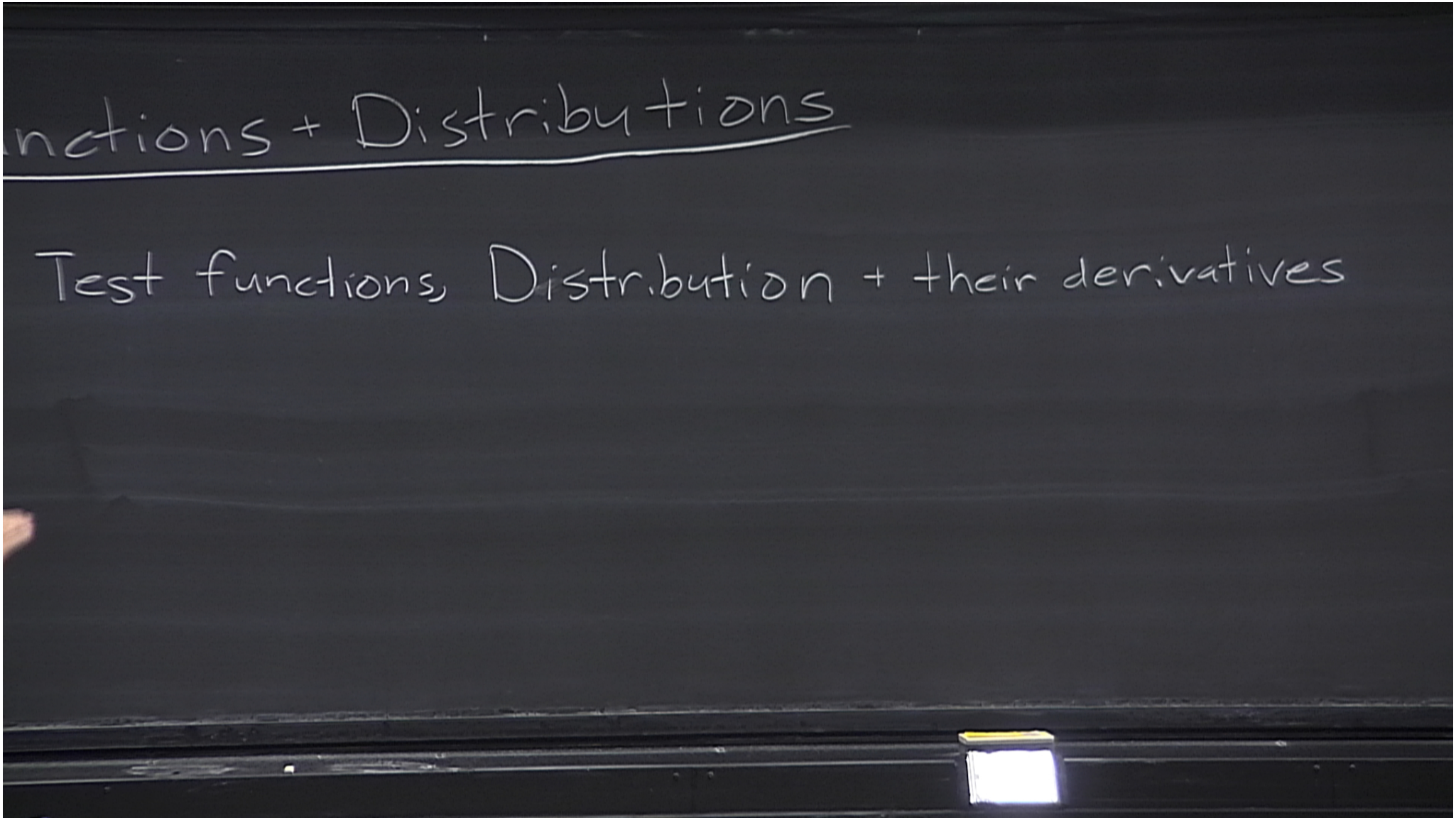
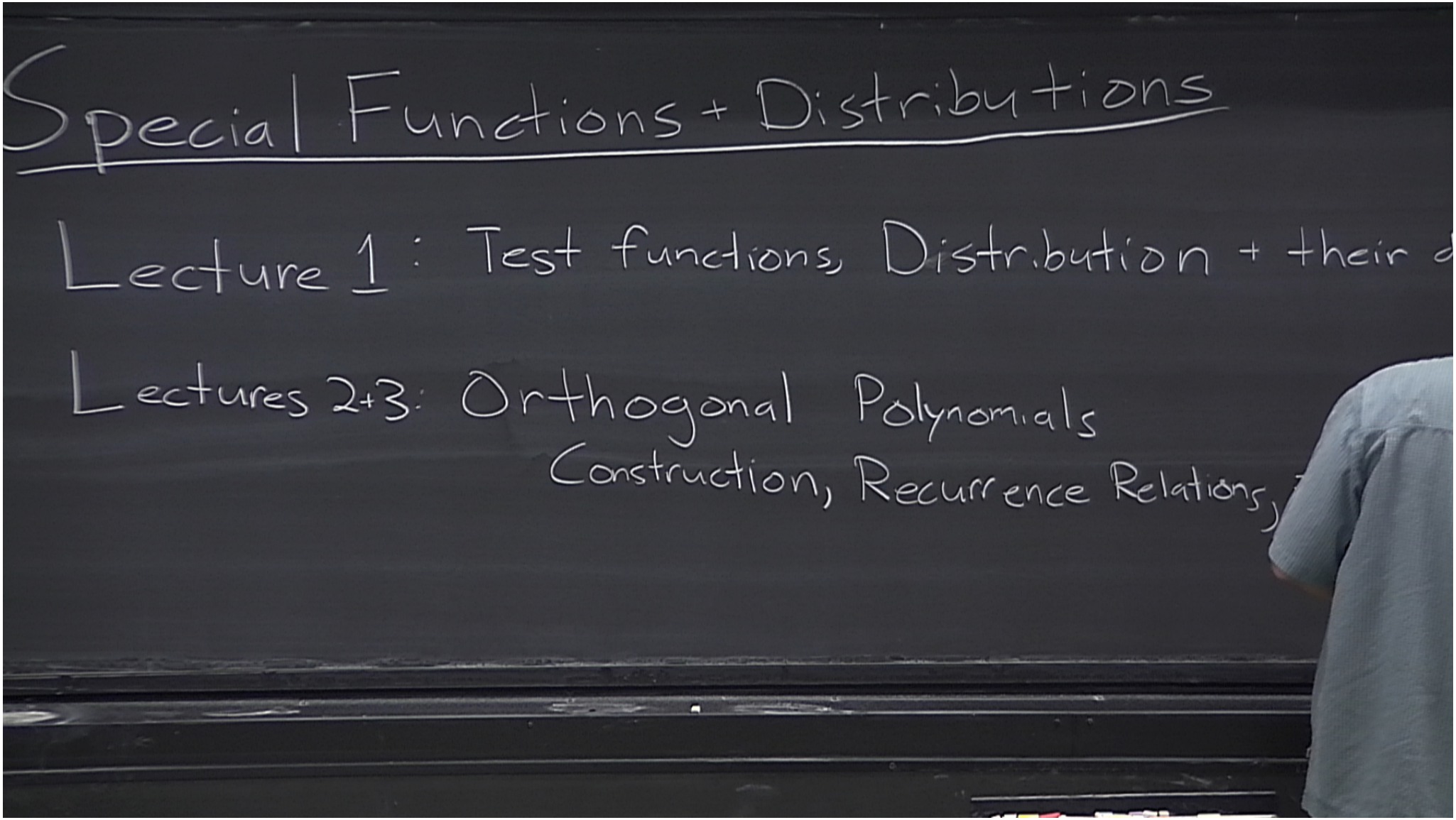
Title: Special Functions and Distributions - Lecture 1

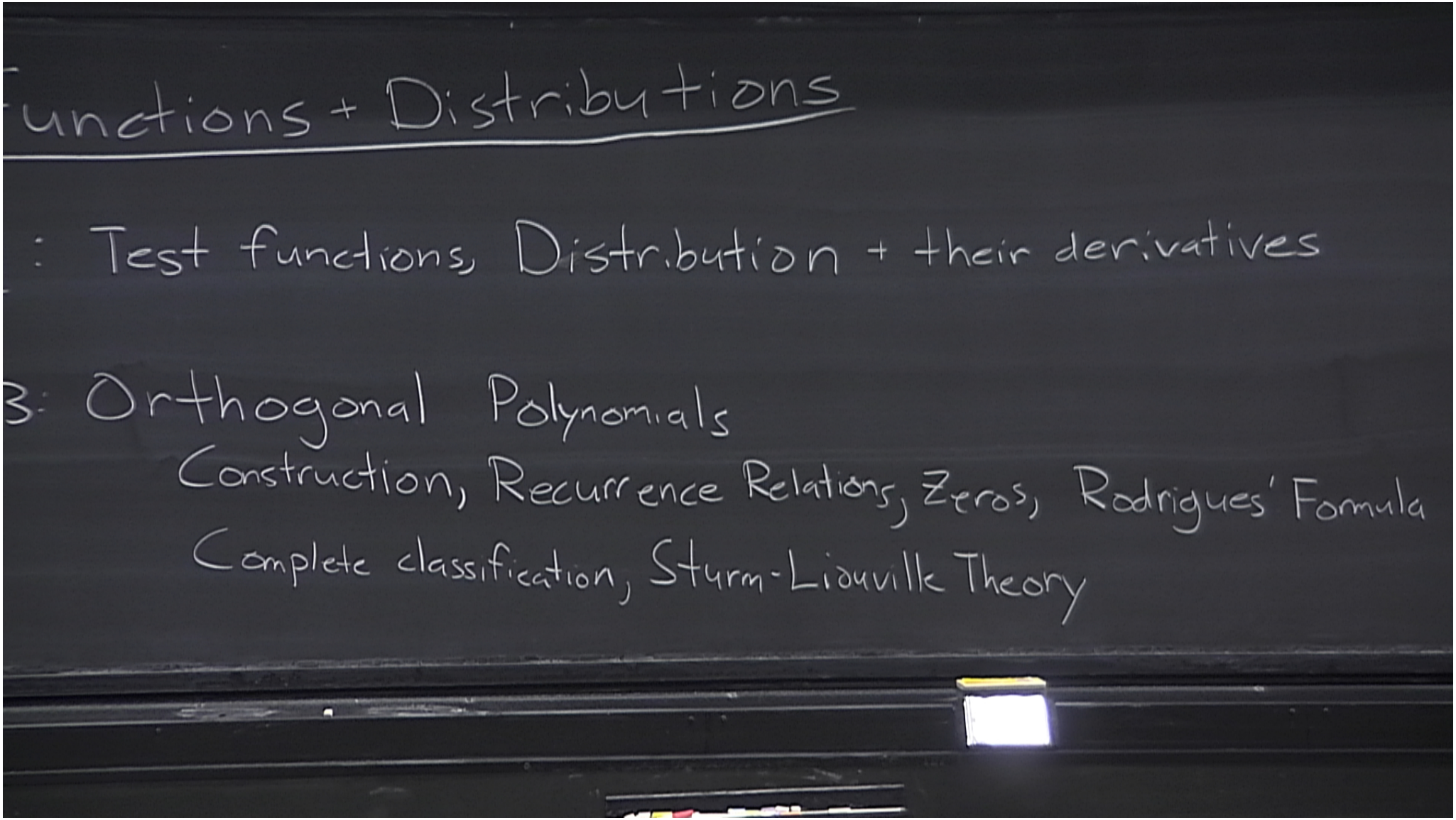
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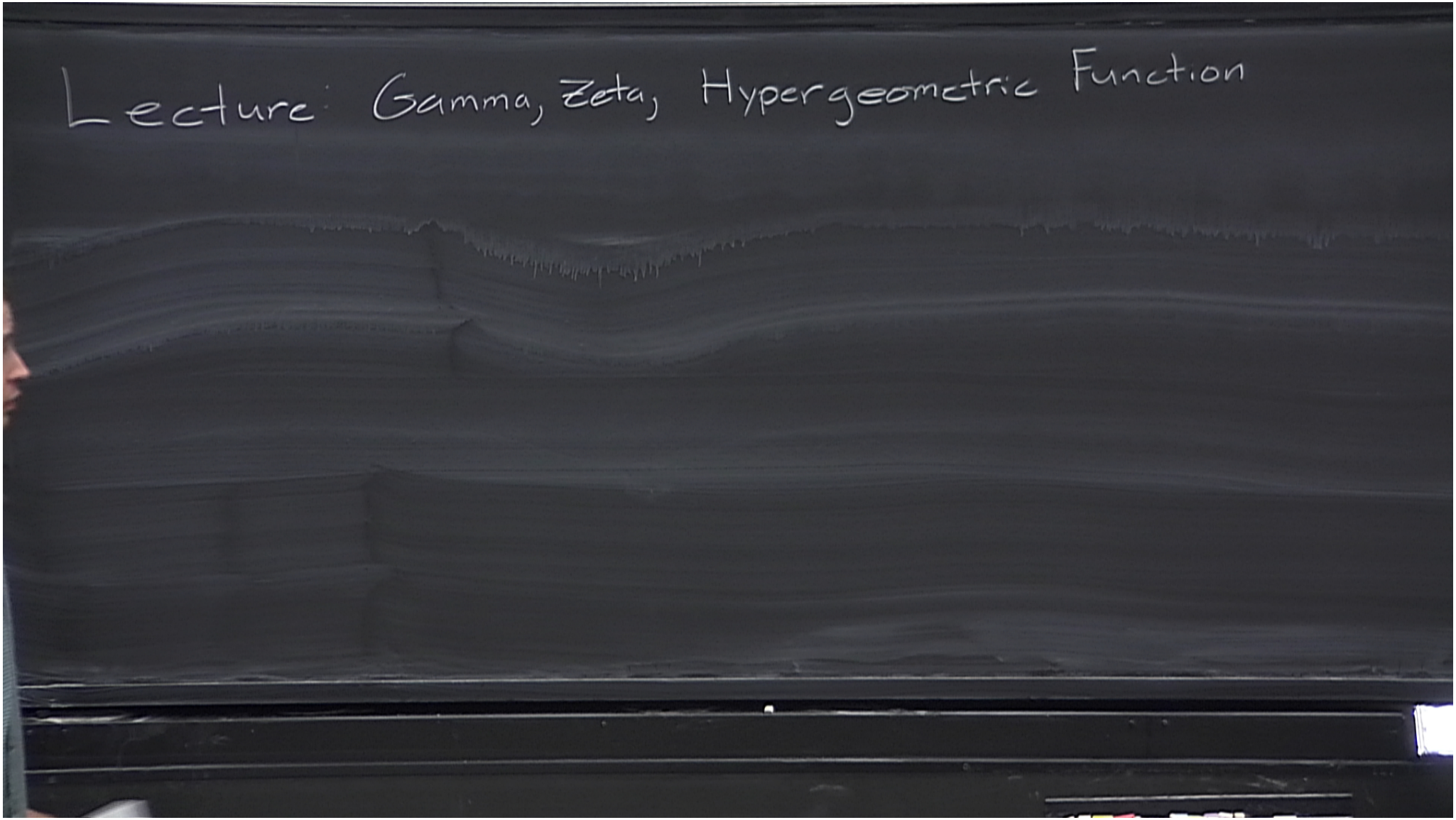
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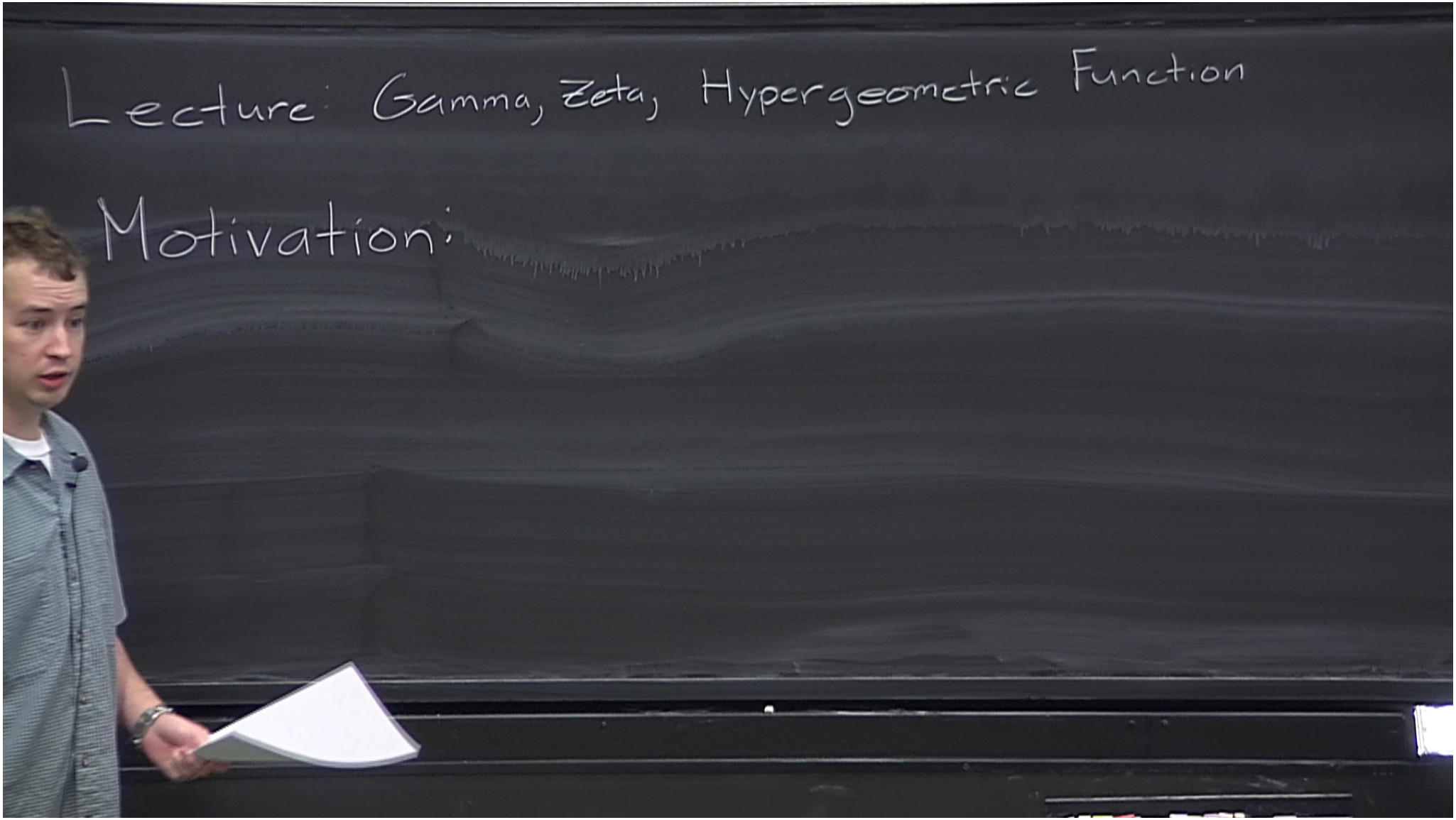
Abstract:

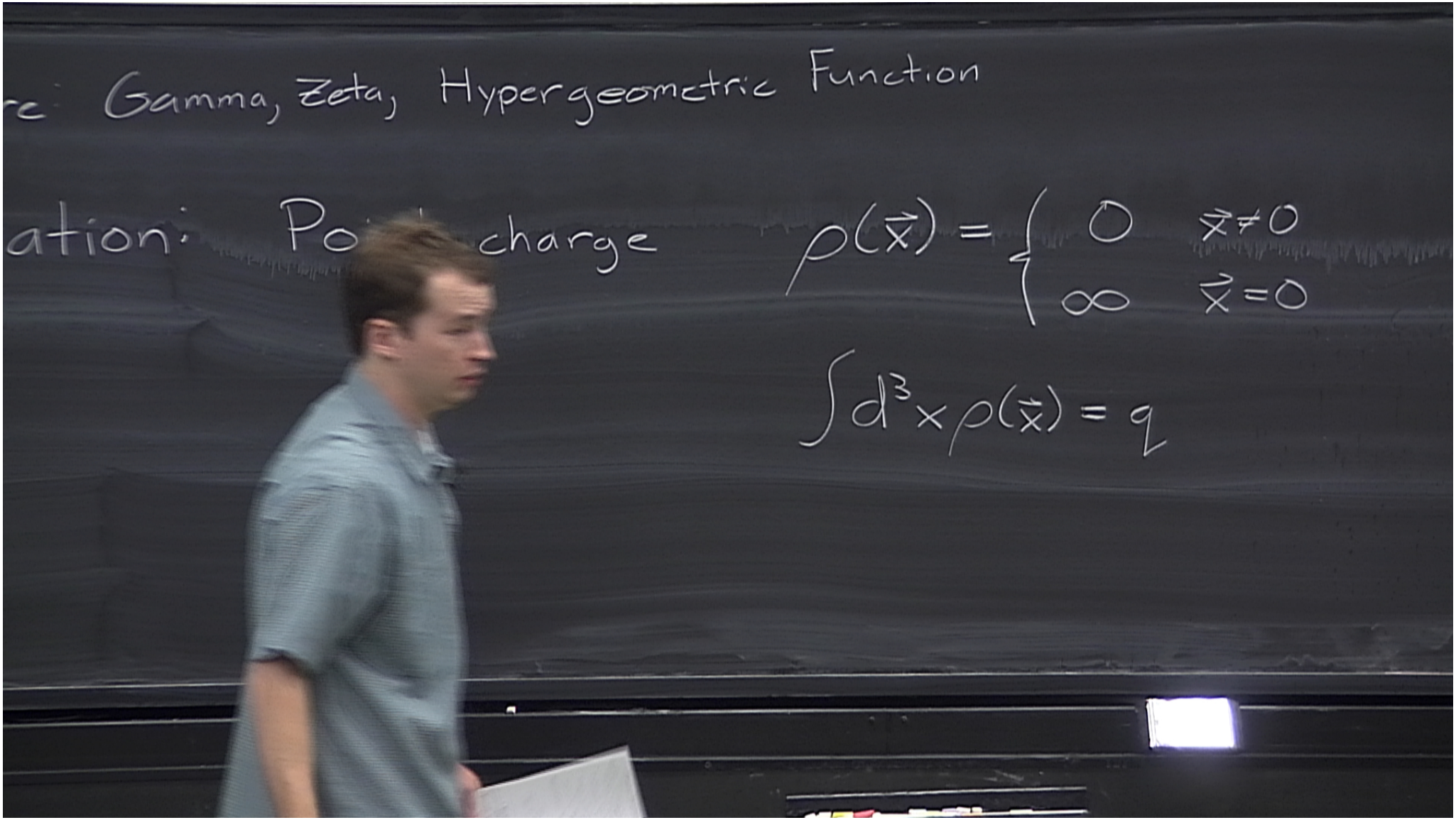

nctions + Distributions
Test functions, Distribution + their derivatives

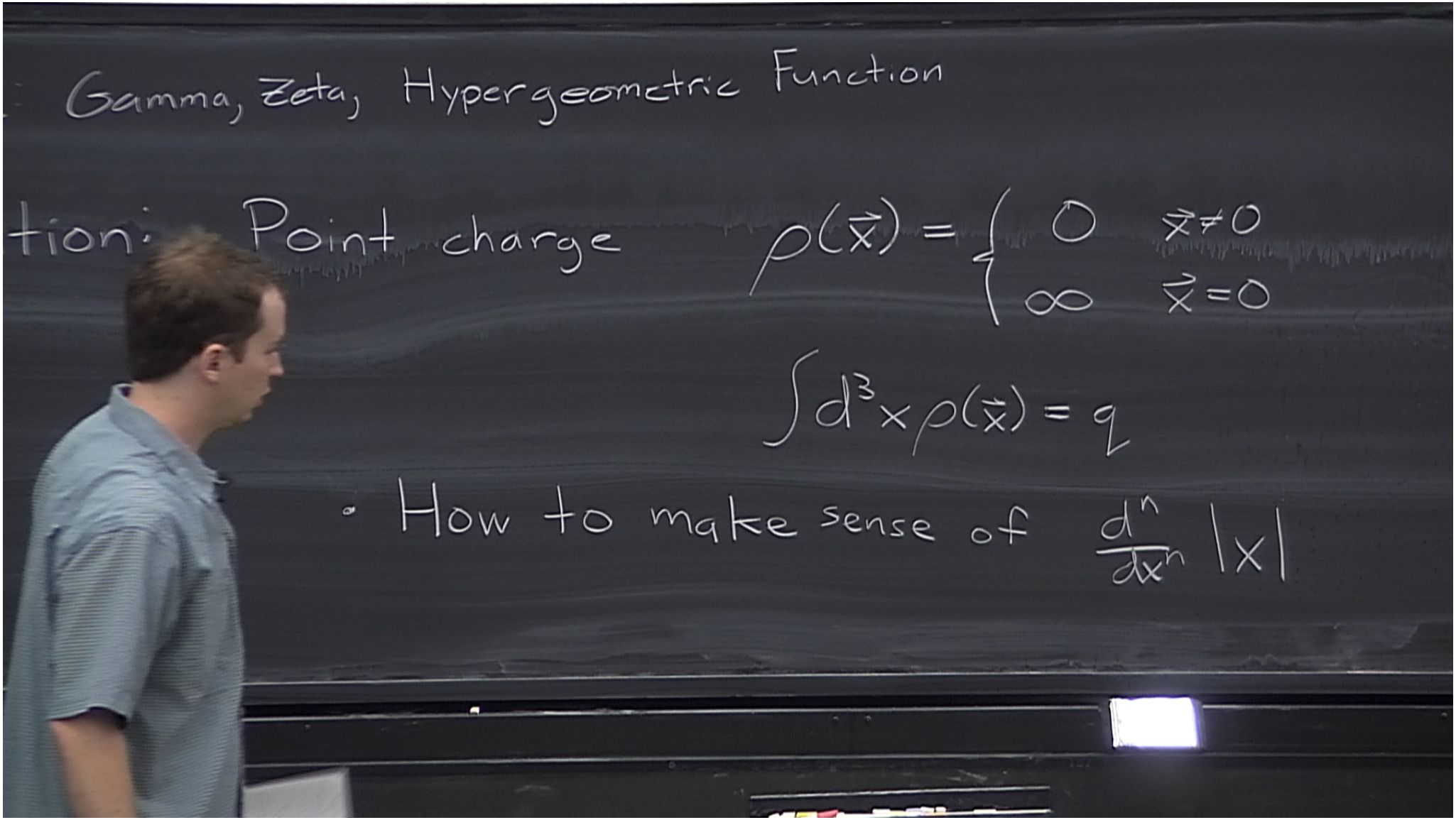

Special Functions + Distributions
Lecture 1: Test functions, Distribution + their
Lectures 2+3: Orthogonal Polynomials
Construction, Recurrence Relations,

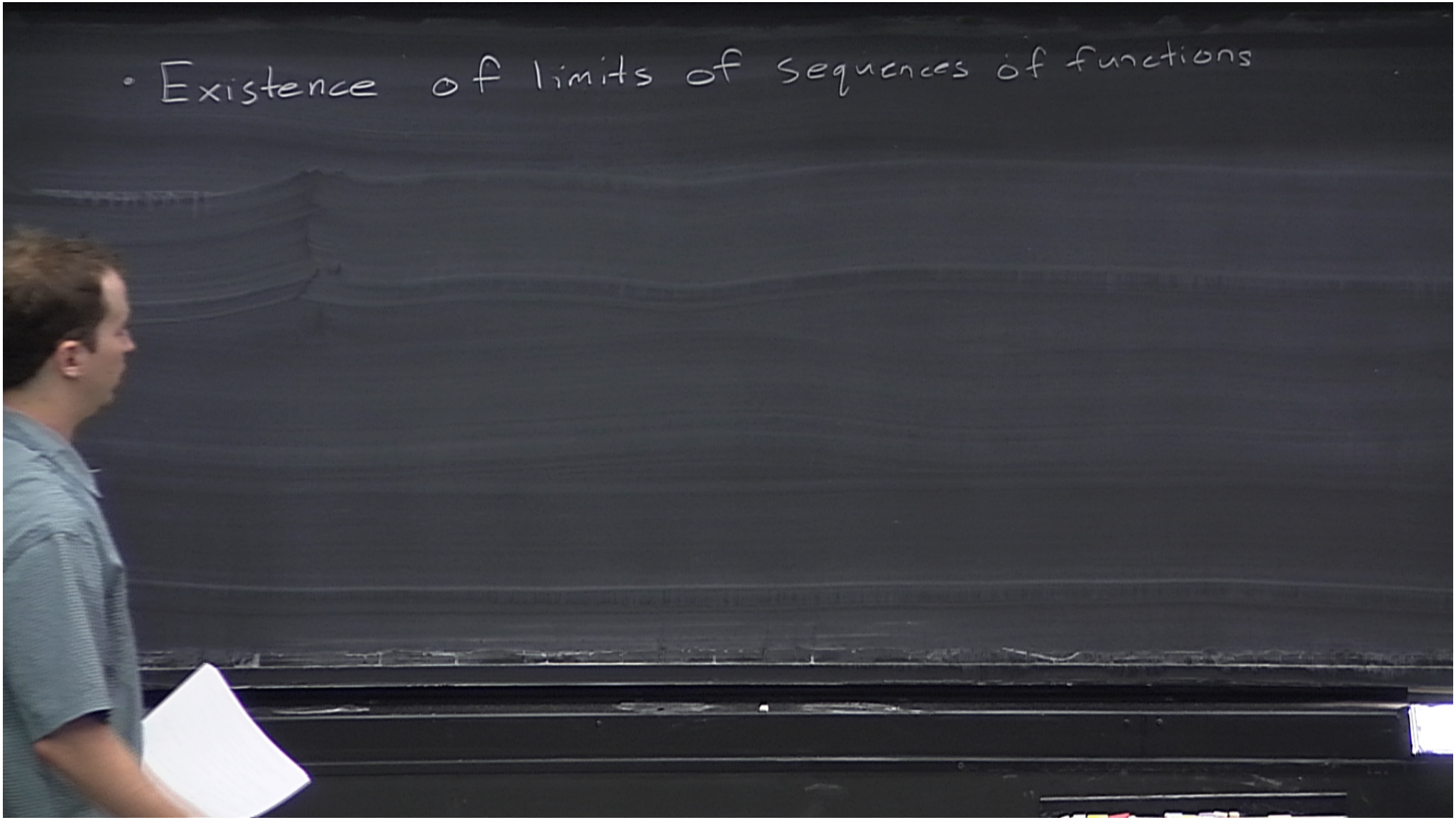

Functions + Distributions
: Test functions, Distribution + their derivatives
3: Orthogonal Polynomials
Construction, Recurrence Relations, Zeros, Rodrigues' Formula
Complete classification, Sturm-Liouville Theory

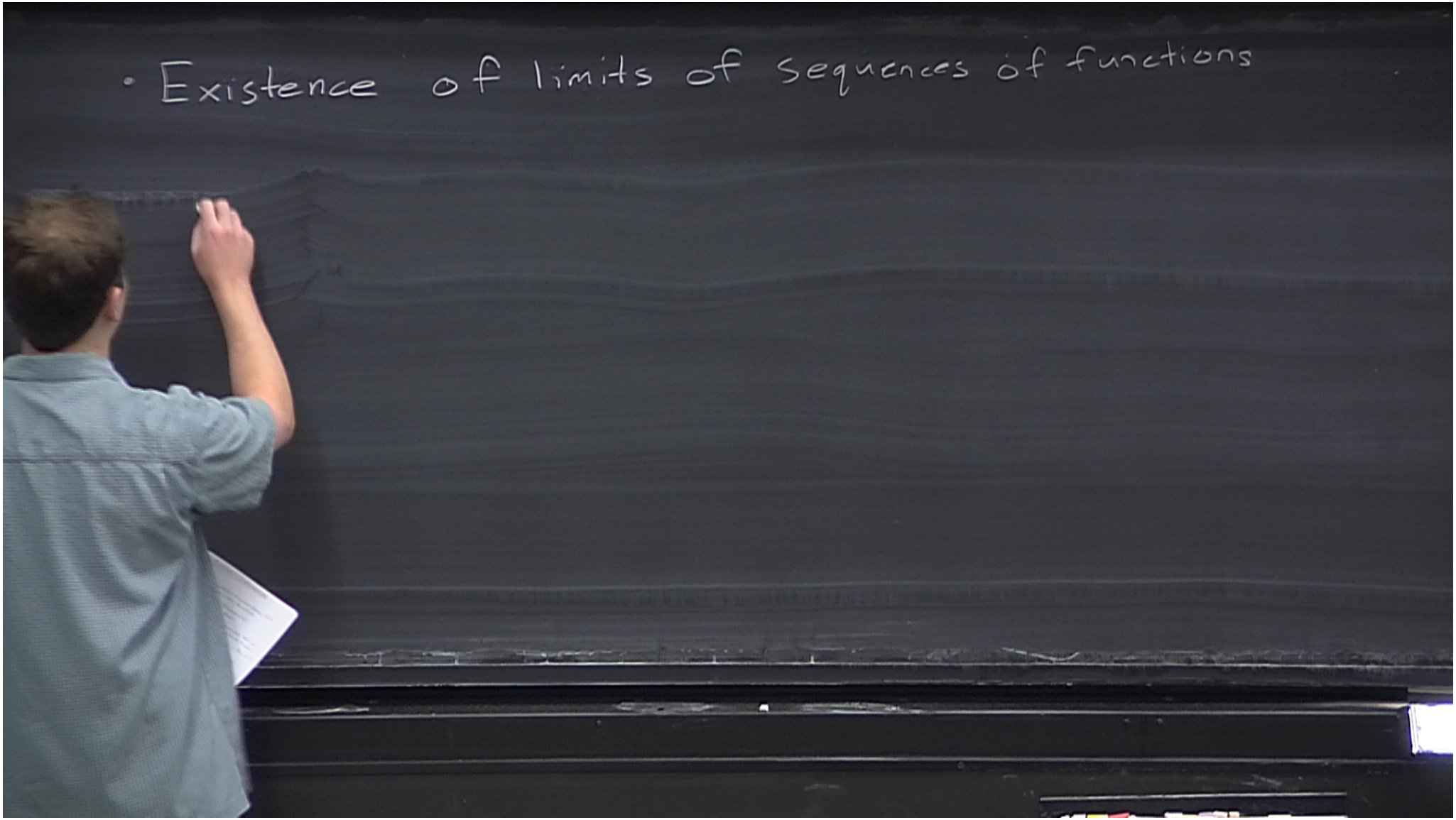

Lecture: Gamma, Zeta, Hypergeometric Function

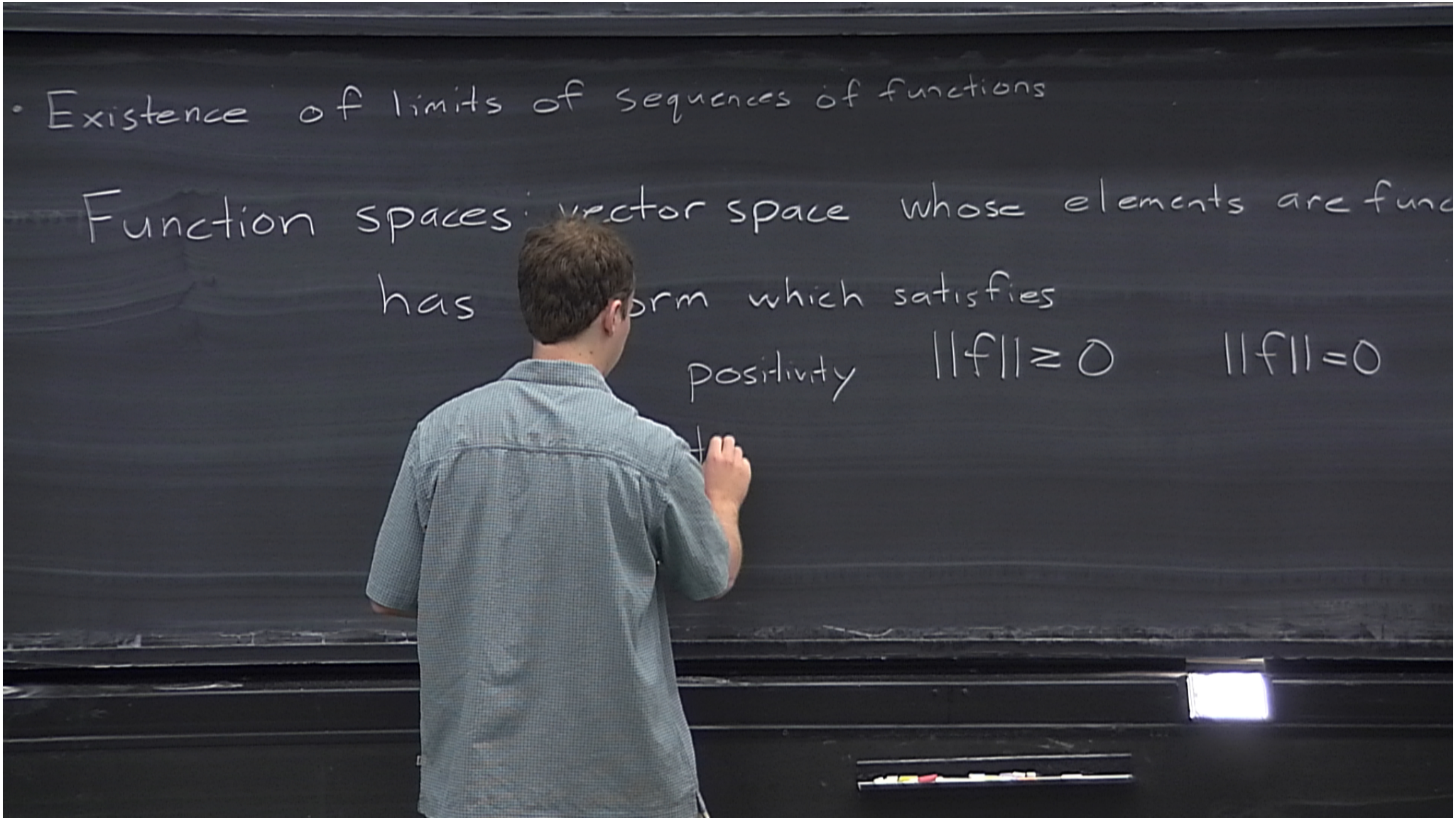

Lecture: Gamma, Zeta, Hypergeometric Function
Motivation:

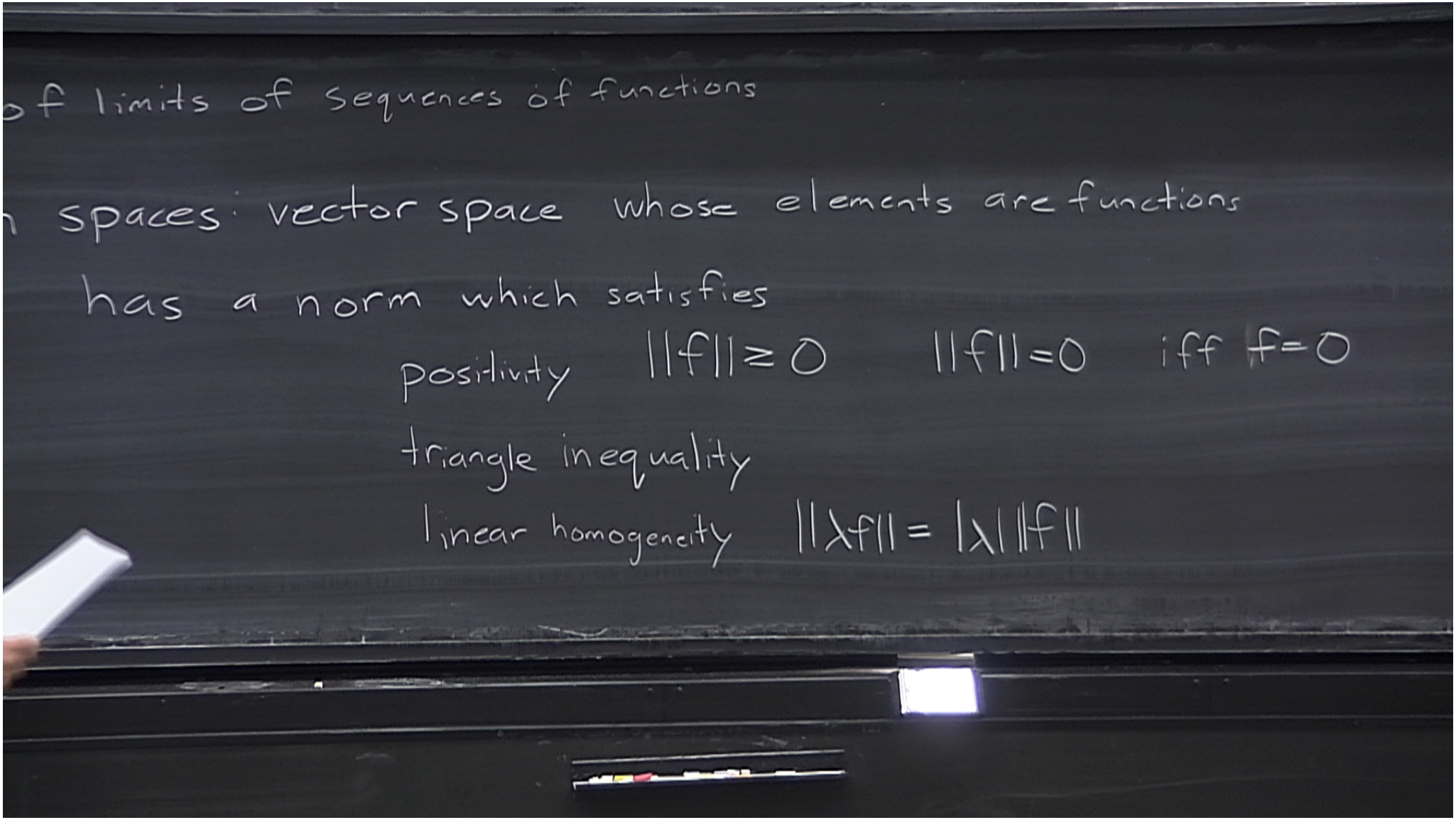

re: Gamma, Zeta, Hypergeometric Function
ation: Po charge
$\rho(\vec{x}) = \begin{cases} 0 & \vec{x} \neq 0 \\ \infty & \vec{x} = 0 \end{cases}$
$\int d^3x \rho(\vec{x}) = q$

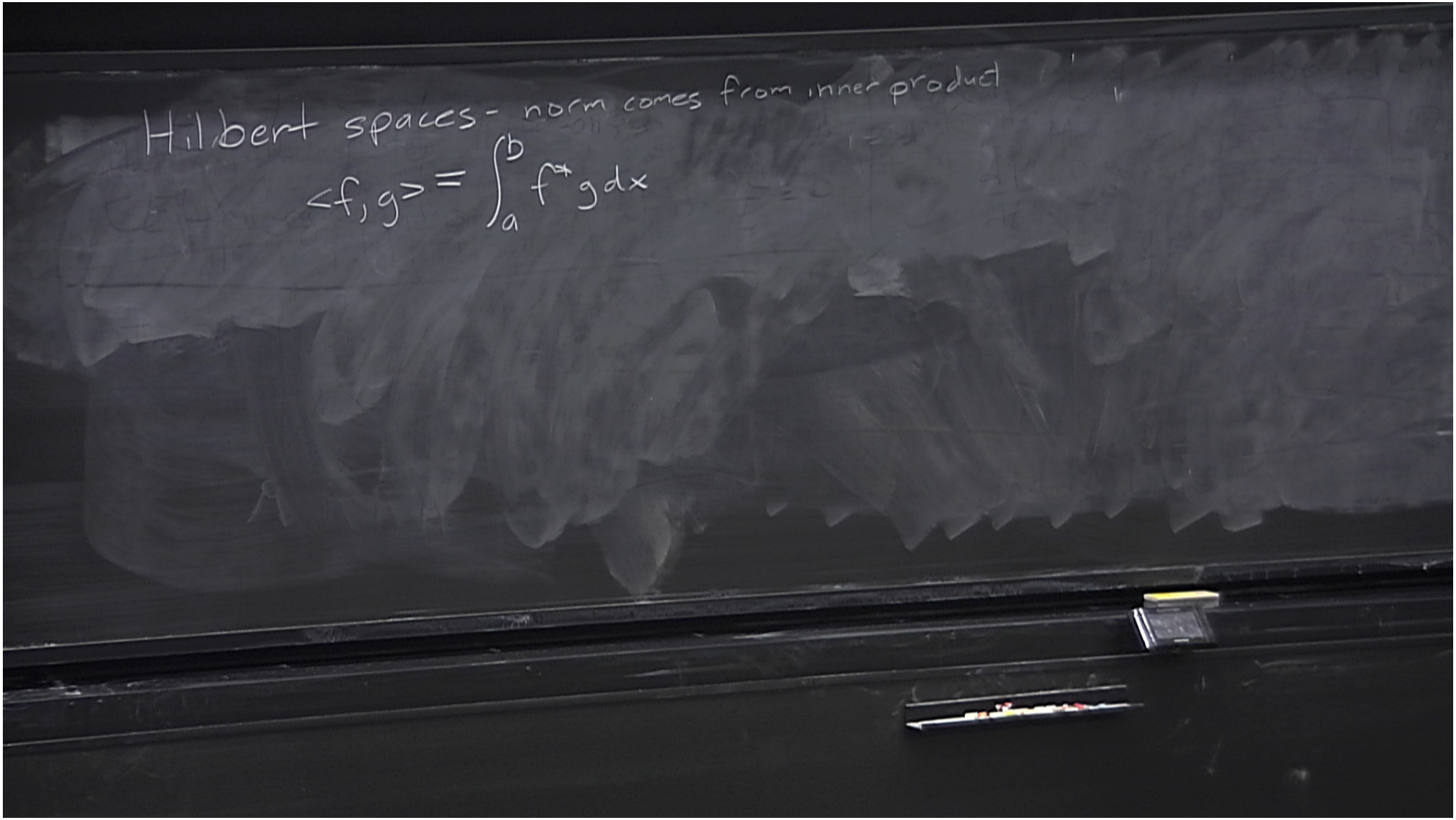

Gamma, Zeta, Hypergeometric Function
tion: Point charge
$\rho(\vec{x}) = \begin{cases} 0 & \vec{x} \neq 0 \\ \infty & \vec{x} = 0 \end{cases}$
$\int d^3x \rho(\vec{x}) = q$
• How to make sense of $\frac{d^n}{dx^n}|x|$

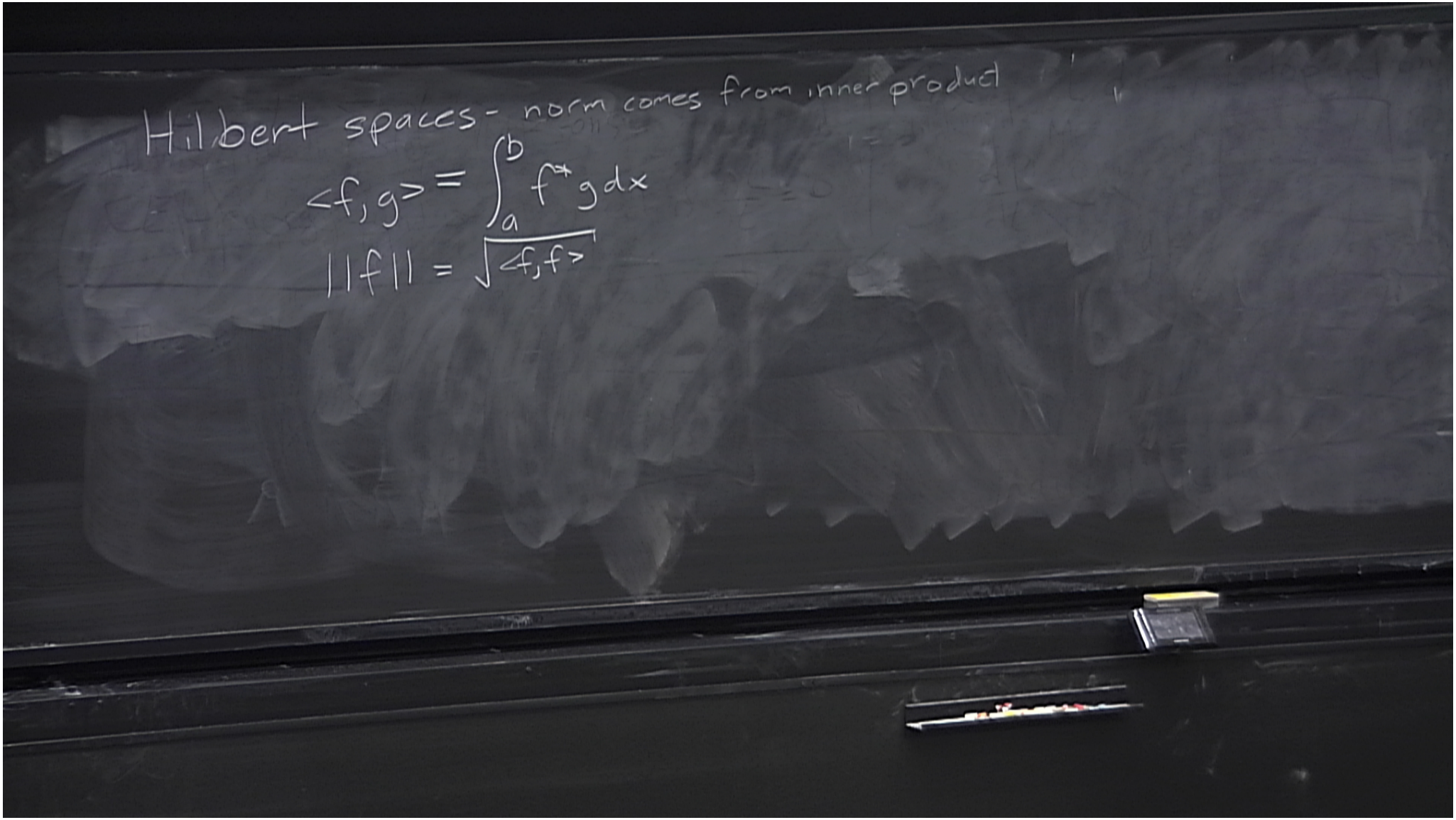

• Existence of limits of sequences of functions

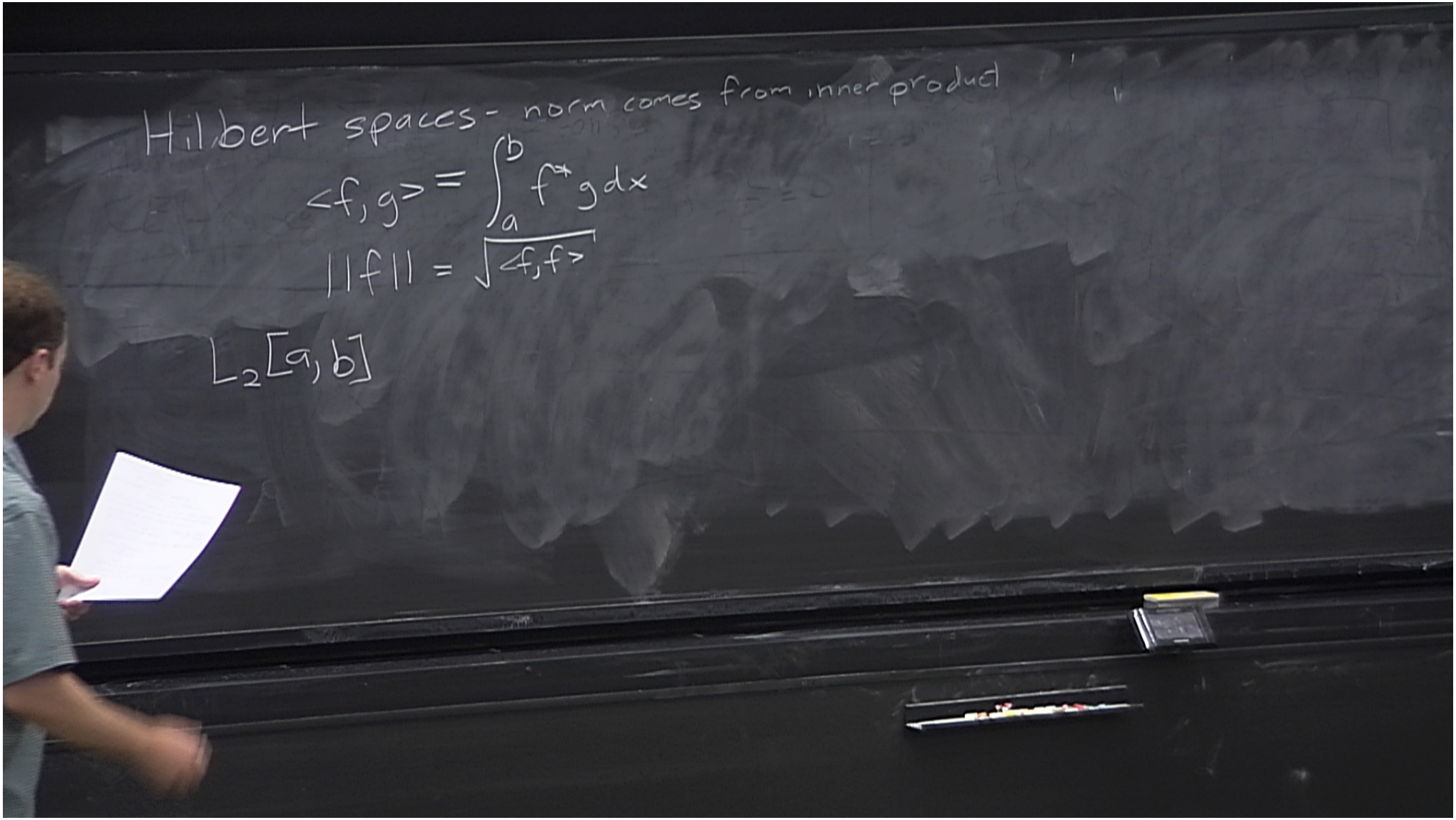

• Existence of limits of sequences of functions

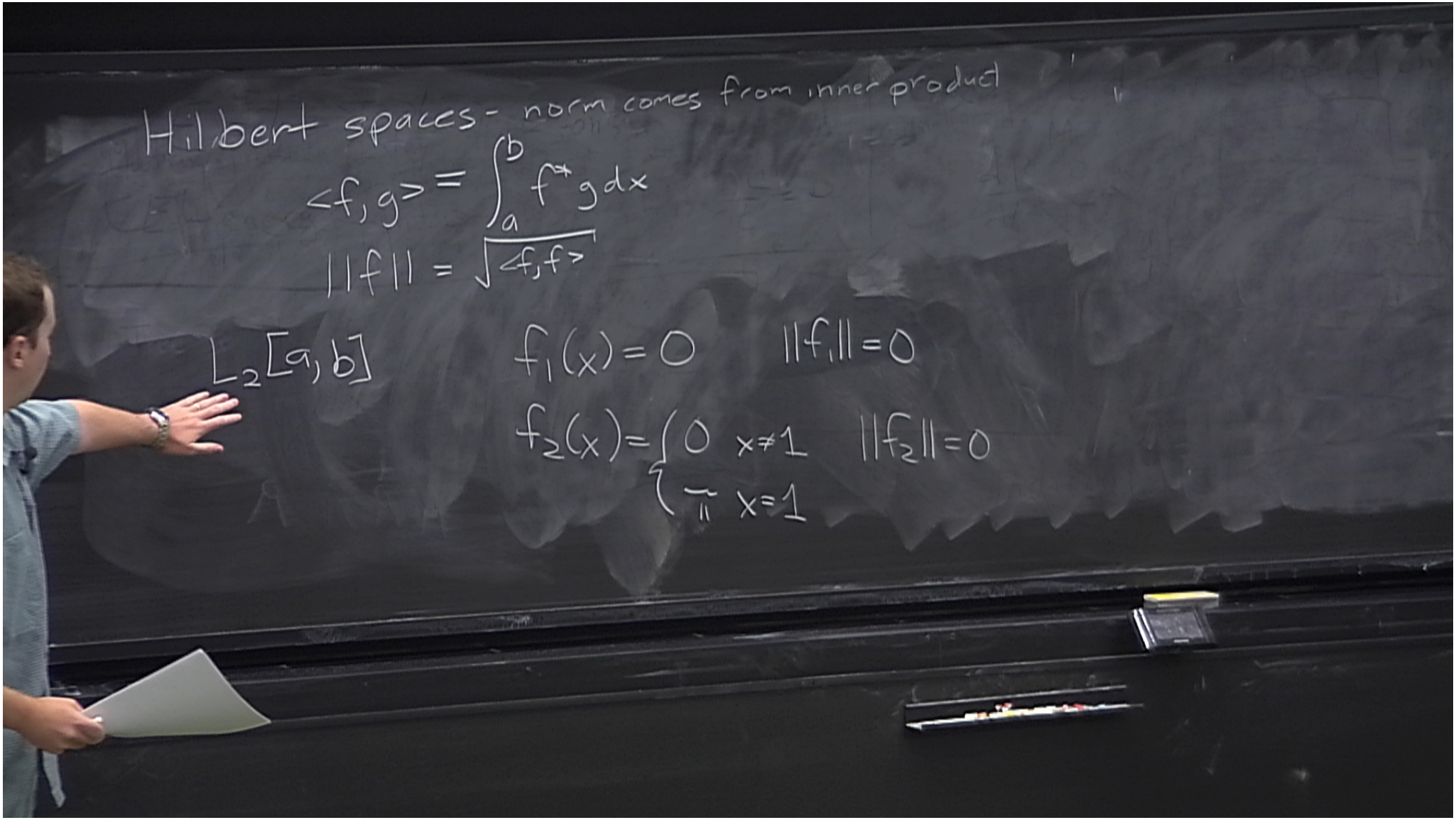

• Existence of limits of sequences of functions
Function spaces: vector space whose elements are func
has orm which satisfies
positivity ||f|| ≥ 0 ||f|| = 0

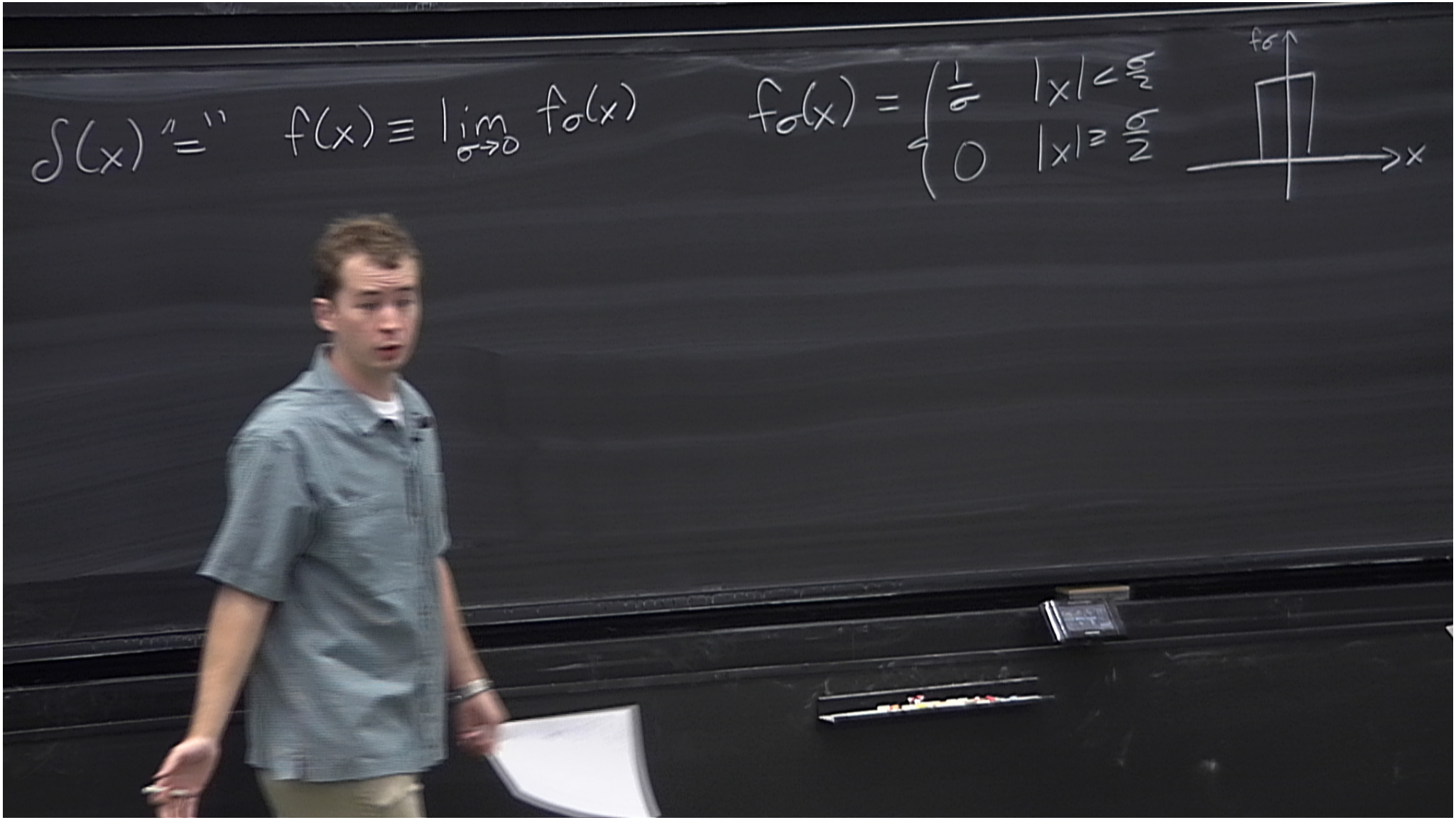

of limits of sequences of functions
spaces: vector space whose elements are functions
has a norm which satisfies
positivity ||f|| ≥ 0 ||f|| = 0 iff f = 0
triangle inequality
linear homogeneity ||λf|| = |λ| ||f||


Hilbert spaces - norm comes from inner product
$\langle f, g \rangle = \int_a^b f^* g \, dx$


Hilbert spaces - norm comes from inner product
$\langle f, g \rangle = \int_a^b f^* g \, dx$
$\|f\| = \sqrt{\langle f, f \rangle}$

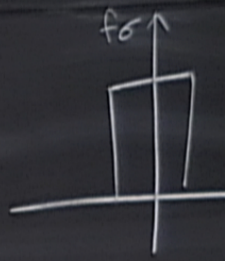

Hilbert spaces - norm comes from inner product
$\langle f, g \rangle = \int_a^b f^* g \, dx$
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$L_2[a, b]$


Hilbert spaces - norm comes from inner product
$\langle f, g \rangle = \int_a^b f^* g \, dx$
$\|f\| = \sqrt{\langle f, f \rangle}$
$L_2[a, b]$
$f_1(x) = 0$ $\|f_1\| = 0$
$f_2(x) = \begin{cases} 0 & x \neq 1 \\ \pi & x = 1 \end{cases}$ $\|f_2\| = 0$


$\delta(x)$ "$=$" $f(x) \equiv \lim_{\sigma \to 0} f_\sigma(x)$
$f_\sigma(x) = \begin{cases} \frac{1}{\sigma} & |x| < \frac{\sigma}{2} \\ 0 & |x| \geq \frac{\sigma}{2} \end{cases}$
f_σ
x

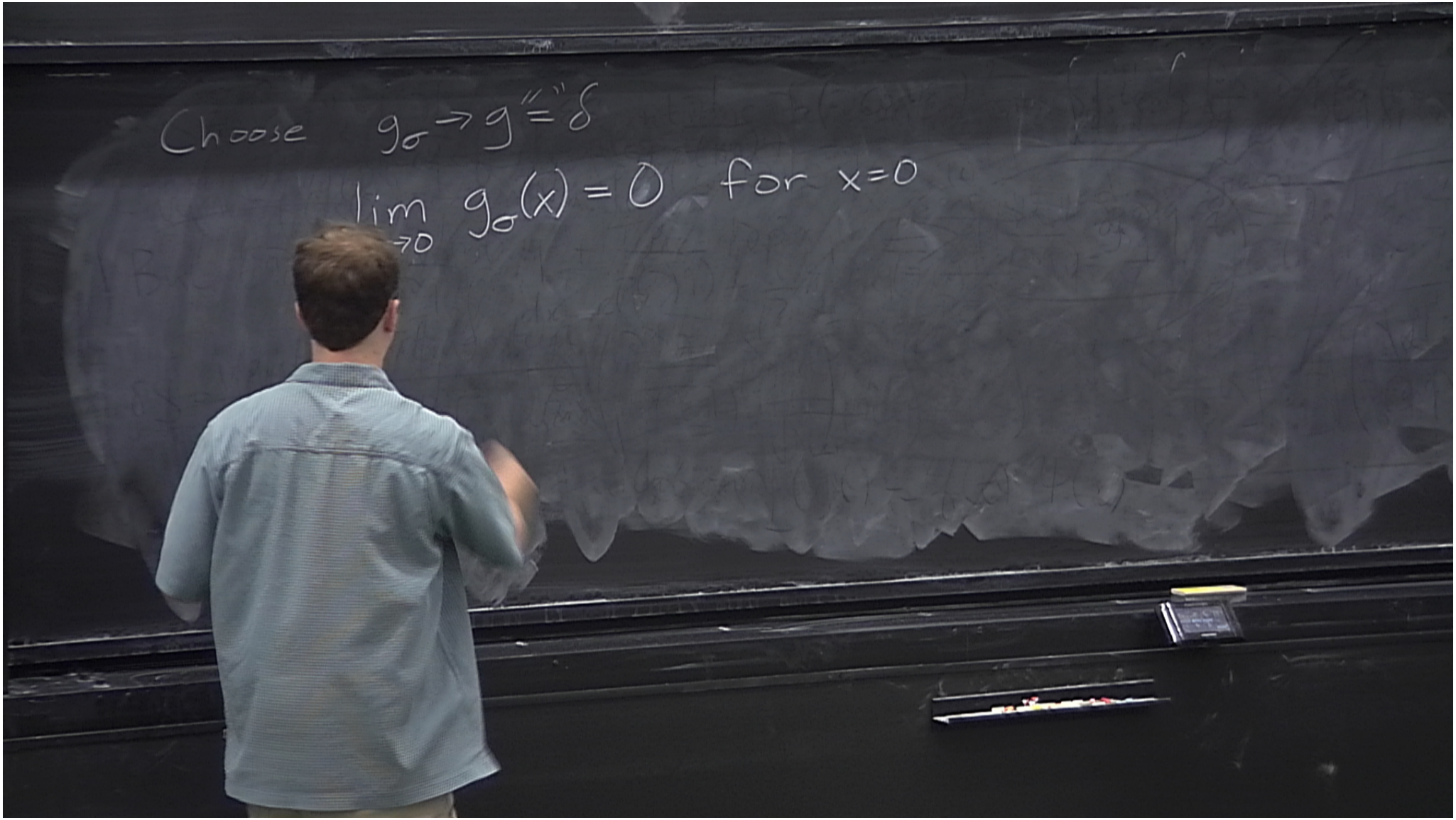
$\delta(x)$ "$=$" $f(x) \equiv \begin{cases} \lim_{\sigma \to 0} f_\sigma(x) \\ \text{or any other definitions} \end{cases}$

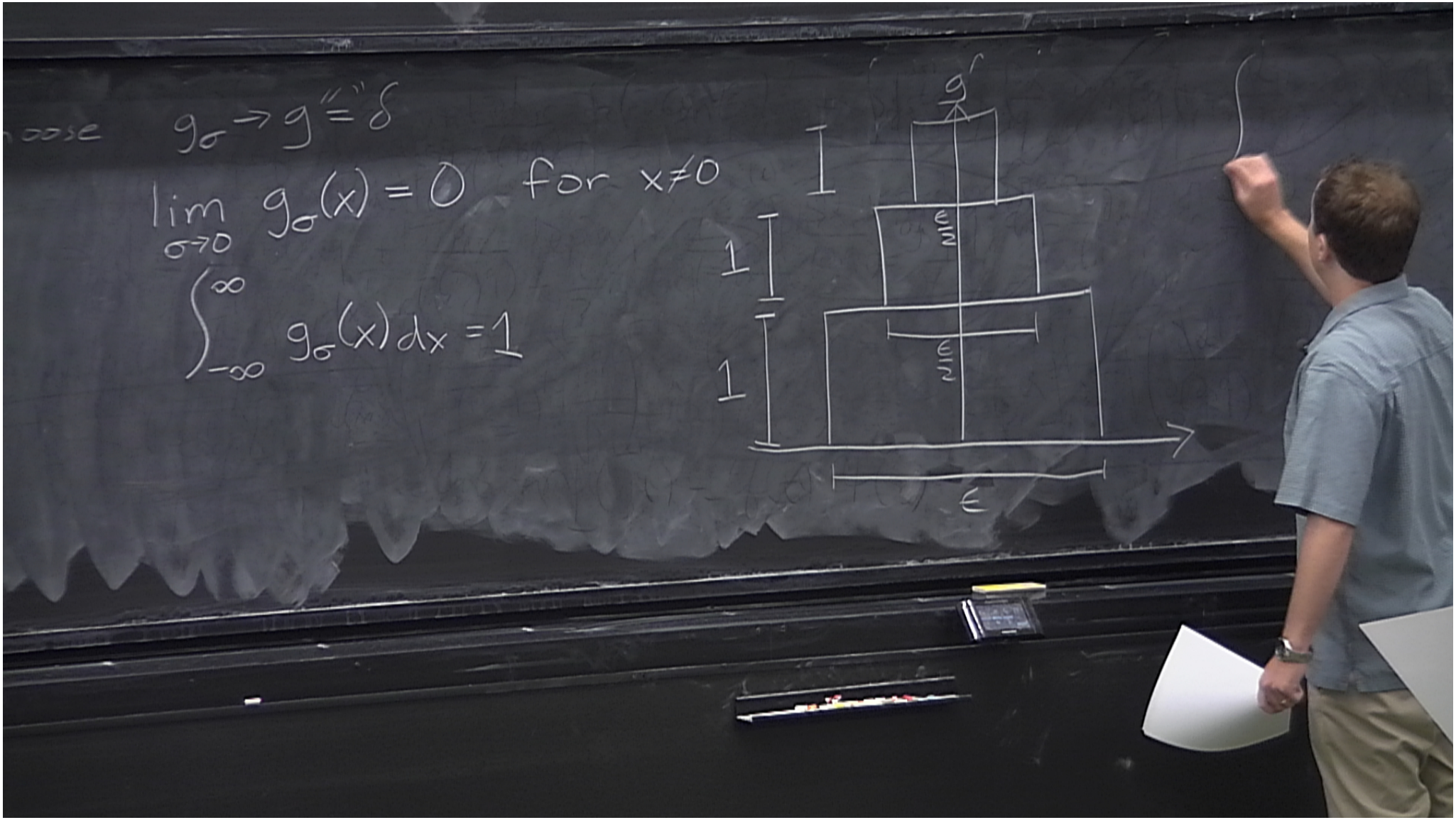
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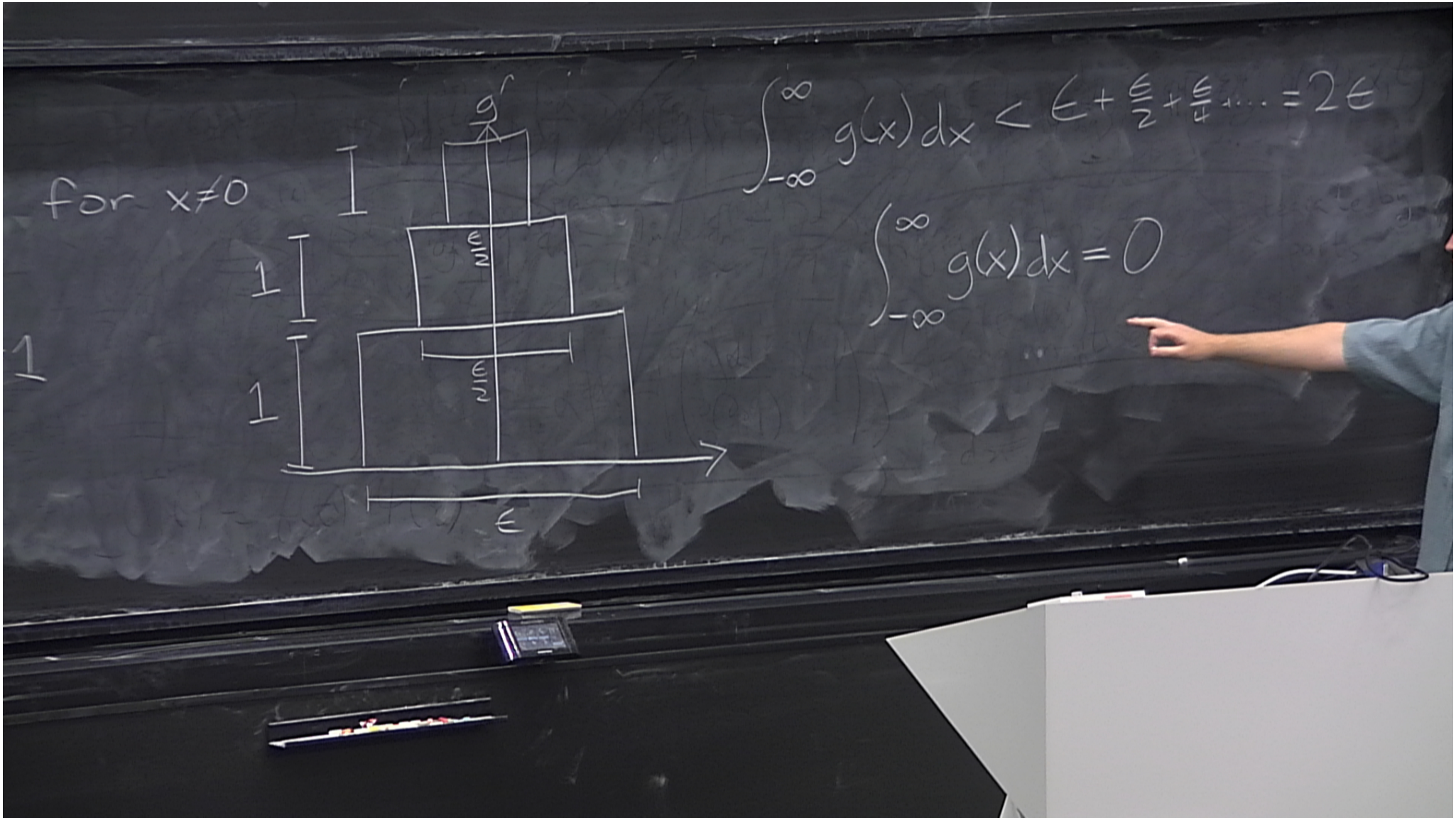


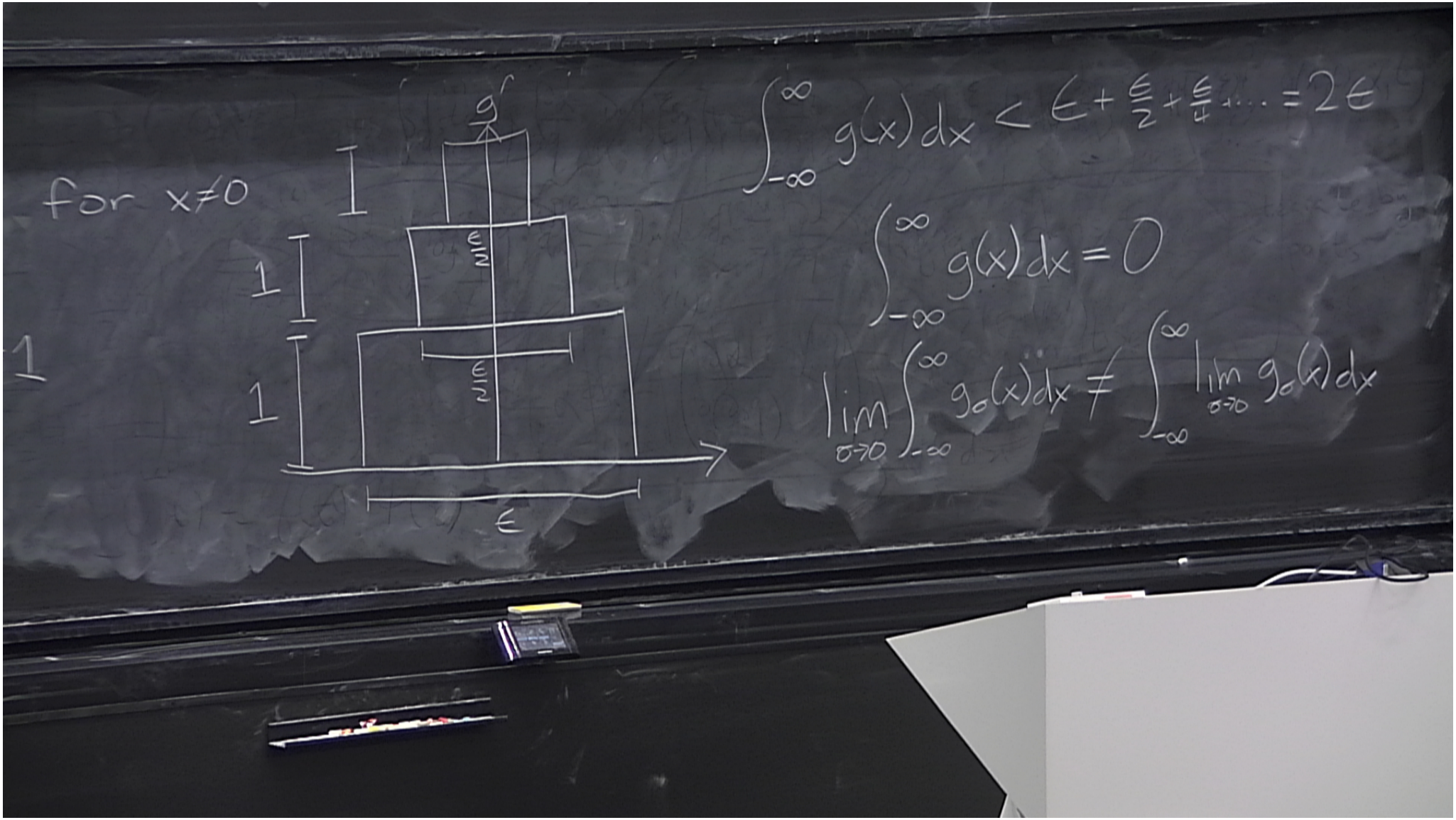
In $L_2[a,b]$ $\|f_\sigma\|^2 = \int_{-\infty}^{\infty} |f_\sigma(x)|^2 dx$

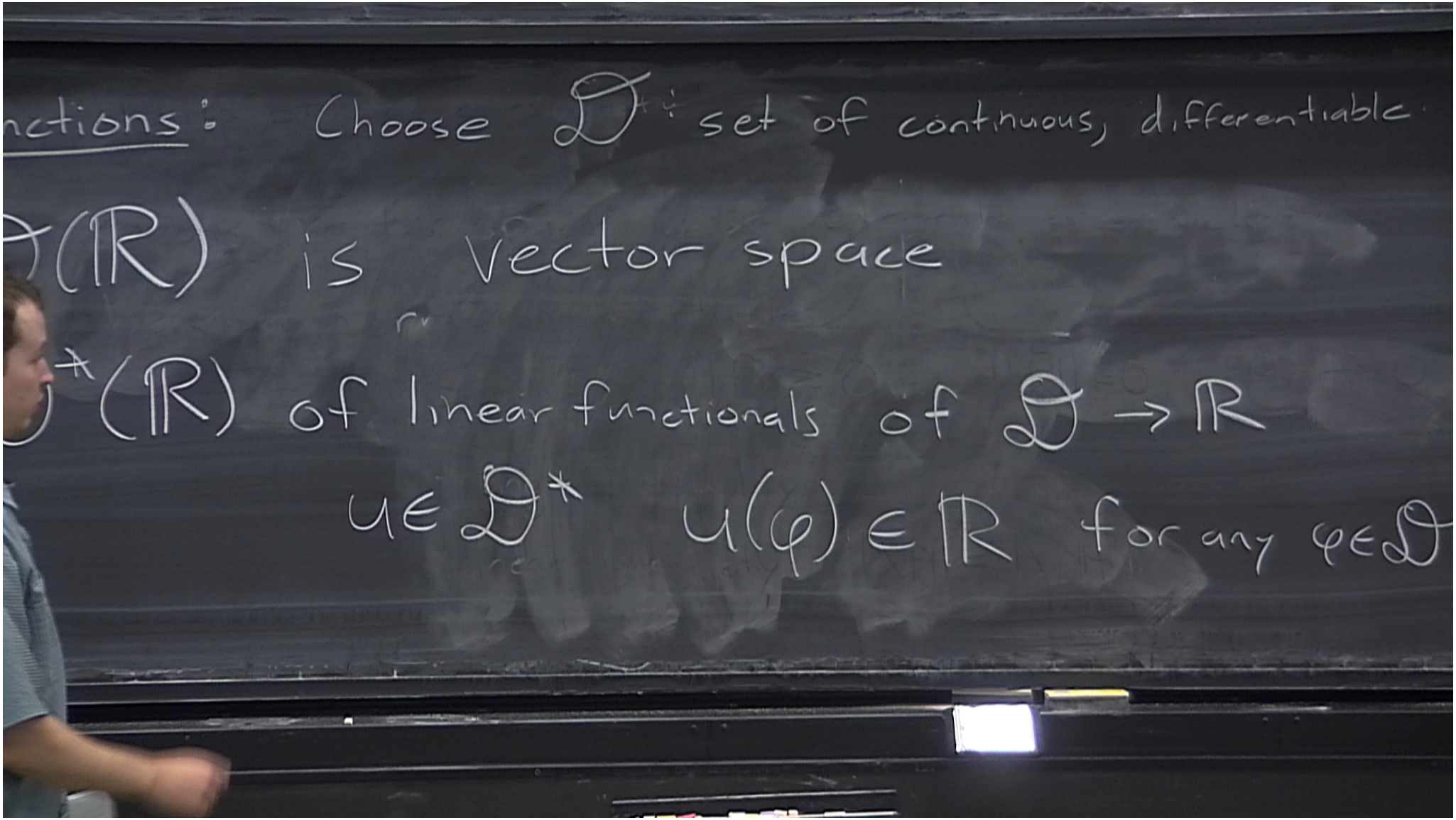
$= \int_{-\frac{\sigma}{2}}^{\frac{\sigma}{2}} \frac{1}{\sigma^2} dx = \frac{1}{\sigma}$

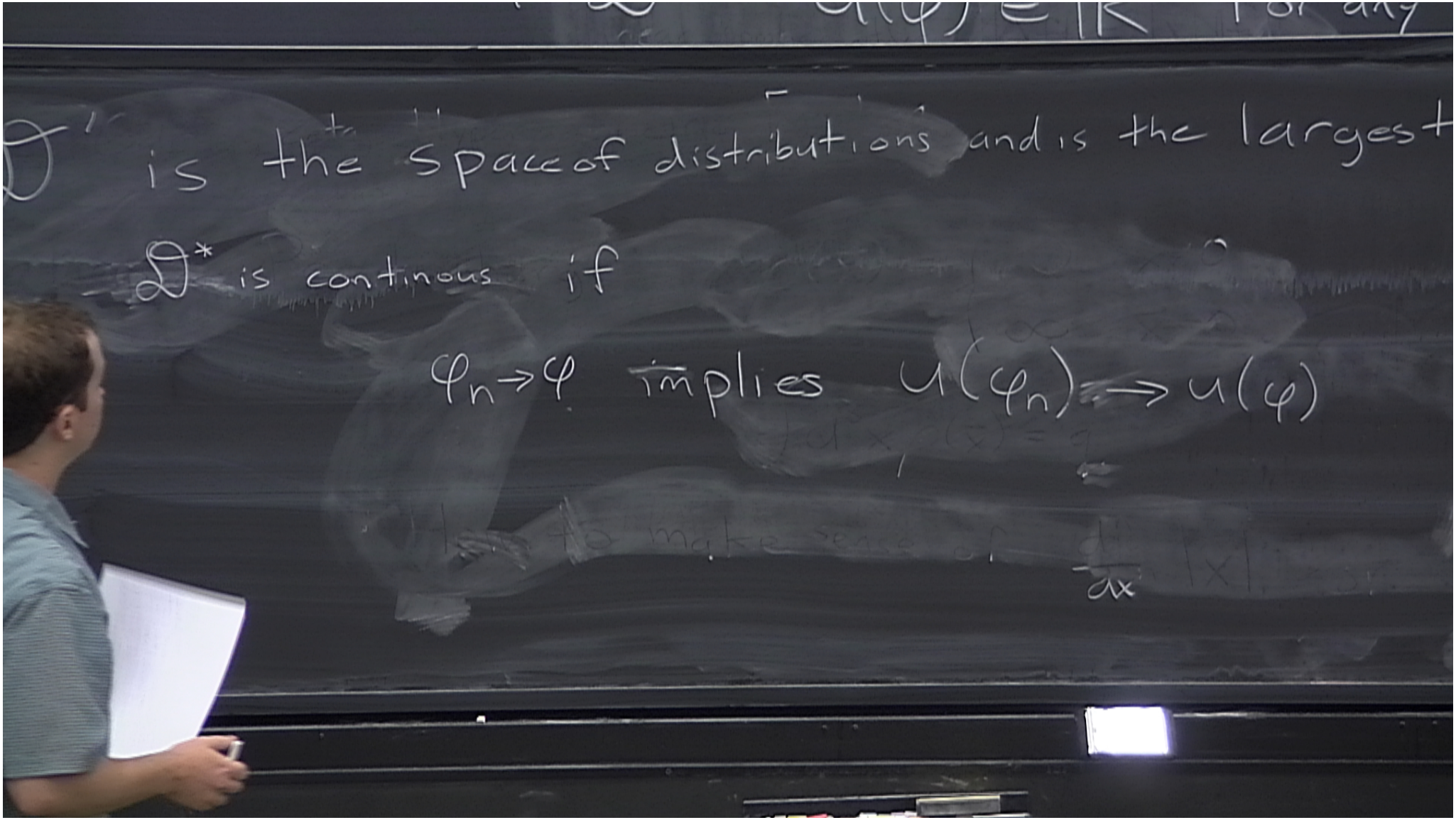

Choose $g_\sigma \to g'' = \delta$
$\lim_{\to 0} g_\sigma(x) = 0$ for $x=0$

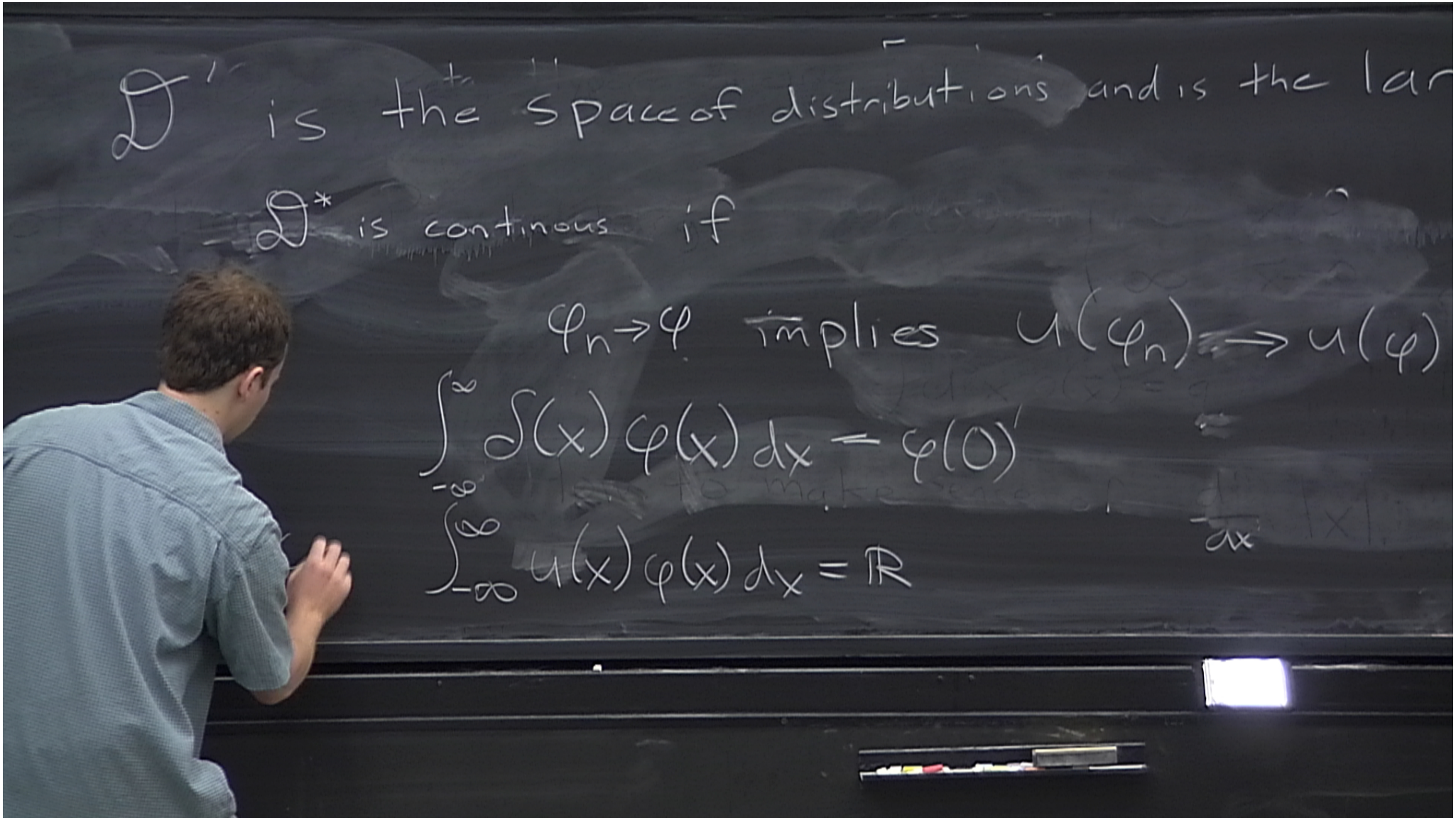

oose $g_\sigma \to g = \delta$
$\lim_{\sigma \to 0} g_\sigma(x) = 0$ for $x \neq 0$
$\int_{-\infty}^{\infty} g_\sigma(x)\,dx = 1$
g_σ
1
1
$\epsilon/2$
$\epsilon/2$
ϵ


for x≠0
g
1
1
1
ε/2
ε/2
ε
$\int_{-\infty}^{\infty} g(x)\,dx < \epsilon + \frac{\epsilon}{2} + \frac{\epsilon}{4} \cdots = 2\epsilon$
$\int_{-\infty}^{\infty} g(x)\,dx = 0$


for x≠0
g
1
1
1
ε/2
ε/4
ε
$\int_{-\infty}^{\infty} g(x)\,dx < \epsilon + \frac{\epsilon}{2} + \frac{\epsilon}{4} + \ldots = 2\epsilon$
$\int_{-\infty}^{\infty} g(x)\,dx = 0$
$\lim_{\sigma\to 0} \int_{-\infty}^{\infty} g_\sigma(x)\,dx \neq \int_{-\infty}^{\infty} \lim_{\sigma\to 0} g_\sigma(x)\,dx$


nctions: Choose $\mathcal{D}$: set of continuous, differentiable
$\mathcal{D}(\mathbb{R})$ is vector space
$\mathcal{D}^*(\mathbb{R})$ of linear functionals of $\mathcal{D} \to \mathbb{R}$
$u \in \mathcal{D}^*$ $u(\varphi) \in \mathbb{R}$ for any $\varphi \in \mathcal{D}$


$\mathcal{D}'$ is the space of distributions and is the largest
$\mathcal{D}^*$ is continous if
$\varphi_n \to \varphi$ implies $u(\varphi_n) \to u(\varphi)$


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$\mathcal{D}^*$ is continous if
$\varphi_n \to \varphi$ implies $u(\varphi_n) \to u(\varphi)$
$\int_{-\infty}^{\infty} \delta(x)\varphi(x)\,dx = \varphi(0)$
$\int_{-\infty}^{\infty} u(x)\varphi(x)\,dx = \mathbb{R}$

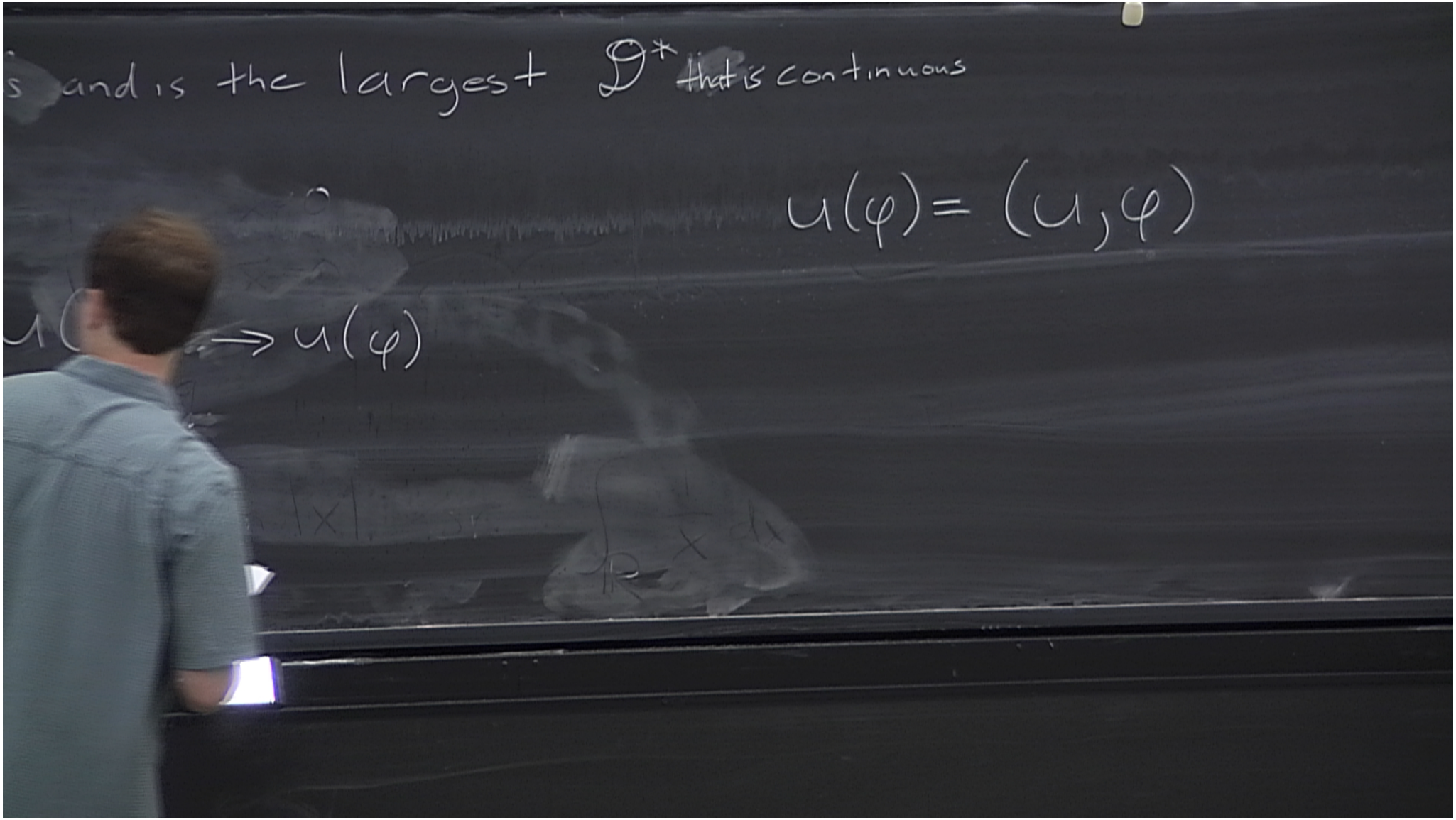
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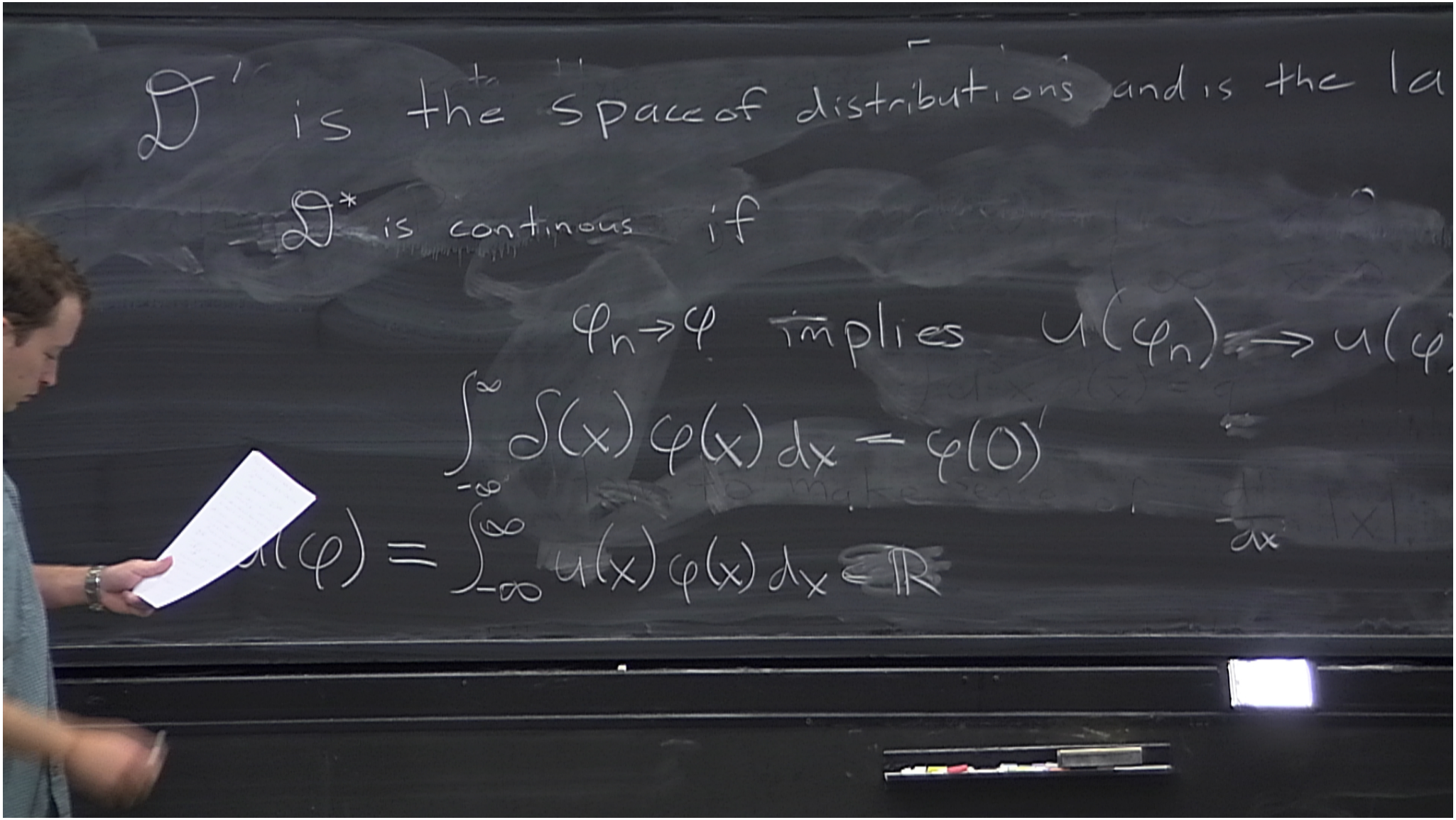
$\mathcal{D}^*$ is continous if

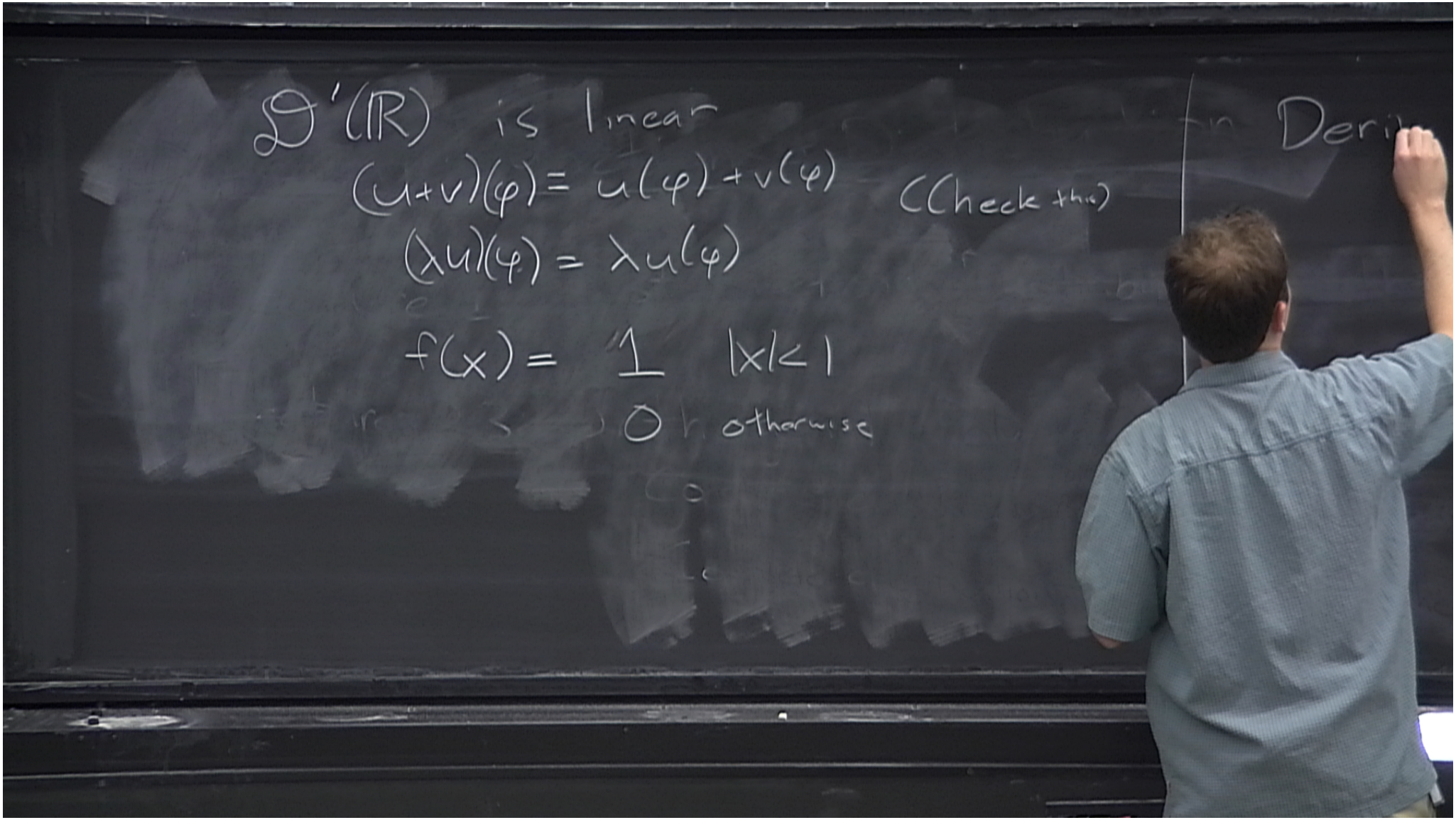
$$\varphi_n \to \varphi \text{ implies } u(\varphi_n) \to u(\varphi)$$

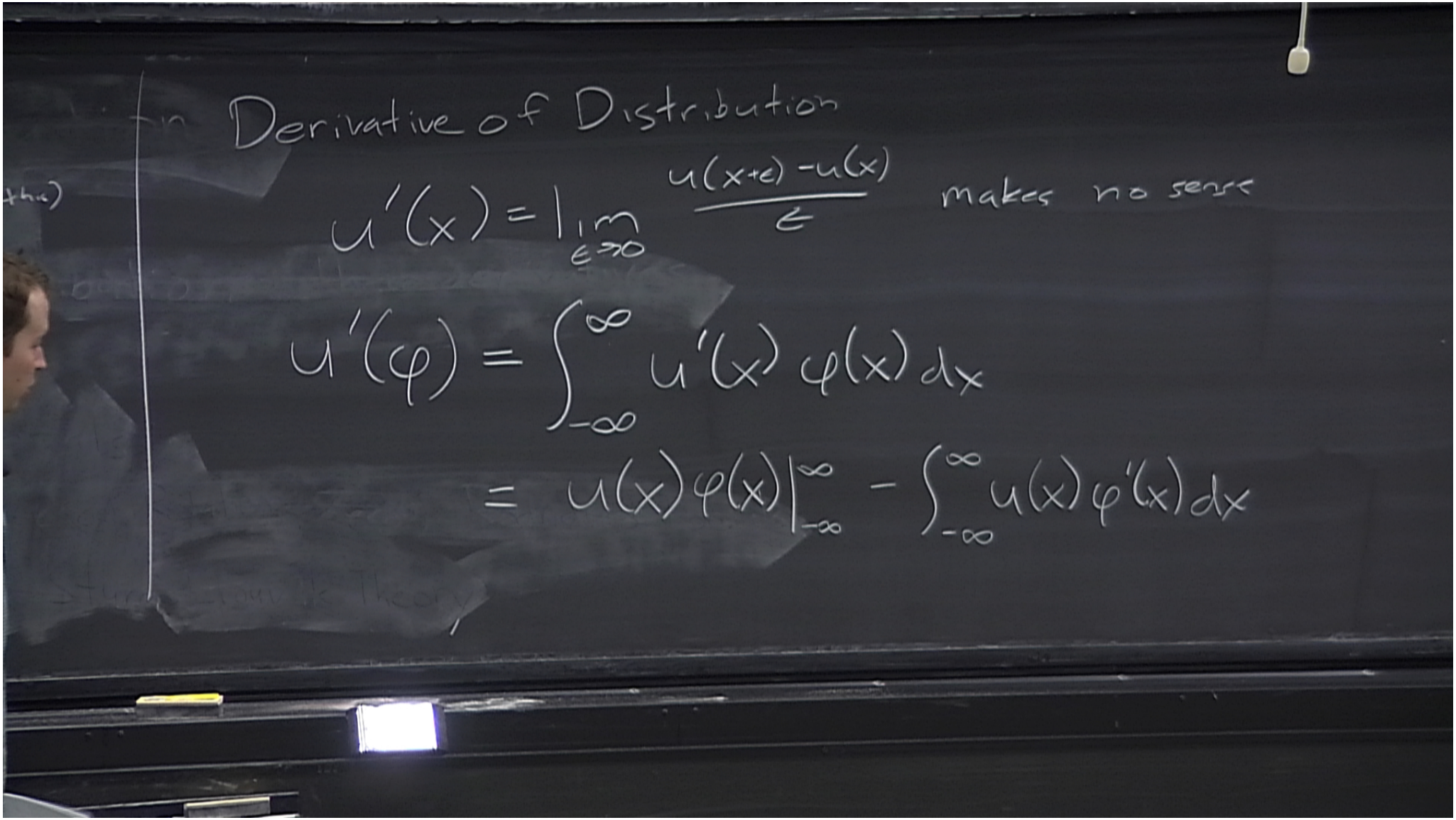
$$\int_{-\infty}^{\infty} \delta(x)\,\varphi(x)\,dx = \varphi(0)$$

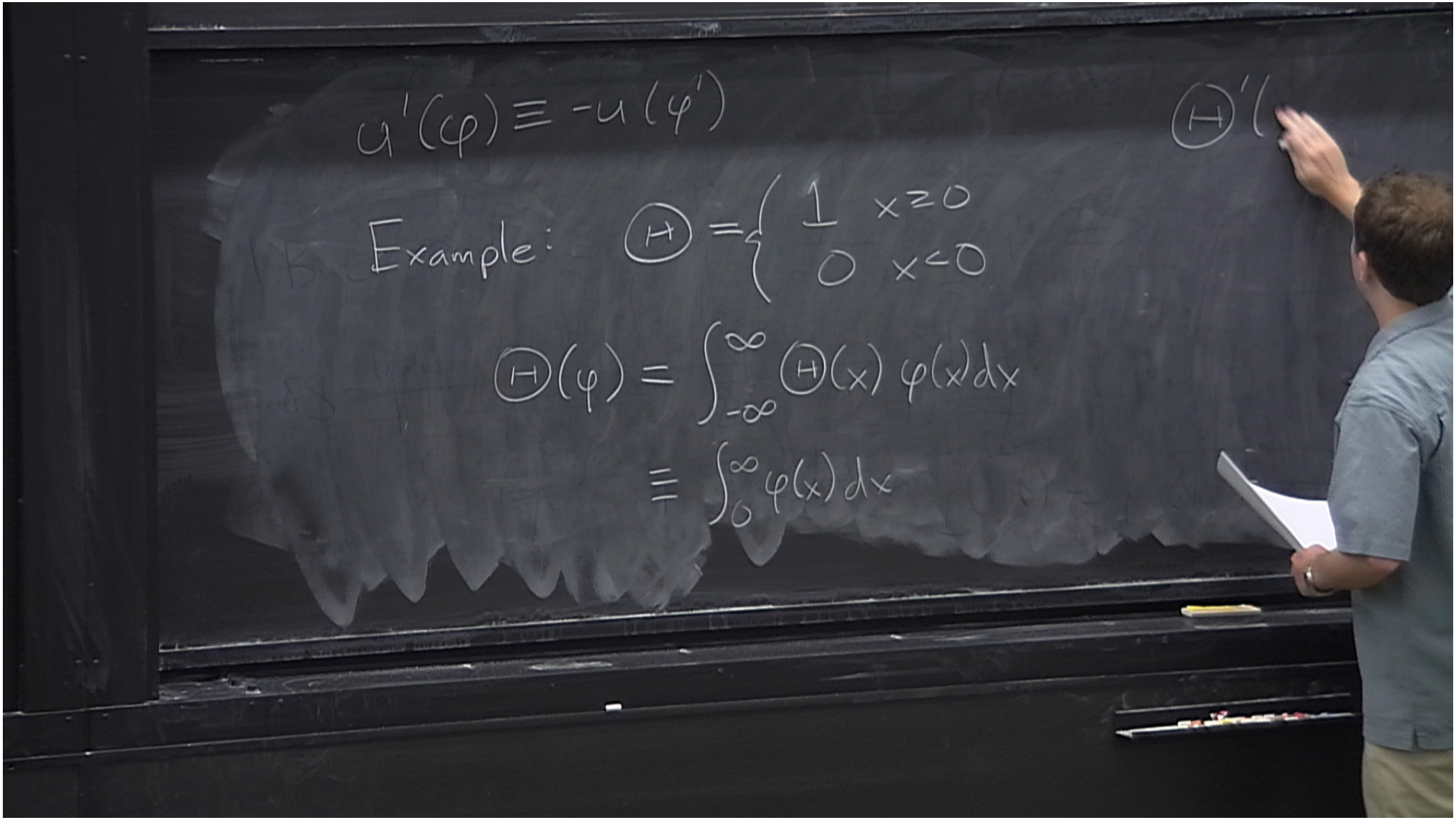
$$u(\varphi) = \int_{-\infty}^{\infty} u(x)\,\varphi(x)\,dx \in \mathbb{R}$$

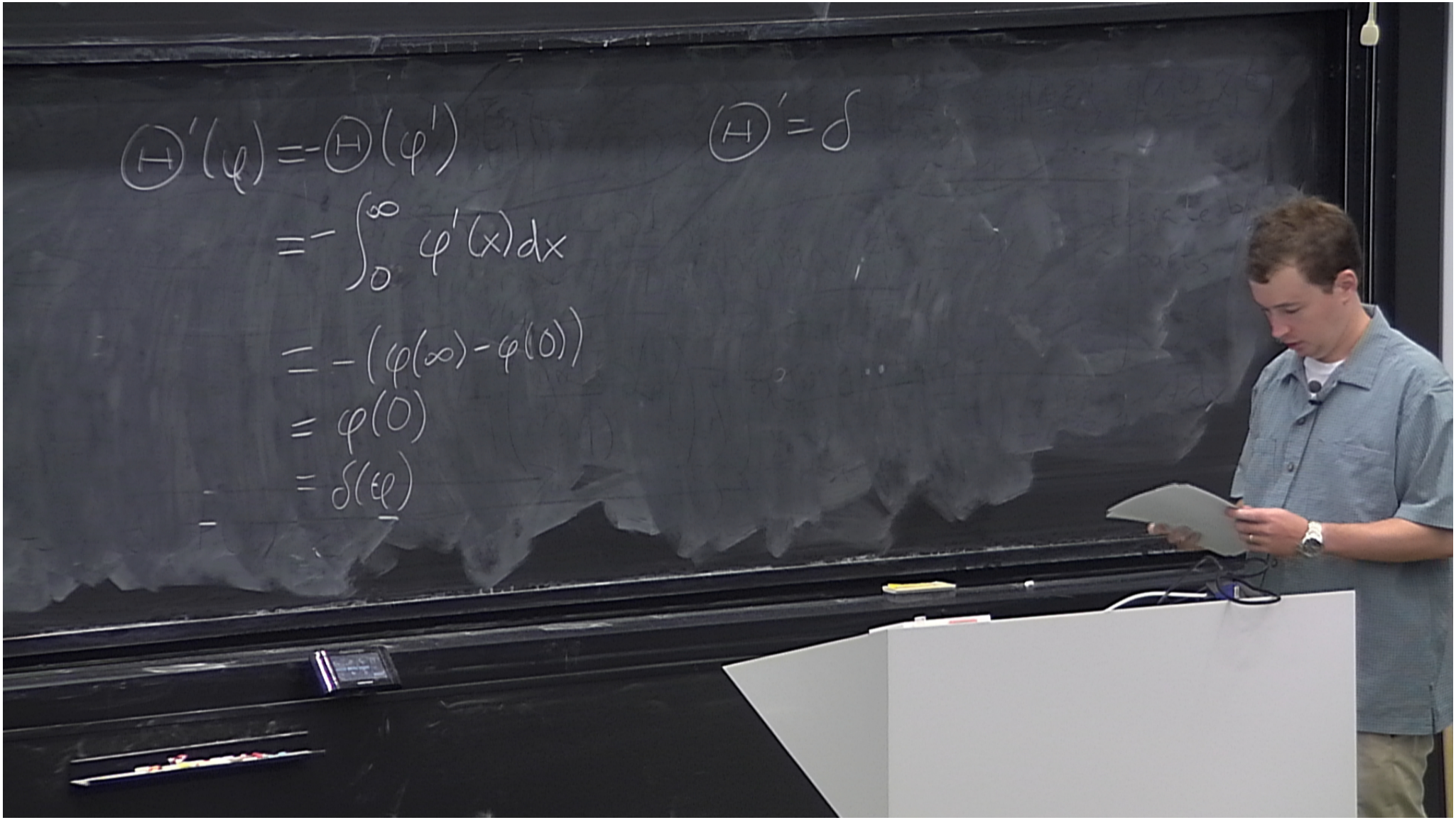

and is the largest $\mathcal{D}^*$ that is continuous
$u(\varphi) = (u, \varphi)$
$\to u(\varphi)$

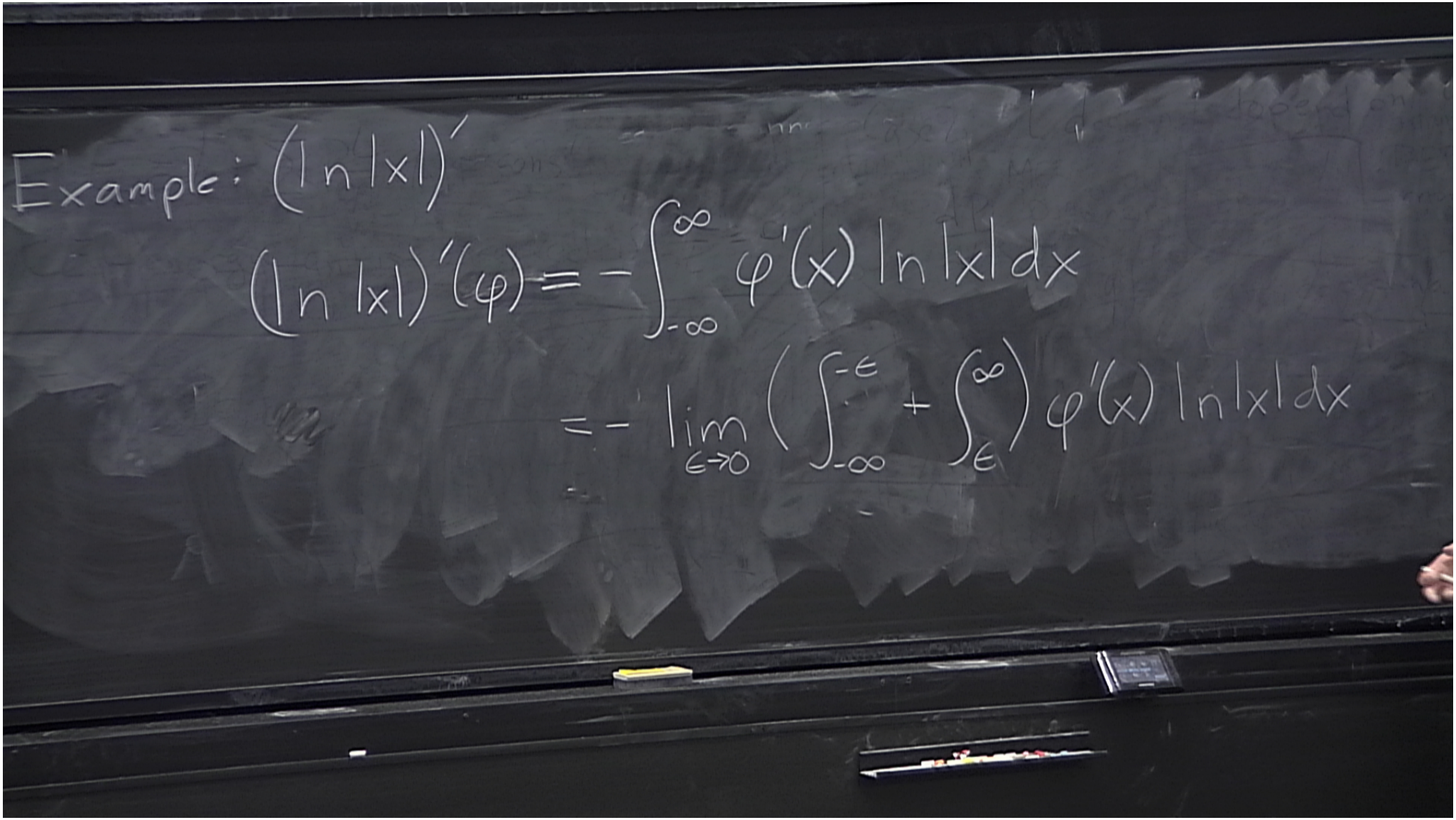

$\mathcal{D}'$ is the space of distributions and is the la
$\mathcal{D}^*$ is continous if
$\varphi_n \to \varphi$ implies $u(\varphi_n) \to u(\varphi$
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$u(\varphi) = \int_{-\infty}^{\infty} u(x)\,\varphi(x)\,dx \in \mathbb{R}$

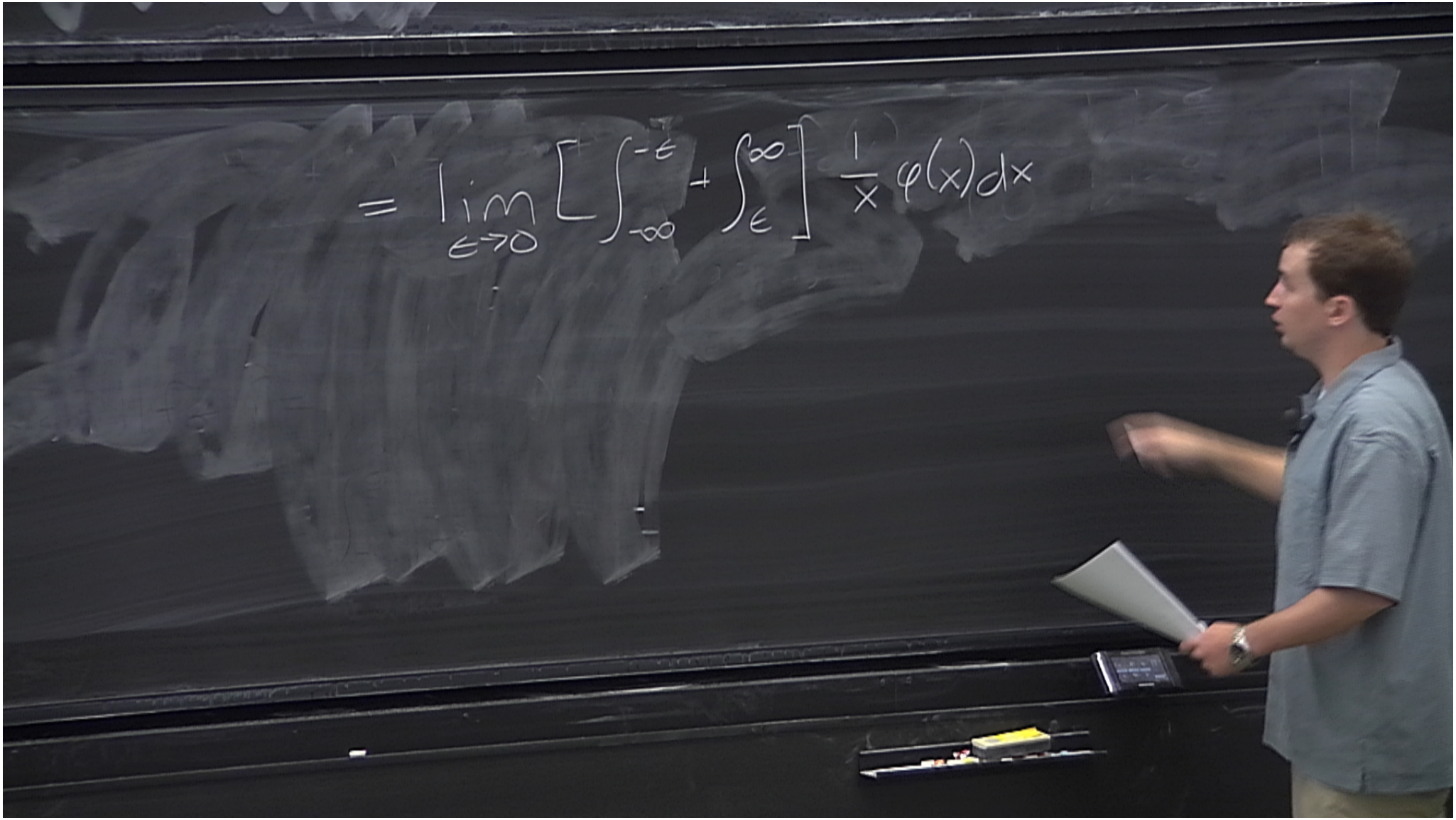

$\mathcal{D}'(\mathbb{R})$ is linear
$(u+v)(\varphi) = u(\varphi) + v(\varphi)$
$(\lambda u)(\varphi) = \lambda u(\varphi)$
(Check this)
$f(x) = 1 \quad |x| < 1$
0 otherwise

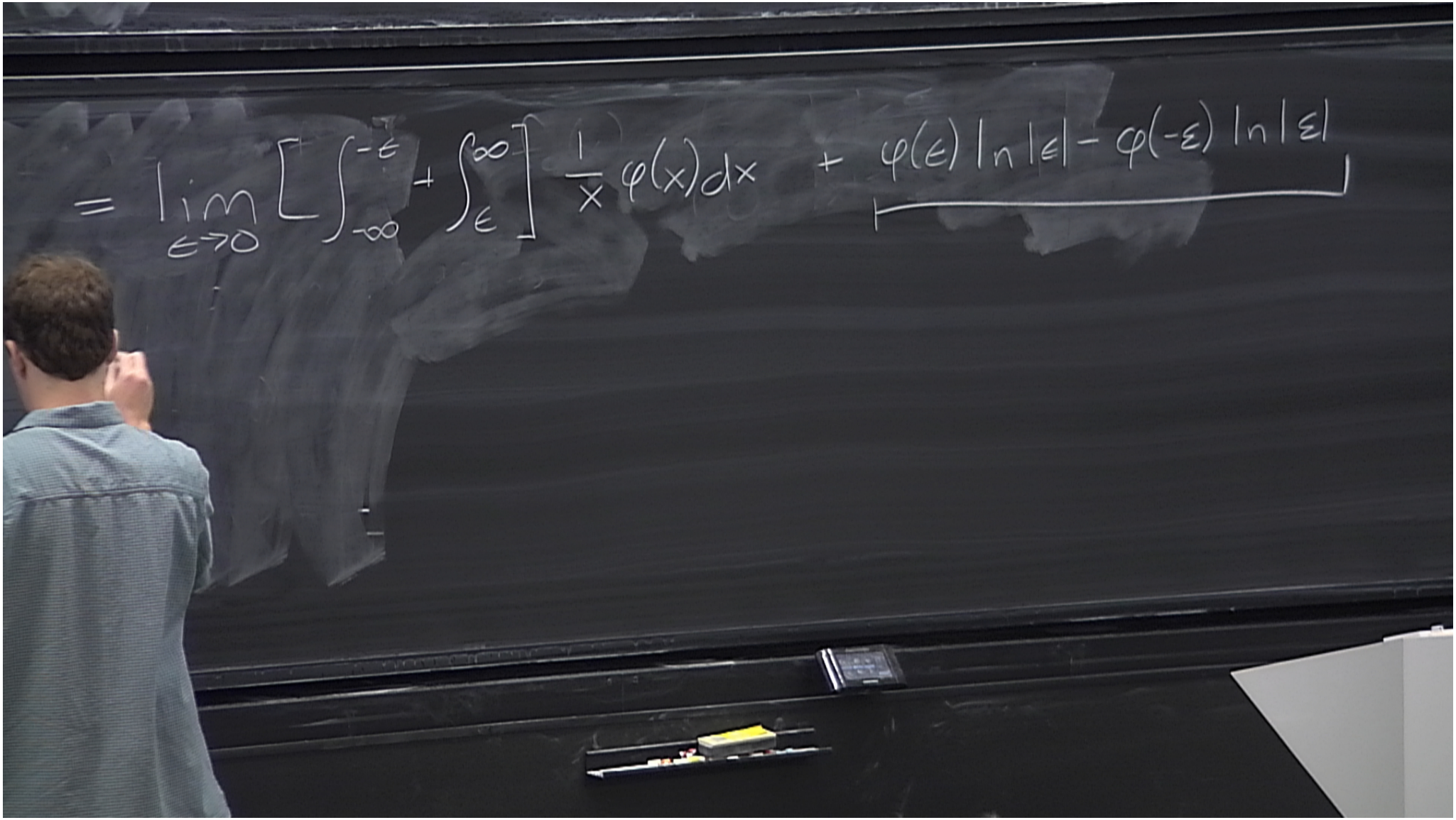

Derivative of Distribution
$u'(x) = \lim_{\epsilon \to 0} \frac{u(x+\epsilon) - u(x)}{\epsilon}$
makes no sense
$u'(\varphi) = \int_{-\infty}^{\infty} u'(x)\,\varphi(x)\,dx$
$= u(x)\varphi(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u(x)\,\varphi'(x)\,dx$

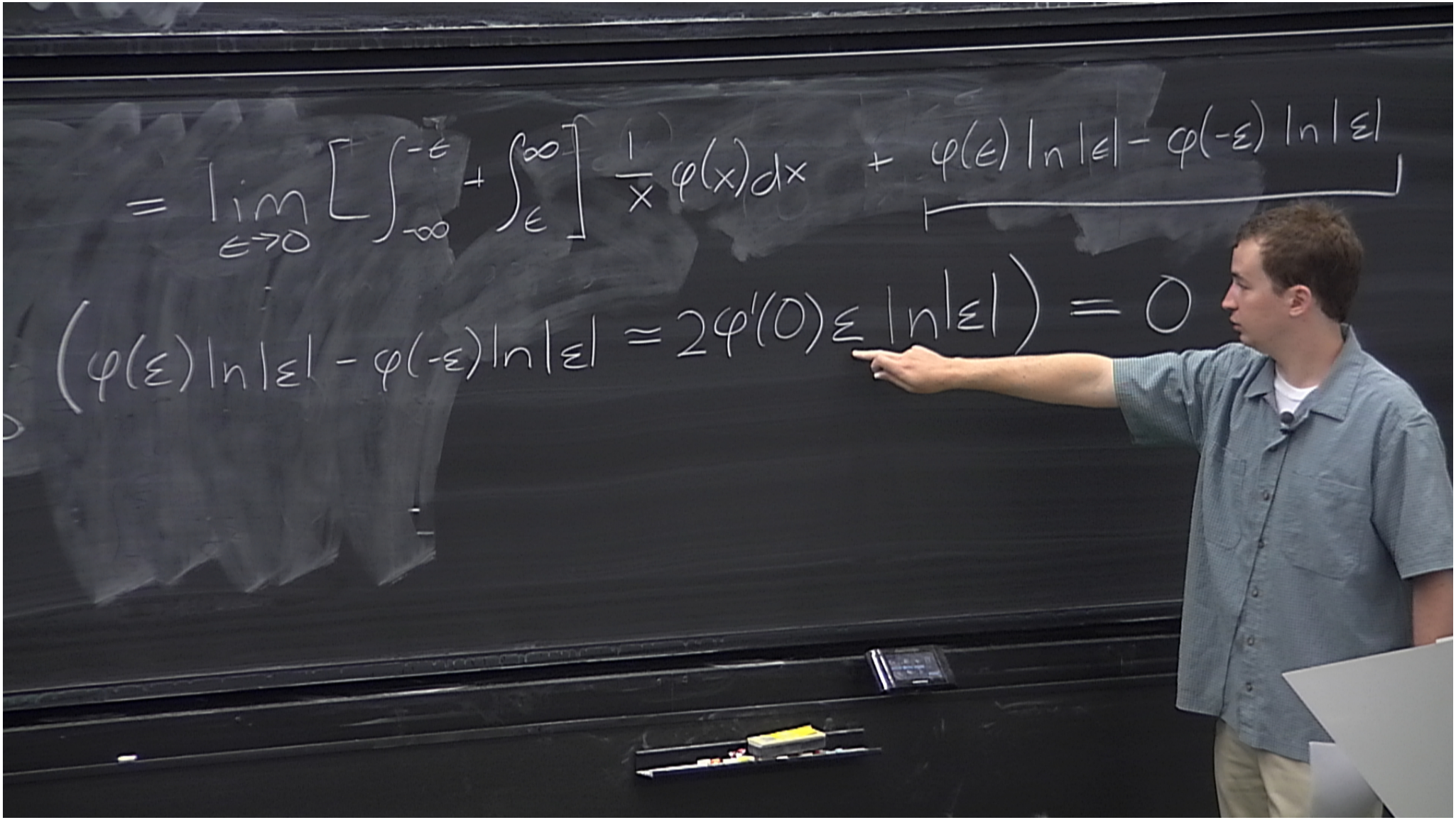

$u'(\varphi) \equiv -u(\varphi')$
Example: $\Theta = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$
$\Theta(\varphi) = \int_{-\infty}^{\infty} \Theta(x)\,\varphi(x)\,dx$
$\equiv \int_{0}^{\infty} \varphi(x)\,dx$

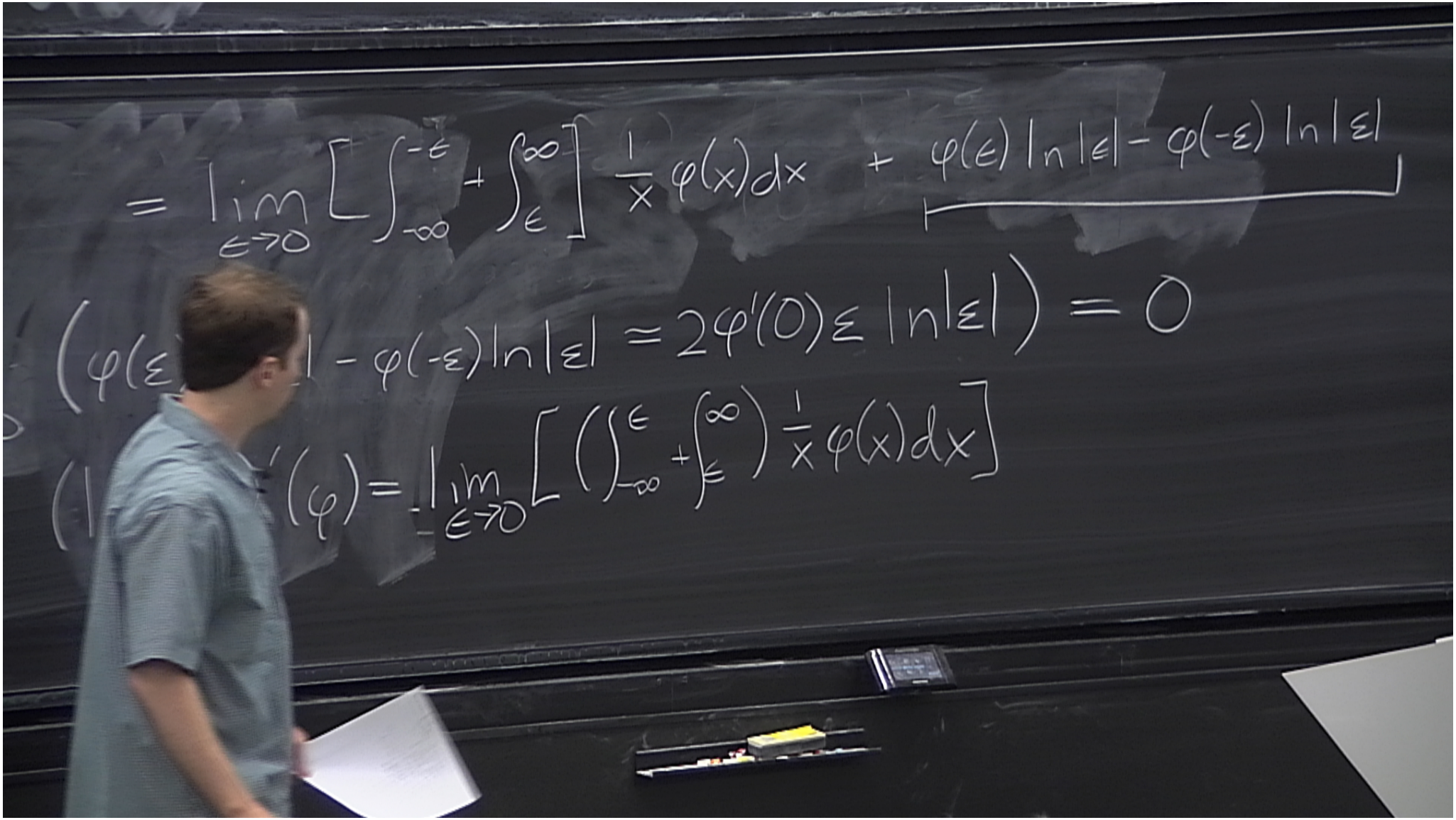

$\Theta'(\varphi) = -\Theta(\varphi')$
$= -\int_0^\infty \varphi'(x)\,dx$
$= -(\varphi(\infty) - \varphi(0))$
$= \varphi(0)$
$= \delta(\varphi)$
$\Theta' = \delta$

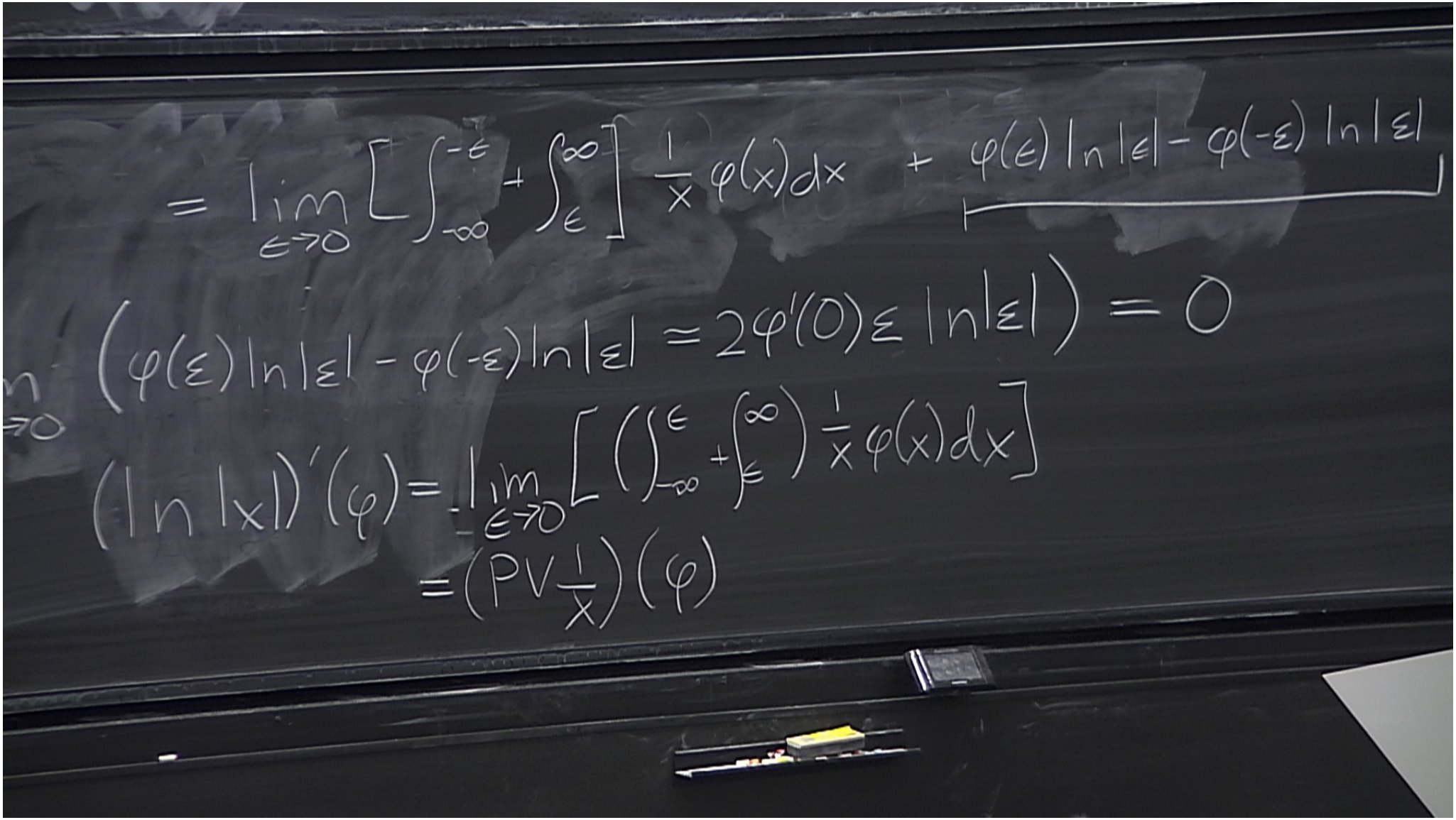

Example: $(\ln|x|)'$
$(\ln|x|)'(\varphi) = -\int_{-\infty}^{\infty} \varphi'(x) \ln|x| dx$
$= -\lim_{\epsilon \to 0} \left(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty} \right) \varphi'(x) \ln|x| dx$

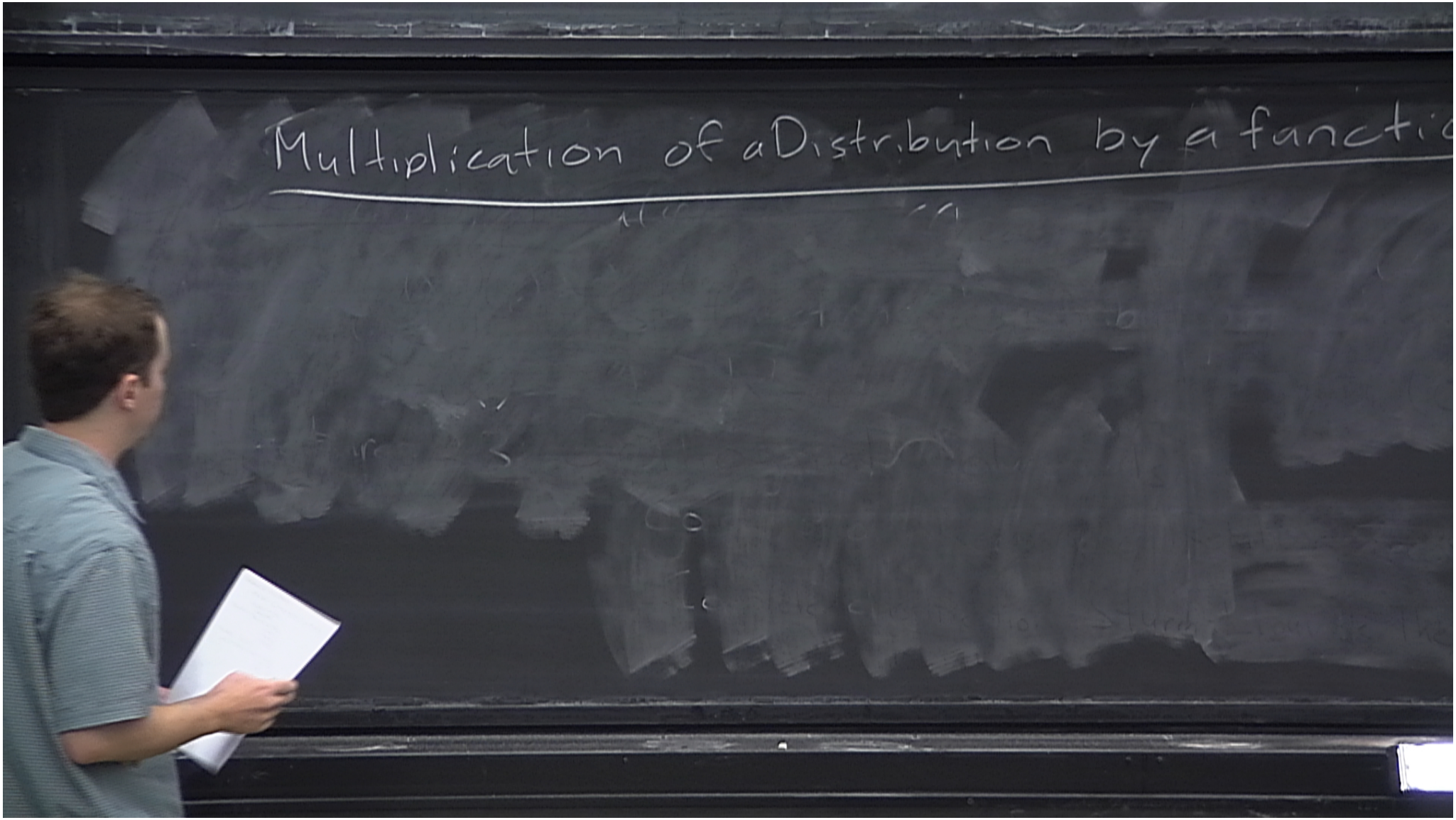

$= \lim_{\epsilon \to 0} \left[\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty} \right] \frac{1}{x} \varphi(x) dx$

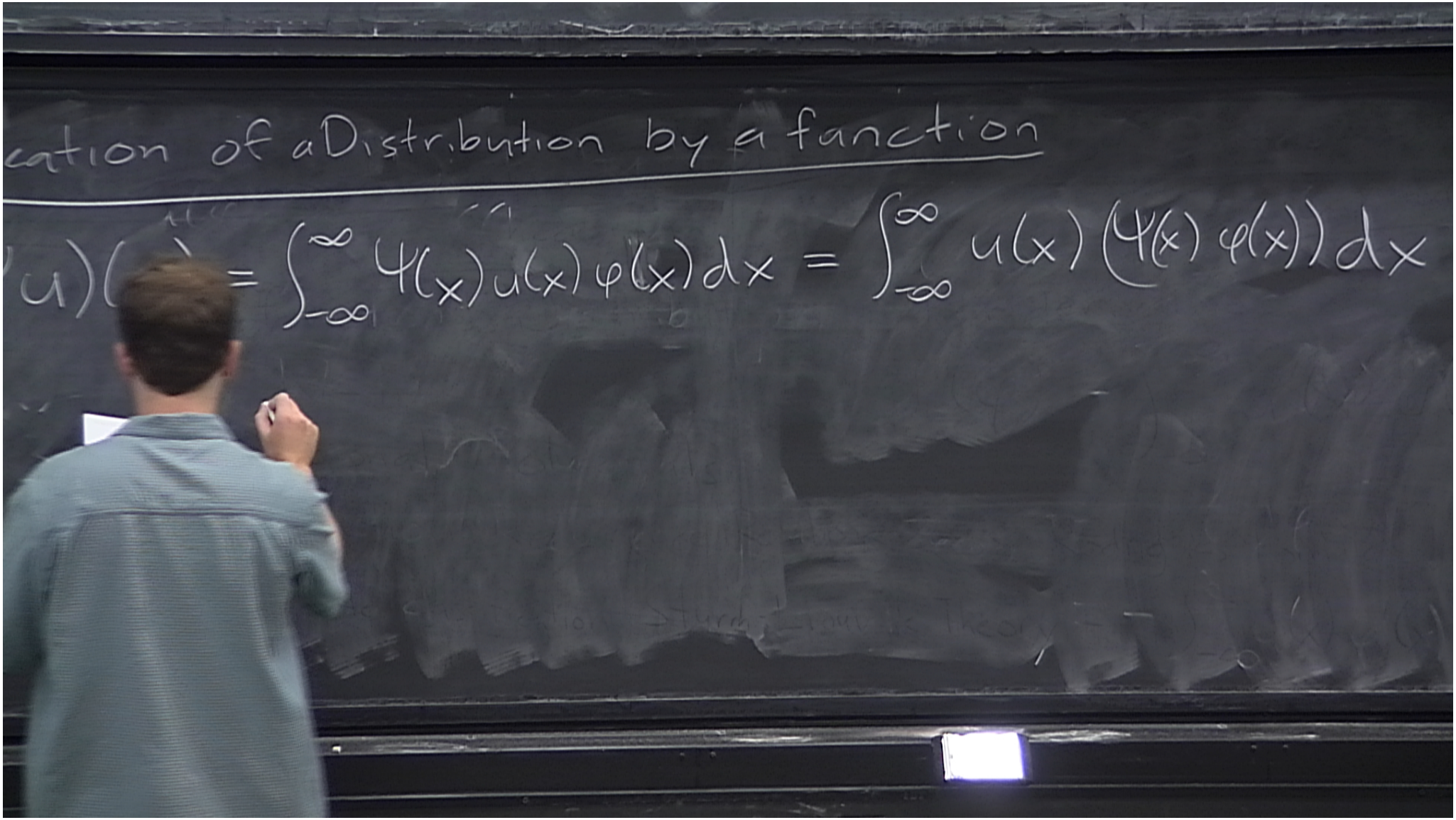

$= \lim_{\epsilon \to 0} \left[\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty} \right] \frac{1}{x} \varphi(x) dx + \varphi(\epsilon) \ln|\epsilon| - \varphi(-\varepsilon) \ln|\varepsilon|$

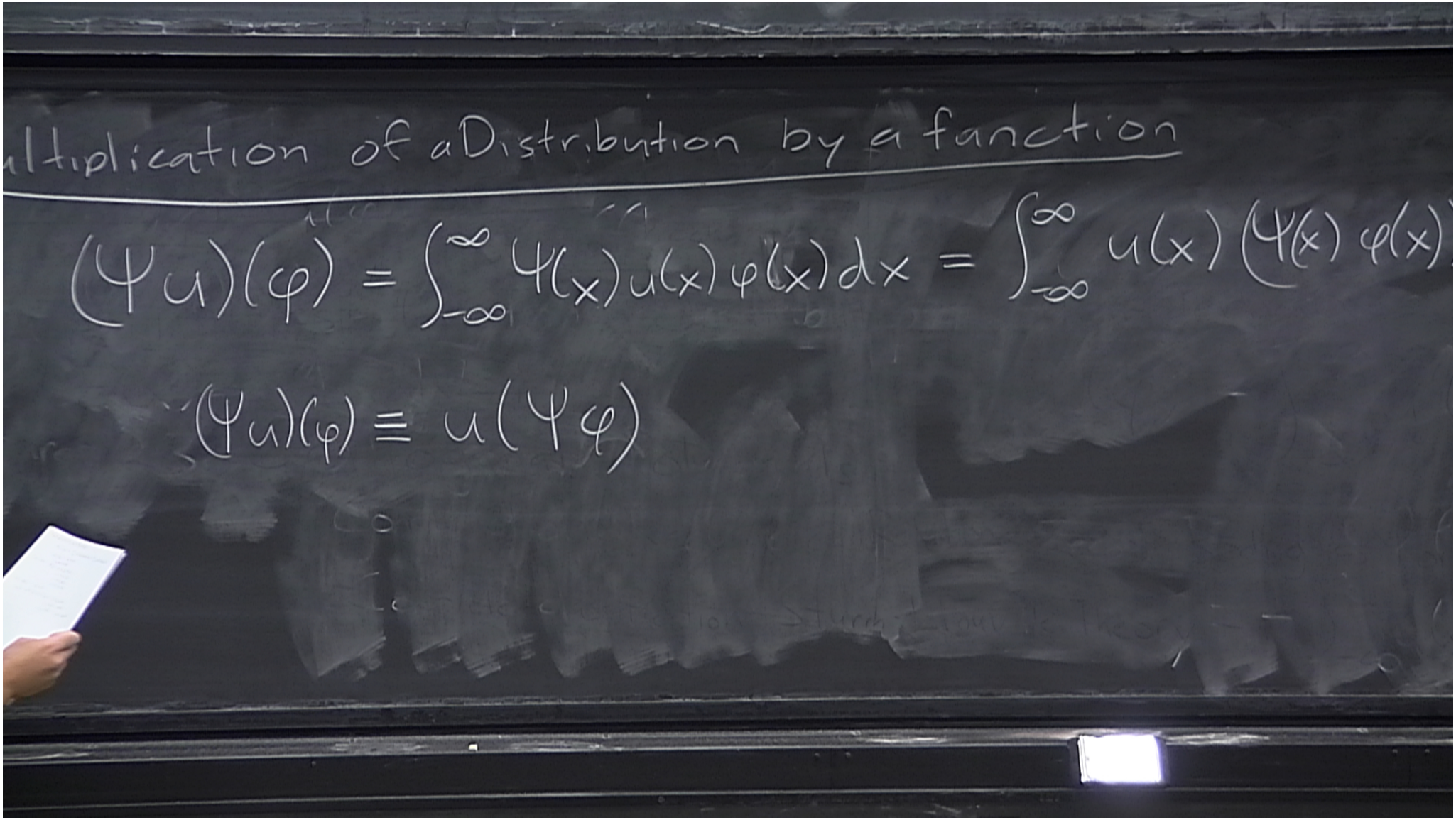

$= \lim_{\epsilon \to 0} \left[\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty} \right] \frac{1}{x} \varphi(x) dx + \varphi(\epsilon) \ln|\epsilon| - \varphi(-\epsilon) \ln|\epsilon|$
$\left(\varphi(\epsilon) \ln|\epsilon| - \varphi(-\epsilon) \ln|\epsilon| \approx 2\varphi'(0) \epsilon \ln|\epsilon| \right) = 0$

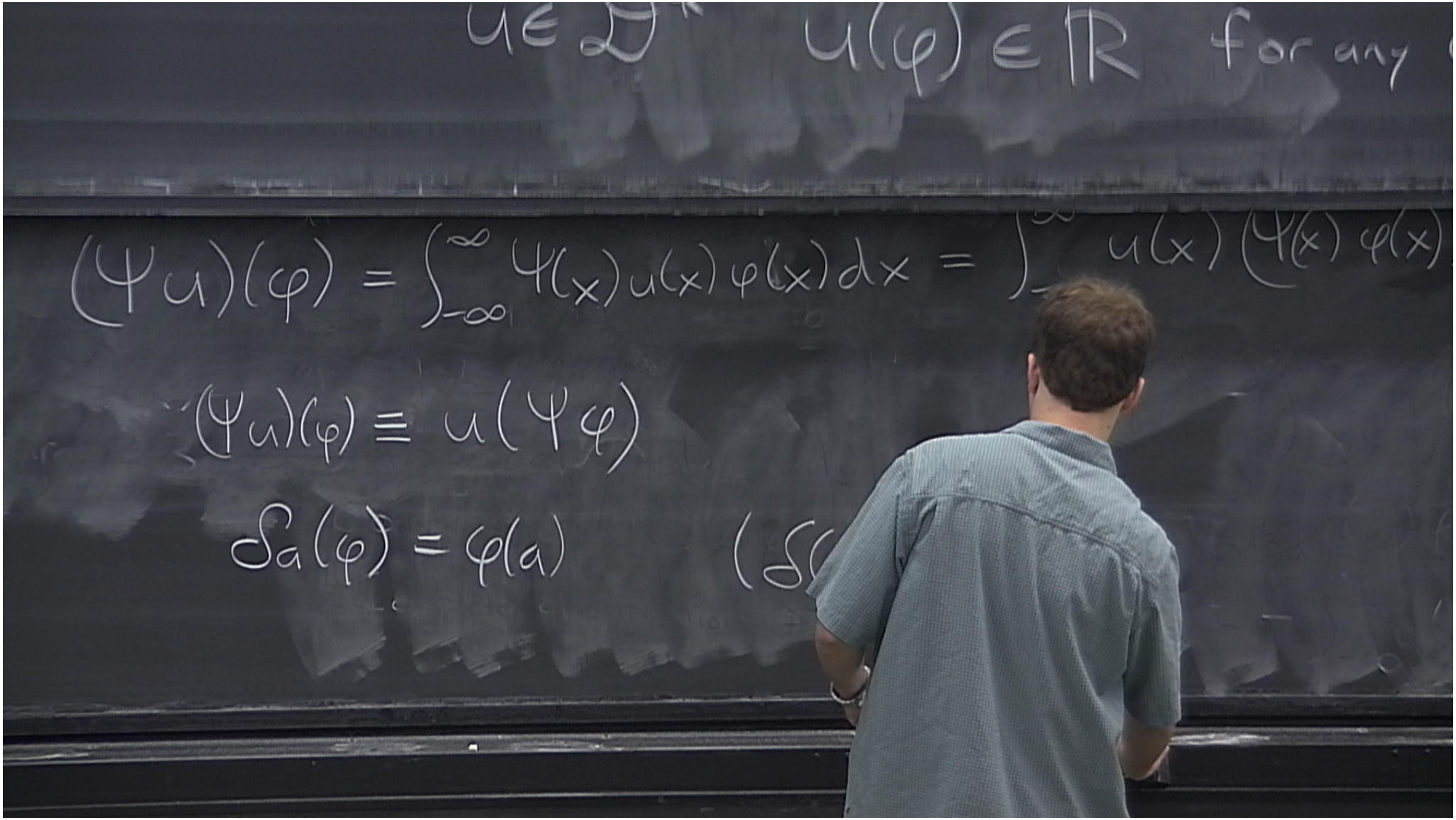

$= \lim_{\epsilon \to 0} \left[\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty} \right] \frac{1}{x} \varphi(x) dx + \varphi(\epsilon) \ln|\epsilon| - \varphi(-\epsilon) \ln|\epsilon|$
$\left(\varphi(\epsilon) \ln|\epsilon| - \varphi(-\epsilon) \ln|\epsilon| \approx 2\varphi'(0)\epsilon \ln|\epsilon| \right) = 0$
$\left(\right)' (\varphi) = -\lim_{\epsilon \to 0} \left[\left(\right)_{-\infty}^{\epsilon} + \int_{\epsilon}^{\infty} \right) \frac{1}{x} \varphi(x) dx \Big]$

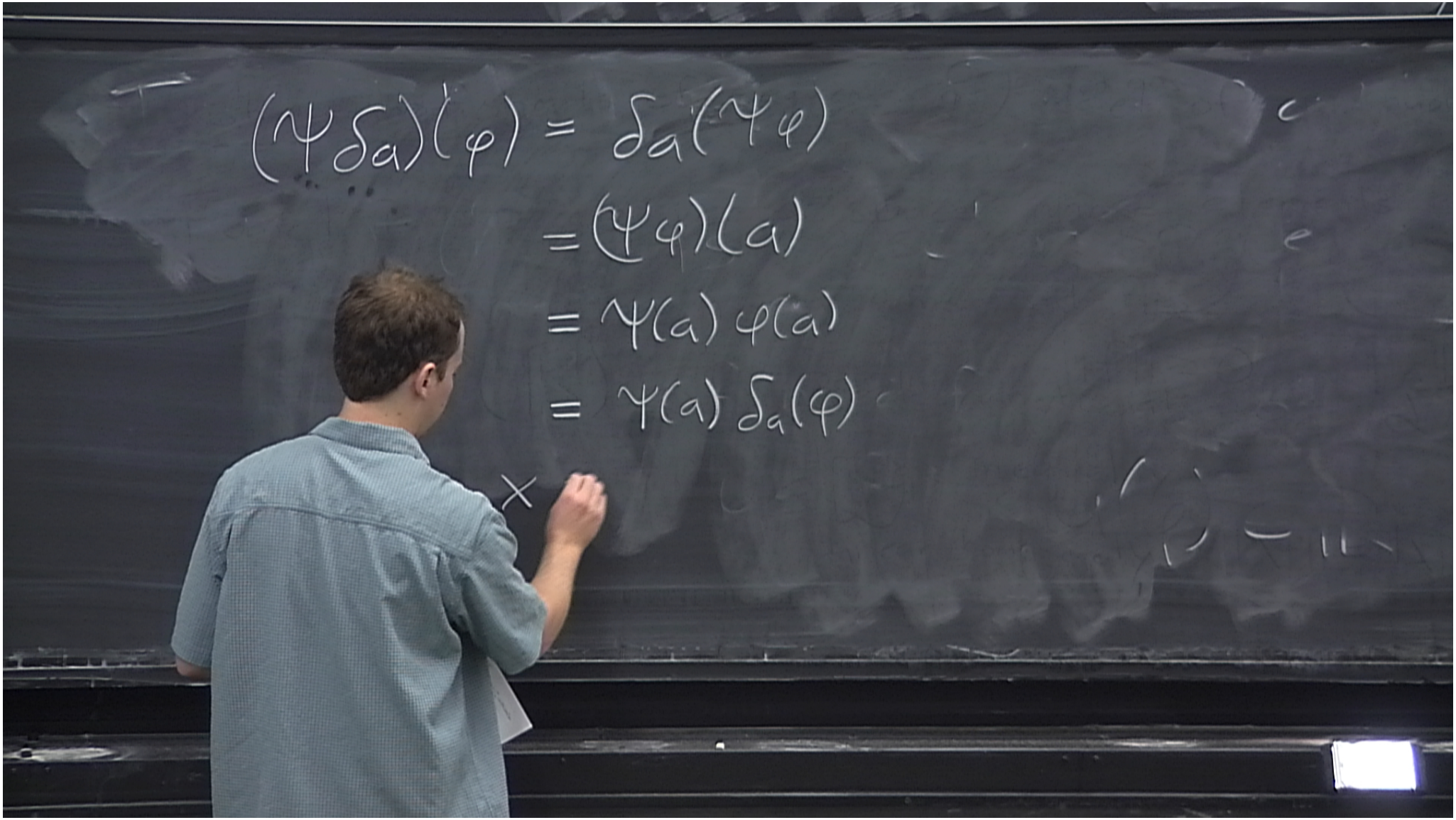

$= \lim_{\epsilon \to 0} \left[\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty} \right] \frac{1}{x} \varphi(x) dx + \varphi(\epsilon) \ln|\epsilon| - \varphi(-\epsilon) \ln|\epsilon|$
$\left(\varphi(\epsilon) \ln|\epsilon| - \varphi(-\epsilon) \ln|\epsilon| \approx 2\varphi'(0)\epsilon \ln|\epsilon| \right) = 0$
$(\ln|x|)'(\varphi) = -\lim_{\epsilon \to 0} \left[()_{-\infty}^{\epsilon} + \int_{\epsilon}^{\infty} \right) \frac{1}{x} \varphi(x) dx \Big]$
$= (PV \frac{1}{x})(\varphi)$

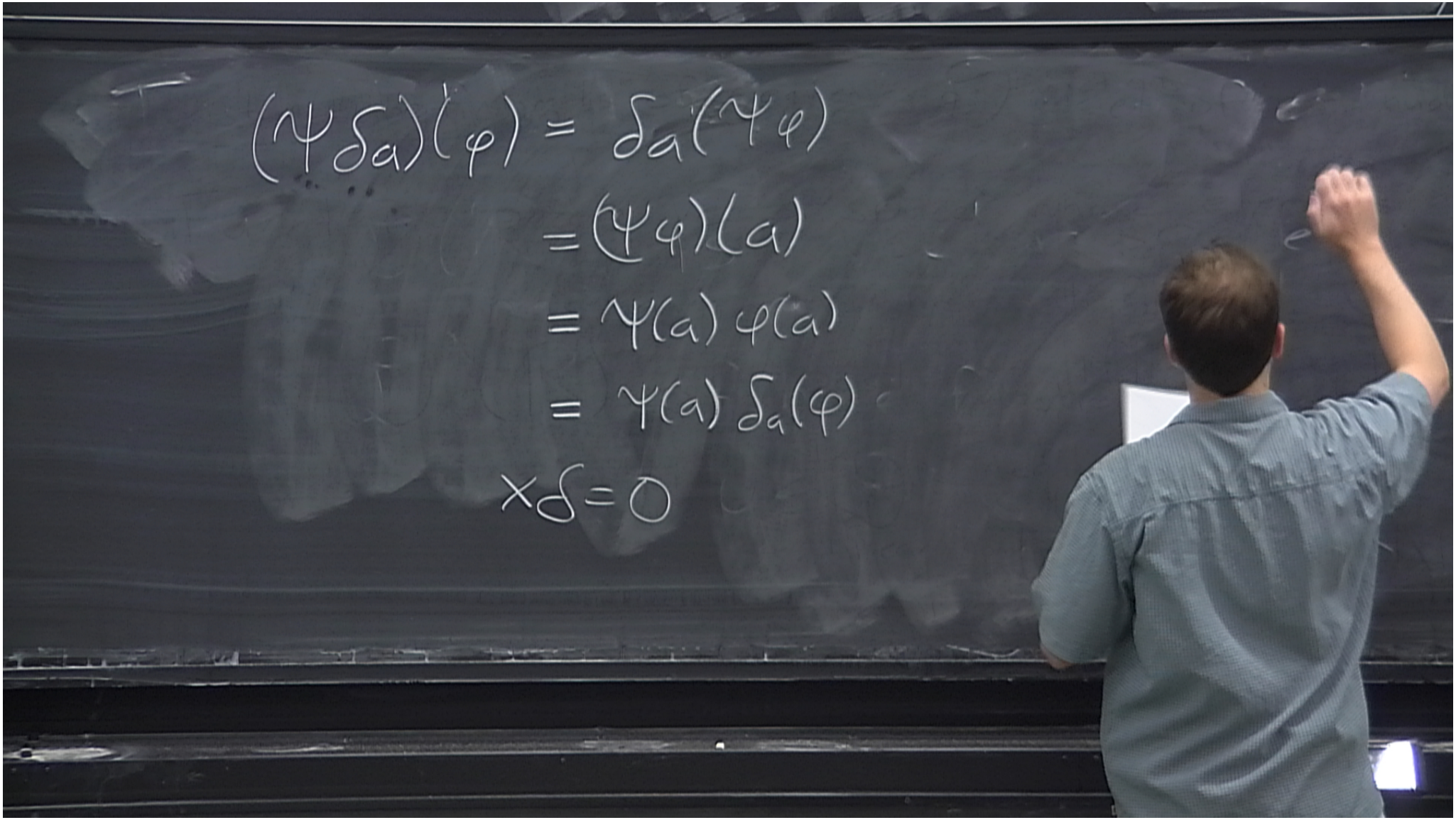

Multiplication of a Distribution by a functi

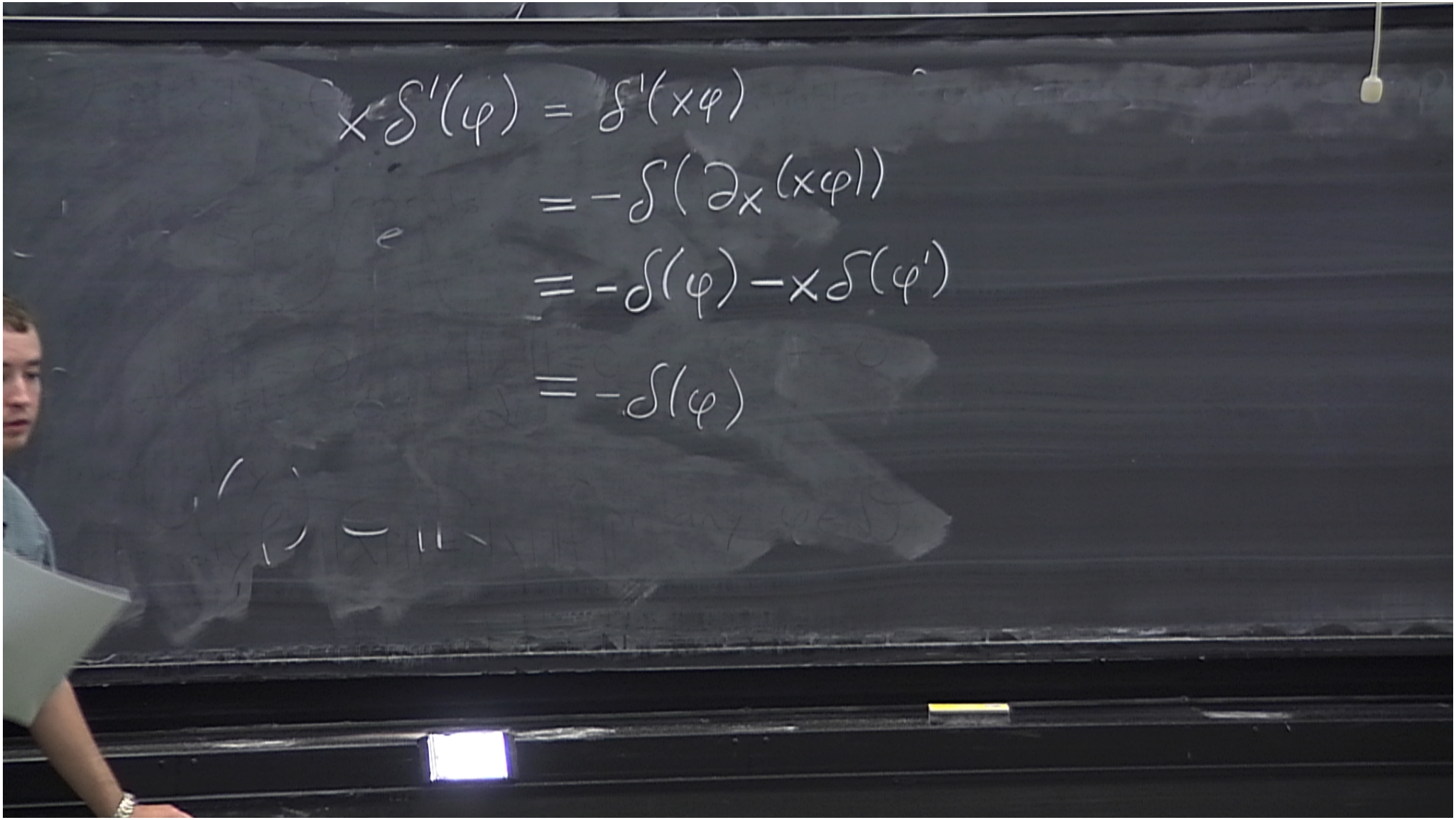

cation of a Distribution by a function
$= \int_{-\infty}^{\infty} \psi(x)\, u(x)\, \varphi(x)\, dx = \int_{-\infty}^{\infty} u(x) \left(\psi(x)\, \varphi(x)\right) dx$


ltiplication of a Distribution by a function
$(\psi u)(\varphi) = \int_{-\infty}^{\infty} \psi(x)\, u(x)\, \varphi(x)\, dx = \int_{-\infty}^{\infty} u(x) \left(\psi(x)\, \varphi(x)\right.$
$(\psi u)(\varphi) \equiv u(\psi\varphi)$


$u \in \mathcal{D}'$ $u(\varphi) \in \mathbb{R}$ for any
$(\psi u)(\varphi) = \int_{-\infty}^{\infty} \psi(x) u(x) \varphi(x) dx =$
$u(x)$ $(\psi(x)$ $\varphi(x)$
$(\psi u)(\varphi) \equiv u(\psi\varphi)$
$\delta_a(\varphi) = \varphi(a)$


$(\Psi \delta_a)(\varphi) = \delta_a(\Psi\varphi)$
$= (\Psi\varphi)(a)$
$= \Psi(a)\,\varphi(a)$
$= \Psi(a)\,\delta_a(\varphi)$


$(\psi\delta_a)(\varphi) = \delta_a(\psi\varphi)$
$= (\psi\varphi)(a)$
$= \psi(a)\,\varphi(a)$
$= \psi(a)\,\delta_a(\varphi)$
$\times\delta = 0$


$x\delta'(\varphi) = \delta'(x\varphi)$
$= -\delta(\partial_x(x\varphi))$
$= -\delta(\varphi) - x\delta(\varphi')$
$= -\delta(\varphi)$