Title: Integral Transforms and Green's Functions - Lecture 3

Date: Sep 08, 2011 10:30 AM

URL: http://pirsa.pi.local/11090050

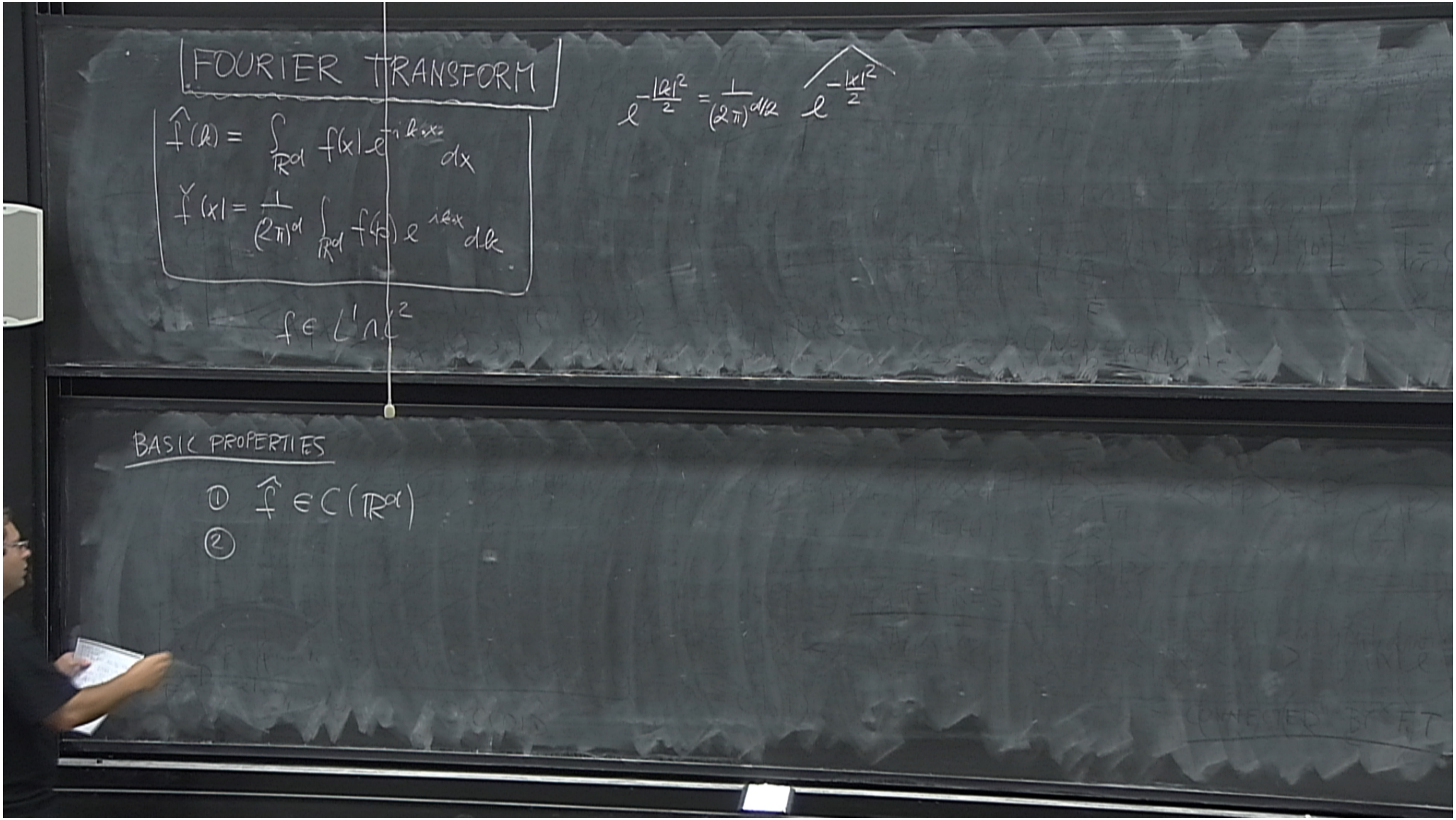
Abstract:

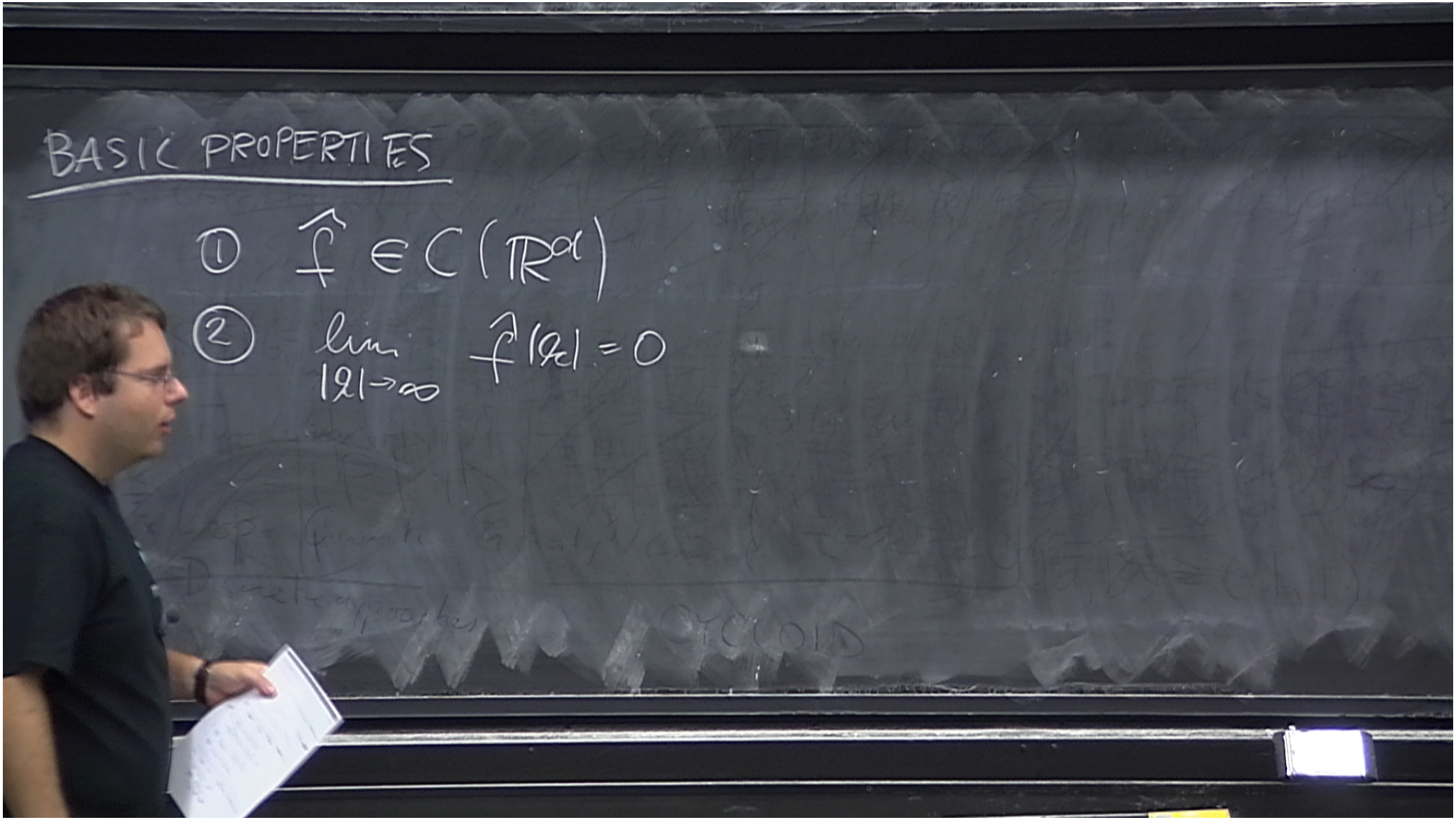
FOURIER TRANSFORM

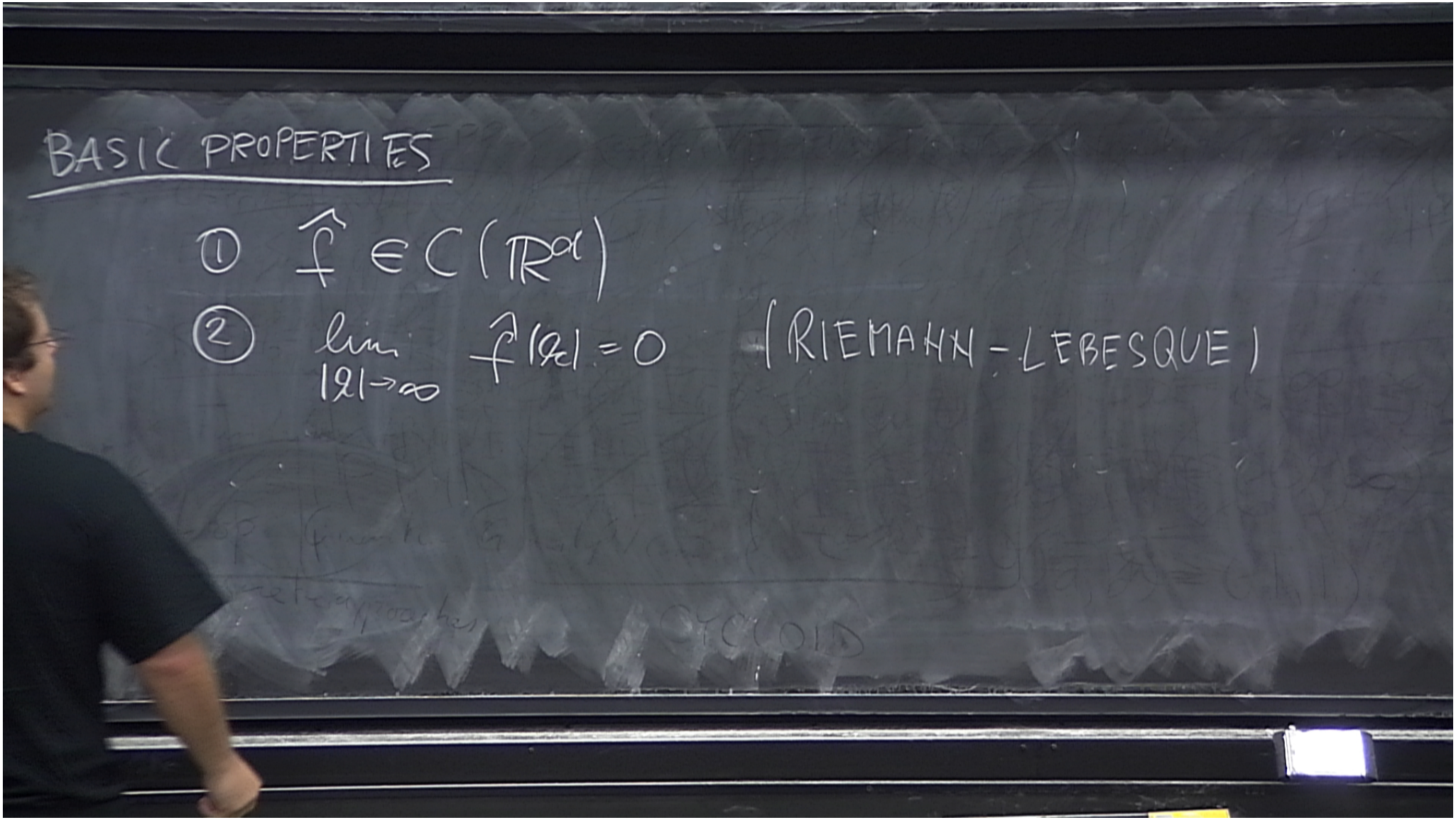
$$\hat{f}(k) = \int_{\mathbb{R}^d} f(x)\, e^{-ik\cdot x}\, dx$$

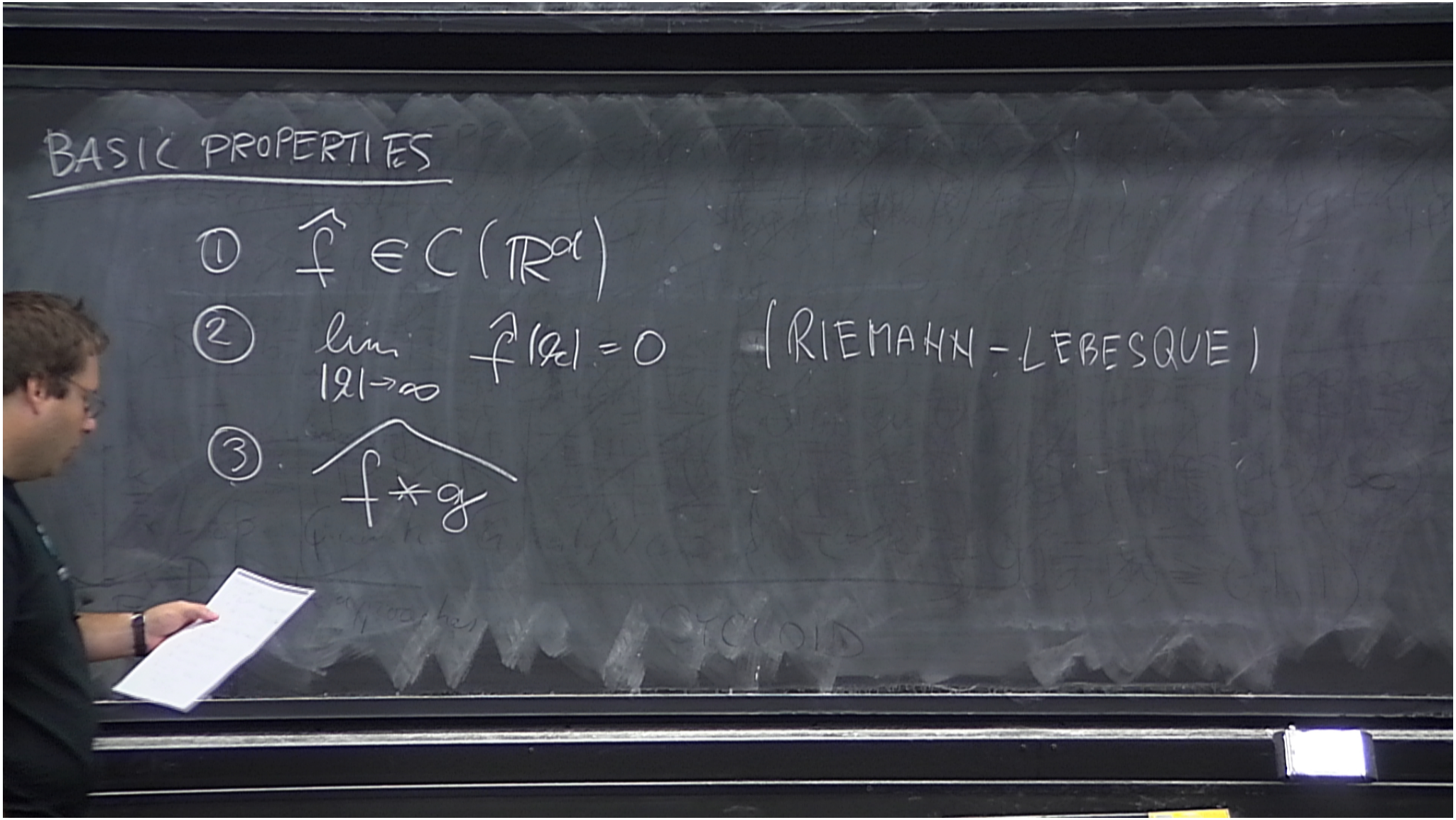
$$\check{f}(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} f(k)\, e^{ik\cdot x}\, dk$$

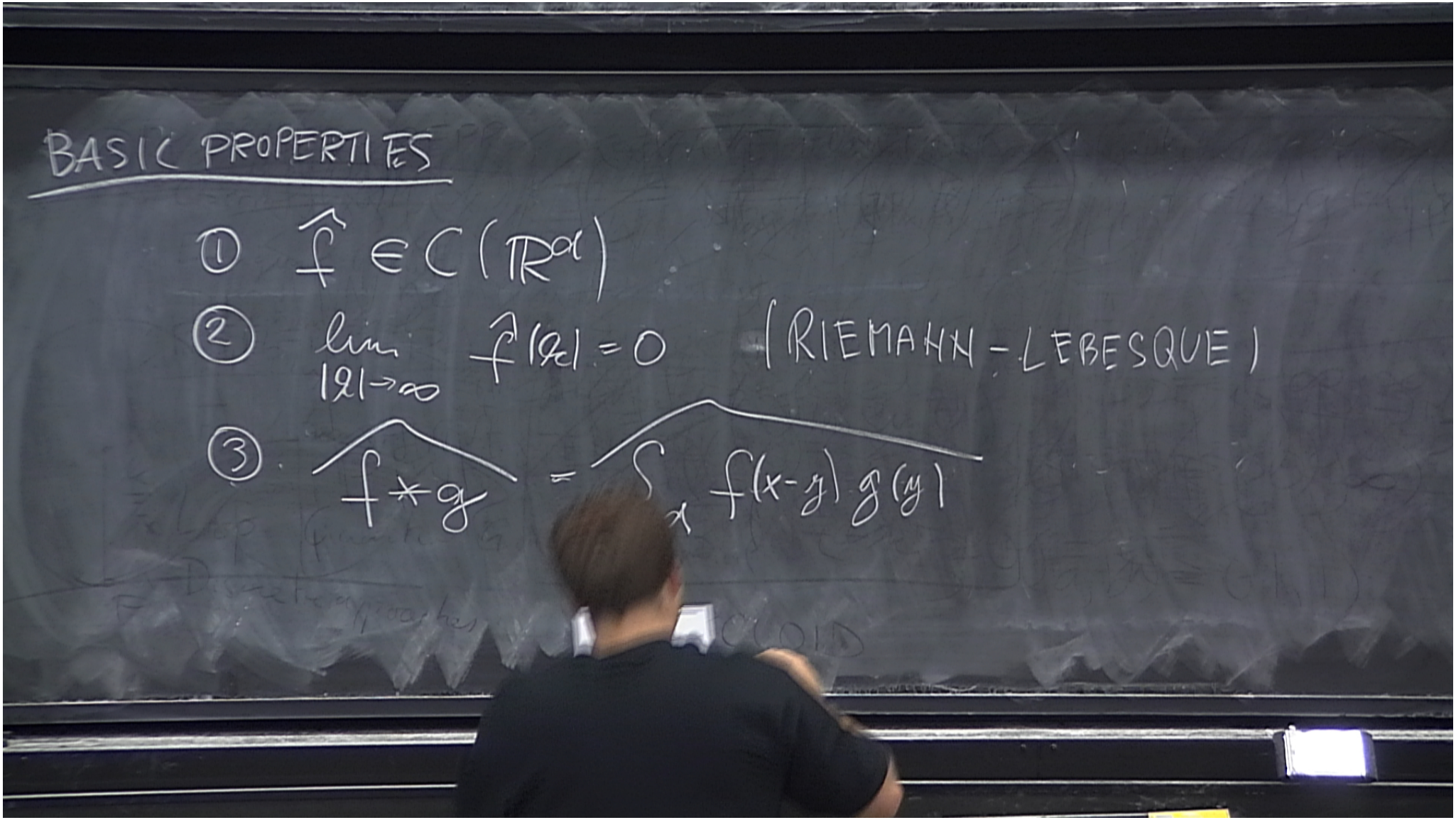
$$e^{-\frac{|k|^2}{2}} = \frac{1}{(2\pi)^d} \widehat{e^{-\frac{|x|^2}{2}}}$$


FOURIER TRANSFORM
$\hat{f}(k) = \int_{\mathbb{R}^d} f(x) e^{-ik\cdot x} dx$
$\check{f}(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} f(k) e^{ik\cdot x} dk$
$f \in L^1 \cap L^2$
$e^{-\frac{|x|^2}{2}} = \frac{1}{(2\pi)^d} \widehat{e^{-\frac{|x|^2}{2}}}$
BASIC PROPERTIES
① $\hat{f} \in C(\mathbb{R}^d)$
②


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BASIC PROPERTIES
① $\hat{f} \in C(\mathbb{R}^d)$
② $\lim_{|\xi| \to \infty} \hat{f}(\xi) = 0$ (RIEMANN-LEBESQUE)
③ $f * g$

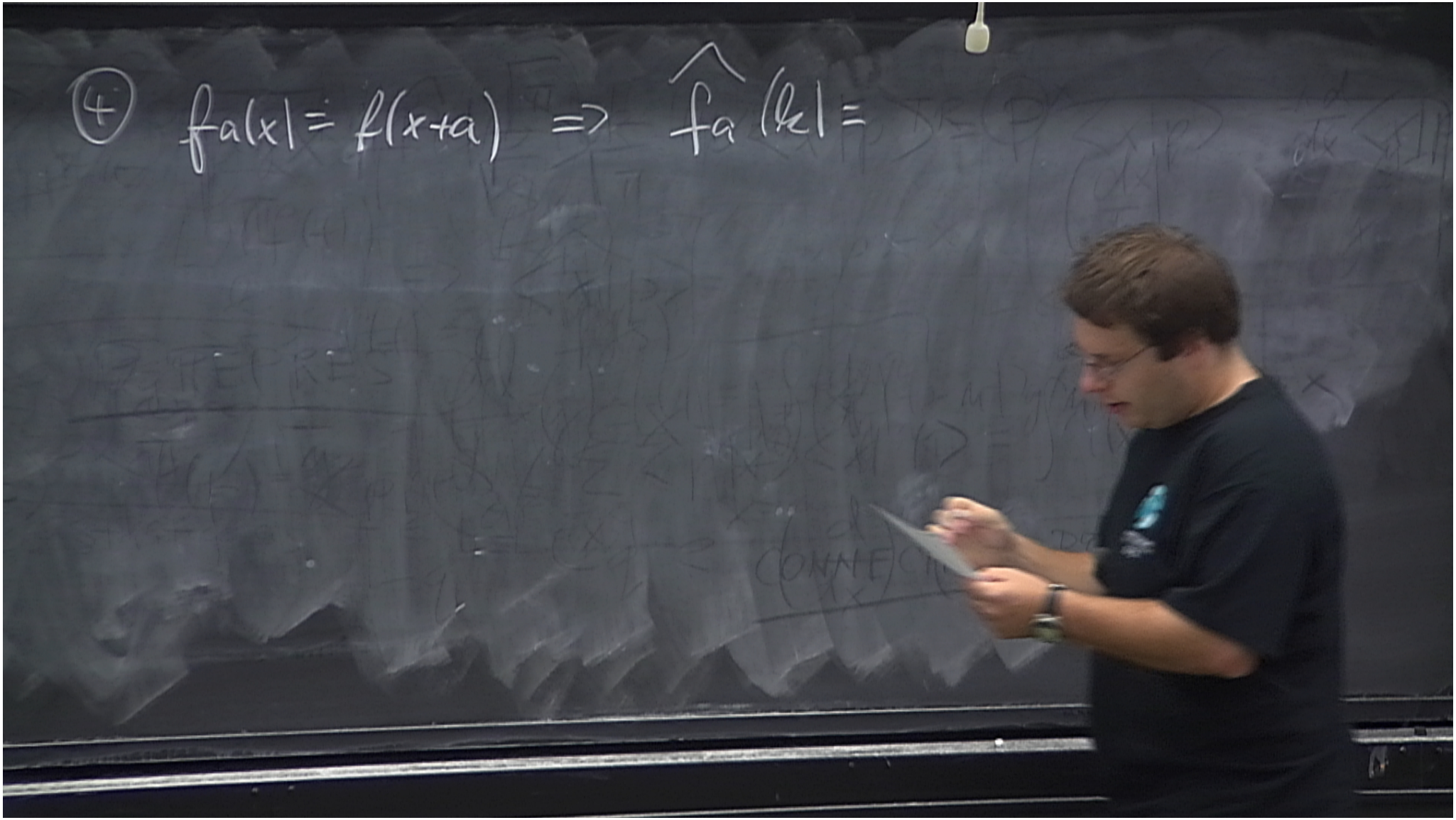

BASIC PROPERTIES
① $\hat{f} \in C(\mathbb{R}^d)$
② $\lim_{|\xi| \to \infty} \hat{f}(\xi) = 0$ (RIEMANN-LEBESQUE)
③ $\widehat{f * g} = \int_x f(x-y)\, g(y)$

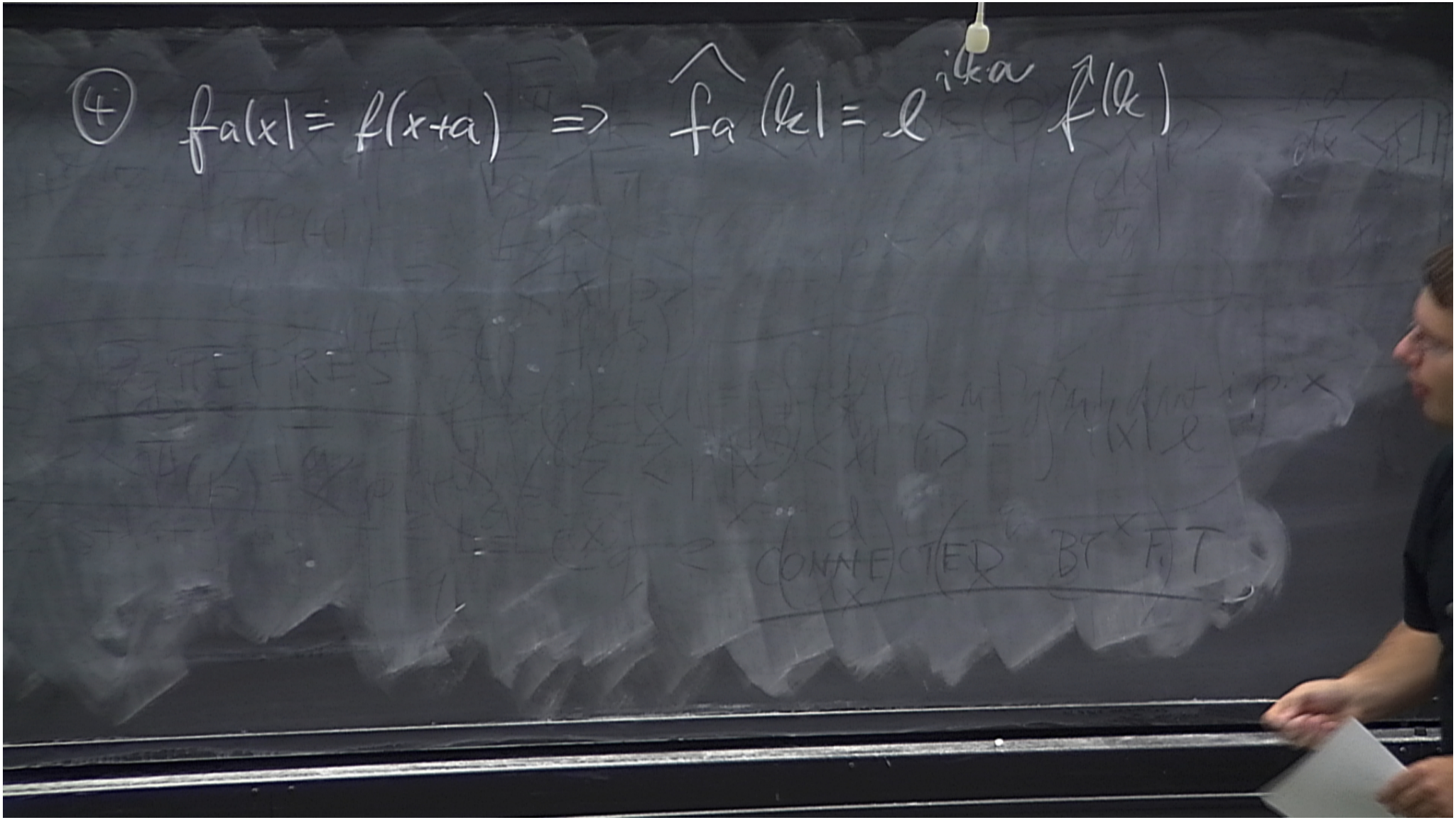
BASIC PROPERTIES

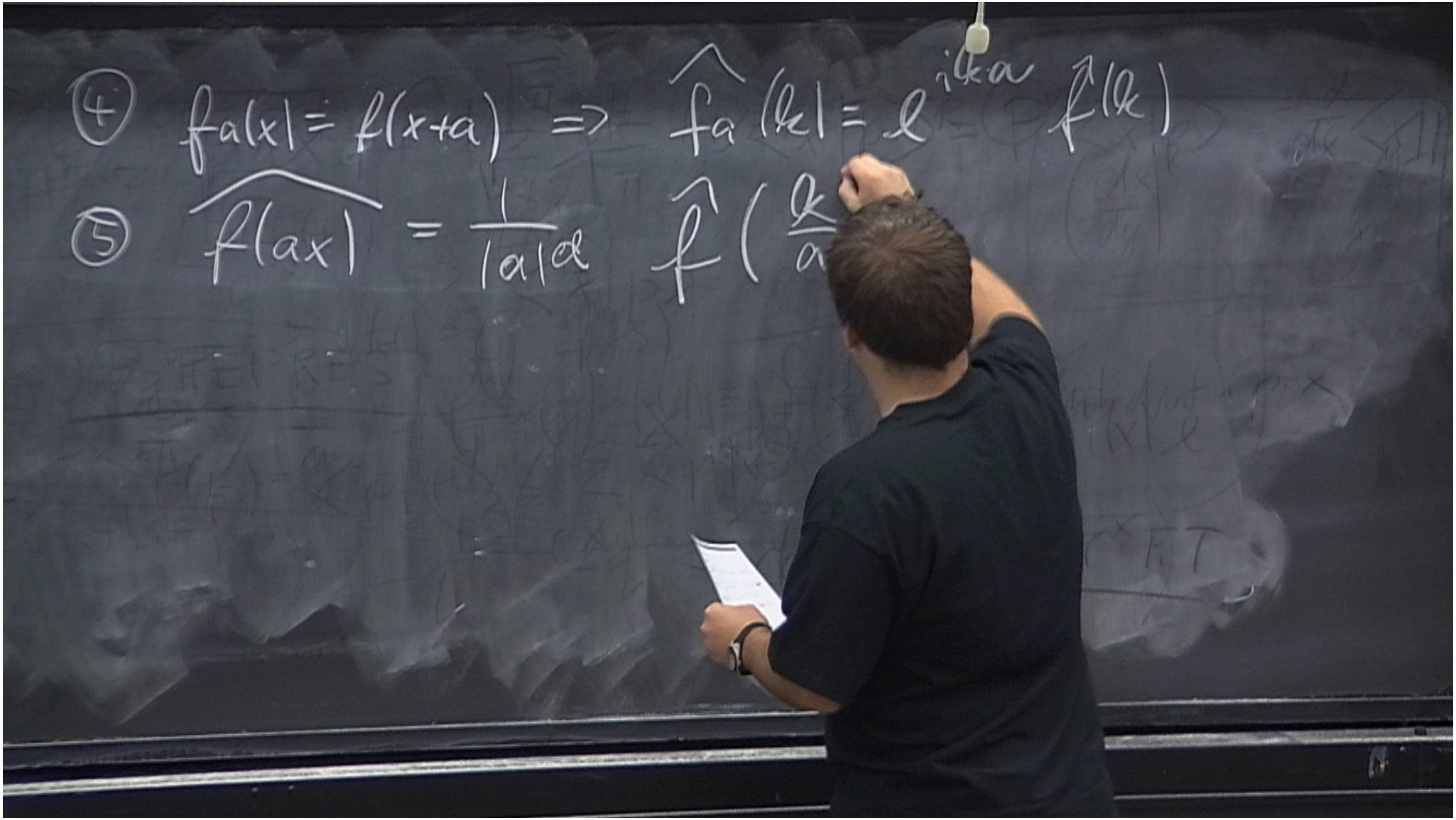
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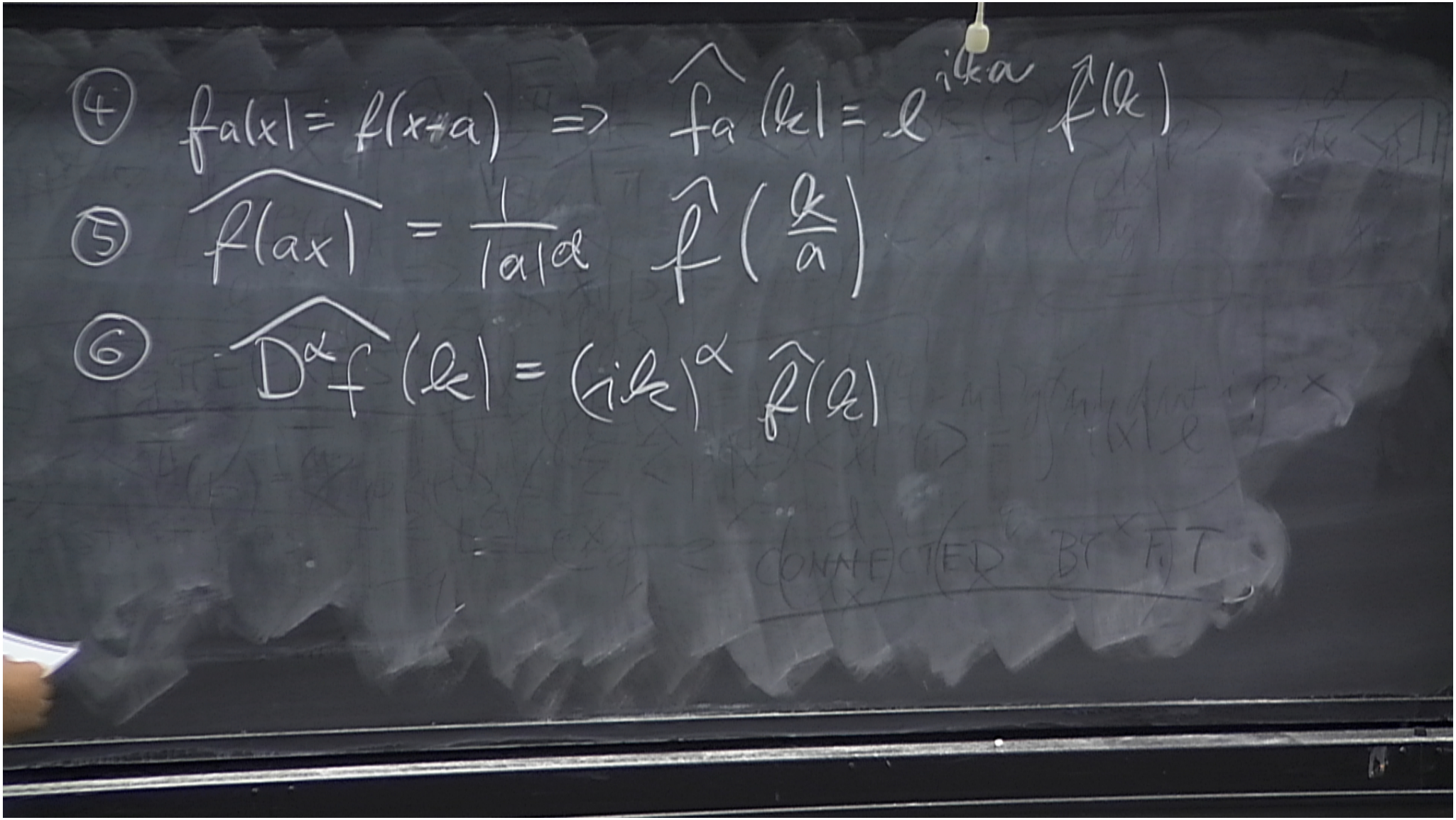
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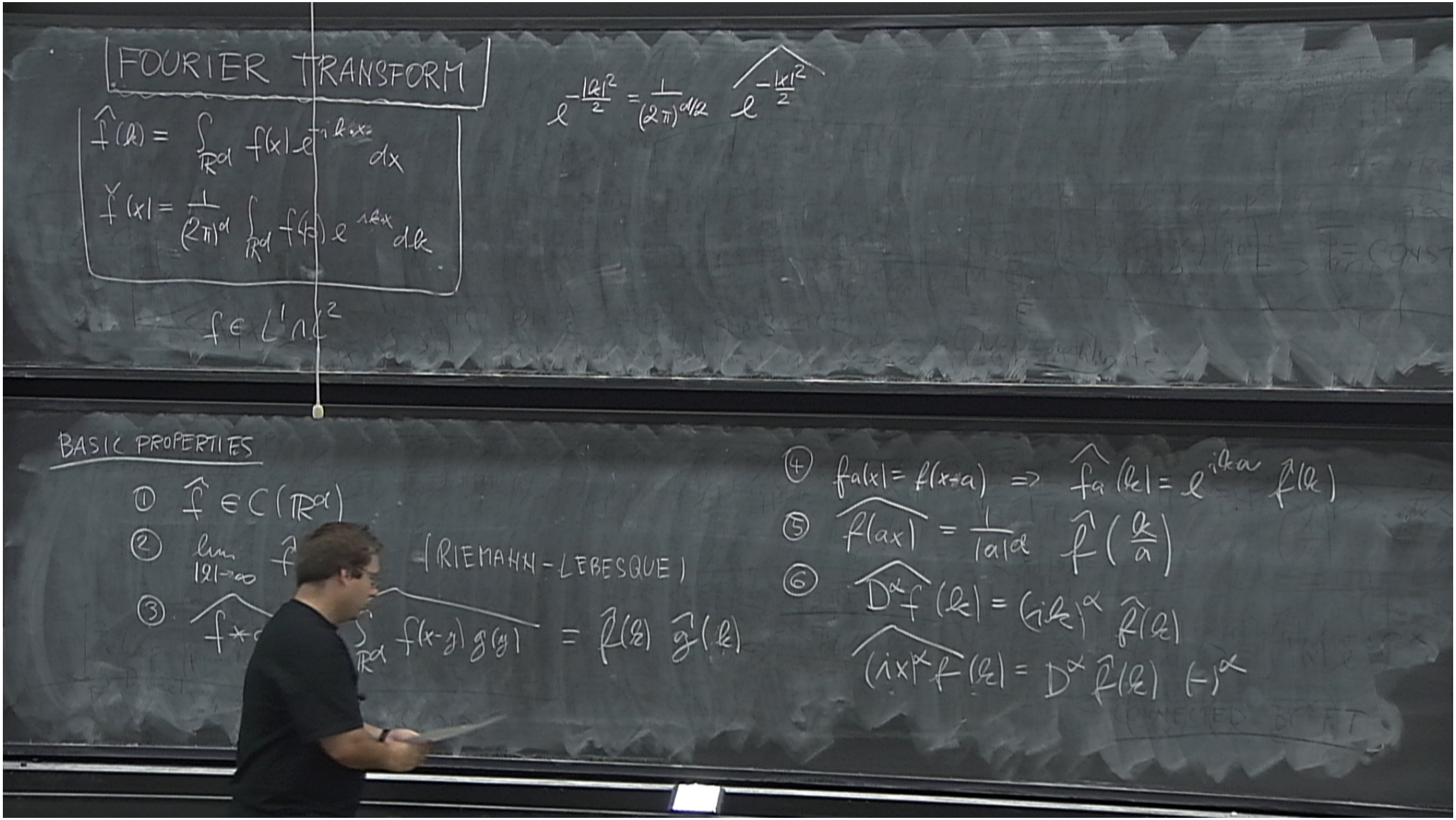
③ $\widehat{f * g} = \int_{\mathbb{R}^d} f(x-y)\, g(y) = \hat{f}(\xi)\, \hat{g}(\xi)$

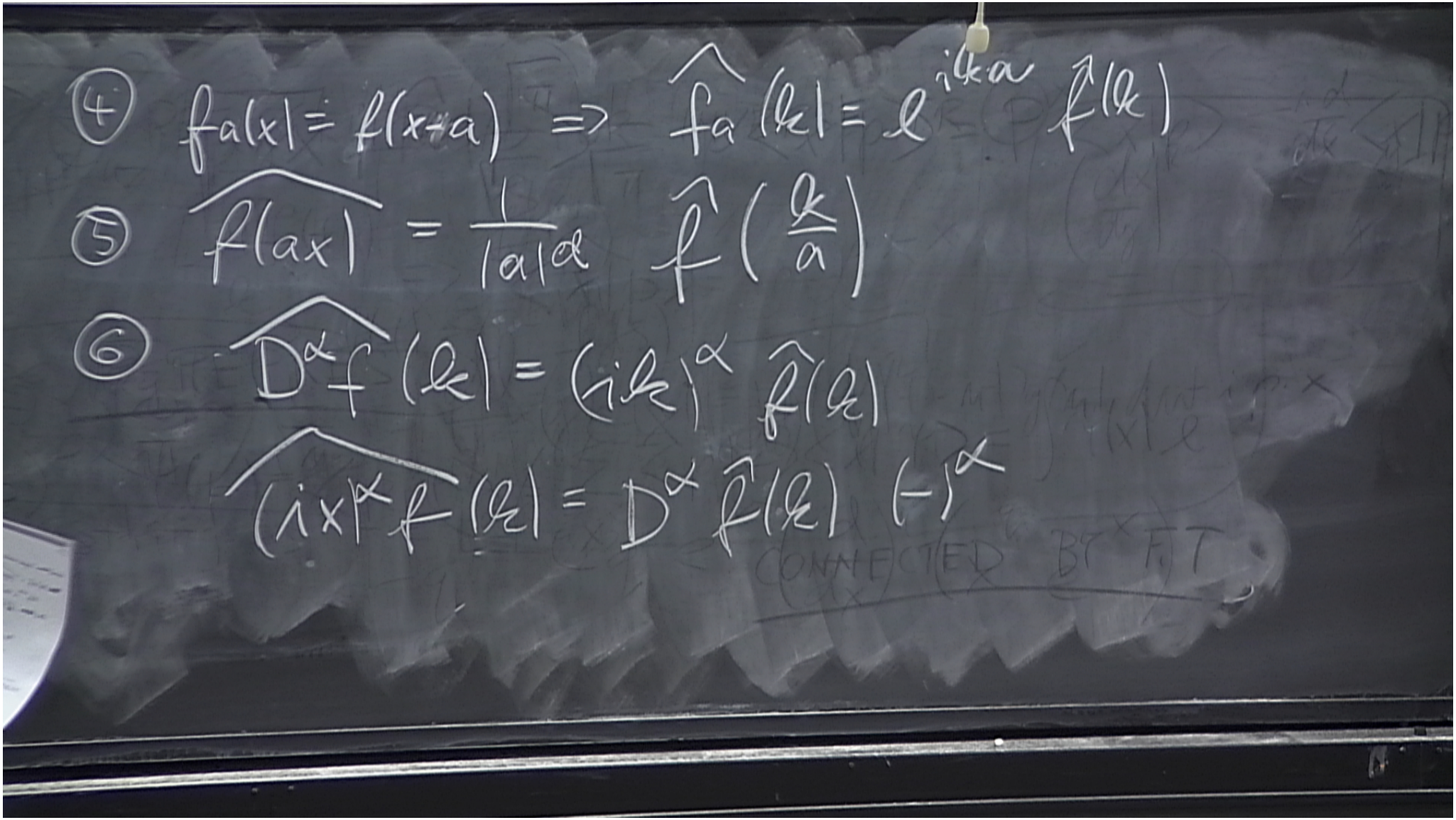

④ $f_a(x) = f(x+a) \Rightarrow \widehat{f_a}(k) =$

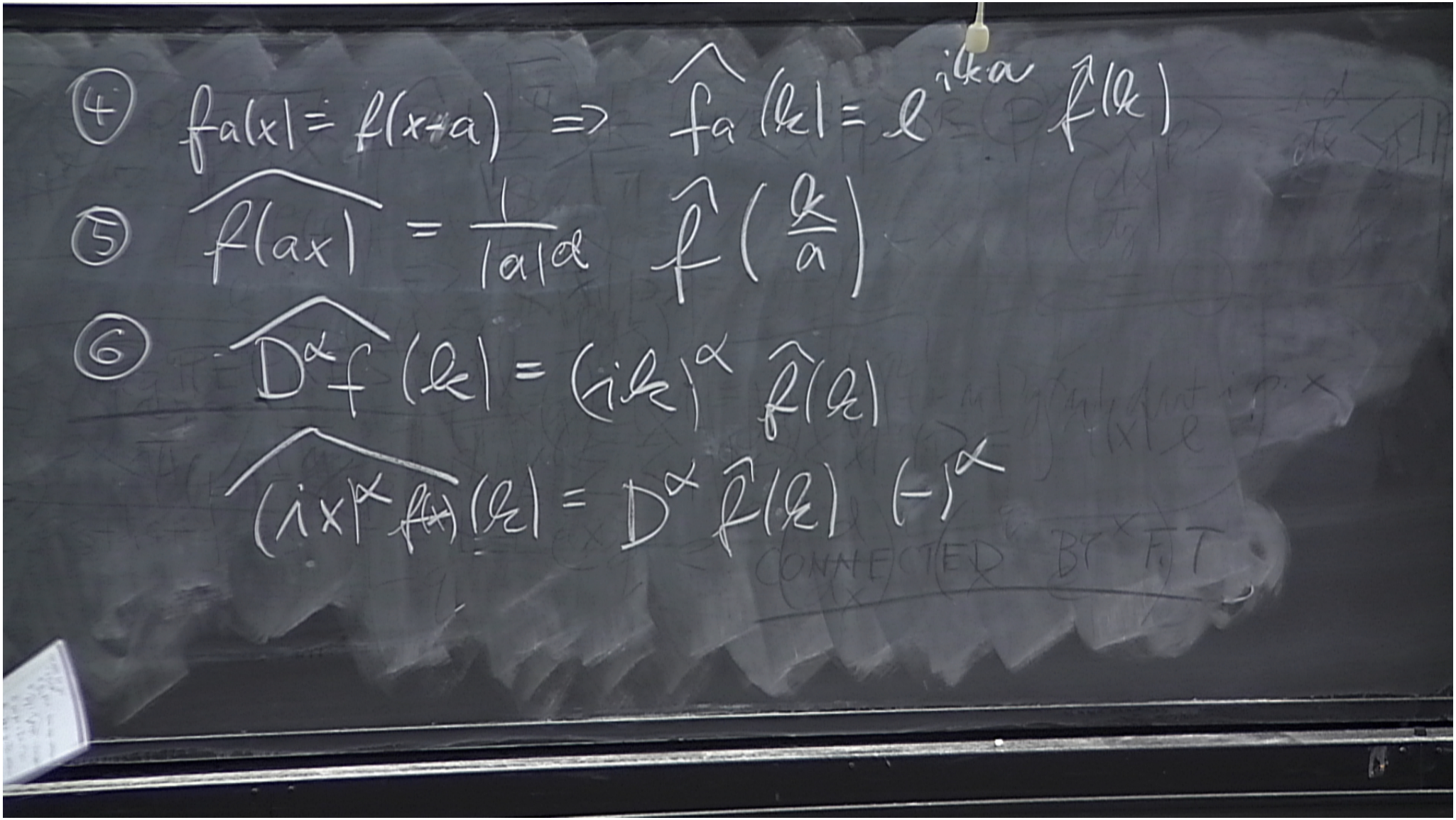

④ $f_a(x) = f(x+a) \Rightarrow \widehat{f_a}(k) = e^{ika}\hat{f}(k)$
CONNECTED BY FT

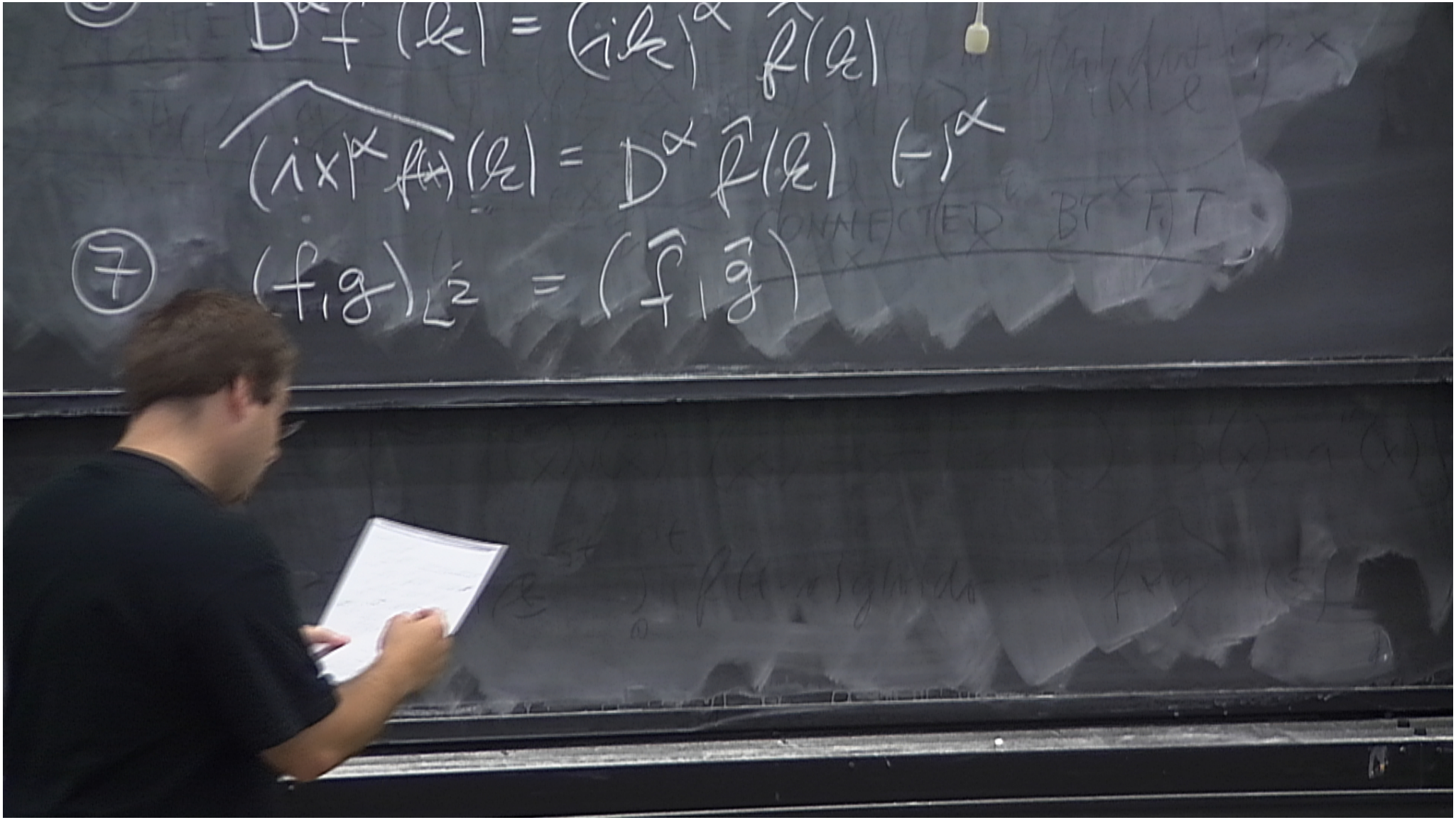

④ $f_a(x) = f(x+a) \Rightarrow \widehat{f_a}(k) = e^{ik\cdot a}\hat{f}(k)$
⑤ $\widehat{f(ax)} = \frac{1}{|a|^d}\hat{f}\left(\frac{k}{a}\right.$

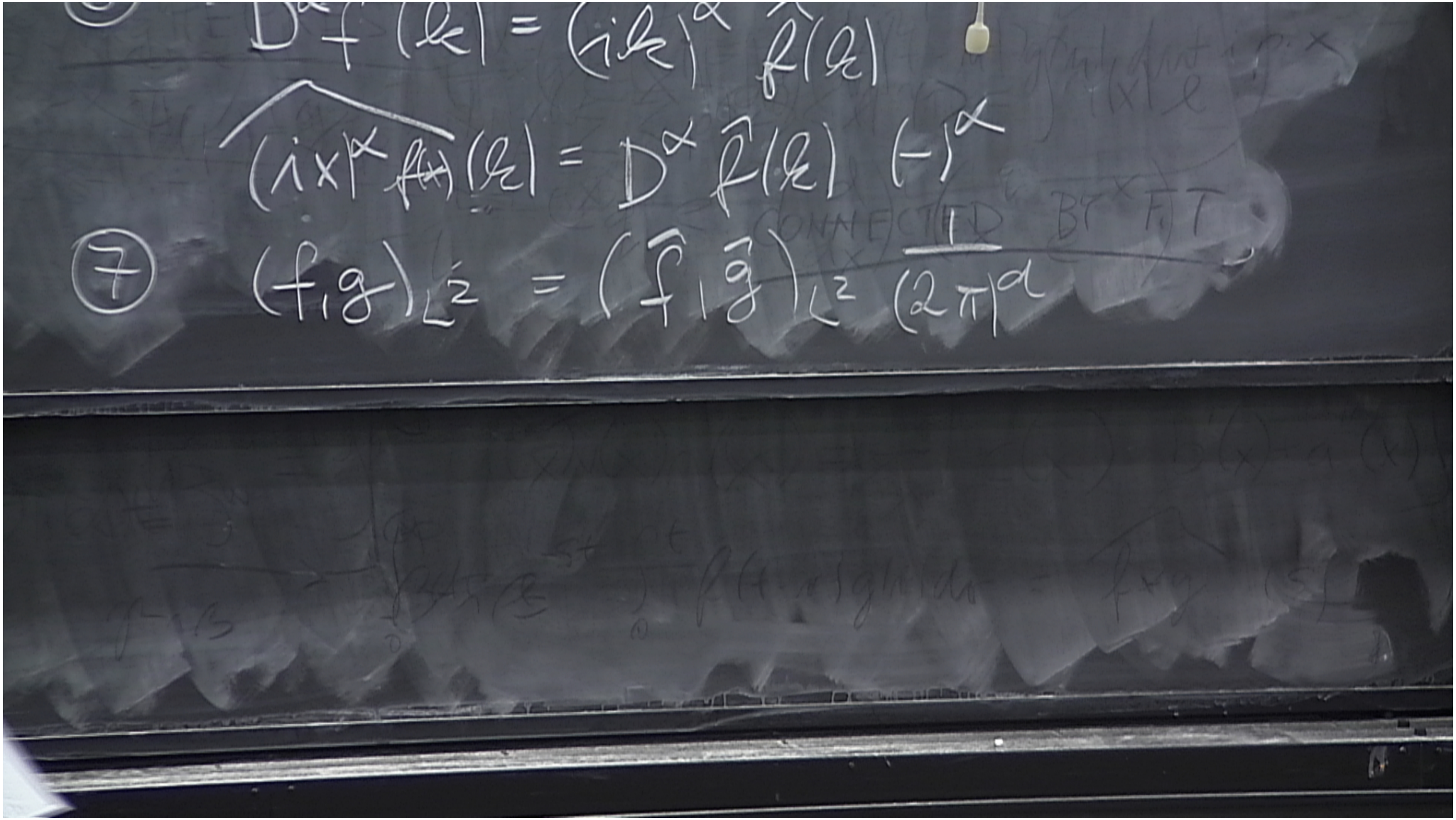

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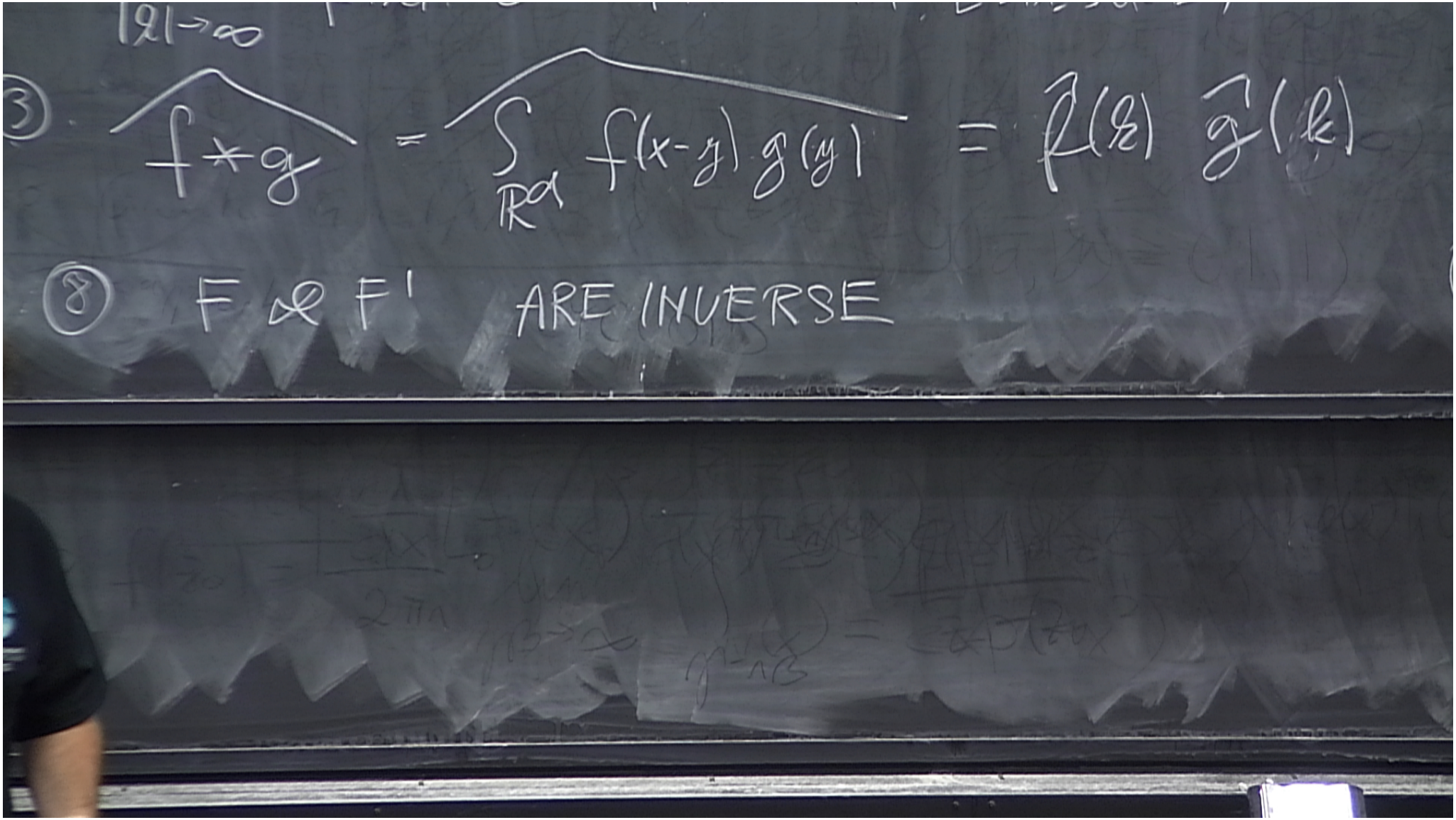

FOURIER TRANSFORM
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BASIC PROPERTIES
① $\hat{f} \in C(\mathbb{R}^d)$
② $\lim_{|x|\to\infty} \hat{f}$ (RIEMANN-LEBESQUE)
③ $\widehat{f * g}(k) = \int_{\mathbb{R}^d} f(x-y) g(y)\, dy = \hat{f}(k)\, \hat{g}(k)$
④ $f_a(x) = f(x+a) \Rightarrow \widehat{f_a}(k) = e^{ika} \hat{f}(k)$
⑤ $\widehat{f(ax)} = \frac{1}{|a|^d} \hat{f}\left(\frac{k}{a}\right)$
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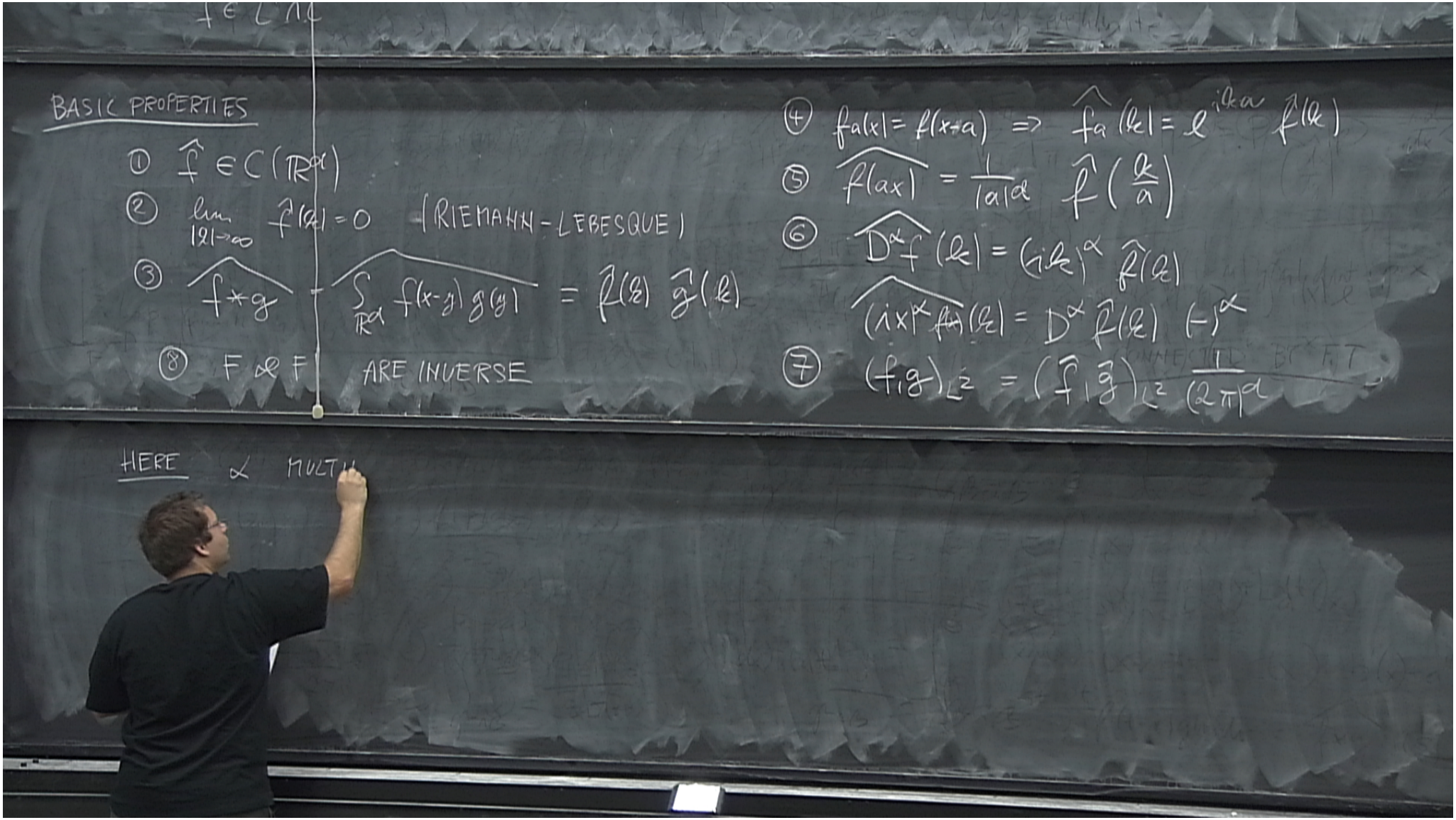

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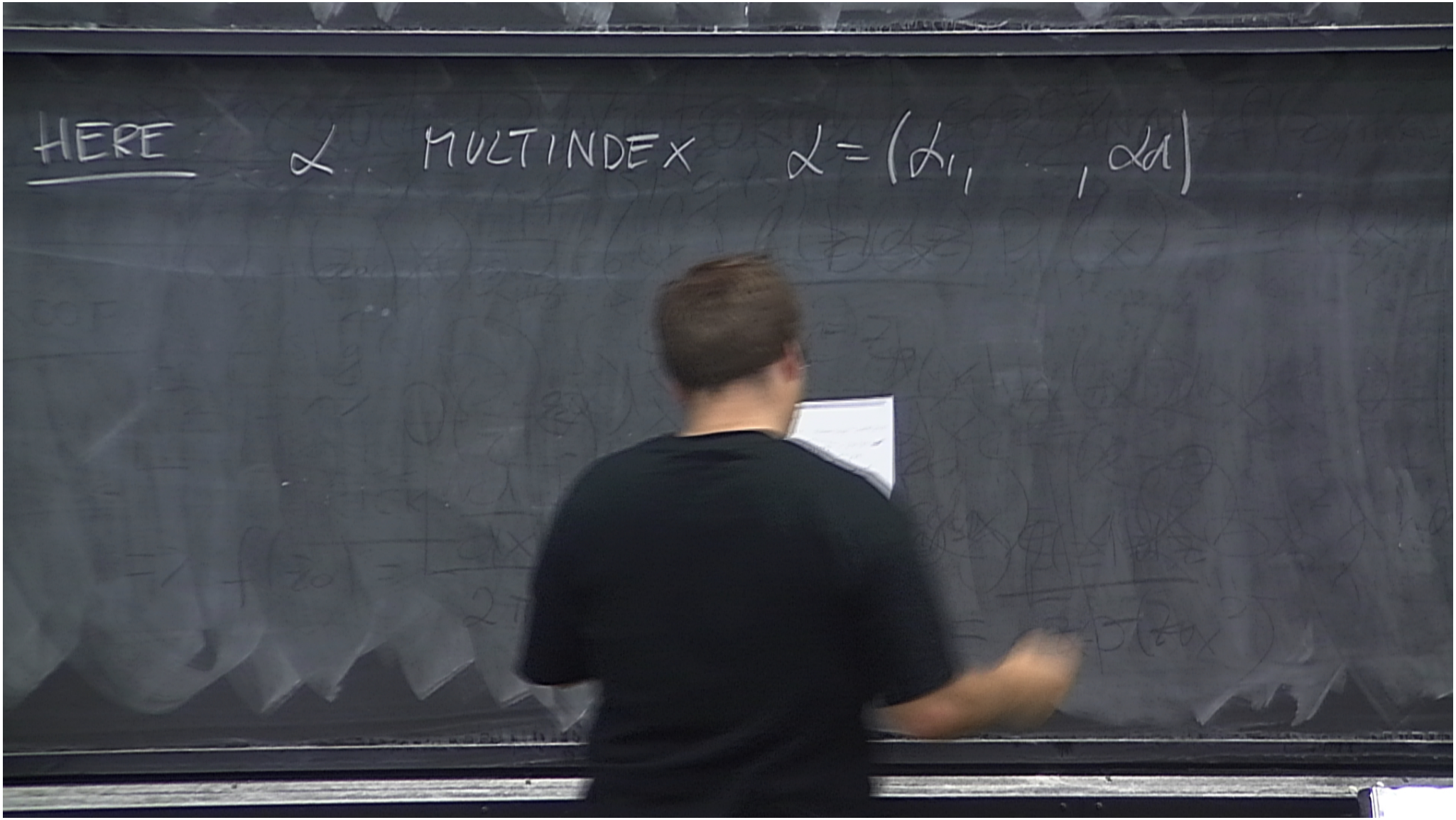

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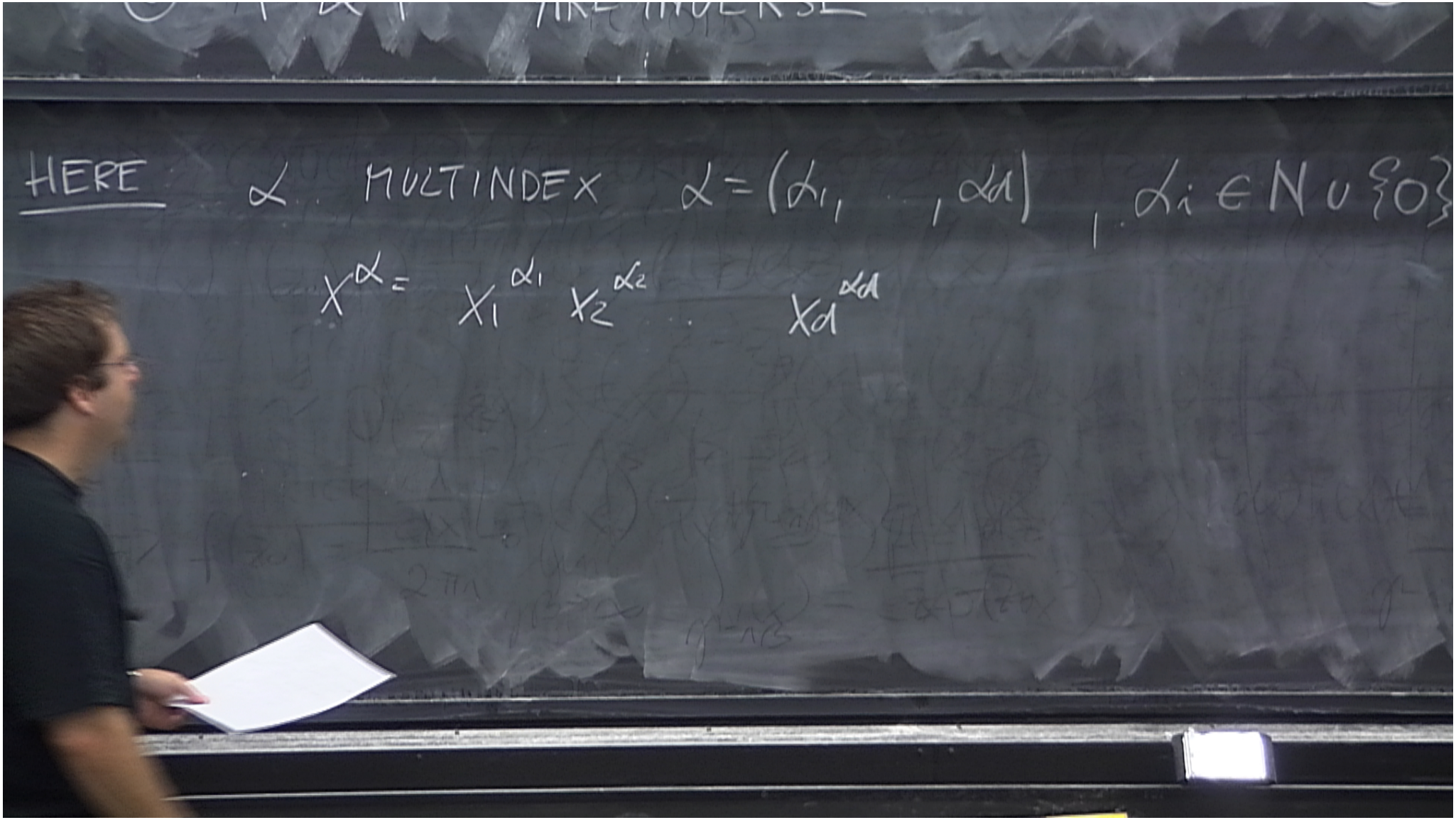

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(7) $(f,g)_{L^2} = (\bar{f}, \bar{g})$

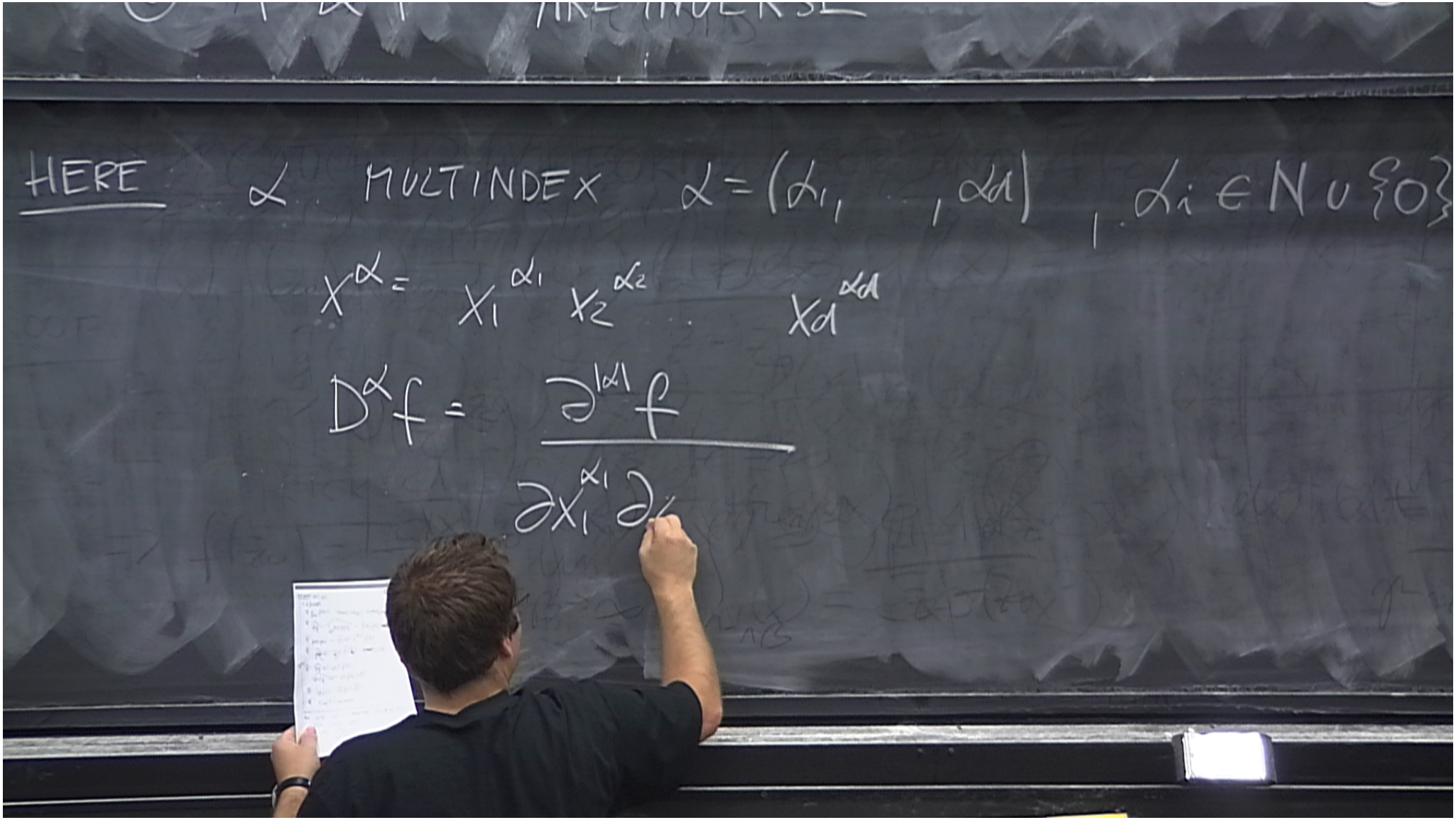

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(7) $(f,g)_{L^2} = (\hat{f},\hat{g})_{L^2}$ $\frac{1}{(2\pi)^d}$

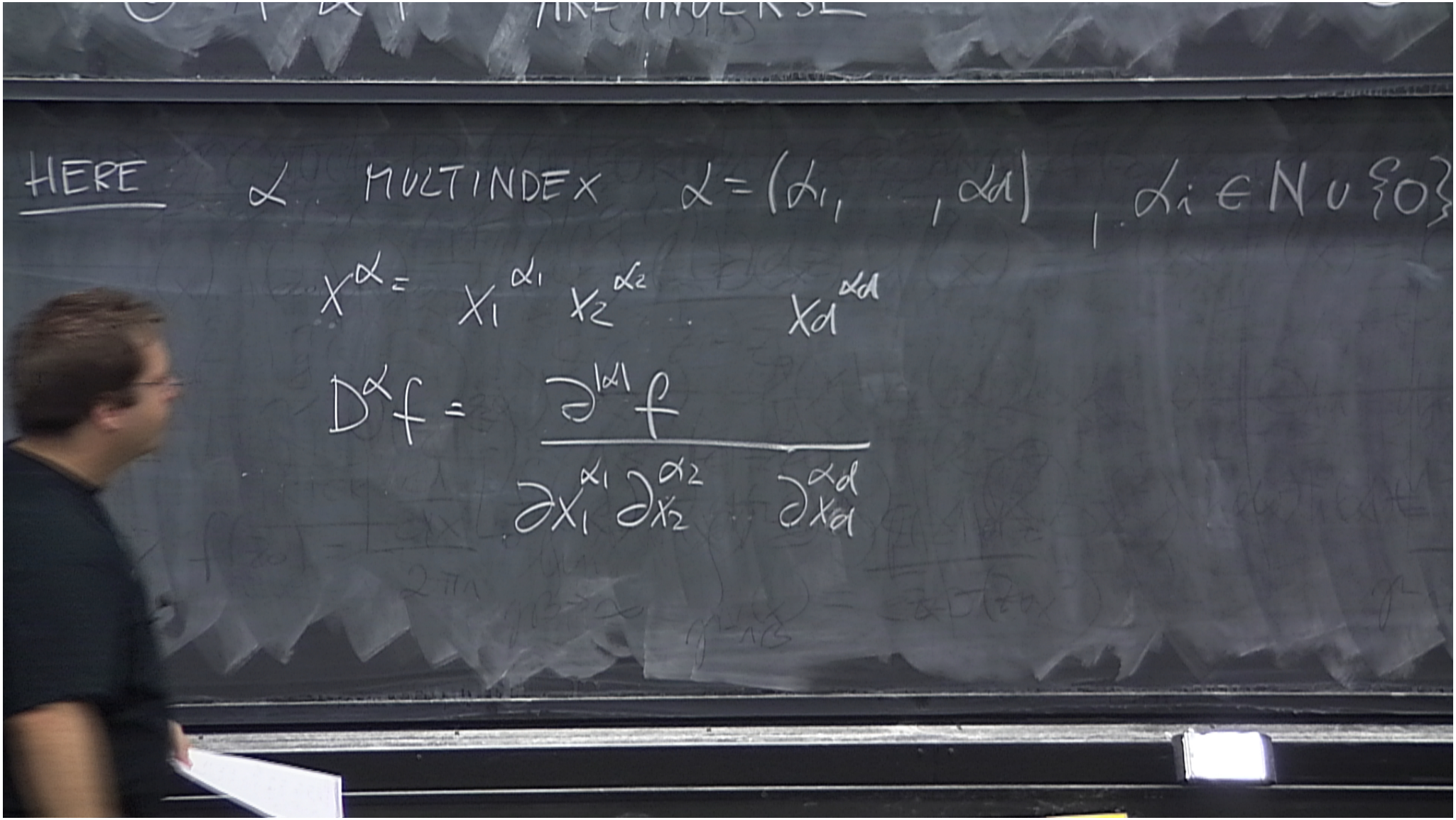

|x|→∞
③ $\widehat{f*g} = \int_{\mathbb{R}^d} f(x-y)\,g(y) = \hat{f}(x)\,\hat{g}(x)$
⑧ F & F' ARE INVERSE

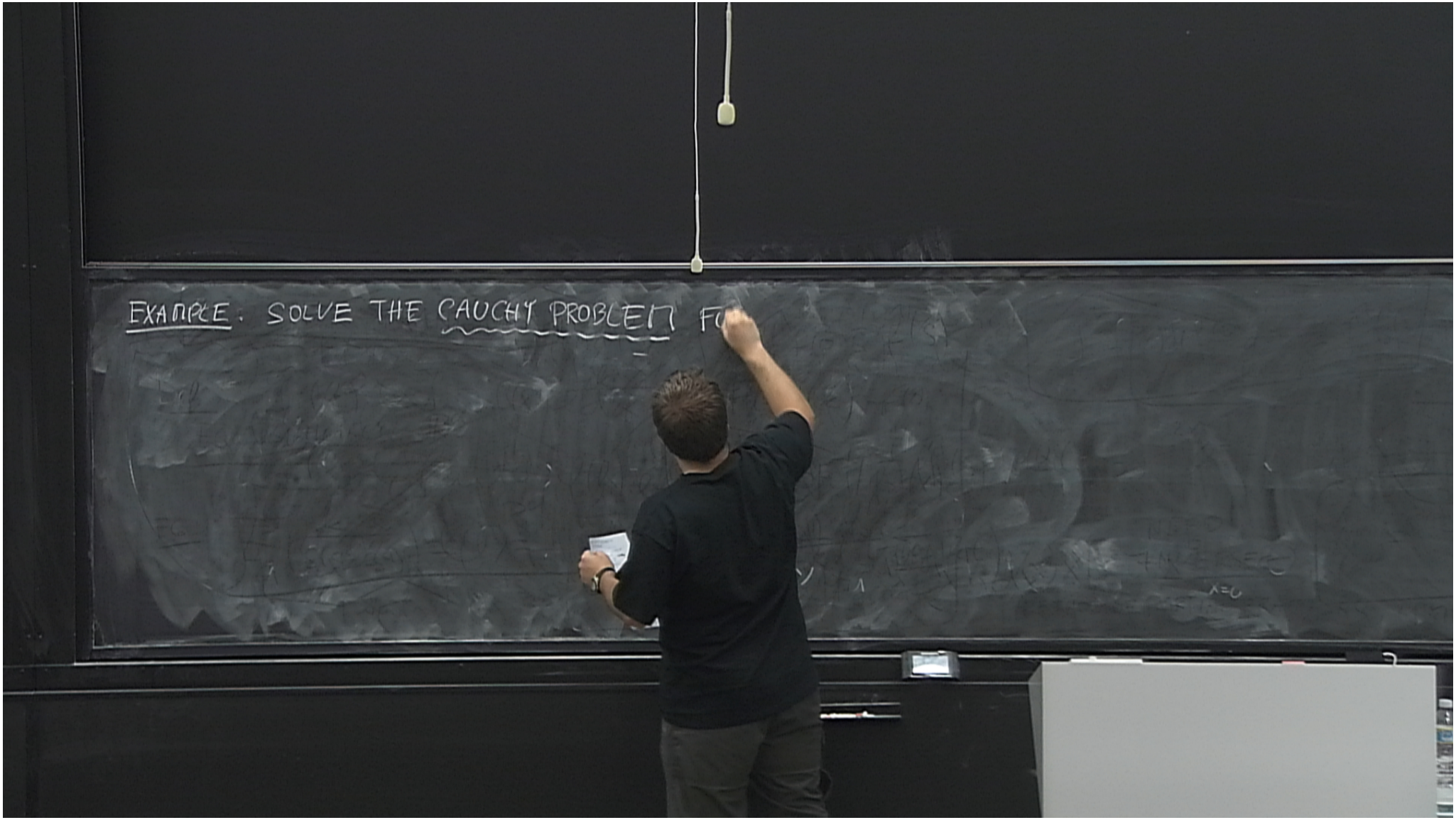

BASIC PROPERTIES
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$\widehat{(ix)^\alpha f(x)}(\xi) = D^\alpha \hat{f}(\xi)\,(-)^\alpha$
⑦ $(f,g)_{L^2} = \frac{1}{(2\pi)^d}(\hat{f},\hat{g})_{L^2}$
⑧ F & F ARE INVERSE
HERE α MULTI

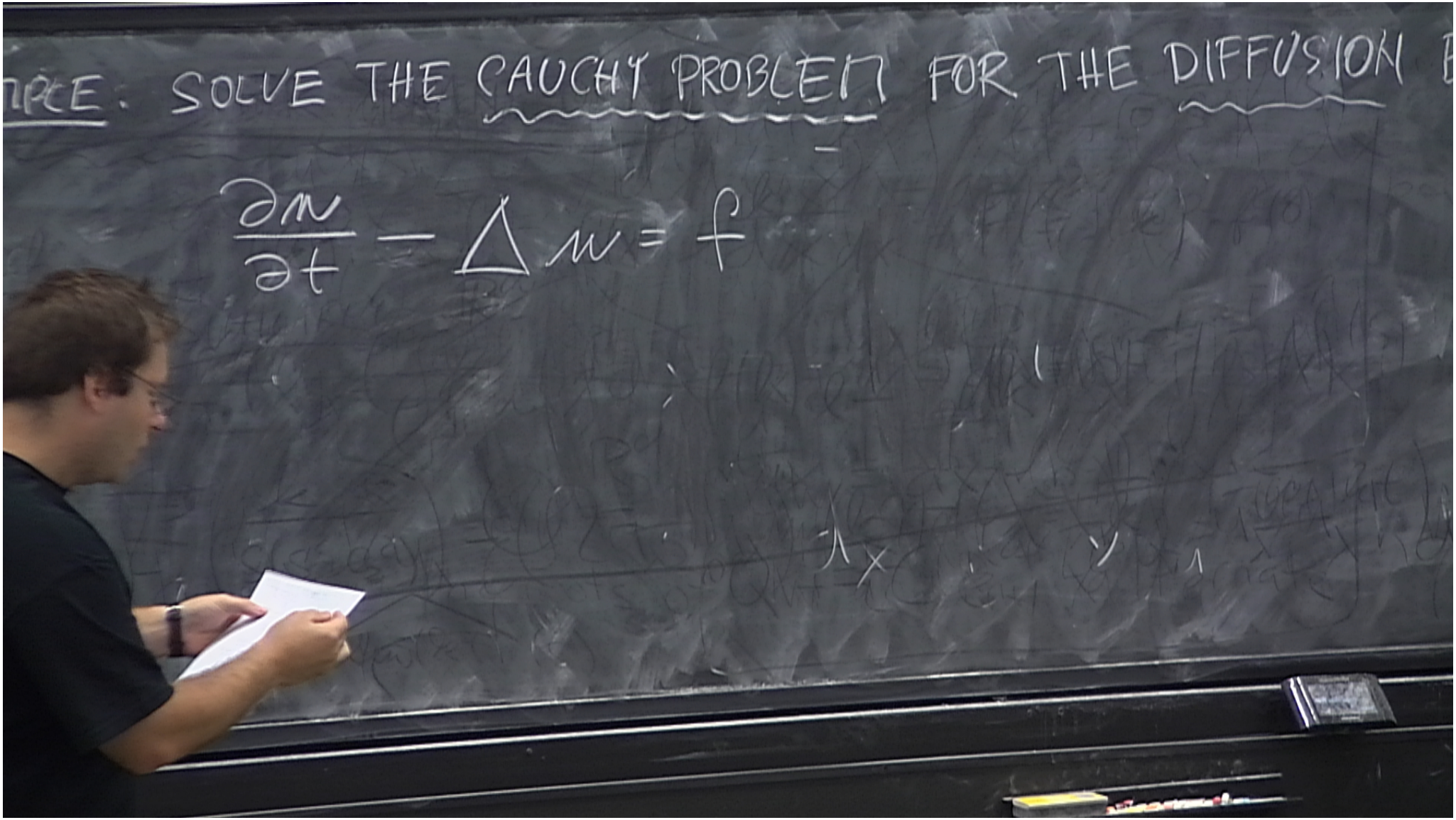

HERE
α MULTINDEX α = (α1, , αd)

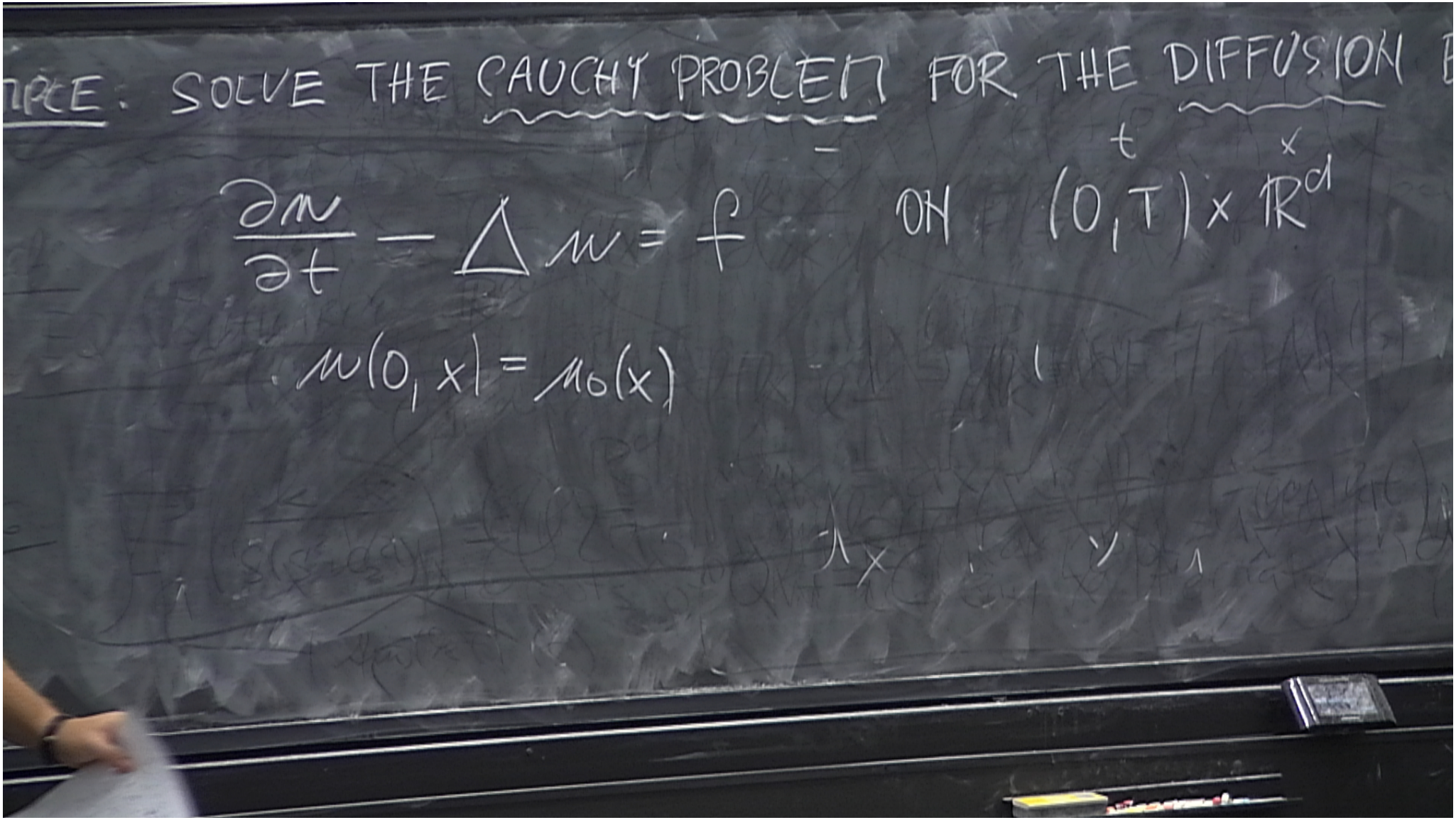

HERE
α MULTINDEX
$\alpha = (\alpha_1, \ldots, \alpha_d)$, $\alpha_i \in \mathbb{N} \cup \{0\}$
$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d}$

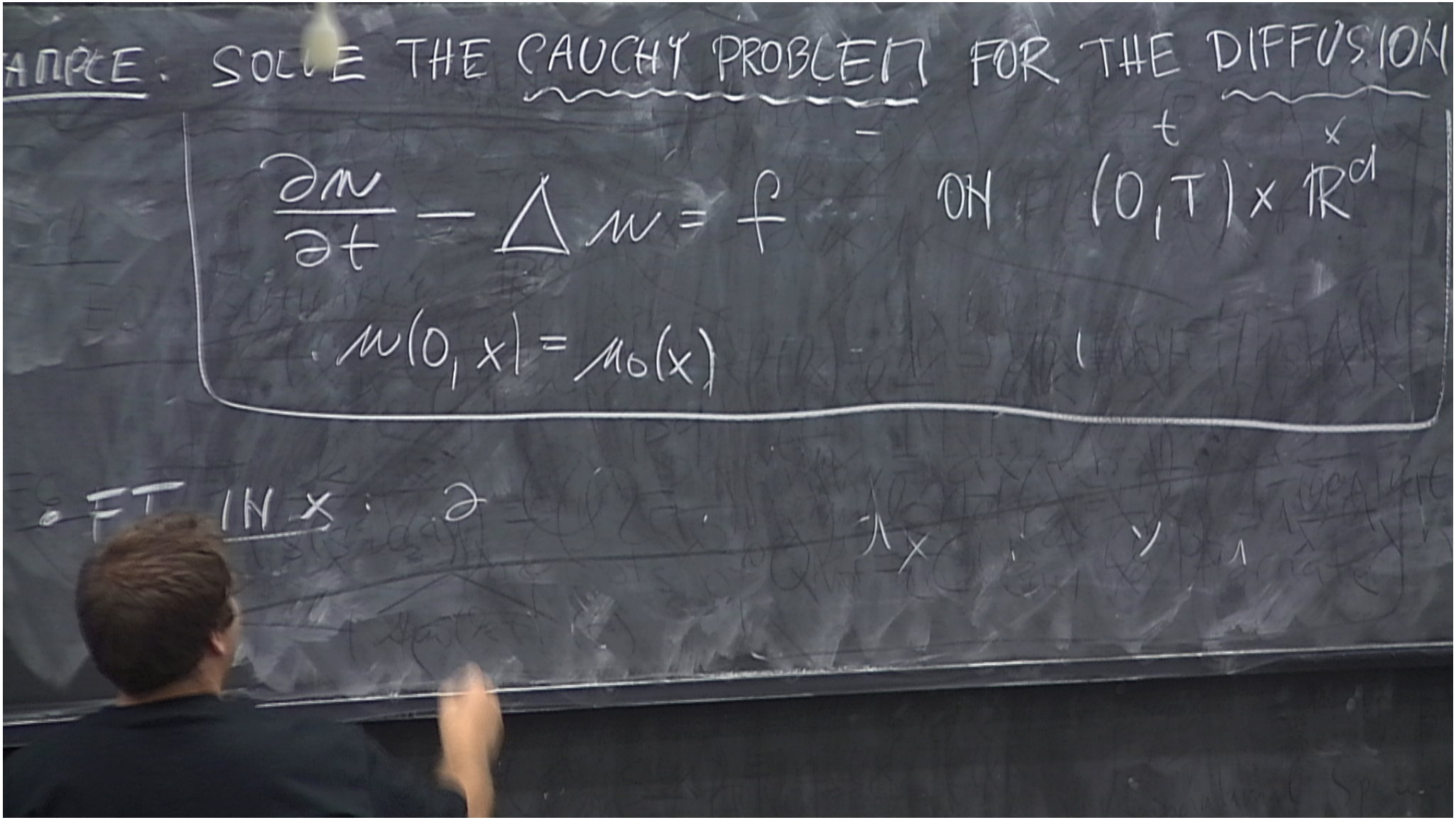

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$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d}$
$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}}$

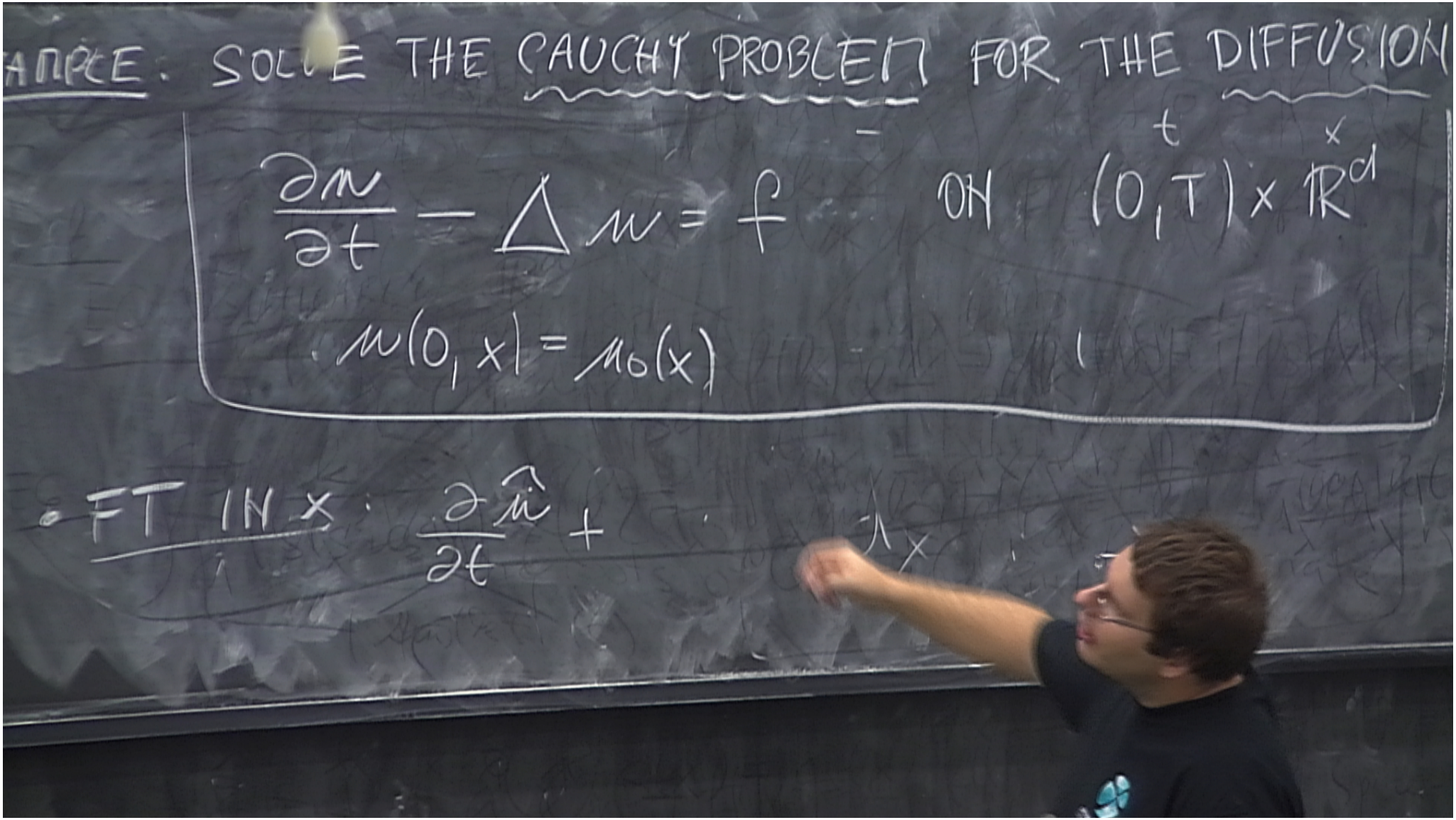

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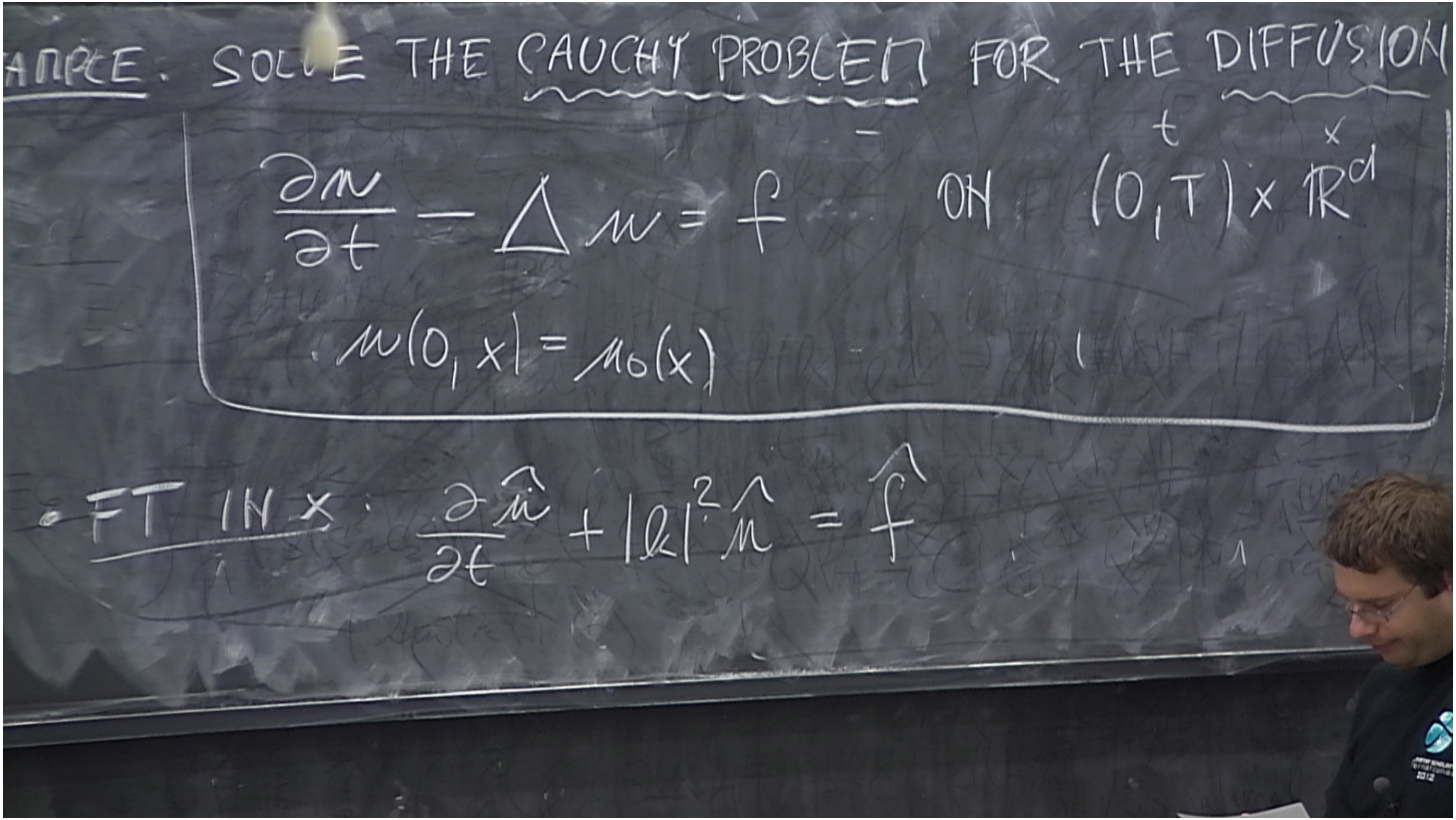

EXAMPLE. SOLVE THE CAUCHY PROBLEM

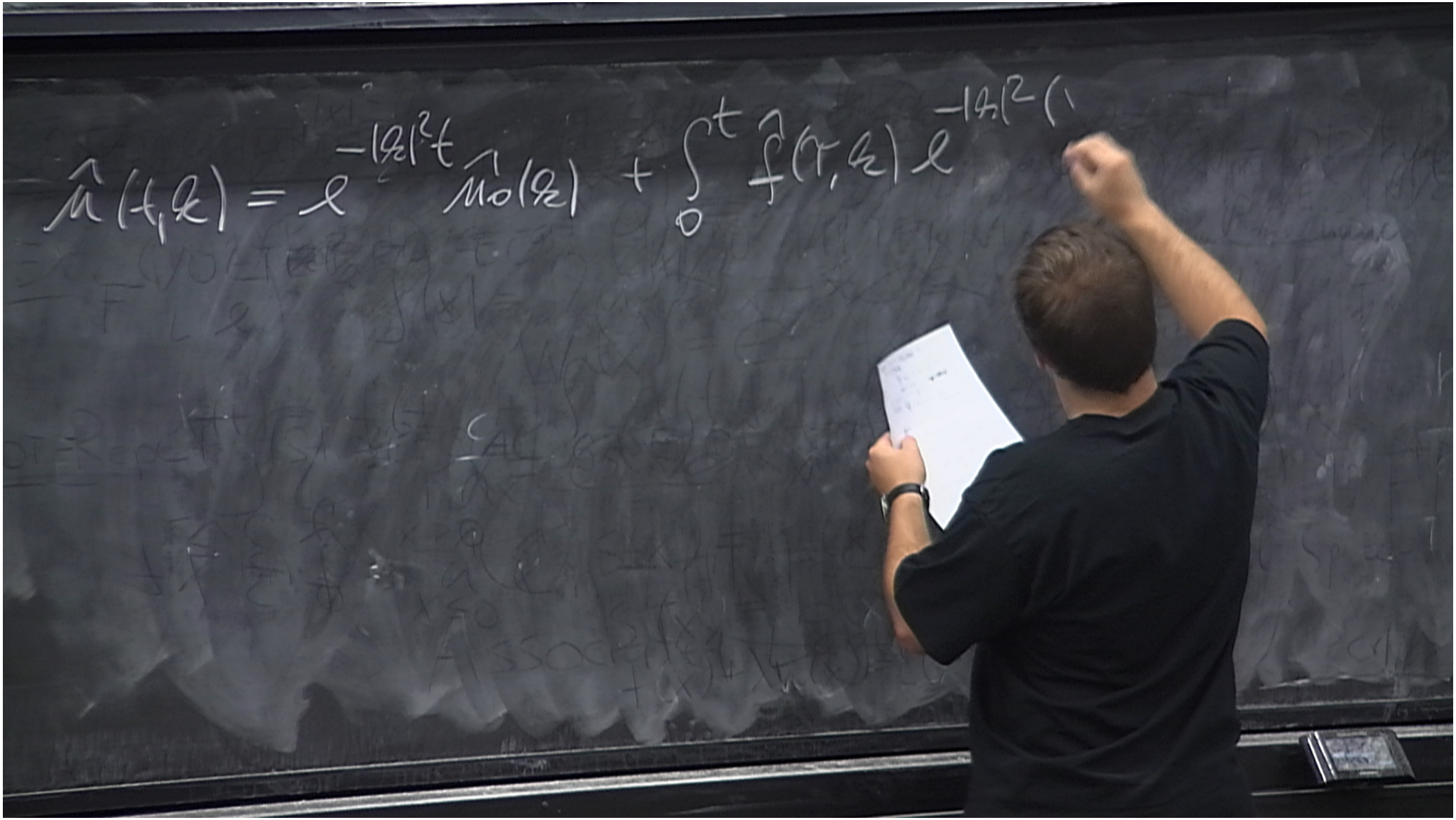

MPLE. SOLVE THE CAUCHY PROBLEM FOR THE DIFFUSION
$\frac{\partial u}{\partial t} - \Delta u = f$


SOLVE THE CAUCHY PROBLEM FOR THE DIFFUSION
$\frac{\partial u}{\partial t} - \Delta u = f$ ON $(0,T) \times \mathbb{R}^d$
$u(0,x) = u_0(x)$

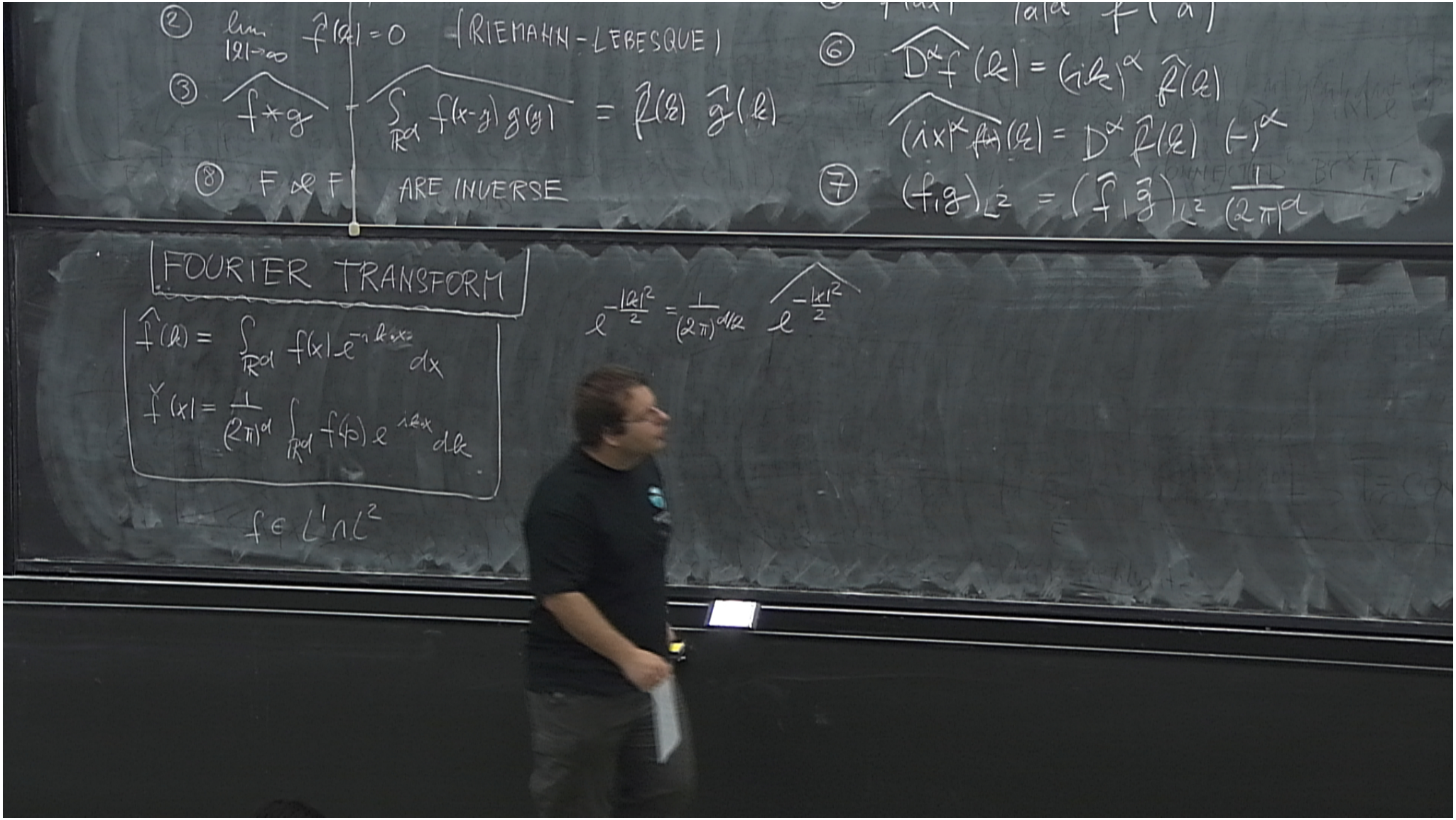

EXAMPLE. SOLVE THE CAUCHY PROBLEM FOR THE DIFFUSION
$\frac{\partial u}{\partial t} - \Delta u = f$ ON $(0,T) \times \mathbb{R}^d$
$u(0,x) = u_0(x)$
FT IN x: ∂
λ_x
ν
1


EXAMPLE: SOLVE THE CAUCHY PROBLEM FOR THE DIFFUSION
$\frac{\partial u}{\partial t} - \Delta u = f$ ON $(0,T) \times \mathbb{R}^d$
$u(0,x) = u_0(x)$
FT IN x: $\frac{\partial \hat{u}}{\partial t}$ +
$-\lambda x$


EXAMPLE: SOLVE THE CAUCHY PROBLEM FOR THE DIFFUSION
$\frac{\partial u}{\partial t} - \Delta u = f$ ON $(0,T) \times \mathbb{R}^d$
$u(0,x) = u_0(x)$
FT IN x: $\frac{\partial \hat{u}}{\partial t} + |k|^2 \hat{u} = \hat{f}$


$\hat{u}(t,k) = e^{-|k|^2 t}\hat{u}_0(k) + \int_0^t \hat{f}(\tau,k)\, e^{-|k|^2(}$

$$\hat{u}(t,k) = e^{-|k|^2 t}\hat{u}_0(k) + \int_0^t \hat{f}(\tau,k)\, e^{-|k|^2(t-\tau)}\, d\tau$$


② $\lim_{|\xi|\to\infty} \hat{f}(\xi) = 0$ (RIEMANN-LEBESGUE)
③ $\widehat{f*g} = \int_{\mathbb{R}^d} f(x-y)\,g(y) = \hat{f}(\xi)\,\hat{g}(\xi)$
⑥ $\widehat{D^\alpha f}(\xi) = (i\xi)^\alpha \hat{f}(\xi)$
$\widehat{(ix)^\alpha f}(\xi) = D^\alpha \hat{f}(\xi)\,(-)^\alpha$
⑦ $(f,g)_{L^2} = (\hat{f},\hat{g})_{L^2}\,\frac{1}{(2\pi)^d}$
⑧ $\mathcal{F}$ & $\mathcal{F}^*$ ARE INVERSE
FOURIER TRANSFORM
$\hat{f}(k) = \int_{\mathbb{R}^d} f(x)\,e^{-ik\cdot x}\,dx$
$\check{f}(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} f(\xi)\,e^{ix\cdot\xi}\,d\xi$
$f \in L^1 \cap L^2$
$e^{-\frac{|x|^2}{2}} = \frac{1}{(2\pi)^{d/2}}\,\widehat{e^{-\frac{|x|^2}{2}}}$

BASIC PROPERTIES

① $\hat{f} \in C(\mathbb{R}^d)$

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③ $\widehat{f * g} = \widehat{\int_{\mathbb{R}^d} f(x-y)\, g(y)} = \hat{f}(\xi)\, \hat{g}(\xi)$

④ $f_a(x) = f(x+a) \Rightarrow \widehat{f_a}(\xi) = e^{i\xi a} \hat{f}(\xi)$

⑤ $\widehat{f(ax)} = \frac{1}{|a|^d} \hat{f}\left(\frac{\xi}{a}\right)$

⑥ $\widehat{D^\alpha f}(\xi) = (i\xi)^\alpha \hat{f}(\xi)$

$\widehat{(ix)^\alpha f(x)}(\xi) = D^\alpha \hat{f}(\xi)\, (-)^\alpha$

⑦ $(f, g)_{L^2} = \left(\hat{f}, \hat{g}\right)_{L^2} \frac{1}{(2\pi)^d}$

⑧ $\mathcal{F}$ & $\mathcal{F}^{-1}$ ARE INVERSE

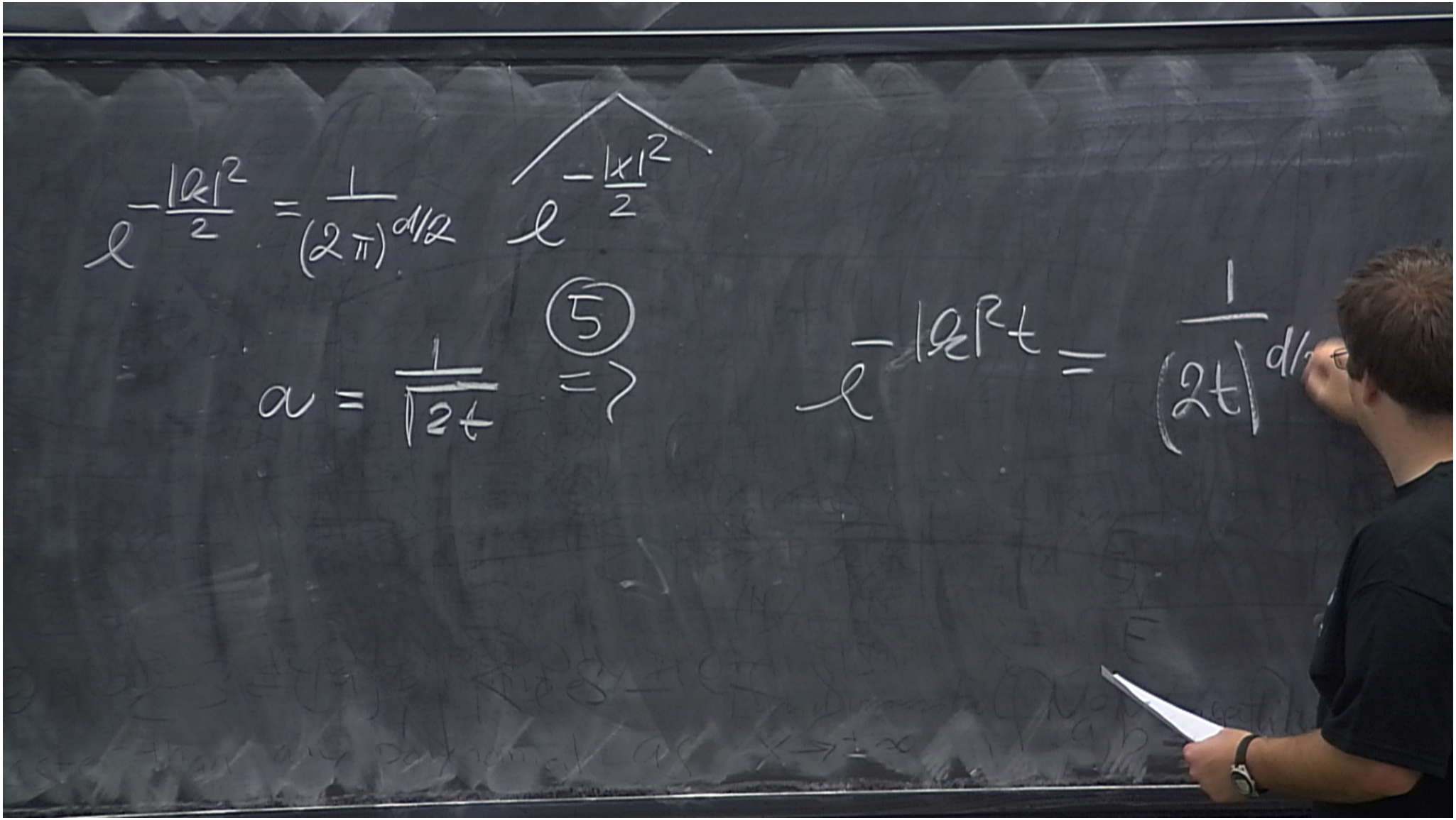
FOURIER TRANSFORM

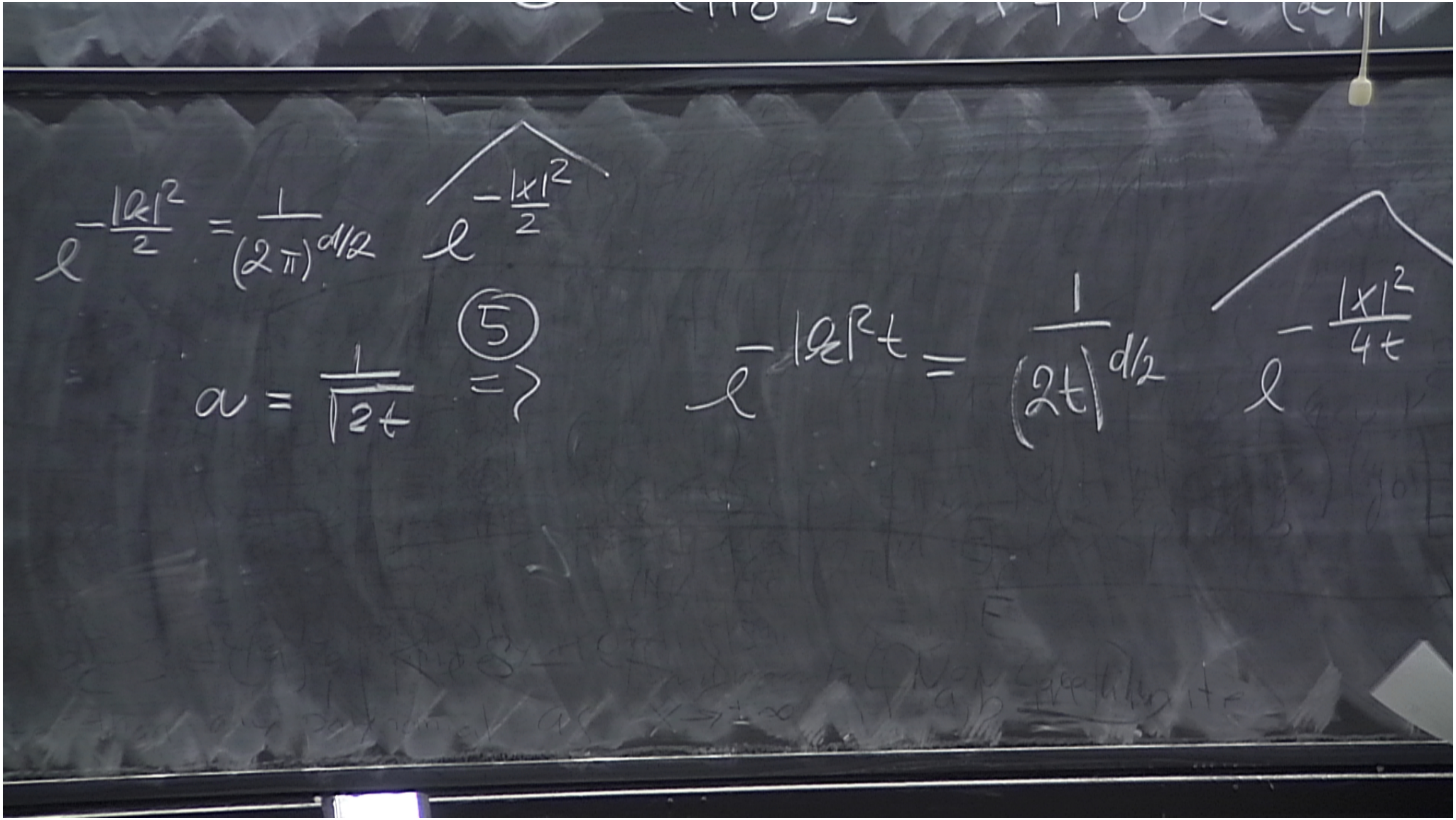
$$\hat{f}(k) = \int_{\mathbb{R}^d} f(x)\, e^{-i k \cdot x}\, dx$$

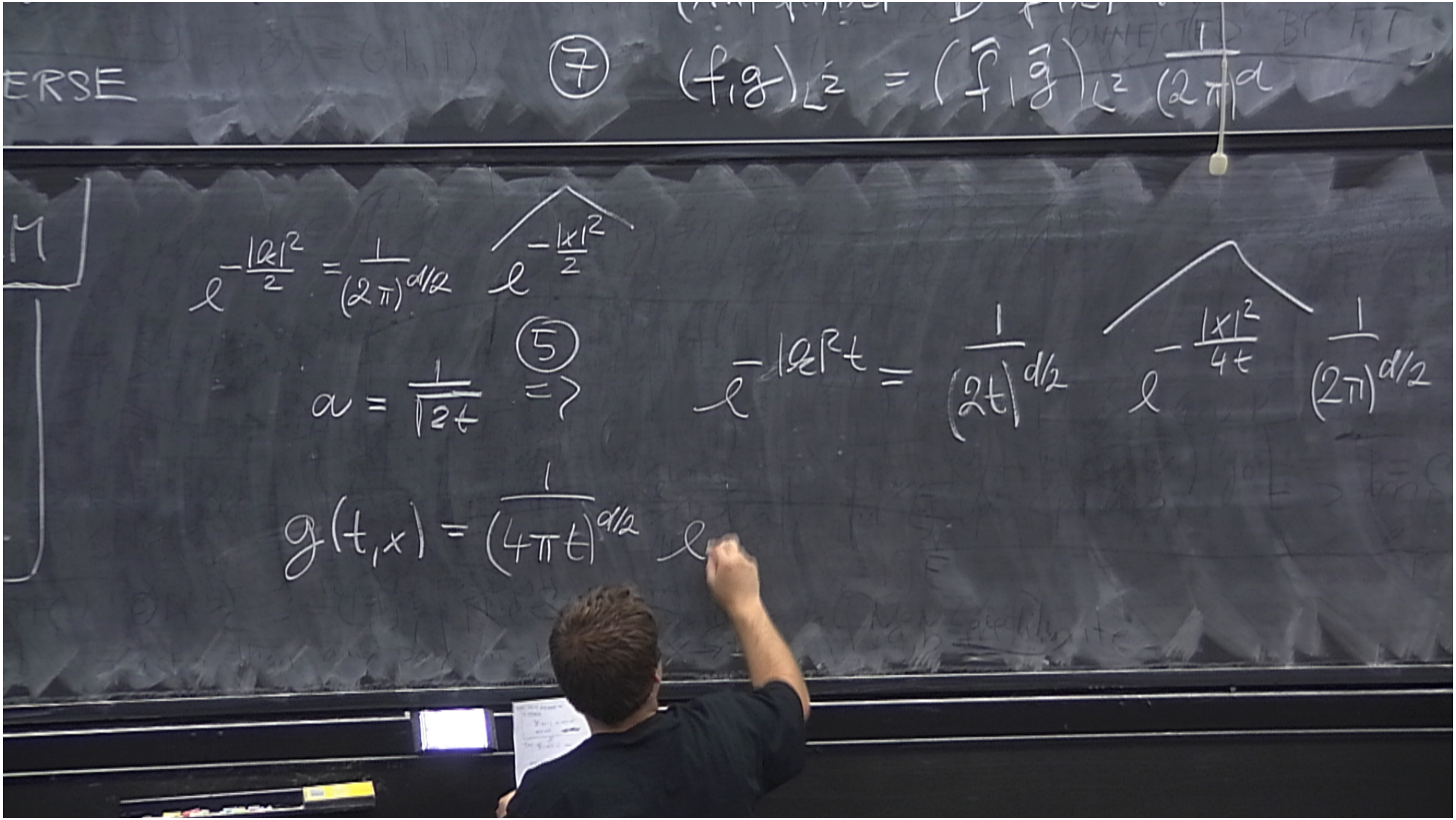
$$f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{f}(\xi)\, e^{i \xi \cdot x}\, d\xi$$

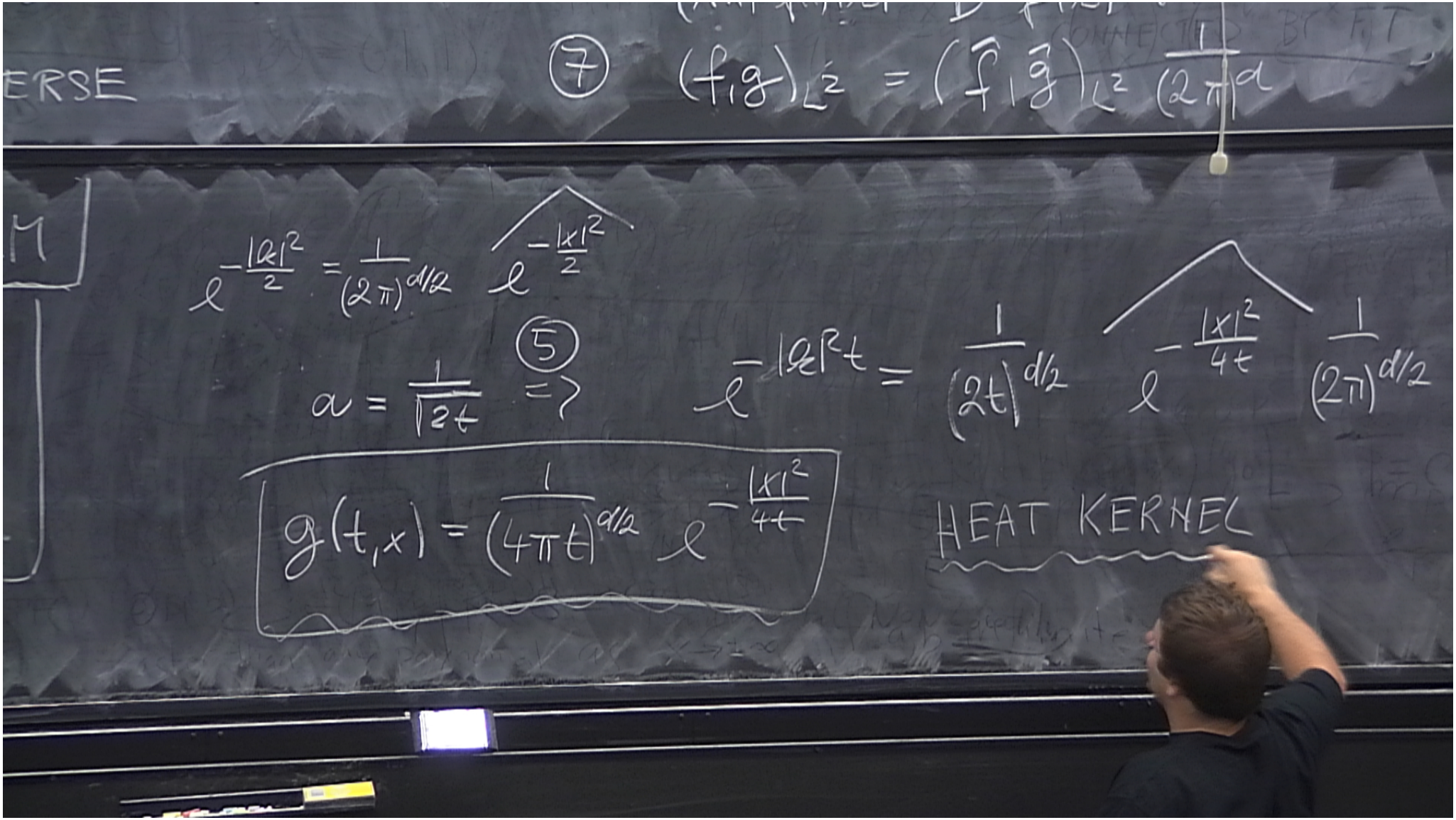
$f \in L^1 \cap L^2$

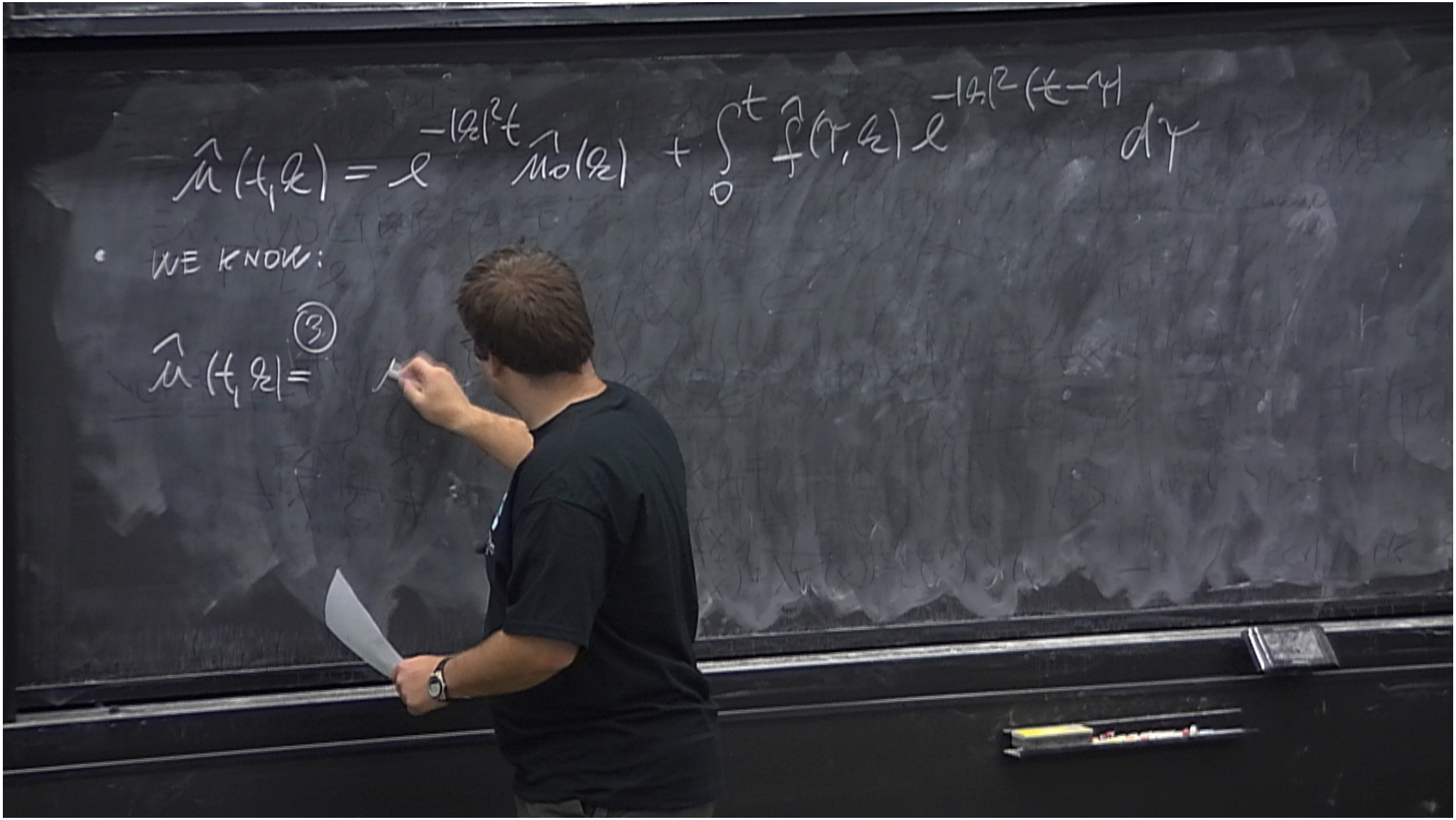
$$e^{-\frac{|\xi|^2}{2}} = \frac{1}{(2\pi)^{d/2}} \widehat{e^{-\frac{|x|^2}{2}}}$$

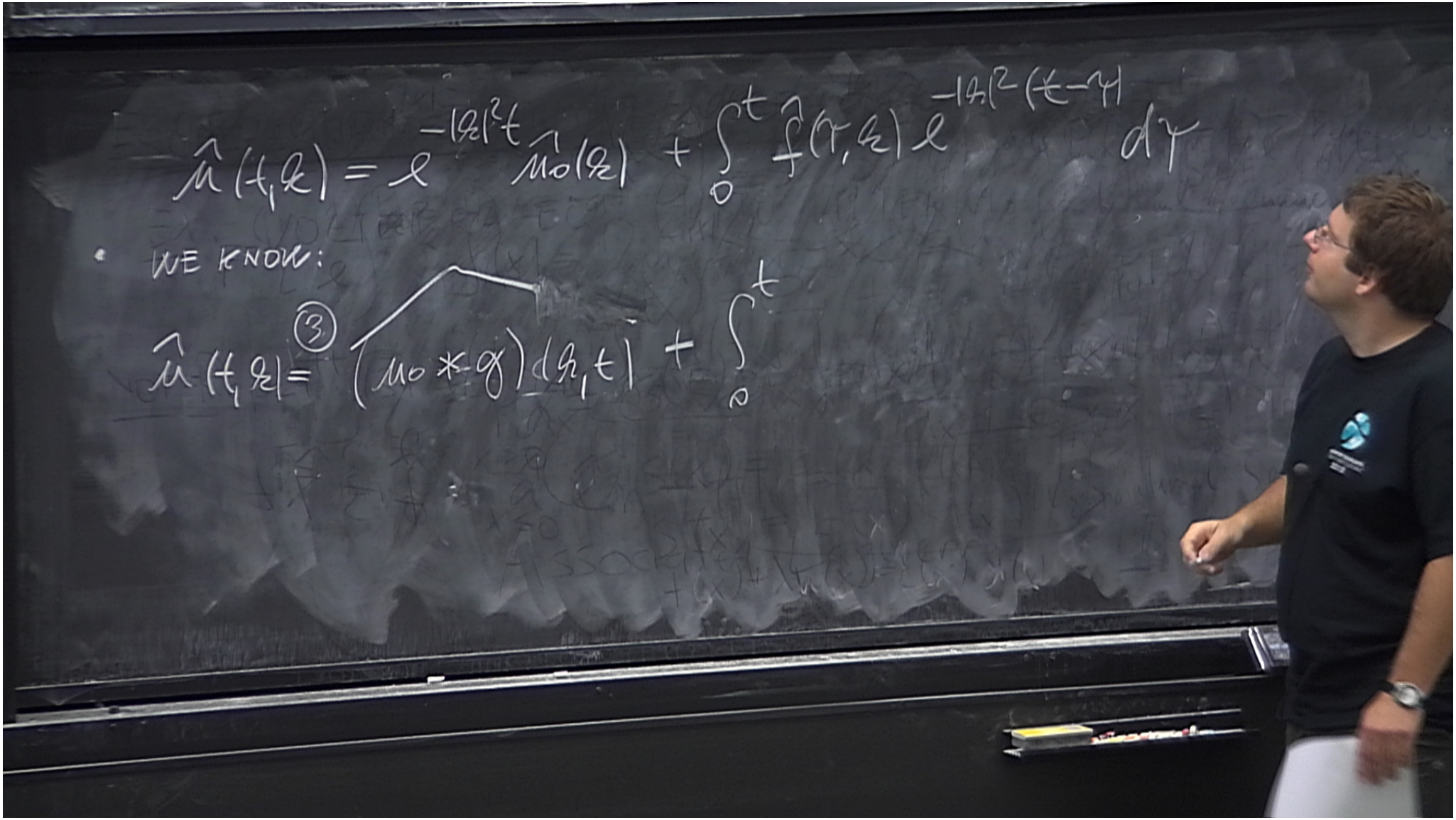

$e^{-\frac{|k|^2}{2}} = \frac{1}{(2\pi)^{d/2}} \int e^{-\frac{|x|^2}{2}}$
$a = \frac{1}{\sqrt{2t}}$
(5)
$\Rightarrow$
$e^{-|k|^2 t} = \frac{1}{(2t)^{d/2}}$


$e^{-\frac{|\xi|^2}{2}} = \frac{1}{(2\pi)^{d/2}} e^{-\frac{|x|^2}{2}}$
$a = \frac{1}{\sqrt{2t}}$
(5)
$\Rightarrow$
$e^{-|\xi|^2 t} = \frac{1}{(2t)^{d/2}} e^{-\frac{|x|^2}{4t}}$


ERSE
(7) $(f,g)_{L^2} = (\hat{f},\hat{g})_{L^2}\frac{1}{(2\pi)^d}$
$e^{-\frac{|\xi|^2}{2}} = \frac{1}{(2\pi)^{d/2}} e^{-\frac{|x|^2}{2}}$
$a = \frac{1}{\sqrt{2t}}$ (5) $\Rightarrow$
$e^{-|\xi|^2 t} = \frac{1}{(2t)^{d/2}} e^{-\frac{|x|^2}{4t}} \frac{1}{(2\pi)^{d/2}}$
$g(t,x) = \frac{1}{(4\pi t)^{d/2}}$ e


ERSE
⑦ $(f,g)_{L^2} = (\hat{f},\hat{g})_{L^2}\,\frac{1}{(2\pi)^d}$
$e^{-\frac{|\xi|^2}{2}} = \frac{1}{(2\pi)^{d/2}}\, e^{-\frac{|x|^2}{2}}$
$a = \frac{1}{\sqrt{2t}}$ ⑤ ⇒
$e^{-|\xi|^2 t} = \frac{1}{(2t)^{d/2}}\, e^{-\frac{|x|^2}{4t}}\, \frac{1}{(2\pi)^{d/2}}$
$g(t,x) = \frac{1}{(4\pi t)^{d/2}}\, e^{-\frac{|x|^2}{4t}}$
HEAT KERNEL

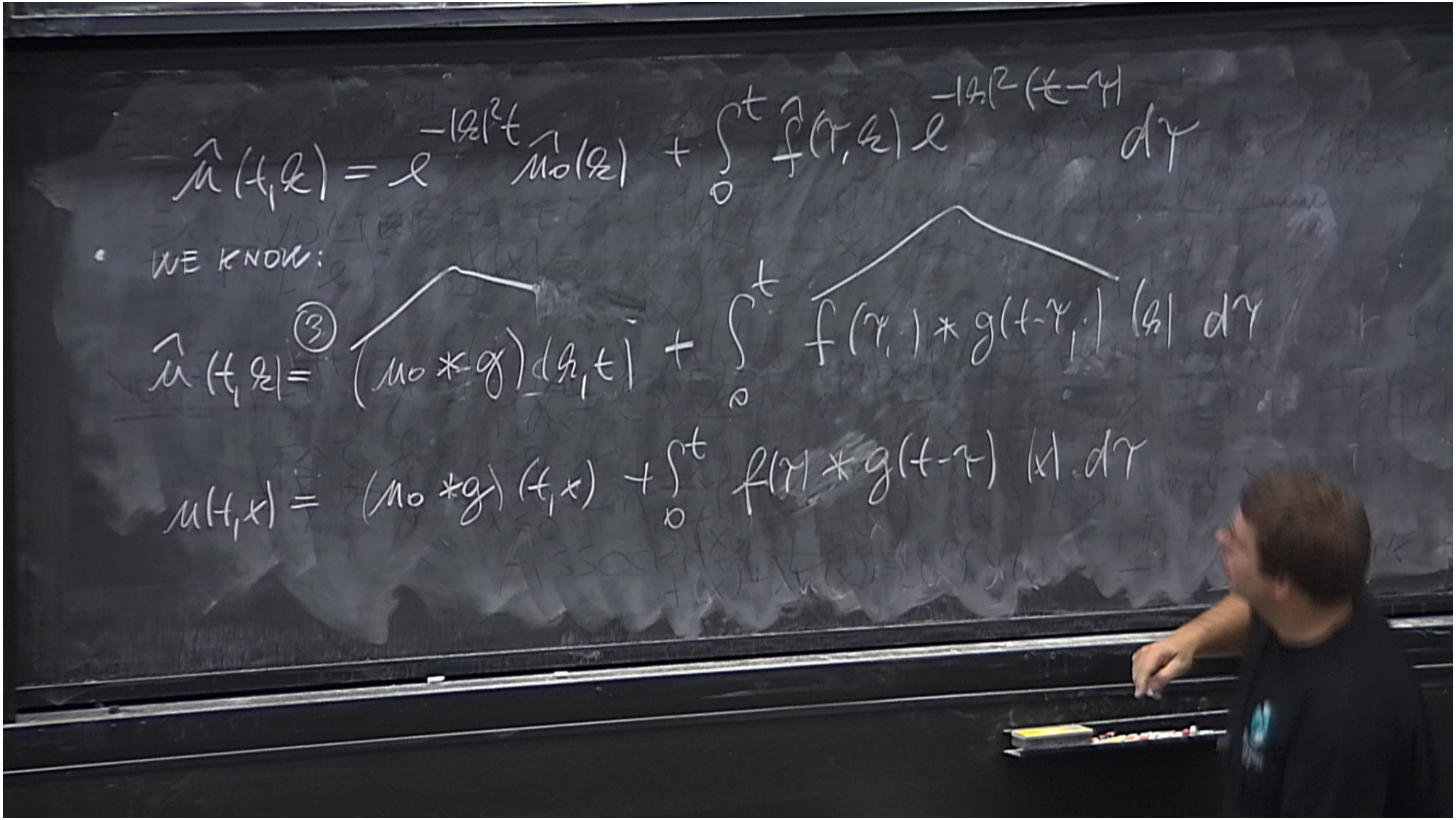

$\hat{u}(t,k) = e^{-|k|^2 t}\hat{u}_0(k) + \int_0^t \hat{f}(\tau,k)\, e^{-|k|^2(t-\tau)}\, d\tau$
• WE KNOW:
$\hat{u}(t,k) =$ ③

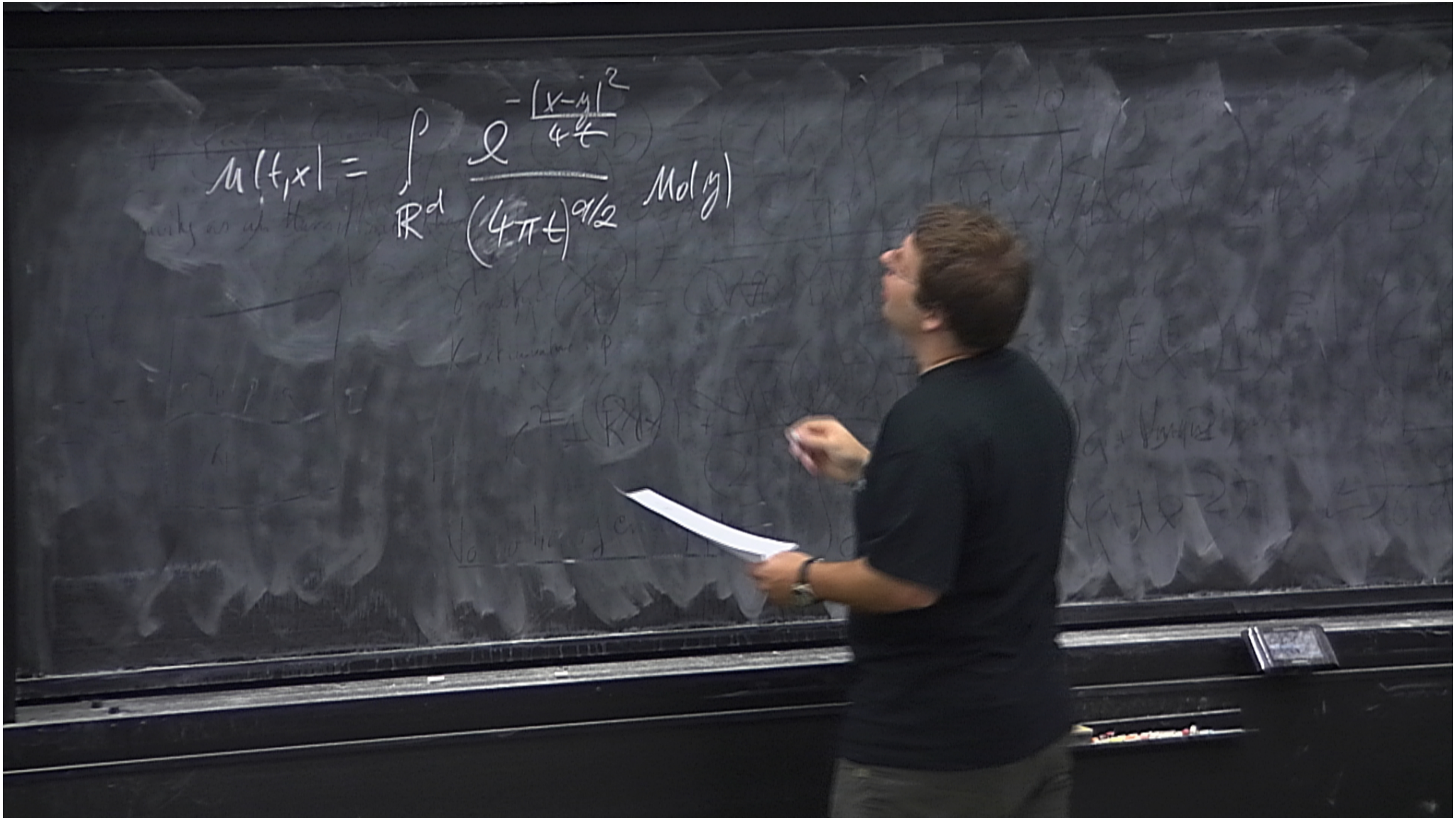

$\hat{u}(t,k) = e^{-|k|^2 t}\hat{u}_0(k) + \int_0^t \hat{f}(\tau,k)\, e^{-|k|^2(t-\tau)}\, d\tau$
• WE KNOW:
$\hat{u}(t,k) \overset{③}{=} (u_0 * g)(k,t) + \int_0^t$

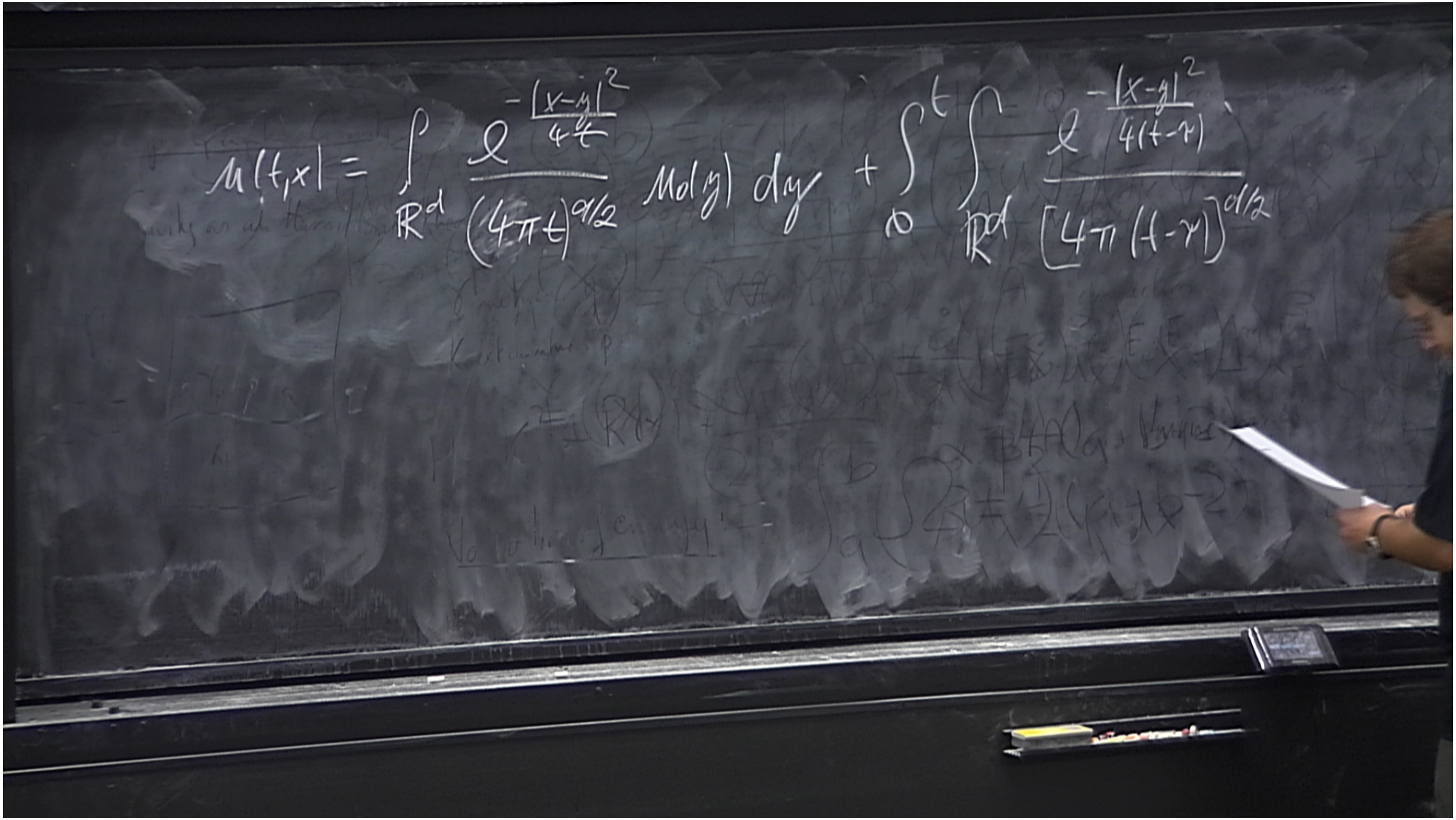
$$\hat{u}(t,k) = e^{-|k|^2 t}\hat{u}_0(k) + \int_0^t \hat{f}(\tau,k)\, e^{-|k|^2(t-\tau)}\, d\tau$$

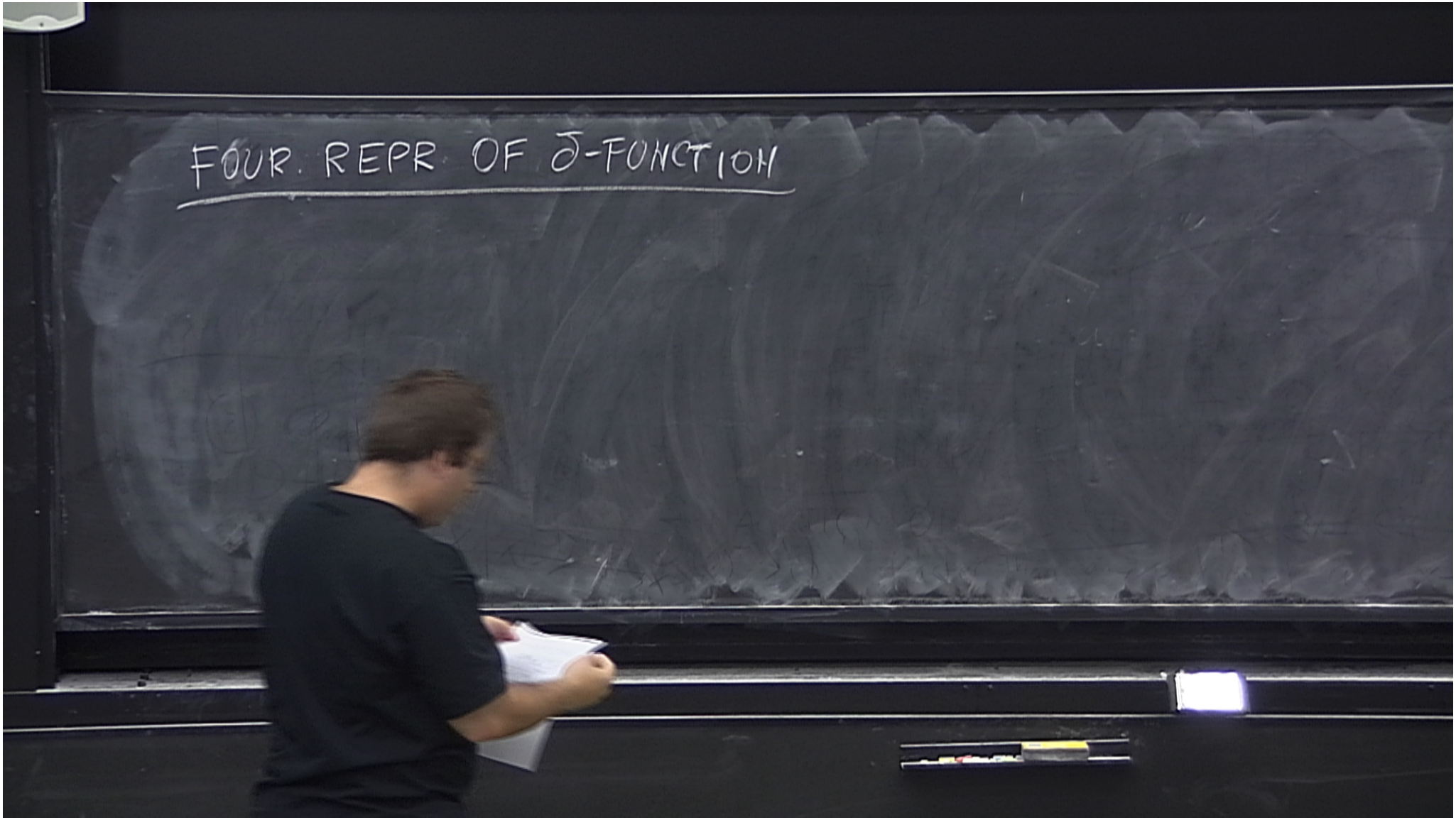
- WE KNOW:

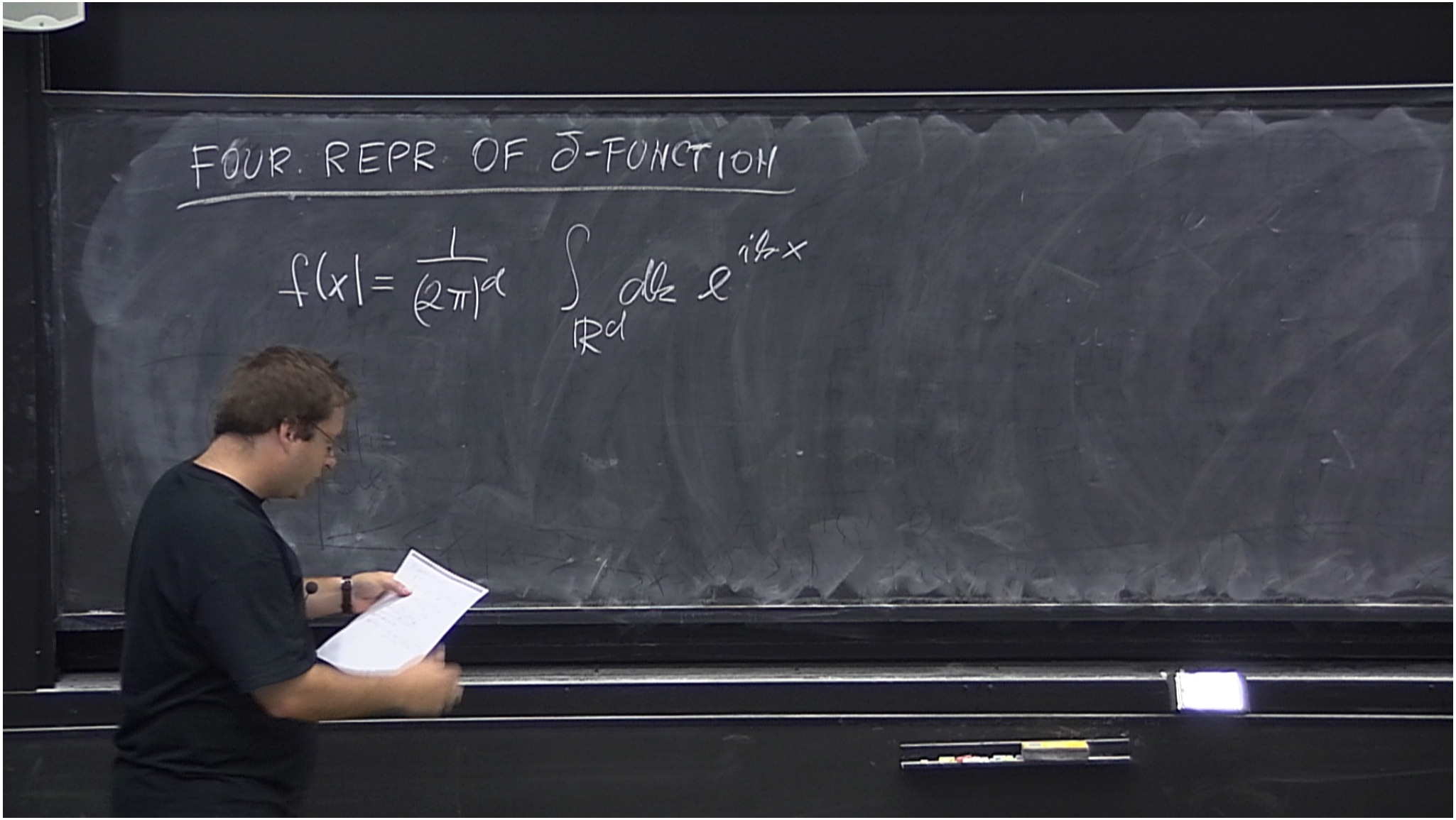
$$\hat{u}(t,k) \overset{③}{=} (u_0 * g)(k,t) + \int_0^t f(\tau,\cdot) * g(t-\tau,\cdot)(k)\, d\tau$$

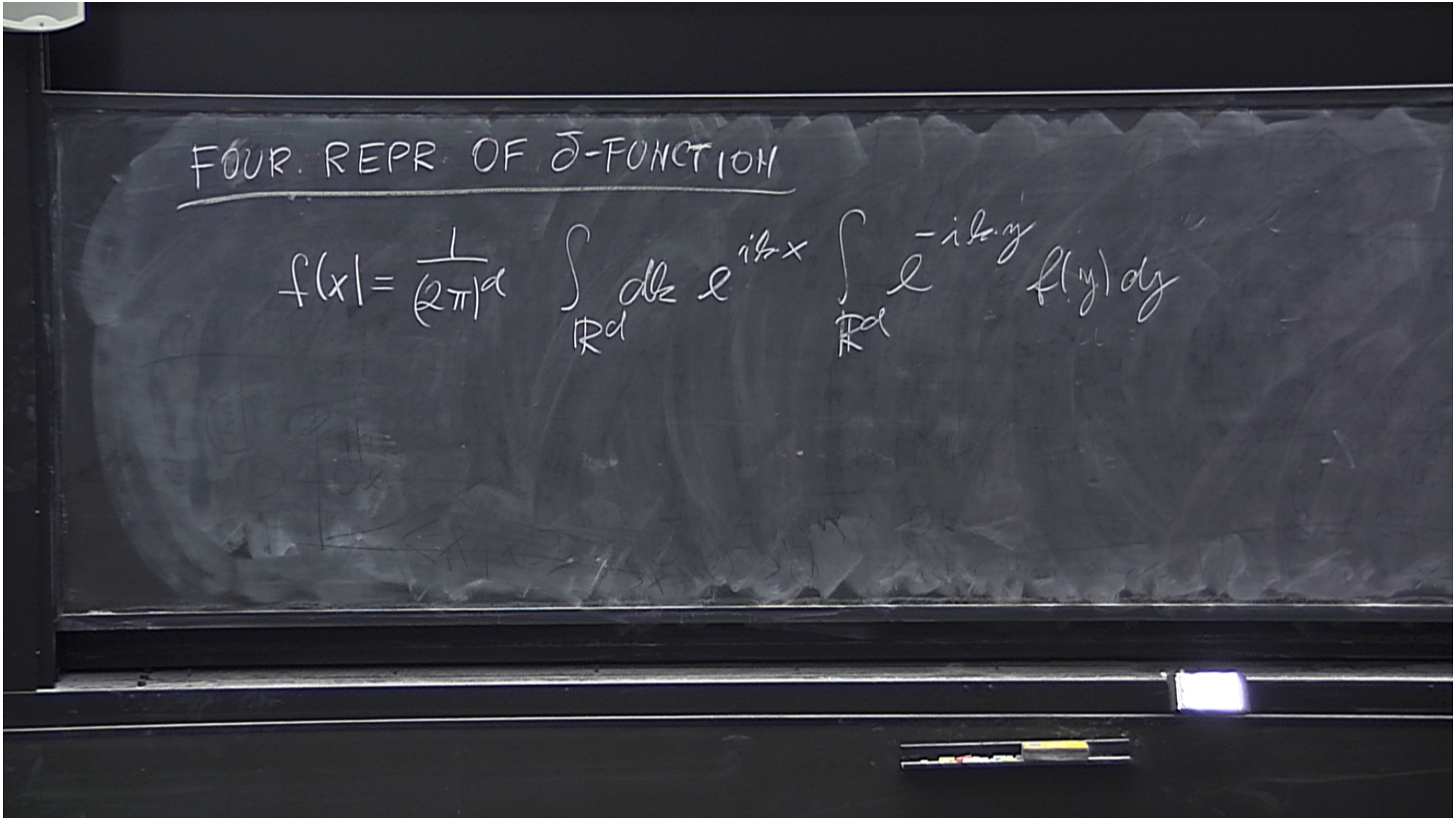

$\hat{u}(t,\xi) = e^{-|\xi|^2 t}\hat{u}_0(\xi) + \int_0^t \hat{f}(\tau,\xi)\, e^{-|\xi|^2(t-\tau)}\, d\tau$
WE KNOW:
$\hat{u}(t,\xi) = (u_0 * g)(\xi,t) + \int_0^t f(\tau) * g(t-\tau)(\xi)\, d\tau$
$u(t,x) = (u_0 * g)(t,x) + \int_0^t f(\tau) * g(t-\tau)(x)\, d\tau$

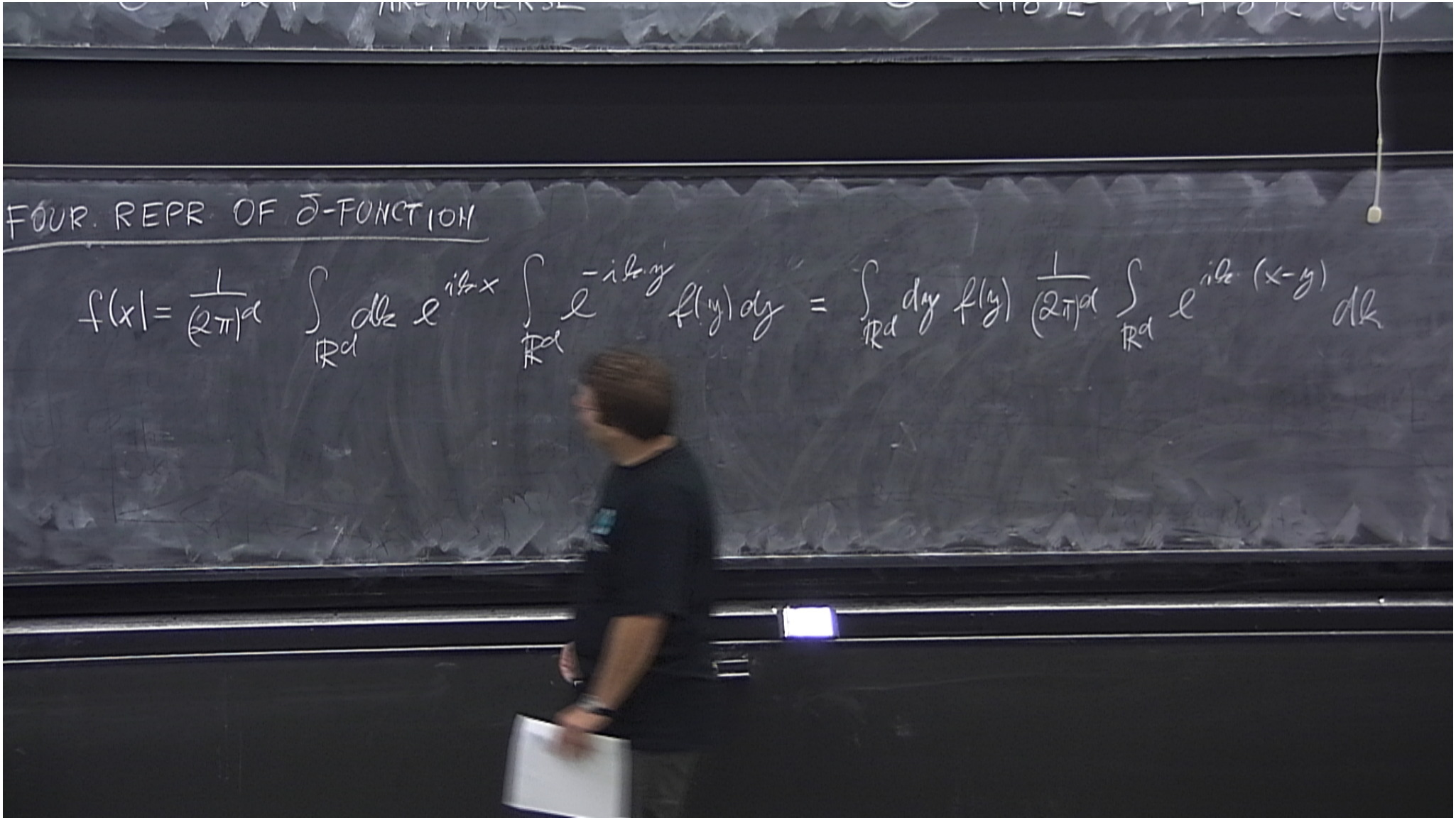

$u(t,x) = \int_{\mathbb{R}^d} \frac{e^{-\frac{|x-y|^2}{4t}}}{(4\pi t)^{d/2}} u_0(y)$

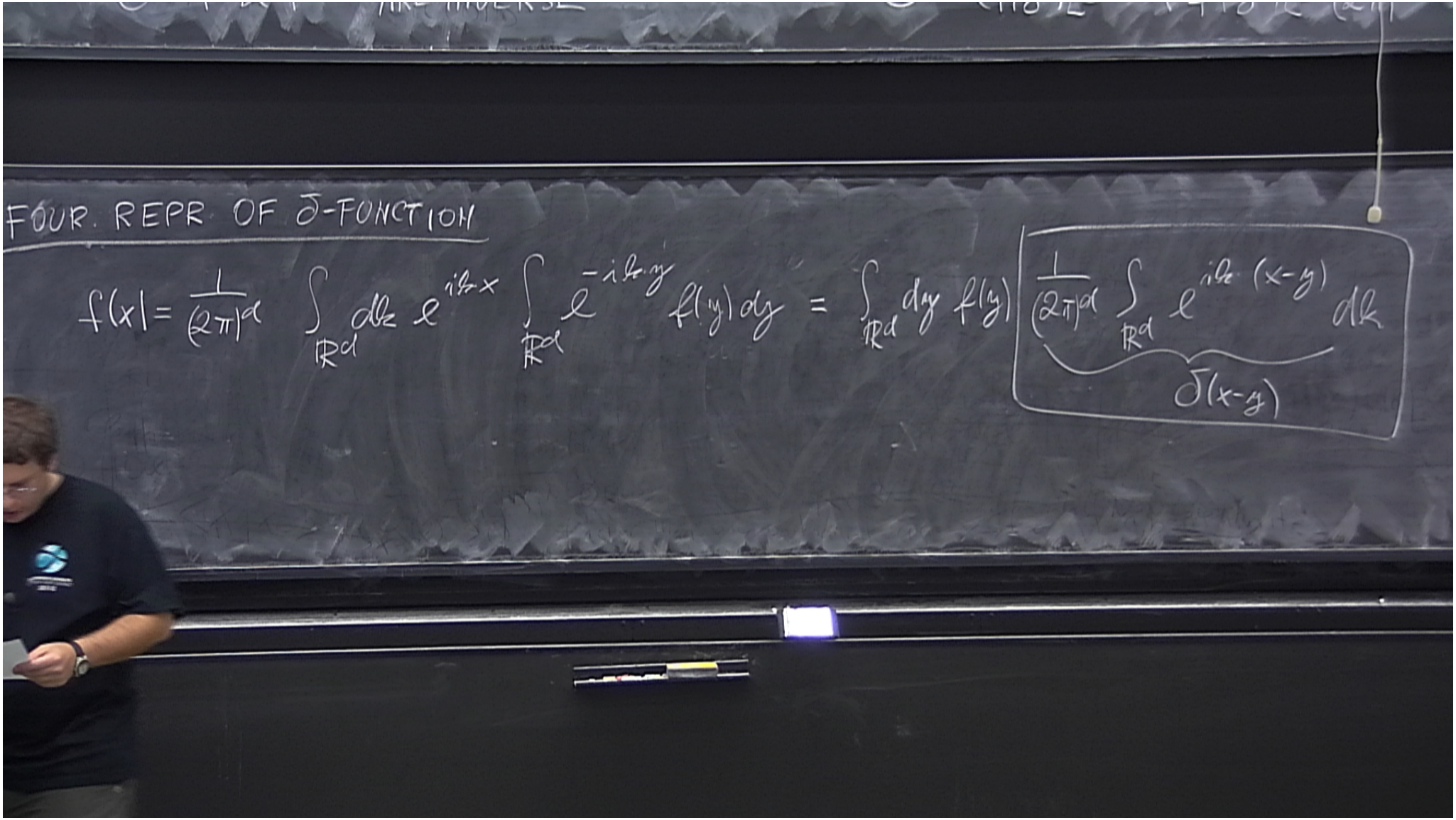

$u(t,x) = \int_{\mathbb{R}^d} \frac{e^{-\frac{|x-y|^2}{4t}}}{(4\pi t)^{d/2}} u_0(y)\,dy + \int_0^t \int_{\mathbb{R}^d} \frac{e^{-\frac{|x-y|^2}{4(t-\tau)}}}{[4\pi(t-\tau)]^{d/2}}$

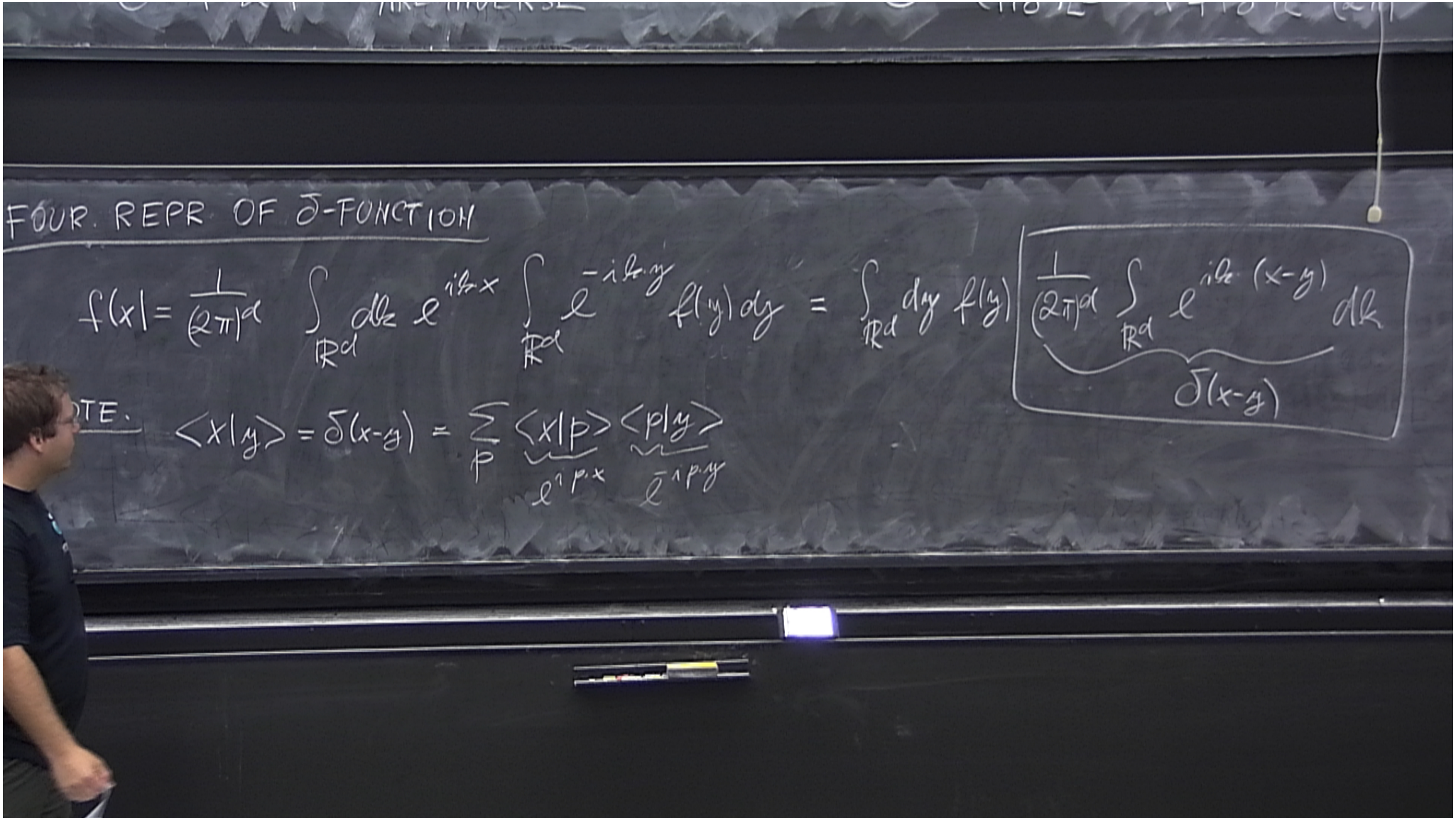

FOUR. REPR OF ϑ-FUNCTION

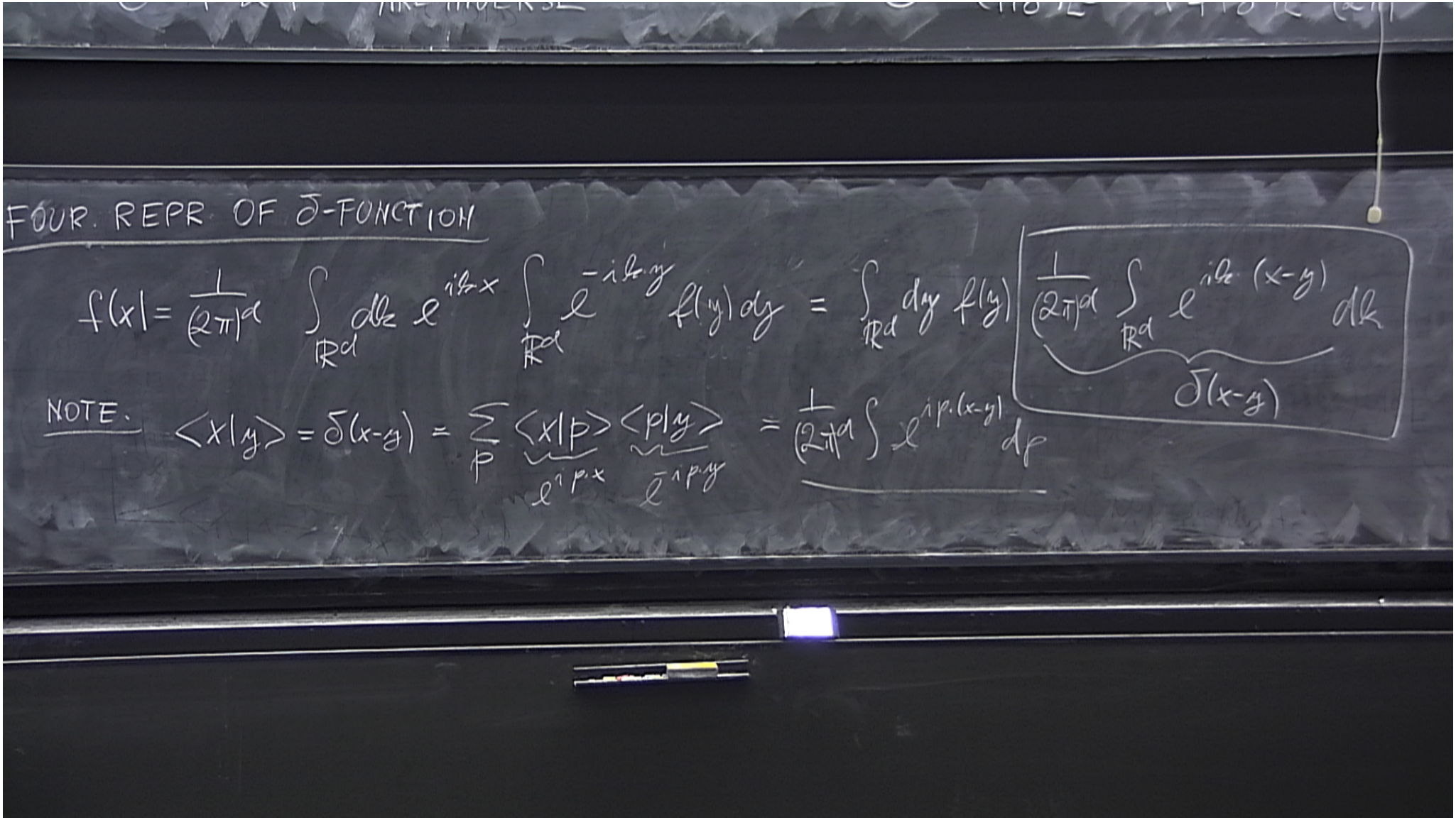

FOUR. REPR OF δ-FUNCTION
$f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} dk\, e^{ik\cdot x}$

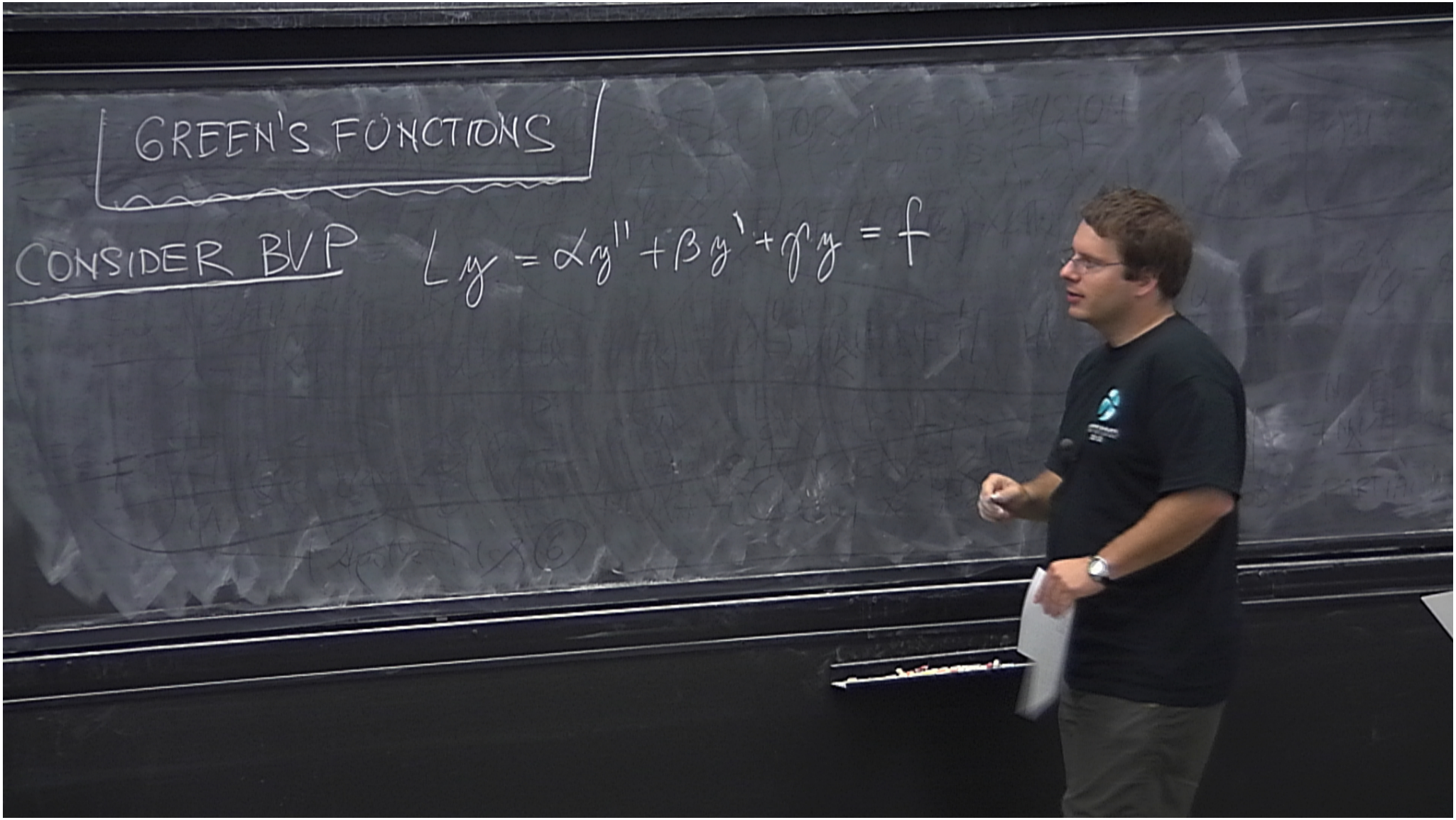

FOUR. REPR OF δ-FUNCTION
$f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} dk\, e^{ik\cdot x} \int_{\mathbb{R}^d} e^{-ik\cdot y} f(y)\, dy$

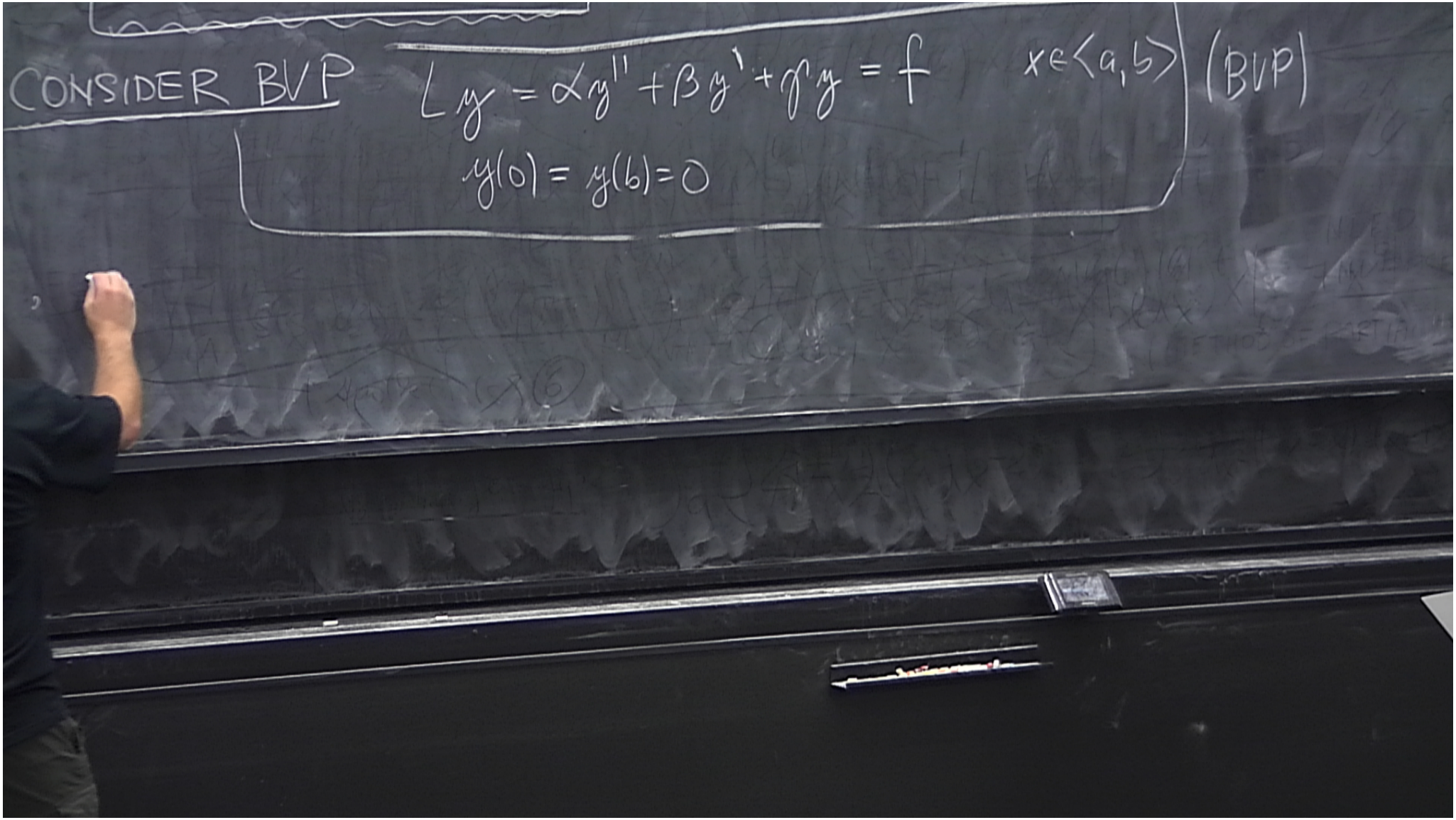

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$f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} dk\, e^{ik\cdot x} \int_{\mathbb{R}^d} e^{-ik\cdot y} f(y)\, dy = \int_{\mathbb{R}^d} dy\, f(y) \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ik\cdot(x-y)}\, dk$

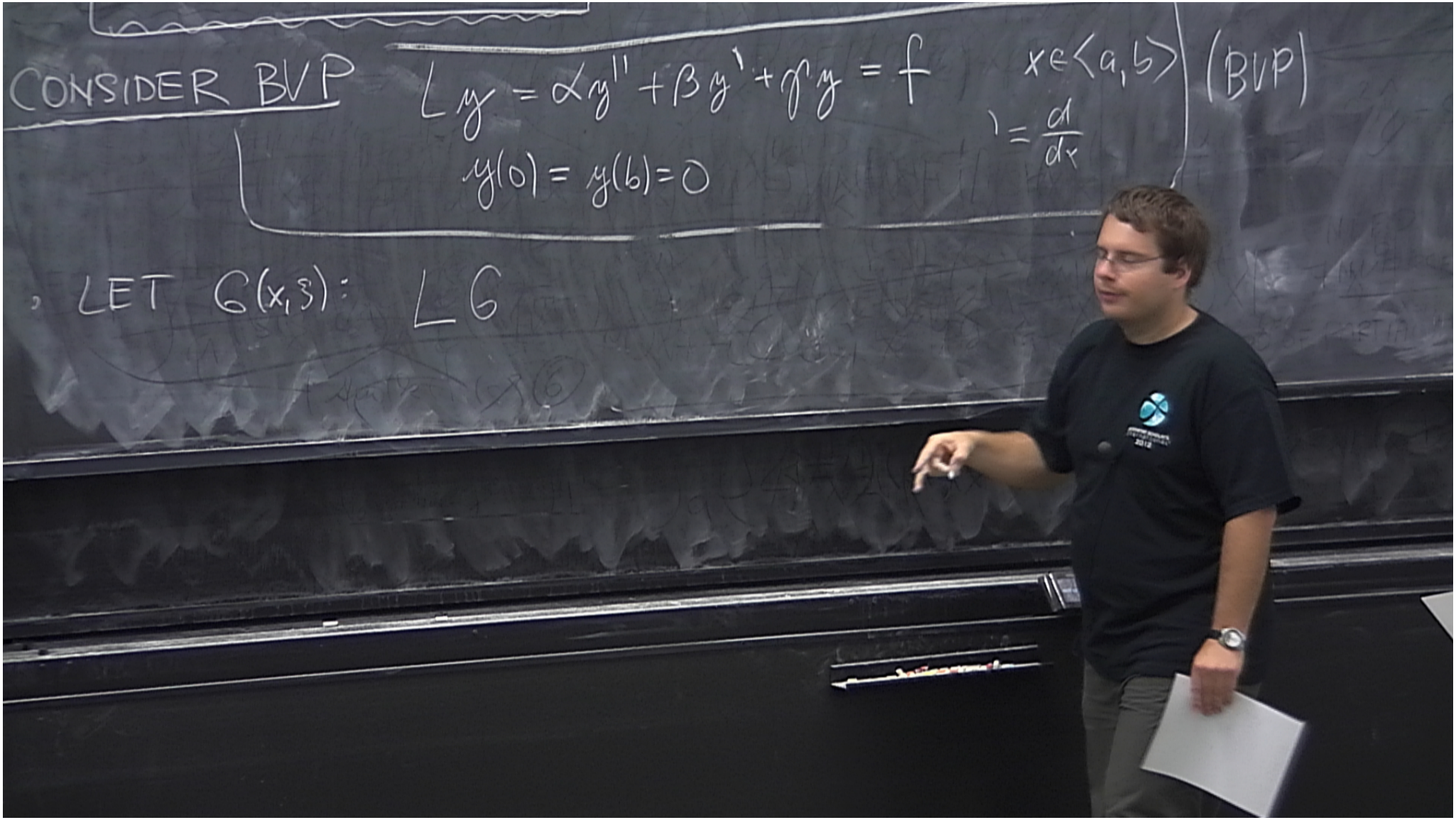

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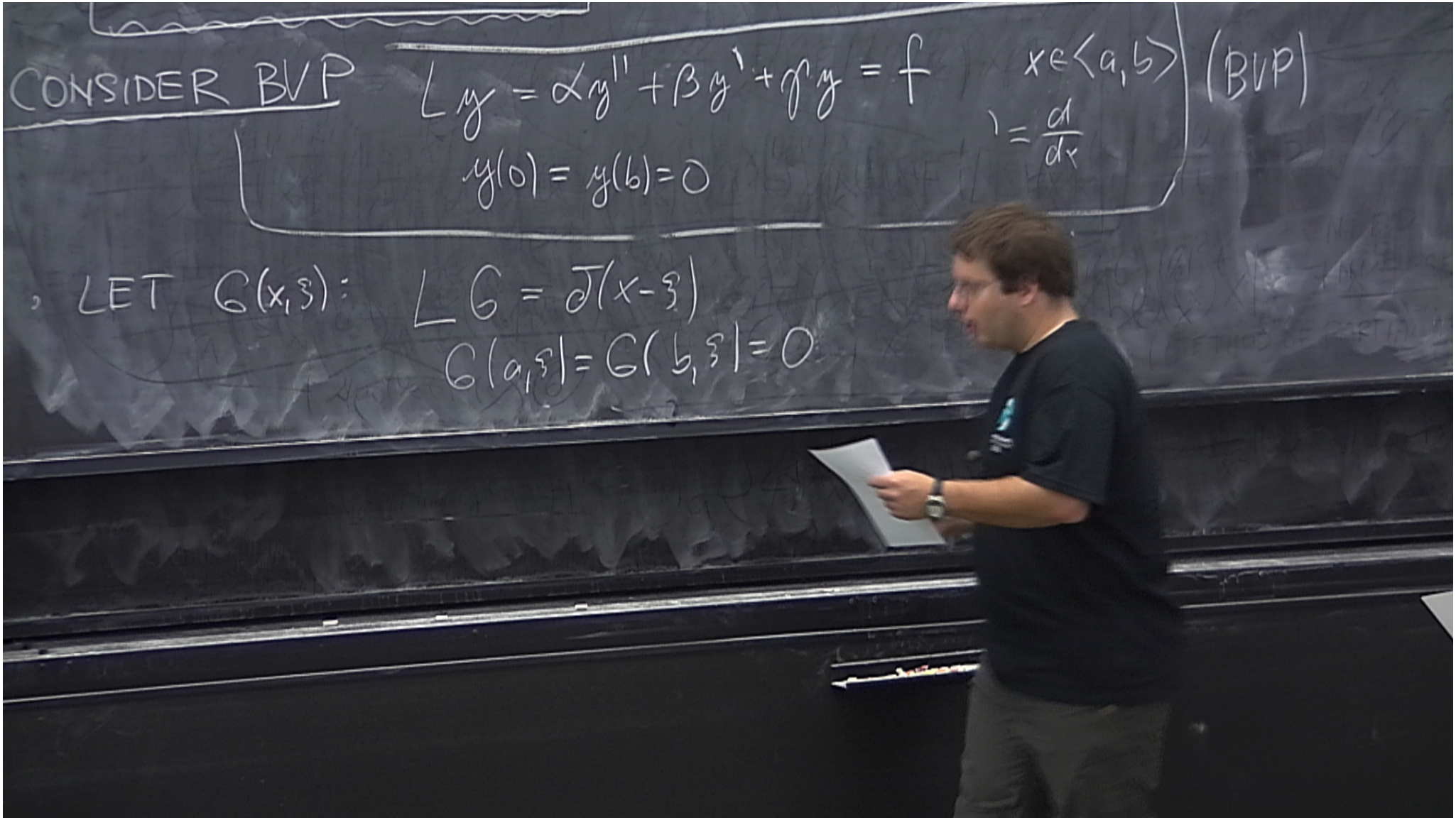

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TE. $\langle x|y\rangle = \delta(x-y) = \sum_p \underbrace{\langle x|p\rangle}_{e^{ip\cdot x}} \underbrace{\langle p|y\rangle}_{\bar{e}^{ip\cdot y}}$

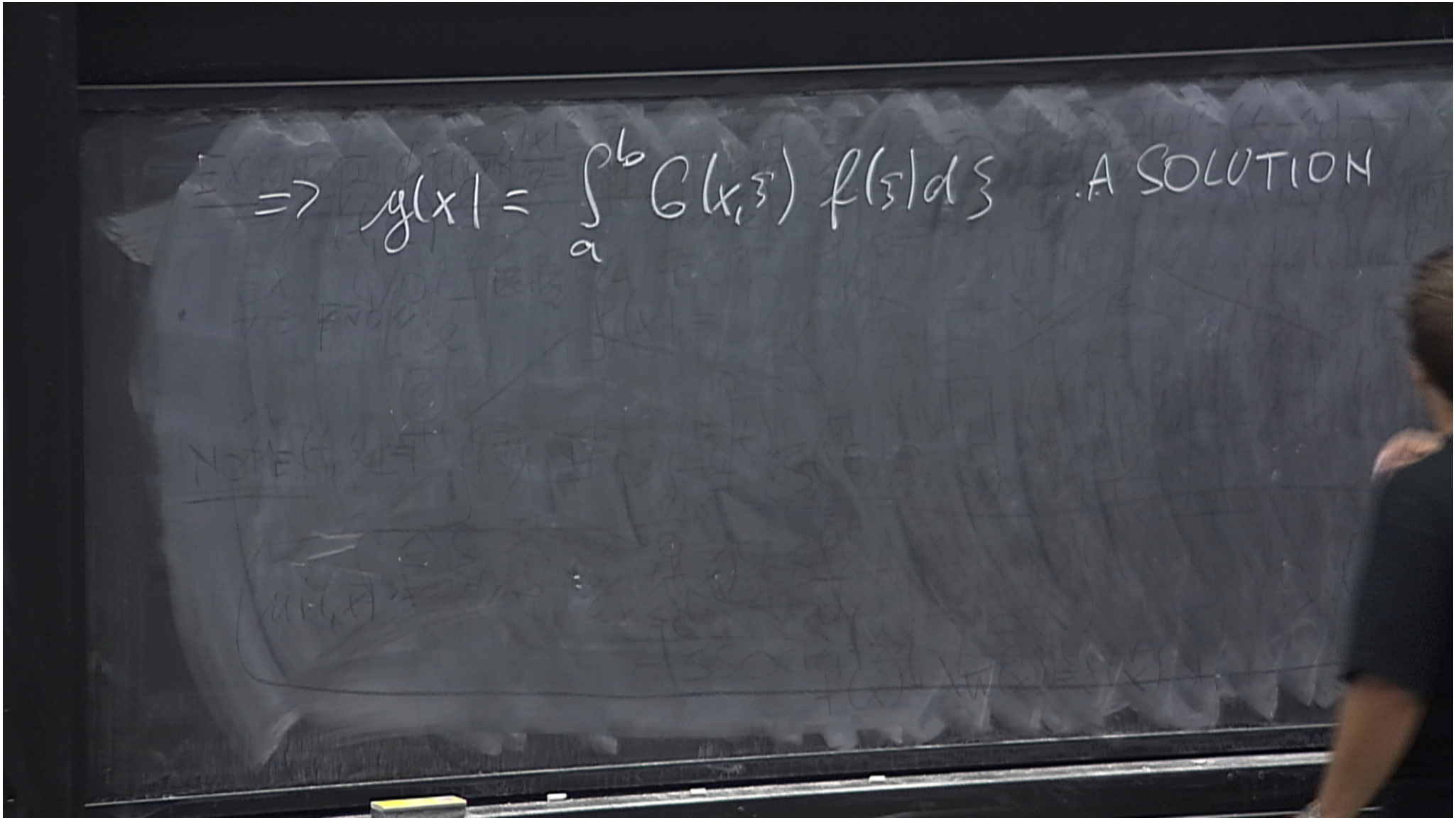

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NOTE. $\langle x|y\rangle = \delta(x-y) = \sum_p \underbrace{\langle x|p\rangle}_{e^{ip\cdot x}} \underbrace{\langle p|y\rangle}_{e^{-ip\cdot y}} = \frac{1}{(2\pi)^d} \int e^{ip\cdot(x-y)}\, dp$

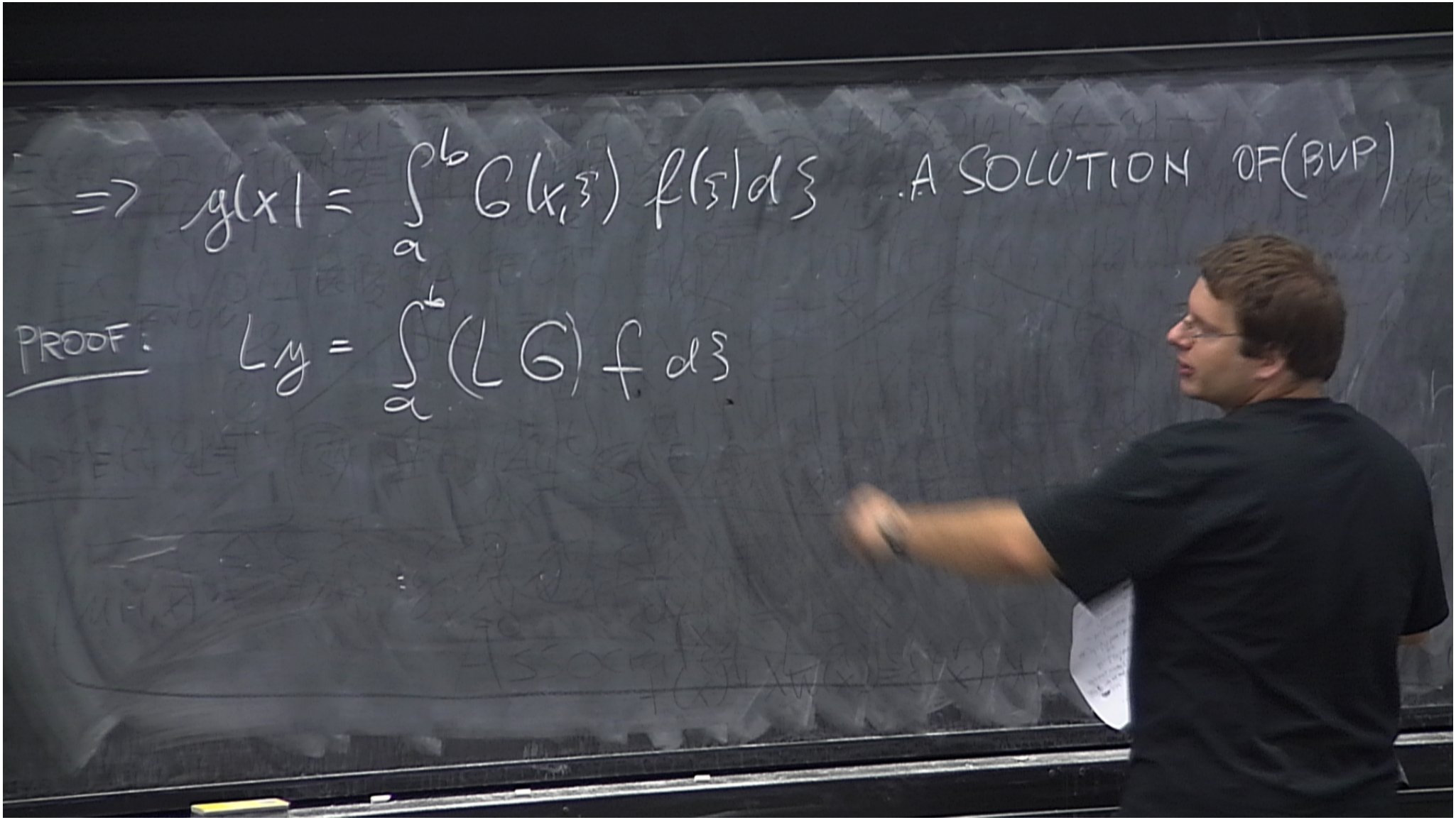

GREEN'S FUNCTIONS
CONSIDER BVP
$Ly = \alpha y'' + \beta y' + \gamma y = f$


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LET $G(x, s)$: LG


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, LET $G(x,s)$:
$LG = \delta(x-s)$
$G(a,s) = G(b,s) = 0$


$\Rightarrow y(x) = \int_a^b G(x,\xi) f(\xi) d\xi$ .A SOLUTION


=> $y(x) = \int_a^b G(x,\xi) f(\xi) d\xi$. A SOLUTION OF (BVP)
PROOF: $Ly = \int_a^b (LG) f \, d\xi$

$\Rightarrow y(x) = \int_a^b G(x,\xi)\, f(\xi)\, d\xi$. A SOLUTION OF (BVP)

PROOF: • $Ly = \int_a^b (LG)\, f\, d\xi = f(x)$

• $y(a) = \int G(a,\xi)\, f\, d\xi$

GREEN'S FUNCTIONS

CONSIDER BVP $Ly = \alpha y'' + \beta y' + \gamma y = f \quad x \in \langle a,b \rangle$ (BVP)

$y(a) = y(b) = 0$

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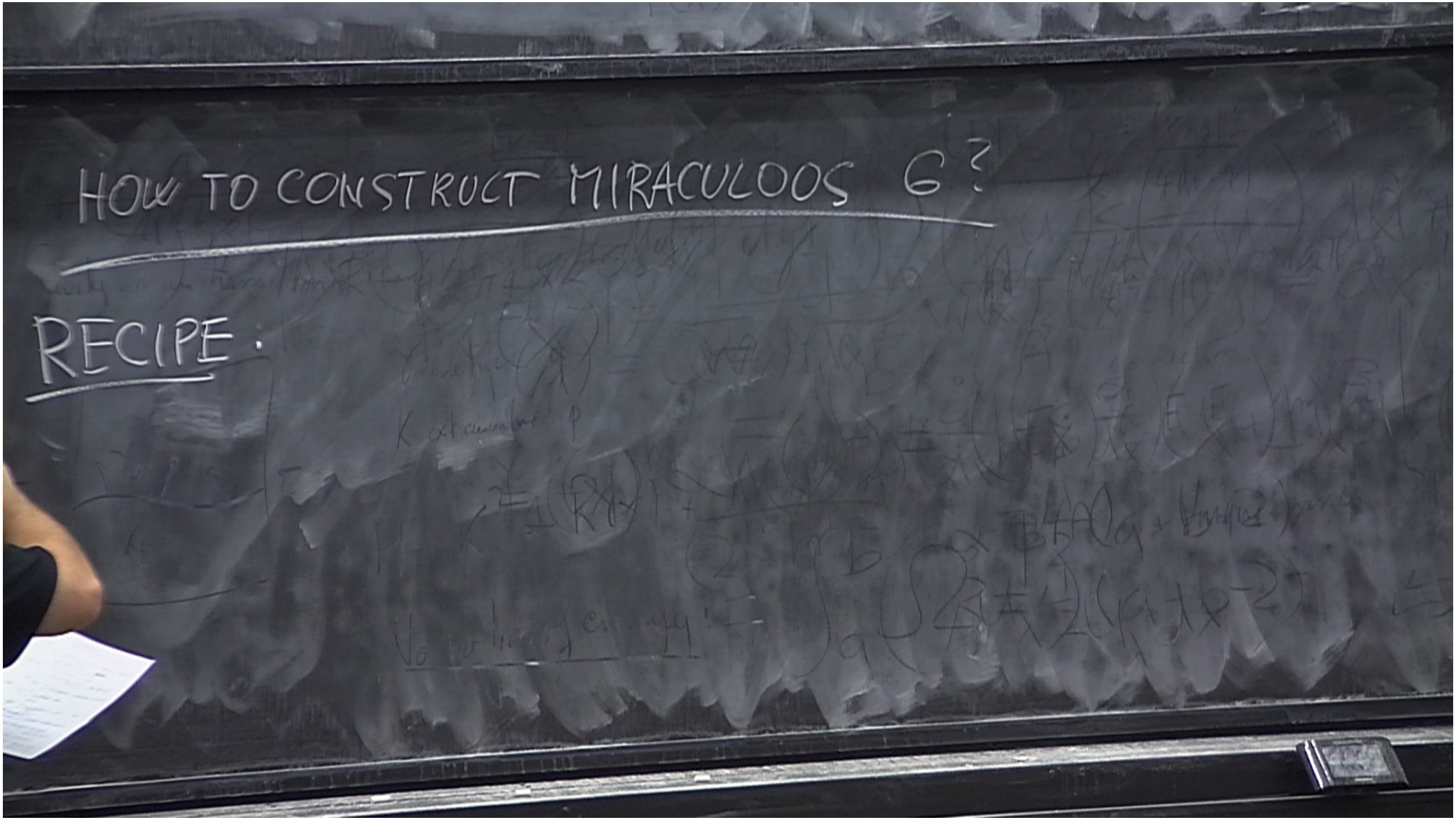
LET $G(x,\xi)$: $LG = \delta(x-\xi)$, $G(a,\xi) = G(b,\xi) = 0$ (G) ... Green's Function

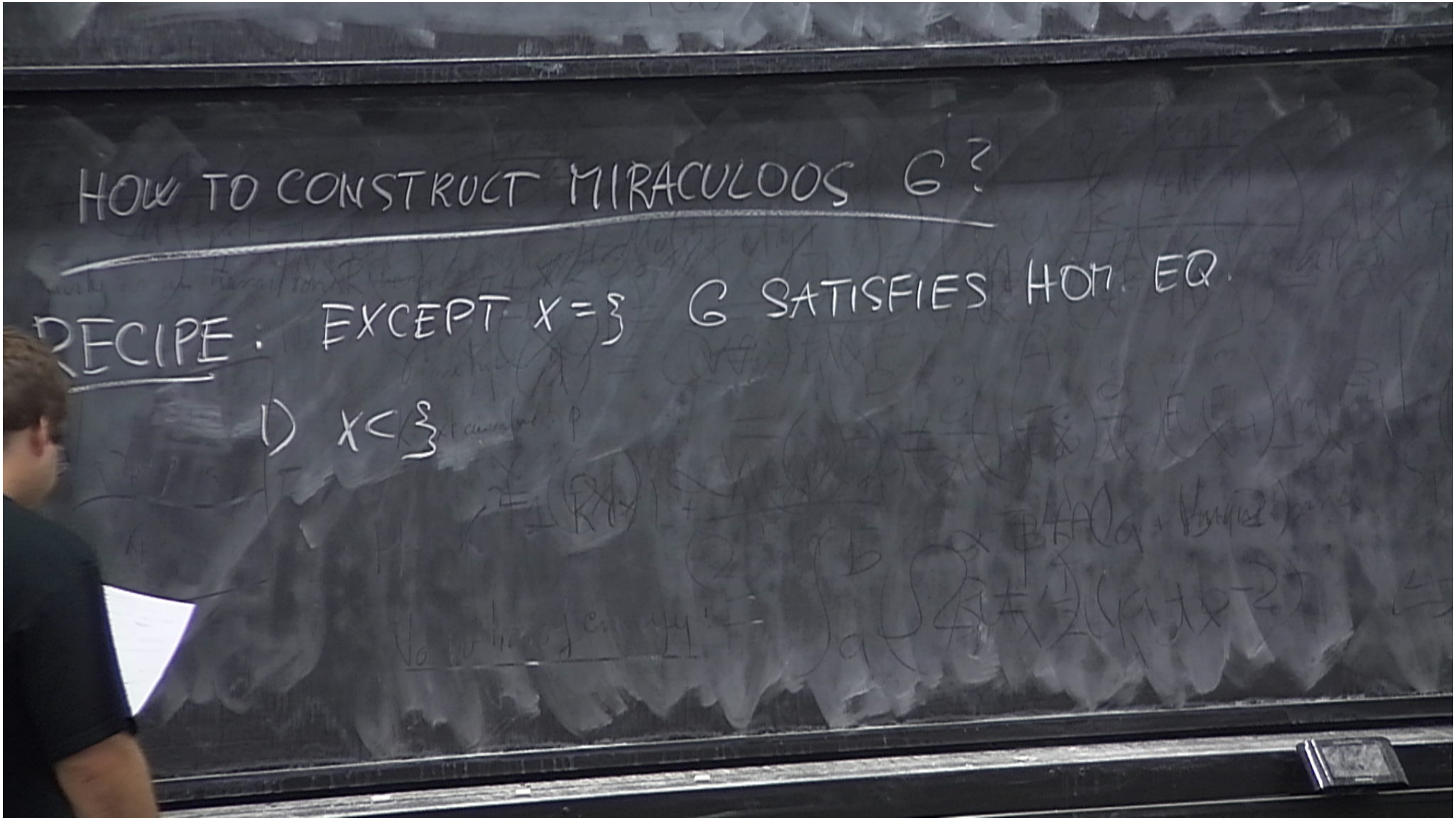
$\Rightarrow y(x) = \int_a^b G(x,\xi) f(\xi) d\xi$ A SOLUTION OF (BVP)

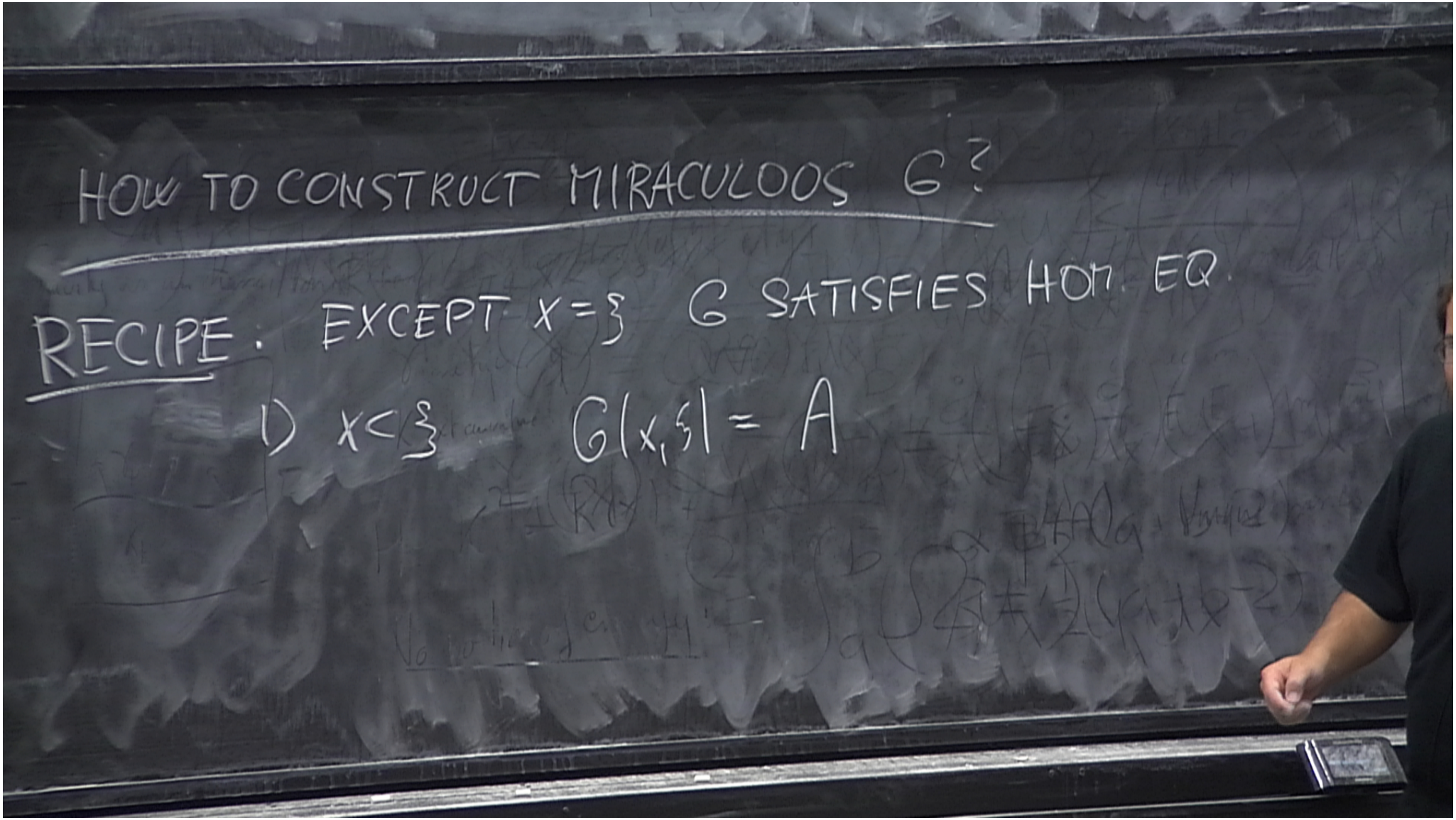
PROOF: $Ly = \int_a^b (LG) f \, d\xi = f(x)$

$y(a) = \int G(a,\xi) f \, d\xi = 0 = y(b)$




HOW TO CONSTRUCT MIRACULOOS G?
RECIPE:


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RECIPE: EXCEPT X=3 G SATISFIES HOM. EQ.
1) X<3

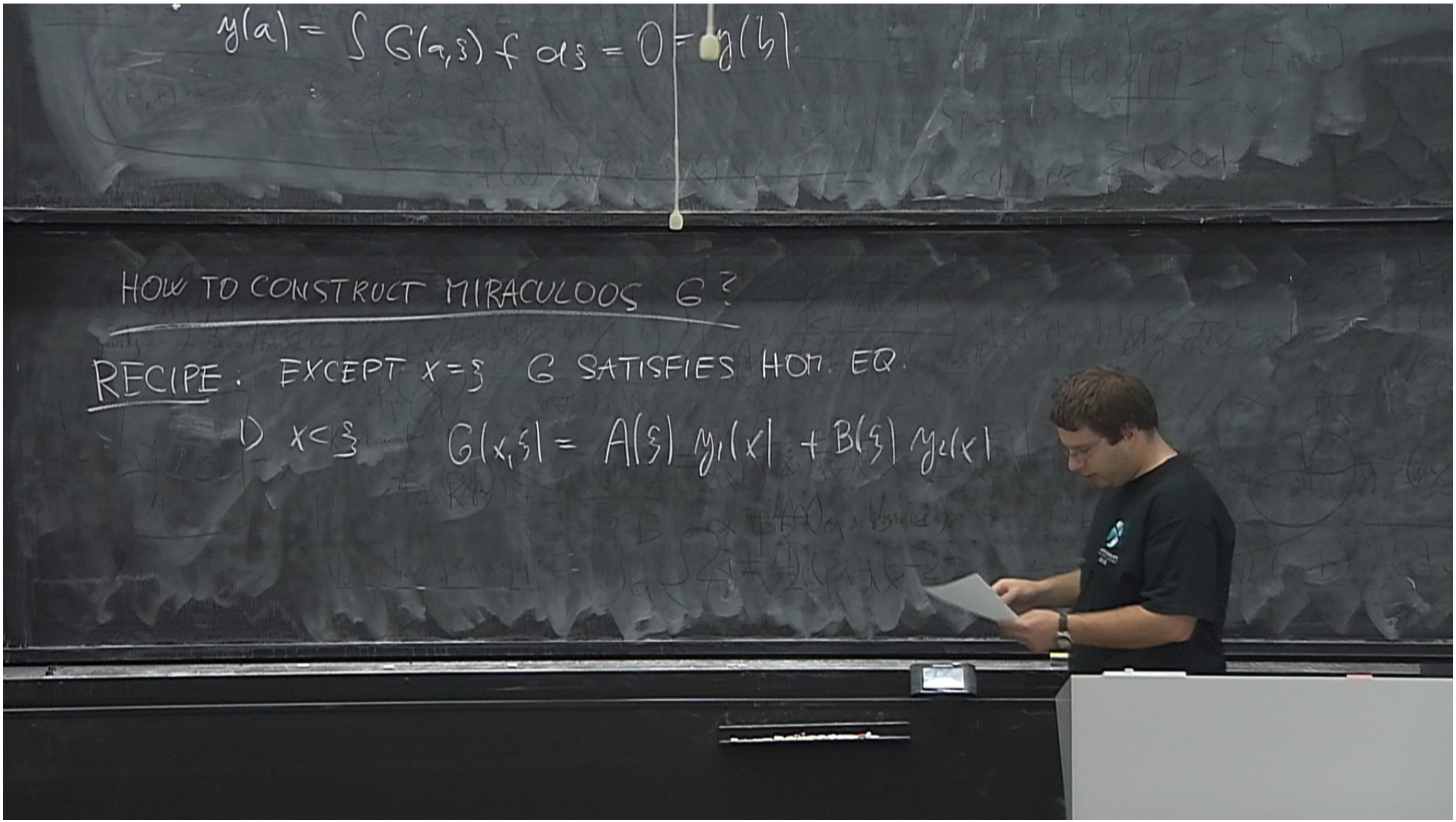

HOW TO CONSTRUCT MIRACULOOS G?
RECIPE. EXCEPT X=S G SATISFIES HOM. EQ.
1) X<S G(X,S| = A

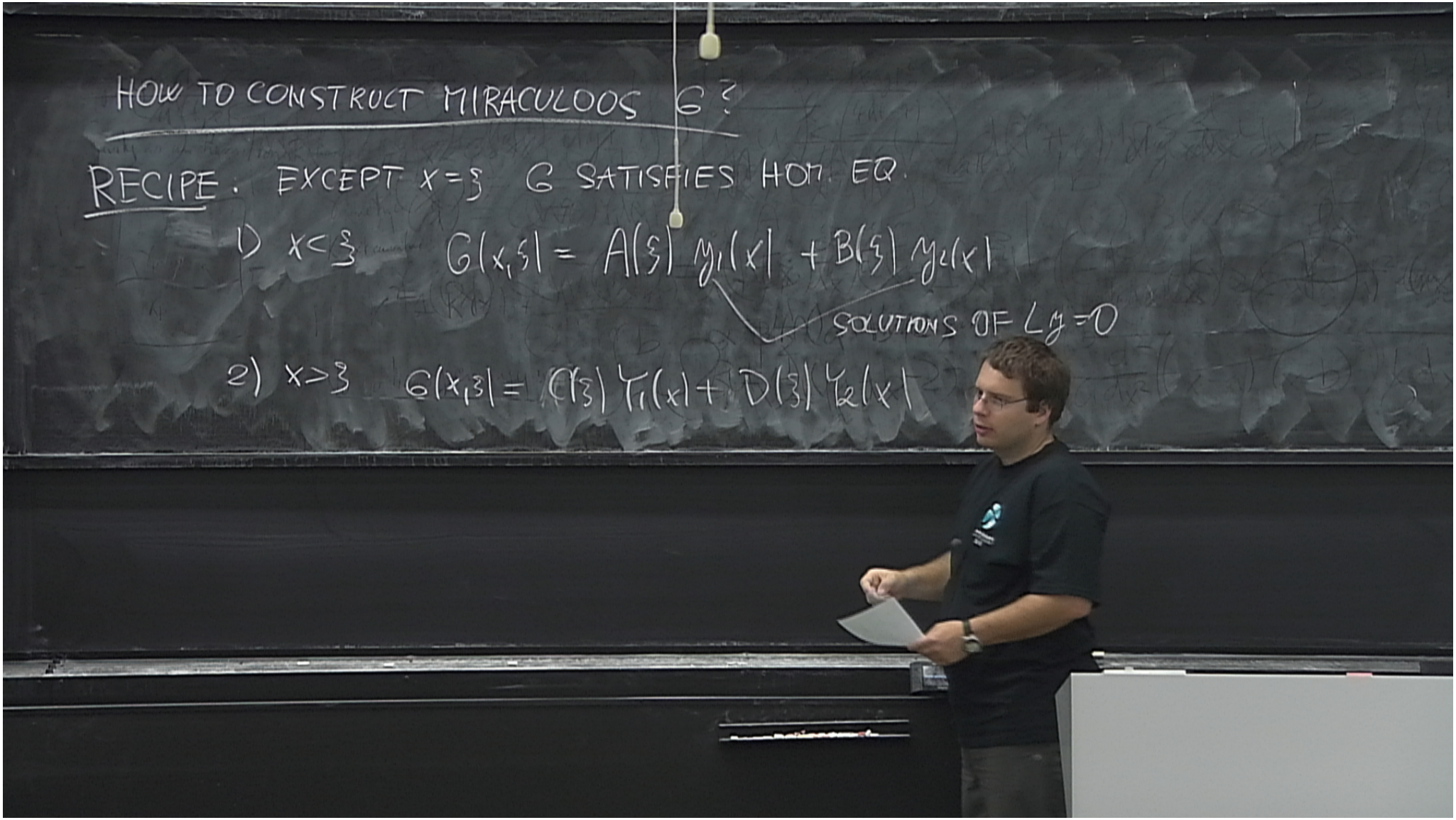
$y(a) = \int G(a,s) f \, ds = 0 = y(b)$

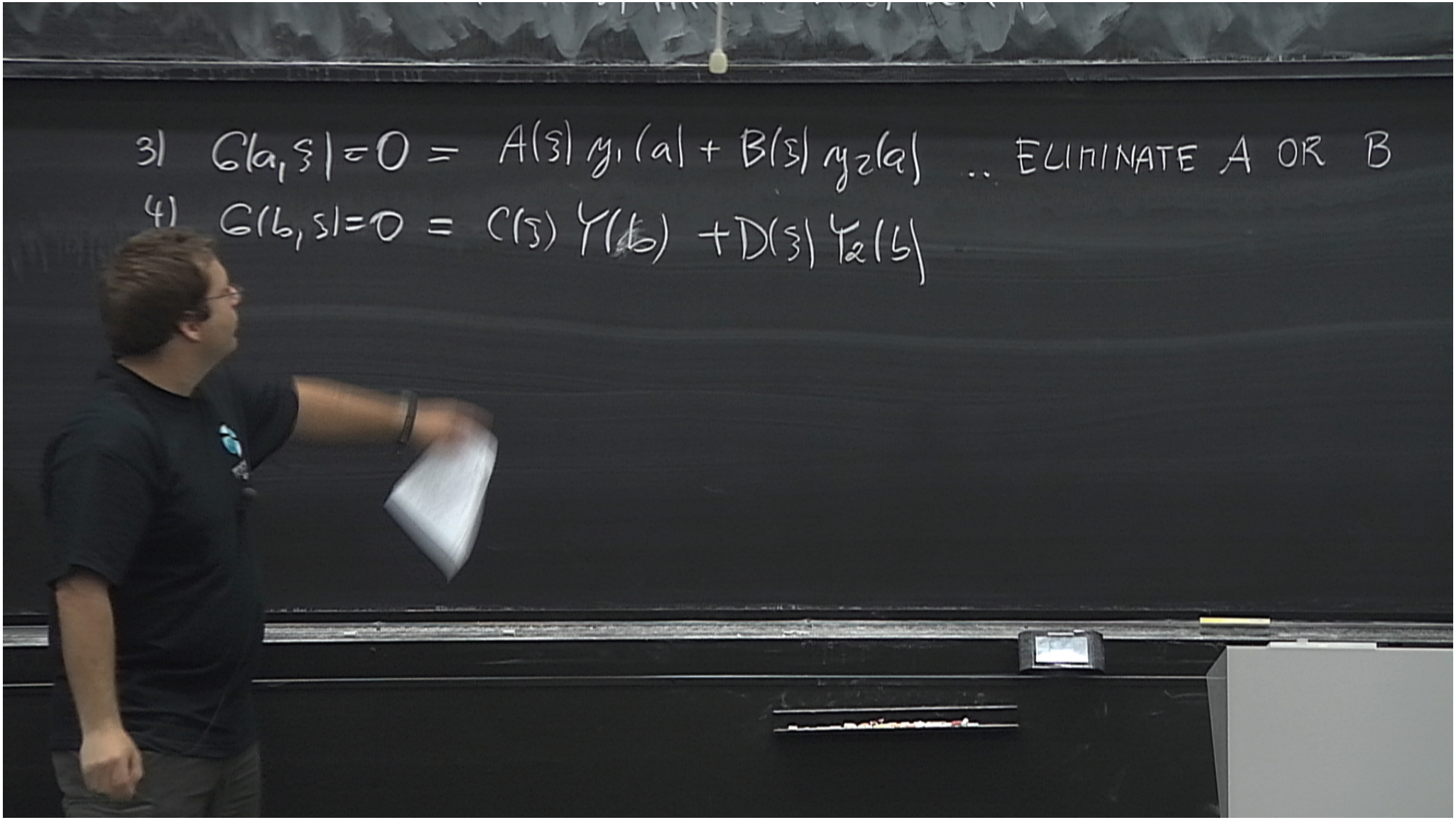
HOW TO CONSTRUCT MIRACULOOS G?

RECIPE: EXCEPT $x=s$ G SATISFIES HOM. EQ.

1) $x<s$ $\quad G(x,s) = A(s)\, y_1(x) + B(s)\, y_2(x)$




HOW TO CONSTRUCT MIRACULOOS G?
RECIPE. EXCEPT x=ξ G SATISFIES HOM. EQ.
1) x<ξ G(x,ξ) = A(ξ) y1(x) + B(ξ) y2(x)
SOLUTIONS OF Ly=0
2) x>ξ G(x,ξ) = C(ξ) Y1(x) + D(ξ) Y2(x)

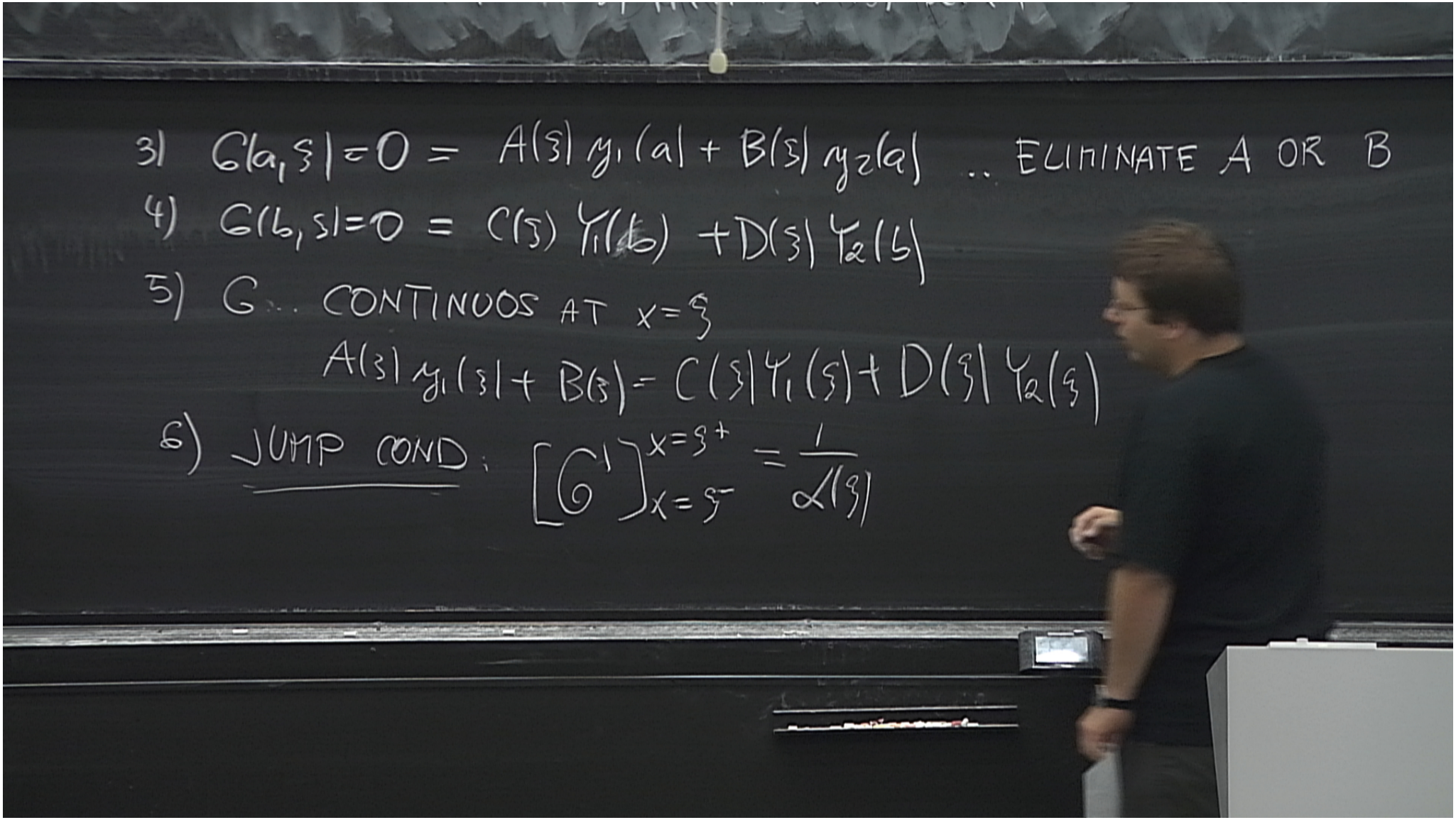

3) G(a,s)=0 = A(s) y1(a) + B(s) y2(a) .. ELIMINATE A OR B
4) G(b,s)=0 = C(s) Y(b) + D(s) Y2(b)

3) $G(a,\xi)=0 = A(\xi)\,y_1(a) + B(\xi)\,y_2(a)$ .. ELIMINATE A OR B

4) $G(b,\xi)=0 = C(\xi)\,Y_1(b) + D(\xi)\,Y_2(b)$

5) G .. CONTINUOS AT $x=\xi$

$$A(\xi)\,y_1(\xi) + B(\xi) = C(\xi)\,Y_1(\xi) + D(\xi)\,Y_2(\xi)$$


3) $G(a,\xi|=0 = A(\xi)\,y_1(a| + B(\xi)\,y_2(a|$ .. ELIMINATE A OR B
4) $G(b,\xi|=0 = C(\xi)\,Y_1(b) + D(\xi|\,Y_2(b|$
5) G .. CONTINUOS AT $x=\xi$
$A(\xi|\,y_1(\xi| + B(\xi) = C(\xi|\,Y_1(\xi) + D(\xi|\,Y_2(\xi)$
6) JUMP COND: $[G']_{x=\xi^-}^{x=\xi^+} = \frac{1}{\alpha(\xi|}$

2) $x>\xi$ $\quad G(x,\xi) = C(\xi)\,y_1(x) + D(\xi)\,y_2(x)$

3) $G(a,\xi)=0 = A(\xi)\,y_1(a) + B(\xi)\,y_2(a)$ .. ELIMINATE A OR B

4) $G(b,\xi)=0 = C(\xi)\,y_1(b) + D(\xi)\,y_2(b)$

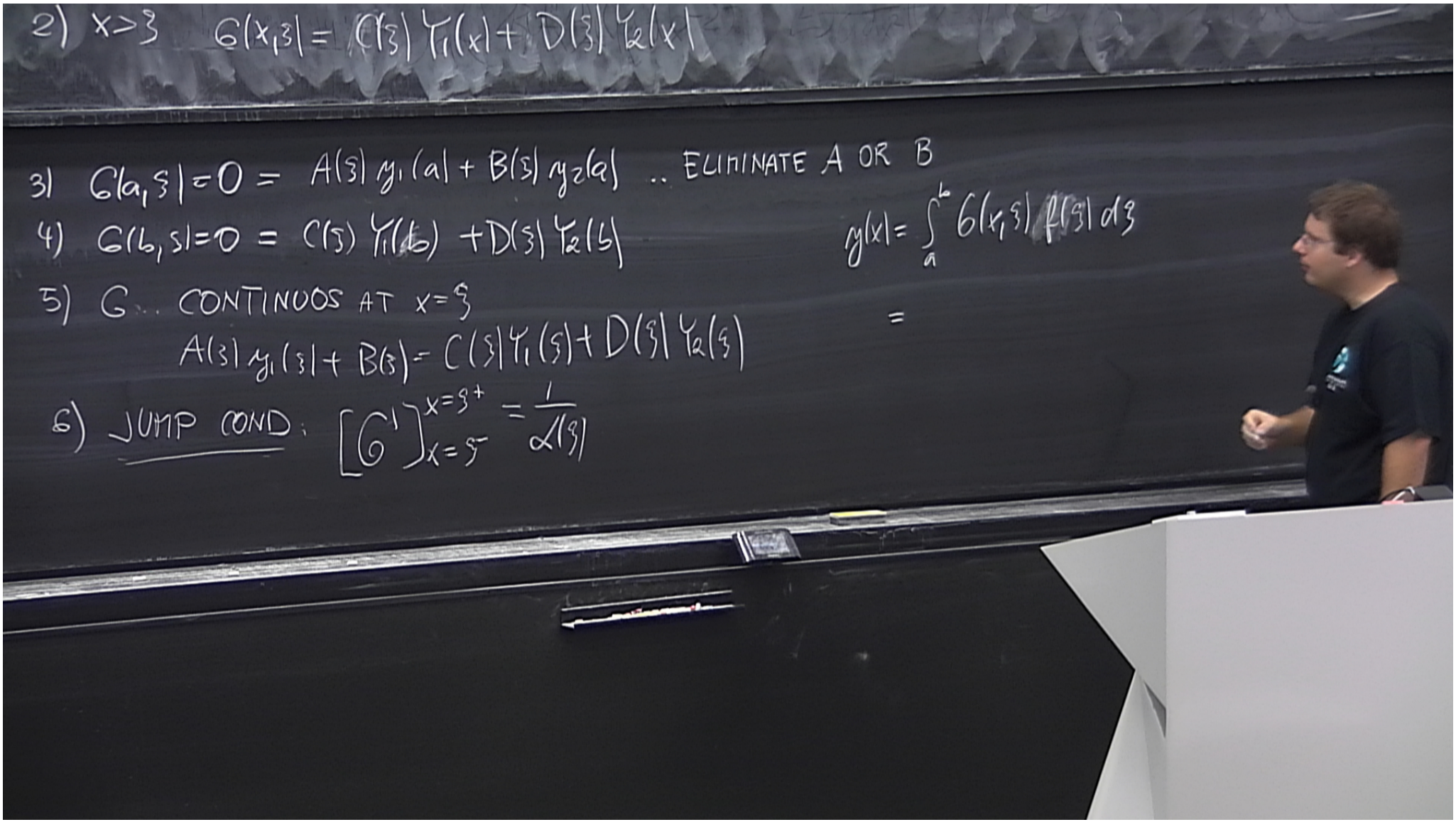
5) G .. CONTINUOS AT $x=\xi$

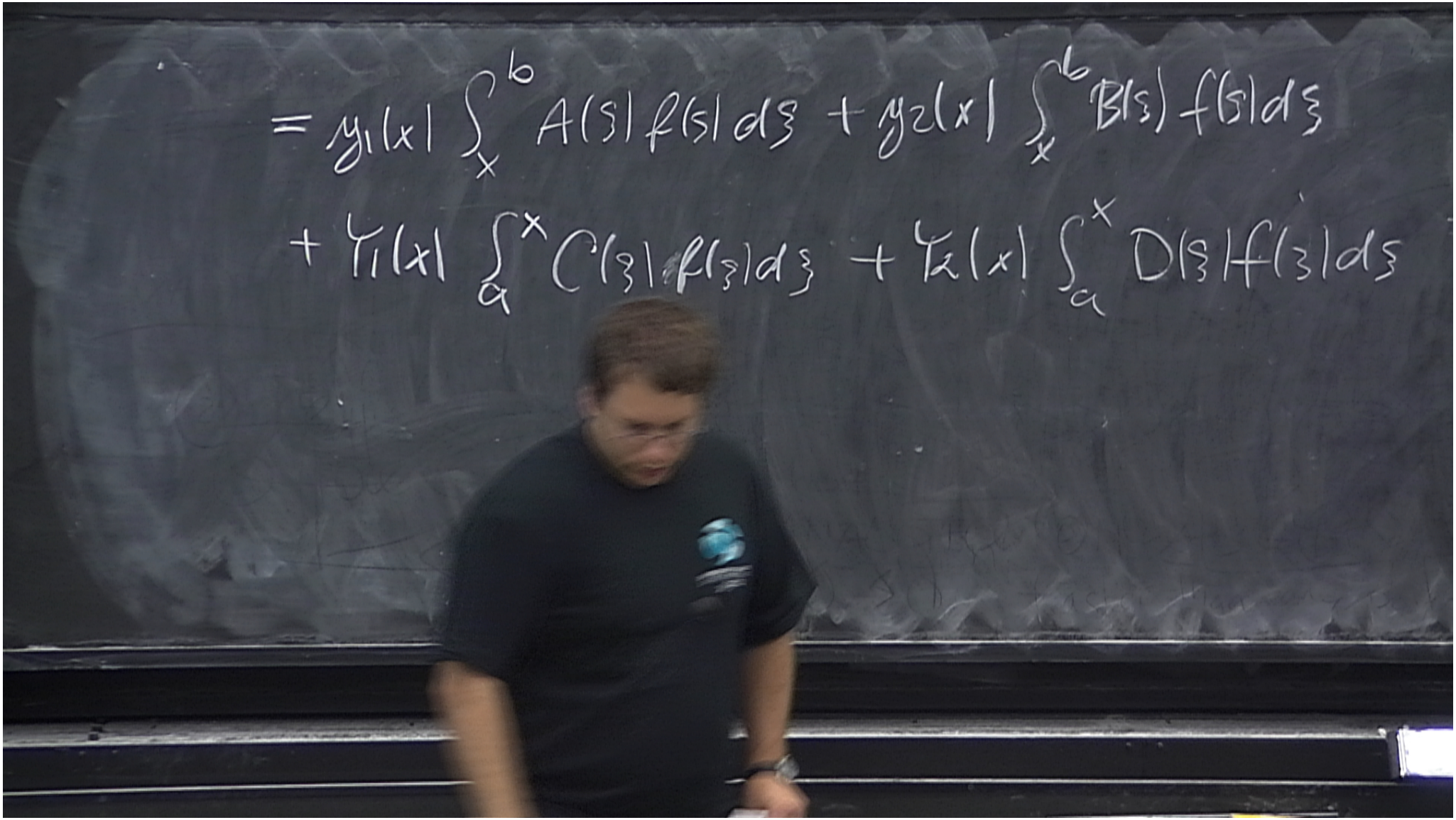
$$A(\xi)\,y_1(\xi) + B(\xi) = C(\xi)\,y_1(\xi) + D(\xi)\,y_2(\xi)$$

6) JUMP COND: $\left[G'\right]_{x=\xi^-}^{x=\xi^+} = \frac{1}{\alpha(\xi)}$

$$y(x) = \int_a^b G(x,\xi)\,f(\xi)\,d\xi$$

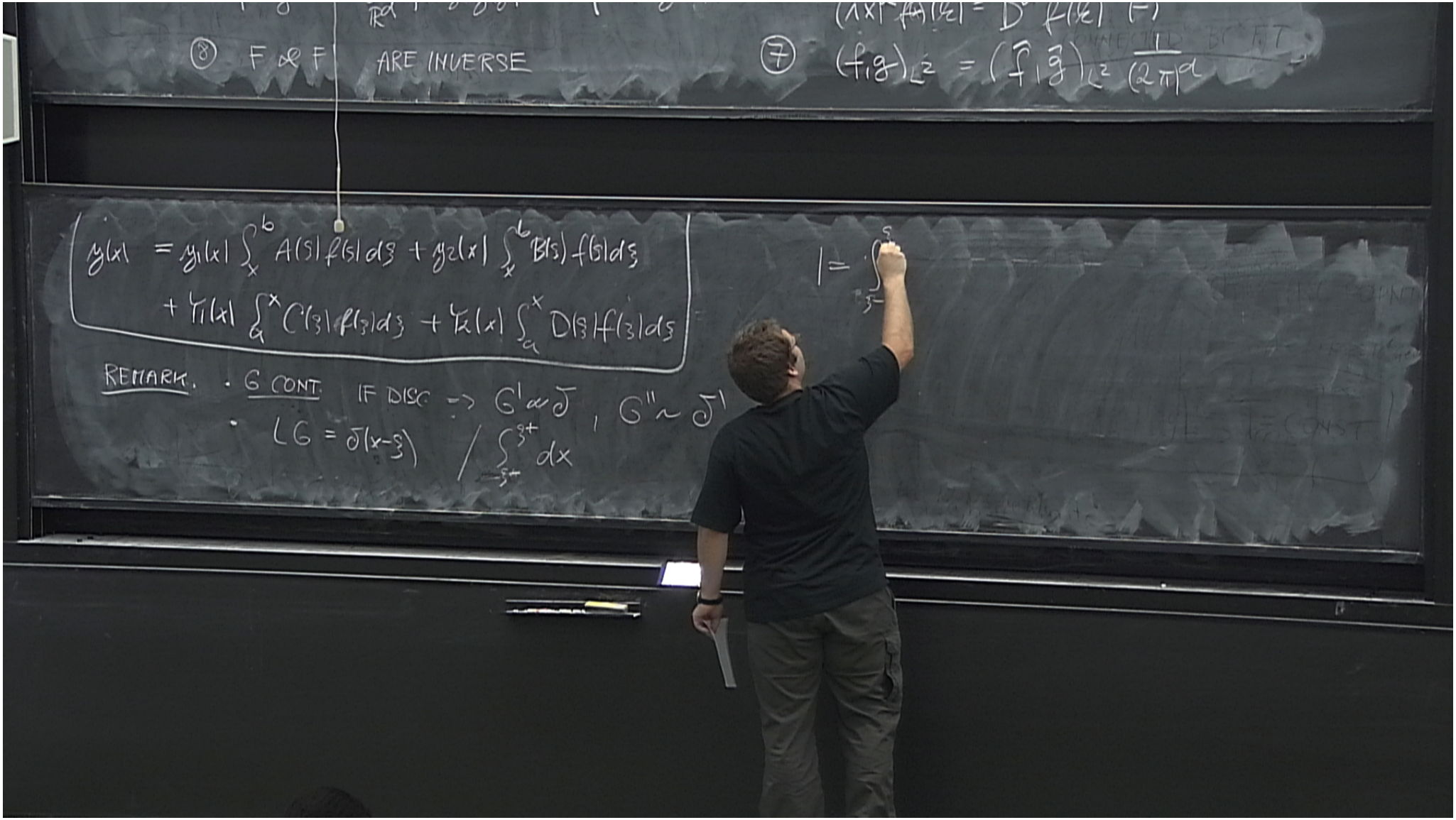
$=$

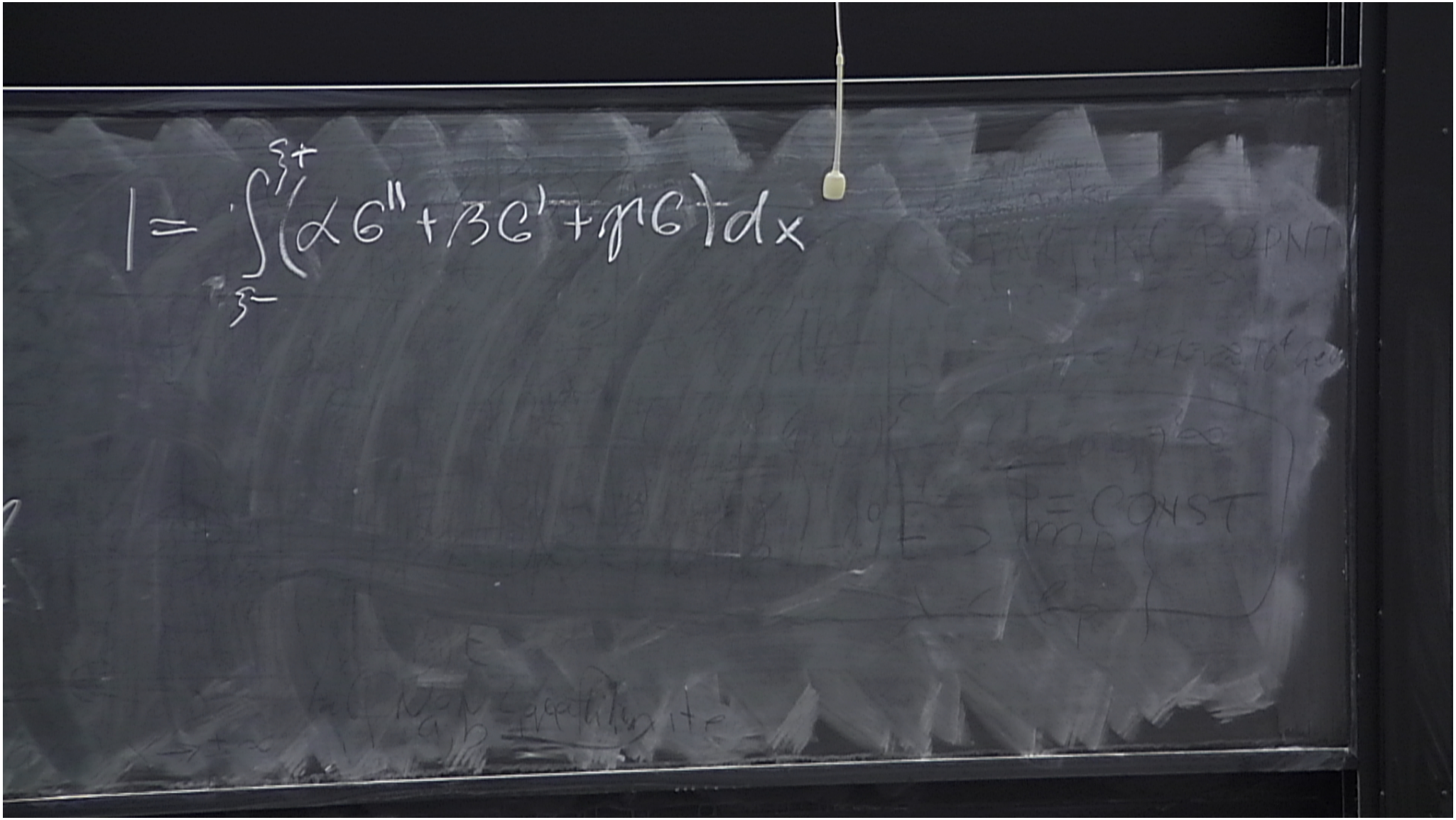


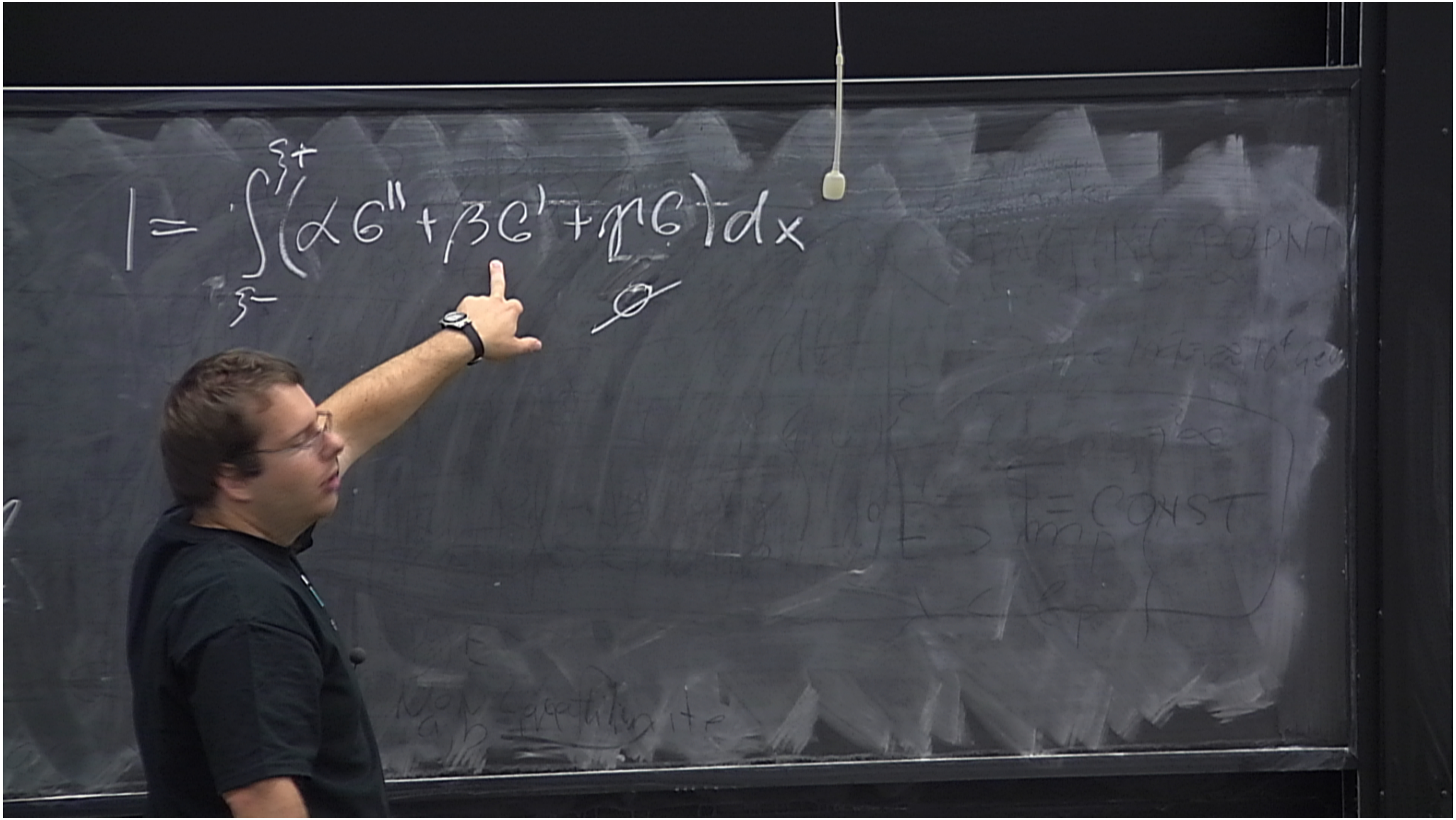

$= y_1(x) \int_x^b A(\xi) f(\xi)\, d\xi + y_2(x) \int_x^b B(\xi) f(\xi)\, d\xi$
$+ Y_1(x) \int_a^x C(\xi) f(\xi)\, d\xi + Y_2(x) \int_a^x D(\xi) f(\xi)\, d\xi$

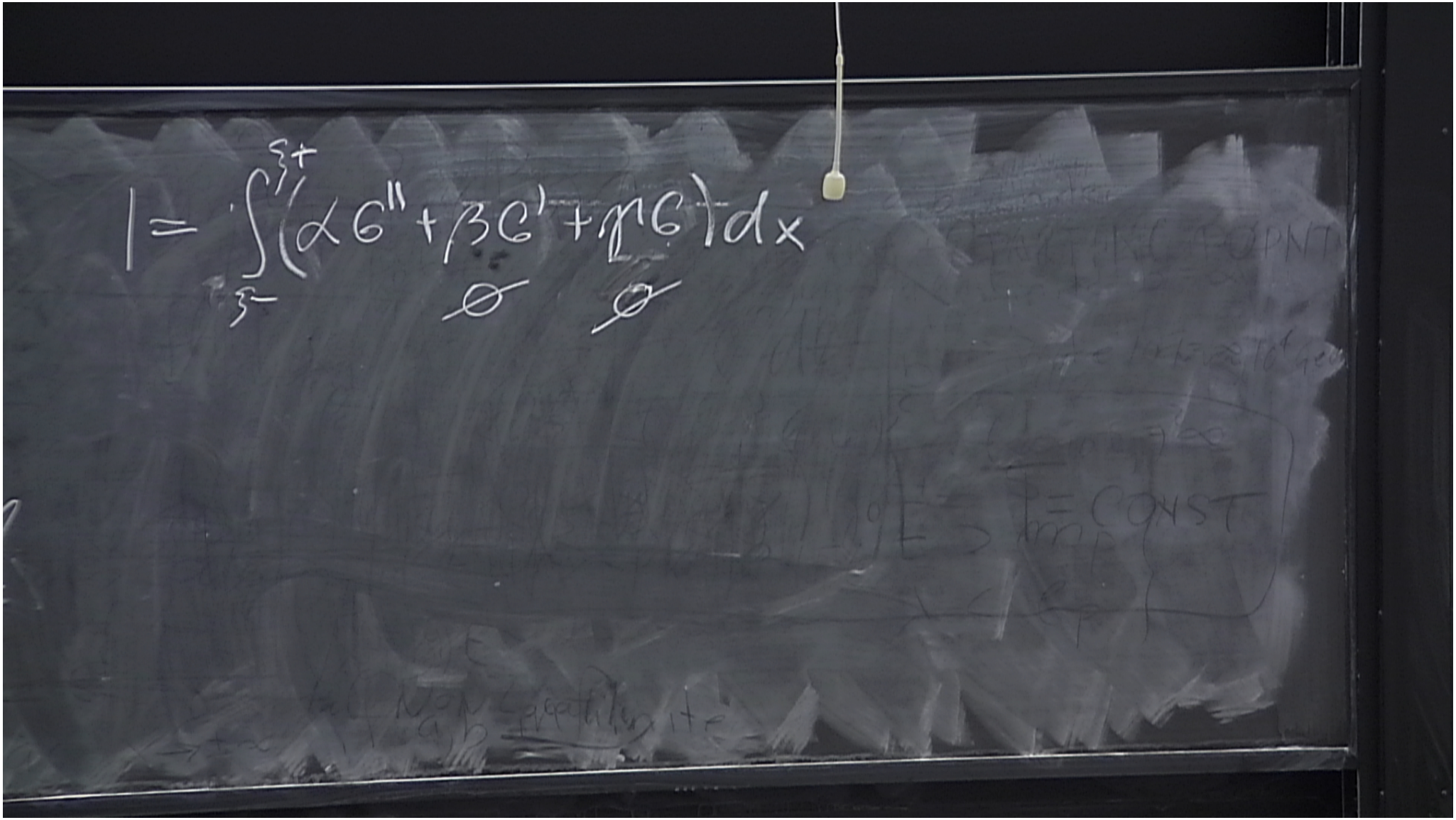
$$y(x) = y_1(x)\int_x^b A(\xi) f(\xi)\,d\xi + y_2(x)\int_x^b B(\xi) f(\xi)\,d\xi$$
$$+ y_1(x)\int_a^x C(\xi) f(\xi)\,d\xi + y_2(x)\int_a^x D(\xi) f(\xi)\,d\xi$$

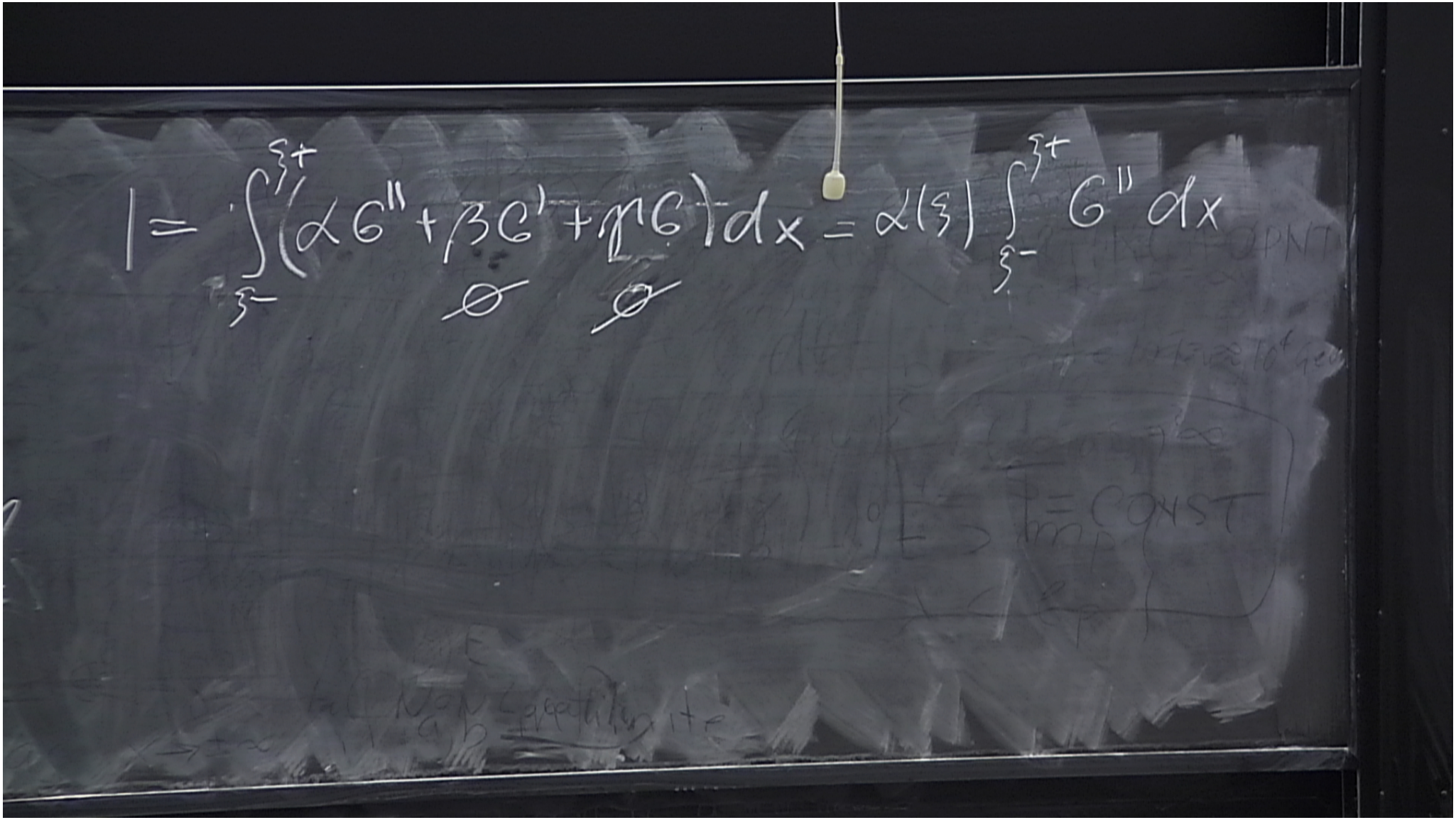
REMARK. · G CONT. IF DISC. $\Rightarrow G' \sim J$, $G'' \sim J'$

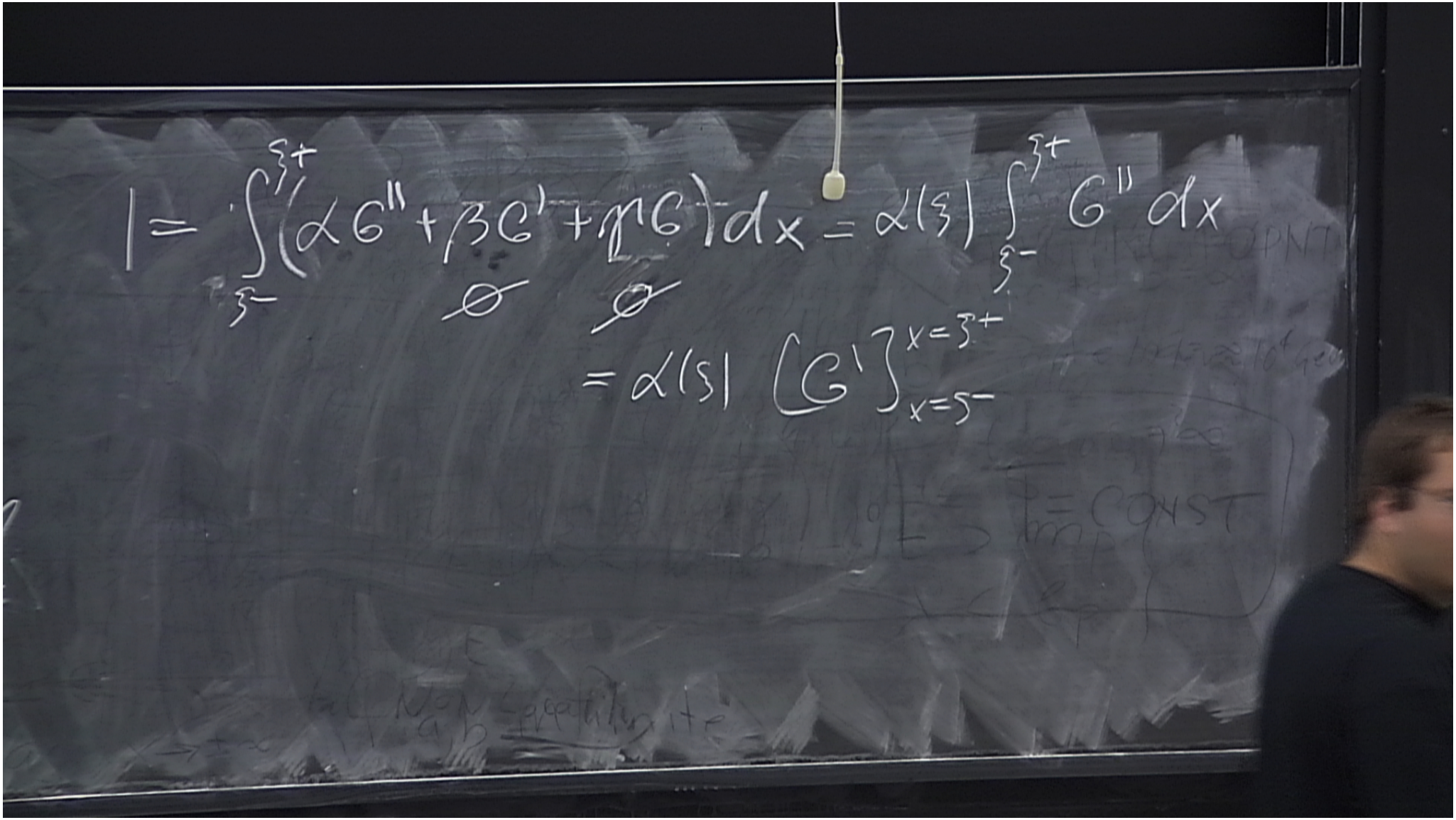

⑧ F & F ARE INVERSE
⑦ $(f,g)_{L^2} = (\hat{f},\hat{g})_{L^2} \frac{1}{(2\pi)^d}$
$y(x) = y_1(x)\int_x^b A(\xi) f(\xi)\,d\xi + y_2(x)\int_x^b B(\xi) f(\xi)\,d\xi$
$+ Y_1(x)\int_a^x C(\xi) f(\xi)\,d\xi + Y_2(x)\int_a^x D(\xi) f(\xi)\,d\xi$
REMARK. · G CONT. IF DISC → $G' \sim \delta$, $G'' \sim \delta'$
· $LG = \delta(x-\xi)$ / $\int_{\xi-}^{\xi+} dx$

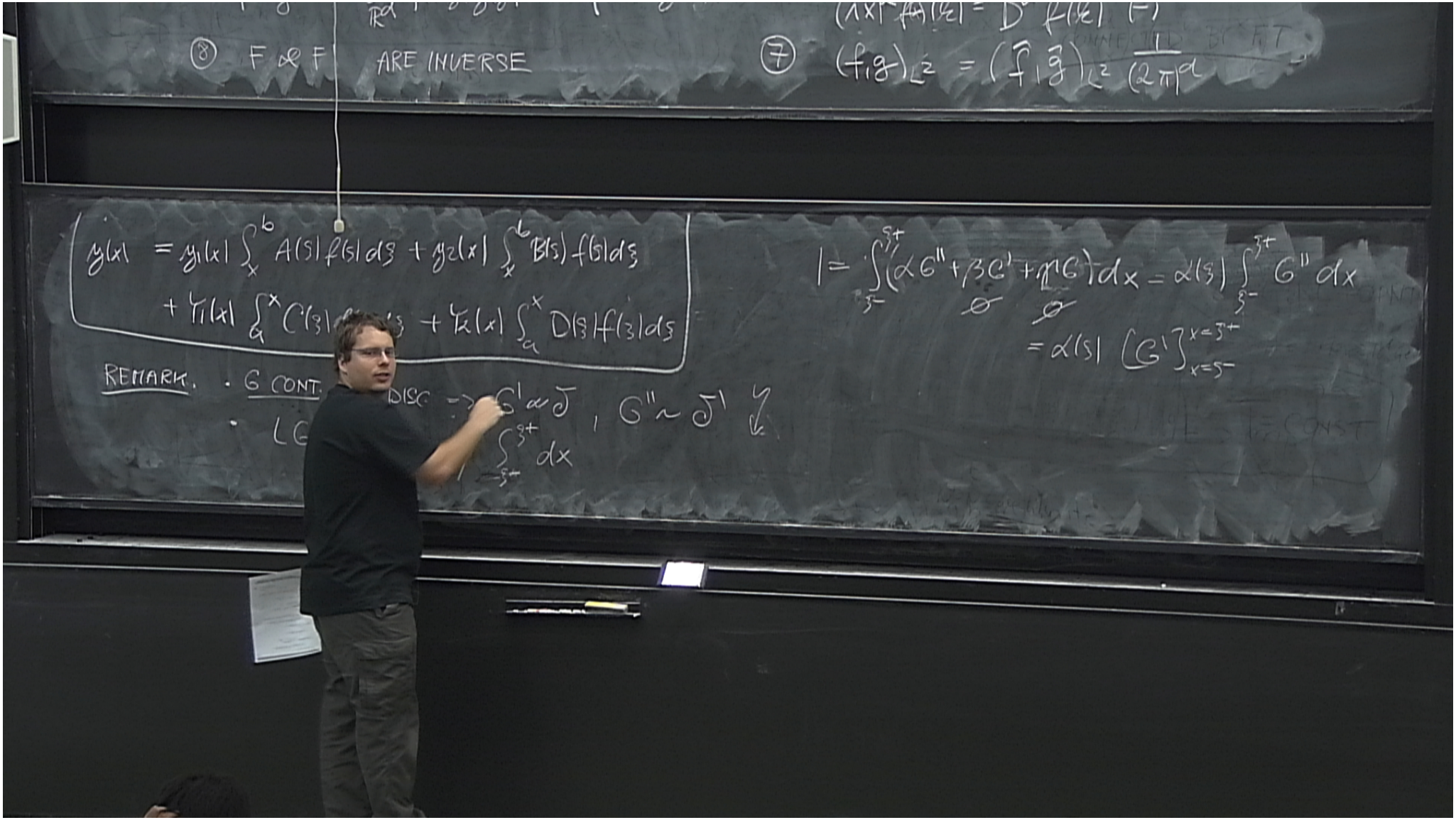

$I = \int_{\xi^-}^{\xi^+} (\alpha G'' + \beta G' + \gamma G)\, dx$

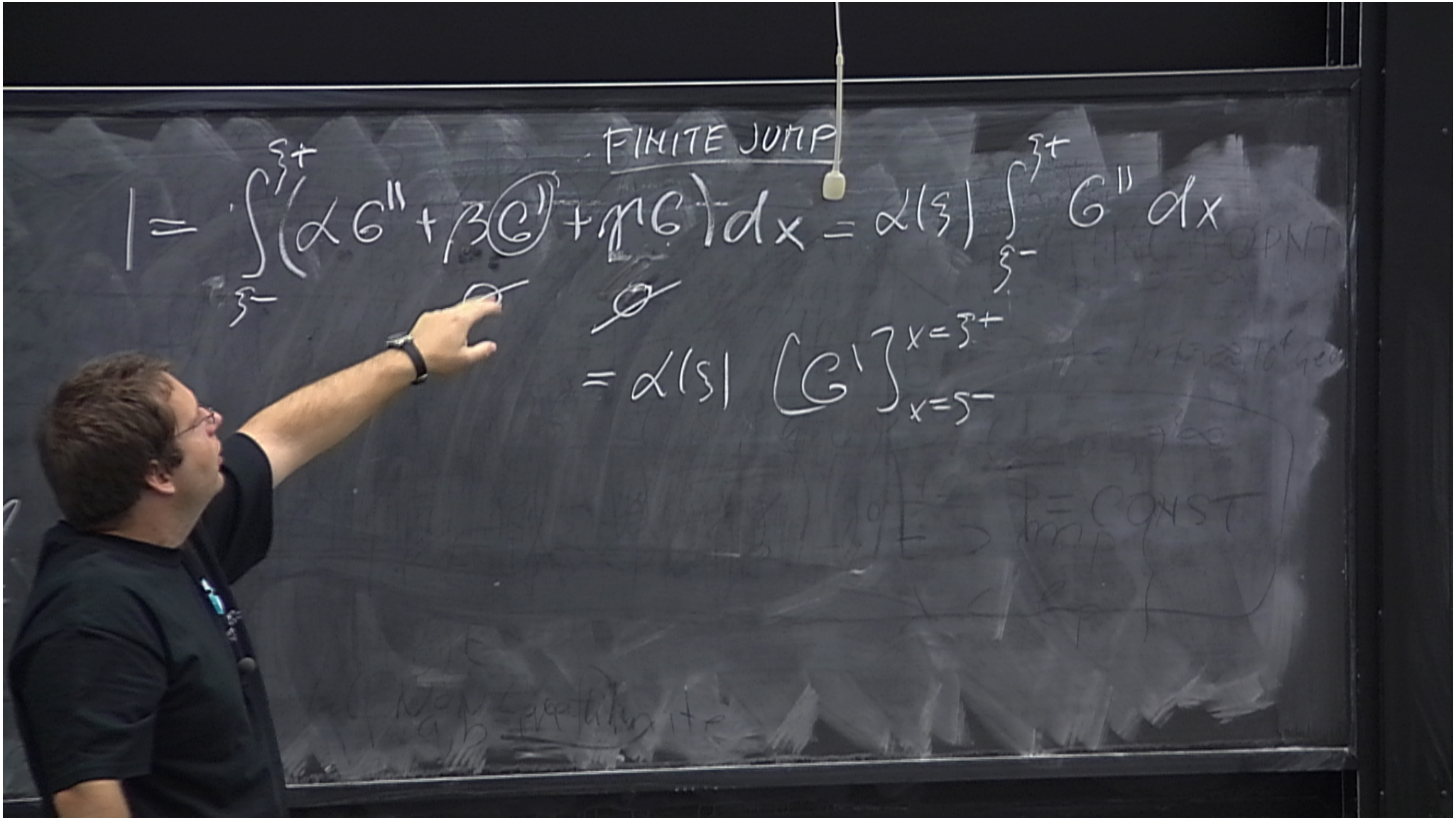

$I = \int_{\xi^-}^{\xi^+} (\alpha G'' + \beta G' + \gamma G)\, dx$
$= 0$
T = CONST

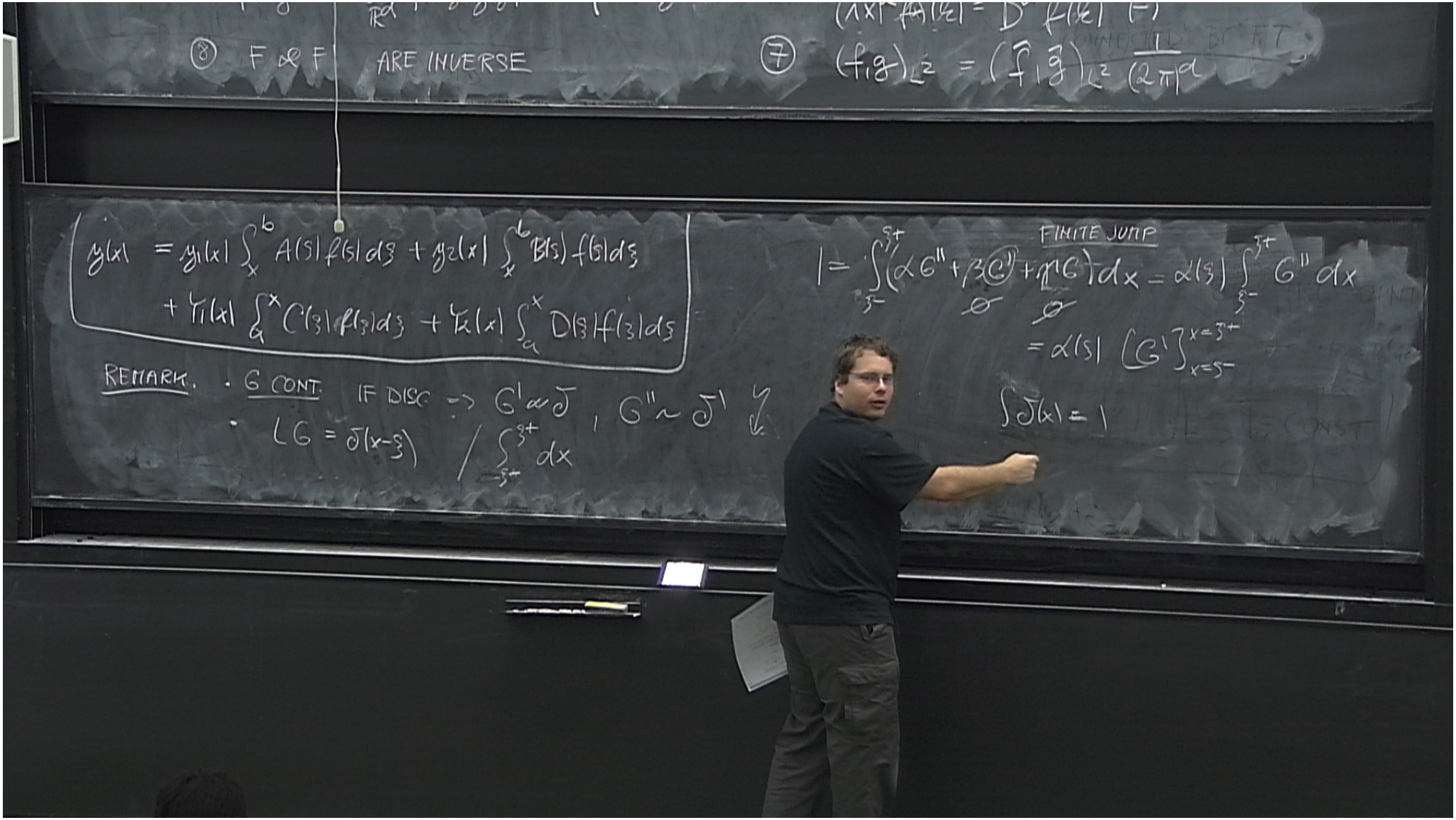

$I = \int_{\xi^-}^{\xi^+} (\alpha G'' + \beta G' + \gamma G)\, dx$
$\alpha = 0$
$\beta = 0$

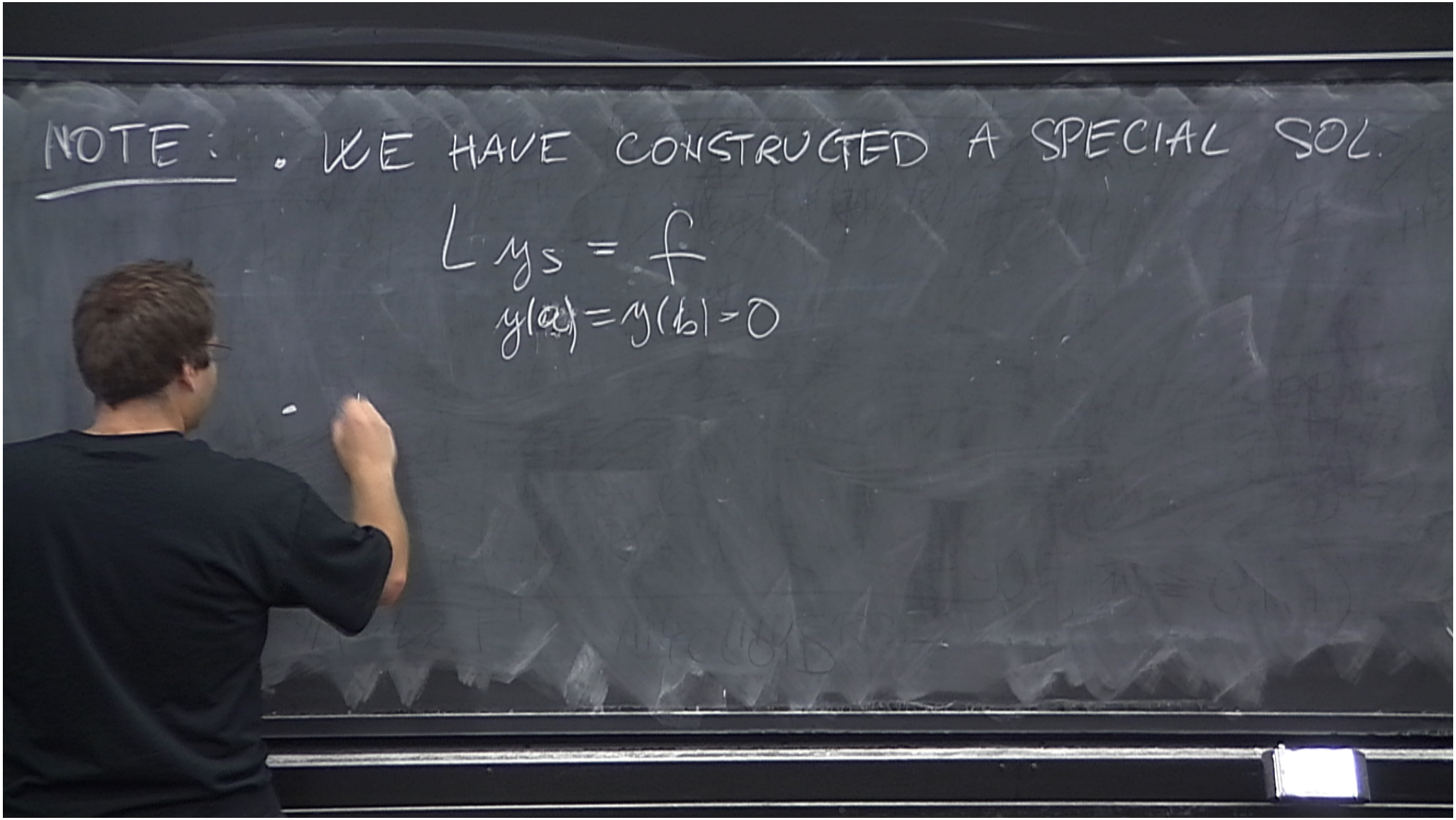

$I = \int_{\xi^-}^{\xi^+} (\alpha G'' + \beta G' + \gamma G)\, dx = \alpha(\xi) \int_{\xi^-}^{\xi^+} G''\, dx$
β
γ

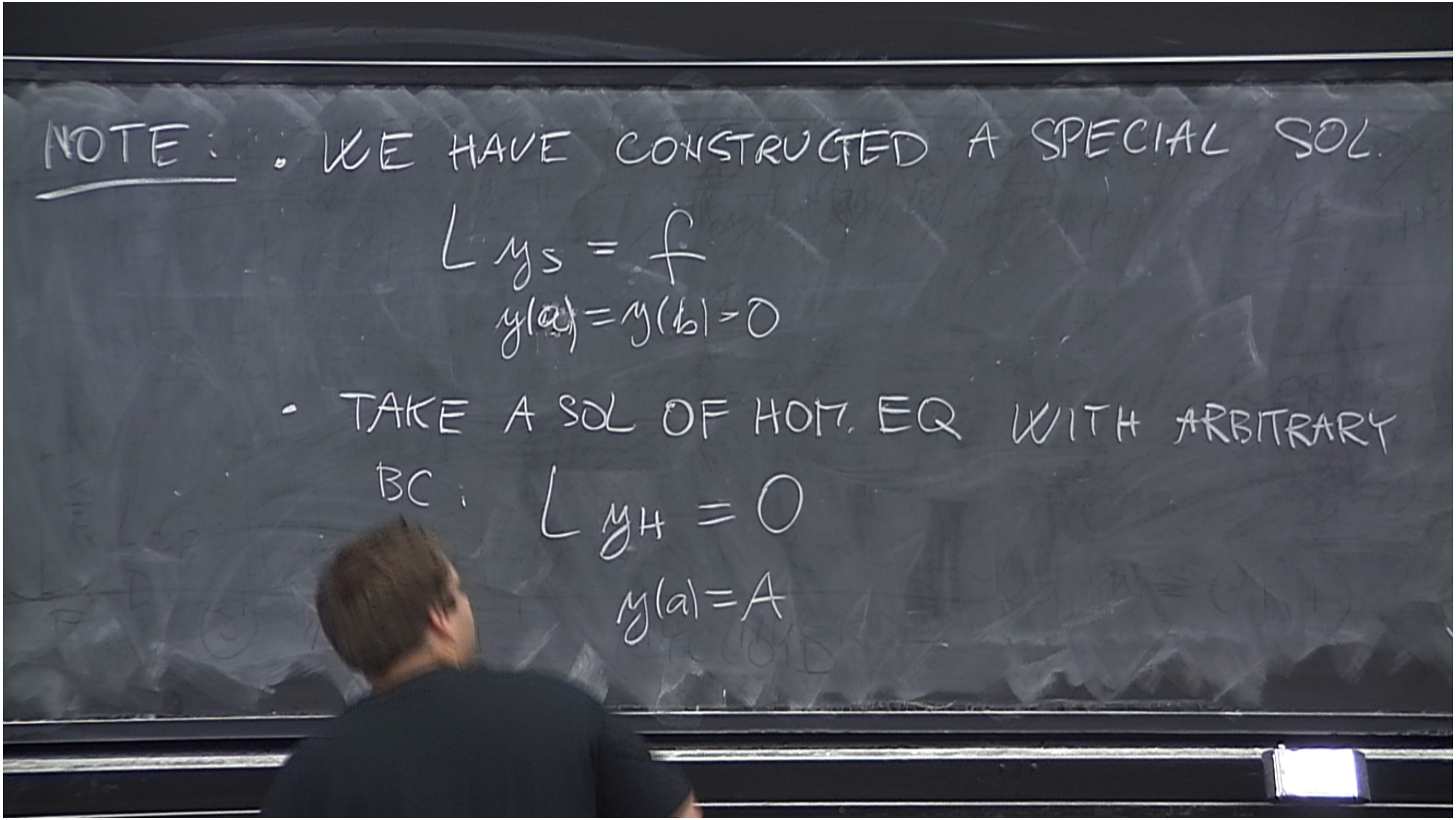

$I = \int_{\xi^-}^{\xi^+} (\alpha G'' + \beta G' + \gamma G)\, dx = \alpha(\xi) \int_{\xi^-}^{\xi^+} G''\, dx$
$= \alpha(\xi) \left[G' \right]_{x=\xi^-}^{x=\xi^+}$


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$y(x) = y_1(x)\int_x^b A(s)f(s)ds + y_2(x)\int_x^b B(s)f(s)ds$
$+ y_1(x)\int_a^x C(s)f(s)ds + y_2(x)\int_a^x D(s)f(s)ds$
REMARK.
$= \alpha(s)\,[G']_{x=s-}^{x=s+}$


FINITE JUMP
$1 = \int_{\xi^-}^{\xi^+} (\alpha G'' + \beta G' + \gamma G)\, dx = \alpha(\xi) \int_{\xi^-}^{\xi^+} G''\, dx$
$= \alpha(\xi) \left[G'\right]_{x=\xi^-}^{x=\xi^+}$


⑧ F & F ARE INVERSE
⑦ $(f,g)_{L^2} = (\hat{f},\hat{g})_{L^2}\frac{1}{(2\pi)^d}$
$y(x) = y_1(x)\int_x^b A(\xi)f(\xi)d\xi + y_2(x)\int_x^b B(\xi)f(\xi)d\xi$
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REMARK. · G CONT. IF DISC -> $G' \sim \delta$, $G'' \sim \delta'$
· $LG = \delta(x-\xi)$ / $\int_{\xi-}^{\xi+} dx$
FINITE JUMP
$1 = \int_{\xi-}^{\xi+}(\alpha G'' + \beta G' + \gamma G)dx = \alpha(\xi)\int_{\xi-}^{\xi+} G''\,dx$
$= \alpha(\xi)\,[G']_{x=\xi-}^{x=\xi+}$
$\int \delta(x) = 1$


NOTE: . WE HAVE CONSTRUCTED A SPECIAL SOL.
$L y_s = f$
$y(a) = y(b) = 0$


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- TAKE A SOL OF HOM. EQ WITH ARBITRARY BC:
$L y_H = 0$
$y(a) = A$

NOTE: . WE HAVE CONSTRUCTED A SPECIAL SOL.

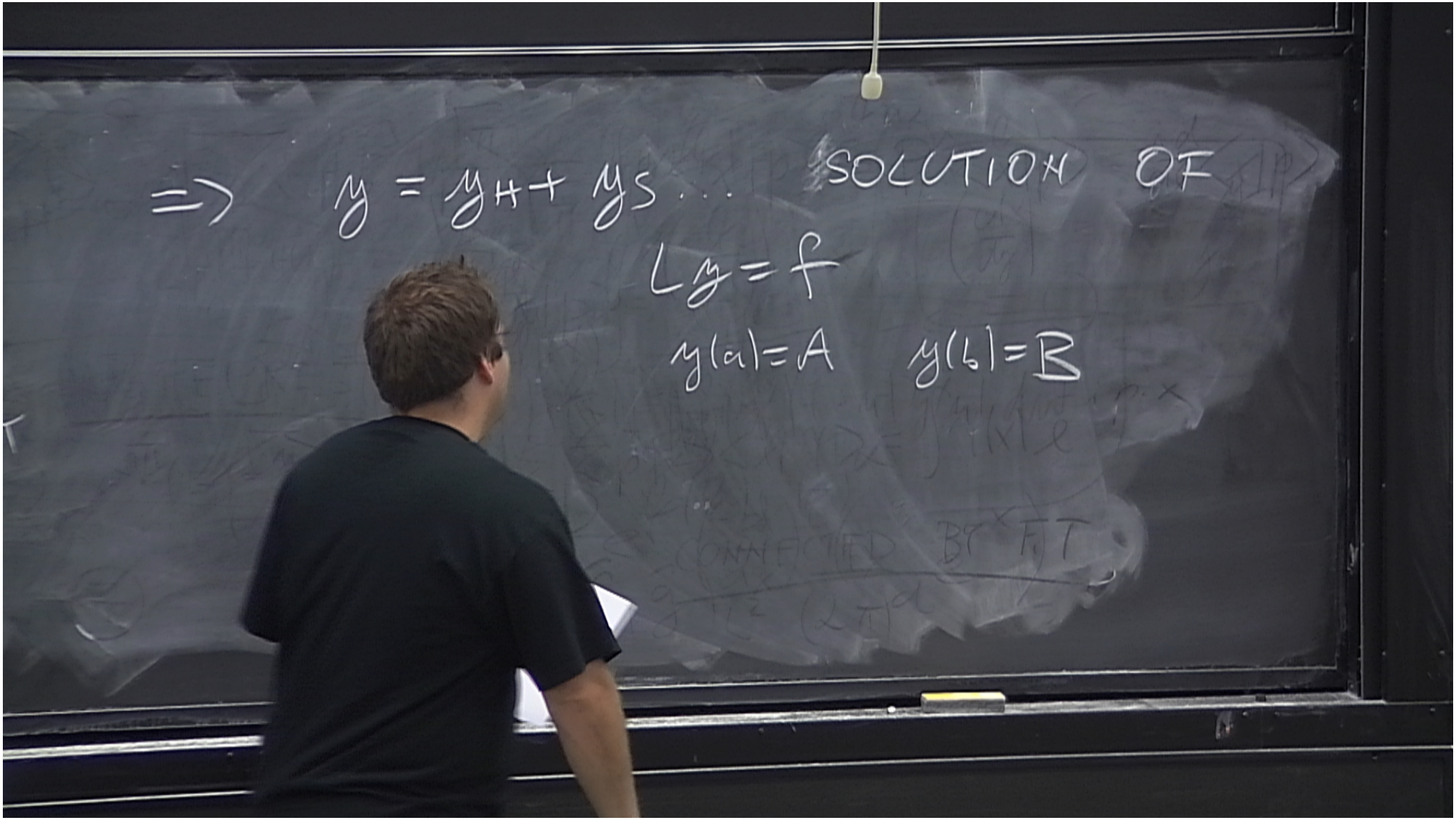
$$L y_s = f$$

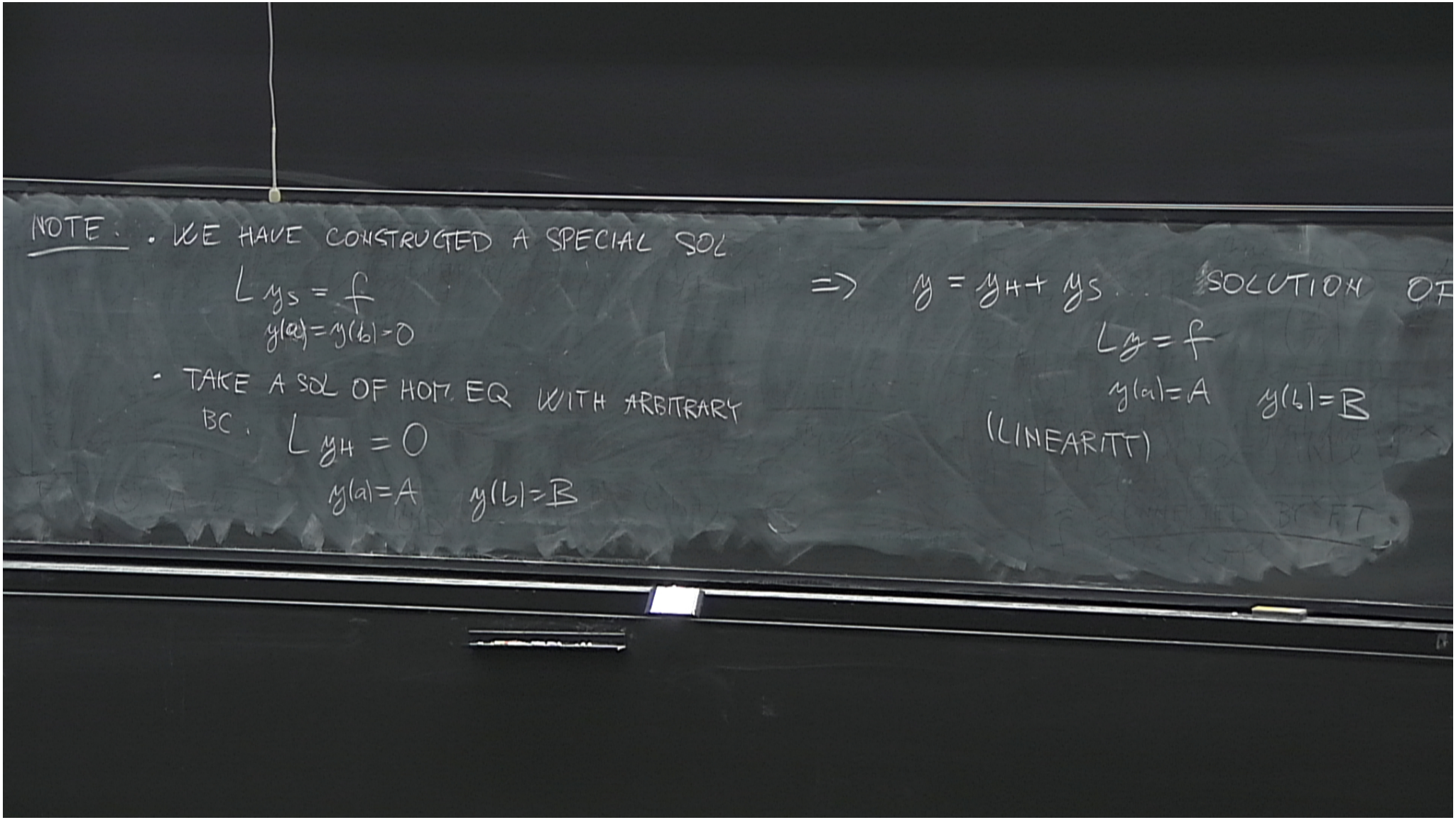
$$y(a) = y(b) = 0$$

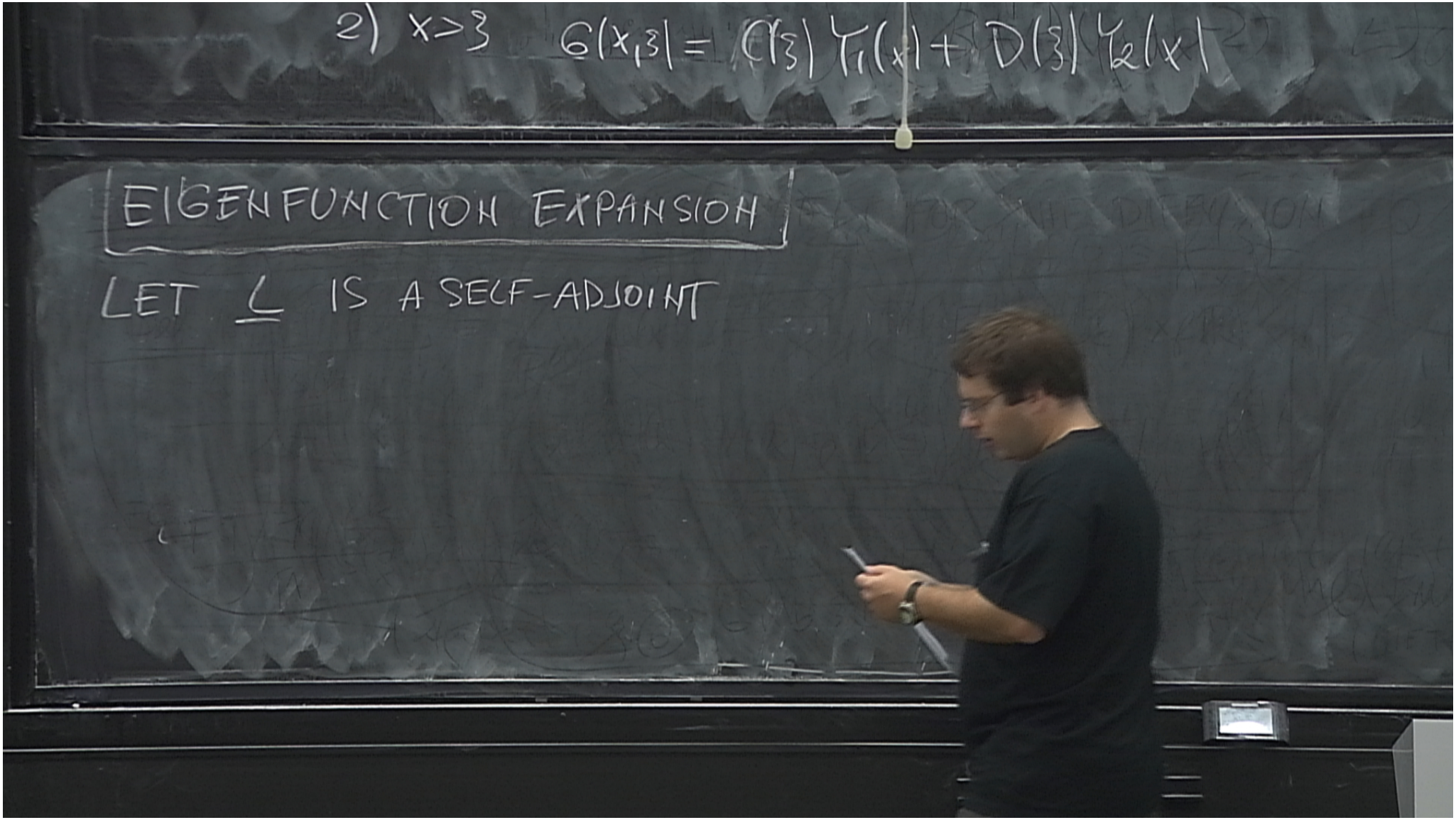
- TAKE A SOL OF HOM. EQ WITH ARBITRARY BC:

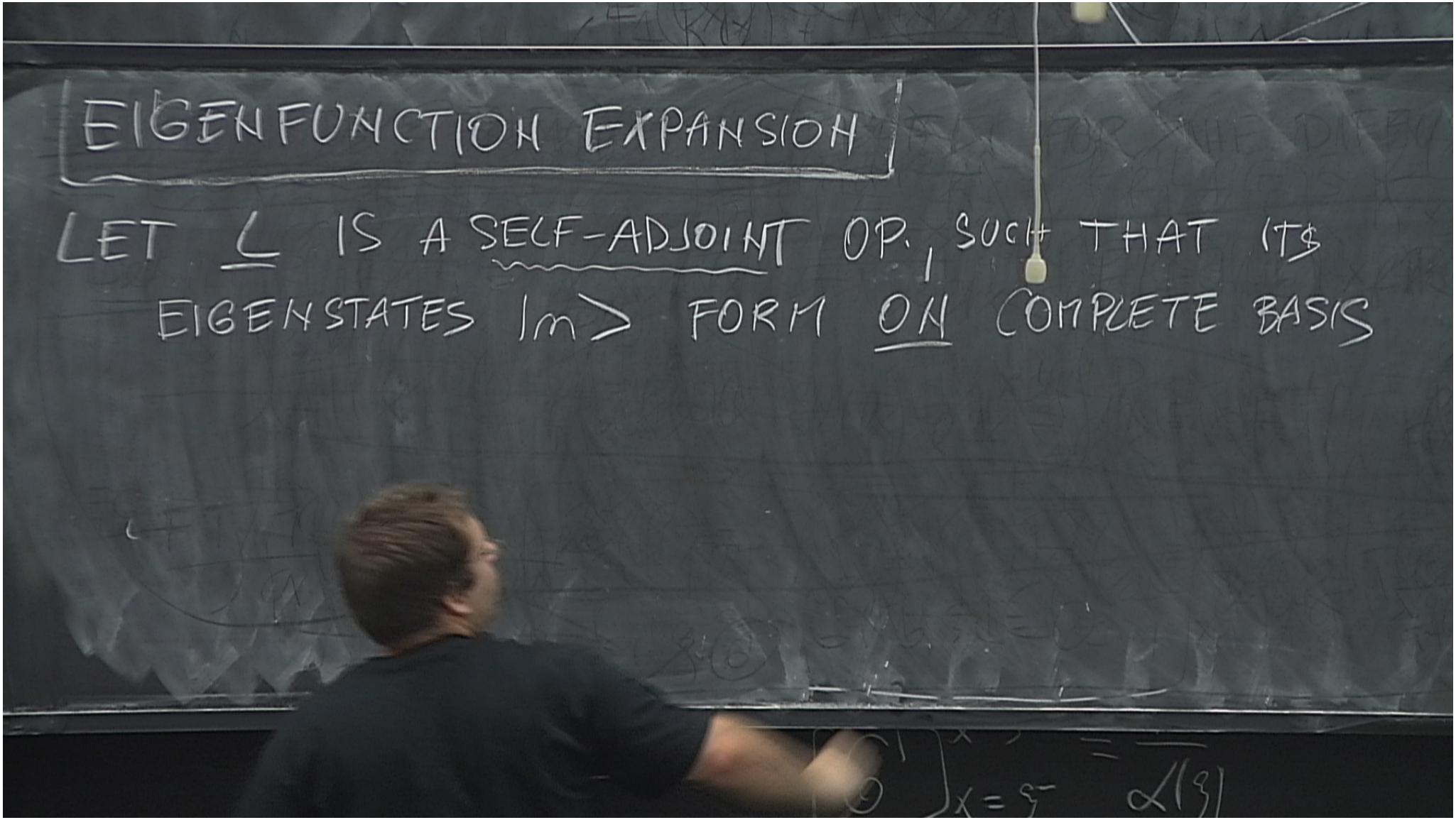
$$L y_H = 0$$

$$y(a) = A \qquad y(b) = B$$


=> y = y_H + y_S ... SOLUTION OF
Ly = f
y(a) = A y(b) = B


NOTE: • WE HAVE CONSTRUCTED A SPECIAL SOL
$L y_S = f$
$y(a) = y(b) = 0$
• TAKE A SOL OF HOM. EQ WITH ARBITRARY BC.
$L y_H = 0$
$y(a) = A \quad y(b) = B$
$\Rightarrow \quad y = y_H + y_S$ SOLUTION OF
$Ly = f$
$y(a) = A \quad y(b) = B$
(LINEARITY)


2) x>ξ G(x,ξ) = C(ξ) y1(x) + D(ξ) y2(x)
EIGENFUNCTION EXPANSION
LET L IS A SELF-ADJOINT


EIGENFUNCTION EXPANSION
LET L IS A SELF-ADJOINT OP., SUCH THAT ITS
EIGENSTATES |m> FORM ON COMPLETE BASIS

EIGENFUNCTION EXPANSION

LET $\mathcal{L}$ IS A SELF-ADJOINT OP., SUCH THAT ITS EIGENSTATES $|m\rangle$ FORM AN COMPLETE BASIS

$$\mathcal{L}|m\rangle = \lambda_m |m\rangle$$

$$\langle m|m\rangle = \delta_{mm}$$

JUMP COND.: $\left[G\right]_{x=\xi^-}^{x=\xi^+} = \frac{1}{\mathcal{L}(\xi)}$

EIGENFUNCTION EXPANSION

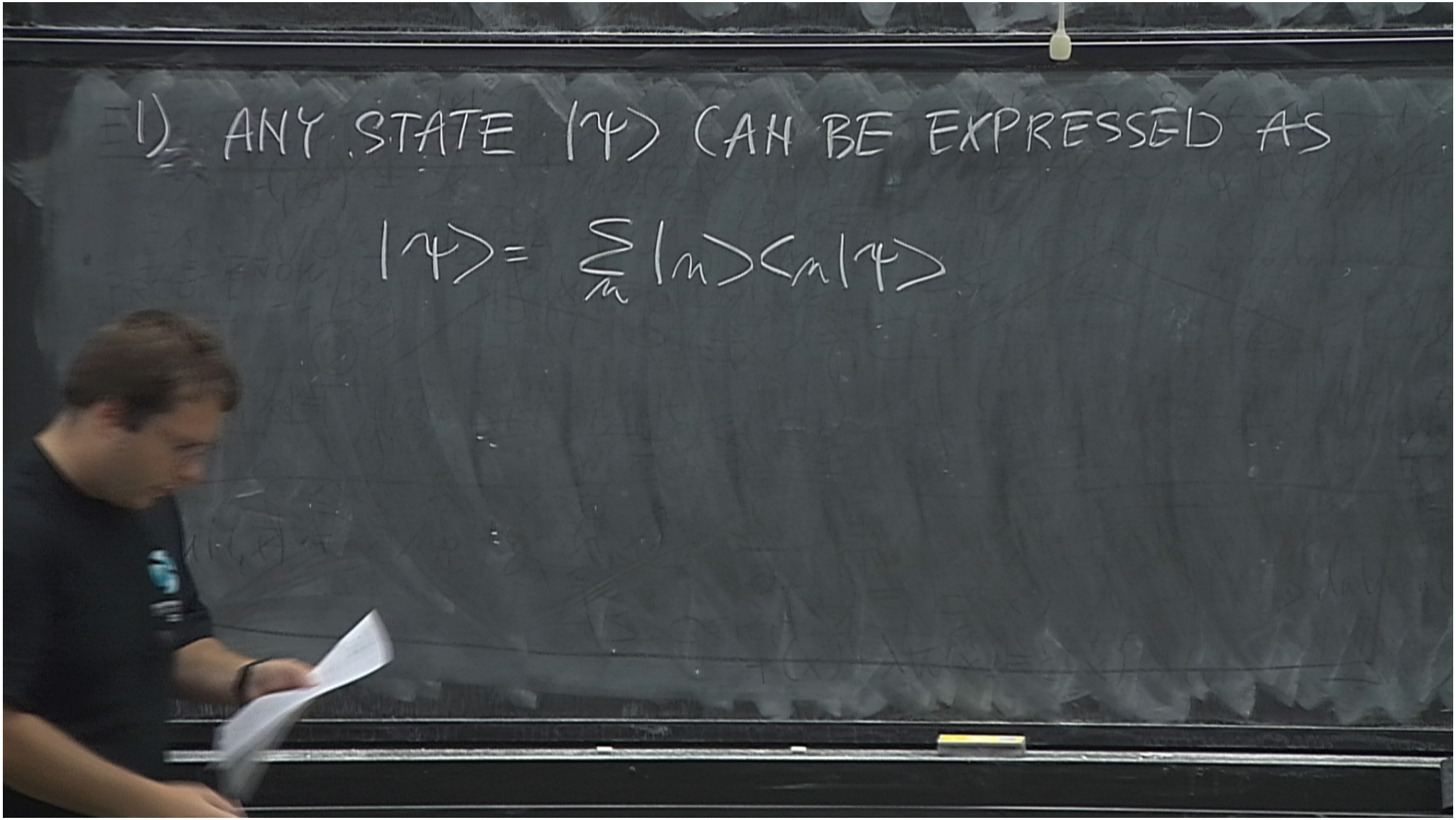
LET $\mathcal{L}$ IS A SELF-ADJOINT OP., SUCH THAT ITS EIGENSTATES $|m\rangle$ FORM ON COMPLETE BASIS

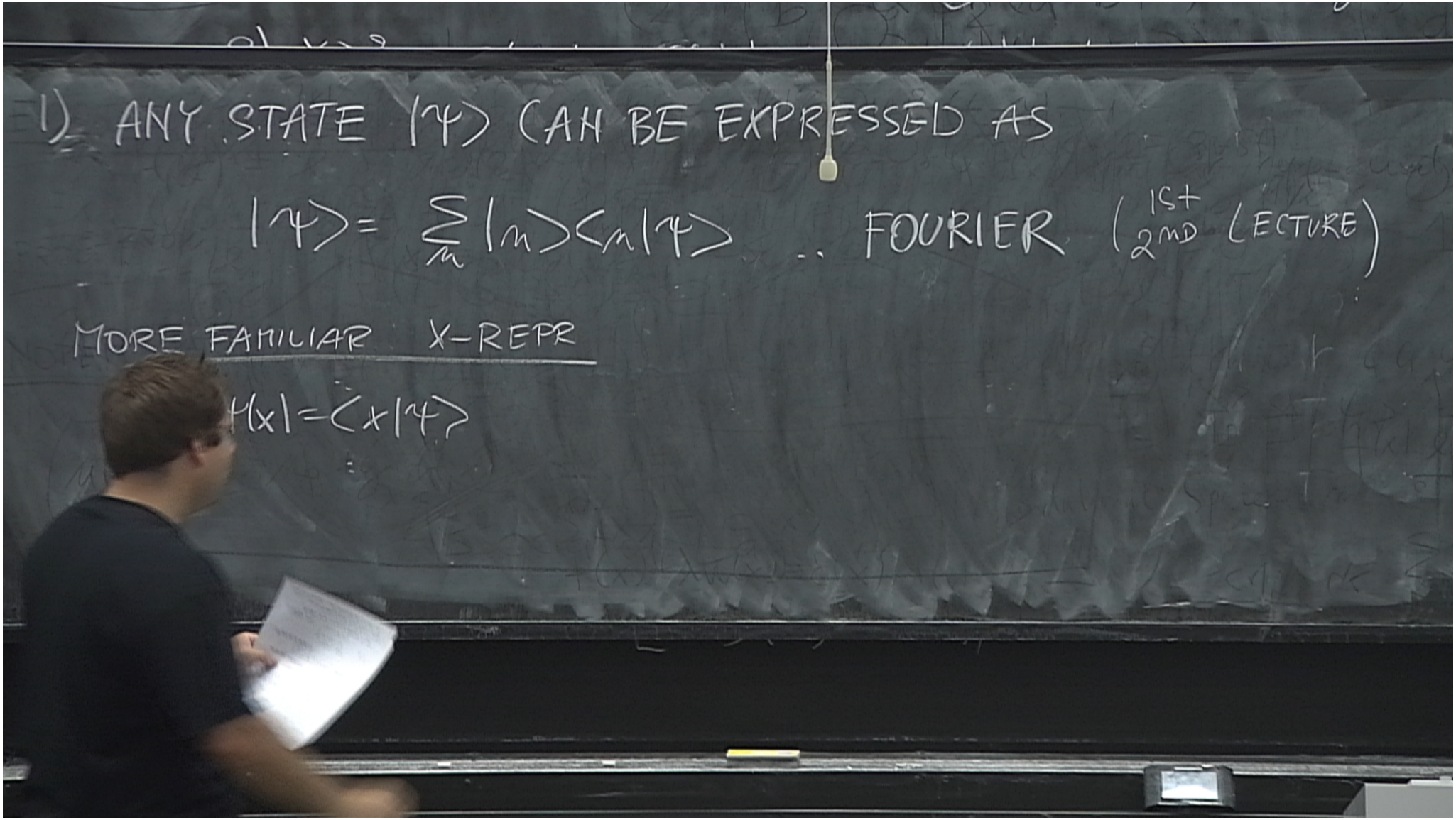
$$\mathcal{L}|m\rangle = \lambda_m |m\rangle$$

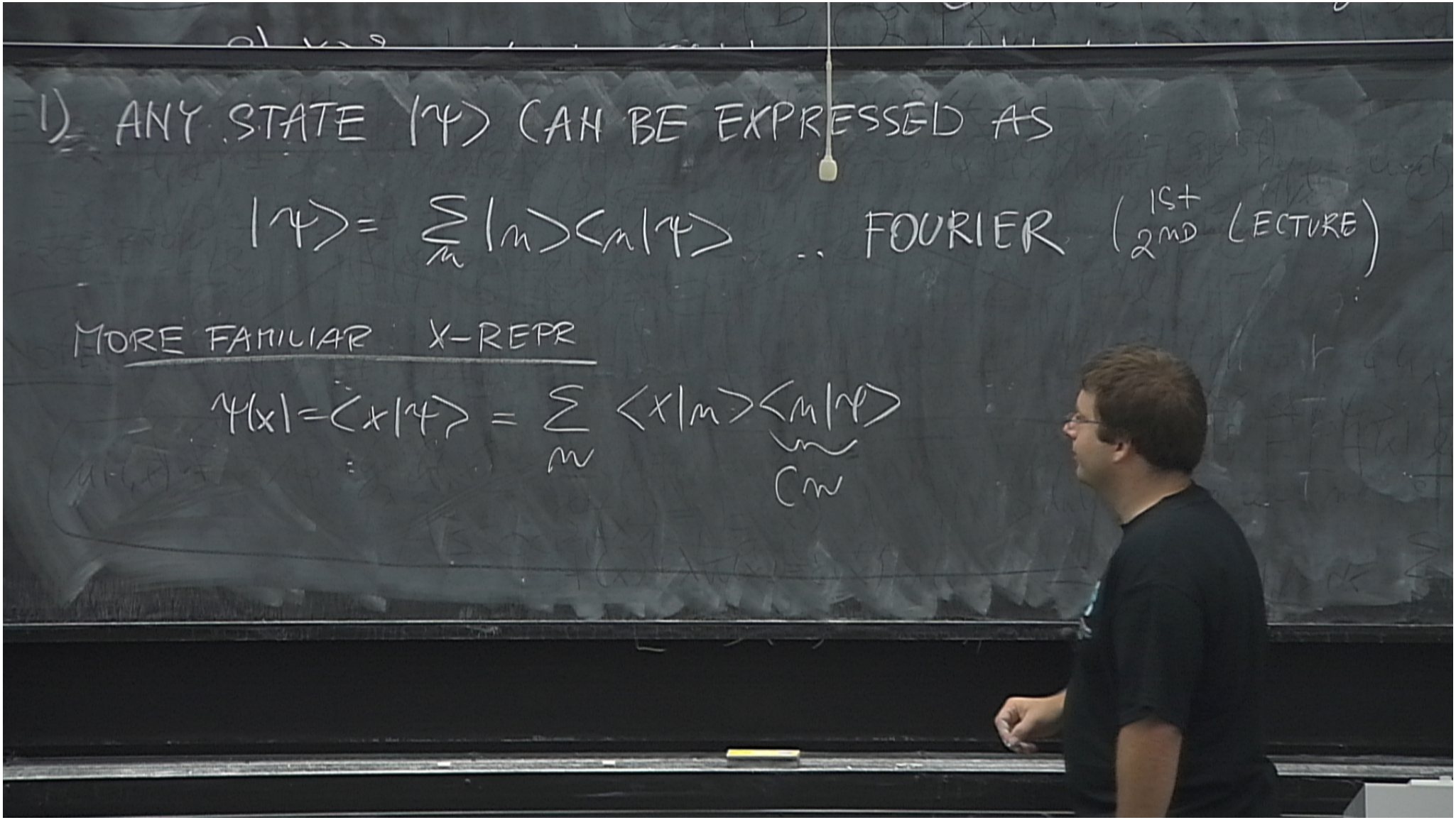
$$\langle m|m'\rangle = \delta_{m m'}$$

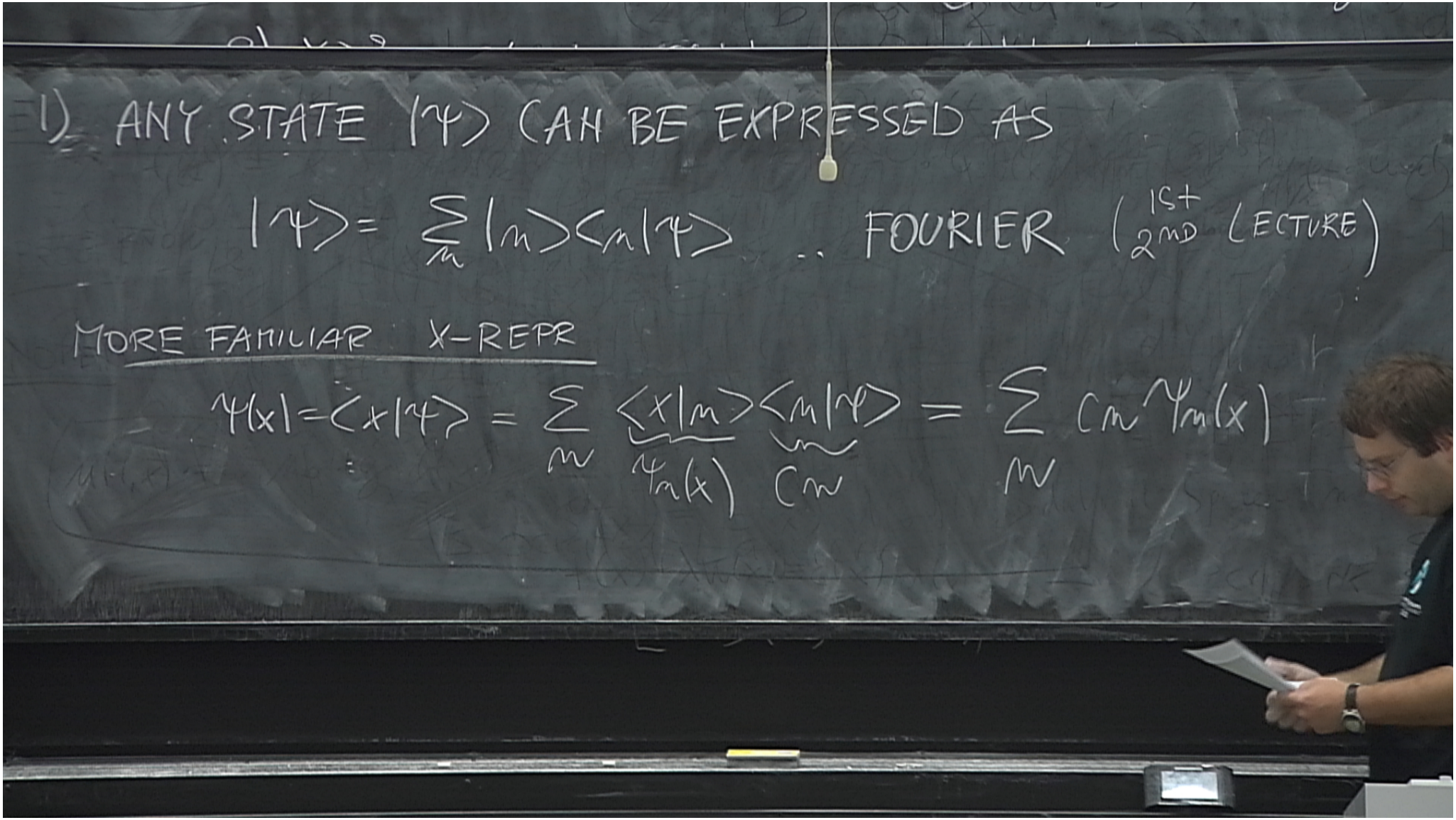
$$\sum_m |m\rangle\langle m| = \mathbb{1}$$

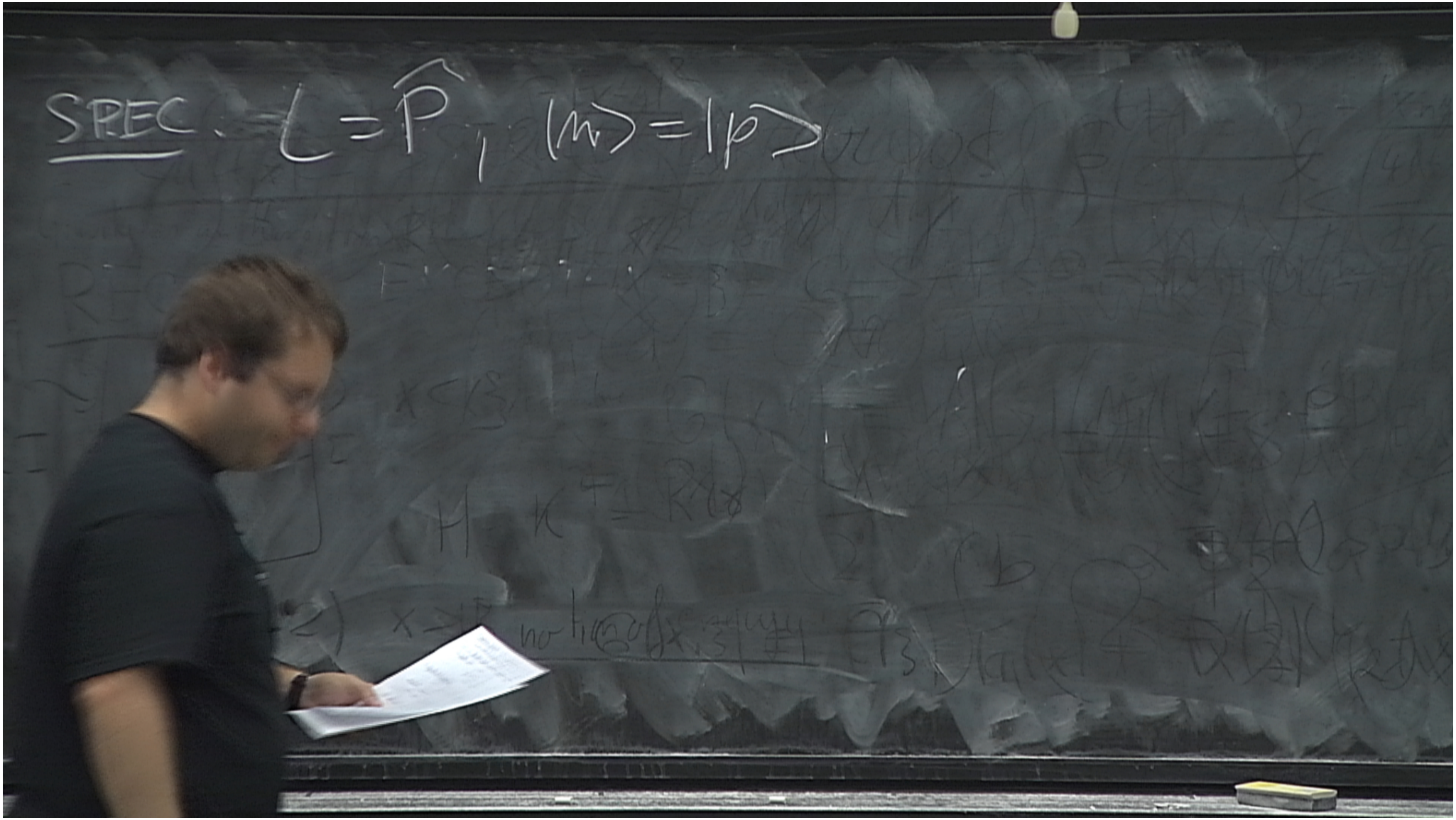
JUMP COND: $\left[G\right]_{x=\xi^-}^{x=\xi^+} = \frac{1}{\alpha(\xi)}$

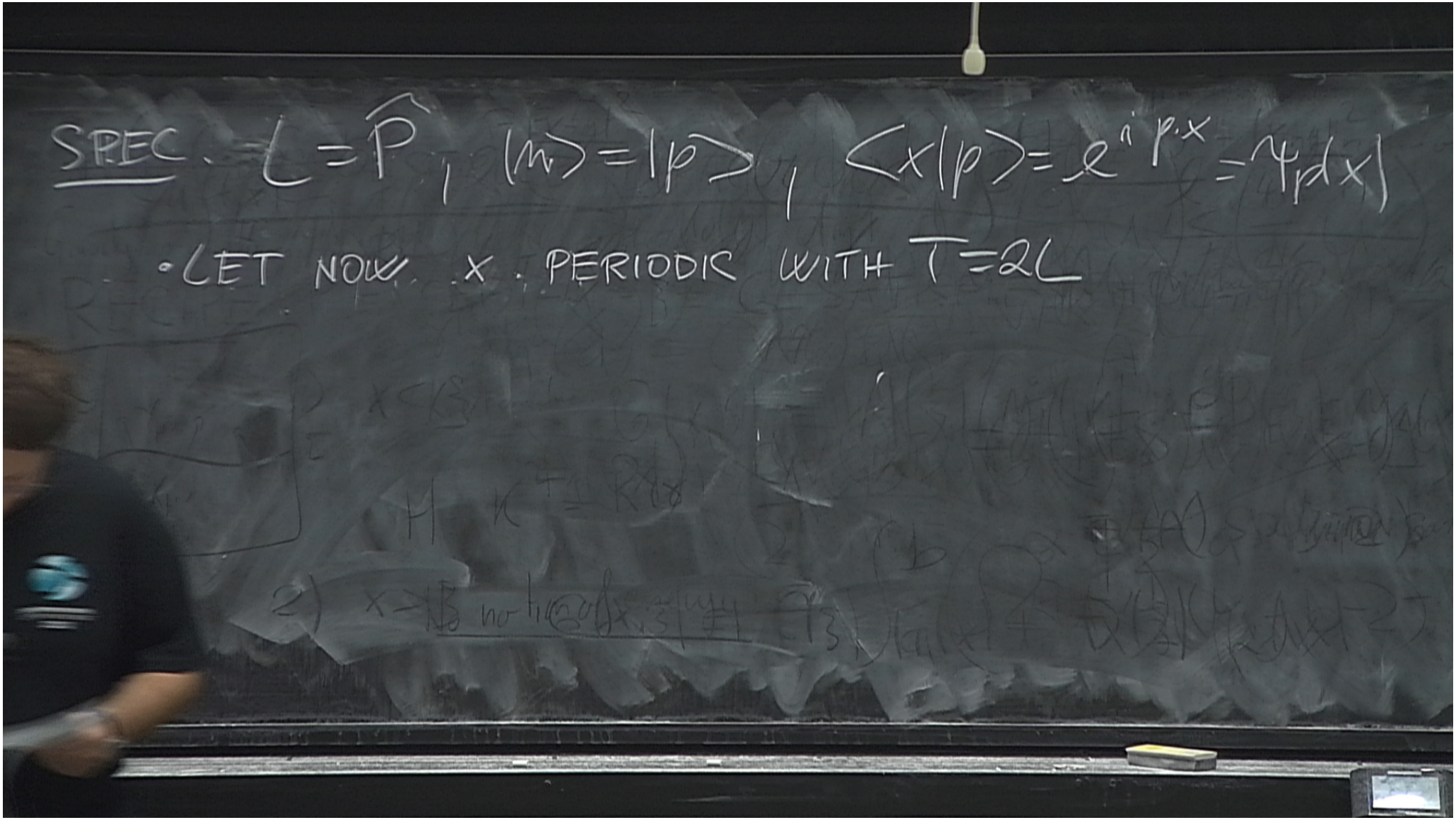

1) ANY STATE $|\psi\rangle$ CAN BE EXPRESSED AS
$|\psi\rangle = \sum_m |m\rangle\langle m|\psi\rangle$

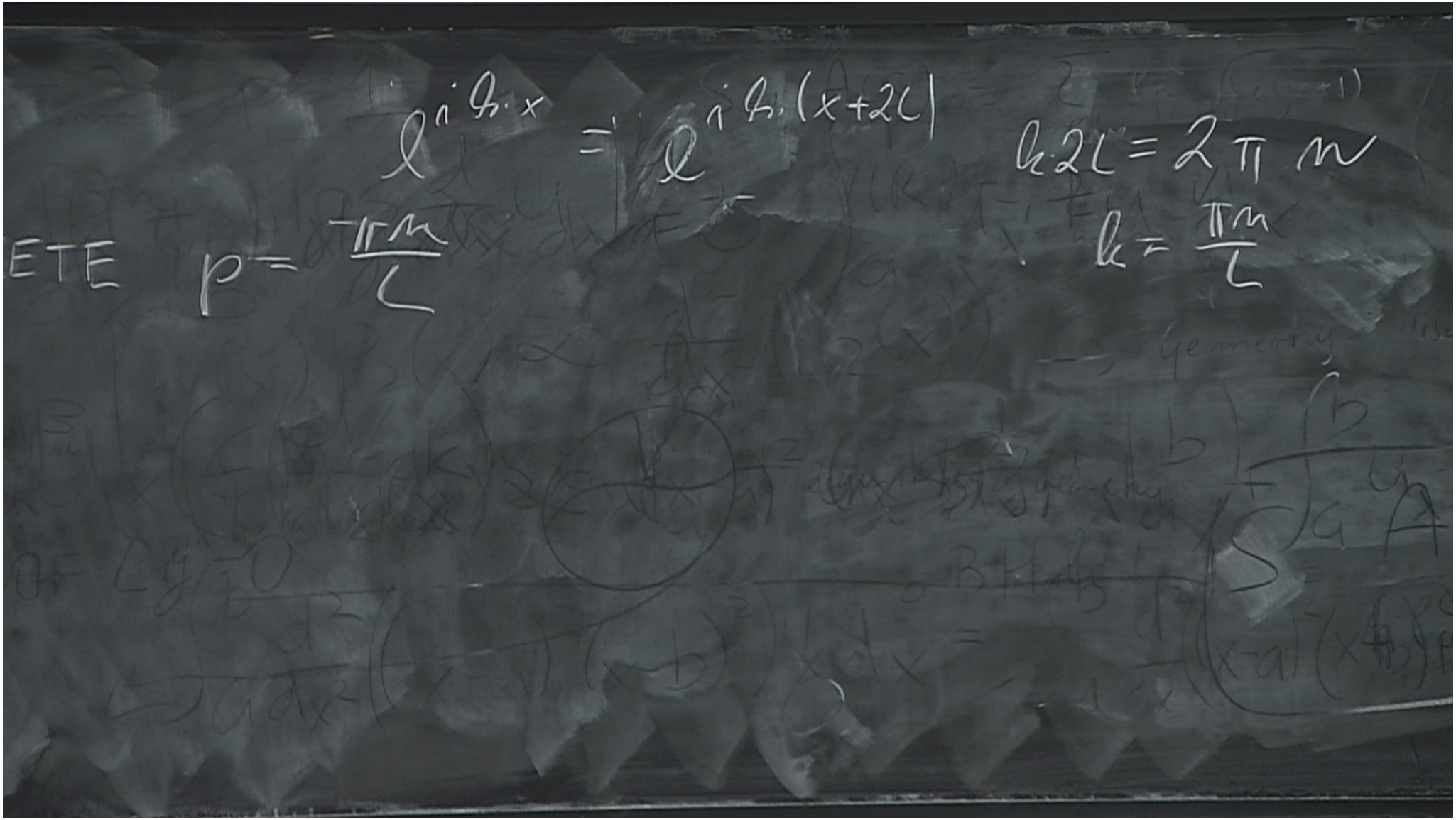

1) ANY STATE |ψ⟩ CAN BE EXPRESSED AS
|ψ⟩ = Σ_n |n⟩⟨n|ψ⟩ .. FOURIER (1st 2nd LECTURE)
MORE FAMILIAR X-REPR
ψ(x) = ⟨x|ψ⟩


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C_m


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MORE FAMILIAR X-REPR
ψ(x) = ⟨x|ψ⟩ = Σ_m ⟨x|m⟩⟨m|ψ⟩ = Σ_m c_m ψ_m(x)
ψ_m(x) c_m
m


SPEC
L = P̂
|w> = |p>

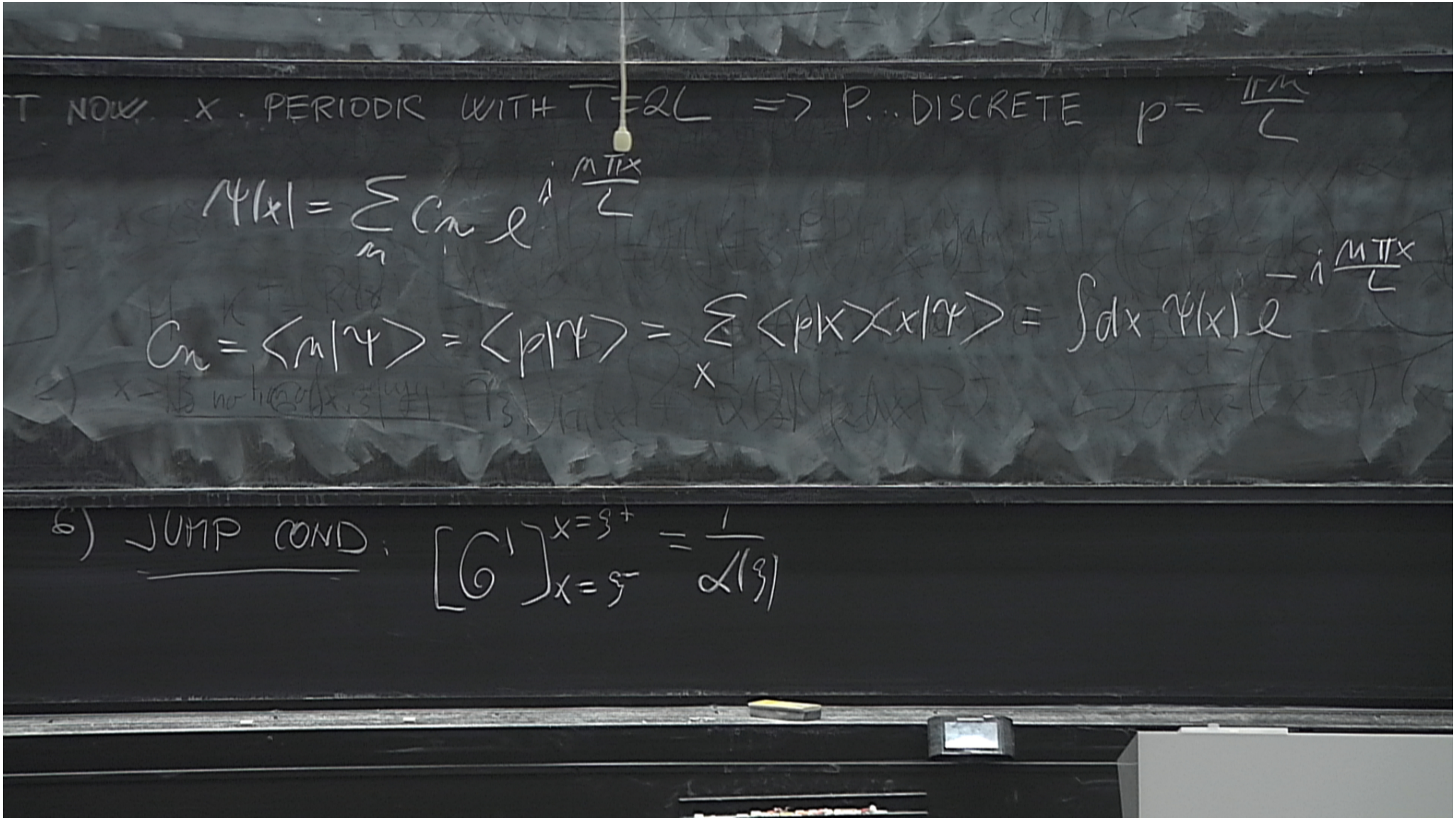

SPEC. $L = \hat{P}$; $|x\rangle = |p\rangle$; $\langle x|p\rangle = e^{i p \cdot x} = \psi_p(x)$
• LET NOW x : PERIODIC WITH $T = 2L$


$e^{ik\cdot x} = e^{ik\cdot(x+2L)}$
$k\cdot 2L = 2\pi n$
$k = \frac{\pi n}{L}$
$p = \frac{\pi n}{L}$

NOW x PERIODIC WITH $T = 2L$ $\Rightarrow$ P... DIS

$$\psi(x) = \sum_n c_n e^{i \frac{n\pi x}{L}}$$

$$c_n = \langle n|\psi\rangle = \langle p|\psi\rangle = \sum_x \langle p|x\rangle\langle x|\psi$$


T NOW. x . PERIODIC WITH T=2L => P... DISCRETE p = πn/L
ψ(x) = Σ_n C_n e^{i nπx/L}
C_n = ⟨n|ψ⟩ = ⟨p|ψ⟩ = Σ_x ⟨p|x⟩⟨x|ψ⟩ = ∫dx ψ(x) e^{-i nπx/L}
6) JUMP COND: [G']_{x=s⁻}^{x=s⁺} = 1/α(s)

NOW x PERIODIC WITH $T = 2L \Rightarrow p$ ... DISCRETE

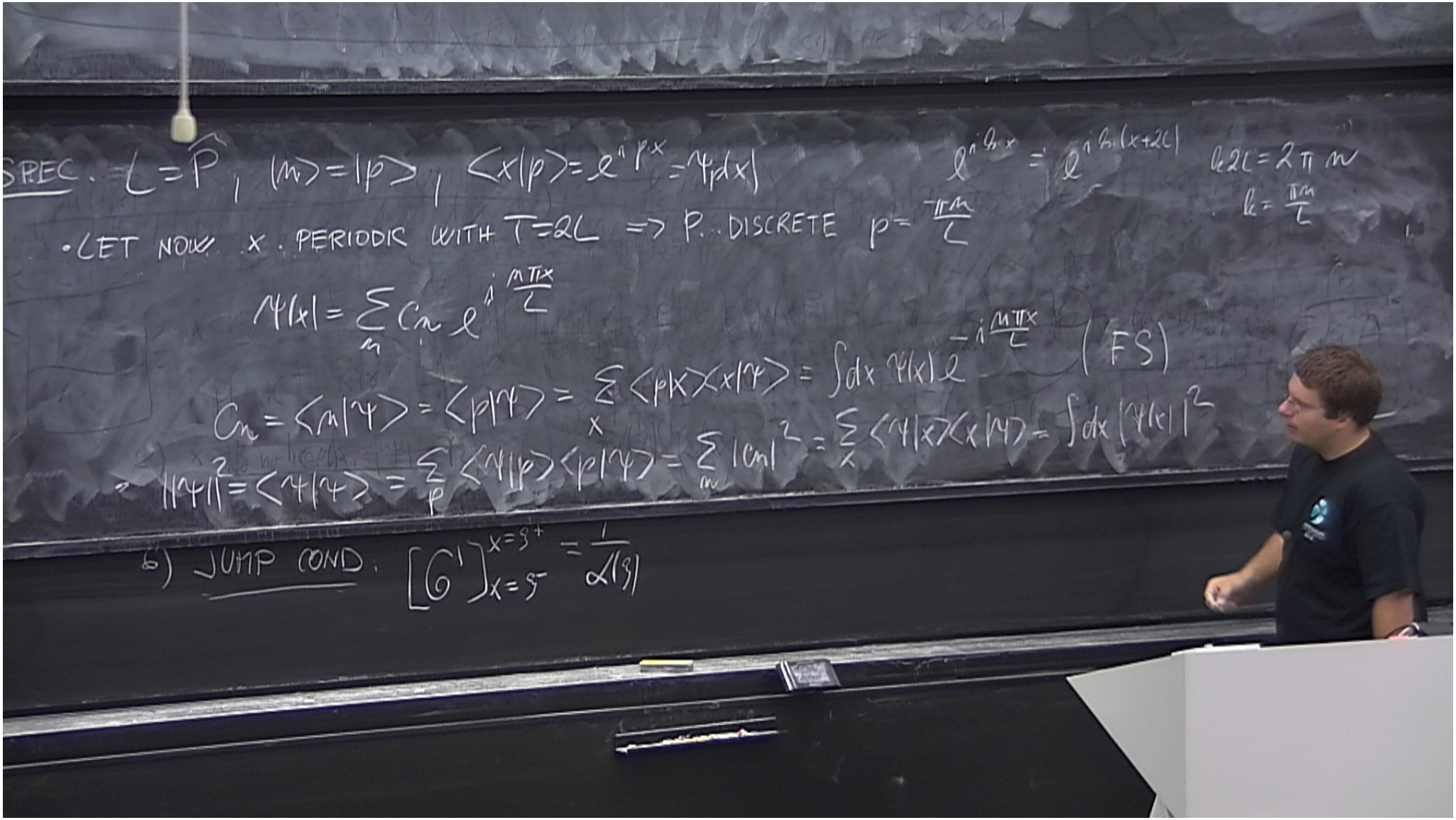
$$\psi(x) = \sum_n c_n e^{i \frac{n\pi x}{L}}$$

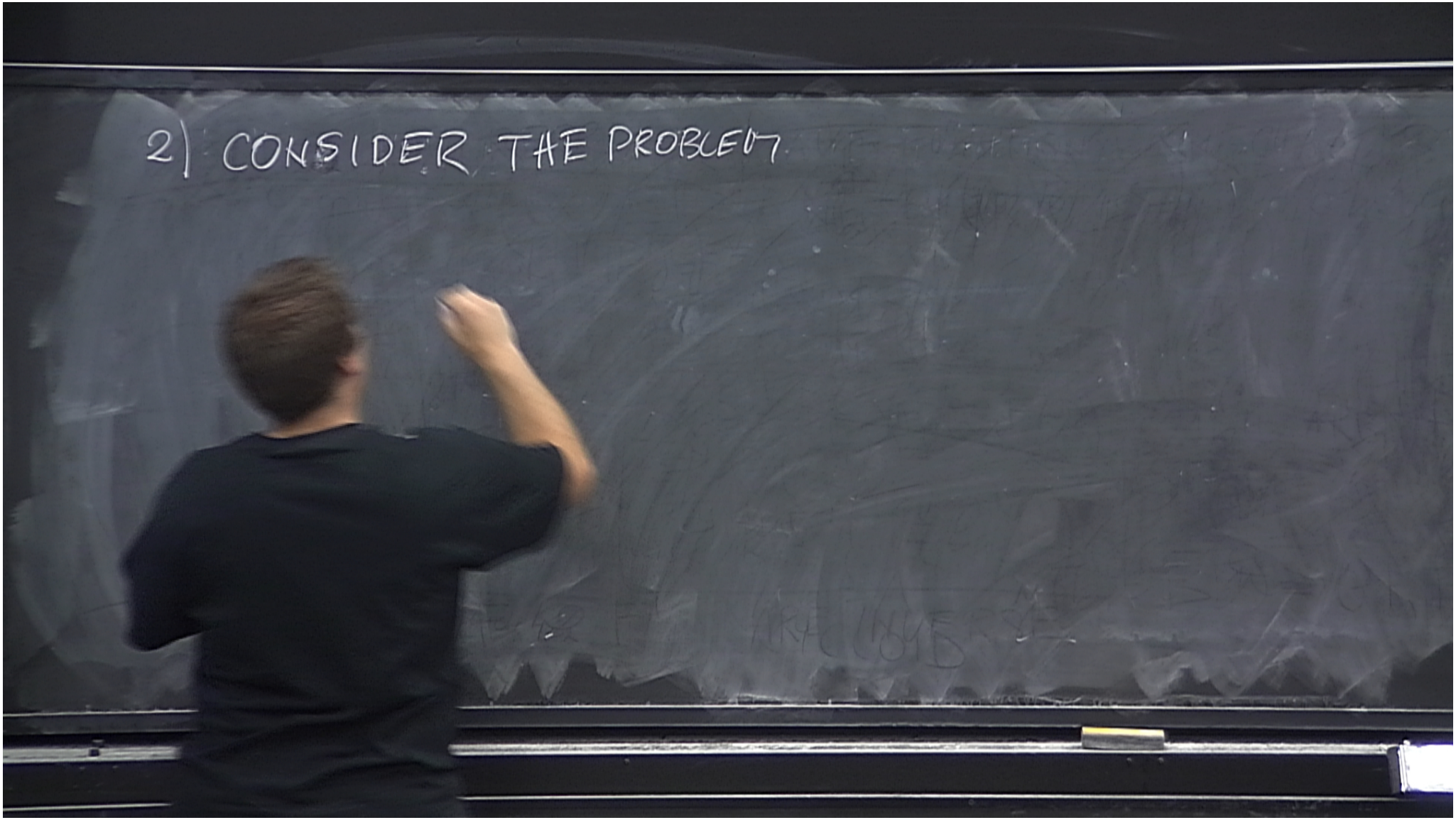
$$c_n = \langle n|\psi\rangle = \langle p|\psi\rangle = \sum_x \langle p|x\rangle\langle x|\psi\rangle = \int dx$$

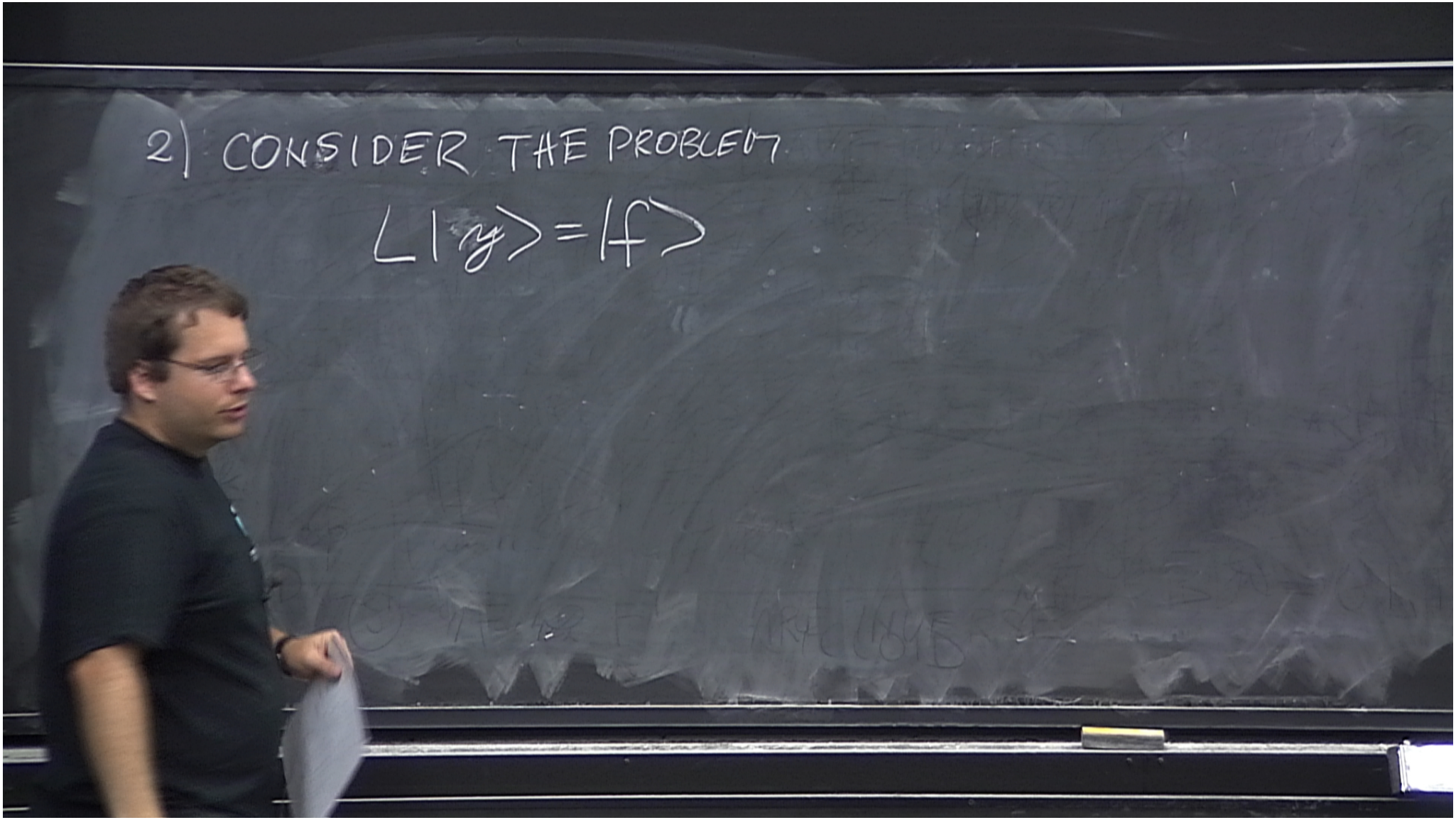
$$\||\psi\|^2 = \langle\psi|\psi\rangle = \sum_p \langle\psi|p\rangle\langle p|\psi\rangle = \sum_n |c_n|^2$$

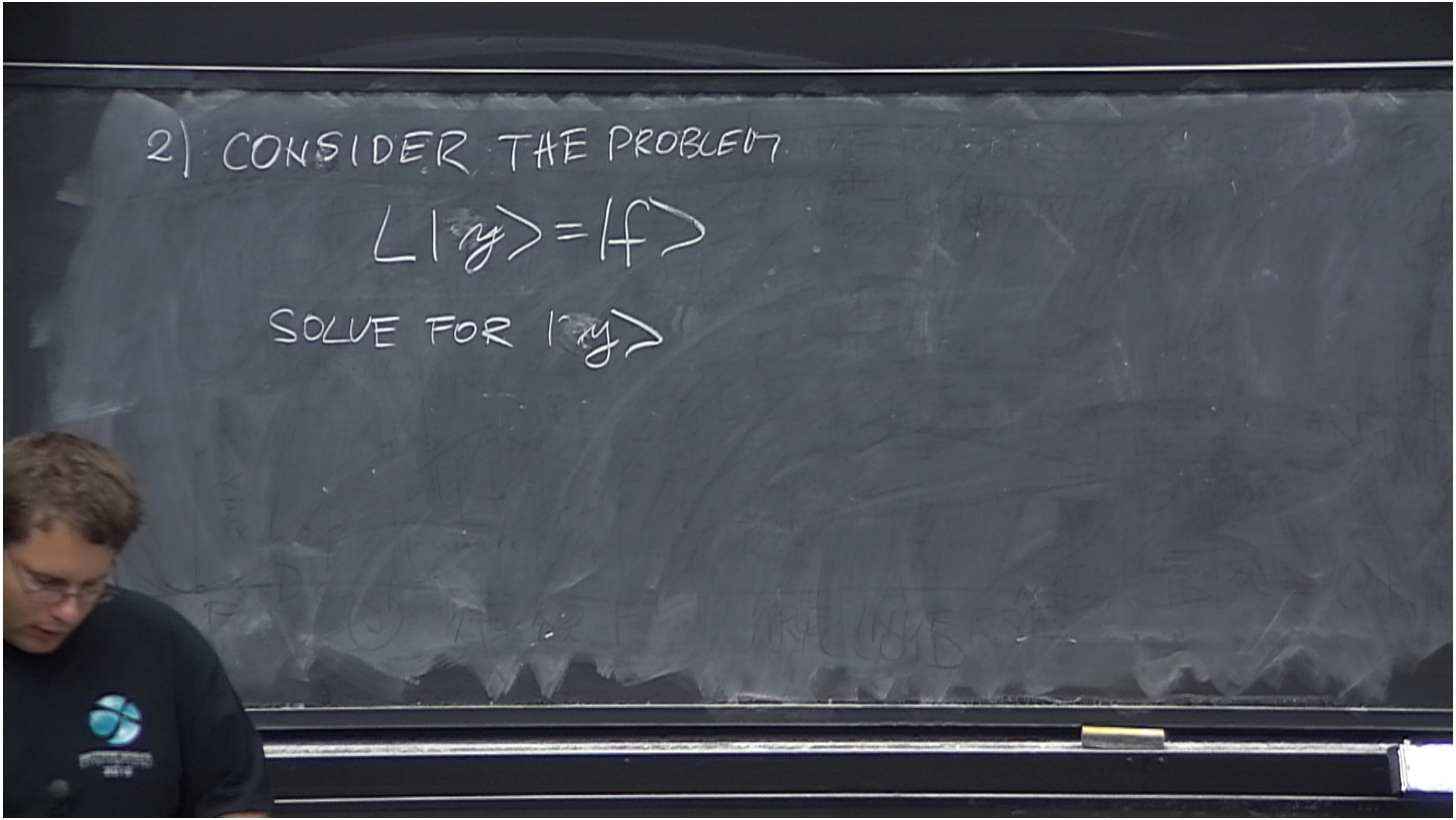
6) JUMP COND:

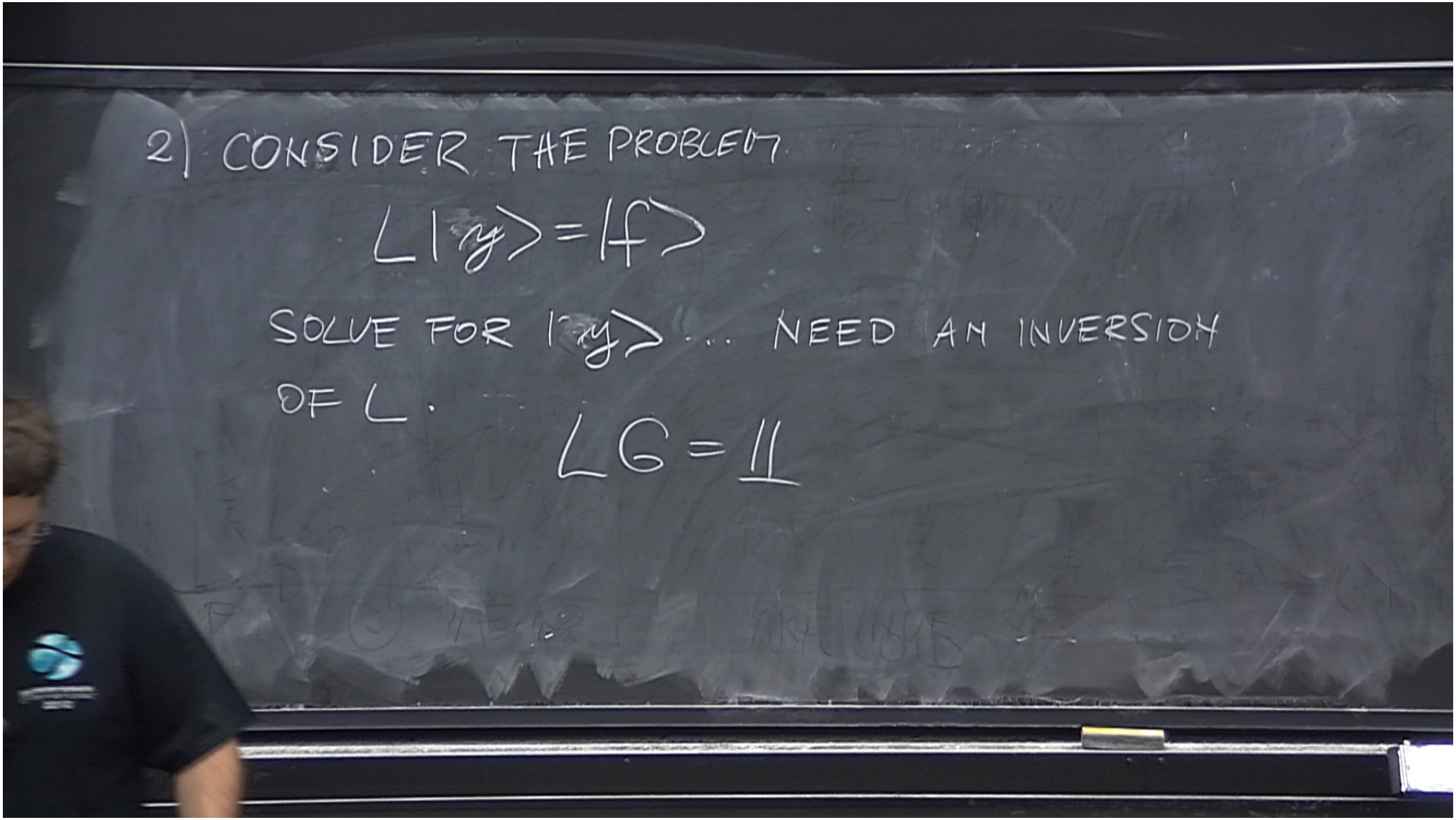
$$[G']_{x=s^-}^{x=s^+} = \frac{1}{\alpha(s)}$$

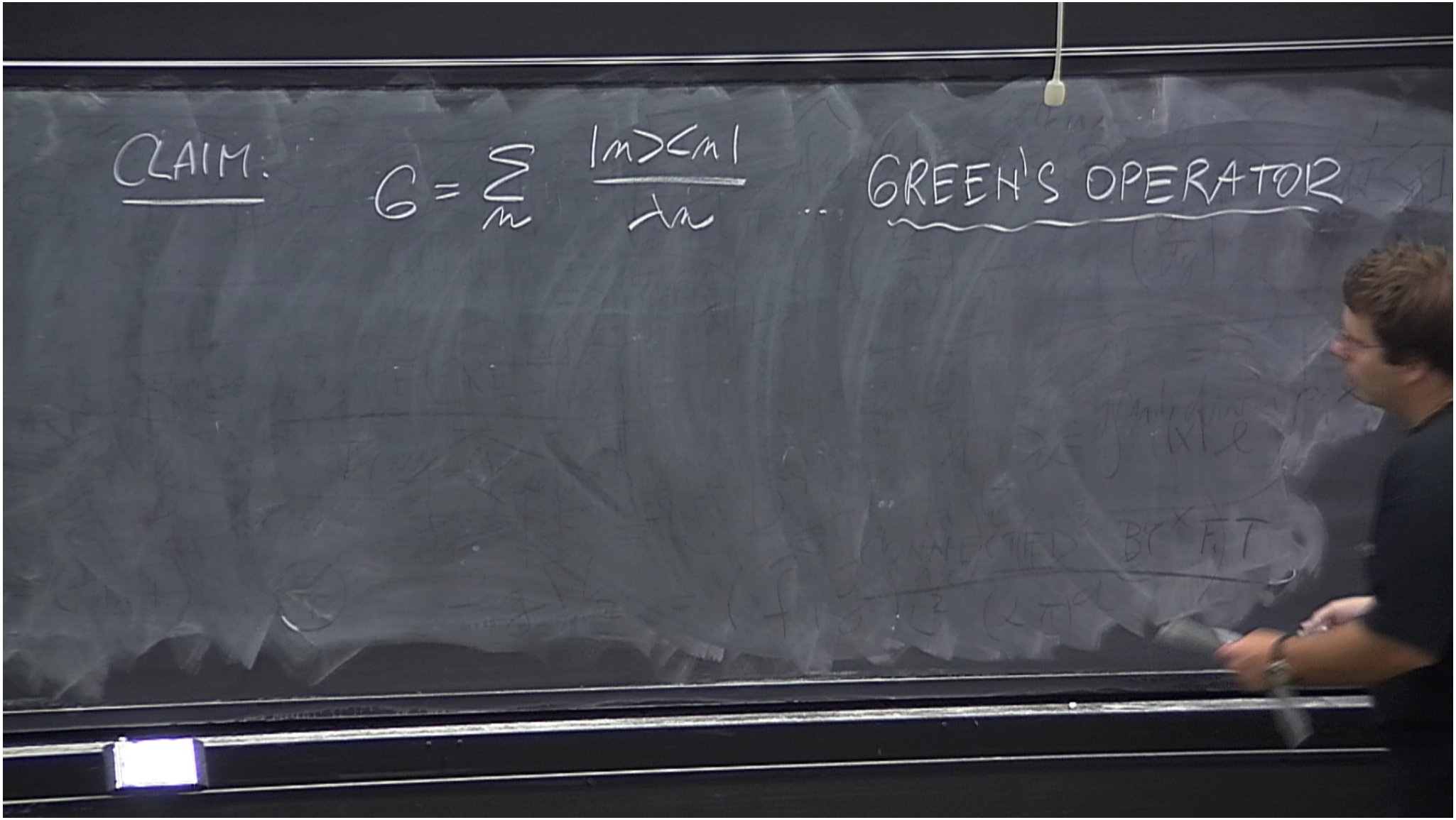

SPEC. $\hat{L}=\hat{P}$, $|n\rangle=|p\rangle$, $\langle x|p\rangle=e^{ipx}=\psi_p(x)$
$e^{ik_n x}=e^{ik_n(x+2L)}$ $k_n 2L=2\pi n$ $k_n=\frac{\pi n}{L}$
• LET NOW x PERIODIC WITH $T=2L$ => P.. DISCRETE $p=\frac{\pi n}{L}$
$\psi(x)=\sum_n C_n e^{i\frac{n\pi x}{L}}$
$C_n=\langle n|\psi\rangle=\langle p|\psi\rangle=\sum_x\langle p|x\rangle\langle x|\psi\rangle=\int dx\,\psi(x)e^{-i\frac{n\pi x}{L}}$ (FS)
$\|\psi\|^2=\langle\psi|\psi\rangle=\sum_p\langle\psi|p\rangle\langle p|\psi\rangle=\sum_n|c_n|^2=\sum_x\langle\psi|x\rangle\langle x|\psi\rangle=\int dx\,|\psi(x)|^2$
6) JUMP COND. $[G']_{x=s^-}^{x=s^+}=\frac{1}{\alpha(s)}$

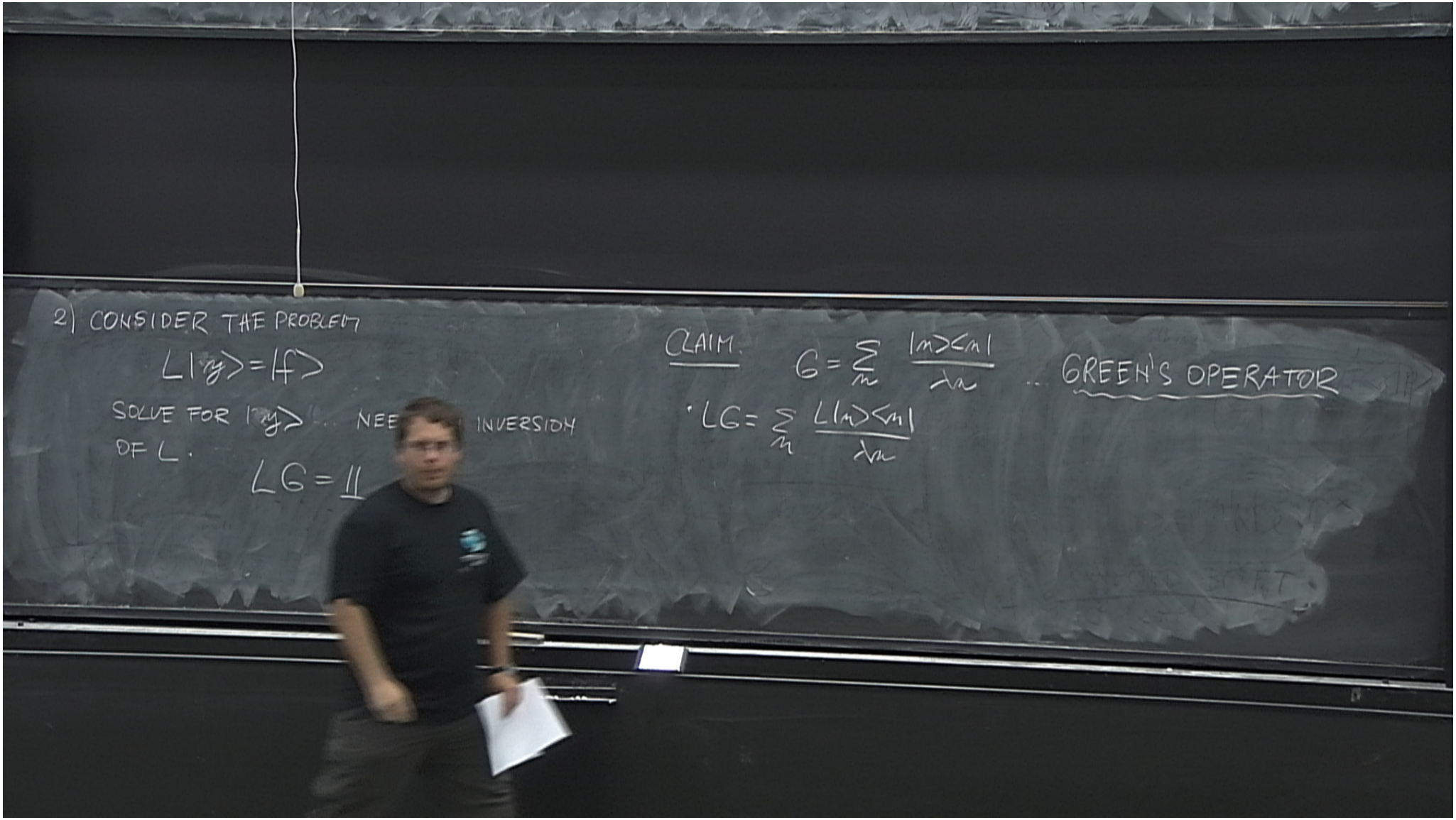

2) CONSIDER THE PROBLEM

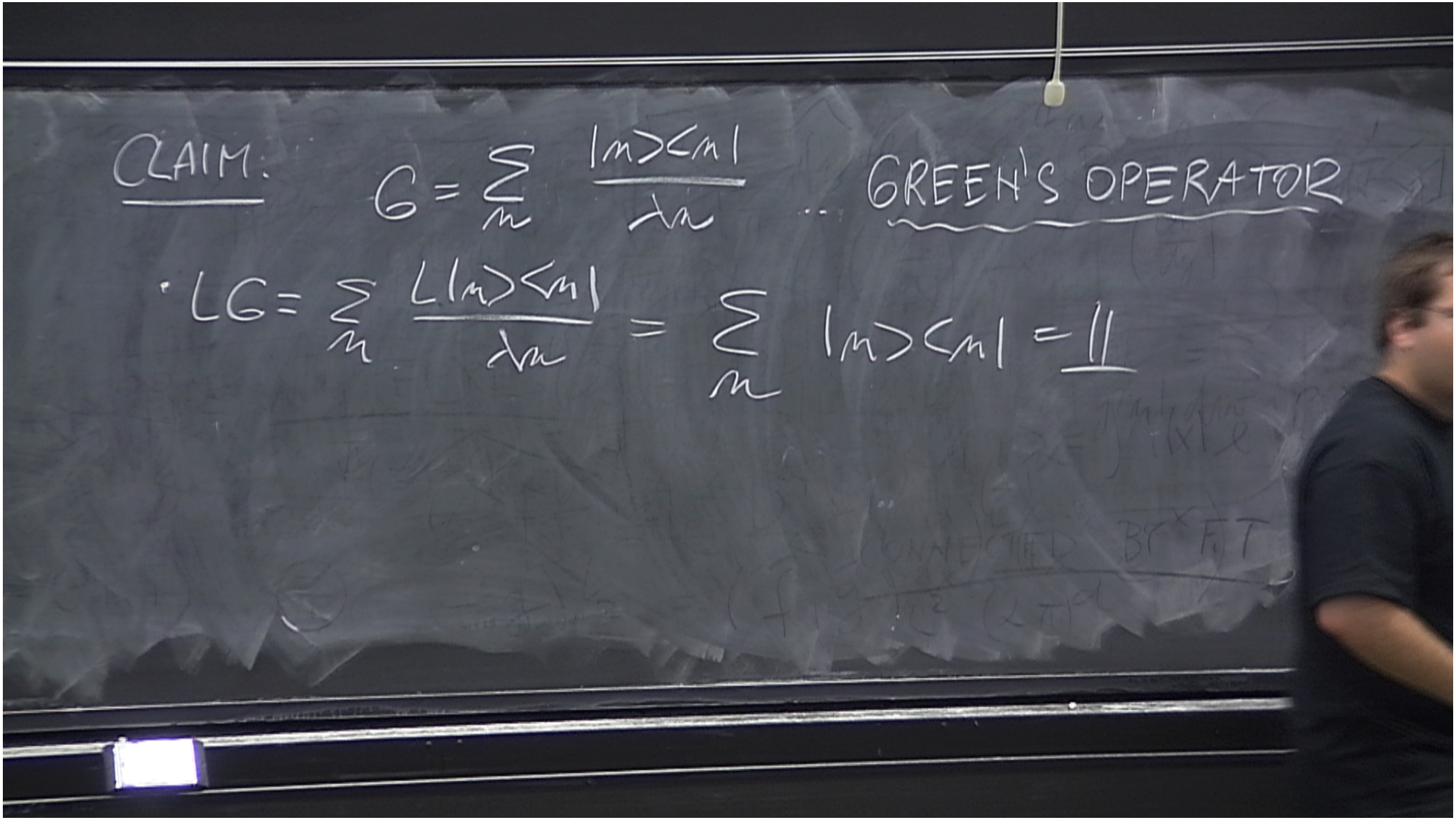

2) CONSIDER THE PROBLEM
L|y> = |f>

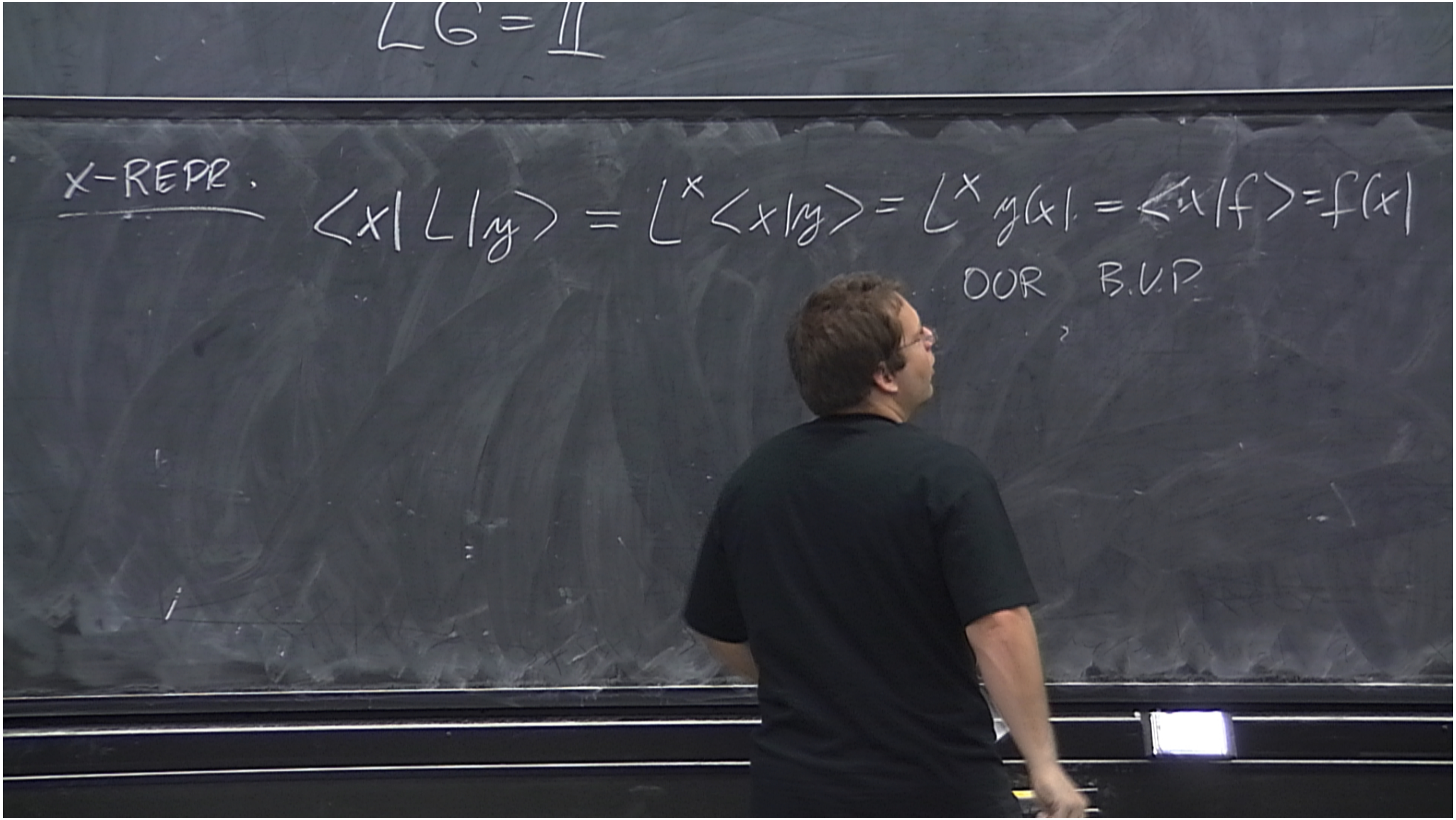

2) CONSIDER THE PROBLEM
$L|y\rangle = |f\rangle$
SOLVE FOR $|y\rangle$

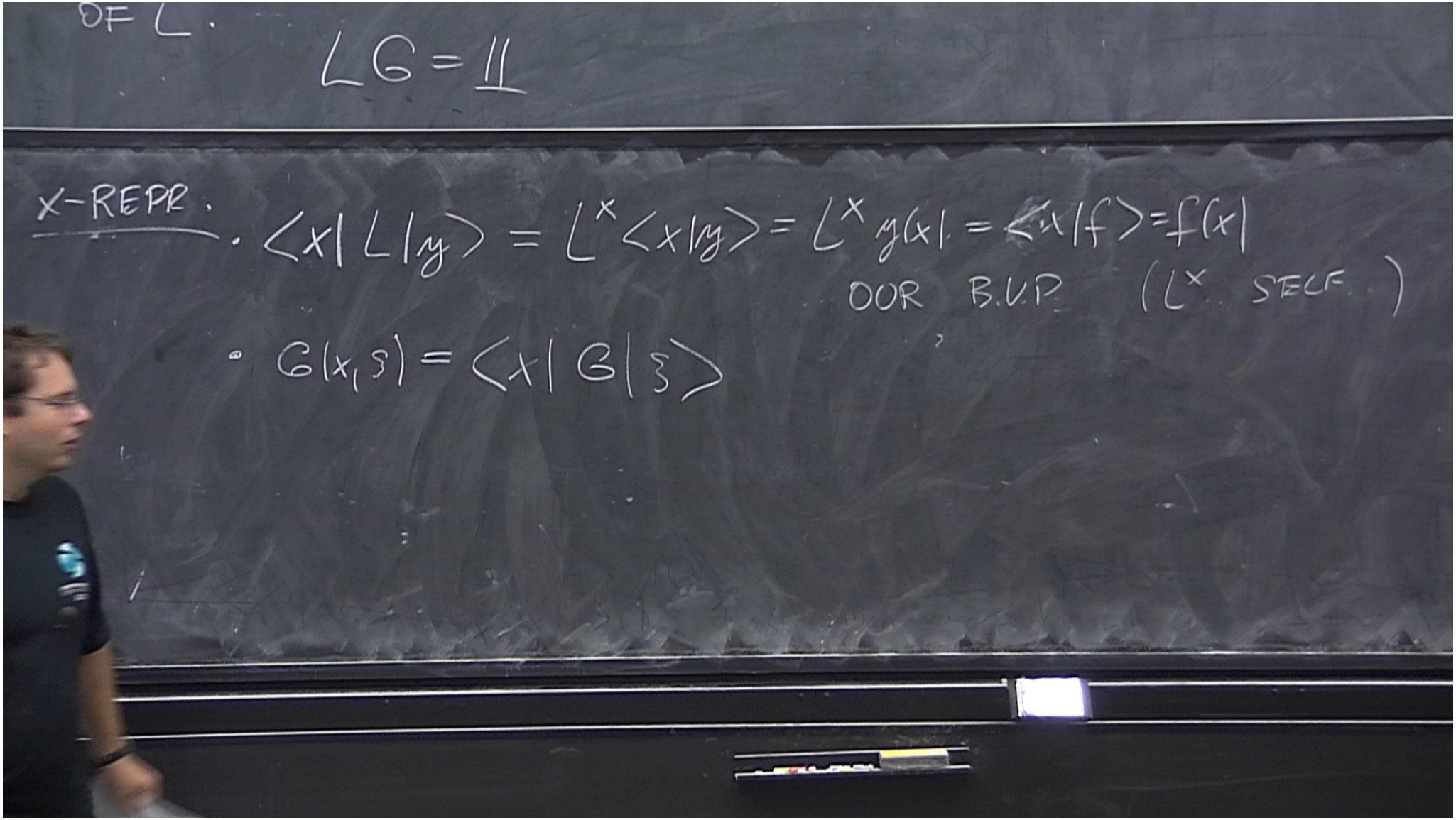

2) CONSIDER THE PROBLEM
$L|y\rangle = |f\rangle$
SOLVE FOR $|y\rangle$ ... NEED AN INVERSION OF L.
$LG = \mathbb{1}$


CLAIM:
$G = \sum_n \frac{|n\rangle\langle n|}{\lambda_n}$
... GREEN'S OPERATOR


2) CONSIDER THE PROBLEM
$L|y\rangle = |f\rangle$
SOLVE FOR $|y\rangle$ .. NEE INVERSION
OF L.
$LG = \mathbb{1}$
CLAIM $G = \sum_n \frac{|n\rangle\langle n|}{\lambda_n}$ .. GREEN'S OPERATOR
$LG = \sum_n \frac{L|n\rangle\langle n|}{\lambda_n}$


CLAIM: $G = \sum_n \frac{|n\rangle\langle n|}{\lambda_n}$ ... GREEN'S OPERATOR
$LG = \sum_n \frac{L|n\rangle\langle n|}{\lambda_n} = \sum_n |n\rangle\langle n| = \mathbb{1}$


LG = II
X-REPR.
<x| L |y> = L^x <x|y> = L^x y(x) = <x|f> = f(x)
OOR B.U.P


OF L.
$LG = \mathbb{1}$
X-REPR.
$\langle x|L|y\rangle = L^x\langle x|y\rangle = L^x y(x) = \langle x|f\rangle = f(x)$
OOR B.U.P (L^x SELF.)
$G(x,s) = \langle x|G|s\rangle$

OF L.

$LG = \mathbb{1}$

X-REPR.

- $\langle x|L|y\rangle = L^{*}\langle x|y\rangle = L^{*}\, y(x) = \langle x|f\rangle = f(x)$

  OOR B.V.P (L^{*} SELF.)

- $G(x,s) = \langle x|G|s\rangle = \sum_{n} \frac{\varphi_n(x)\,\varphi_n^{*}(s)}{\lambda_n}$

2) CONSIDER THE PROBLEM

$$L|y\rangle = |f\rangle$$

SOLVE FOR $|y\rangle$ ... NEED AN INVERSION OF L.

$$LG = \mathbb{1}$$

CLAIM. $G = \sum_n \frac{|n\rangle\langle n|}{\lambda_n}$ .. GREEN'S OPERATOR

$$\cdot LG = \sum_n \frac{L|n\rangle\langle n|}{\lambda_n} = \sum_n |n\rangle\langle n| = \mathbb{1}$$

X-REPR.

$$\cdot \langle x|L|y\rangle = L^x\langle x|y\rangle = L^x y(x) = \langle x|f\rangle = f(x)$$

OOR B.V.P (L^x SELF.)

$$\cdot G(x,s) = \langle x|G|s\rangle = \sum_n \frac{\varphi_n(x)\,\varphi_n^*(s)}{\lambda_n}$$ .. EIGENFUNCTION EXPANSION OF GREEN'S FUNCTION

$$\langle x|LG|s\rangle$$

OF L.

$$LG = \mathbb{1}$$

X-REPR.

- $\langle x|L|y\rangle = L^x\langle x|y\rangle = L^x\, y(x) = \langle x|f\rangle = f(x)$

  OOR B.U.P. (L^x ITSELF...)

- $G(x,s) = \langle x|G|s\rangle = \sum_n \frac{\varphi_n(x)\,\varphi_n^*(s)}{\lambda_n}$ ... EIGENFUN

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... EIGENFUNCTION EXPANSION OF GREEN'S FUNCTION

$$\langle x|LG|s\rangle = L^x\langle x|G|s\rangle = L^x G(x,s) = \langle x|\mathbb{1}|s\rangle = \delta(x-s)$$