

Title: Special Functions and Distributions - Lecture 3

Date: Sep 08, 2011 09:00 AM

URL: <http://pirsa.org/11090061>

Abstract:

Orthogonal Polynomials Continued

$$P_n(x)$$

Orthogonal Polynomials Continued

$$P_n(x) = \frac{c_n}{w(x)} \frac{d^n}{dx^n} (w(x) s(x)^n)$$

$$s(x) = \begin{cases} (x-a)(x-b) & \text{if } a, b \text{ finite} \\ x-a & \text{if } a \text{ finite} \\ x-b & \text{if } b \text{ finite} \\ \text{constant} & \text{if } a = -\infty, b = \infty \end{cases}$$

Orthogonal Polynomials Continued

$$P_n(x) = \frac{c_n}{w(x)} \frac{d^n}{dx^n} (w(x) s(x)^n)$$

$$\langle P_n, P_m \rangle_w = \delta_{nm}$$

Orthogonal Polynomials Continued

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if w is finite and differentiable

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 $w(a)s(a) = 0 = w(b)s(b)$ if a, b finite
 $w(x)s(x) \rightarrow 0$

Orthogonal Polynomials Continued

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if w is finite and differentiable
 $w(a)s(a) = 0 = w(b)s(b)$ if (a, b)
 $w(x)s(x) \rightarrow 0$ faster than any poly

$$w(x) \leq A x^n \quad \text{for } x > x_0(n)$$

$w(x)s(x) < Ax^n$ for $x > x_0(n)$ for any n

When c these polynomials

3 c case 1 $(a,b) = (-\infty, \infty)$

-2 $(a,b) = (0, \infty)$

3 $(a,b) = (-1, 1)$

$\ln, \ln w \sim \ln m$
 if w is finite and differentiable
 $w(a)s(a) = 0 = w(b)s(b)$ if a, b finite
 $w(x)s(x) \rightarrow 0$ faster than any polynomial as $x \rightarrow \pm\infty$ if a, b infinite

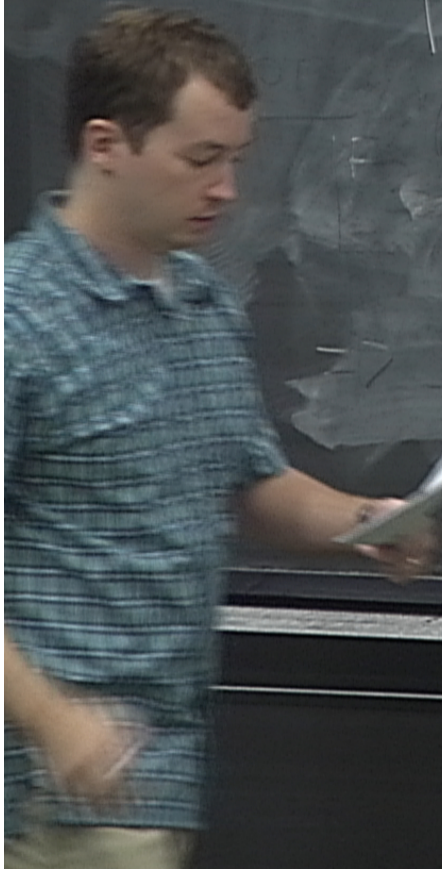
$w(x)s(x) \leq Ax^n$ for $x > x_0(n)$ for any n

When are these polynomials

- 3 cases:
- case 1 $(a, b) = (-\infty, \infty)$
 - 2 $(a, b) = (0, \infty)$
 - 3 $(a, b) = (-1, 1)$

$P_n(cx+d)$

Case 1: $f(z)$ with $f(z) = \frac{1}{z}$ and $f(z) = \frac{1}{z^2}$
 linear transformation $z \rightarrow P_1(x) = \pm 2x$



$$w(x)s(x) \leq Ax^n \quad \text{for } x > x_0(n) \quad \text{for any } n$$

When are these polynomials

3 cases: case 1 $(a,b) = (-\infty, \infty)$

2 $(a,b) = (0, \infty)$

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$$P_n(cx+d)$$

Case 1:

linear transformation $\rightarrow P_1(x) = \pm 2x$

$$P_1(x) \propto \frac{1}{w_1} \frac{dw}{dx} = \pm 2x$$

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linear transformation $\rightarrow P_1(x) = \pm 2x$

$$P_1(x) \propto \frac{1}{w} \frac{dw}{dx} = \pm 2x$$

$$w \propto \exp(\pm x^2)$$

$$w(x) = \exp(-x^2)$$

$$P_n(x) = C_n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

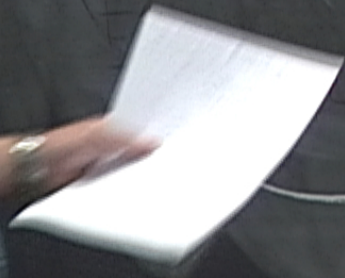
$$P_n(x) = C_n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

$$C_n = (-1)^n$$

Hermite polynomials

Case 2: $(a,b) = (0,\infty)$

$P_1(x)$



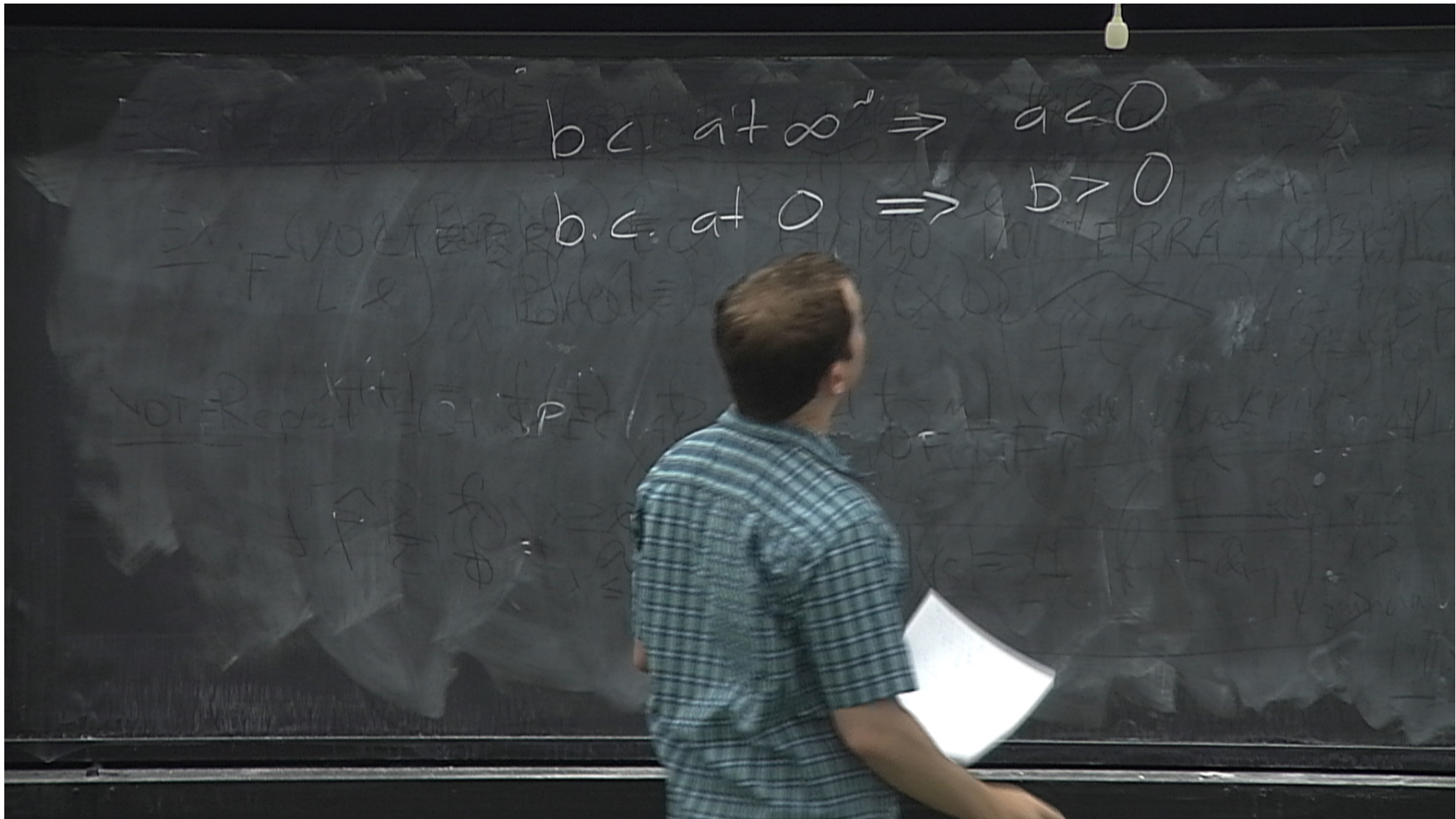
Case 2: $(a, b) = (0, \infty)$

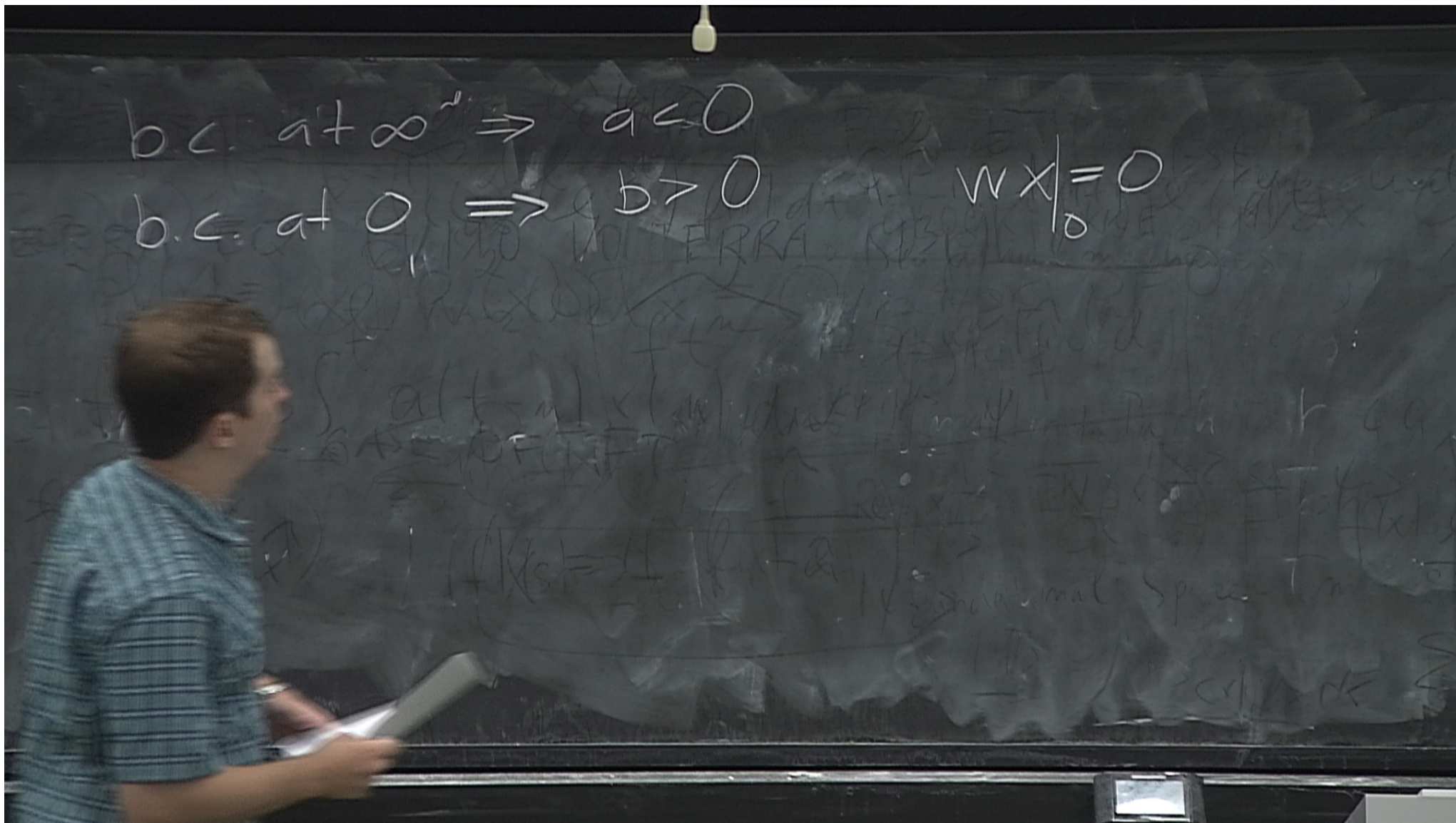
$$P_1(x) = \frac{1}{w} \frac{d(wx)}{dx} = ax + b$$

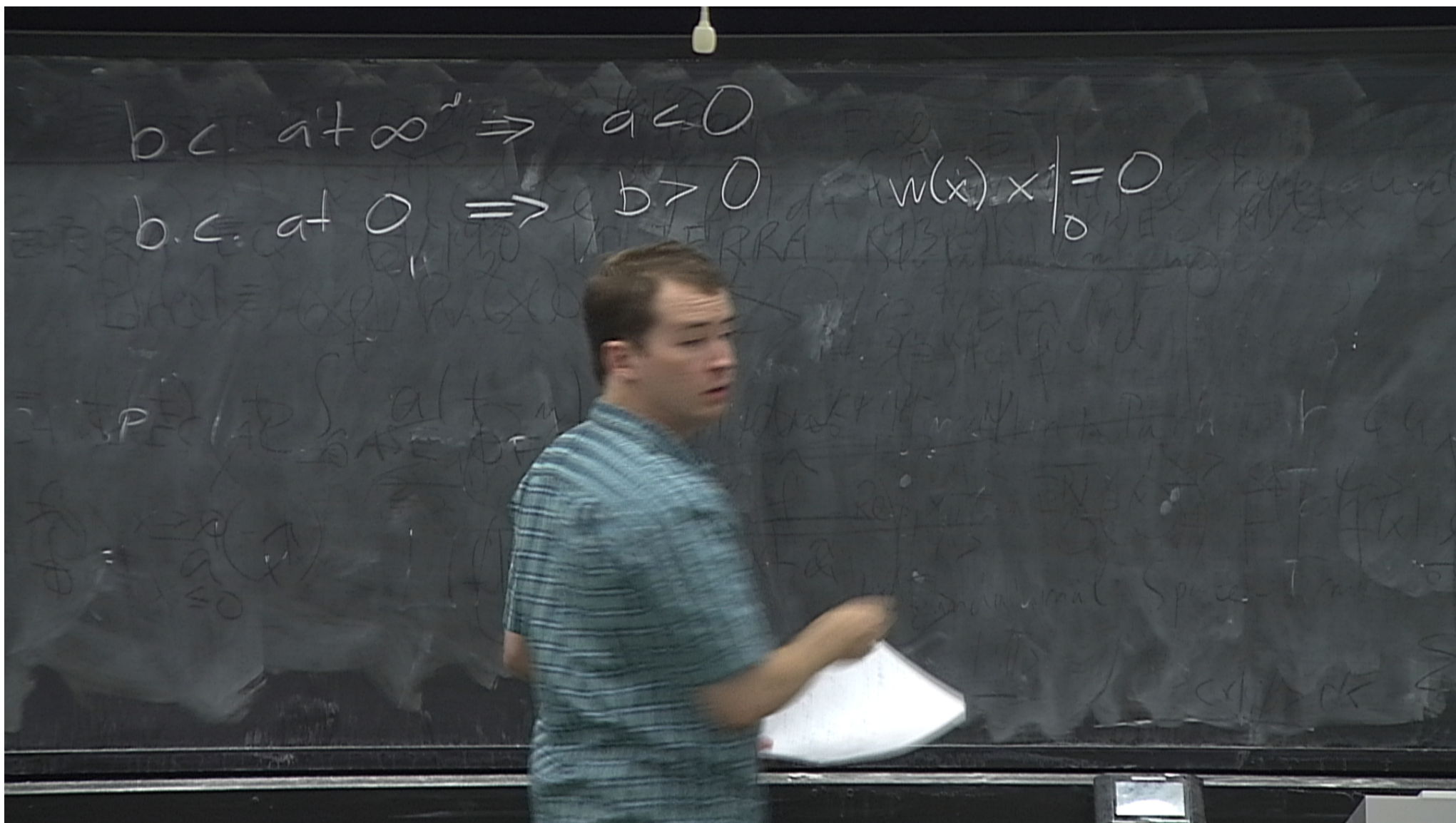
$$\frac{w'}{w} x + 1 = ax + b$$

$$\ln w = ax + (b-1) \ln x$$

$$w = C e^{ax} x^{b-1}$$







$$b.c. \text{ at } \infty \Rightarrow a < 0$$

$$b.c. \text{ at } 0 \Rightarrow b > 0$$

$$w(x) = e^{-x} x^p \quad p > 0$$

$$w(x) x \Big|_0 = 0$$

$$b.c. \text{ at } \infty \Rightarrow a < 0$$

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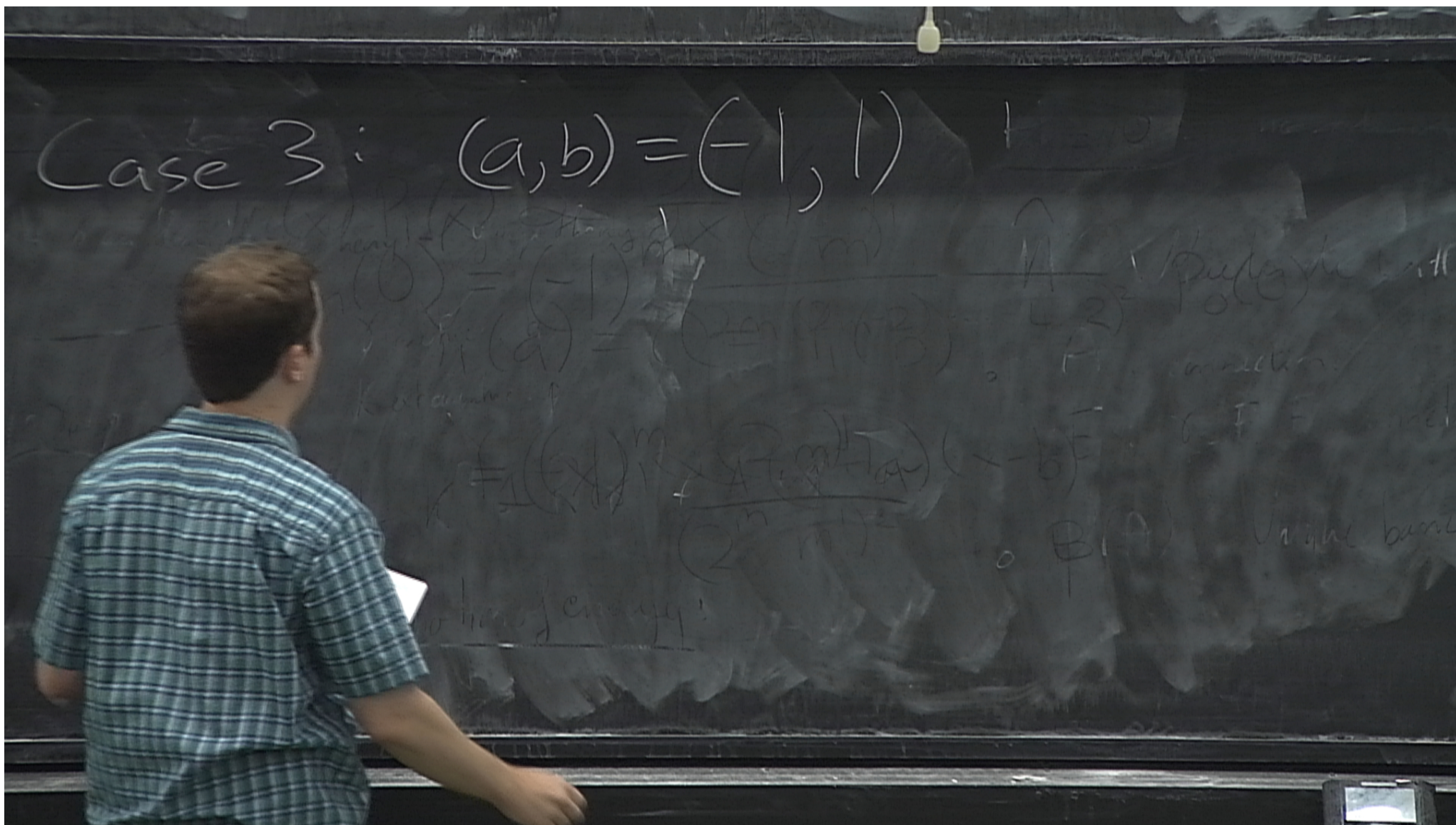
$$w(x) x \Big|_0 = 0$$

$$w(x) = e^{-x} x^p \quad p > 0$$

$$P_n = C_n e^{-x} x^{-p} \frac{d^n}{dx^n} [e^{-x} x^{n+p}]$$

$$C_n = \frac{1}{n!}$$

Associated Laguerre



$$(a, b) = (-1, 1)$$

$$P_1(x) = \frac{1}{w(x)} \frac{d}{dx} [w(x) (x^2 - 1)] = "ax + b$$

$$w(x) = (1-x)^\alpha (x+1)^\beta \quad (\text{Check})$$

$$\alpha = \frac{1}{2}(a+b-2)$$

$$\beta = \frac{1}{2}(a-b-2)$$

$$3 \quad (a, b) = (-1, 1)$$

$$w(x)(x^2-1) \Big|_{x=\pm 1} = 0$$

$$\alpha, \beta > -1$$

$$P_n(x) = C_n (1-x)^{-\alpha} (1+x)^{-\beta} \frac{d^n}{dx^n} \left[(1-x)^\alpha (1+x)^\beta (x^2-1)^n \right]$$

$$C_n = \frac{(-1)^n}{2^n n!}$$

Jacobi polynomials

$$3 \quad (a, b) = (-1, 1)$$

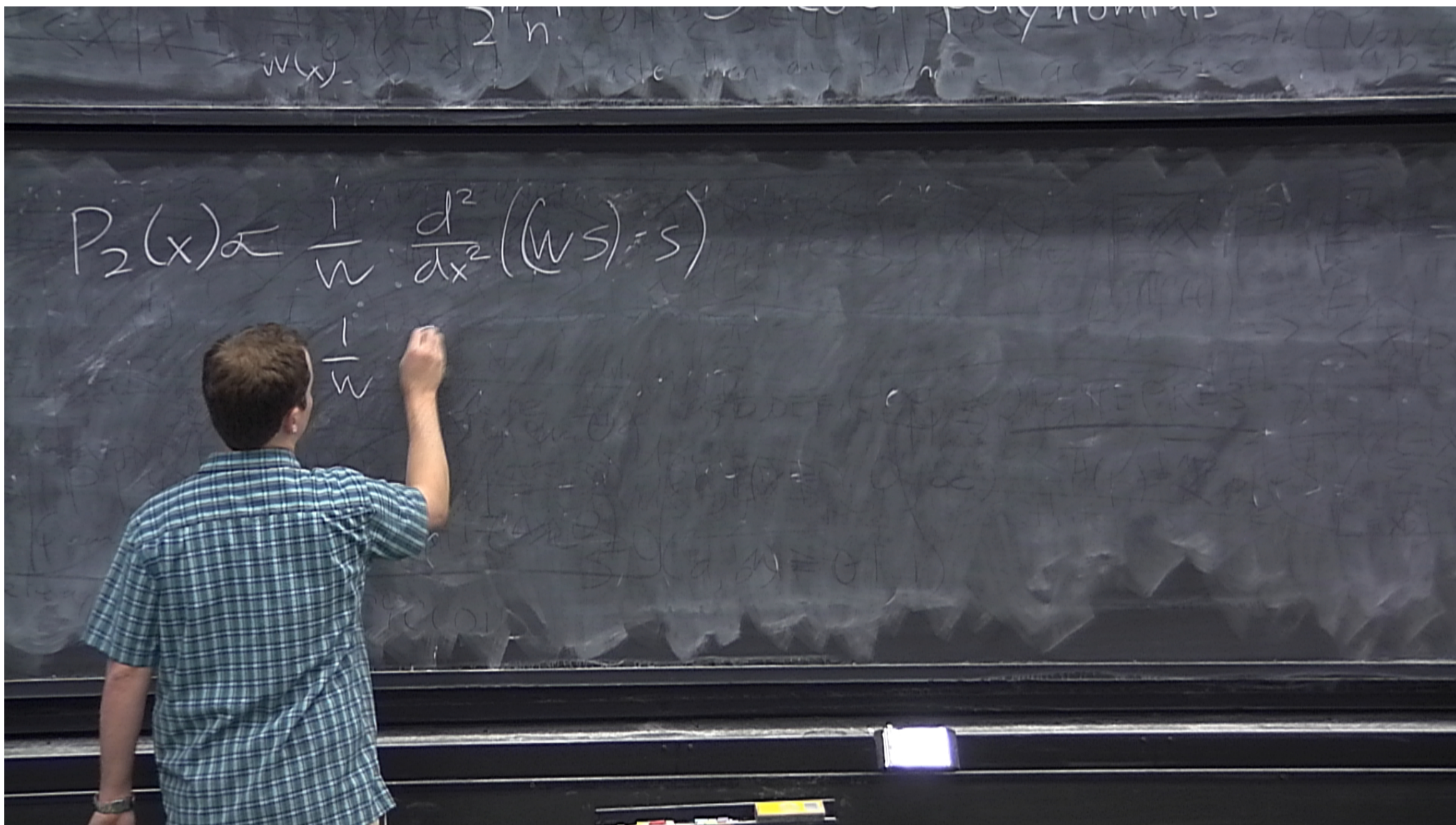
$$w(x)(x^2-1) \Big|_{x=\pm 1} = 0$$

$$\alpha, \beta > -1$$

$$P_n(x) \doteq C_n (1-x)^{-\alpha} (1+x)^{-\beta} \frac{d^n}{dx^n} \left[(1-x)^\alpha (1+x)^\beta (x^2-1)^n \right]$$

$$C_n = \frac{(-1)^n}{2^n n!}$$

Jacobi polynomials



$$\begin{aligned}
 P_2(x) &\propto \frac{1}{w} \frac{d^2}{dx^2} (ws) = s \\
 &= \frac{1}{w} \frac{d}{dx} (ws)'s + ws s' \\
 &= \frac{1}{w} ((ws)''s + 2(ws)'s' + (ws)s'')
 \end{aligned}$$

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Sturm-Liouville Theory

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$$a(x)u''(x) + b(x)u'(x) + c(x)u(x) = 0$$

Sturm-Liouville Theory

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$$\int_{\Omega} u_x(x)$$

Sturm-Liouville Theory

$$a(x) u''(x) + b(x) u'(x) + c(x) u(x) = 0$$

$$\Omega u_x(x) + \lambda w(x) u_x(x) = 0$$

$$\left[\frac{d}{dx} \left[f(x) \frac{d}{dx} + g(x) \right] u(x) \right] + \lambda h(x) u(x) = 0$$

$$r^{-1}(s) = \frac{1}{c(x)}(x)$$

$$= \frac{1}{w} \frac{d}{dx} (ws)'s + ws's'$$

$$= \frac{1}{w} ((ws)''s + 2(ws)'s' + (ws)s'')$$

Sturm-Liouville Theory

$$a(x)u''(x) + b(x)u'(x) + c(x)u(x) = 0$$

$$\int_a^b u_x(x) + \lambda w(x)u_x(x) = 0$$

$$\Rightarrow \left[\frac{d}{dx} \left(f(x) \frac{d}{dx} + g(x) \right) u(x) + \lambda h(x)u(x) = 0 \right]$$

$$f(x) = a(x)$$

$$g(x) = b(x) - a'(x)$$

$$f(x) = a(x)$$

$$g(x) = b(x) - a'(x)$$

$$h(x) = \frac{1}{x} [c(x) - b'(x) + a''(x)]$$

$$= \frac{1}{w} \frac{d}{dx} (ws)'s + ws s'$$

$$= \frac{1}{w} ((ws)''s + 2(ws)'s' + (ws)s'')$$

Sturm-Liouville Theory

$$a(x) u''(x) + b(x) u'(x) + c(x) u(x) = 0$$

$$\int_a^b u_x(x) + \lambda h(x) u_x(x) = 0$$

$$\left[\frac{d}{dx} \left[f(x) \frac{d}{dx} + g(x) \right] u(x) + \lambda h(x) u(x) = 0 \right.$$

$$f(x) = a(x)$$

$$g(x) = b(x) - a'(x)$$

$$h(x) = \frac{1}{\lambda} [c(x) -$$

Sturm-Liouville Theory

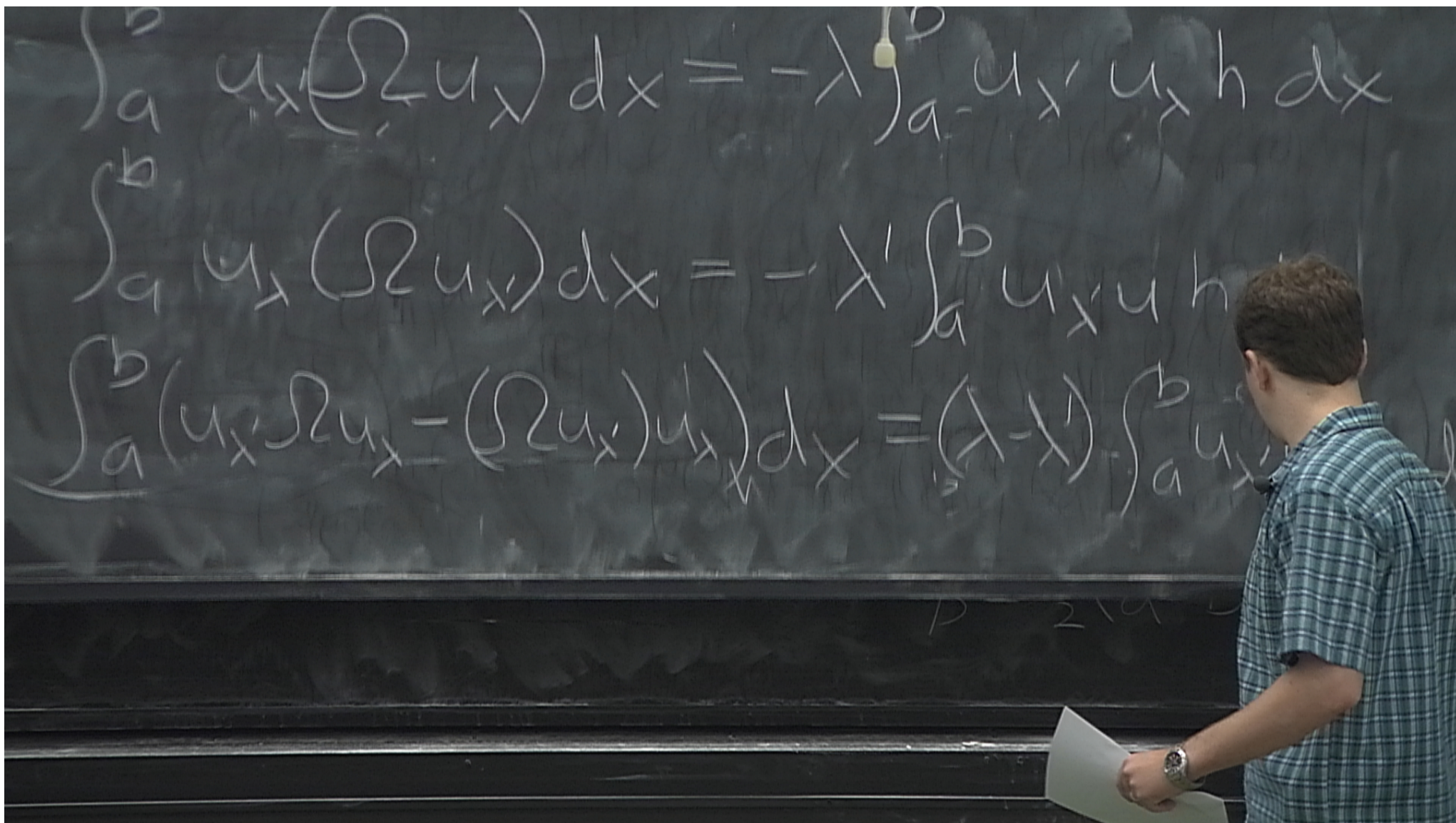
$$a(x) u''(x) + b(x) u'(x) + c(x) u(x) = 0$$

$$\Omega u_x(x) + \lambda h(x) u_x(x) = 0$$

$$\left[\frac{d}{dx} \left[f(x) \frac{d}{dx} + g(x) \right] u(x) + \lambda h(x) u(x) = 0 \right]$$

Associated Laguerre

$$\int_a^b u_x' (\Omega_2 u_x) dx = -\lambda \int_a^b u_x' u_x$$



$$\langle u_\lambda, u_{\lambda'} \rangle_n$$

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$$\langle u_\lambda, \Omega u_{\lambda'} \rangle = \langle \Omega u_\lambda, u_{\lambda'} \rangle$$

Associated Laguerre

$$\langle u_\lambda, u_{\lambda'} \rangle_n = 0 \quad \text{for } \lambda \neq \lambda'$$

$$\langle u_\lambda, \Omega u_{\lambda'} \rangle = \langle \Omega u_\lambda, u_{\lambda'} \rangle$$

$$\Omega = \frac{d}{dx} [S(x)]$$

$$\langle u_\lambda, u_{\lambda'} \rangle_n = 0 \text{ for } \lambda \neq \lambda'$$

$$\langle u_\lambda, \Omega u_{\lambda'} \rangle = \langle \Omega u_\lambda, u_{\lambda'} \rangle$$

$$\Omega = \frac{1}{x} \left[S(x) \frac{d}{dx} \right] + T(x)$$

$$\langle u_\lambda, u_{\lambda'} \rangle_n = 0 \quad \text{for } \lambda \neq \lambda'$$

$$\langle u_\lambda, \Omega u_{\lambda'} \rangle = \langle \Omega u_\lambda, u_{\lambda'} \rangle$$

$$\Omega = \frac{d}{dx} \left[S(x) \frac{d}{dx} \right] + t(x)$$

$$S(x) = a(x)$$

$$S'(x) = b(x)$$

$$t(x) + \lambda w(x) = c(x)$$

$$\langle u_\lambda, u_{\lambda'} \rangle_n = 0 \text{ for } \lambda \neq \lambda'$$

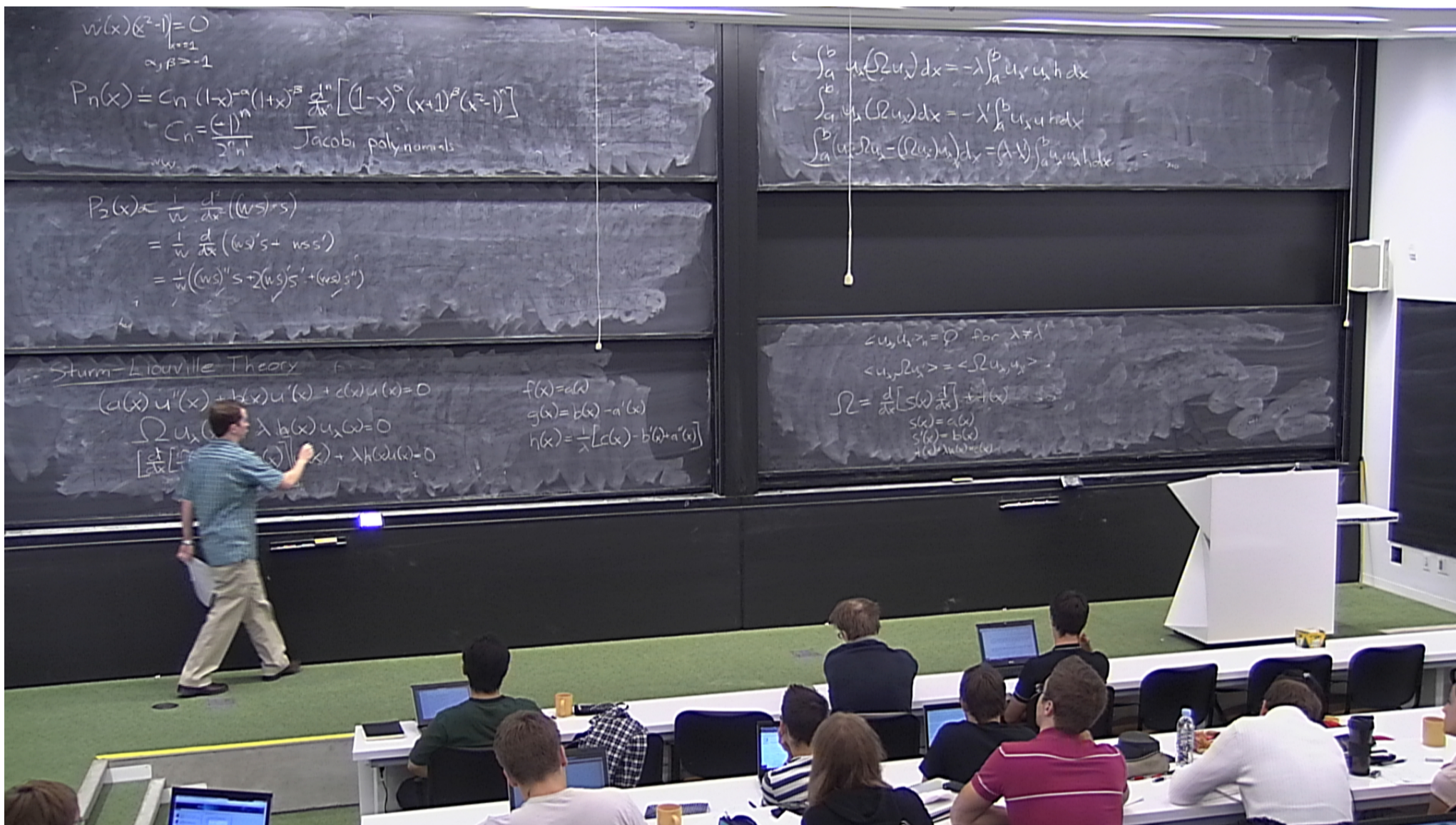
$$\langle u_\lambda, \Omega u_{\lambda'} \rangle = \langle \Omega u_\lambda, u_{\lambda'} \rangle$$

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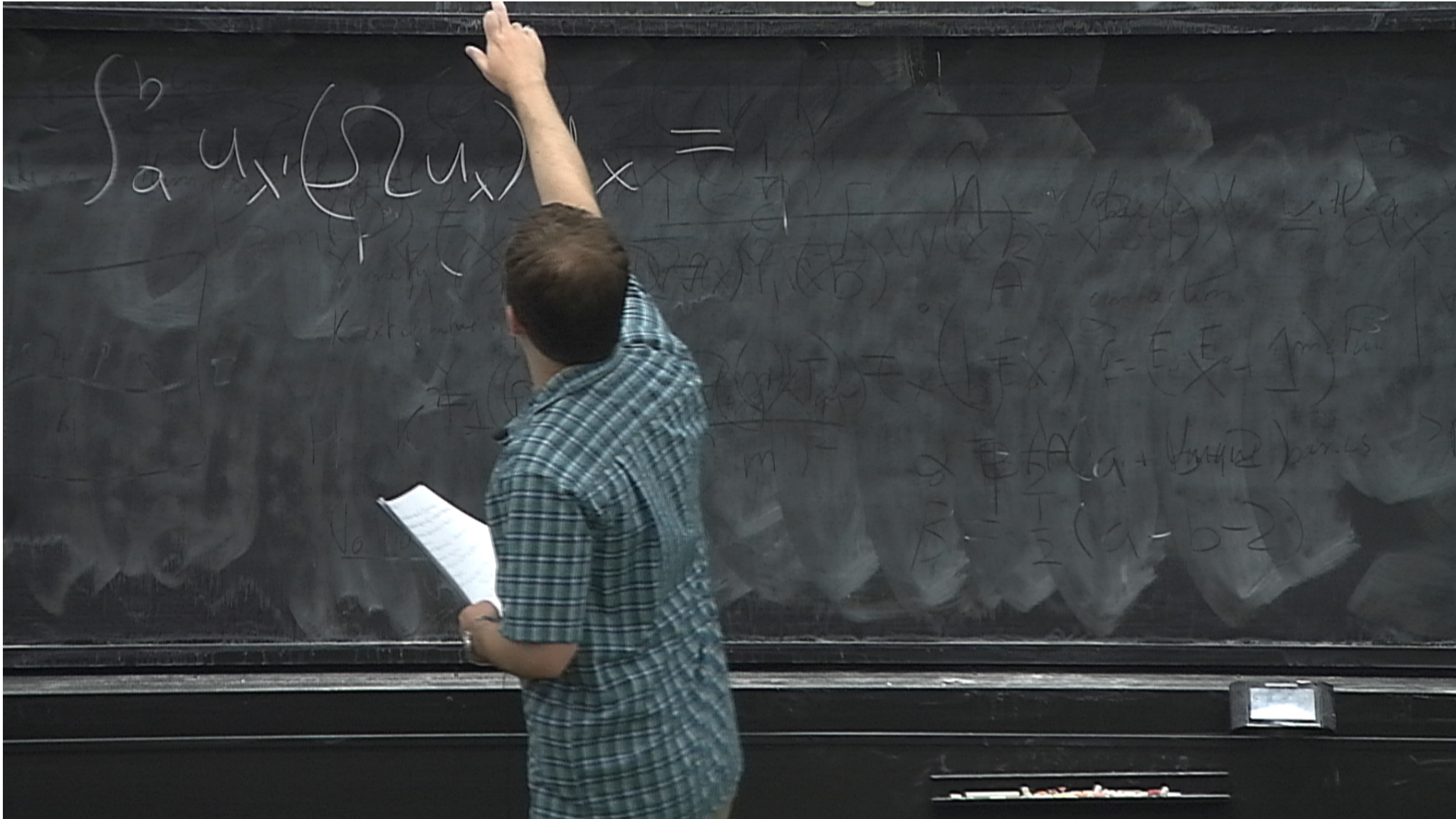
$$\langle u_\lambda, \Omega u_{\lambda'} \rangle = \langle \Omega u_\lambda, u_{\lambda'} \rangle$$

$$\Omega = \frac{d}{dx} \left[s(x) \frac{d}{dx} \right] + t(x)$$

$$s(x) = a(x) d(x)$$

$$s'(x) = b(x) d(x)$$

$$t(x) + \lambda w(x) = c(x) d(x)$$



$$JL = \frac{d}{dx} \left(S(x) \frac{d}{dx} \right) + H(x)$$

$$S(x) = a(x) d(x)$$

$$S'(x) = b(x) d(x)$$

$$f(x) + \lambda u(x) = c(x) d(x)$$

$$\int_a^b u_x' (\Omega u_x) dx = - \int_a^b \left(\frac{d}{dx} u_x \right) S \left(\frac{d}{dx} u_x \right) dx + u_x S \frac{d}{dx} u_x \Big|_a^b + \int_a^b u_x' + u_x dx$$

$$JL = \frac{d}{dx} \left(S(x) \frac{du}{dx} \right) + H(x)$$

$$S(x) = a(x) d(x)$$

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$$\int_a^b u_x' (S u_x) dx = - \int_a^b \left(\frac{d}{dx} u_x \right) S \left(\frac{d}{dx} u_x \right) dx + u_x S \frac{d}{dx} u_x \Big|_a^b + \int_a^b u_x' + u_x dx$$

$$= \int_a^b \left(\frac{d}{dx} \left(S \frac{d}{dx} \right) u_x \right) u_x dx$$

$$\int_a^b (u_\lambda \Omega u_\lambda - (\Omega u_\lambda) u_\lambda) dx = (\lambda - \lambda') \int_a^b u_\lambda u_{\lambda'} dx$$

$$\langle u_\lambda, u_{\lambda'} \rangle_n = 0 \text{ for } \lambda \neq \lambda'$$

$$\langle u_\lambda, \Omega u_{\lambda'} \rangle = \langle \Omega u_\lambda, u_{\lambda'} \rangle$$

$$\Omega = \frac{d}{dx} \left[s(x) \frac{d}{dx} \right] + t(x)$$

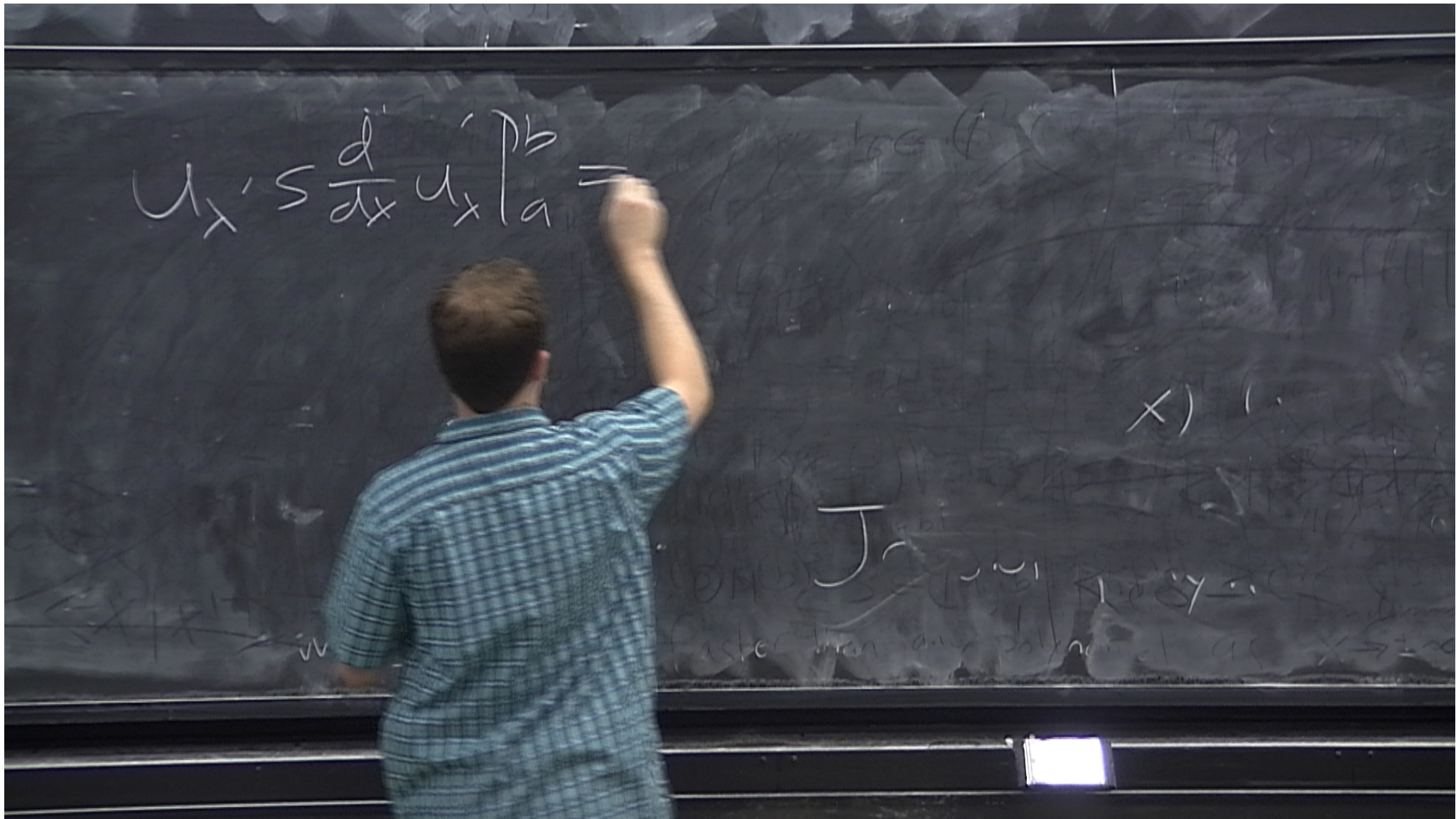
$$s(x) = a(x) dx$$

$$s'(x) = b(x) dx$$

$$t(x) + \lambda u(x) = c(x) dx$$

$$\int_a^b u_{\lambda'} (\Omega u_\lambda) dx = - \int_a^b \left(\frac{d}{dx} u_{\lambda'} \right) s \left(\frac{d}{dx} u_\lambda \right) dx + u_{\lambda'} s \frac{d}{dx} u_\lambda \Big|_a^b + \int_a^b u_{\lambda'} t u_\lambda dx$$

$$\int_a^b \left(\frac{d}{dx} (s \frac{d}{dx} u_{\lambda'}) \right) u_\lambda dx - \left(\frac{d}{dx} u_{\lambda'} \right) s u_\lambda \Big|_a^b + u_{\lambda'} s \frac{d}{dx} u_\lambda \Big|_a^b + \int_a^b u_{\lambda'} t u_\lambda dx$$



$$u_x' \cdot \frac{d}{dx} u_x \Big|_a^b = \left(\frac{d}{dx} u_x \right) \cdot u_x \Big|_a^b \Rightarrow \Omega \text{ is hermitian}$$

$$\Rightarrow \text{solutions orthogonal}$$

x) 1.

$$\Omega = \frac{d}{dx} \left[s(x) \frac{d}{dx} \right] + t(x)$$

$$s(x) = a(x) d(x)$$

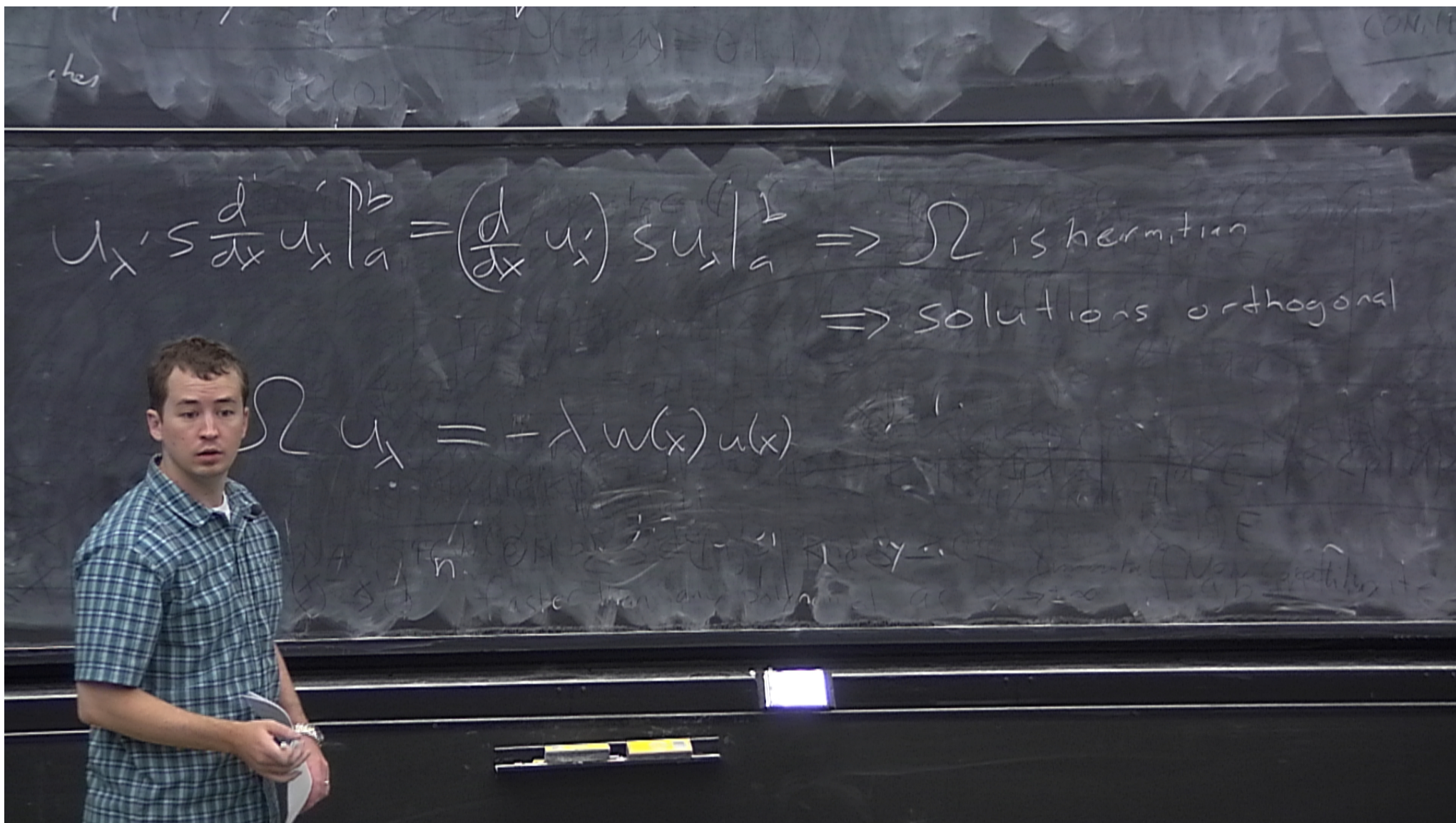
$$s'(x) = b(x) d(x)$$

$$t(x) + \lambda w(x) = c(x) d(x)$$

$$\int_a^b u_x' (\Omega u_x) dx = - \int_a^b \left(\frac{d}{dx} u_x \right) s \left(\frac{d}{dx} u_x \right) dx + u_x s \frac{d}{dx} u_x \Big|_a^b + \int_a^b u_x' t u_x dx$$

$$= \int_a^b \left(\frac{d}{dx} \left(s \frac{d}{dx} \right) u_x \right) u_x dx - \left(\frac{d}{dx} u_x \right) s u_x \Big|_a^b + u_x s \frac{d}{dx} u_x \Big|_a^b + \int_a^b u_x' t u_x dx$$

$$= \int_a^b (\Omega u_x') u_x dx = - \quad \leftarrow \quad 0$$



$$\langle u_\lambda, u_{\lambda'} \rangle_n = 0 \text{ for } \lambda \neq \lambda'$$

$$\langle u_\lambda, \Omega u_{\lambda'} \rangle = \langle \Omega u_\lambda, u_{\lambda'} \rangle$$

$$\Omega = \frac{d}{dx} \left[s(x) \frac{d}{dx} \right] + t(x)$$

$$s(x) = a(x) d(x)$$

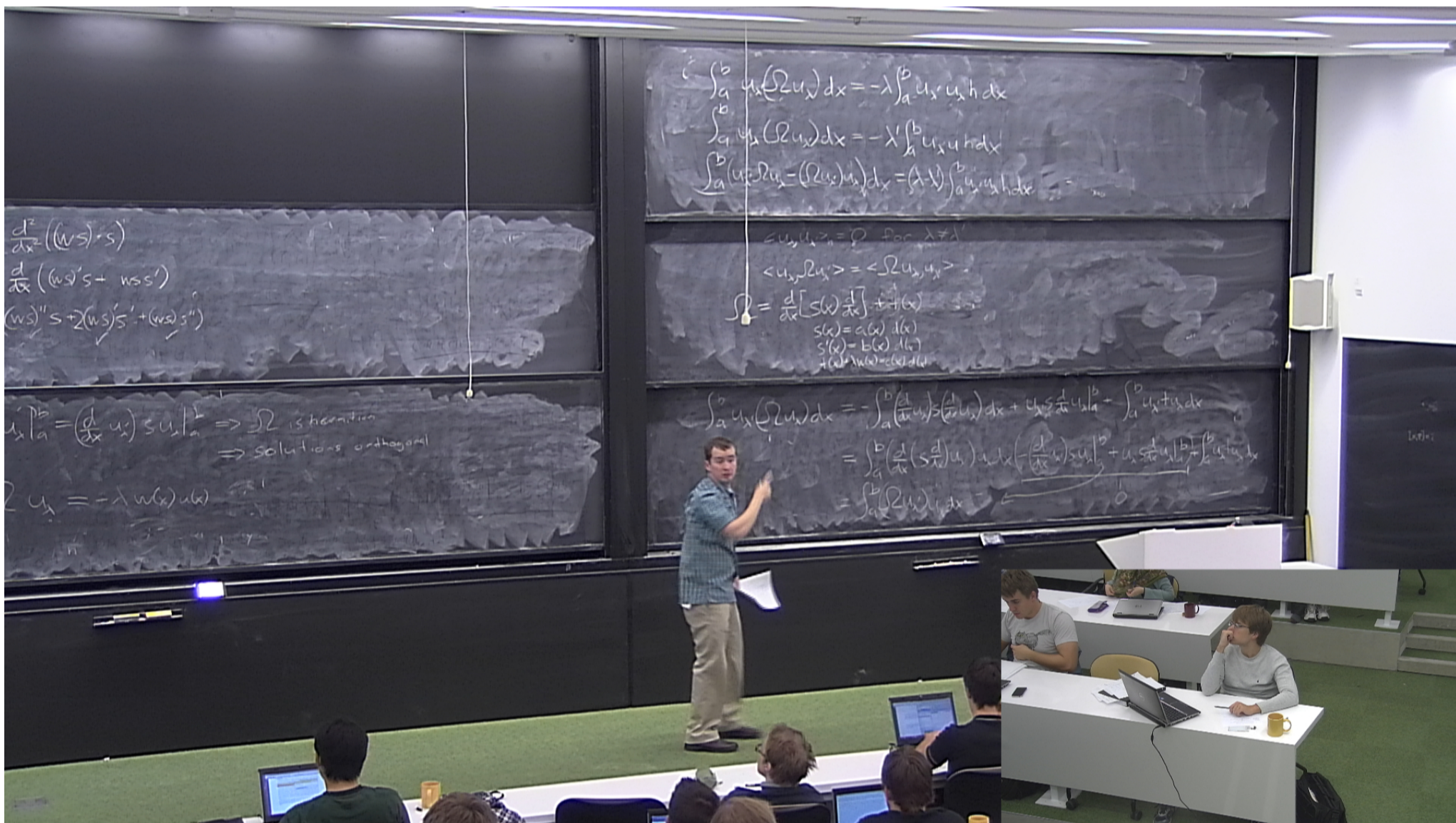
$$s'(x) = b(x) d(x)$$

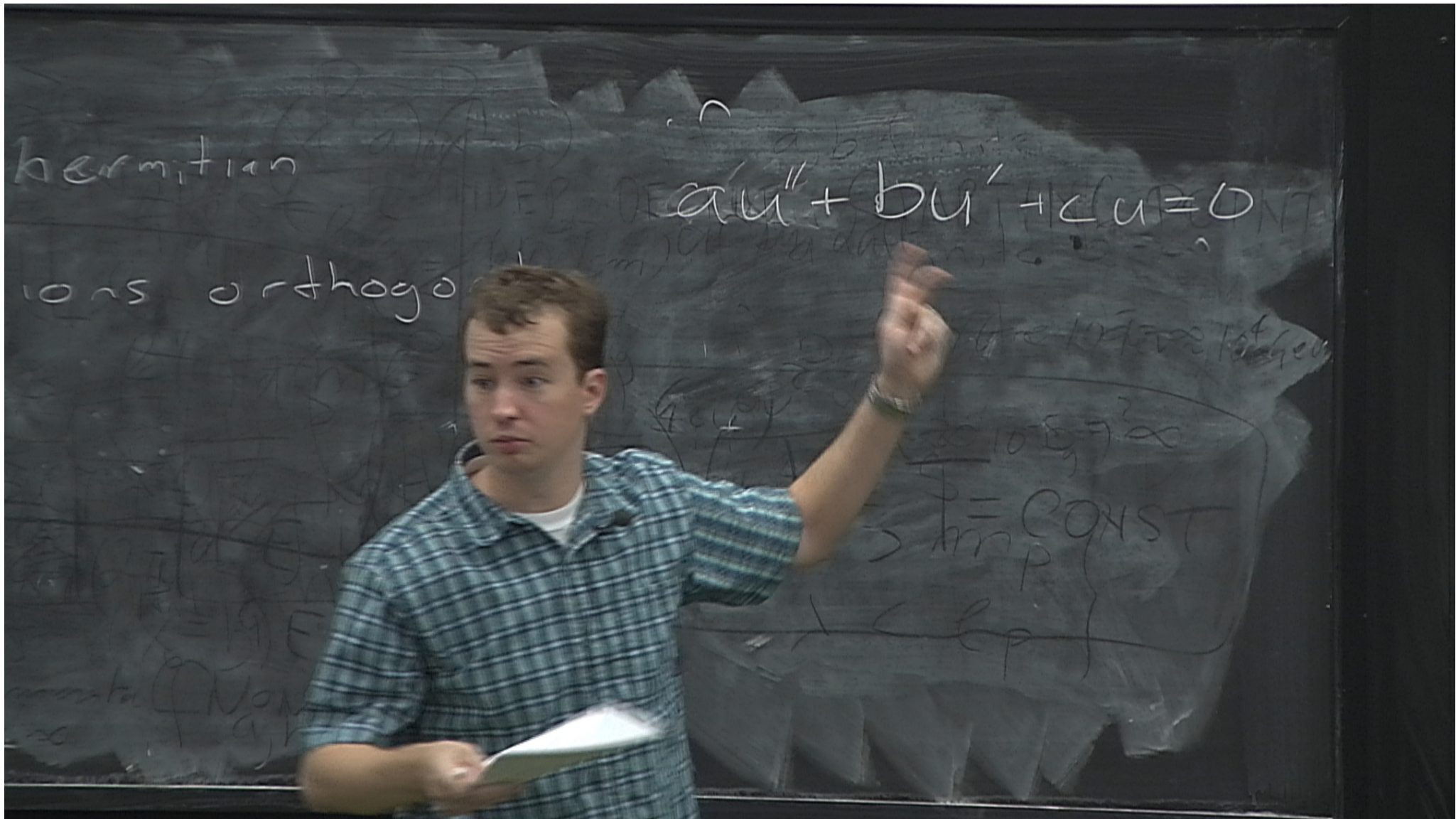
$$t(x) + \lambda u(x) = c(x) d(x)$$

$$\int_a^b u_{\lambda'} (\Omega u_\lambda) dx = - \int_a^b \left(\frac{d}{dx} u_{\lambda'} \right) s \left(\frac{d}{dx} u_\lambda \right) dx + u_{\lambda'} s \frac{d}{dx} u_\lambda \Big|_a^b + \int_a^b u_{\lambda'} t u_\lambda dx$$

$$= \int_a^b \left(\frac{d}{dx} \left(s \frac{d}{dx} \right) u_{\lambda'} \right) u_\lambda dx - \left(\frac{d}{dx} u_{\lambda'} \right) s u_\lambda \Big|_a^b + u_{\lambda'} s \frac{d}{dx} u_\lambda \Big|_a^b + \int_a^b u_{\lambda'} t u_\lambda dx$$

$$= \int_a^b (\Omega u_{\lambda'}) u_\lambda dx - \underbrace{\left(\frac{d}{dx} u_{\lambda'} \right) s u_\lambda \Big|_a^b - u_{\lambda'} s \frac{d}{dx} u_\lambda \Big|_a^b}_{=0}$$





$$x L_q'' + (1-x) L_q' + q L_q = 0$$



$$(x L_q'' + (1-x) L_q' + q L_q = 0) g$$

$$s(x) = x g(x)$$

$$(x L_q'' + (1-x) L_q' + q L_q = 0) g$$

$$s(x) = x g(x)$$

$$s'(x) = x g' + g = (1-x) g$$

$$(x L_q'' + (1-x) L_q' + q L_q = 0) g$$

$$) = x g(x)$$

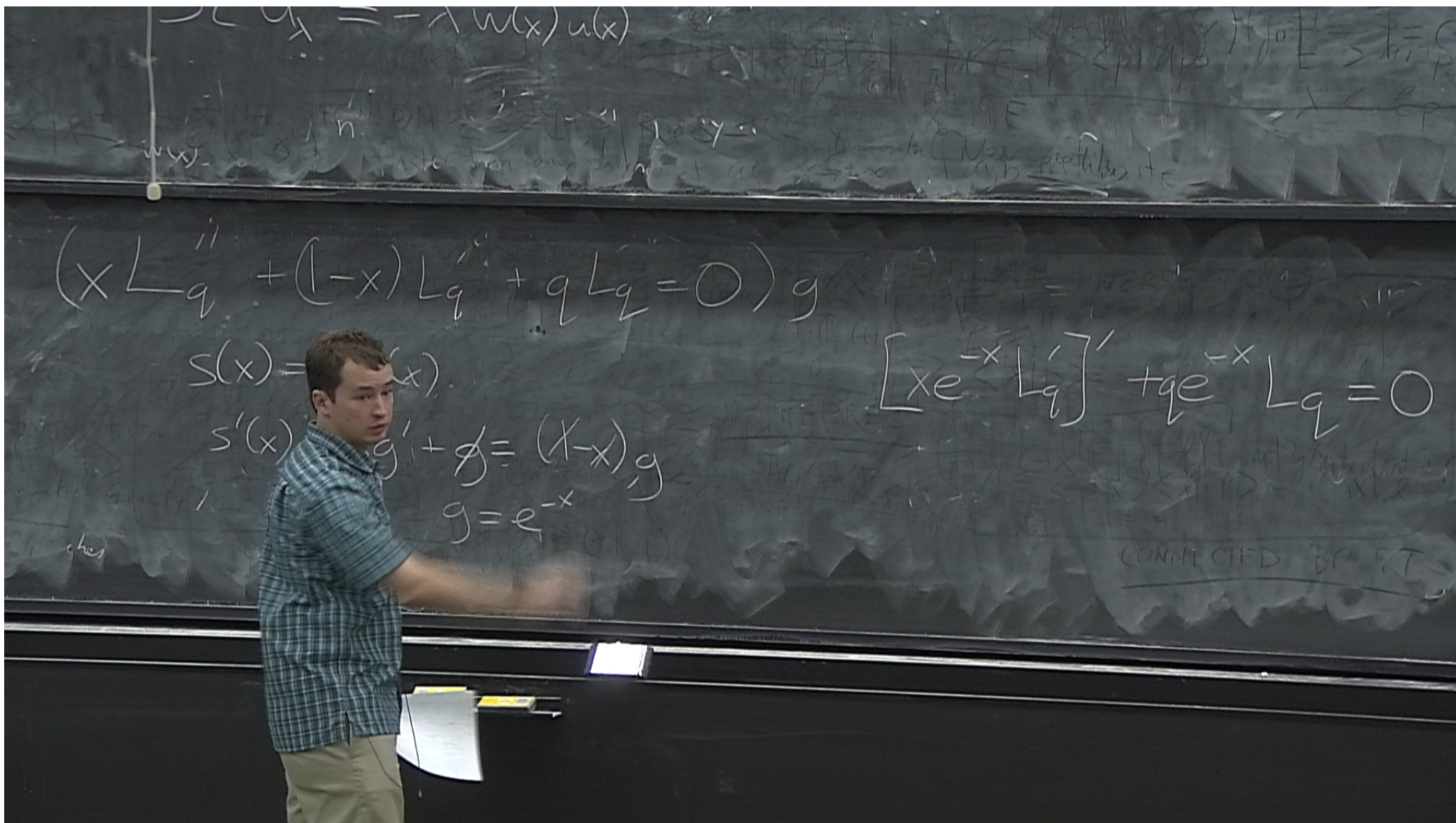
$$g'(x) = x g' + g = (1-x) g$$

$$(x L_q'' + (1-x) L_q' + q L_q = 0) g$$

$$s(x) = x g(x)$$

$$s'(x) = x g' + g = (1-x) g$$

$$g = e^{-x}$$



$$[xe^{-x} L_q]' + qe^{-x} L_q = 0$$

$$w = e^{-x}$$

CONNECTED BY F.T

$$L_q = -\lambda w(x) u(x)$$

$$(x L_q'' + (1-x) L_q' + q L_q = 0) g$$

$$s(x) = (x)$$

$$s'(x) = g' + g = (1-x)g$$

$$g = e^{-x}$$

$$[x e^{-x} L_q']' + q e^{-x} L_q = 0$$

$$w = e^{-x}$$

CONNECTED BY IT

$$[xe^{-x}L_q] + qe^{-x}L_q = 0$$

$$w = e^{-x}$$

$$L_q = C_q e^x \left(\frac{d}{dx} \right)^n (x^n e^{-x})$$

