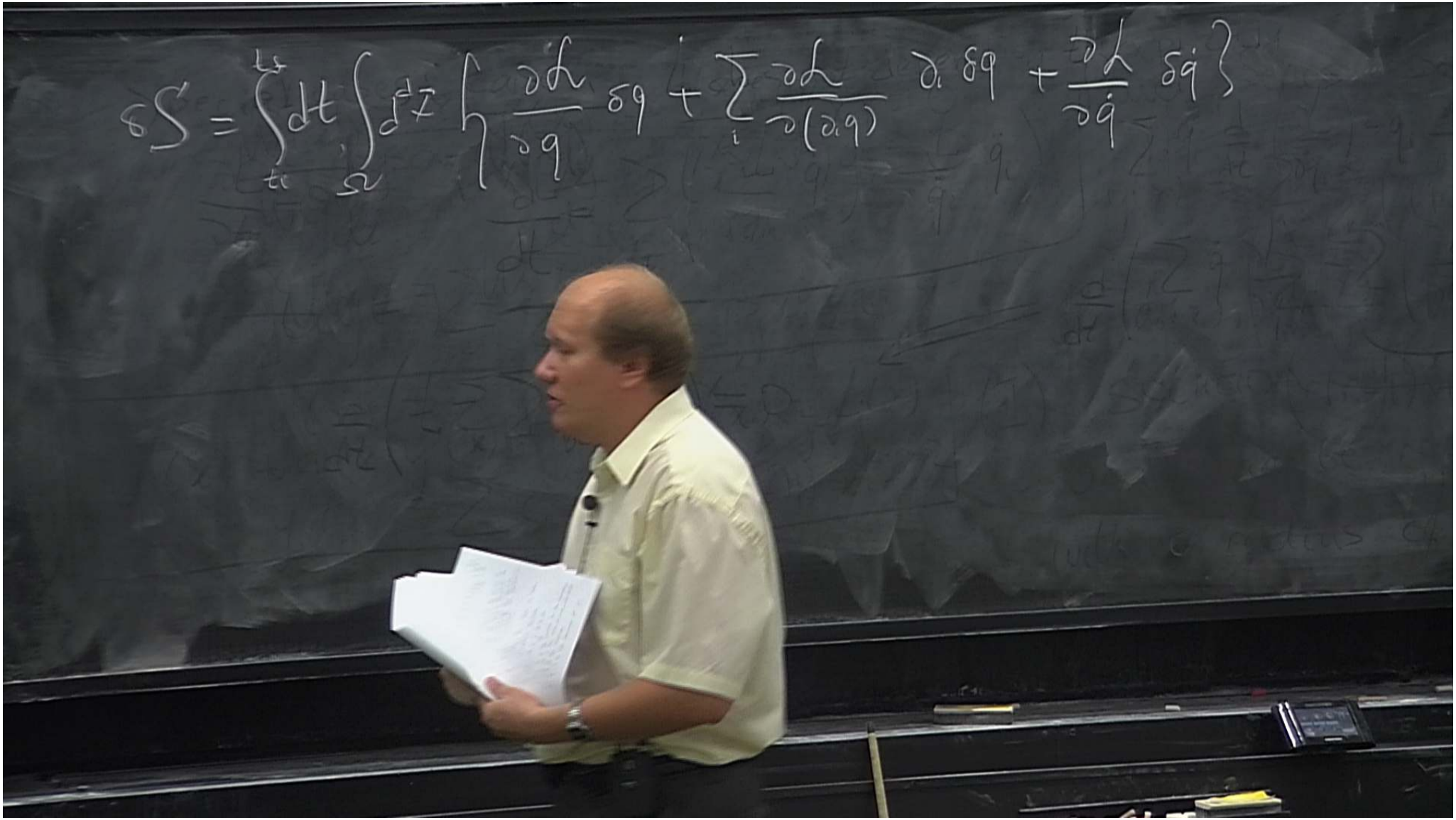
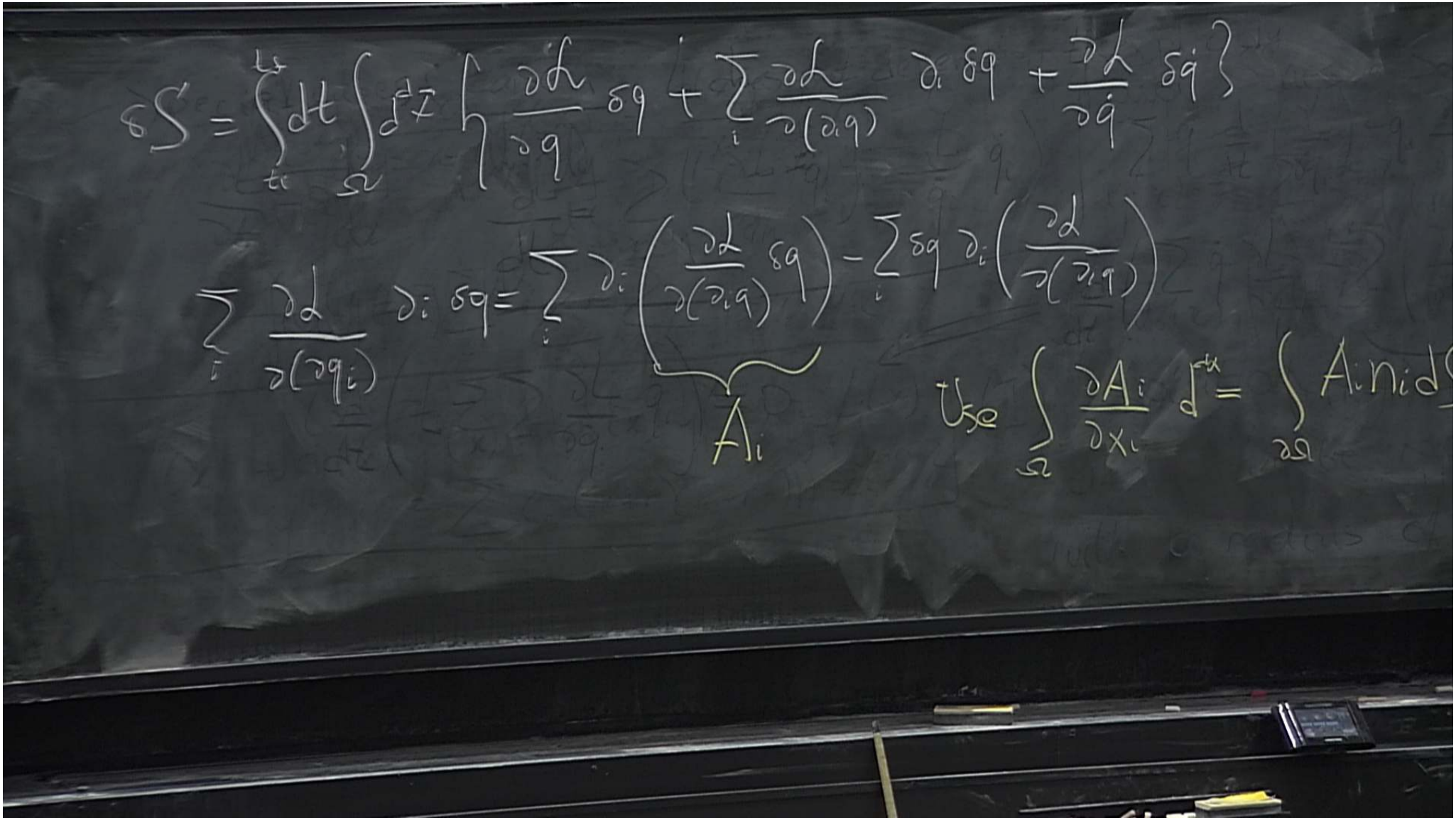
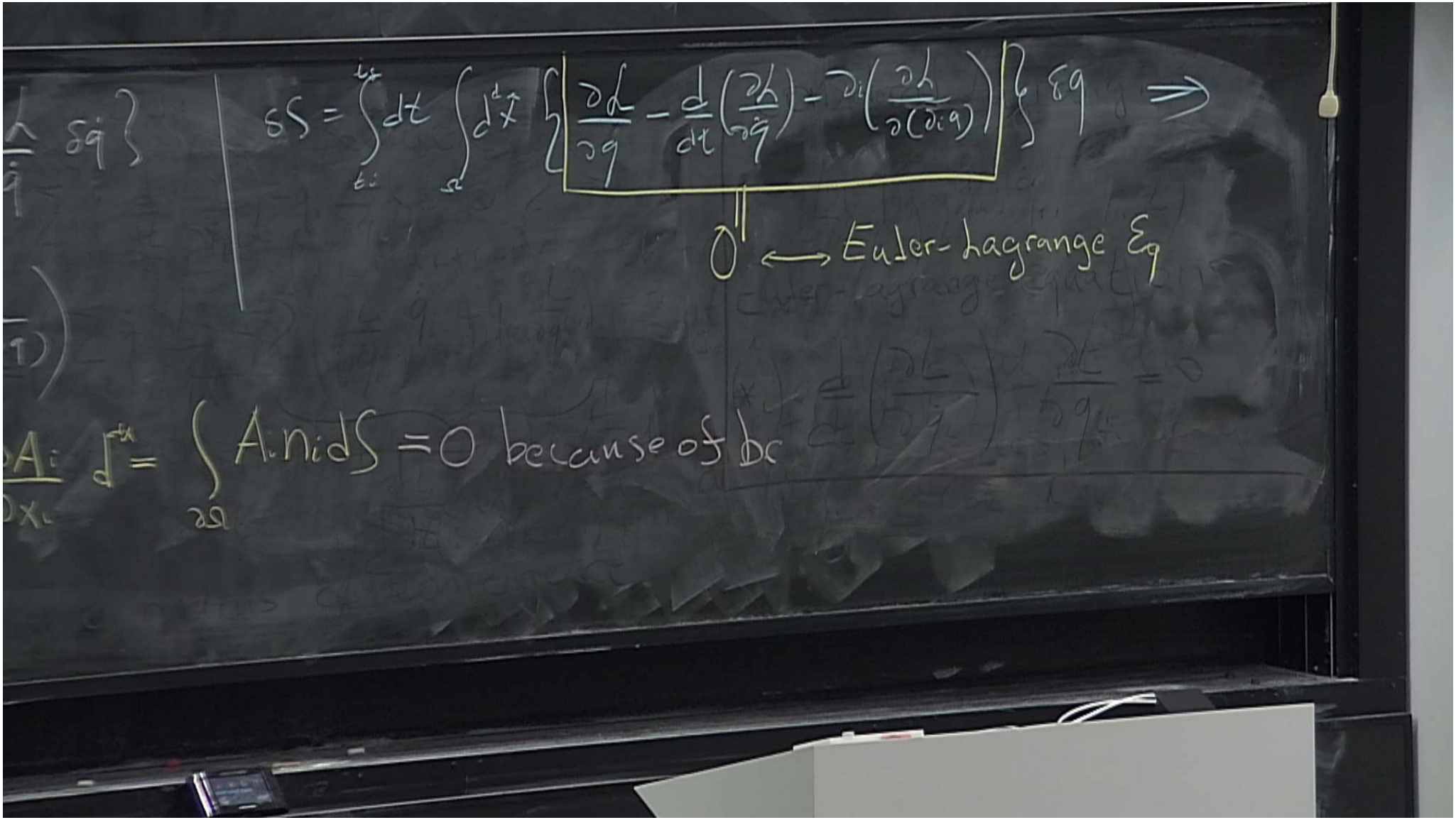
$\dot{q} = \frac{\partial q}{\partial t}$
$\mathcal{L}$ - Langrangian density
$L = \int d^d\vec{x}\, \mathcal{L}$ - Lagrangian itself


$\delta S = \int_{t_i}^{t_f} dt \int_{\Omega} d^3x \left\{ \frac{\partial \mathcal{L}}{\partial q} \delta q + \sum_i \frac{\partial \mathcal{L}}{\partial(\partial_i q)} \partial_i \delta q + \frac{\partial \mathcal{L}}{\partial \dot{q}} \delta \dot{q} \right\}$


$\delta S = \int_{t_i}^{t_f} dt \int_\Omega d^3x \left\{ \frac{\partial \mathcal{L}}{\partial q} \delta q + \sum_i \frac{\partial \mathcal{L}}{\partial(\partial_i q)} \partial_i \delta q + \frac{\partial \mathcal{L}}{\partial \dot{q}} \delta \dot{q} \right\}$
$\sum_i \frac{\partial \mathcal{L}}{\partial(\partial_i q_i)} \partial_i \delta q = \sum_i \partial_i \underbrace{\left(\frac{\partial \mathcal{L}}{\partial(\partial_i q)} \delta q \right)}_{A_i} - \sum_i \delta q \, \partial_i \left(\frac{\partial \mathcal{L}}{\partial(\partial_i q)} \right)$
Use $\int_\Omega \frac{\partial A_i}{\partial x_i} d^4x = \int_{\partial\Omega} A_i n_i d$


$\delta S = \int_{t_i}^{t_f} dt \int_{\Omega} d^3x \left\{ \frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \partial_i\left(\frac{\partial \mathcal{L}}{\partial(\partial_i q)}\right) \right\} \delta q \Rightarrow$
$0 \longleftrightarrow$ Euler-Lagrange Eq
$\int = \int_{\partial\Omega} A_i n_i dS = 0$ because of bc

$= 0$ for any $\vec{x}$

for any t

Remark: $X_\mu = (\vec{x}, t)$ – dtl vector

$$S[q] = \int d^{d+1}x \, \mathcal{L}(q, \partial_\mu q, x_\mu)$$

$$\boxed{\frac{\partial \mathcal{L}}{\partial q} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu q)} \right) = 0}$$

Euler–Lagrange Eq.

$= 0$ for any $\vec{x}$

for any t

Remark: $X_\mu = (\vec{x}, t)$ – dtl vector

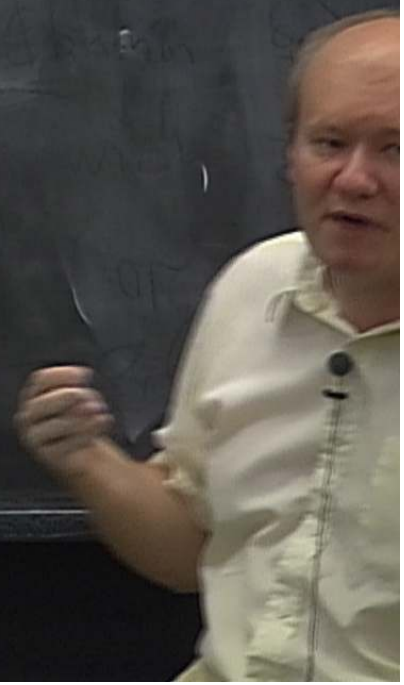
$$S[q] = \int d^{d+1}x \, \mathcal{L}(q, \partial_\mu q, x_\mu)$$

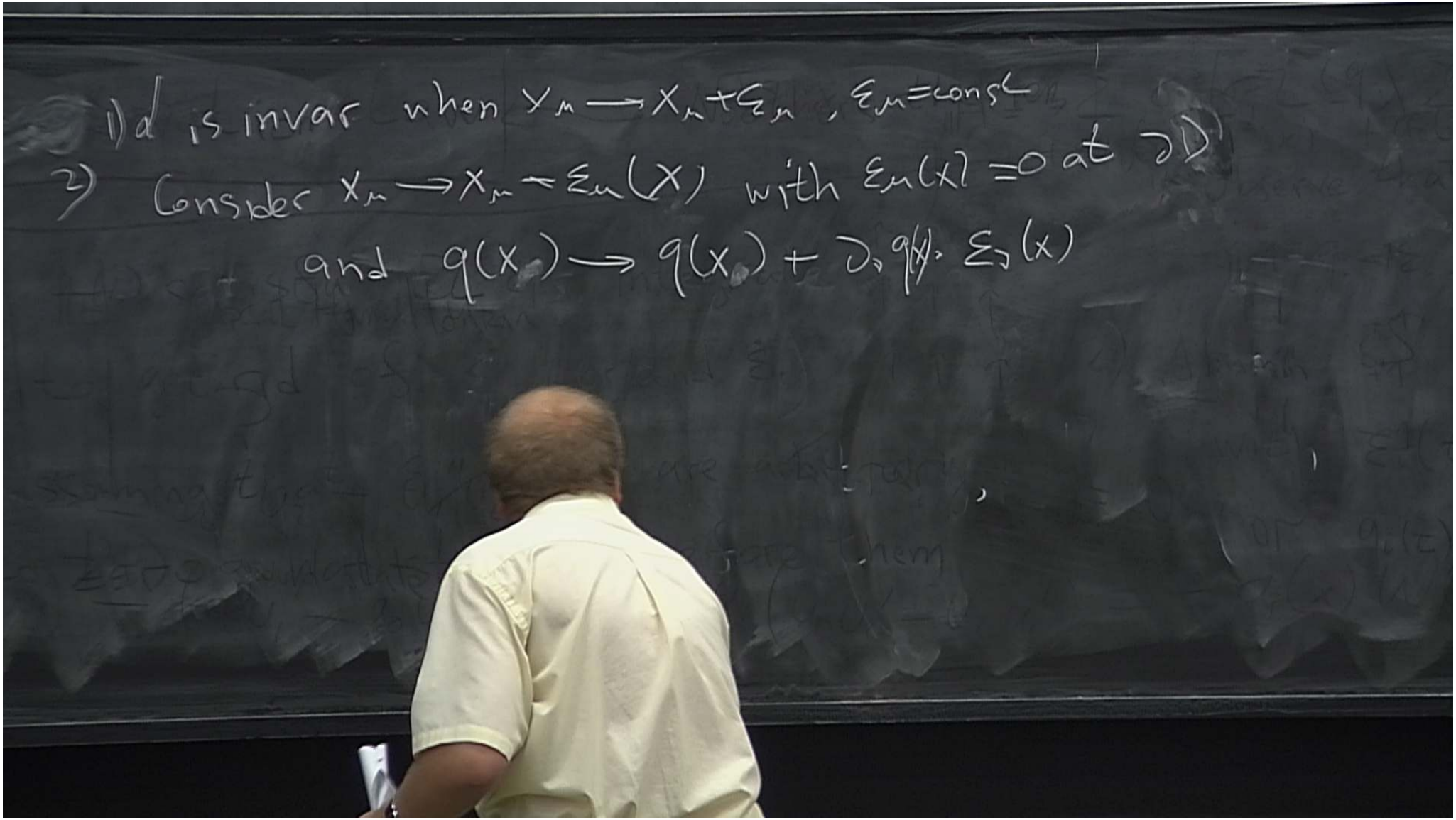
$$\boxed{\frac{\partial \mathcal{L}}{\partial q} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu q)} \right) = 0}$$

Euler–Lagrange Eq.

1) d is invar when $x_\mu \to x_\mu + \varepsilon_\mu$, $\varepsilon_\mu = \text{const}$

2) Consider $x_\mu \to x_\mu - \varepsilon_\mu(x)$ with $\varepsilon_\mu(x) = 0$ at $\partial\mathcal{D}$




1) d is invar when $y_m \to x_m + \varepsilon_m$, $\varepsilon_m = \text{const}$
2) Consider $x_m \to x_m - \varepsilon_m(x)$ with $\varepsilon_m(x) = 0$ at ∂D
and $q(x_m) \to q(x_m) + \partial_j q(x_m) \cdot \varepsilon_j(x)$

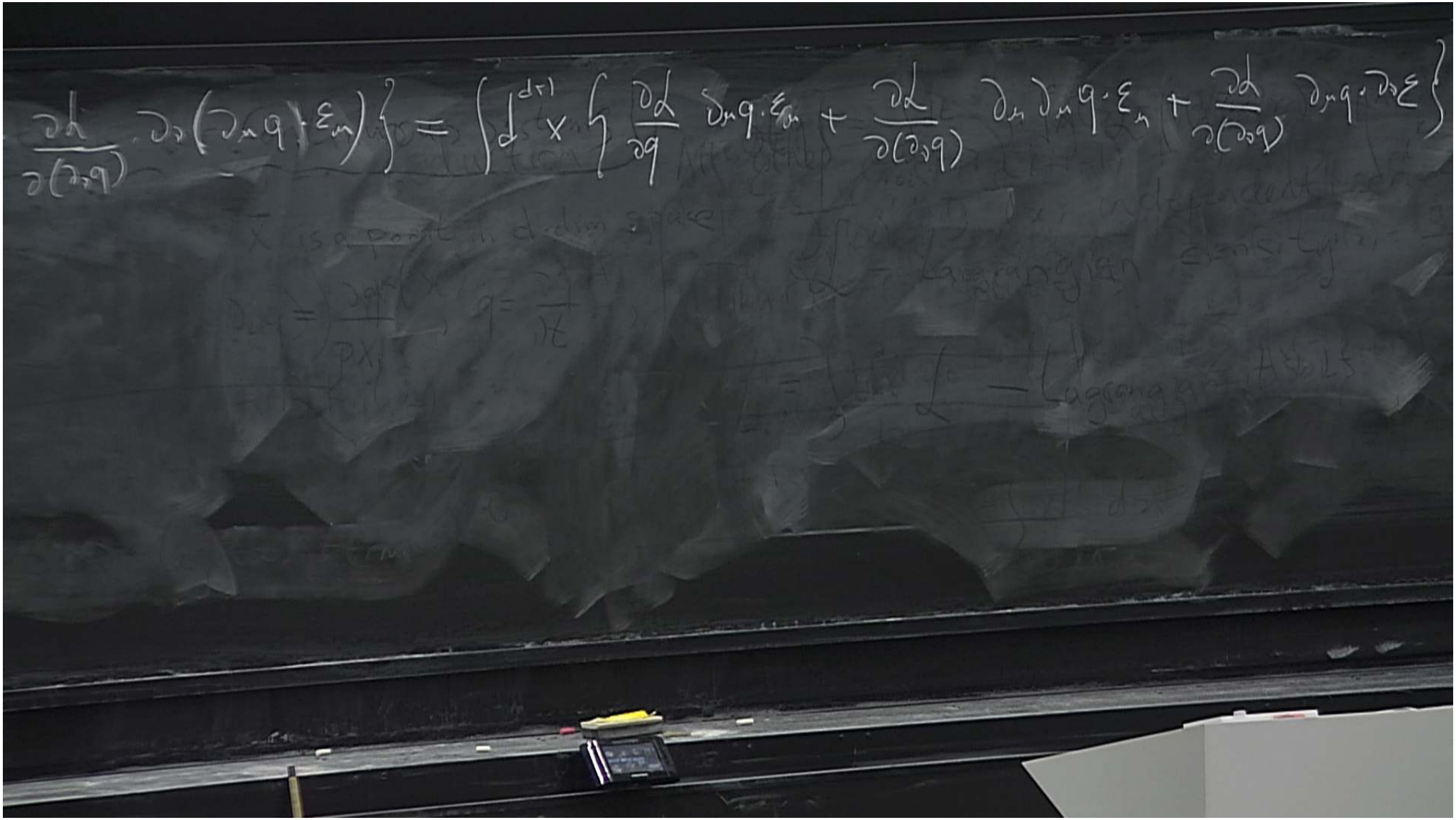
1) d is invar when $y_\mu \to x_\mu + \varepsilon_\mu$, $\varepsilon_\mu = \text{const}$

2) Consider $x_\mu \to x_\mu - \varepsilon_\mu(x)$ with $\varepsilon_\mu(x) = 0$ at $\partial\mathcal{D}$

and $q(x_\mu) \to q(x_\mu) + \partial_\nu q(x_\mu) \cdot \varepsilon_\nu(x)$

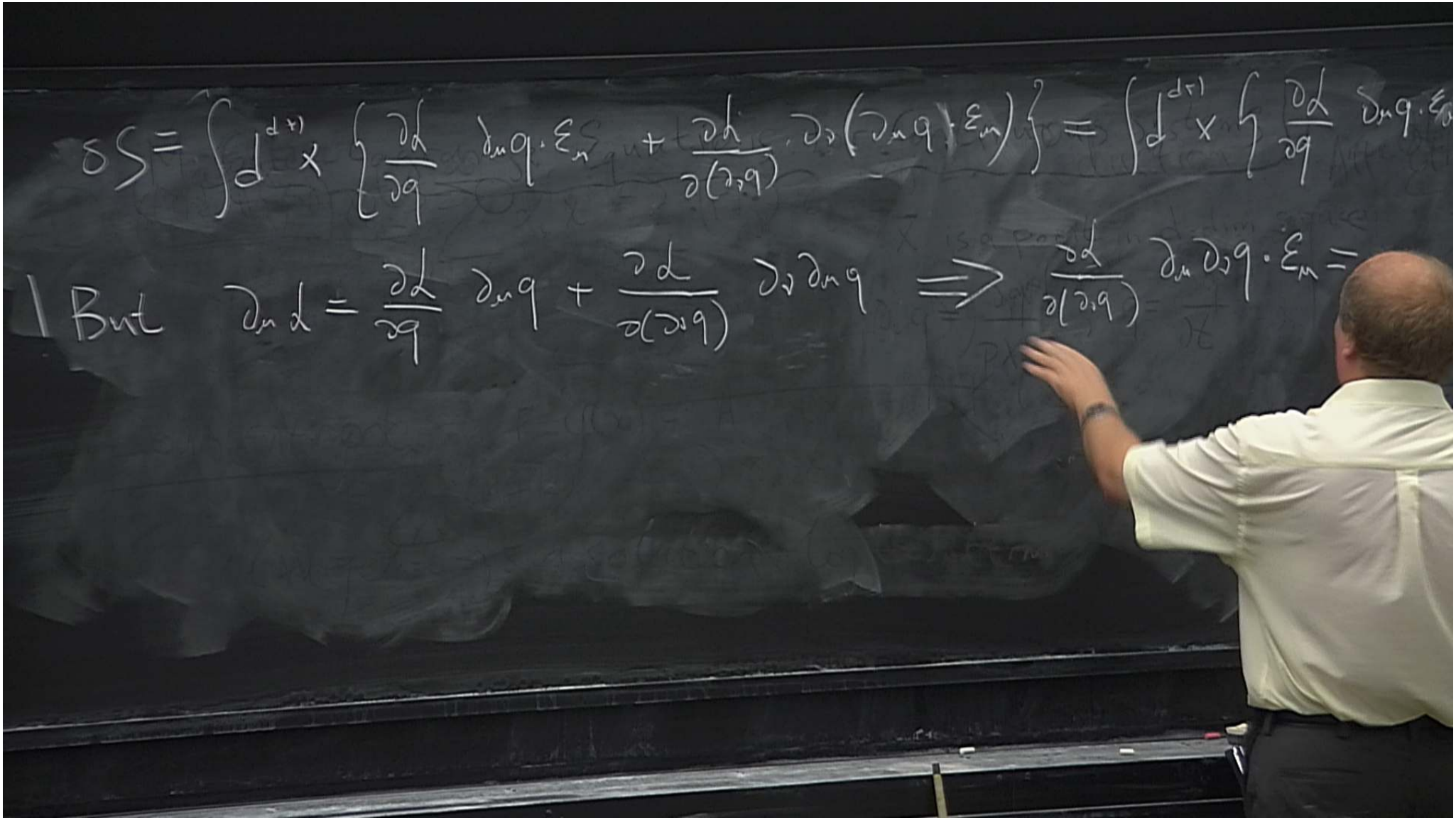
Einstein summation

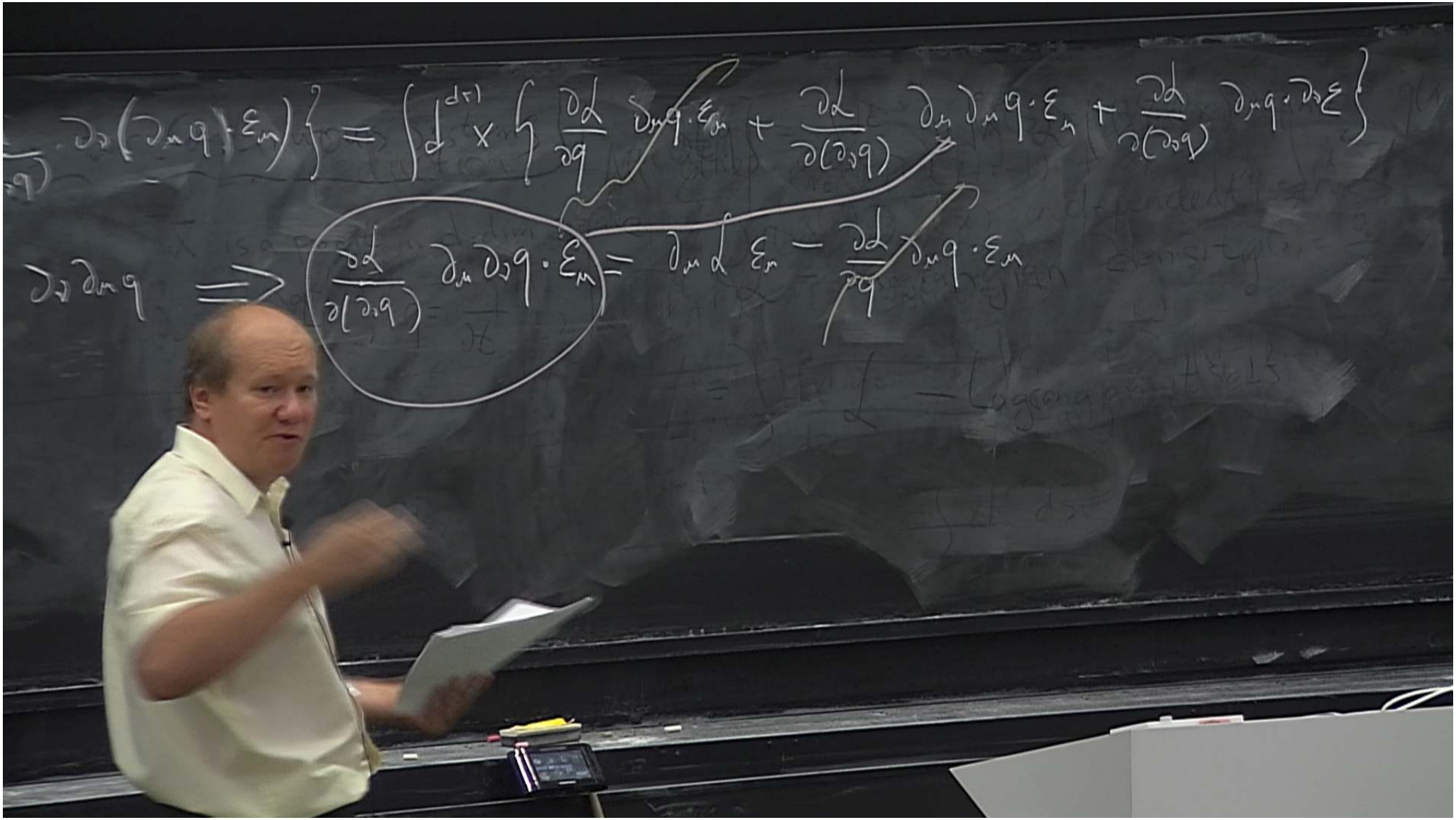
$$\delta S = \int d^{d+1}x \left\{ \frac{\partial \mathcal{L}}{\partial q} \partial_\mu q \cdot \varepsilon_n + \frac{\partial \mathcal{L}}{\partial(\partial_\nu q)} \cdot \partial_\nu (\partial_\mu q \cdot \varepsilon_n) \right\} = \int d^{d+1}x \left\{ \frac{\partial \mathcal{L}}{\partial q} \right.$$

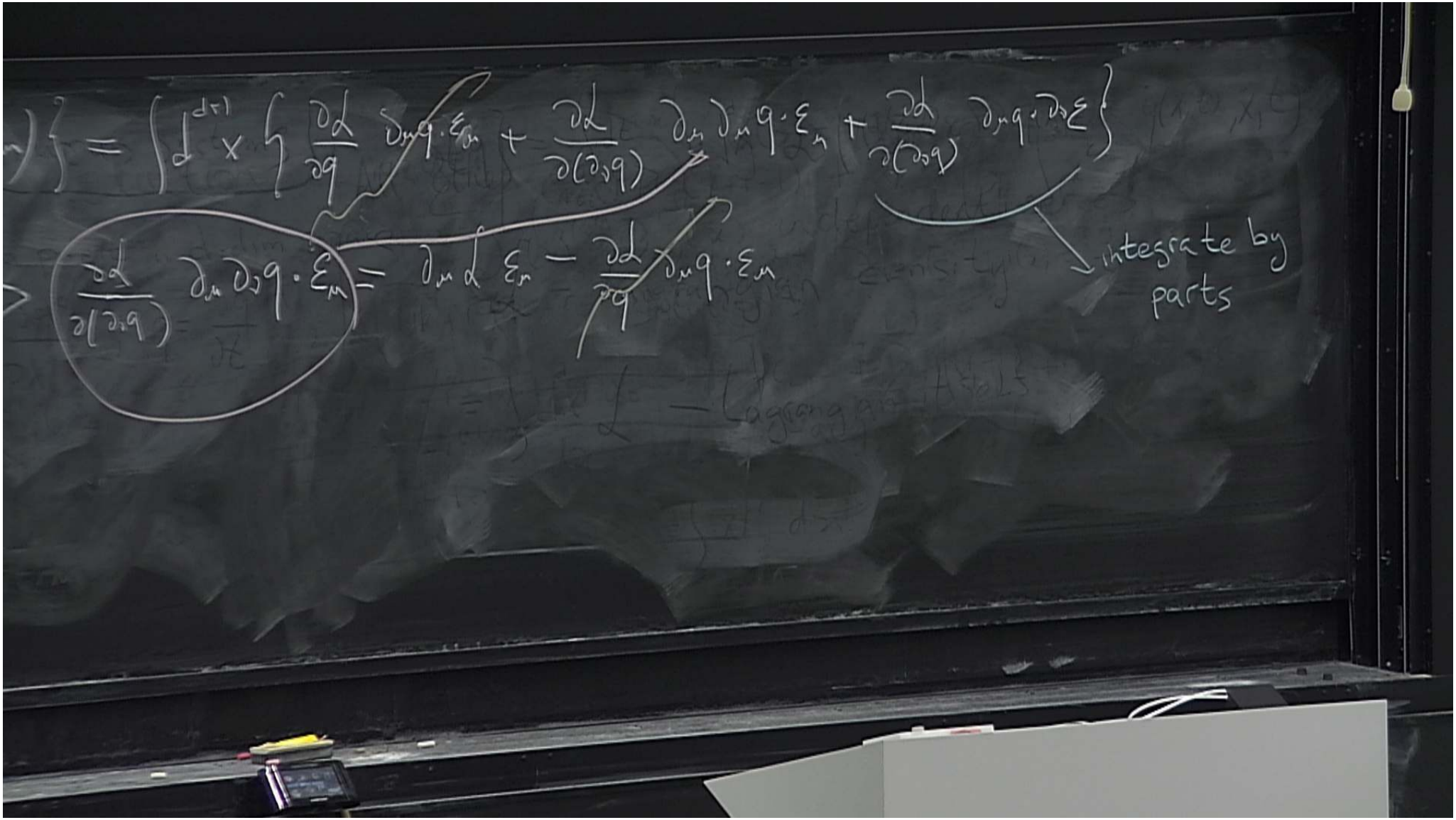


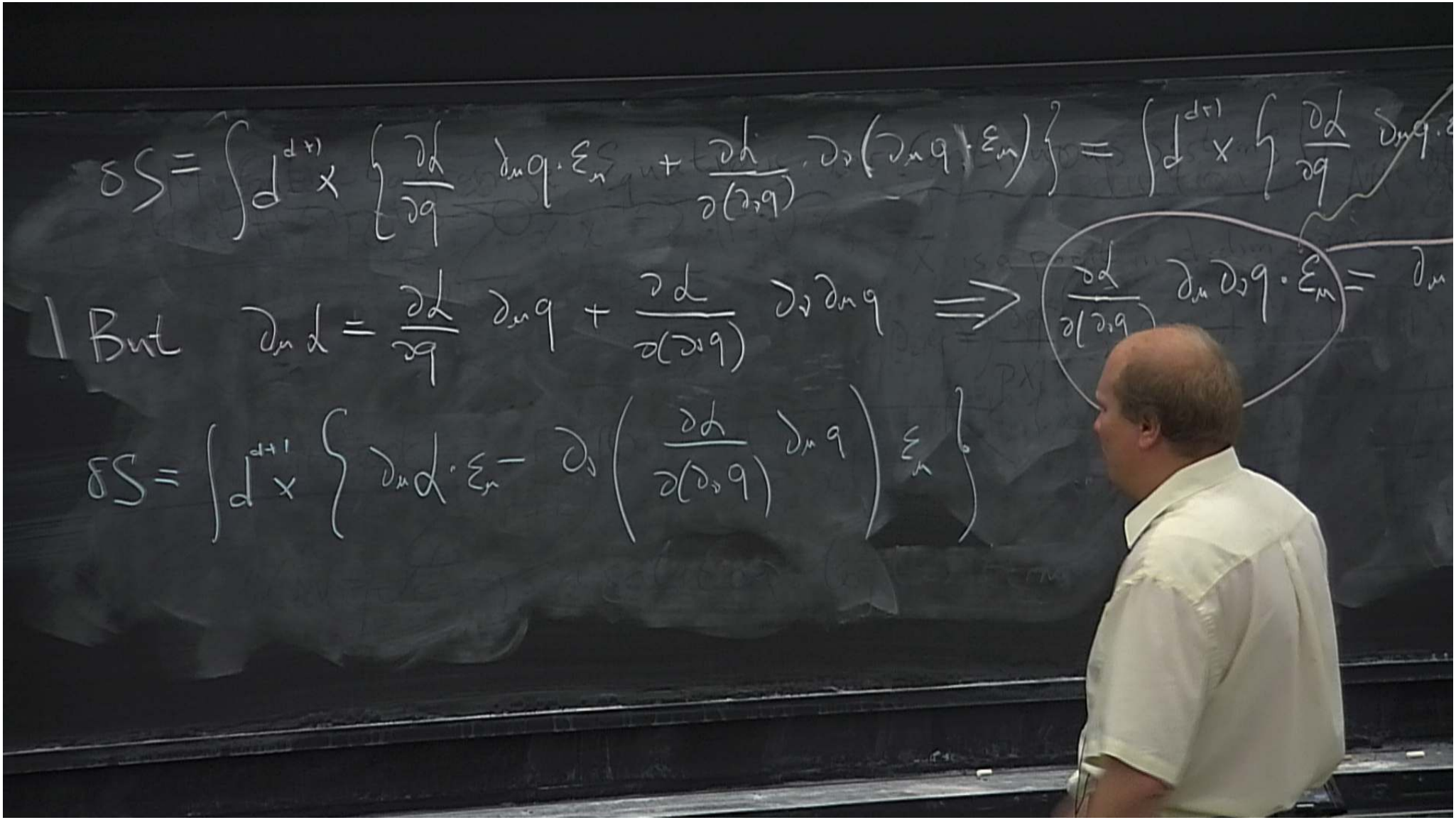
$$\delta S = \int d^{d+1}x \left\{ \frac{\partial \mathcal{L}}{\partial q} \partial_\mu q \cdot \varepsilon_n + \frac{\partial \mathcal{L}}{\partial(\partial_\nu q)} \cdot \partial_\nu (\partial_\mu q \cdot \varepsilon_n) \right\} = \int d^{d+1}x \left\{ \frac{\partial \mathcal{L}}{\partial q} \partial_\mu q \cdot \varepsilon_n \right.$$

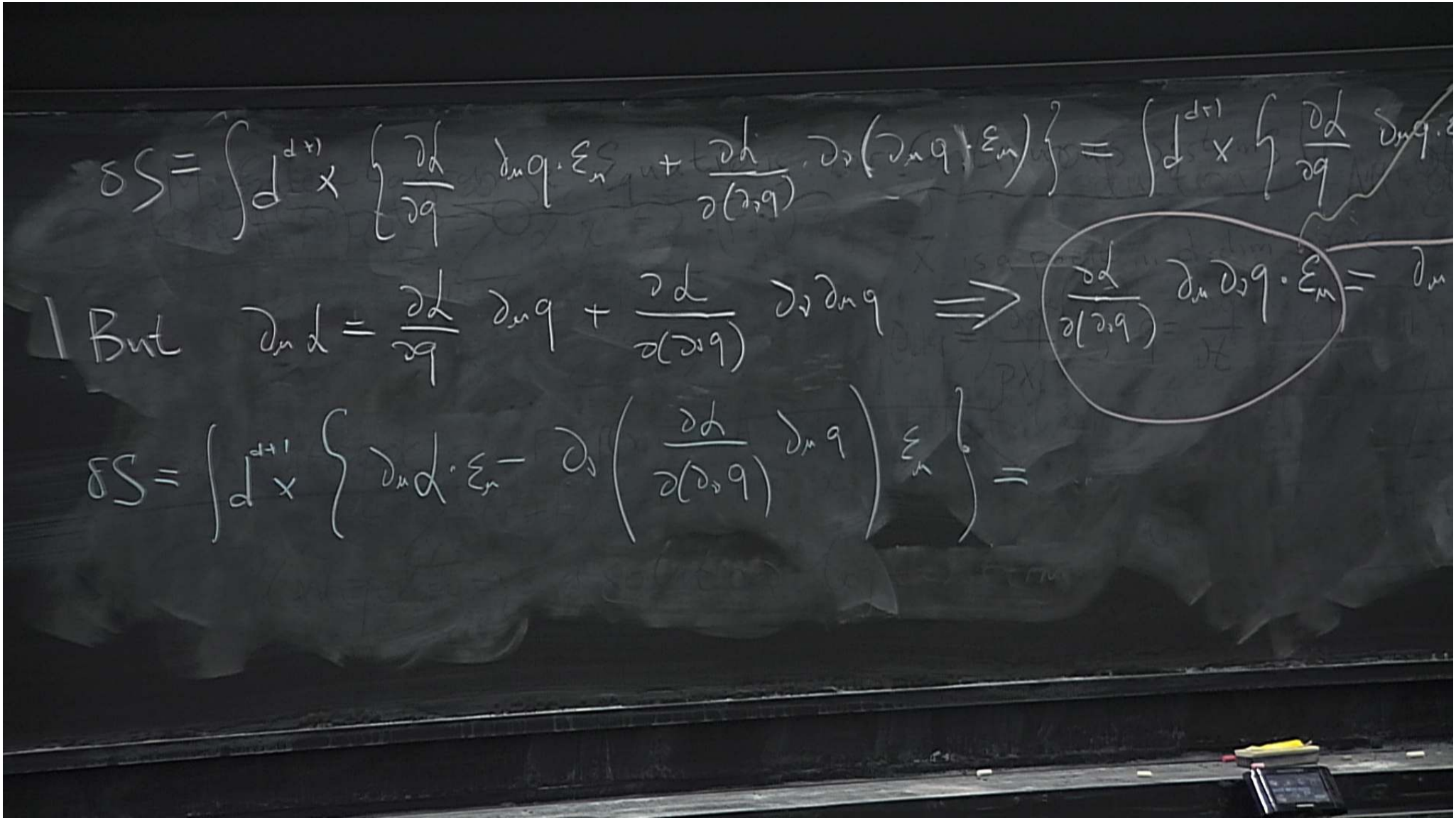
But $\partial_\mu \mathcal{L} = \frac{\partial \mathcal{L}}{\partial q} \partial_\mu q + \frac{\partial \mathcal{L}}{\partial(\partial_\nu q)} \partial_\nu \partial_\mu q$

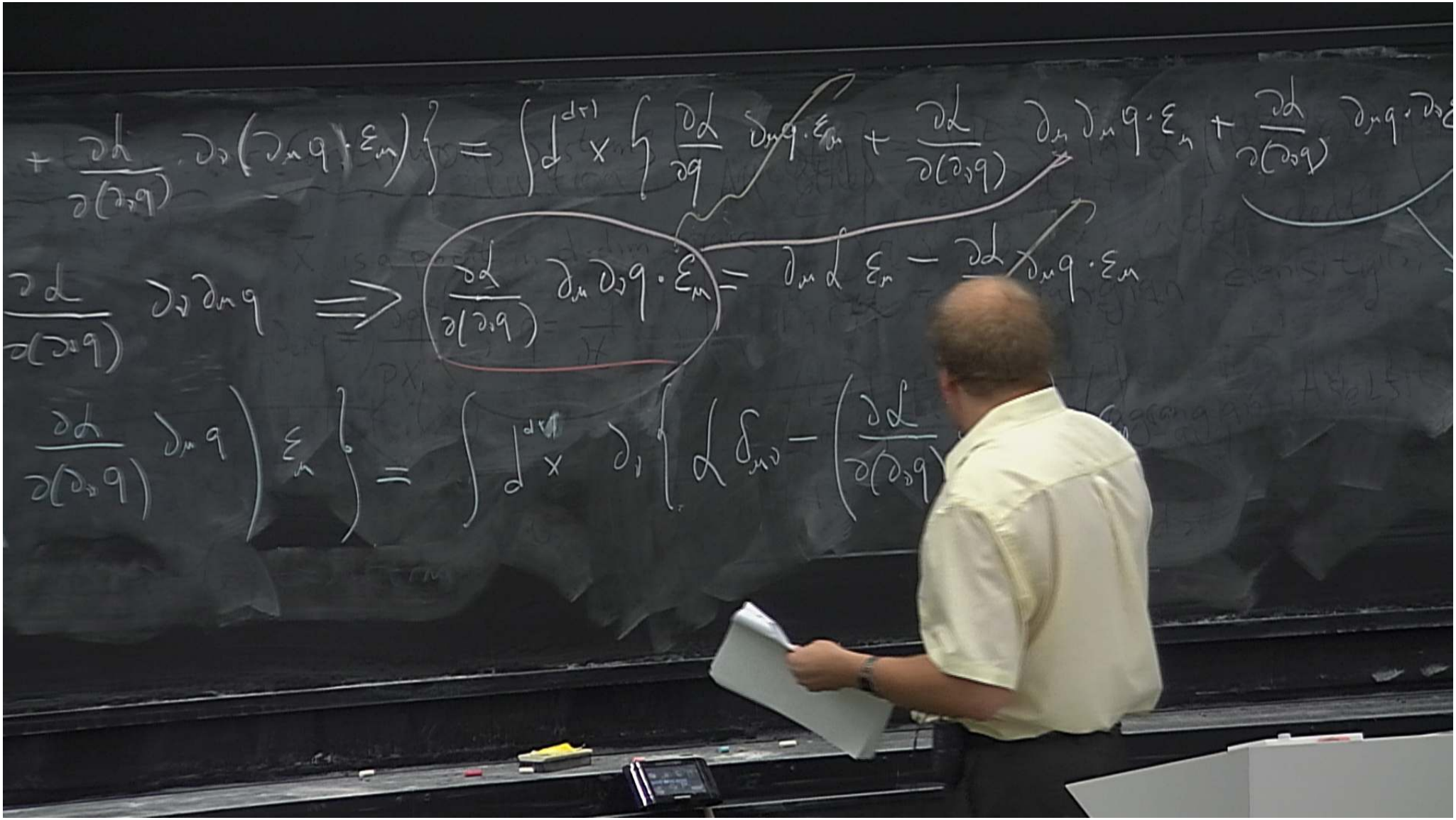

$\delta S = \int d^{d+1}x \left\{ \frac{\partial \mathcal{L}}{\partial q} \partial_\mu q \cdot \varepsilon_n + \frac{\partial \mathcal{L}}{\partial(\partial_\nu q)} \cdot \partial_\nu (\partial_\mu q \cdot \varepsilon_n) \right\} = \int d^{d+1}x \left\{ \frac{\partial \mathcal{L}}{\partial q} \partial_\mu q \cdot \varepsilon$
But $\partial_\mu \mathcal{L} = \frac{\partial \mathcal{L}}{\partial q} \partial_\mu q + \frac{\partial \mathcal{L}}{\partial(\partial_\nu q)} \partial_\nu \partial_\mu q \Rightarrow \frac{\partial \mathcal{L}}{\partial(\partial_\nu q)} \partial_\mu \partial_\nu q \cdot \varepsilon_n =$

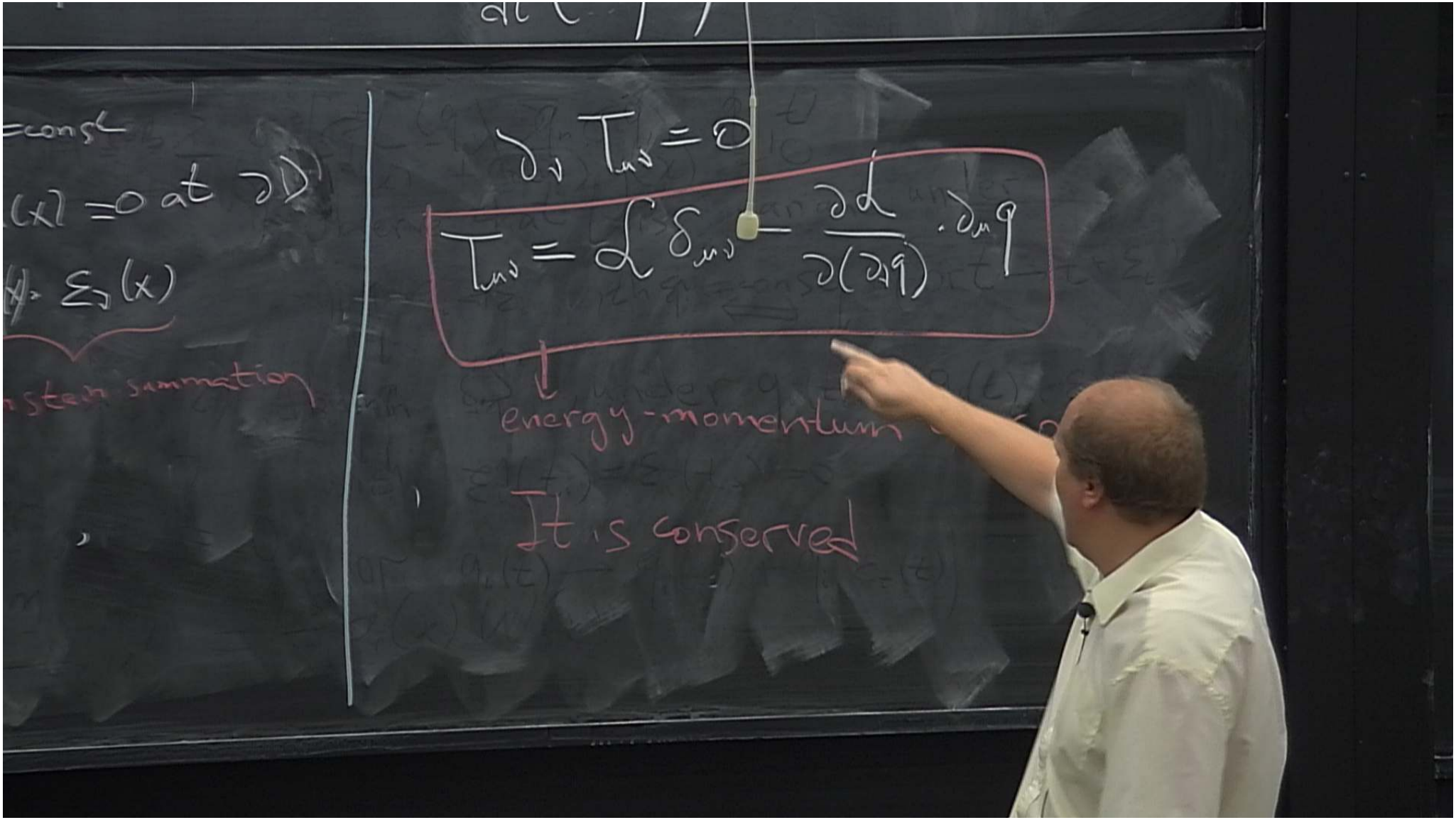

$\partial_\nu(\partial_\mu q \cdot \varepsilon_\mu)\Big\} = \int d^{d+1}x \Big\{ \frac{\partial \mathcal{L}}{\partial q} \partial_\mu q \cdot \varepsilon_\mu + \frac{\partial \mathcal{L}}{\partial(\partial_\nu q)} \partial_\mu \partial_\mu q \cdot \varepsilon_\mu + \frac{\partial \mathcal{L}}{\partial(\partial_\nu q)} \partial_\mu q \cdot \partial_\nu \varepsilon \Big\}$
$\partial_\nu \partial_\mu q \Longrightarrow \frac{\partial \mathcal{L}}{\partial(\partial_\nu q)} \partial_\mu \partial_\nu q \cdot \varepsilon_\mu = \partial_\mu \mathcal{L}\, \varepsilon_\mu - \frac{\partial \mathcal{L}}{\partial q} \partial_\mu q \cdot \varepsilon_\mu$


integrate by parts


$\delta S = \int d^{d+1}x \left\{ \frac{\partial \mathcal{L}}{\partial q} \partial_\mu q \cdot \varepsilon_\mu + \frac{\partial \mathcal{L}}{\partial(\partial_\nu q)} \cdot \partial_\nu (\partial_\mu q \cdot \varepsilon_\mu) \right\} = \int d^{d+1}x \left\{ \frac{\partial \mathcal{L}}{\partial q} \partial_\mu q \cdot$
But $\partial_\mu \mathcal{L} = \frac{\partial \mathcal{L}}{\partial q} \partial_\mu q + \frac{\partial \mathcal{L}}{\partial(\partial_\nu q)} \partial_\nu \partial_\mu q \Rightarrow \frac{\partial \mathcal{L}}{\partial(\partial_\nu q)} \partial_\mu \partial_\nu q \cdot \varepsilon_\mu = \partial_\mu$
$\delta S = \int d^{d+1}x \left\{ \partial_\mu \mathcal{L} \cdot \varepsilon_\mu - \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu q)} \partial_\mu q \right) \varepsilon_\mu \right\}$


$\delta S = \int d^{d+1}x \left\{ \frac{\partial \mathcal{L}}{\partial q} \partial_\mu q \cdot \varepsilon_\mu + \frac{\partial \mathcal{L}}{\partial(\partial_\nu q)} \cdot \partial_\nu (\partial_\mu q \cdot \varepsilon_\mu) \right\} = \int d^{d+1}x \left\{ \frac{\partial \mathcal{L}}{\partial q} \partial_\mu q \cdot$
But $\partial_\mu \mathcal{L} = \frac{\partial \mathcal{L}}{\partial q} \partial_\mu q + \frac{\partial \mathcal{L}}{\partial(\partial_\nu q)} \partial_\nu \partial_\mu q \Rightarrow$
$\frac{\partial \mathcal{L}}{\partial(\partial_\nu q)} \partial_\mu \partial_\nu q \cdot \varepsilon_\mu = \partial_\mu$
$\delta S = \int d^{d+1}x \left\{ \partial_\mu \mathcal{L} \cdot \varepsilon_\mu - \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu q)} \partial_\mu q \right) \varepsilon_\mu \right\} =$




$\partial_\nu T_{\mu\nu} = 0$
$T_{\mu\nu} = \mathcal{L}\,\delta_{\mu\nu} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu q)}\,\partial_\mu q$
energy-momentum
It is conserved