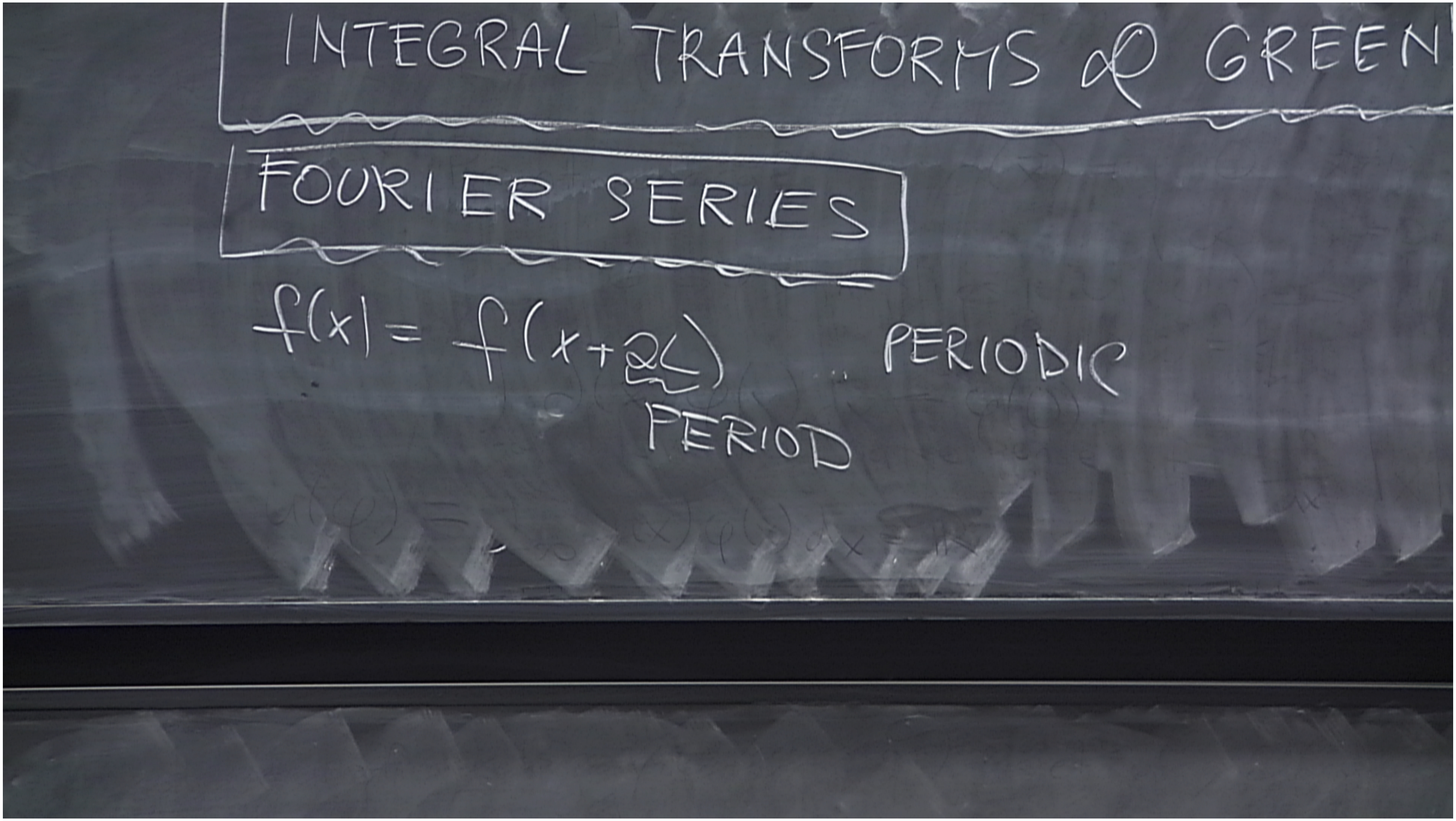
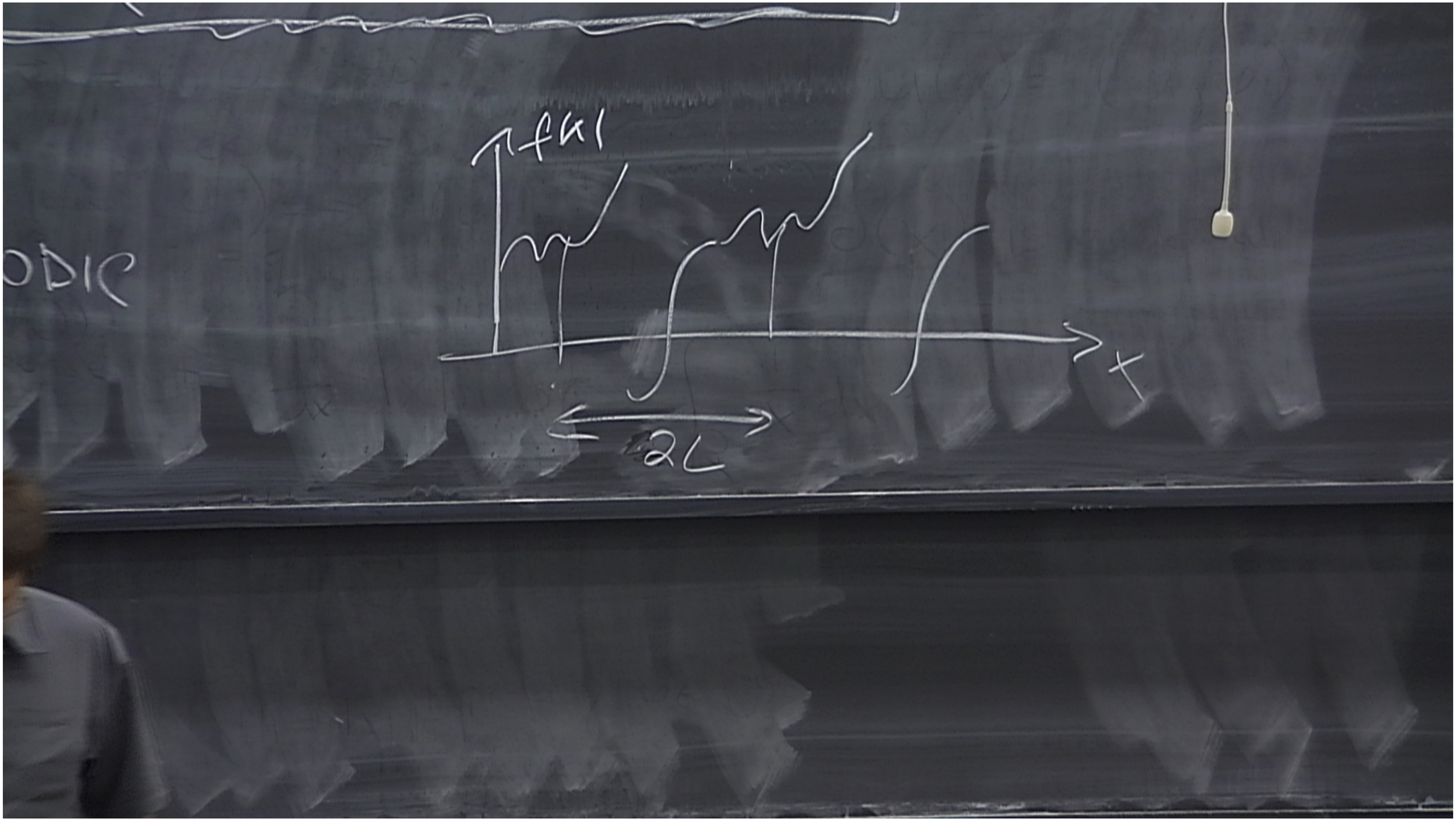
Title: Integral Transforms and Green's Functions - Lecture 1

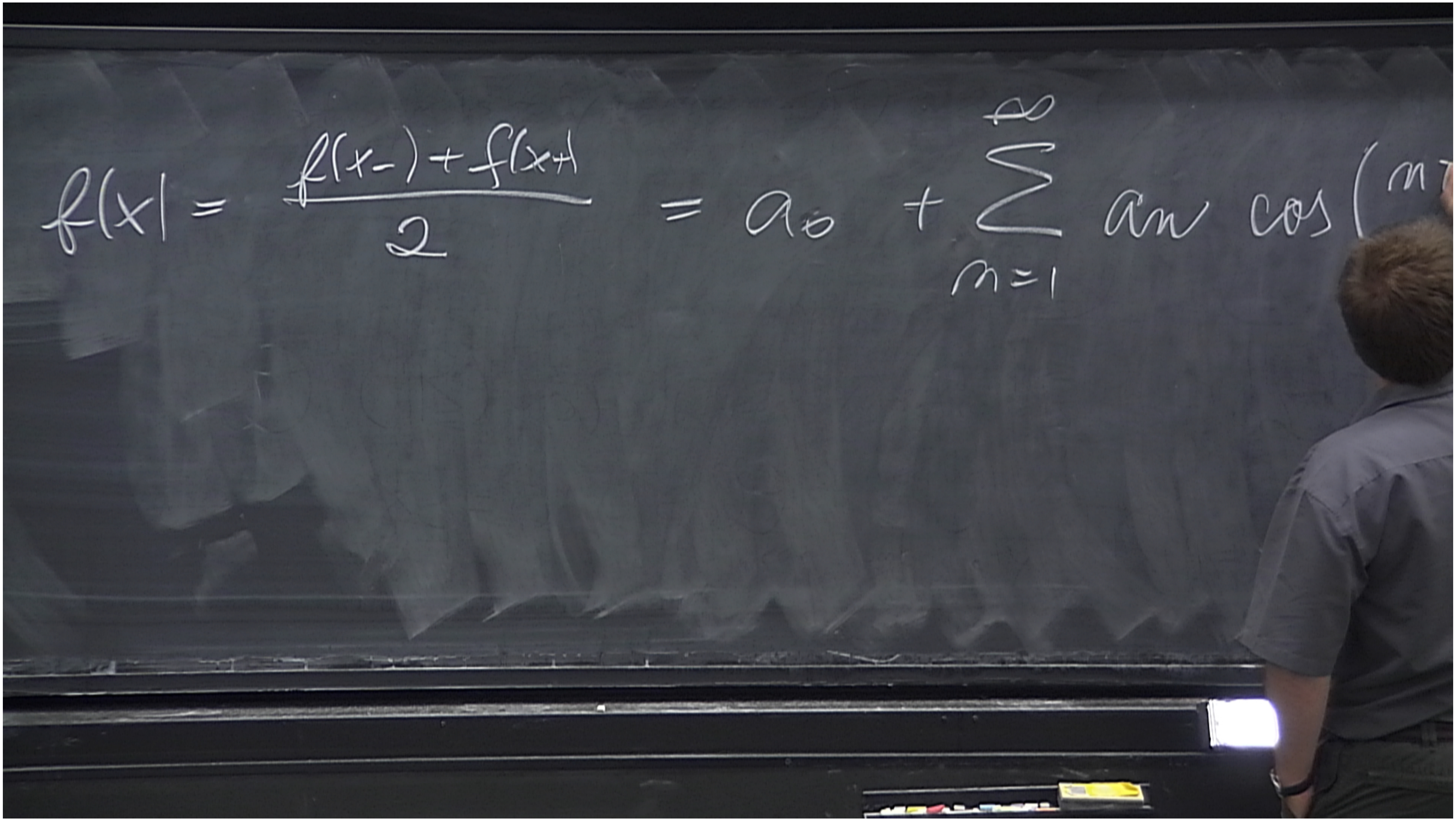
Date: Sep 06, 2011 10:30 AM

URL: http://pirsa.pi.local/11090048

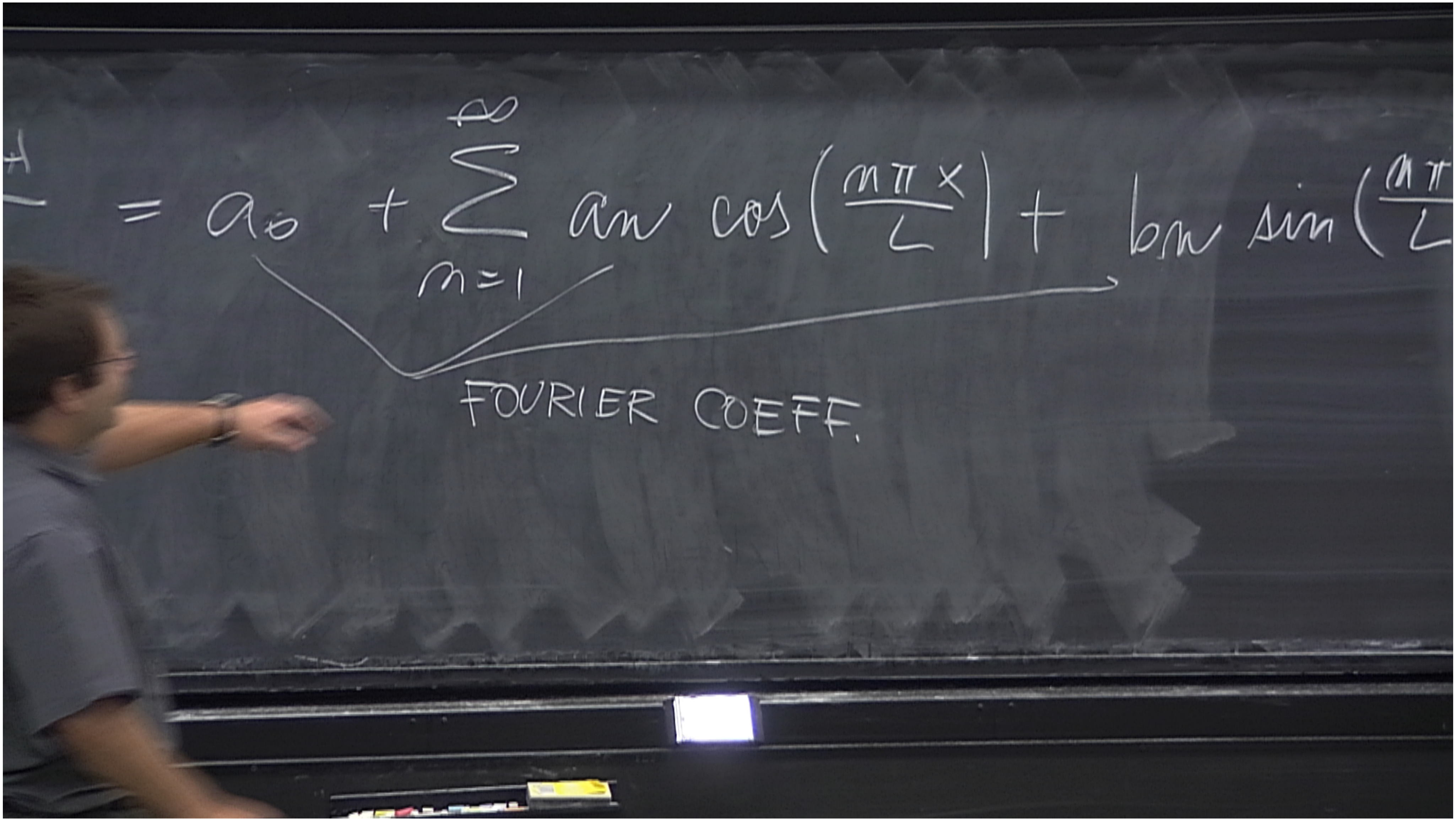
Abstract:

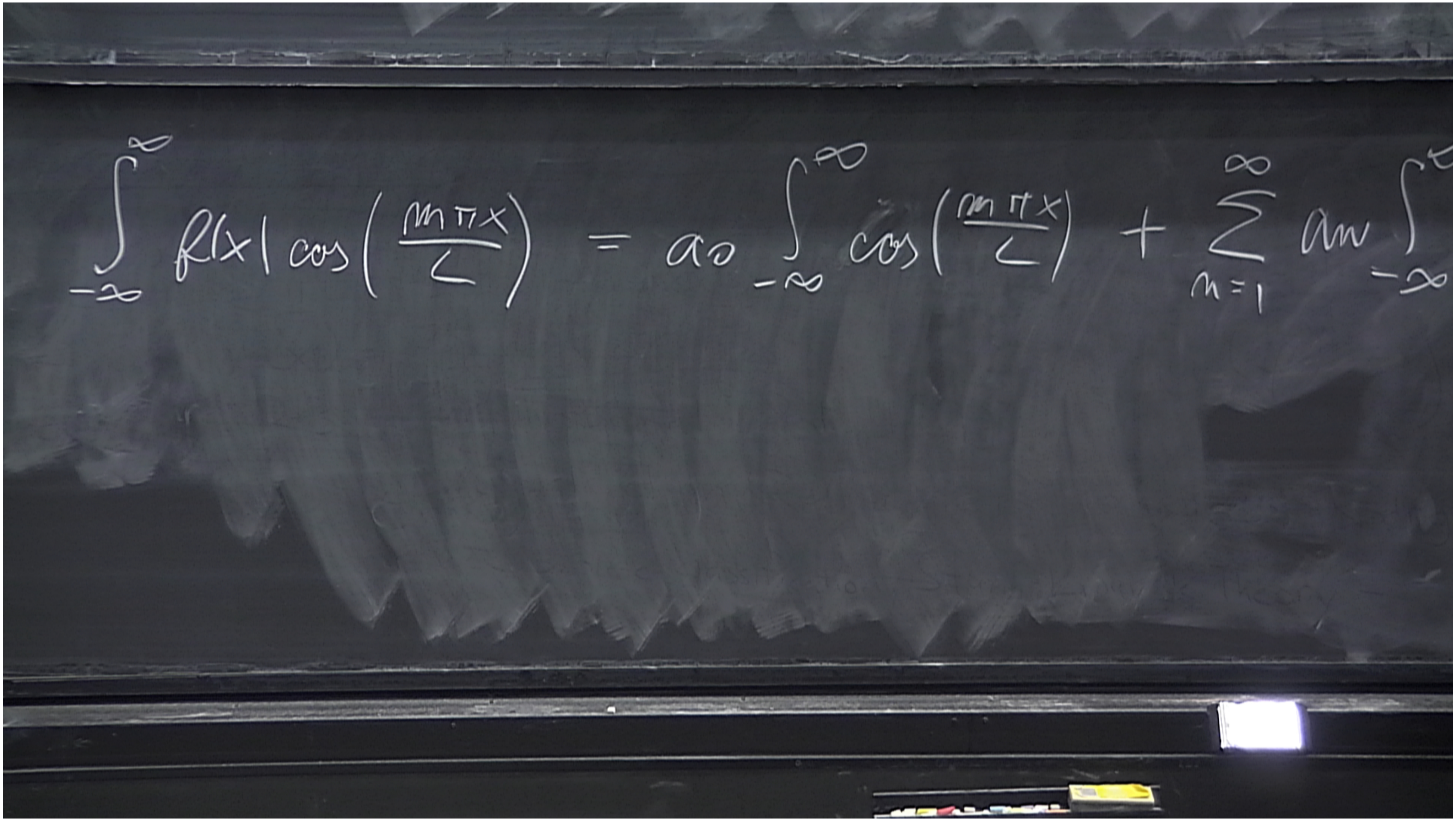

INTEGRAL TRANSFORMS & GREEN
FOURIER SERIES
$f(x) = f(x+2L)$ PERIODIC
PERIOD

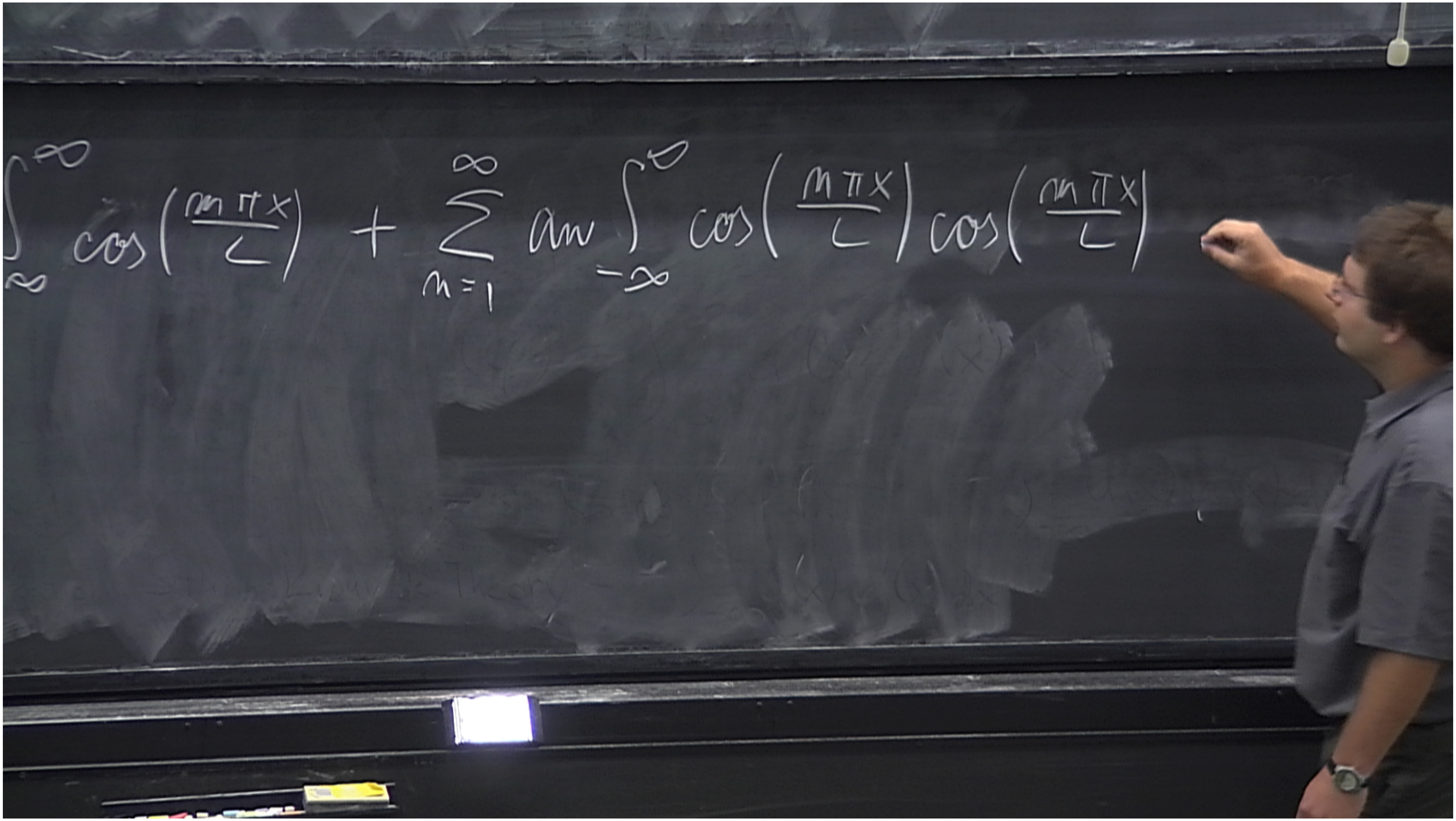

ODIC
f(x)
2L
x

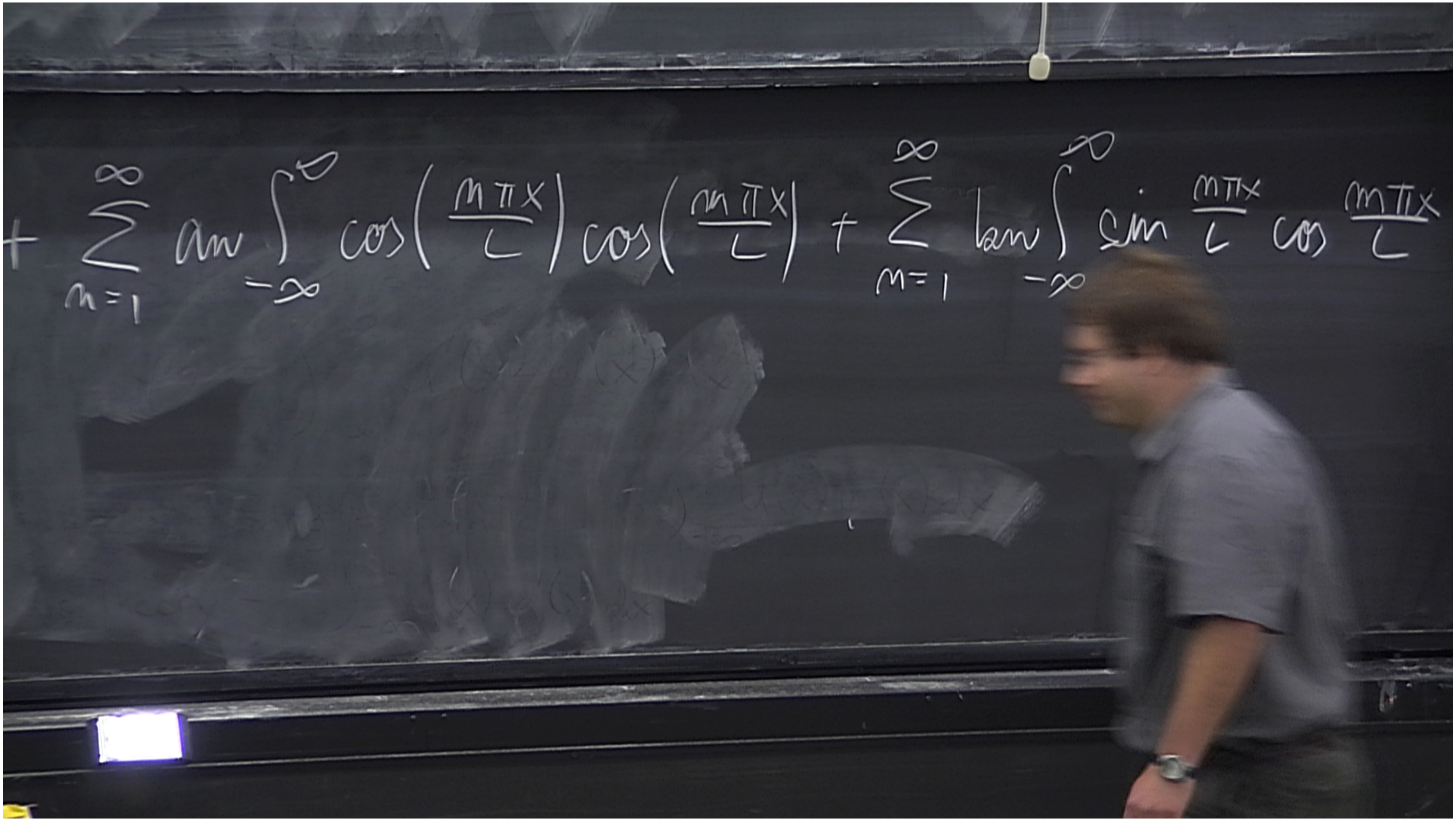

$f(x) = \frac{f(x-) + f(x+)}{2} = a_0 + \sum_{n=1}^{\infty} a_n \cos(n$

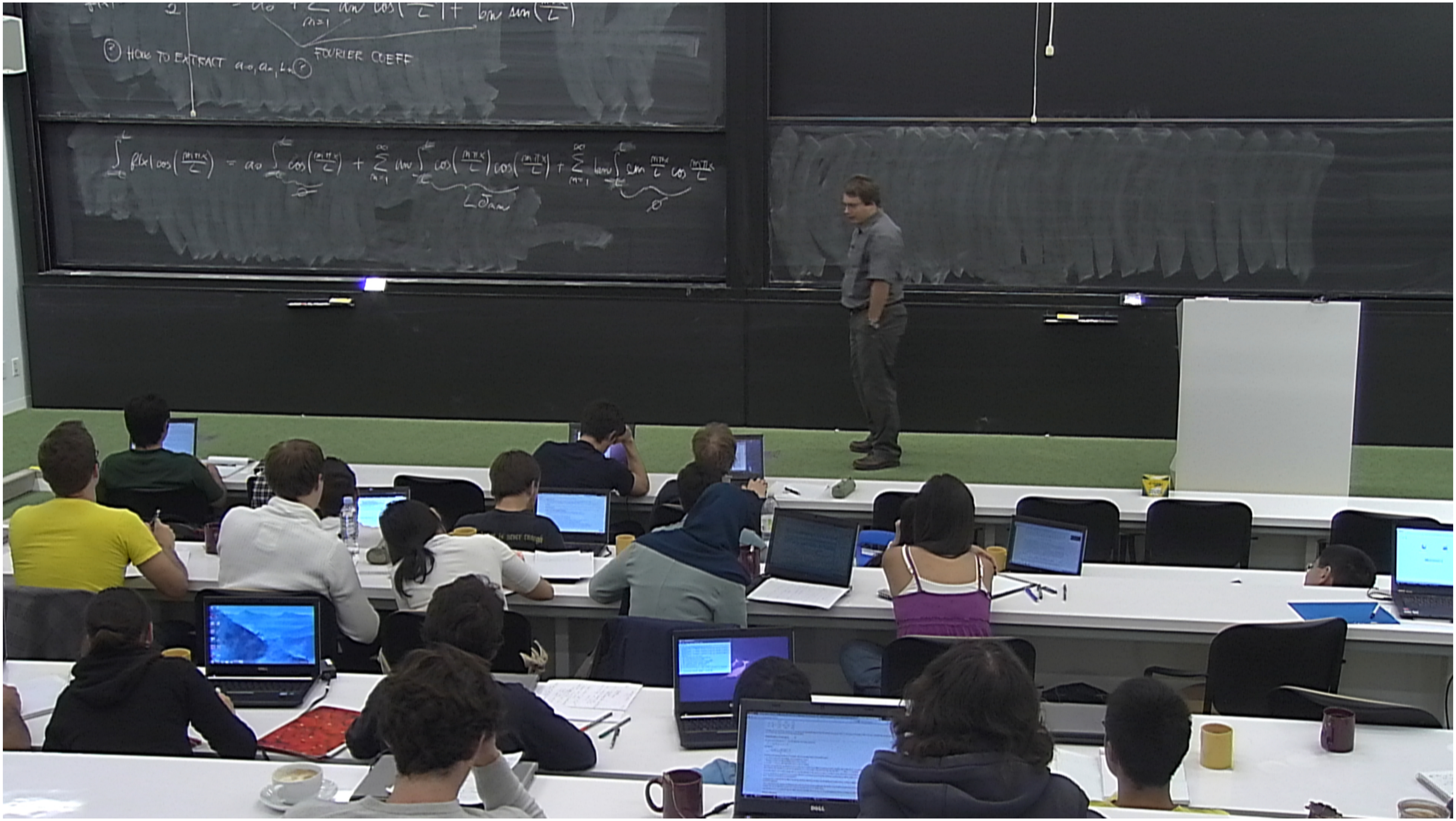
$$+\sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$


$= a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi}{L}\right.$
FOURIER COEFF.


$\int_{-\infty}^{\infty} f(x) \cos\left(\frac{m\pi x}{L}\right) = a_0 \int_{-\infty}^{\infty} \cos\left(\frac{m\pi x}{L}\right) + \sum_{n=1}^{\infty} a_n \int_{-\infty}^{\infty}$

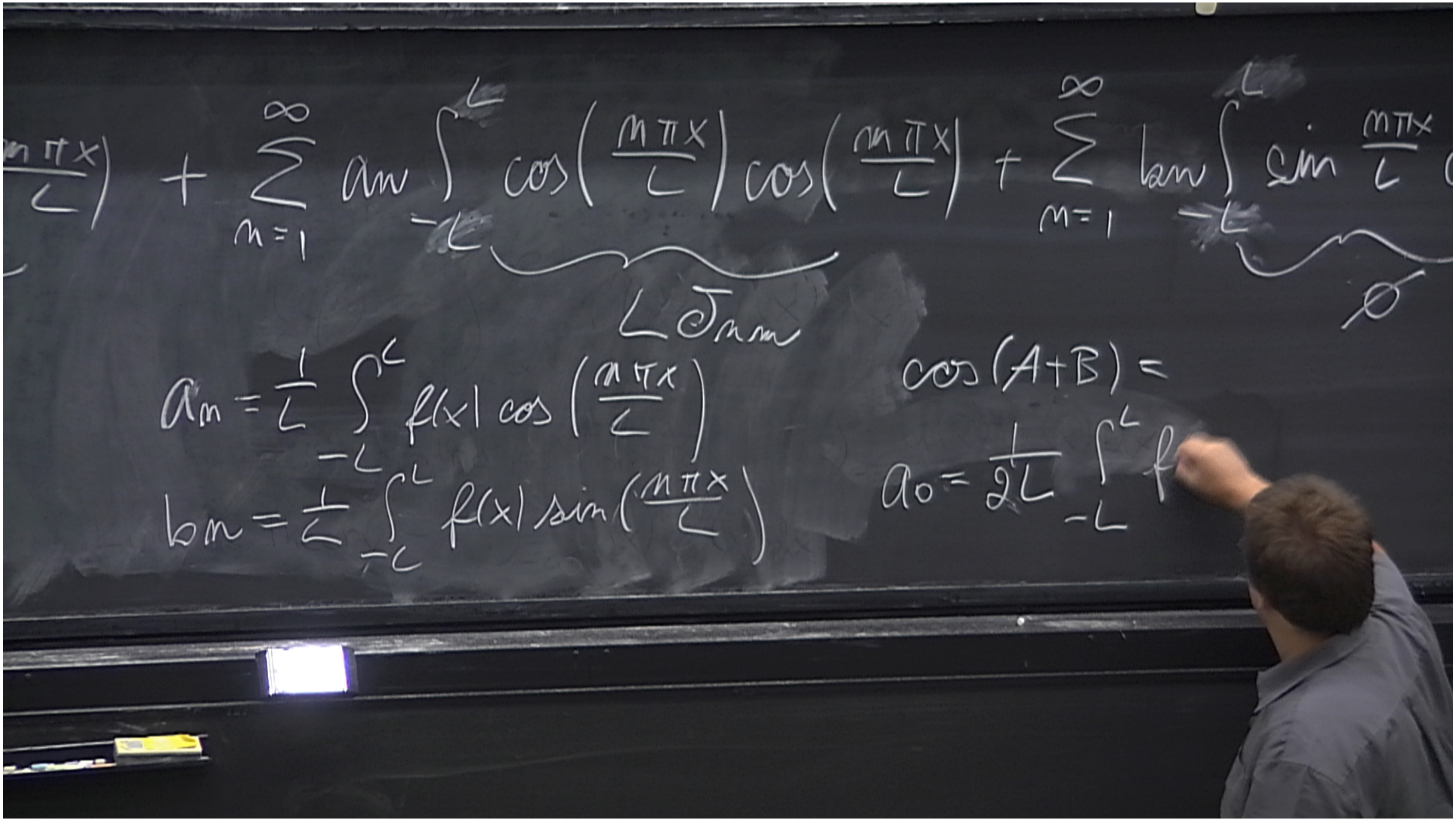

$\int_{-\infty}^{\infty} \cos\left(\frac{m\pi x}{L}\right) + \sum_{n=1}^{\infty} a_n \int_{-\infty}^{\infty} \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right)$


$+\sum_{n=1}^{\infty} a_n \int_{-\infty}^{\infty} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \int_{-\infty}^{\infty} \sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}$


HOW TO EXTRACT a_0, a_m, b_m ?
FOURIER COEFF

$$\ldots m \underbrace{\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)}_{L\delta_{nm}} + \sum_{m=1}^{\infty} b_m \underbrace{\int_{-L}^{L} \sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}}_{\emptyset}$$

$$\cos(A+B) =$$


$\frac{n\pi x}{L}) + \sum_{n=1}^{\infty} a_n \underbrace{\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)}_{L\delta_{nm}} + \sum_{n=1}^{\infty} b_n \underbrace{\int_{-L}^{L} \sin \frac{n\pi x}{L}}_{0}$
$a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right)$
$b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)$
$\cos(A+B) =$
$a_0 = \frac{1}{2L}\int_{-L}^{L} f$

$$\ldots\frac{m\pi x}{L}\Big) + \sum_{n=1}^{\infty} a_n \underbrace{\int_{-L}^{L} \cos\Big(\frac{m\pi x}{L}\Big)\cos\Big(\frac{n\pi x}{L}\Big)}_{L\,\delta_{nm}} + \sum_{n=1}^{\infty} b_n \underbrace{\int_{-L}^{L} \sin\frac{n\pi x}{L}\,(\ldots}_{\emptyset}$$

$$\cos(A+B) =$$

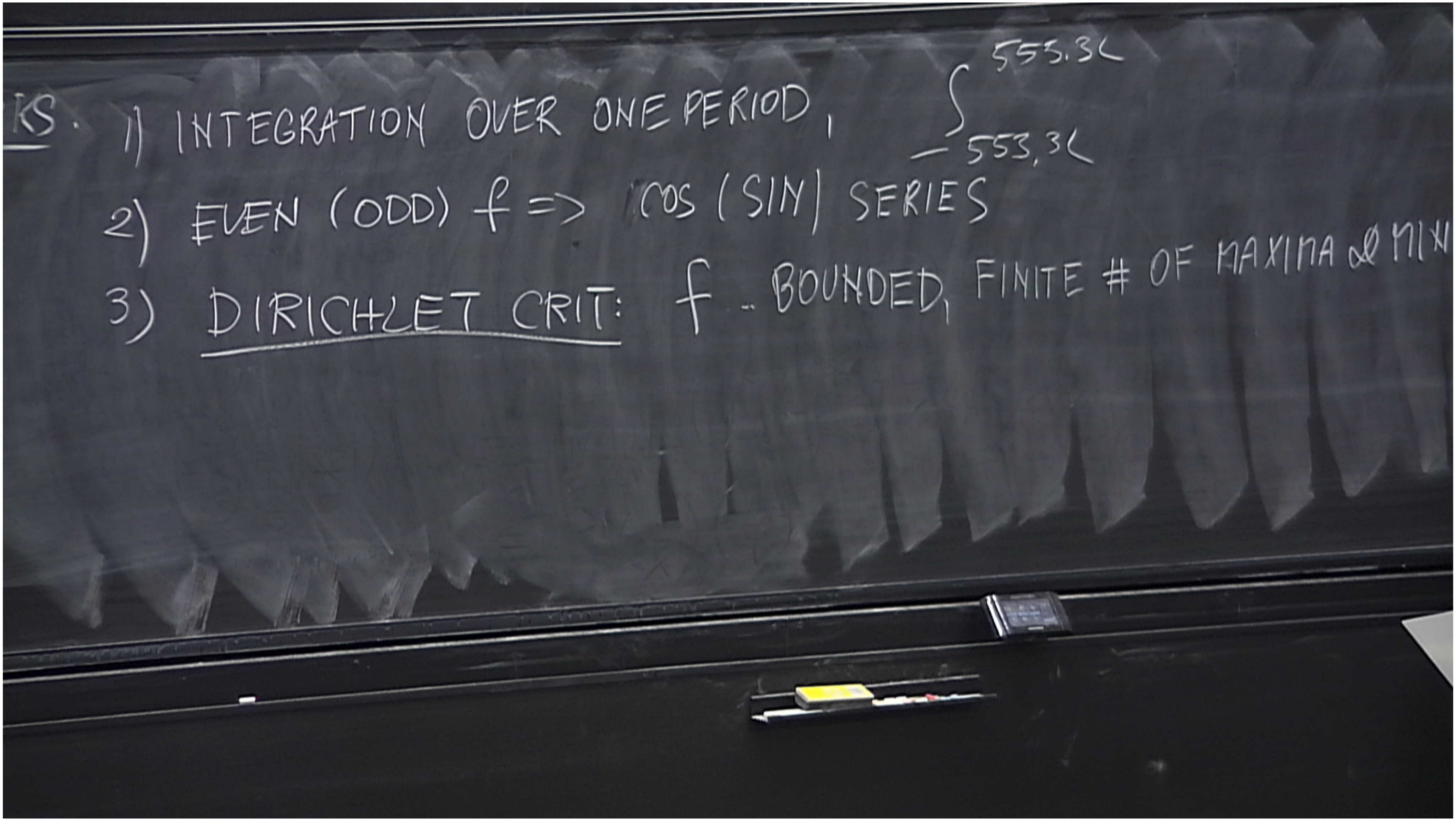
$$\boxed{\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\Big(\frac{n\pi x}{L}\Big) \qquad a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\Big(\frac{n\pi x}{L}\Big) \end{aligned}}$$

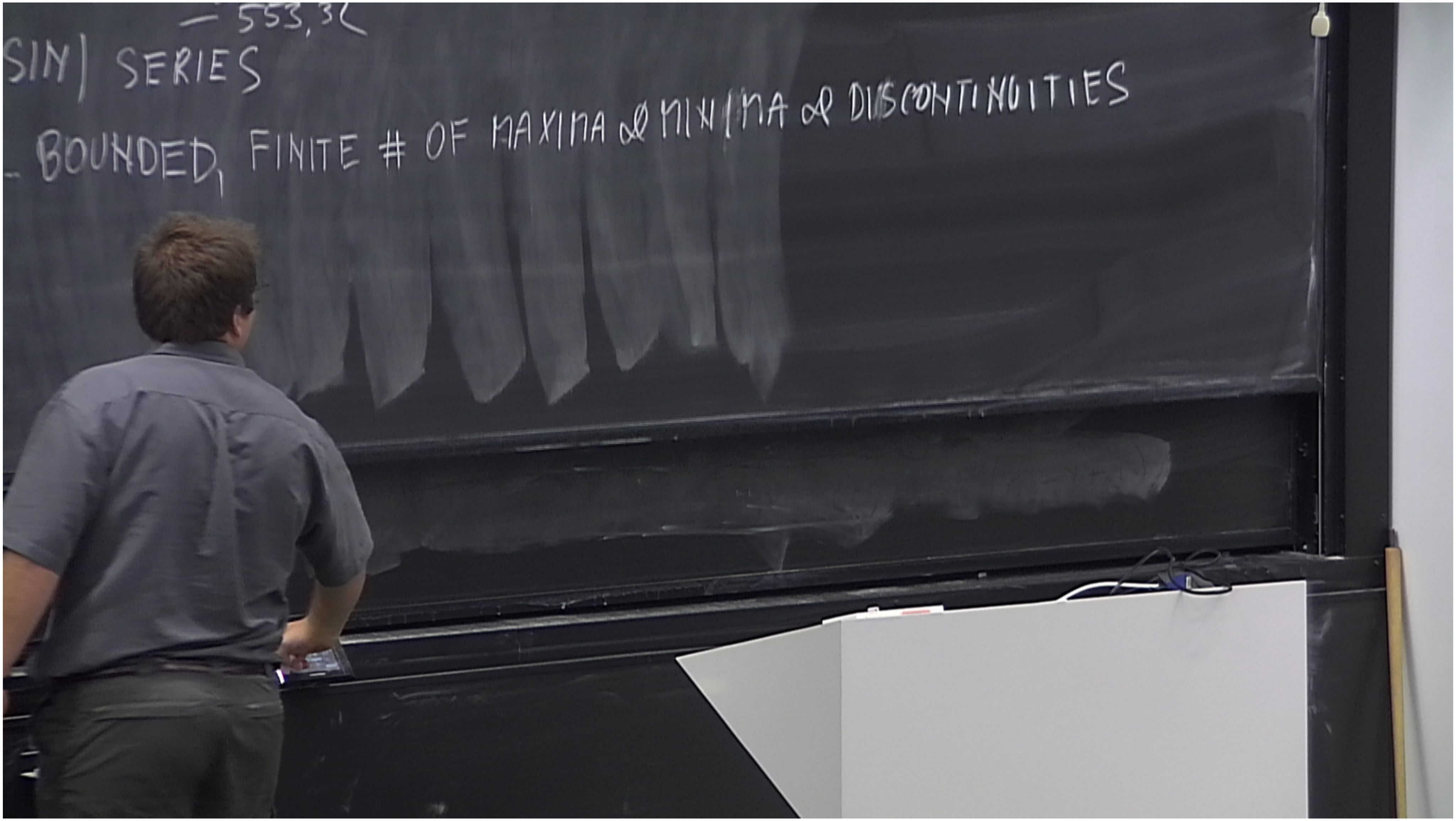
REMARKS. 1) INTEGRATION OVER ONE PERIOD, $\int_{-553.3L}^{553.3L}$

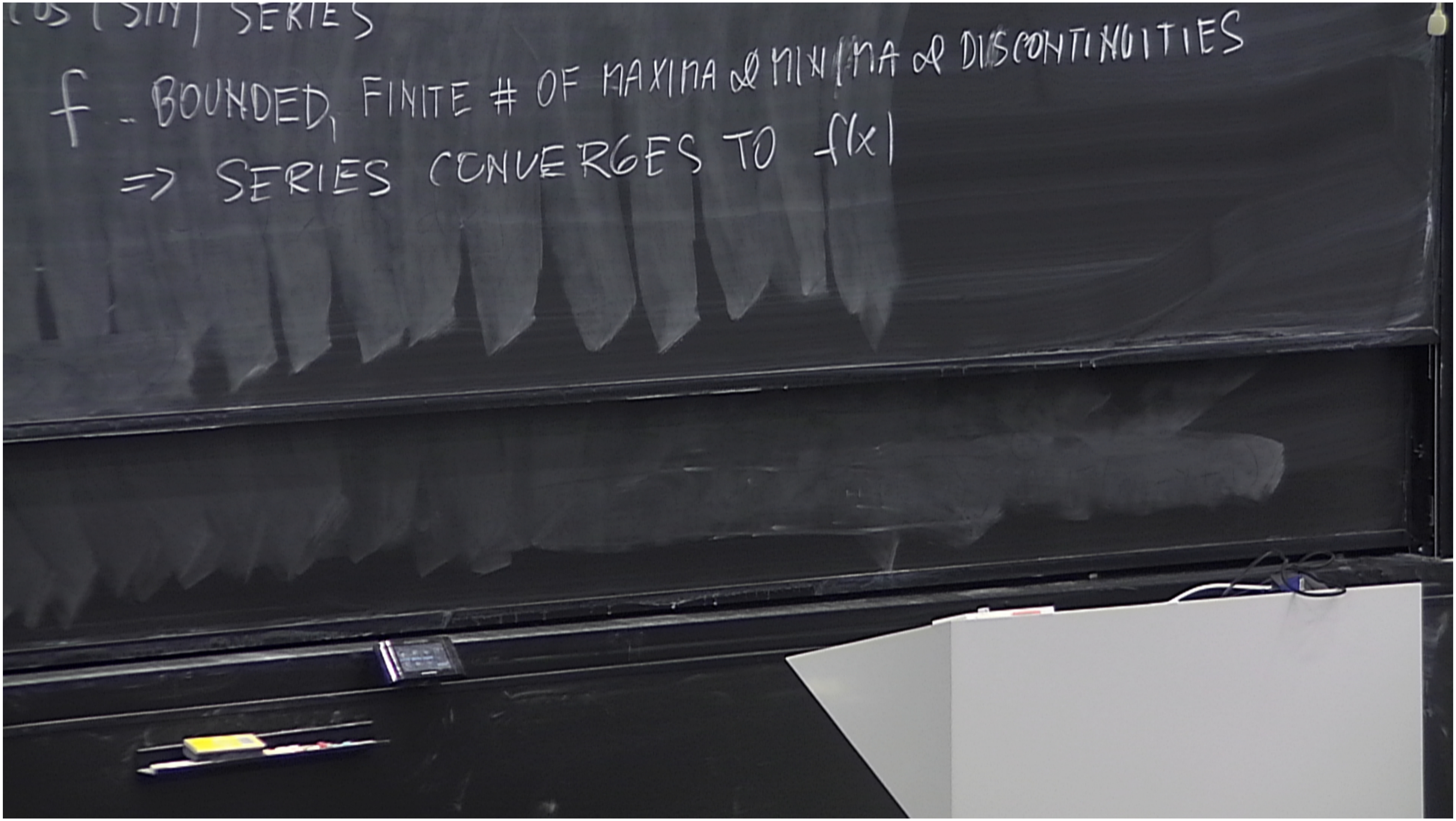
REMARKS. 1) INTEGRATION OVER ONE PERIOD, $\int_{-553.3L}^{553.3L}$

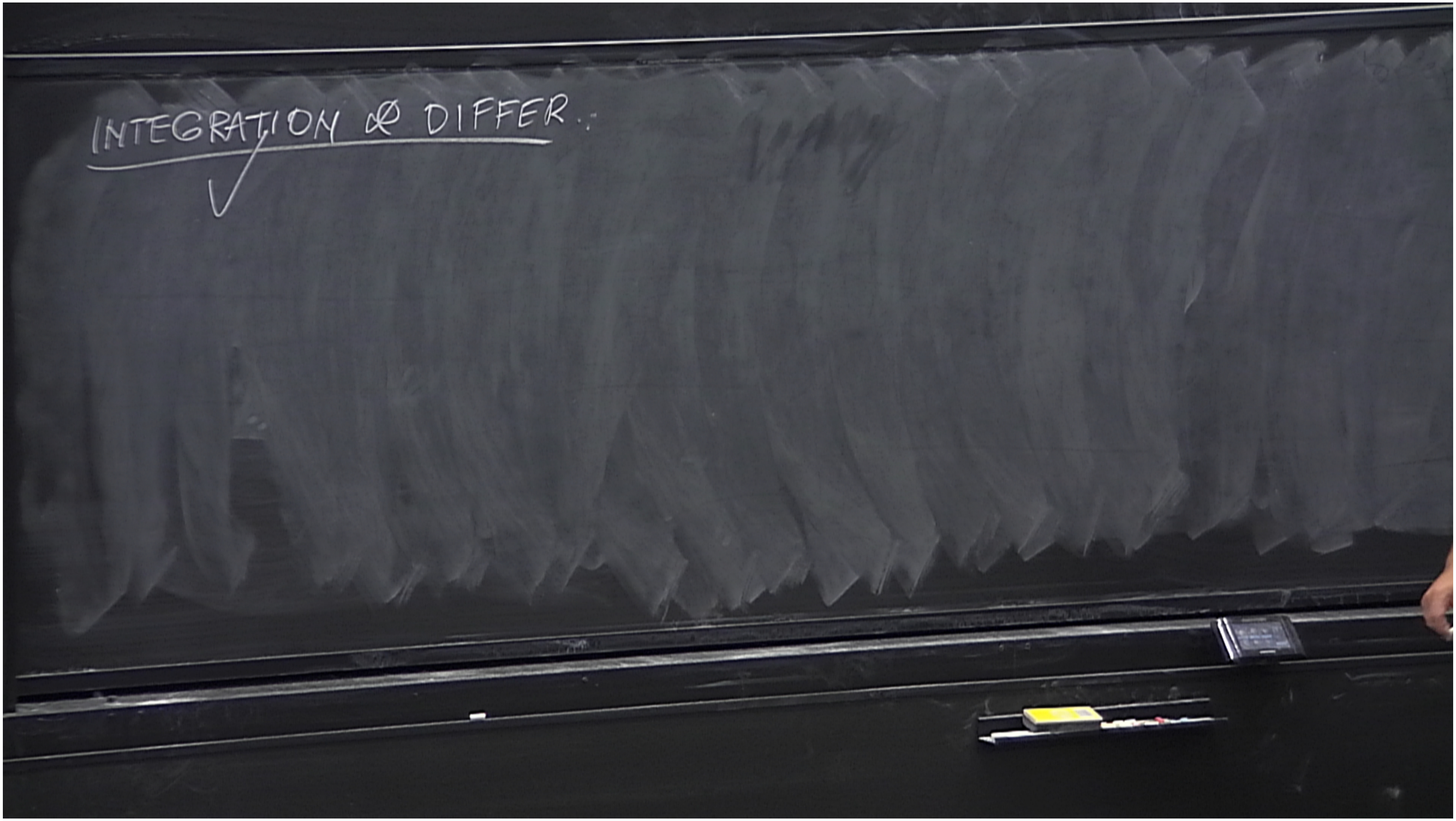
2) EVEN (ODD) $f \Rightarrow$ COS (SIN) SERIES

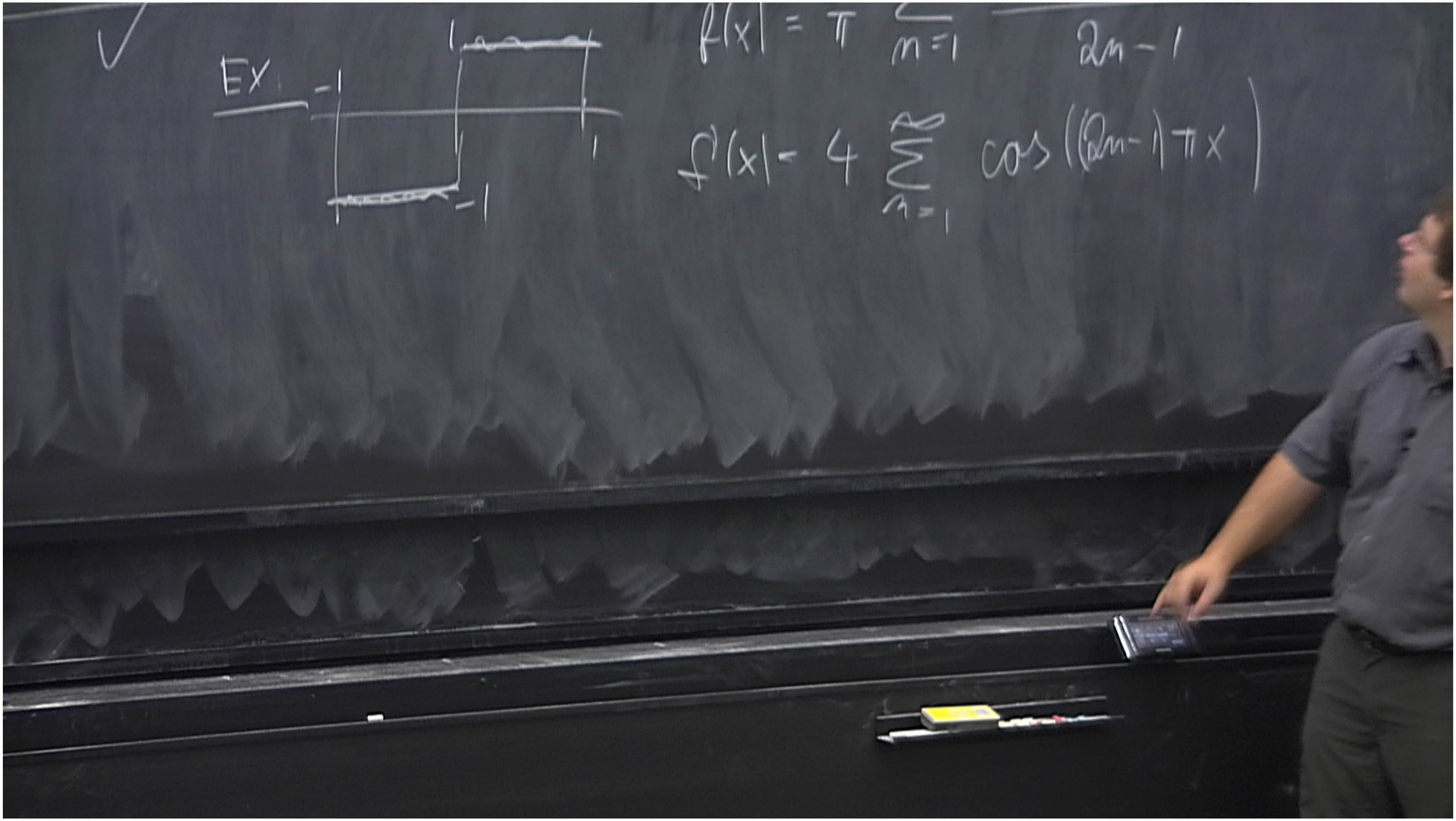
3)

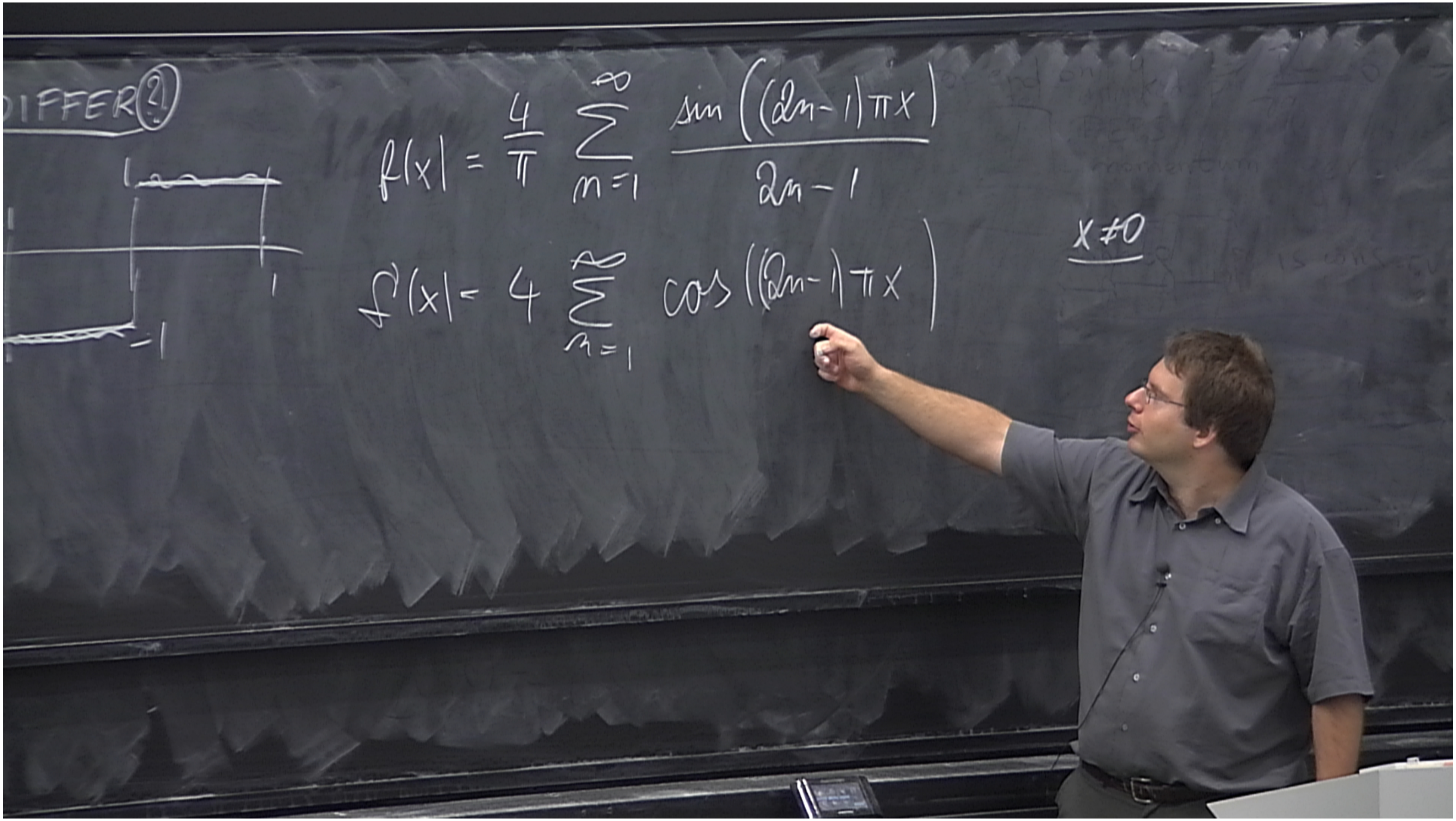

KS.
1) INTEGRATION OVER ONE PERIOD,
553.3L
-553.3L
2) EVEN (ODD) f => COS (SIN) SERIES
3) DIRICHLET CRIT: f .. BOUNDED, FINITE # OF MAXIMA & MIN


= 553,3(
SIN) SERIES
BOUNDED, FINITE # OF MAXIMA & MINIMA & DISCONTINUITIES

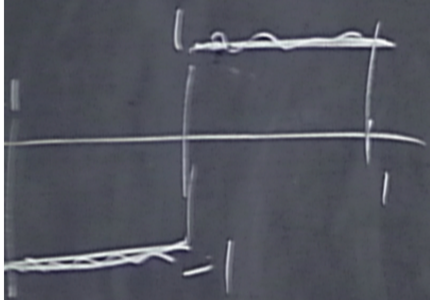

COS (SIN) SERIES
f - BOUNDED, FINITE # OF MAXIMA & MINIMA & DISCONTINUITIES
=> SERIES CONVERGES TO f(x)


INTEGRATION & DIFFER.:


Ex:
f(x) = π
n=1
2n − 1
f'(x) = 4 Σ cos((2n−1)πx)
n=1


DIFFER 2
f(x) = 4/π Σ_{n=1}^{∞} sin((2n-1)πx)/(2n-1)
f'(x) = 4 Σ_{n=1}^{∞} cos((2n-1)πx)
x ≠ 0
1
-1
1

DIFFER (2)

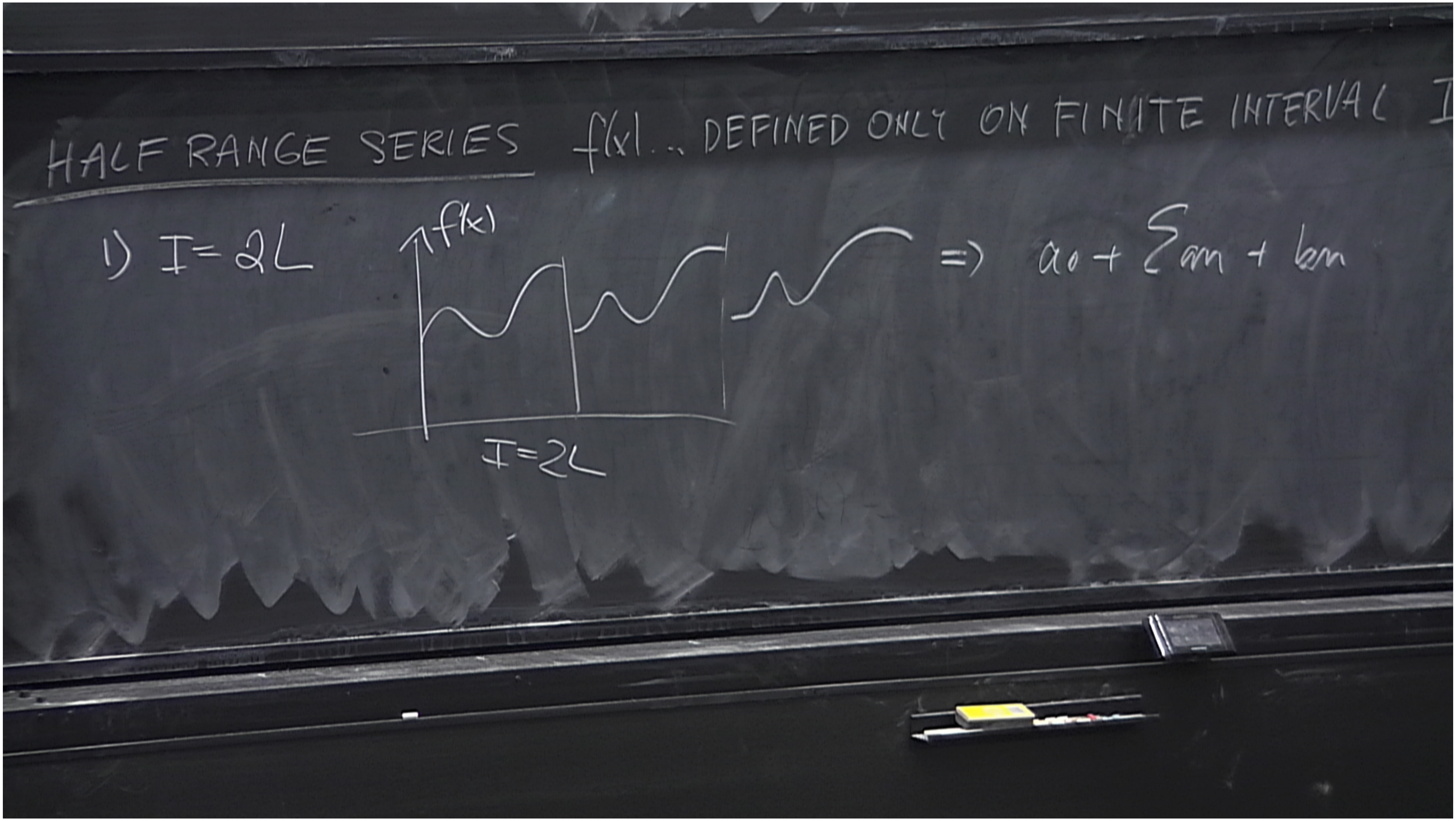


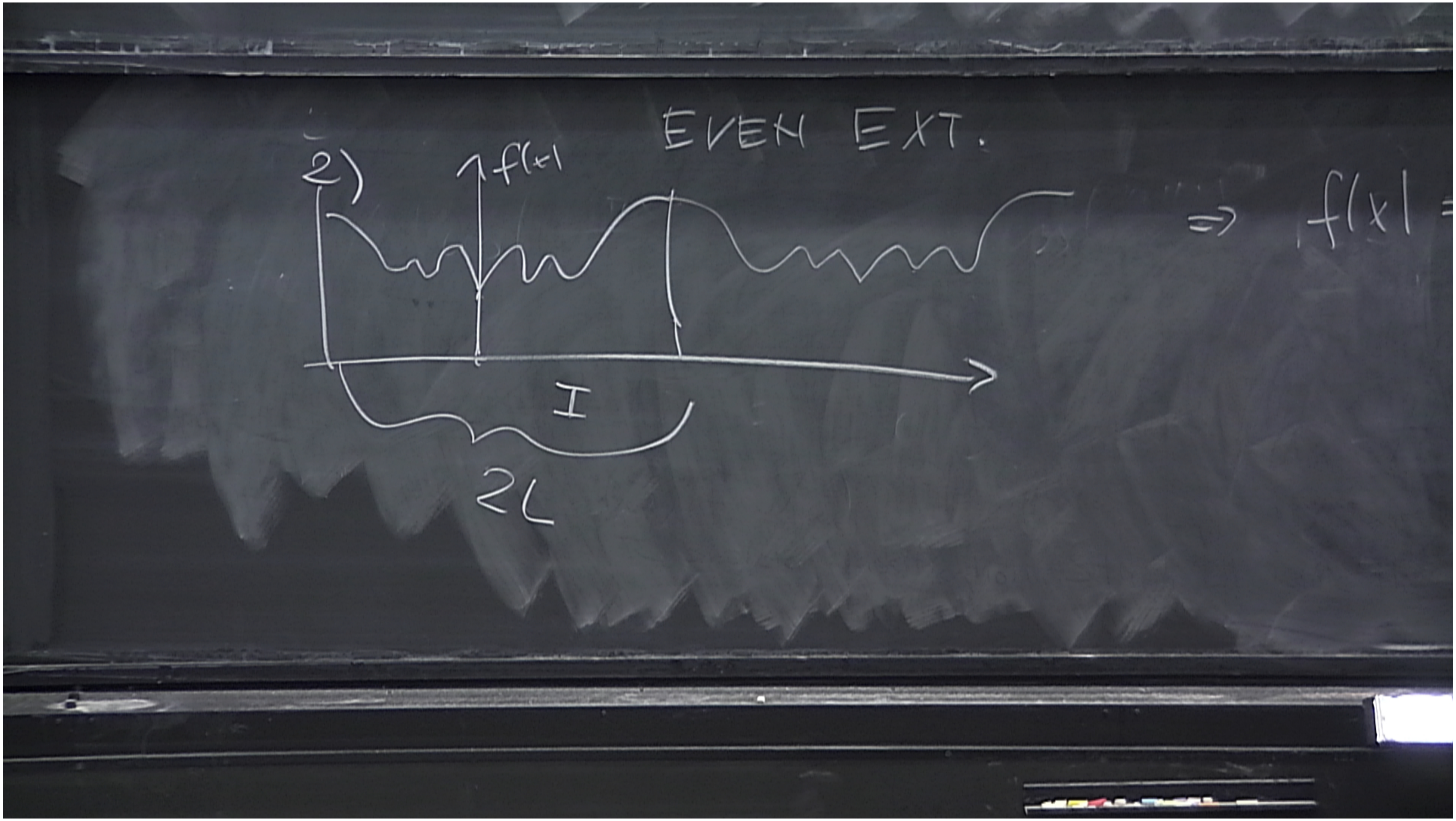
$$f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin\left((2n-1)\pi x\right)}{2n-1}$$

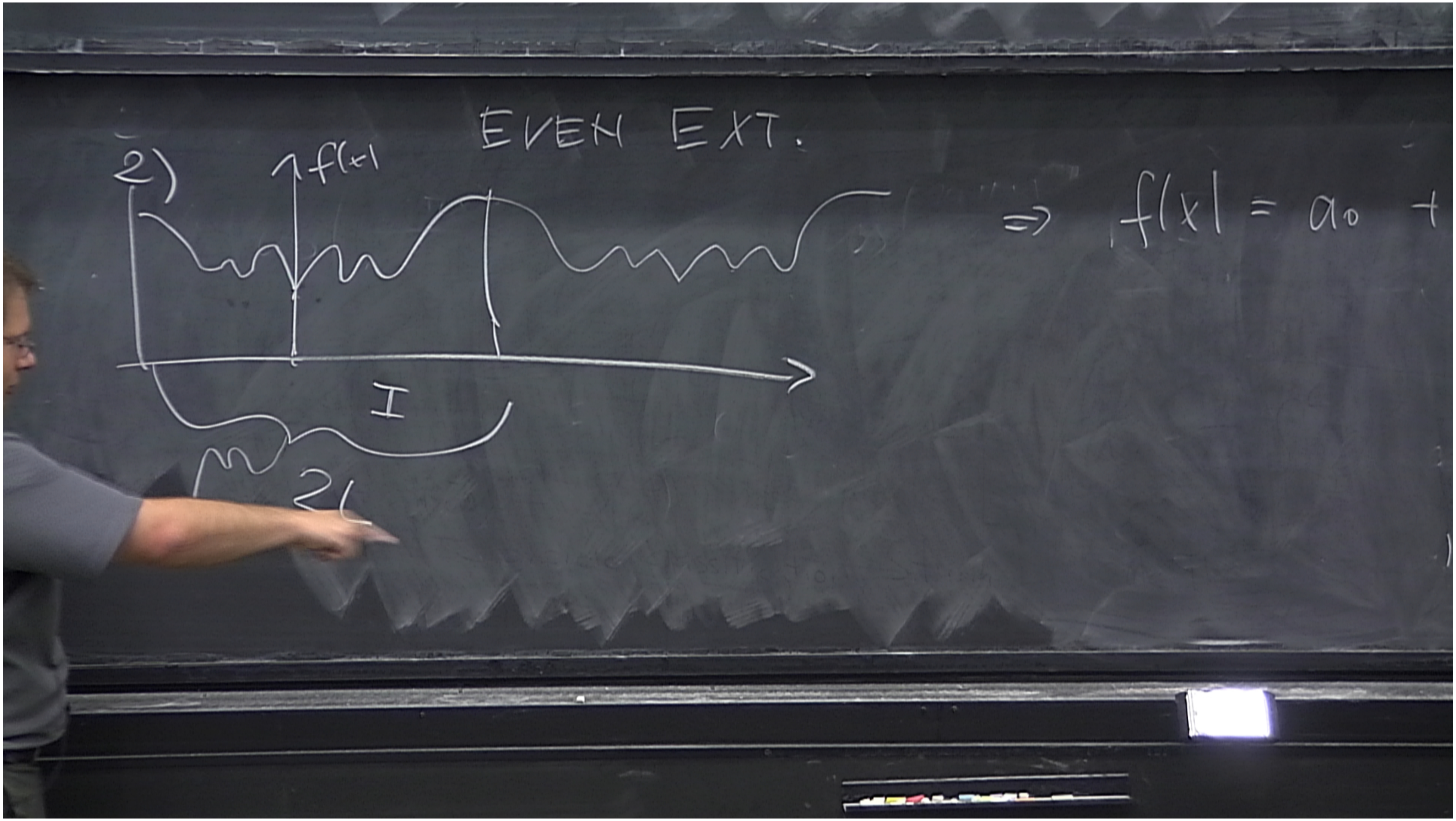
$$f'(x) = 4 \sum_{n=1}^{\infty} \cos\left((2n-1)\pi x\right)$$

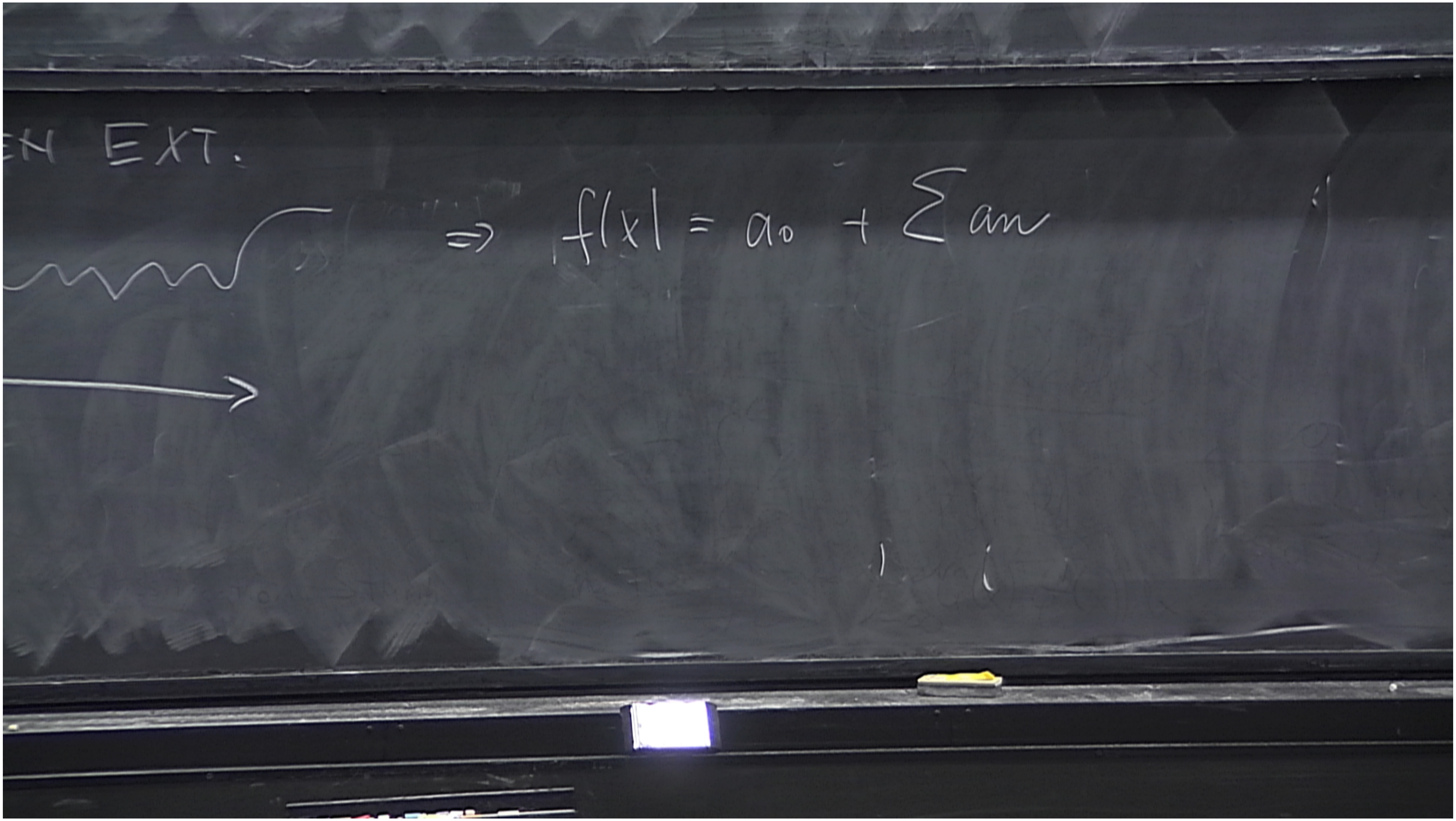
$\underline{x \neq 0}$

$\Sigma \rightarrow$ DIV


HALF RANGE SERIES f(x)... DEFINED ONLY ON FINITE INTERVAL I
1) I = 2L
f(x)
I=2L
=> a0 + Σ an + bn


EVEN EXT.
2)
f(x)
I
2L
⇒ f(x) =


EVEN EXT.
2)
f(x)
I
2l
$\Rightarrow f(x) = a_0 +$


EXT.
$\Rightarrow f(x) = a_0 + \sum a_n$

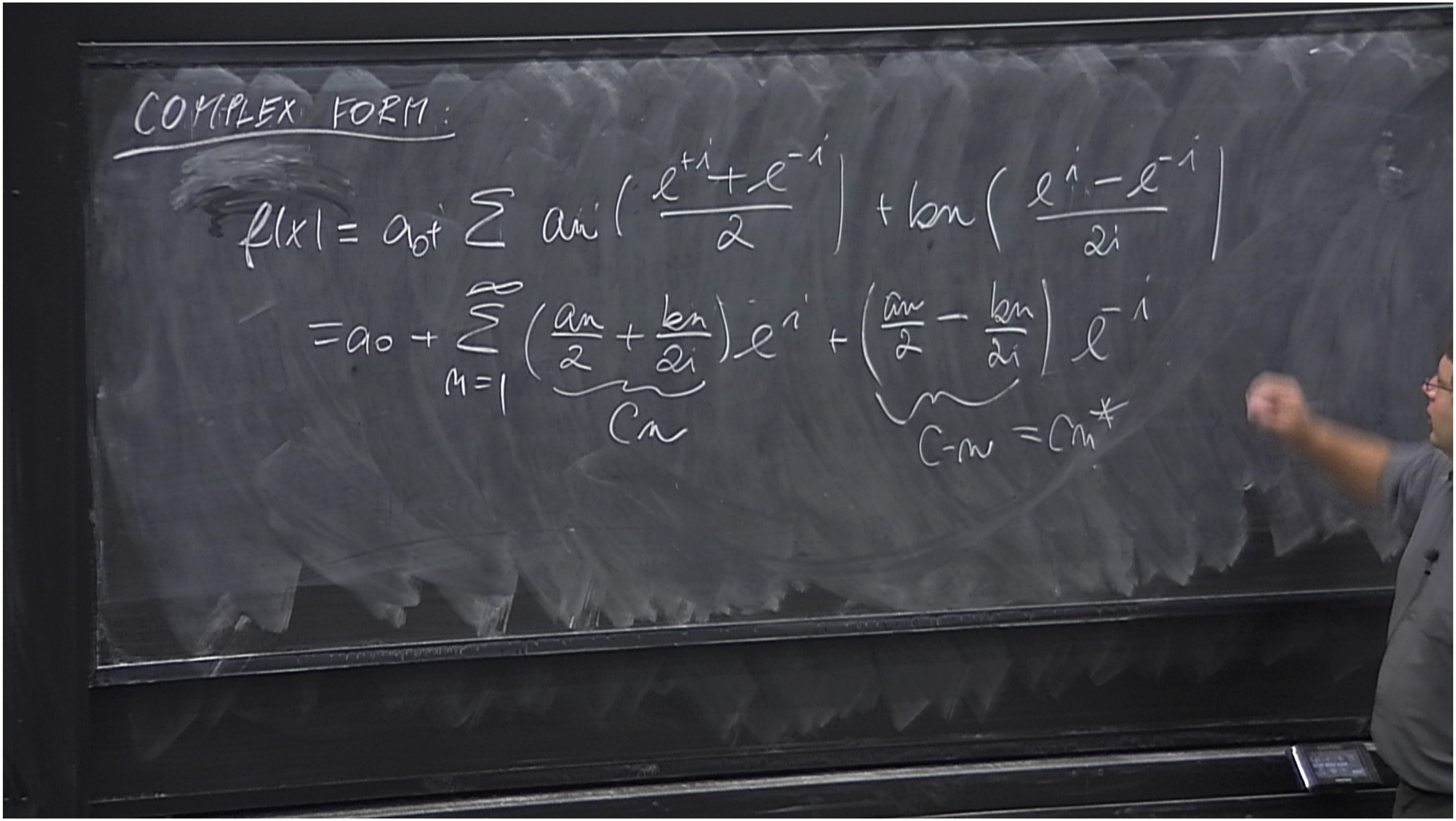
$$\cos\left(\frac{n\pi x}{L}\right) = \frac{e^{i\frac{n\pi x}{L}} + e^{-in\pi x/L}}{2}$$

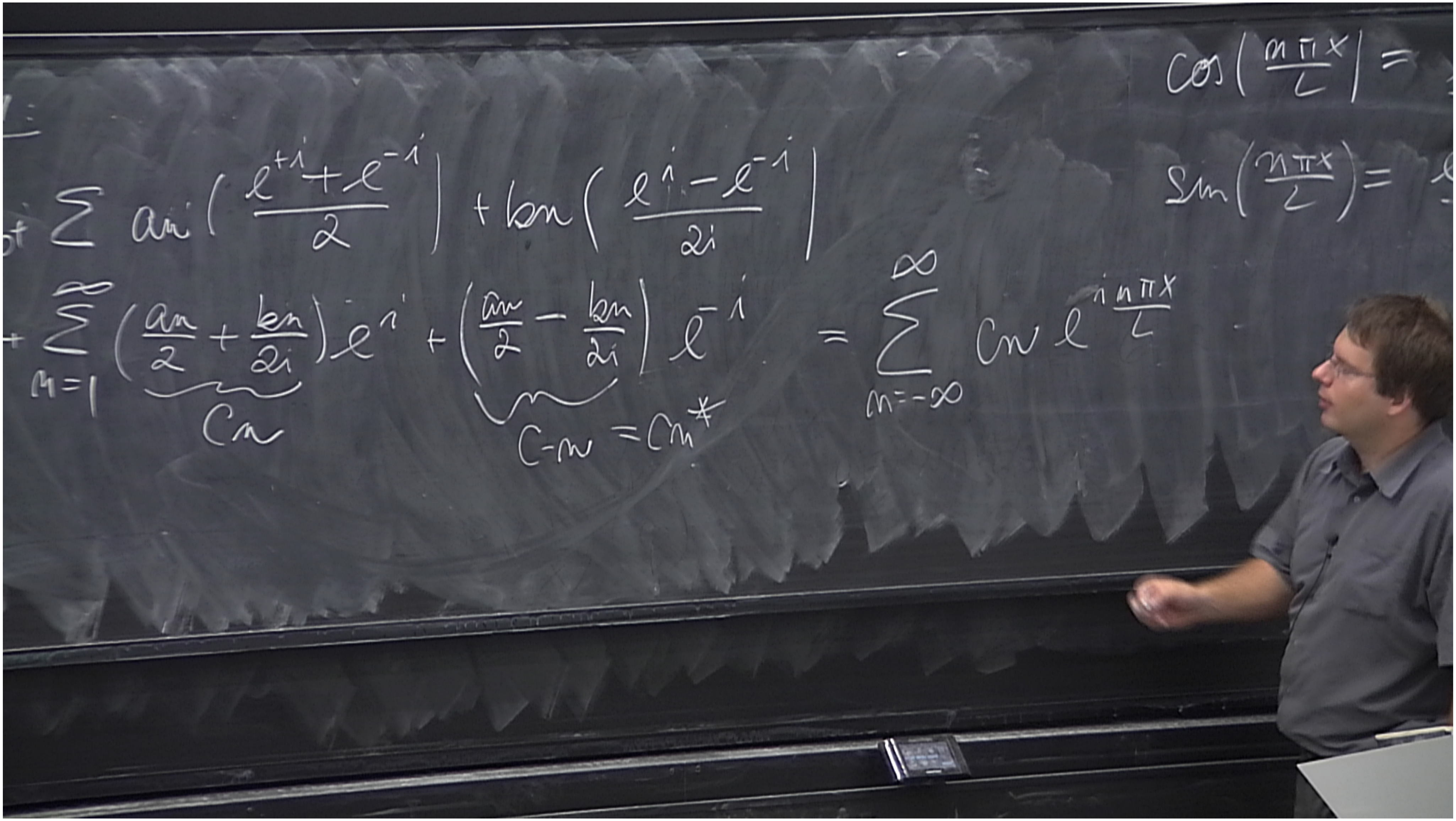
$$\sin\left(\frac{n\pi x}{L}\right) = \frac{e^{i\frac{n\pi x}{L}} - e^{-in\pi x/L}}{2i}$$

COMPLEX FORM:

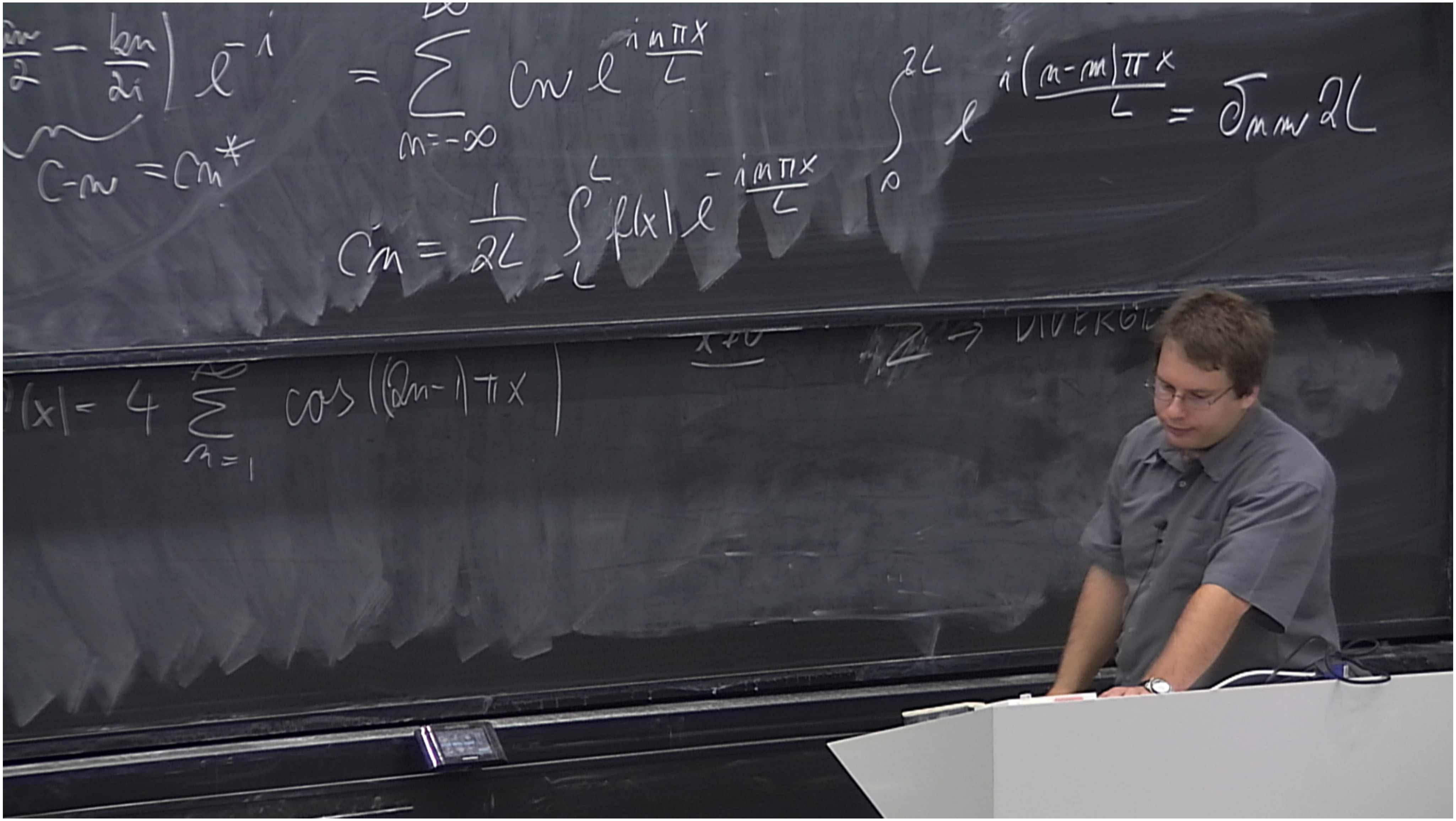
$$f(x) = a_0 + \sum a_n \left(\frac{e^{+i} + e^{-i}}{2} \right) + b_n \left(\frac{e^{i} - e^{-i}}{2i} \right)$$

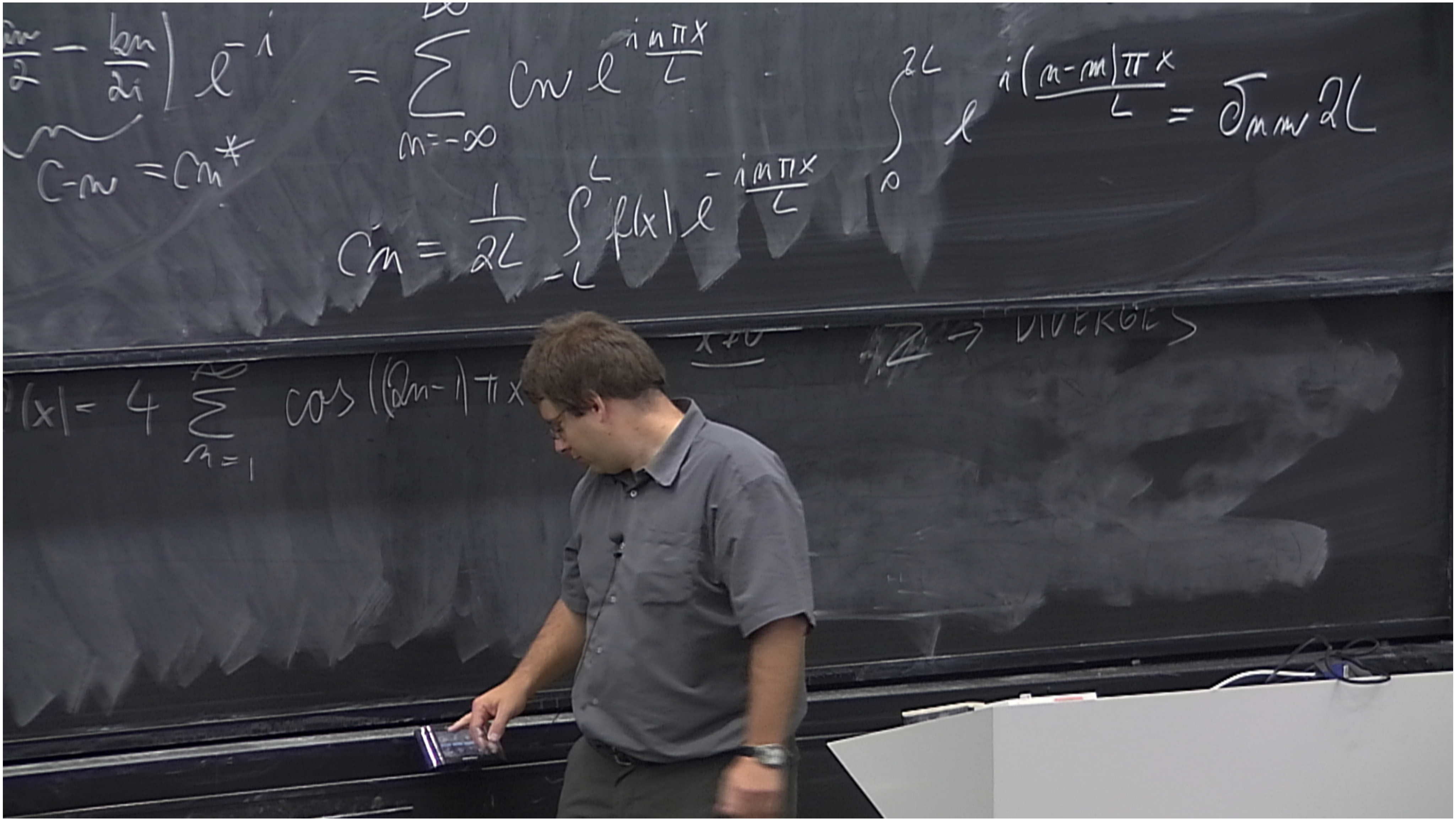
$$= a_0 + \sum_{n=1}^{\infty} \left(\frac{a_n}{2} + \frac{b_n}{2i} \right) e^{i} + \left(\frac{a_n}{2} - \frac{b_n}{2i} \right) e^{-i}$$

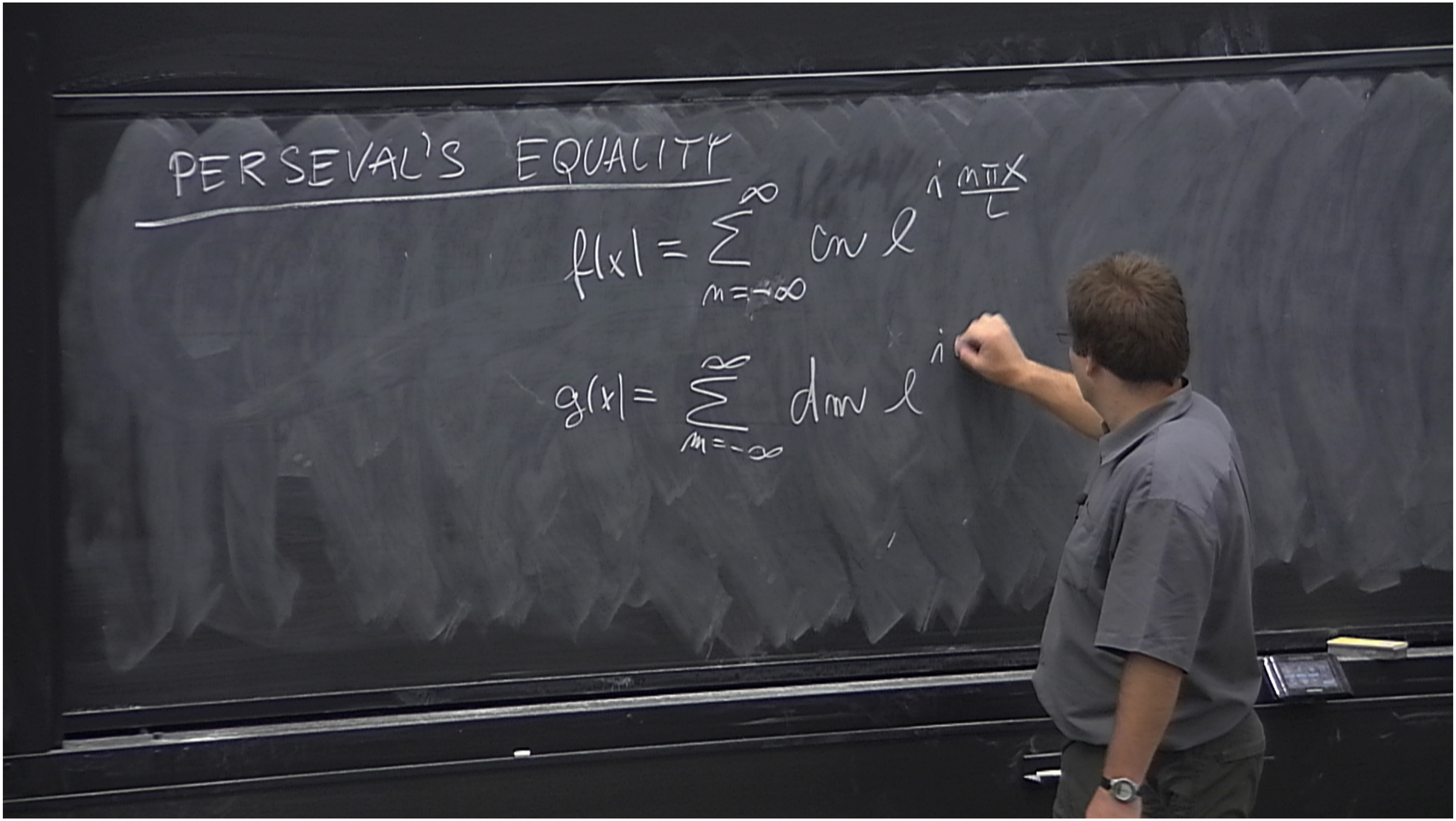

COMPLEX FORM:
$f(x) = a_0 + \sum a_n \left(\frac{e^{+i} + e^{-i}}{2}\right) + b_n\left(\frac{e^{i}-e^{-i}}{2i}\right)$
$= a_0 + \sum_{n=1}^{\infty} \underbrace{\left(\frac{a_n}{2}+\frac{b_n}{2i}\right)}_{c_n} e^{i} + \underbrace{\left(\frac{a_n}{2}-\frac{b_n}{2i}\right)}_{c_{-n} = c_n^*} e^{-i}$

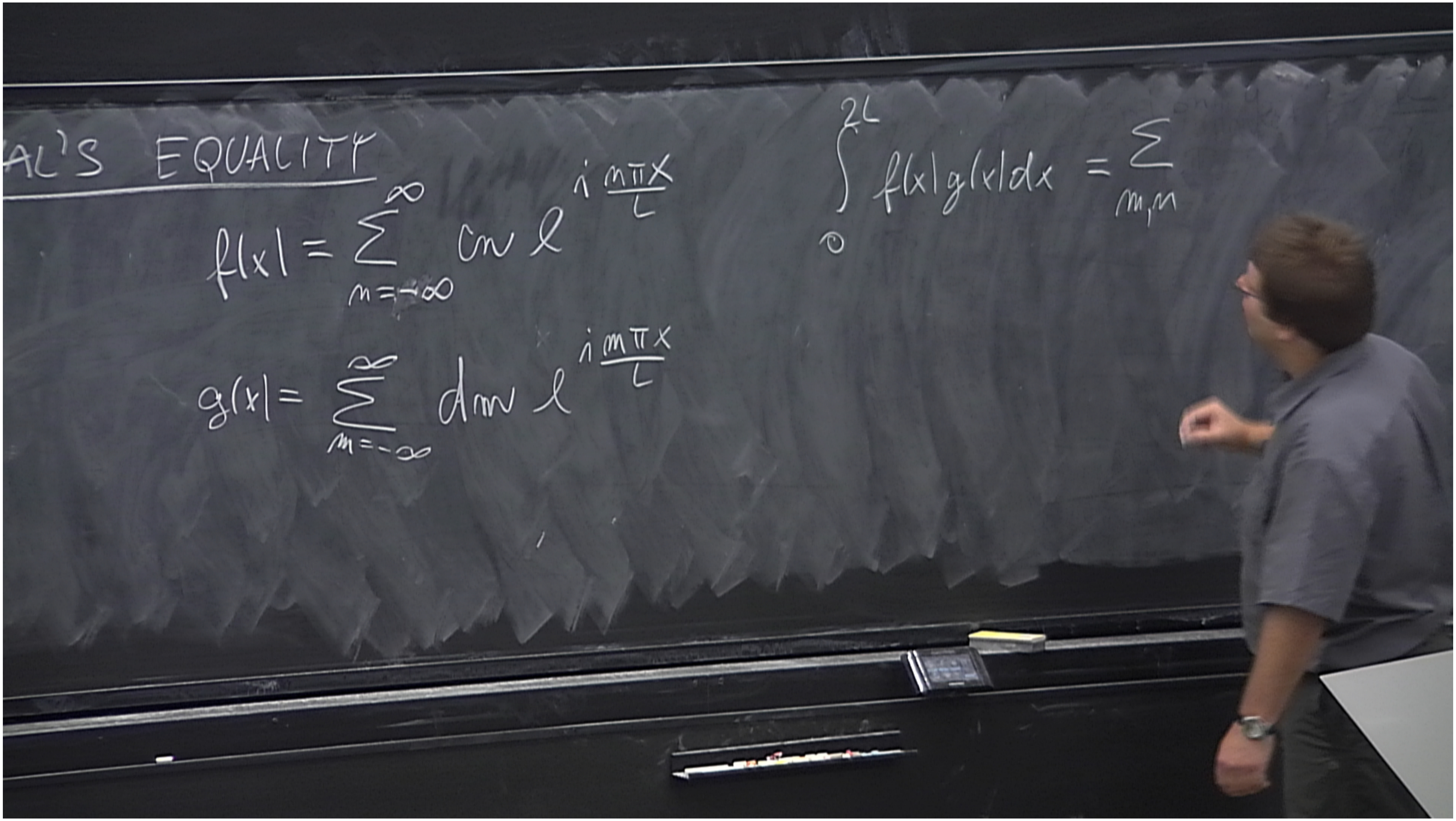


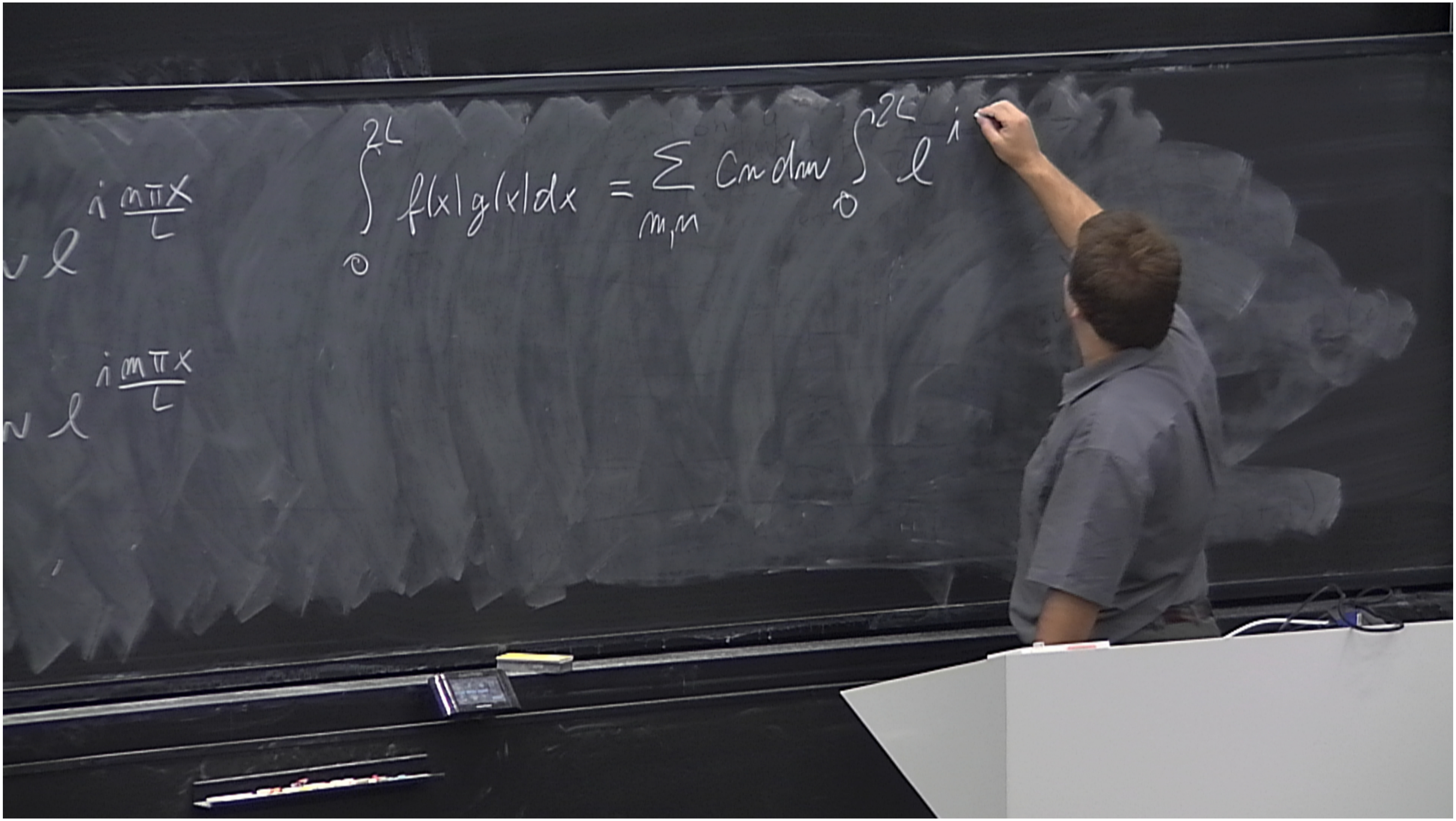
$\cos\left(\frac{n\pi x}{L}\right) =$
$\sin\left(\frac{n\pi x}{L}\right) =$
$\sum a_n \left(\frac{e^{+i} + e^{-i}}{2}\right) + b_n \left(\frac{e^{i} - e^{-i}}{2i}\right)$
$+ \sum_{n=1}^{\infty} \underbrace{\left(\frac{a_n}{2} + \frac{b_n}{2i}\right)}_{c_n} e^{i} + \underbrace{\left(\frac{a_n}{2} - \frac{b_n}{2i}\right)}_{c_{-n} = c_n^*} e^{-i} = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi x}{L}}$

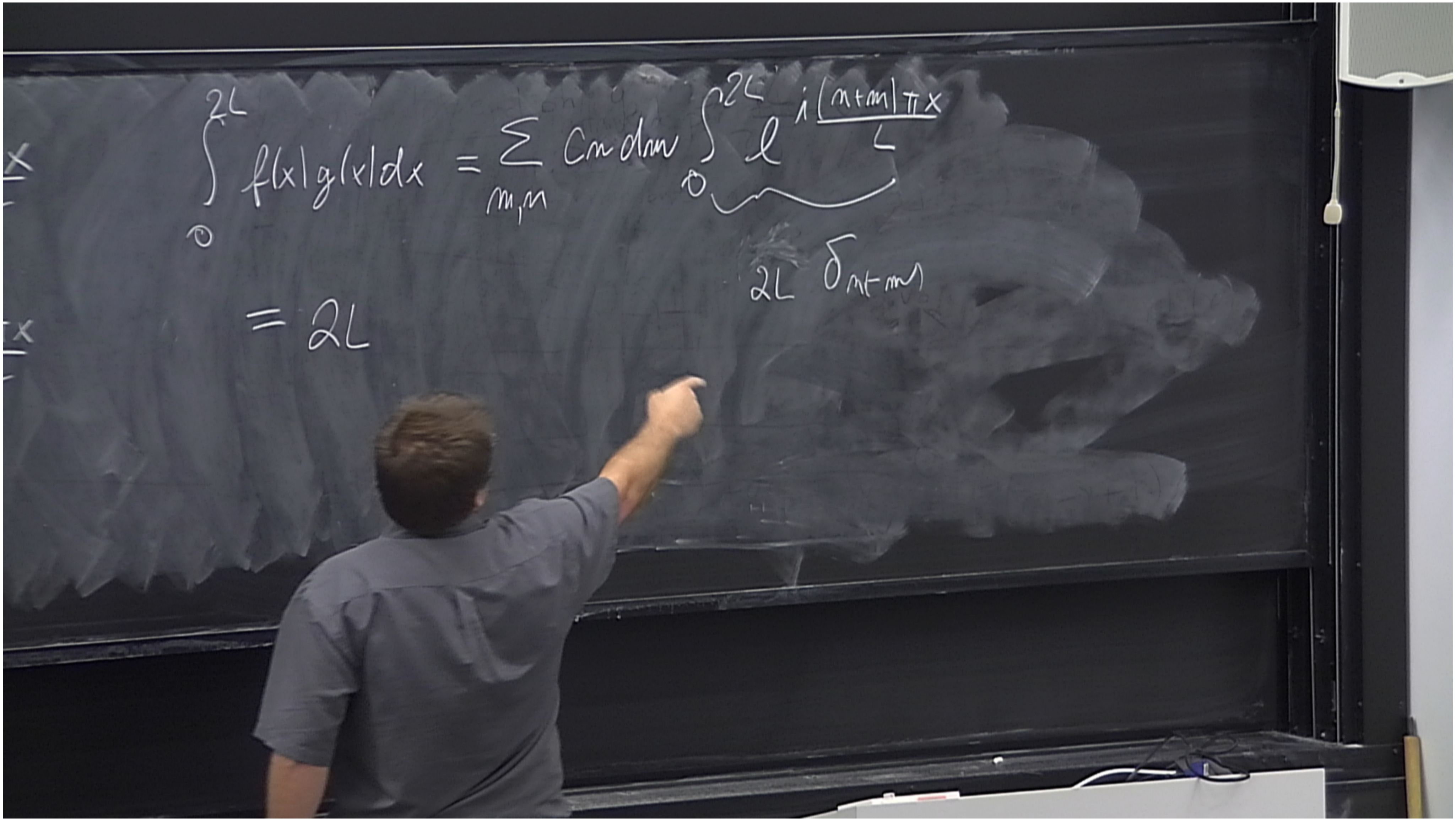

$= \sum_{n=-\infty}^{\infty} c_n e^{\frac{i n \pi x}{L}}$
$c_{-n} = c_n^*$
$\int_0^{2L} e^{\frac{i(n-m)\pi x}{L}} = \delta_{mn} 2L$
$c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-\frac{i n \pi x}{L}}$
$\sum_{n=1}^{\infty} \cos((2n-1)\pi x)$

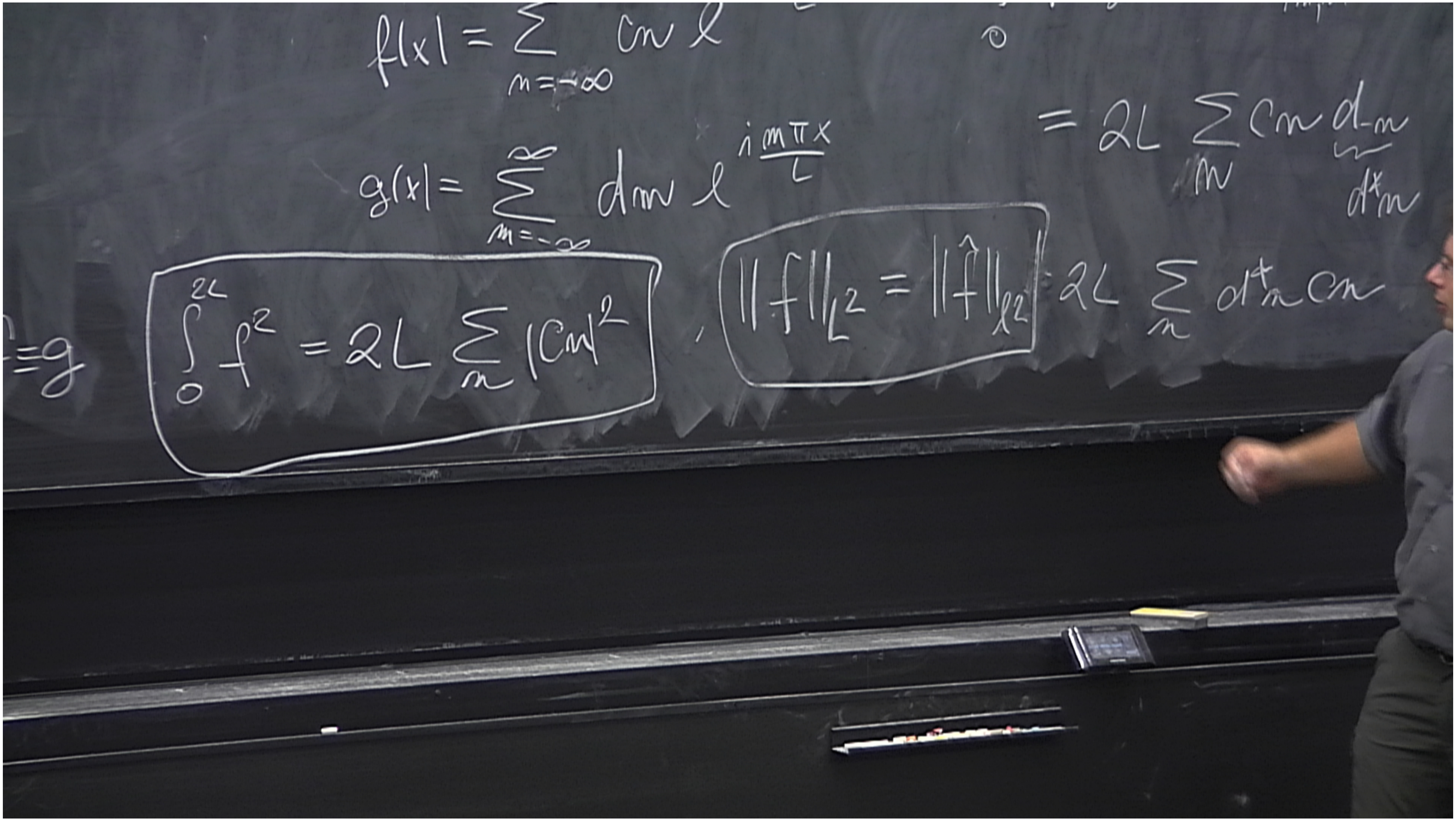

$= \sum_{n=-\infty}^{\infty} C_n e^{\frac{i n \pi x}{L}}$
$C_{-n} = C_n^*$
$\int_0^{2L} e^{\frac{i(n-m)\pi x}{L}} = \delta_{nm} 2L$
$C_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-\frac{i n \pi x}{L}}$
DIVERGES


PERSEVAL'S EQUALITY
$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi x}{L}}$
$g(x) = \sum_{n=-\infty}^{\infty} d_n e^{i}$


AL'S EQUALITY
$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi x}{L}}$
$g(x) = \sum_{m=-\infty}^{\infty} d_m e^{i\frac{m\pi x}{L}}$
$\int_0^{2L} f(x)g(x)dx = \sum_{m,n}$


$e^{i\frac{n\pi x}{L}}$
$e^{i\frac{m\pi x}{L}}$
$\int_0^{2L} f(x)g(x)dx = \sum_{m,n} c_n d_m \int_0^{2L} e^{i}$


$\int_0^{2L} f(x)g(x)dx = \sum_{m,n} c_n d_m \int_0^{2L} e^{i\frac{(n+m)\pi x}{L}}$
$2L\,\delta_{n,-m}$
$= 2L$

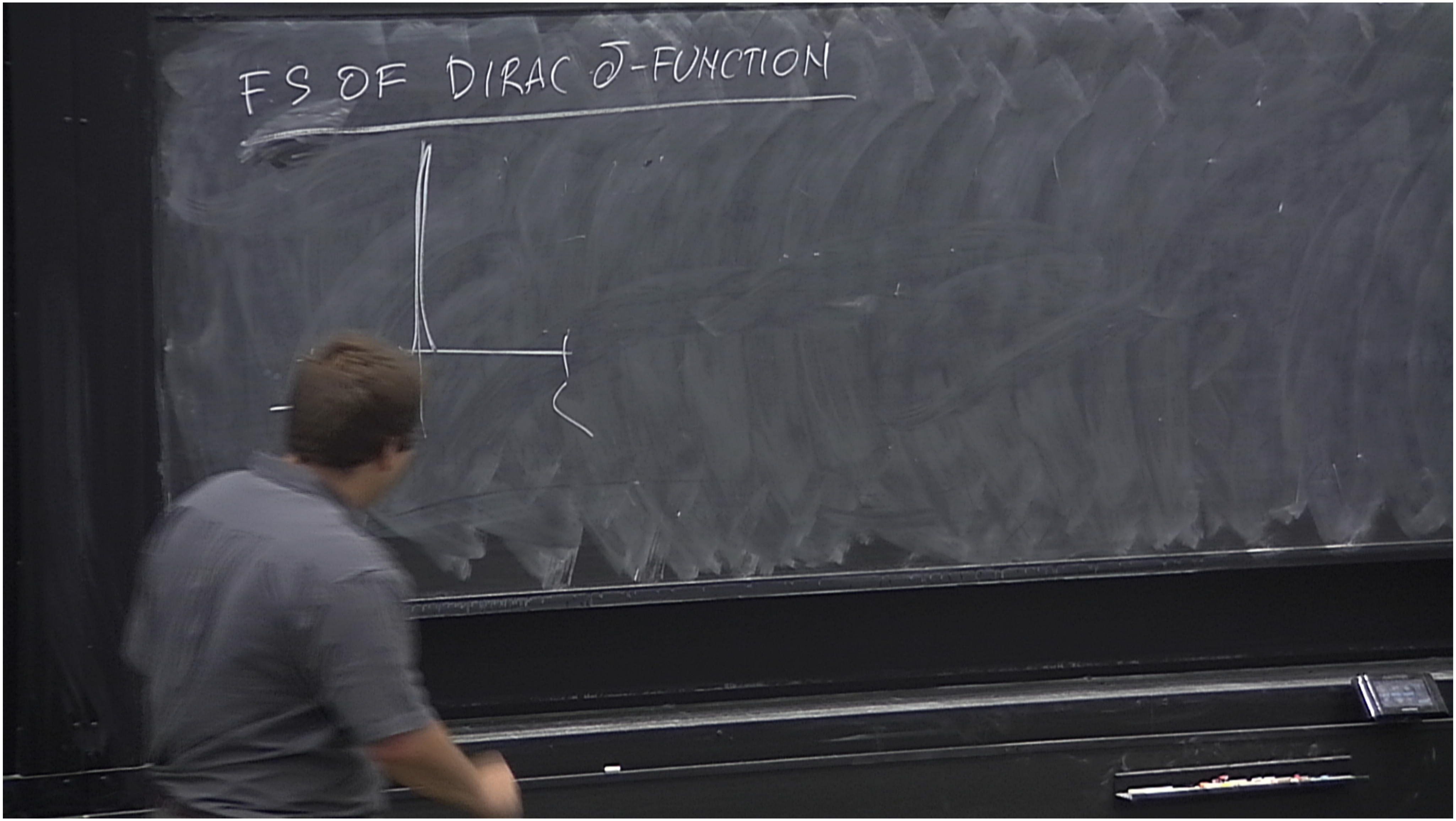

$f(x) = \sum_{n=-\infty} c_n e$
$g(x) = \sum_{n=-\infty}^{\infty} d_n e^{i\frac{n\pi x}{L}}$
$= 2L \sum_n c_n \underbrace{d_{-n}}_{d^*_n}$
$f = g$
$\boxed{\int_0^{2L} f^2 = 2L \sum_n |c_n|^2}$
$\boxed{\|f\|_{L^2} = \|\hat{f}\|_{\ell^2}} = 2L \sum_n d^*_n c_n$

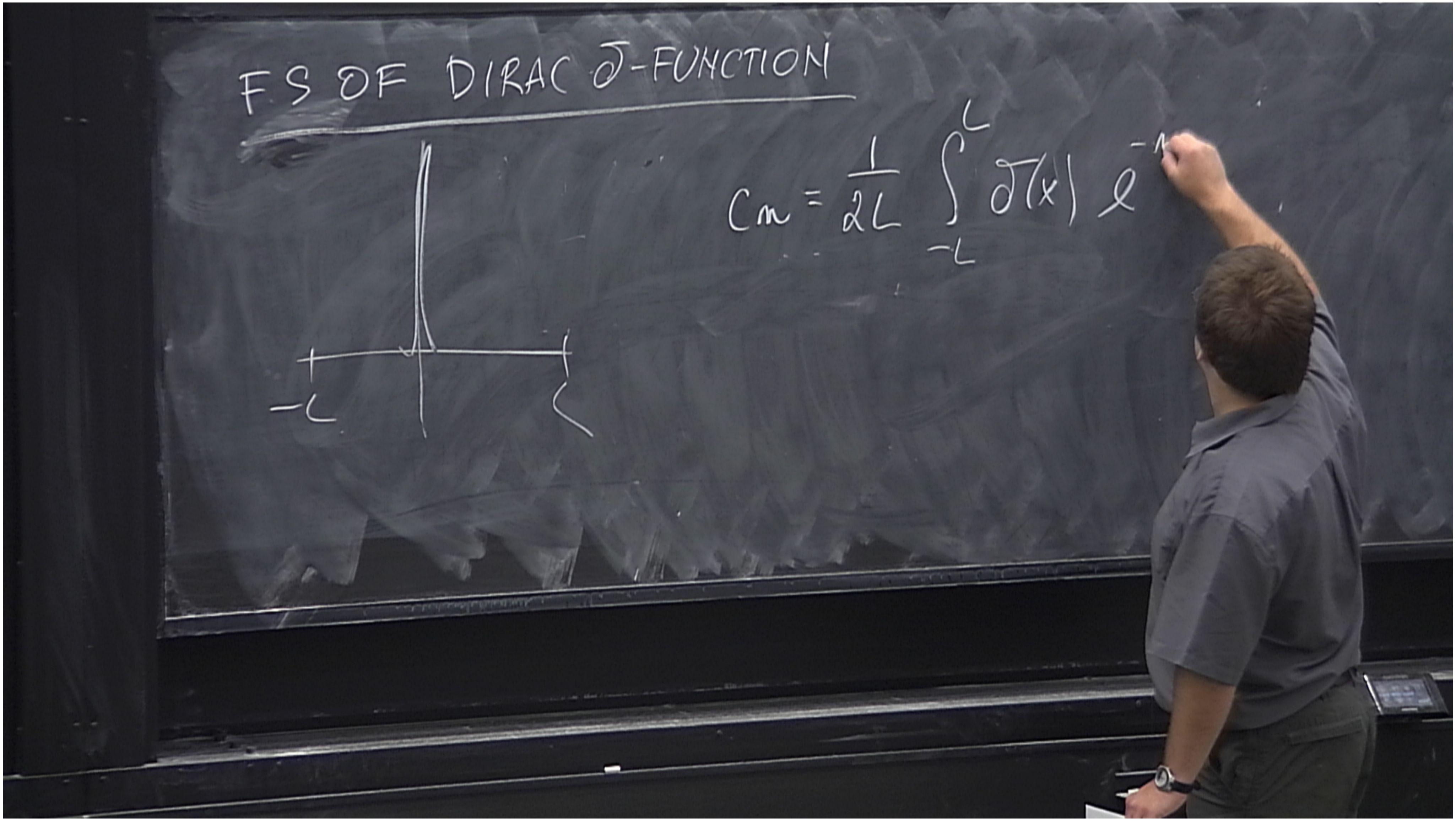
$$f(x) = \sum_{n=-\infty} c_n e$$

$$= 2L \sum_n c_n \underset{d^*_n}{d_{-n}}$$

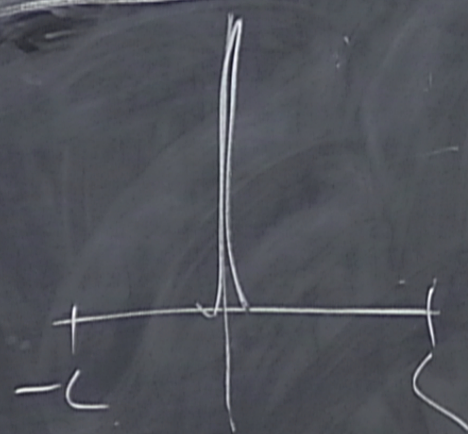
$$g(x) = \sum_{n=-\infty}^{\infty} d_n e^{i\frac{n\pi x}{L}}$$

$$f = g \qquad \boxed{\int_0^{2L} f^2 = 2L \sum_n |c_n|^2}, \qquad \boxed{\|f\|_{L^2} = 2L \|\hat{f}\|_{\ell^2}} = 2L \sum_n d^*_n c_n$$

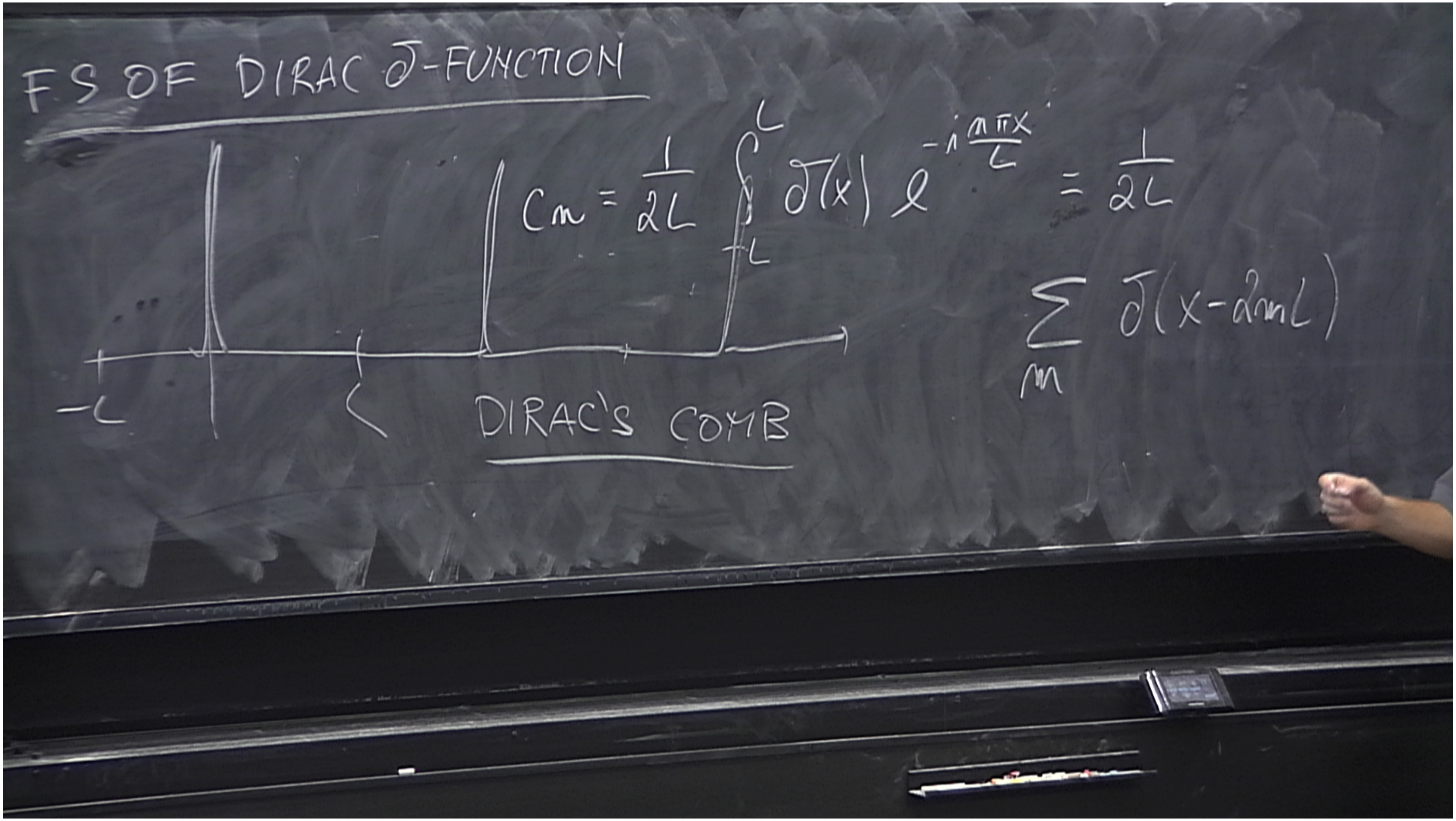

FS OF DIRAC δ-FUNCTION

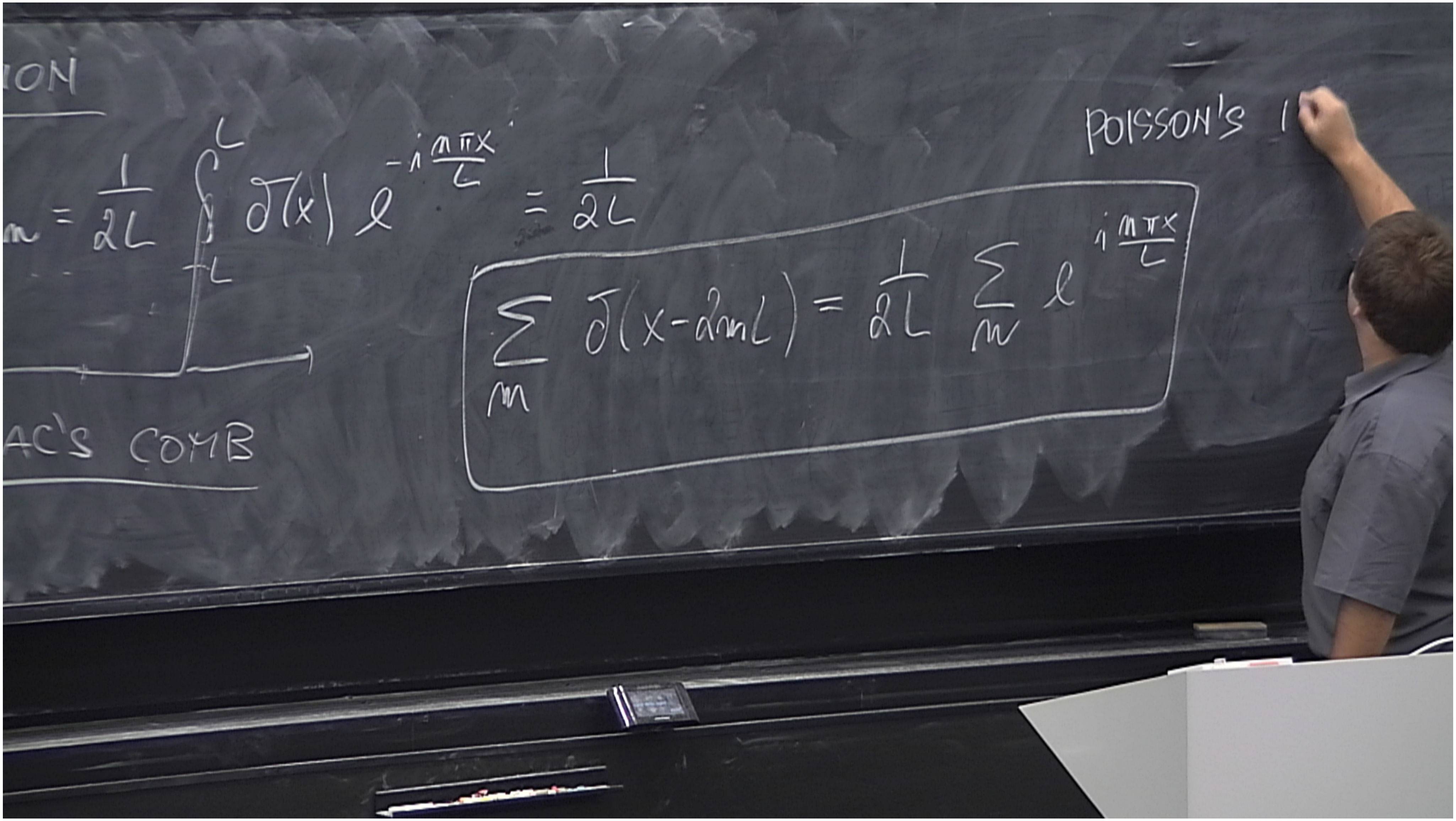

FS OF DIRAC δ-FUNCTION
Cn = 1/2L ∫_{-L}^{L} δ(x) e
-L
L

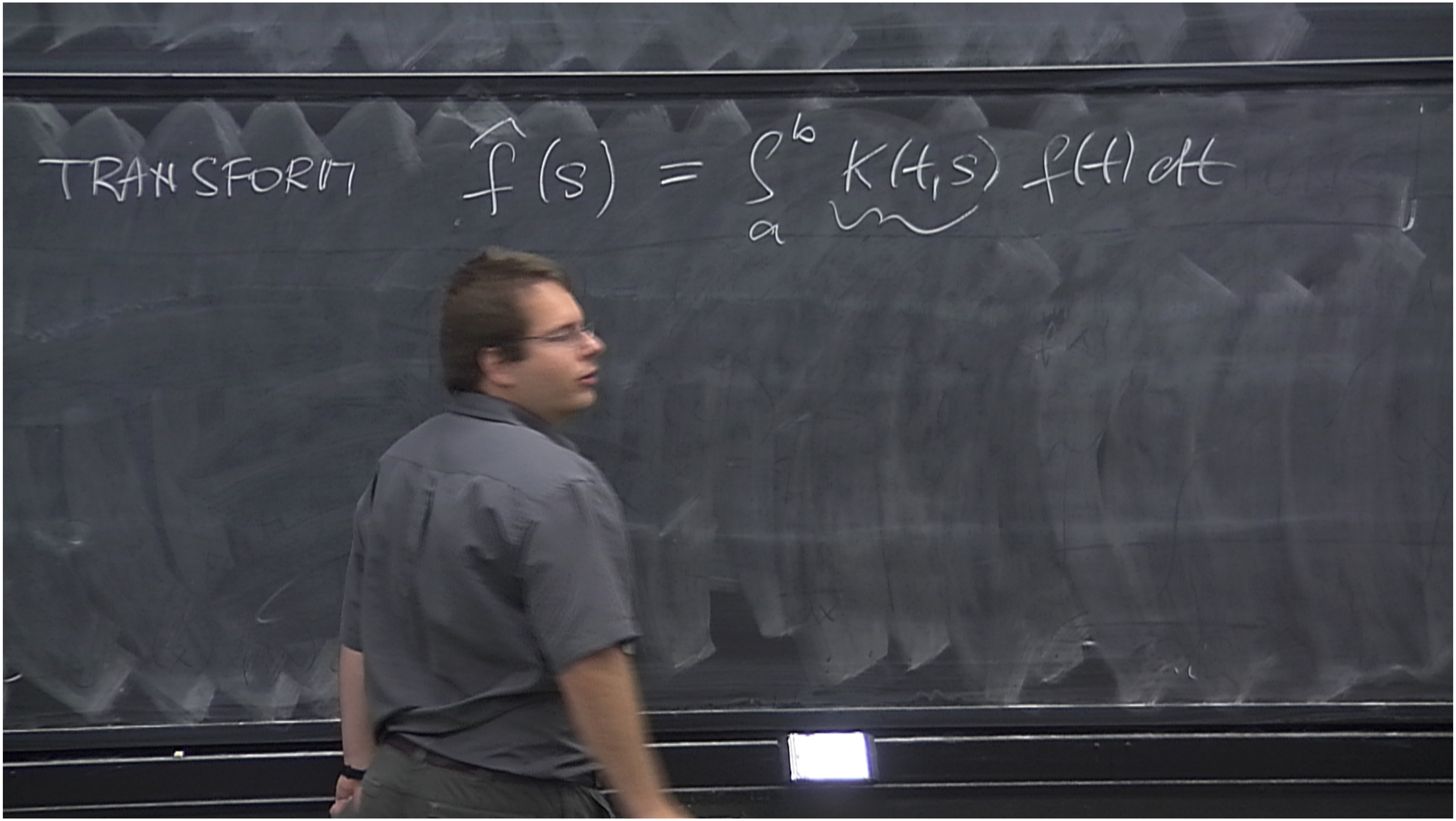
FS OF DIRAC δ-FUNCTION

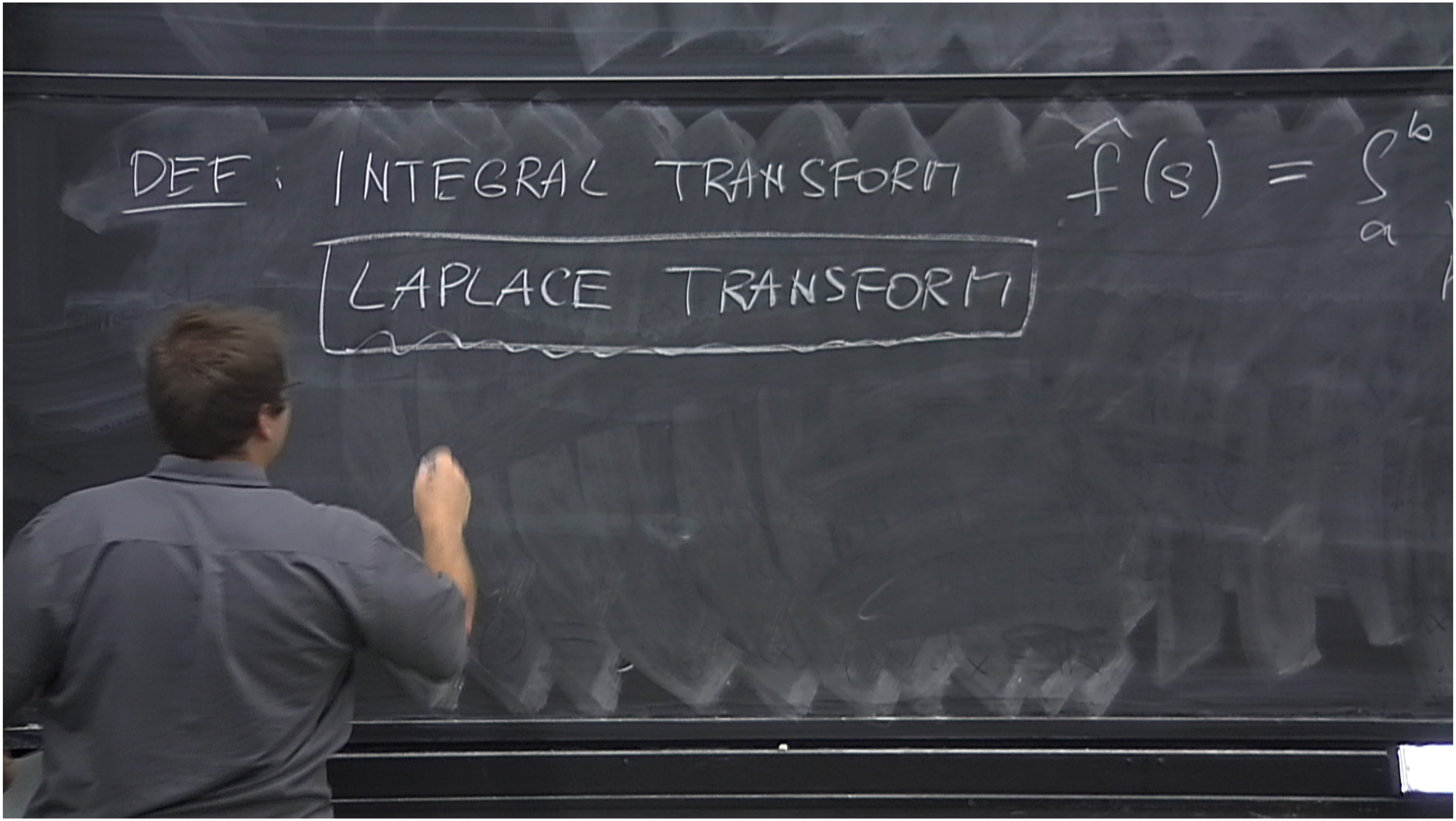


$$C_n = \frac{1}{2L}\int_{-L}^{L} \delta(x)\, e^{-i\frac{n\pi x}{L}} =$$


FS OF DIRAC δ-FUNCTION
$C_m = \frac{1}{2L} \int_{-L}^{L} \delta(x)\, e^{-i\frac{m\pi x}{L}} = \frac{1}{2L}$
$\sum_m \delta(x-2mL)$
$-L$
DIRAC'S COMB


ION
$= \frac{1}{2L} \int_{-L}^{L} \delta(x)\, e^{-i\frac{n\pi x}{L}} = \frac{1}{2L}$
POISSON'S
$\sum_m \delta(x-2mL) = \frac{1}{2L} \sum_n e^{i\frac{n\pi x}{L}}$
AC'S COMB


TRANSFORM
$\hat{f}(s) = \int_a^b K(t,s) f(t)\, dt$

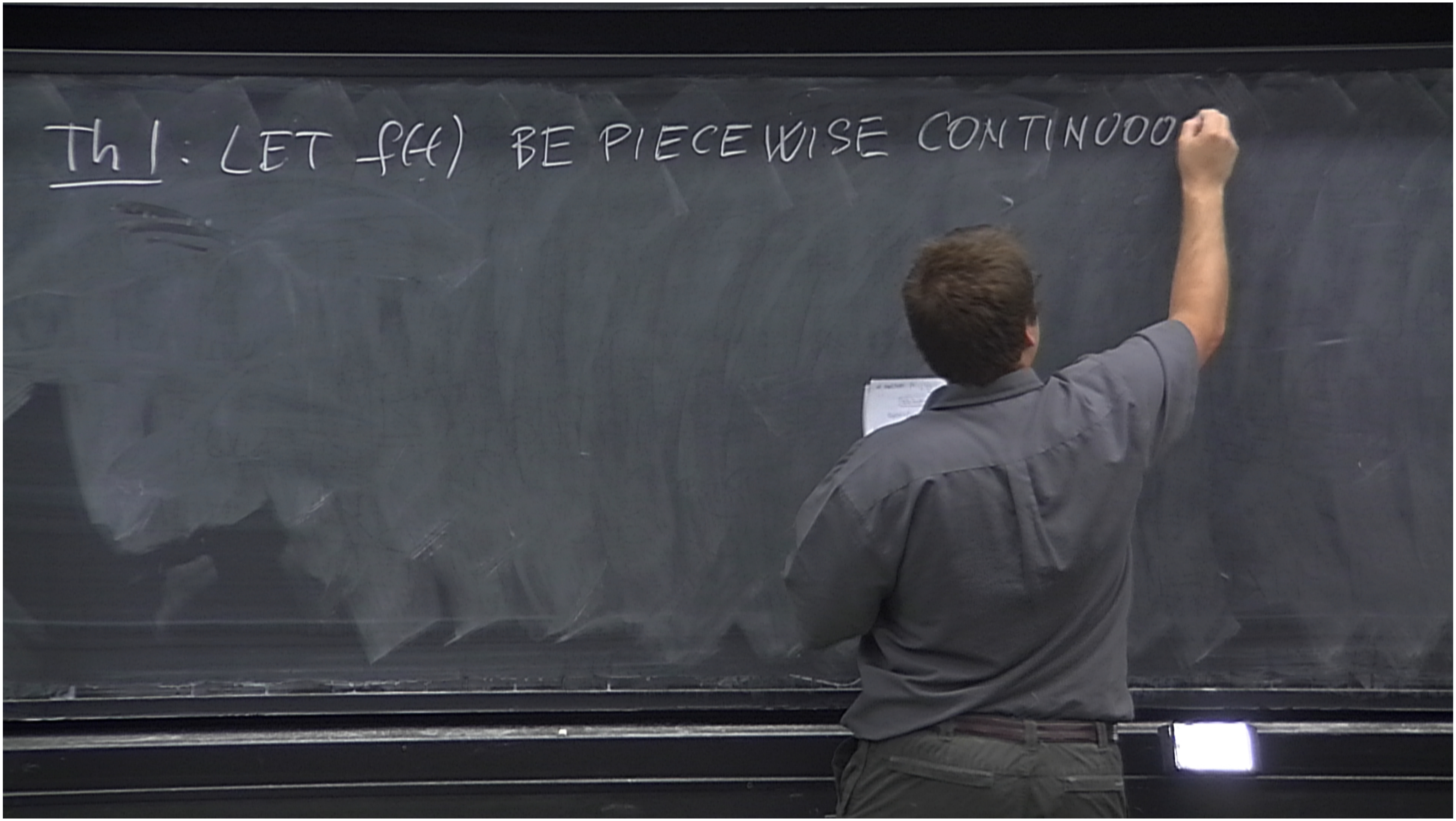

DEF: INTEGRAL TRANSFORM
$\hat{f}(s) = \int_a^b$
LAPLACE TRANSFORM

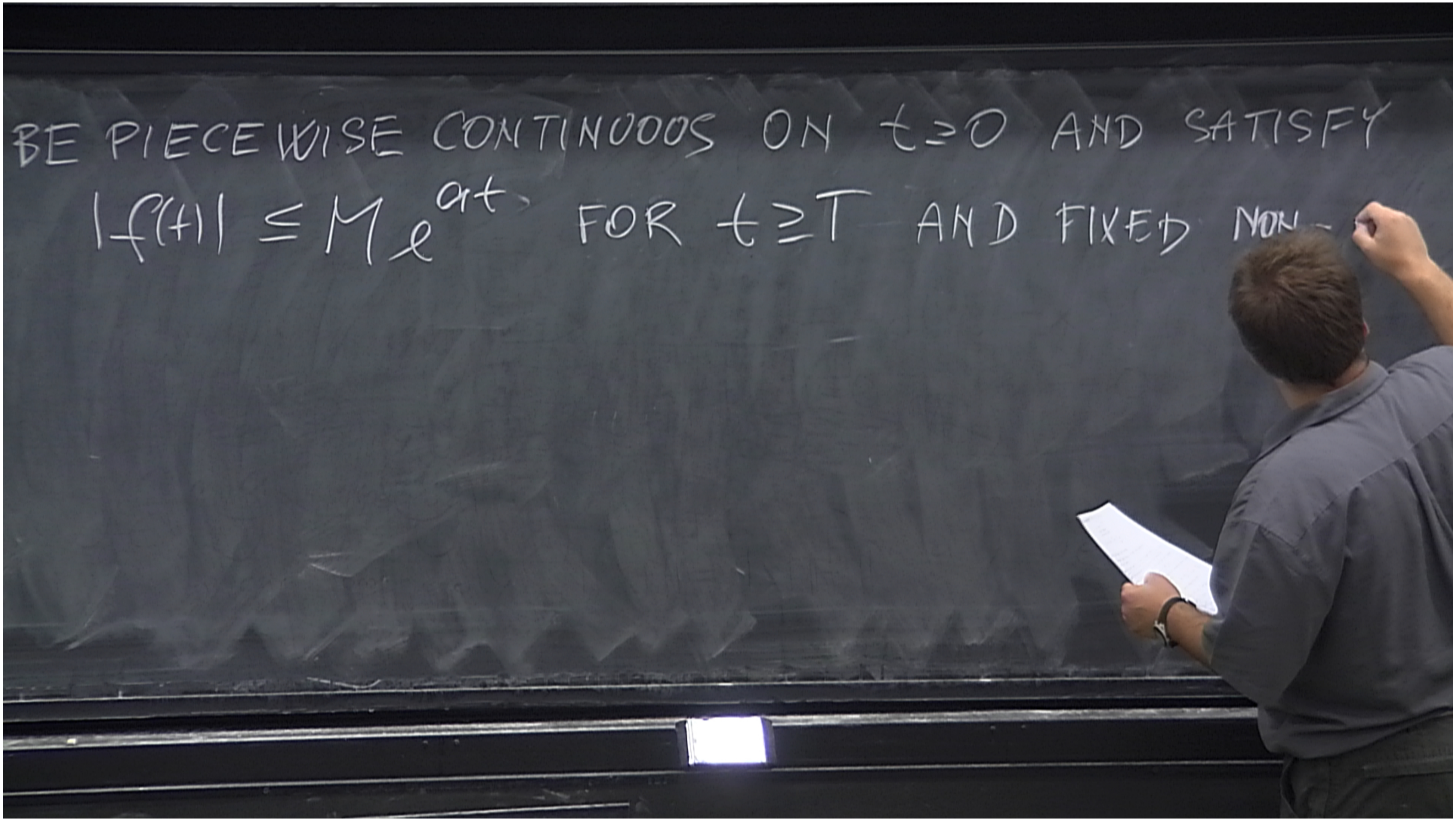

HOW TO EXTRACT
FOURIER COEFF
DIRAC'S COMB
POISSON'S INTEGRA
FORMUL

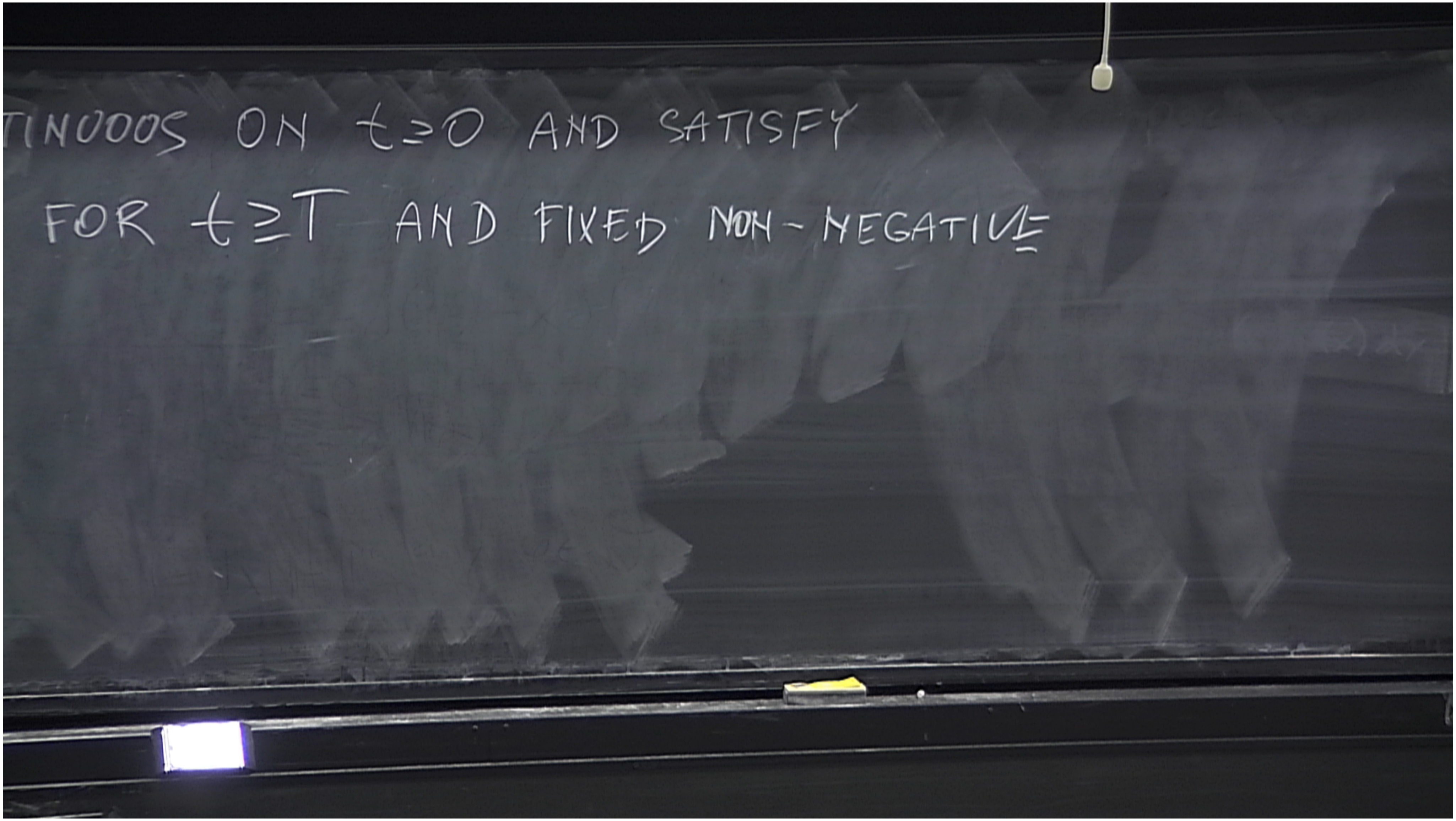
INTEGRAL TRANSFORM $\hat{f}(s) = \int_a^b \underbrace{K(t,s)}_{\text{KERNEL}} f$

$$\boxed{\text{LAPLACE TRANSFORM}}$$

$$\boxed{\mathcal{L}[f](s) \equiv \hat{f}(s) = \int_0^\infty f(t)\, e^{-st}\, dt}$$


Th 1: LET f(t) BE PIECEWISE CONTINUOU

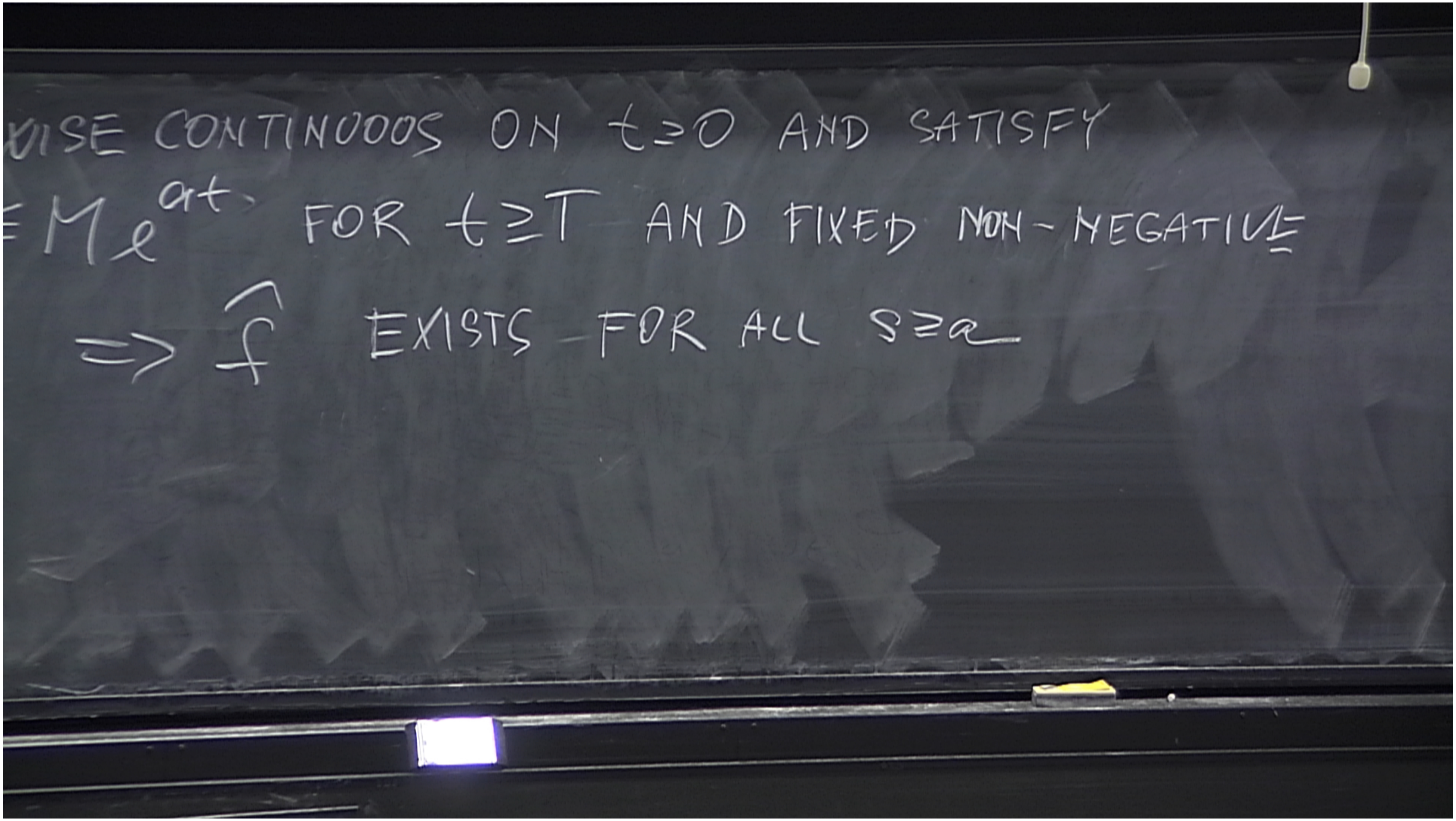

BE PIECEWISE CONTINUOUS ON $t \geq 0$ AND SATISFY
$|f(t)| \leq Me^{at}$ FOR $t \geq T$ AND FIXED NON-

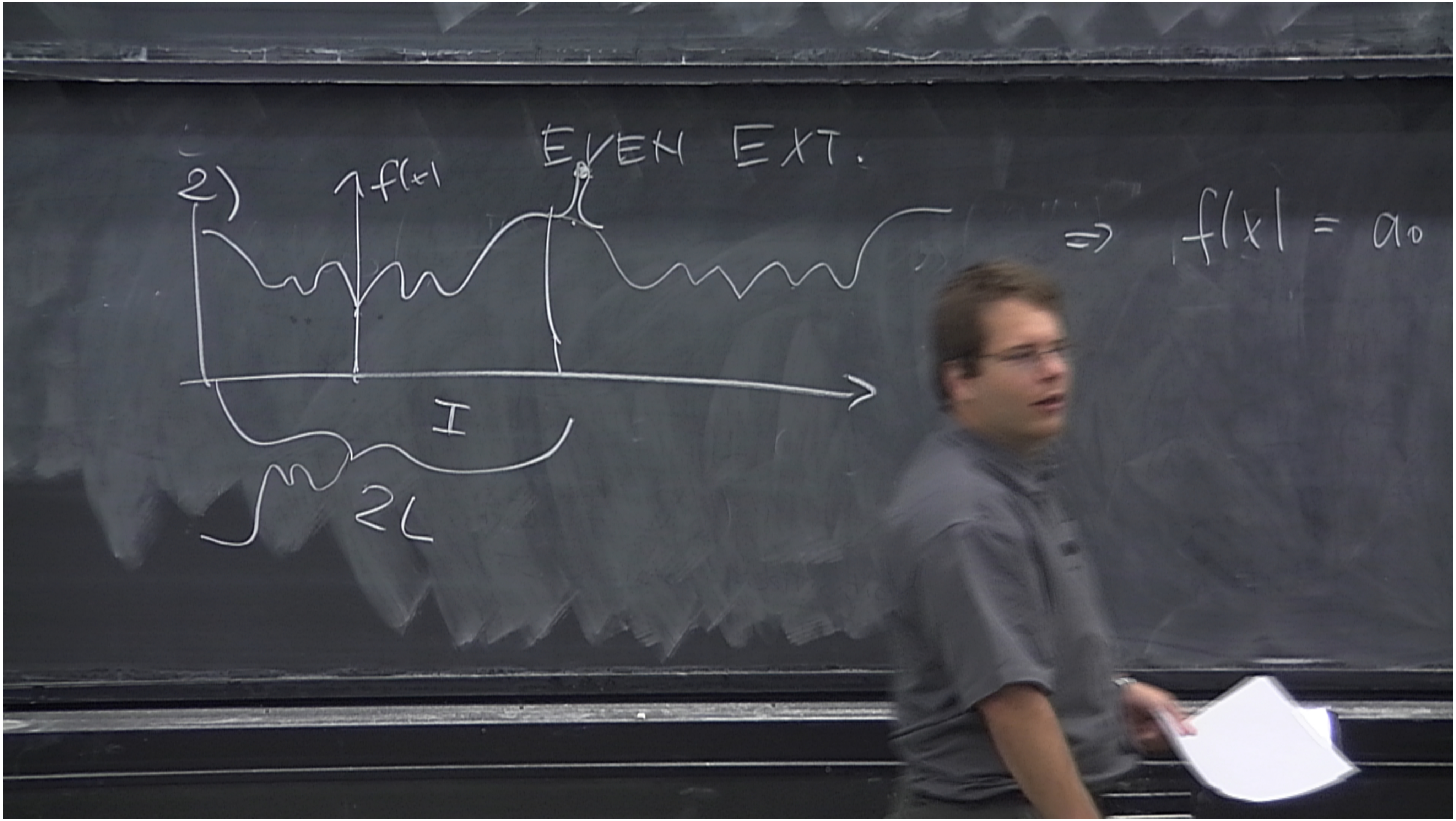

TINUOUS ON t≥0 AND SATISFY
FOR t≥T AND FIXED NON-NEGATIVE

Th 1: LET $f(t)$ BE PIECEWISE CONTINUOUS ON t

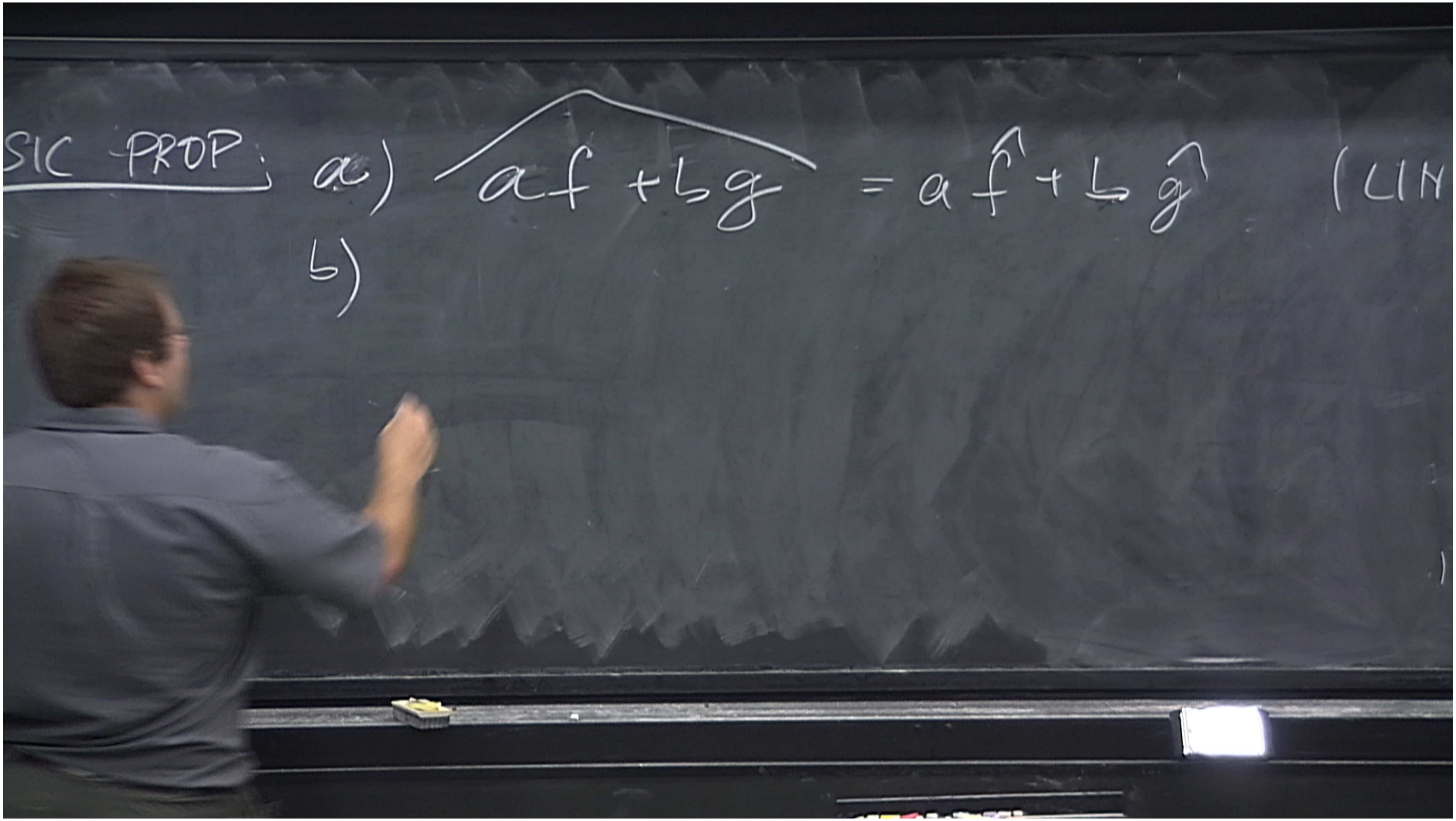
THE COND. $|f(t)| \le M e^{at}$ FOR $t \ge T$

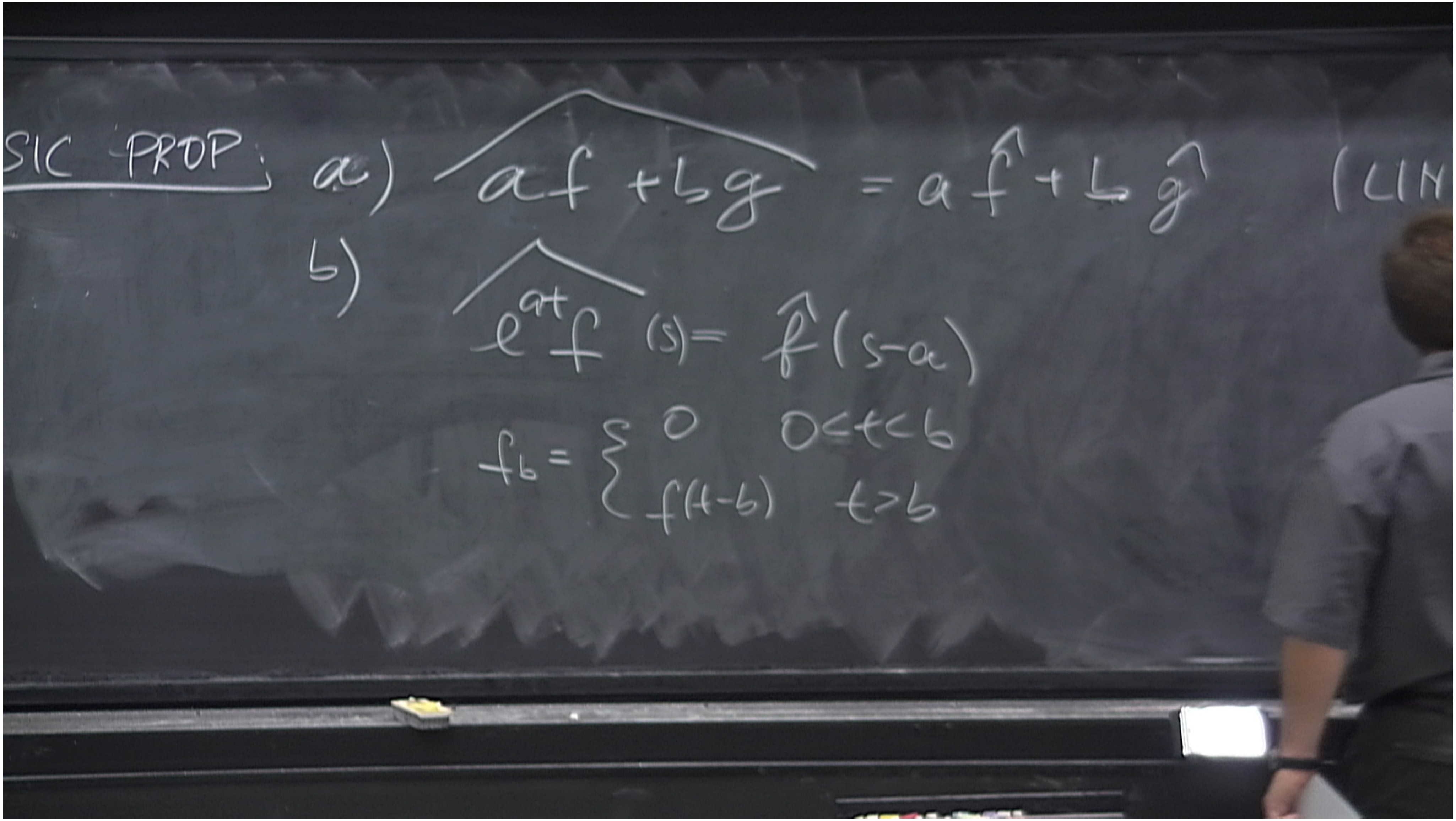
CONSTANTS a, M, AND T $\Rightarrow \hat{f}$

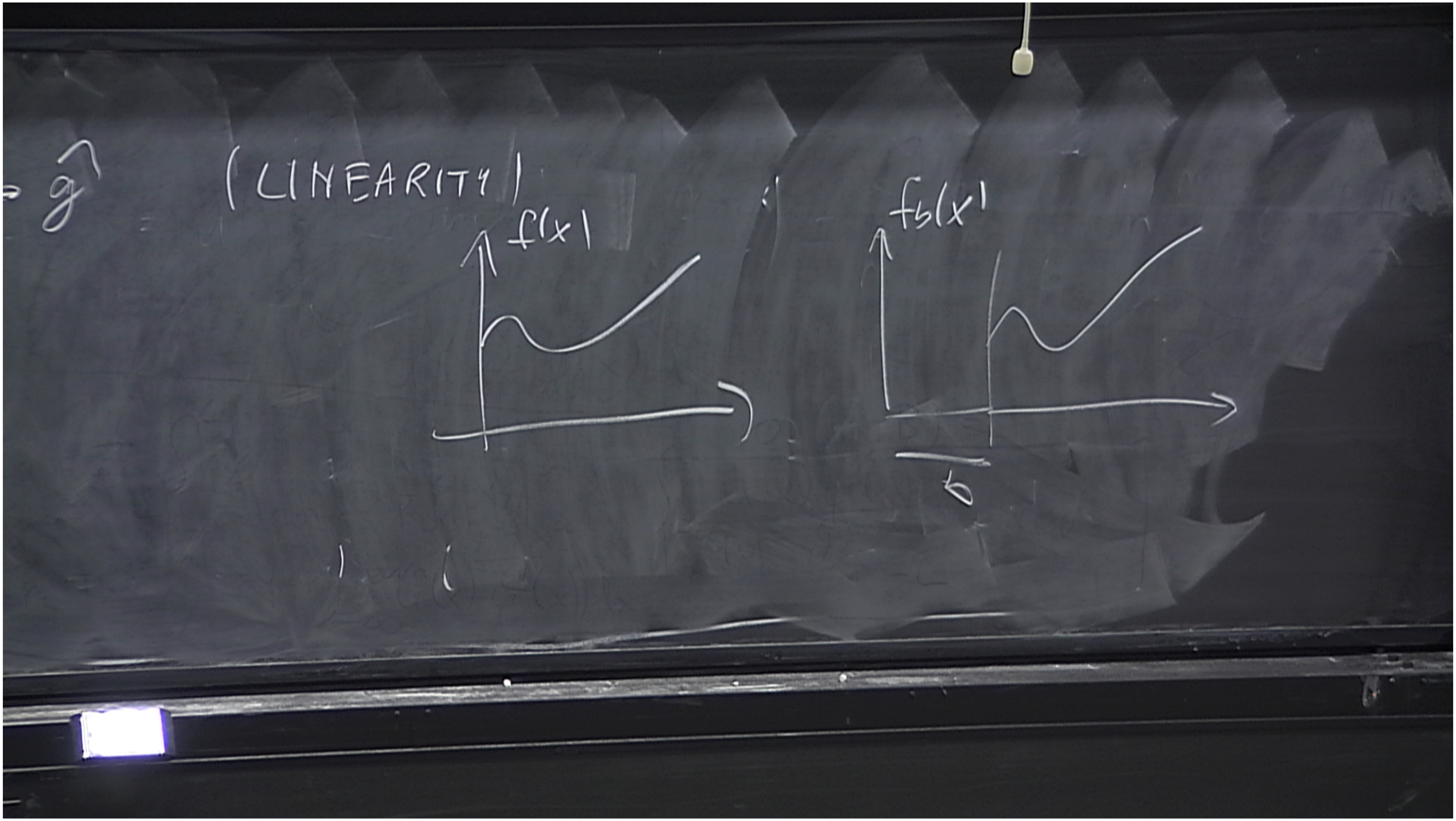

WISE CONTINUOUS ON $t \geq 0$ AND SATISFY
Me^{at} FOR $t \geq T$ AND FIXED NON-NEGATIVE
$\Rightarrow \hat{f}$ EXISTS FOR ALL $s \geq a$



EVEN EXT.
2)
f(x)
I
2L
⇒ f(x) = a₀

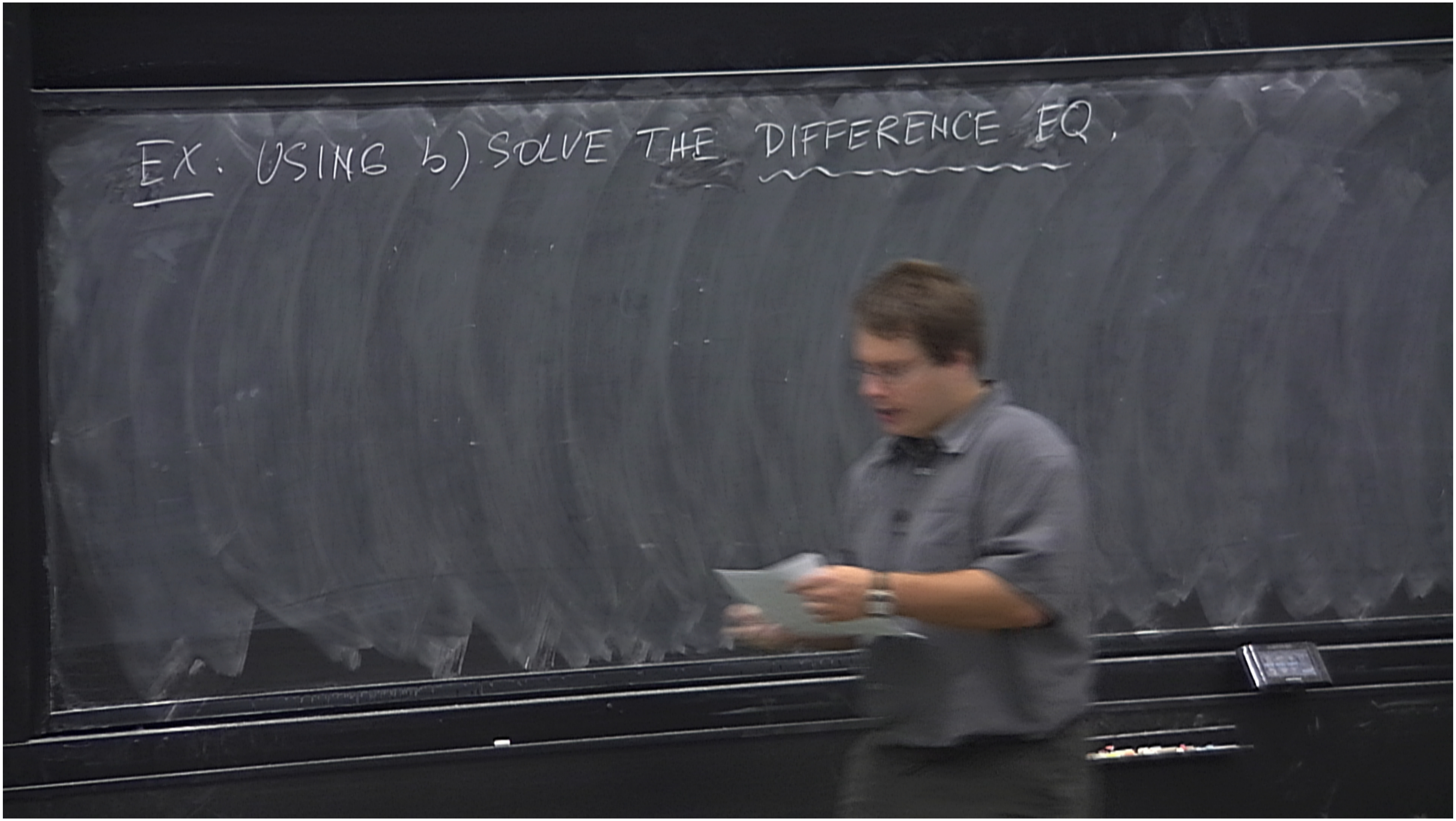

SIC PROP:
a) $\widehat{af+bg} = a\hat{f} + b\hat{g}$ (LIN
b)


SIC PROP:
a) $\widehat{af+bg} = a\hat{f} + b\hat{g}$ (LIN
b) $\widehat{e^{at}f}(s) = \hat{f}(s-a)$
$f_b = \begin{cases} 0 & 0<t<b \\ f(t-b) & t>b \end{cases}$


(LINEARITY)
f(x)
fb(x)
b

$$f_b = \begin{cases} 0 & 0 \le t < b \\ f(t-b) & t > b \end{cases} \qquad \hat{f_b}(s) = e^{-bs}\hat{f}(s)$$

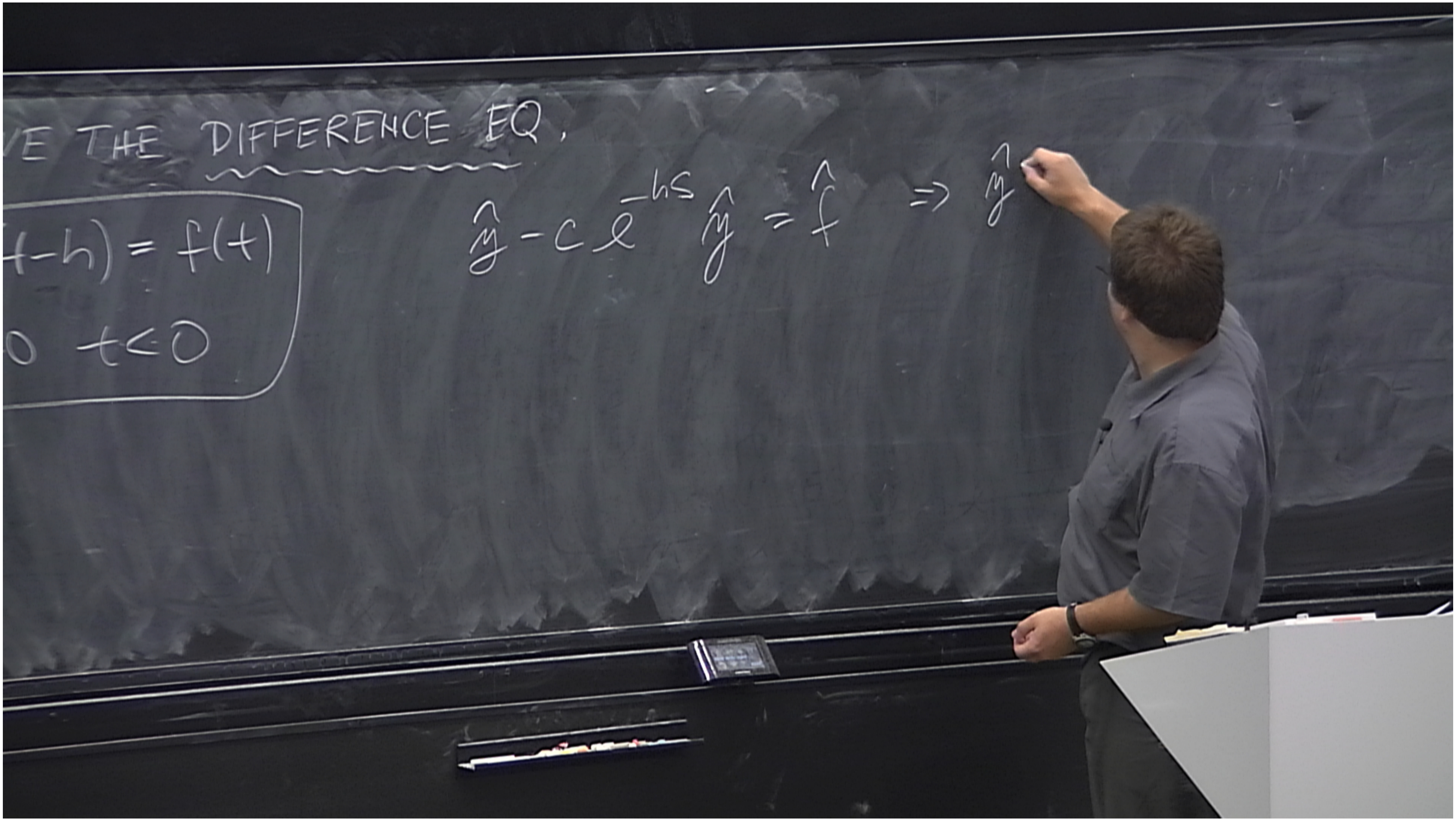
c) $$\widehat{f(at)}(s) = \frac{1}{a}\hat{f}\left(\frac{s}{a}\right)$$

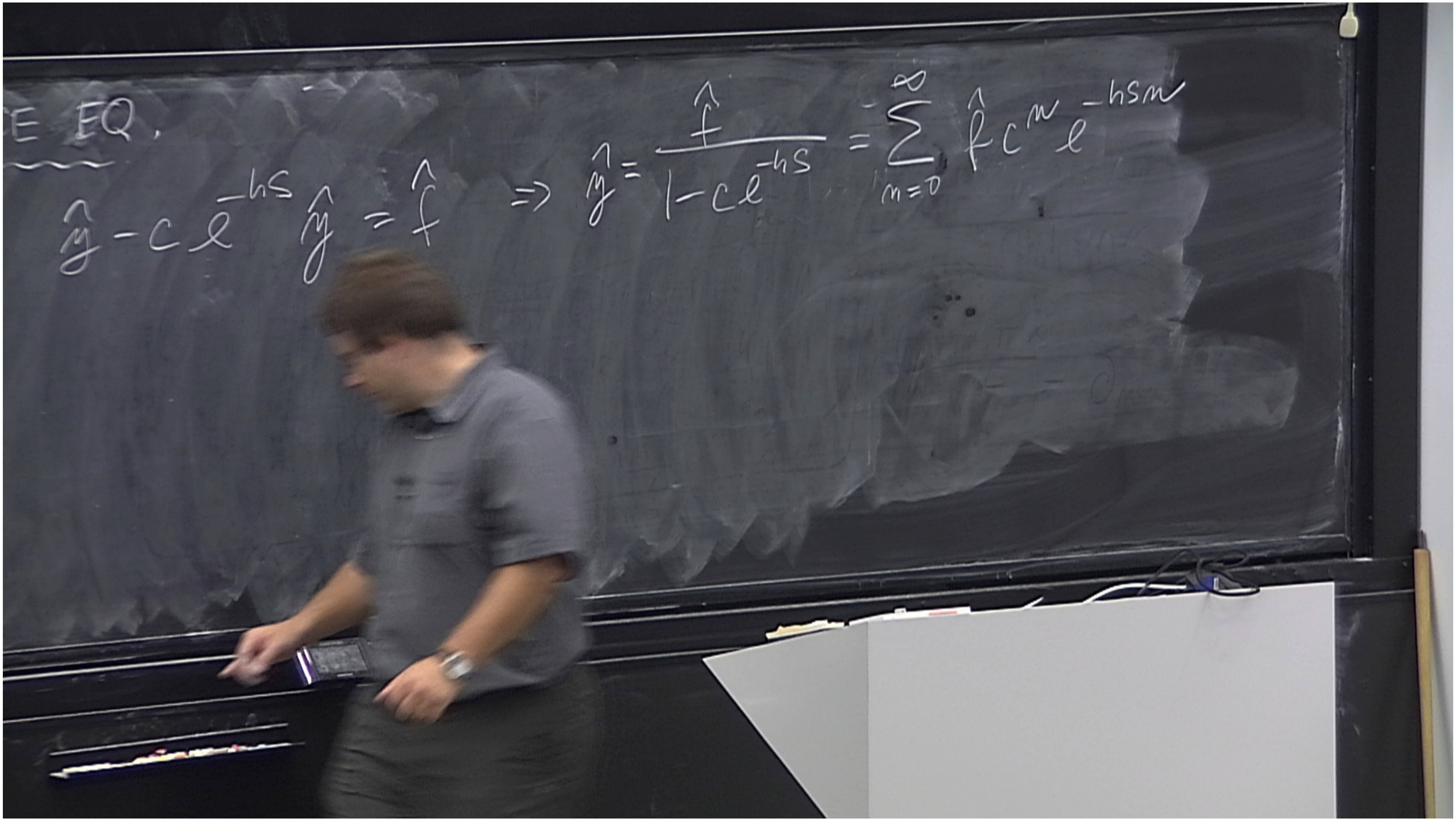

EX: USING b) SOLVE THE DIFFERENCE EQ.

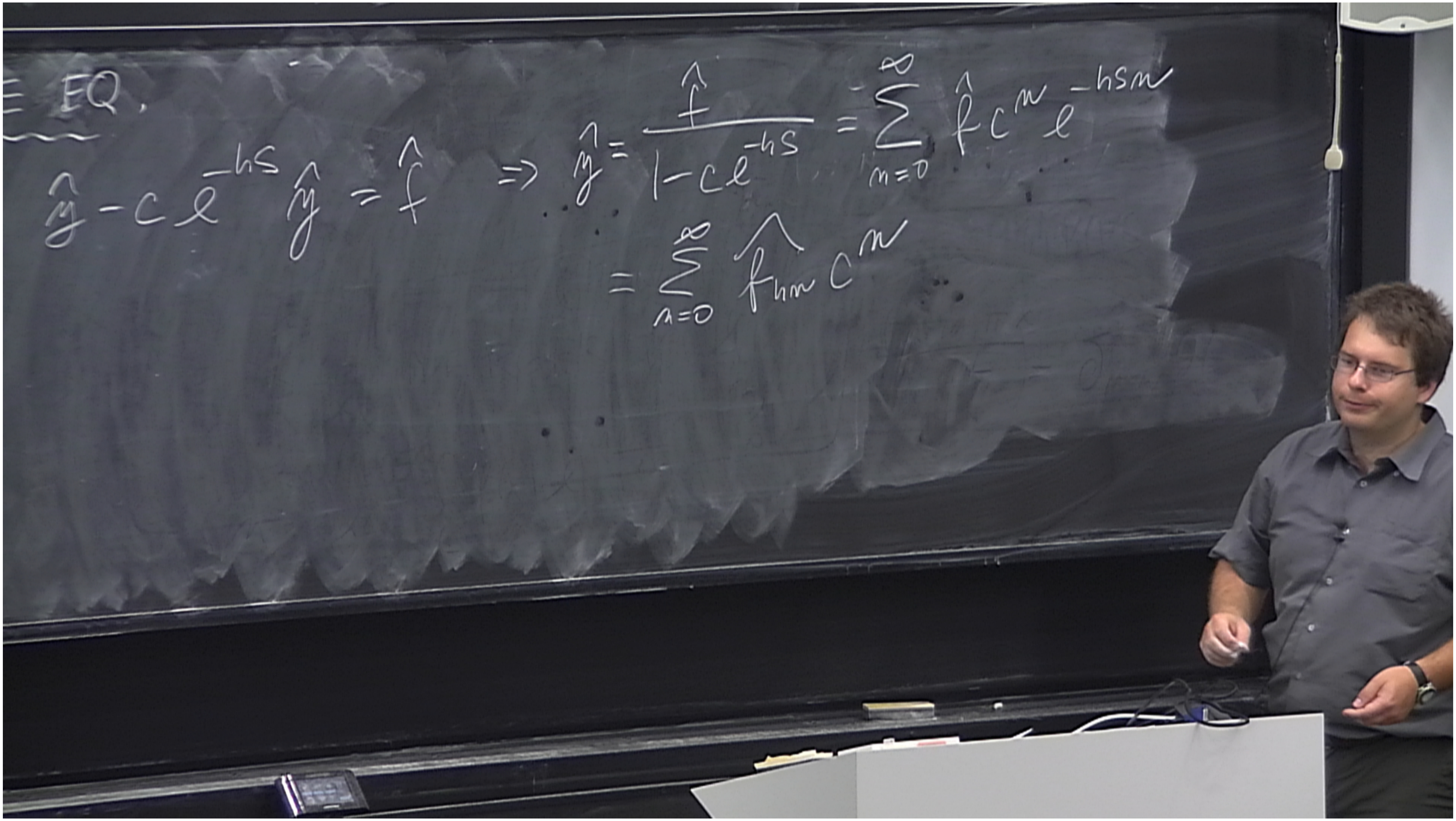
EX: USING b) SOLVE THE DIFFERENCE EQ.

$$\boxed{\begin{gathered} y(t) - c\, y(t-h) = f(t) \\ y(t) = f(t) = 0 \quad t<0 \end{gathered}}$$

$\hat{y} - c$


VE THE DIFFERENCE EQ.
$\hat{y} - c e^{-hs} \hat{y} = \hat{f} \Rightarrow \hat{y}$
$(t-h) = f(t)$
$0 \quad t<0$


E EQ.
$\hat{y} - c e^{-hs} \hat{y} = \hat{f} \Rightarrow \hat{y} = \frac{\hat{f}}{1 - c e^{-hs}} = \sum_{n=0}^{\infty} \hat{f} c^{n} e^{-hsn}$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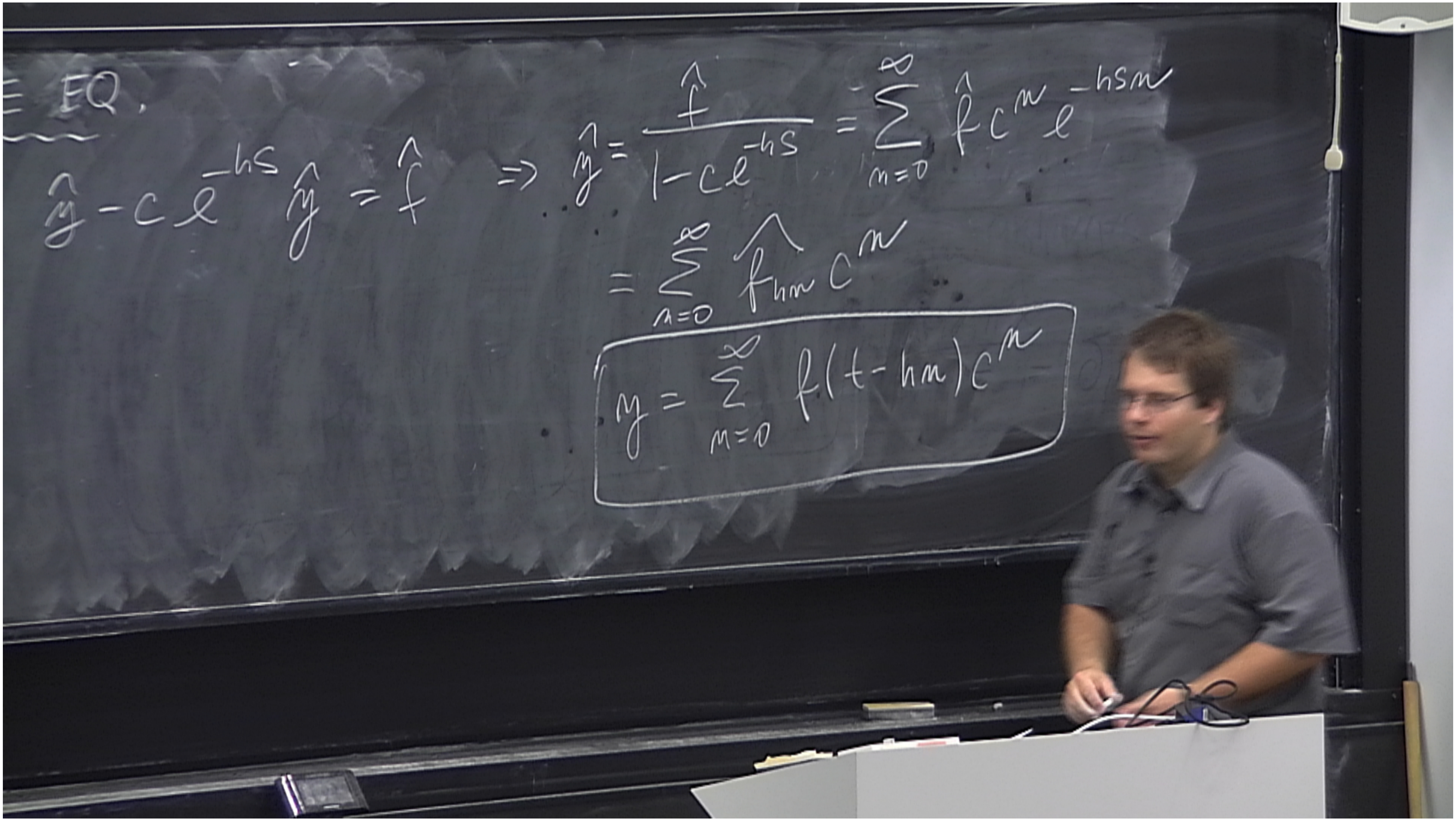
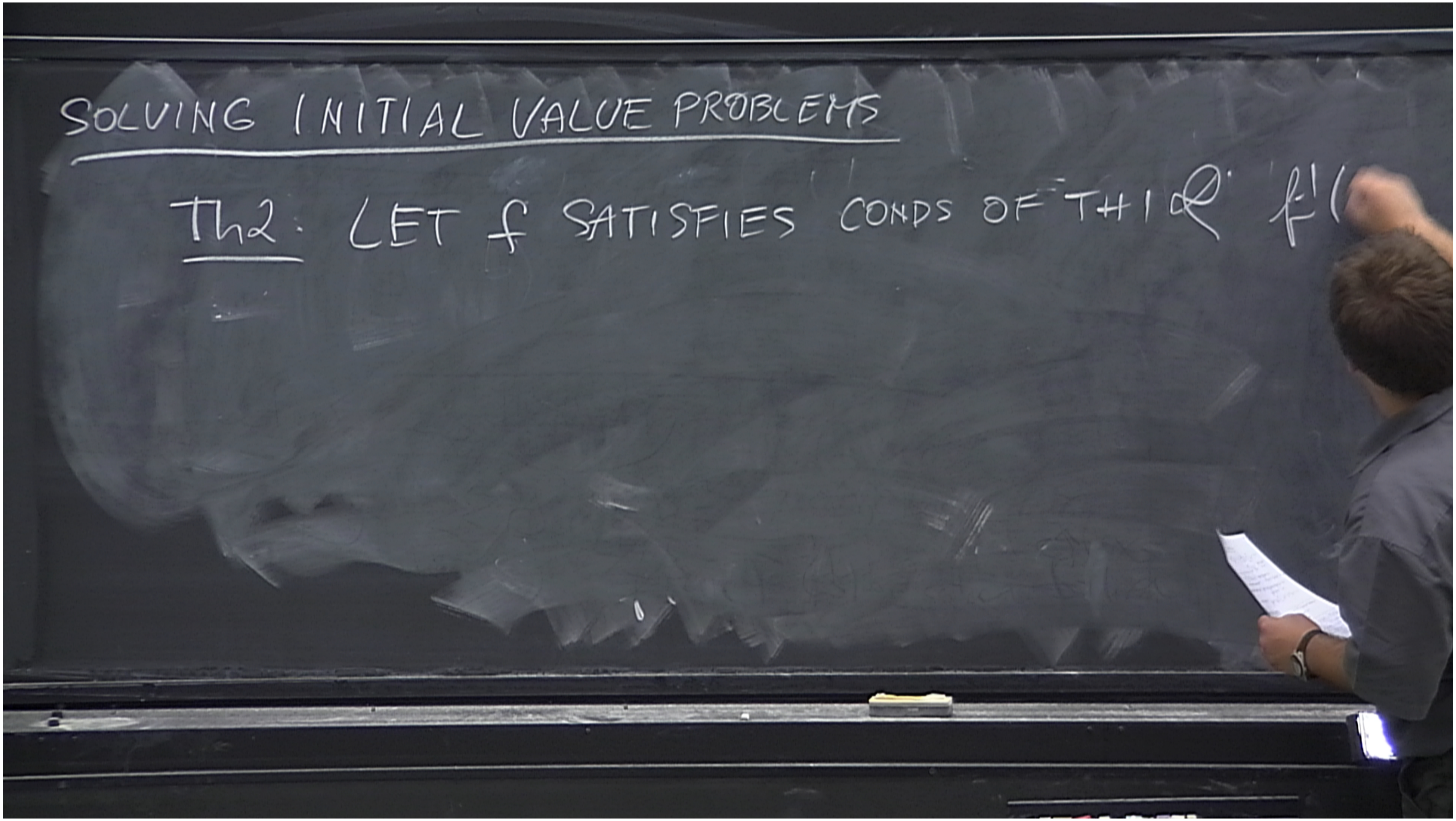
$= \sum_{n=0}^{\infty} \widehat{f_{hn}} c^n$


EQ.
$\hat{y} - c e^{-hs} \hat{y} = \hat{f} \Rightarrow \hat{y} = \frac{\hat{f}}{1 - c e^{-hs}} = \sum_{n=0}^{\infty} \hat{f} c^n e^{-hsn}$
$= \sum_{n=0}^{\infty} \widehat{f_{hn}} c^n$
$y = \sum_{n=0}^{\infty} f(t - hn) c^n$


SOLVING INITIAL VALUE PROBLEMS
Th2: LET f SATISFIES CONDS OF TH1

TS

CONDS OF THM: $f'(t)$ IS PIECEWISE CONT. FOR $t \geq 0$

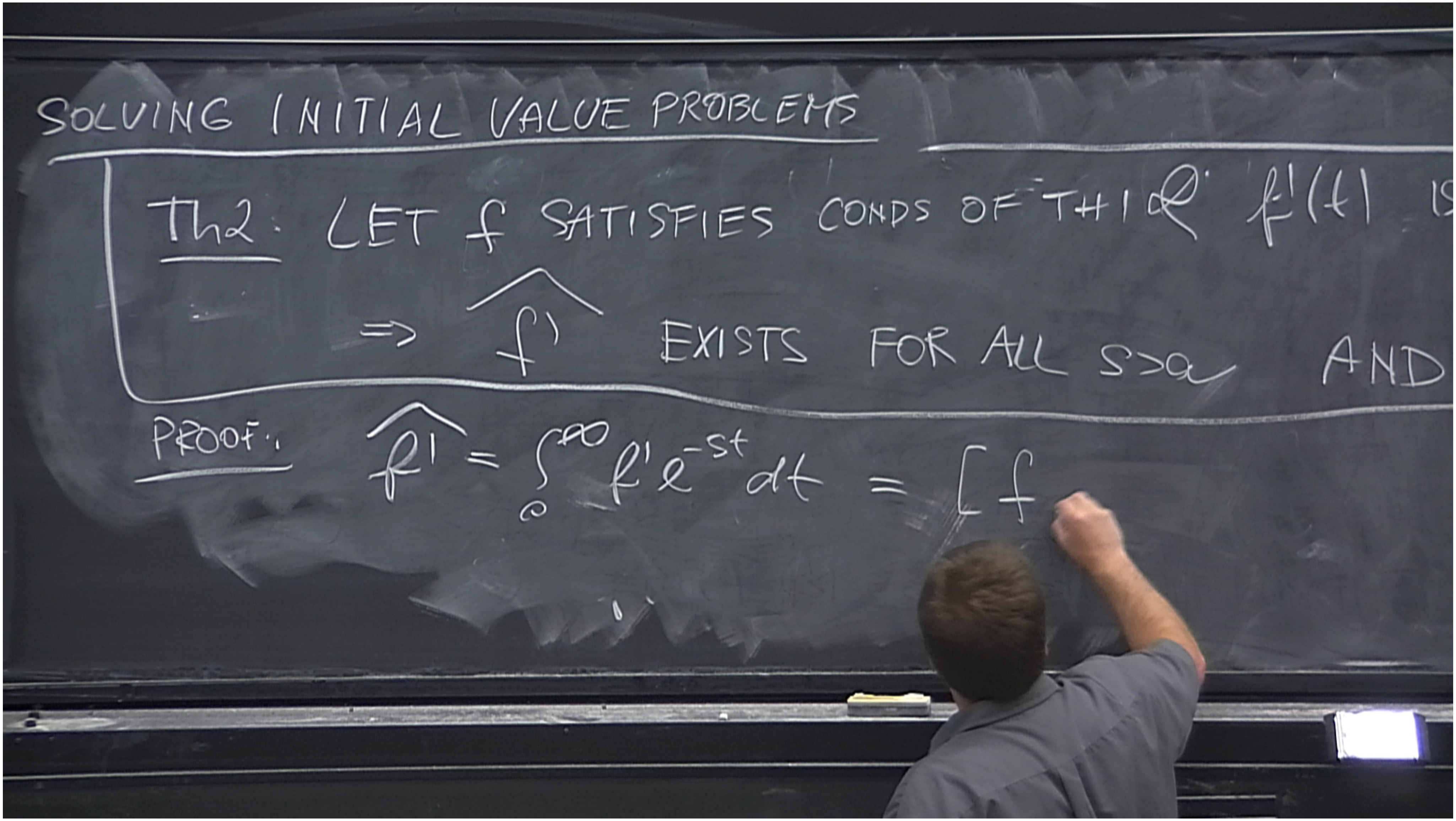
ING INITIAL VALUE PROBLEMS

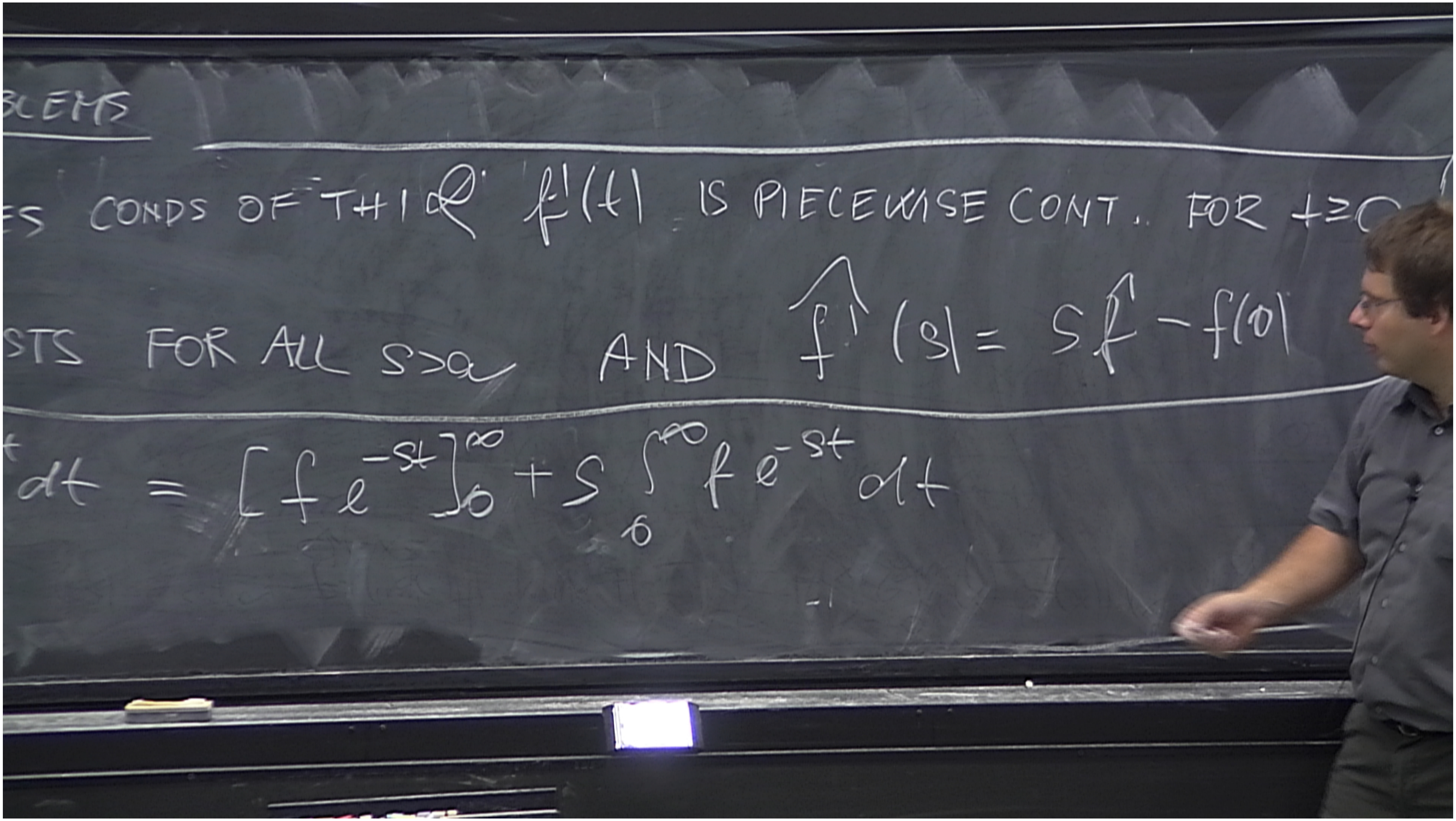
Th2: LET f SATISFIES CONDS OF TH1 & $f'(t)$ IS PIECE

$\Rightarrow$ $\widehat{f'}$ EXISTS FOR ALL $s>a$ AND $\widehat{f'}$

OF THIS $f'(t)$ IS PIECEWISE CONT. FOR $t \geq 0$

ALL $s > a$ AND $\widehat{f'}(s) = s\hat{f} - f(0)$


SOLVING INITIAL VALUE PROBLEMS
Th2: LET f SATISFIES CONDS OF TH1 & f'(t) IS
⇒ $\widehat{f'}$ EXISTS FOR ALL s>a AND
PROOF: $\widehat{f'} = \int_0^\infty f' e^{-st} dt = [f$

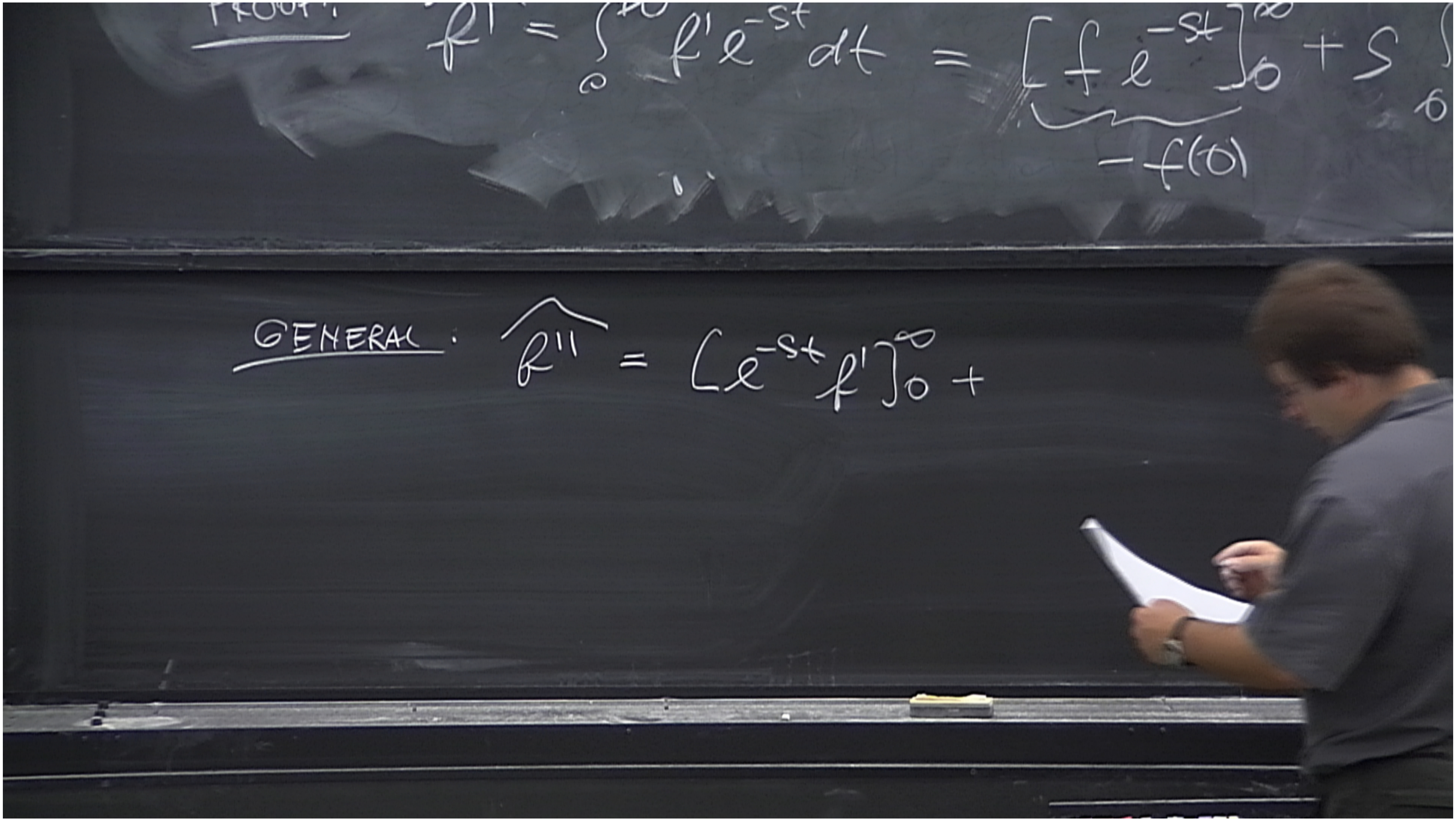

BLEMS
ES CONDS OF THM f'(t) IS PIECEWISE CONT. FOR t≥0
STS FOR ALL s>a AND $\widehat{f'}(s) = s\hat{f} - f(0)$
$dt = [f e^{-st}]_0^\infty + s\int_0^\infty f e^{-st}\,dt$

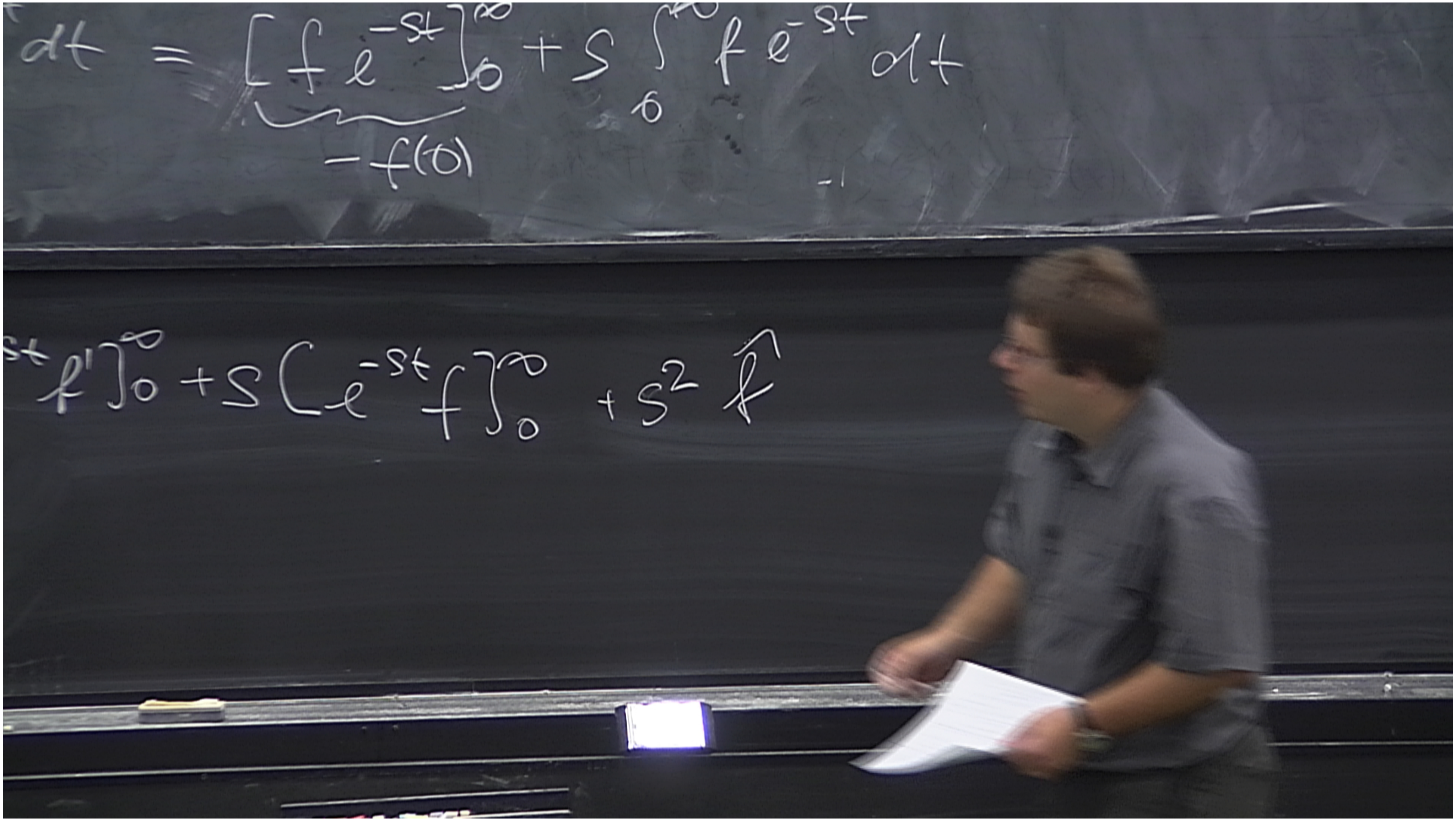
ROBLEMS

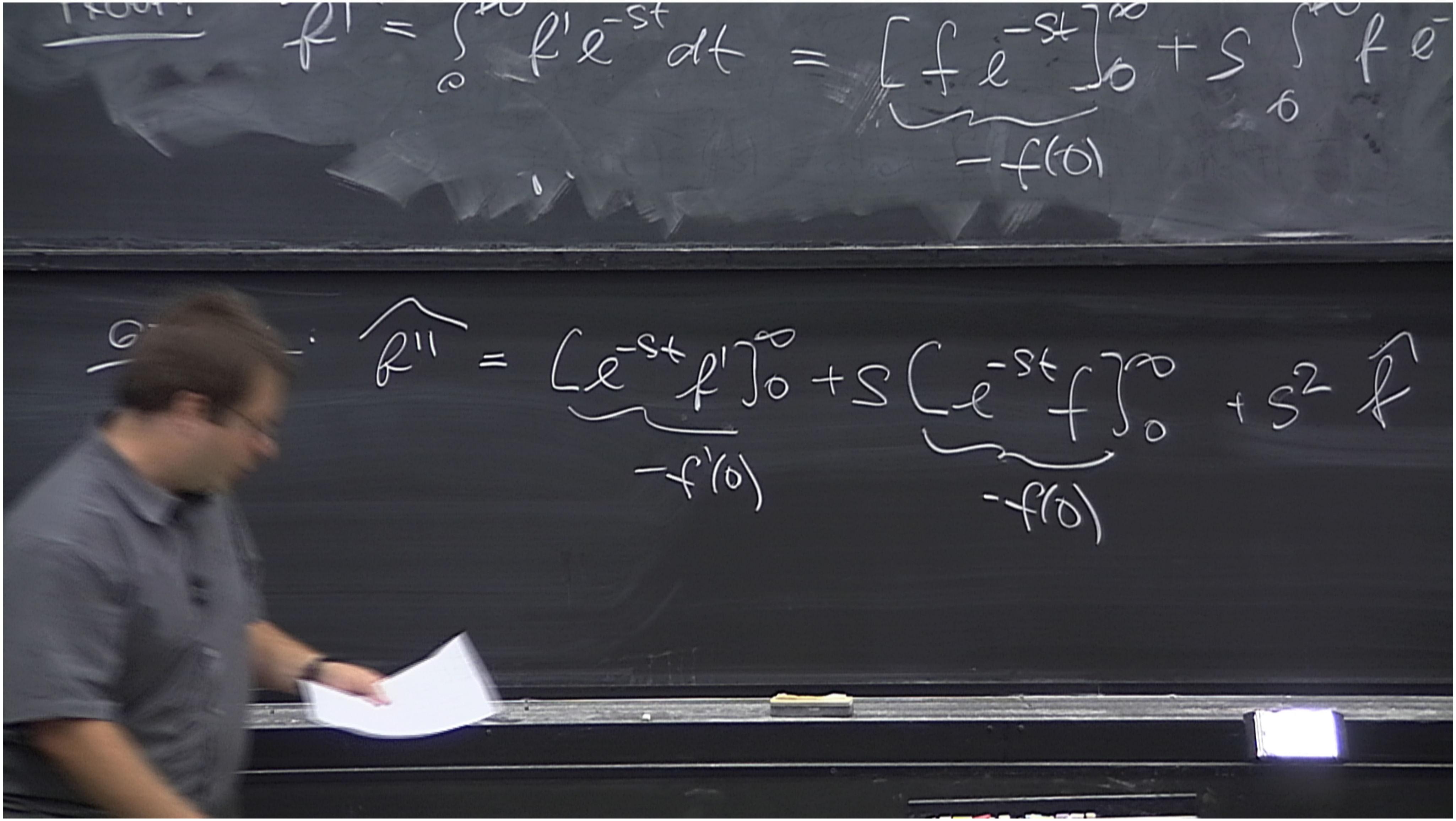
FIES CONDS OF THM $\mathcal{L}$ $f'(t)$ IS PIECEWISE CONT. FOR $t \geq 0$

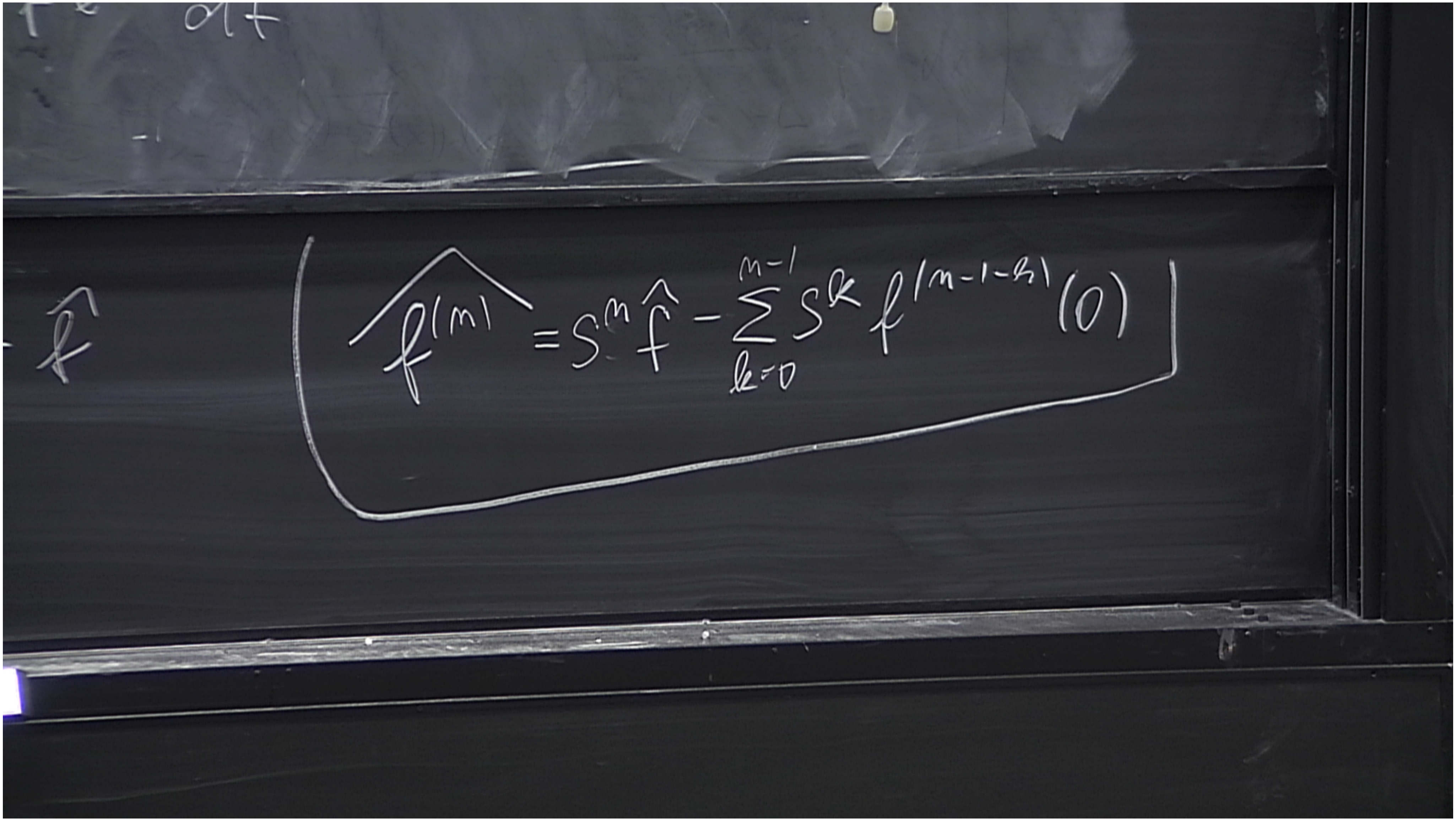
TS FOR ALL $s > a$ AND $\hat{f'}(s) = s\hat{f} - f(0)$

$$= \left[f e^{-st}\right]_0^\infty + s\int_0^\infty f e^{-st}\,dt$$


PROOF:
$\widehat{f'} = \int_0^\infty f' e^{-st} dt = \underbrace{[f e^{-st}]_0^\infty}_{-f(0)} + s \int_0$
GENERAL:
$\widehat{f''} = [e^{-st} f']_0^\infty +$


$dt = \underbrace{[f e^{-st}]_0^\infty}_{-f(0)} + s \int_0^\infty f e^{-st} dt$
$^{st} f']_0^\infty + s[e^{-st} f]_0^\infty + s^2 \hat{f}$


f' = ∫_0^∞ f' e^{-st} dt = [f e^{-st}]_0^∞ + s ∫_0^∞ f e^-
-f(0)
f'' = [e^{-st} f']_0^∞ + s [e^{-st} f]_0^∞ + s^2 f
-f'(0)
-f(0)


$\widehat{f^{(n)}} = s^n \hat{f} - \sum_{k=0}^{n-1} s^k f^{(n-1-k)}(0)$