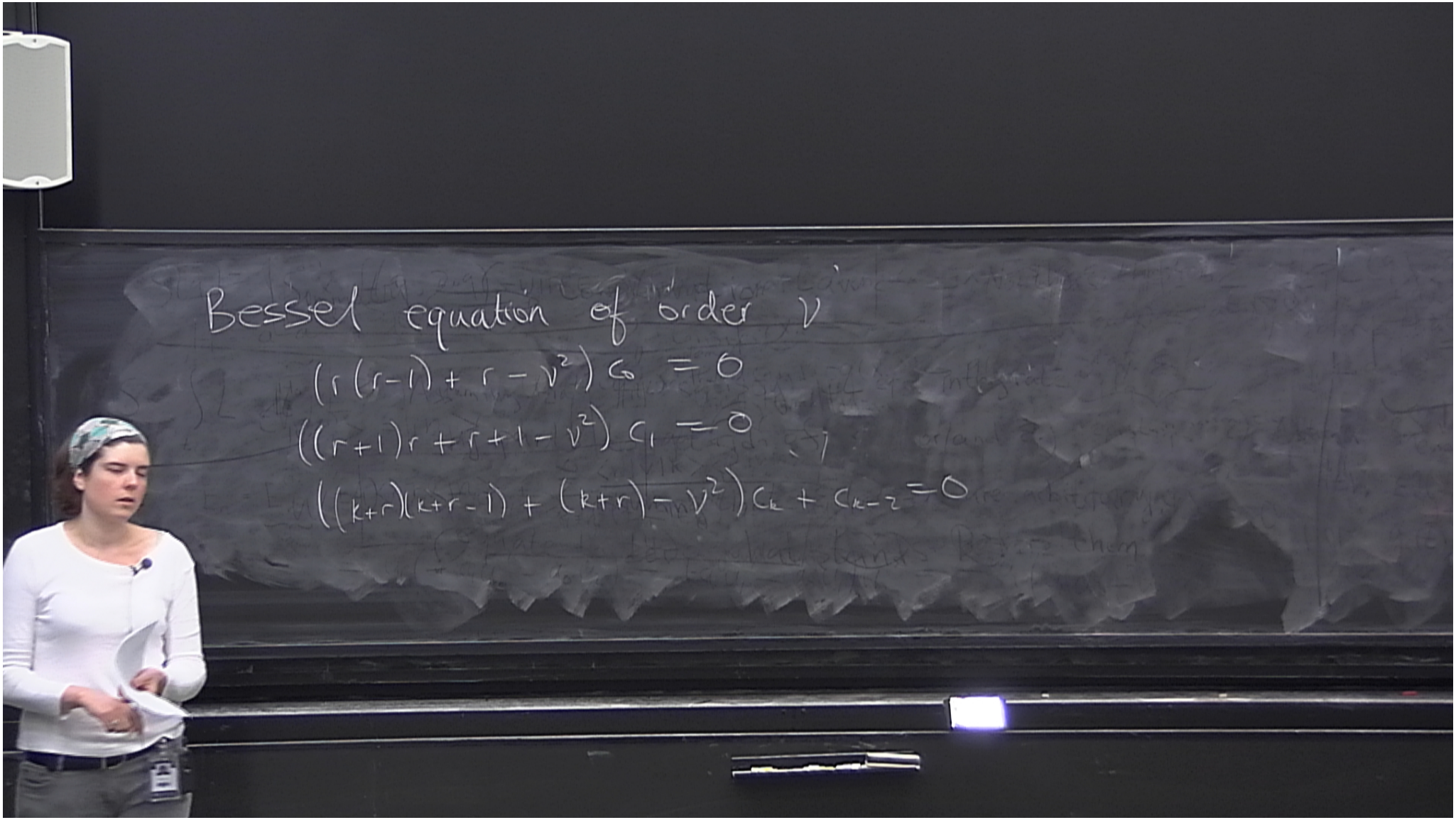
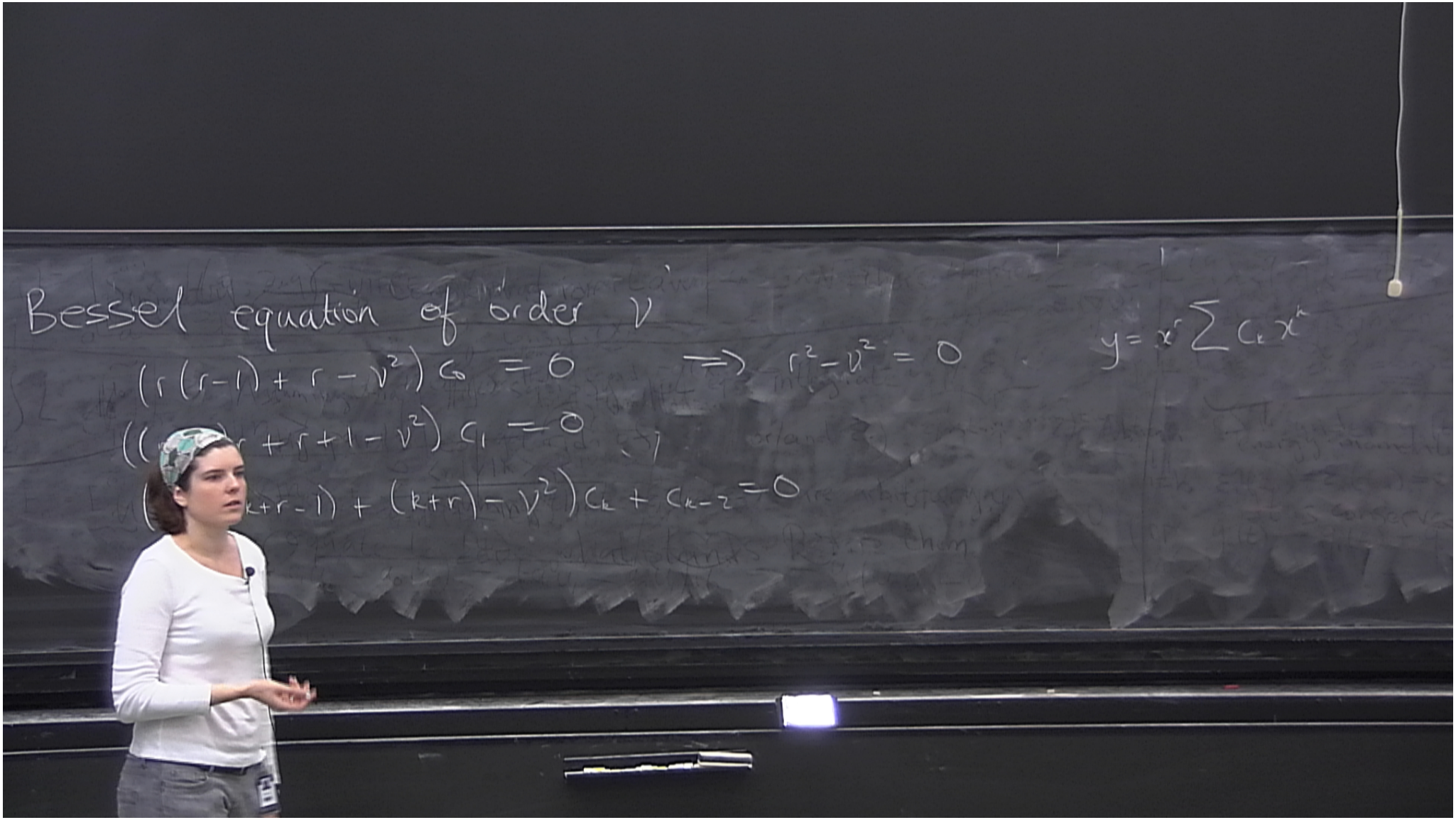
Title: Differential Equations - Lecture 4

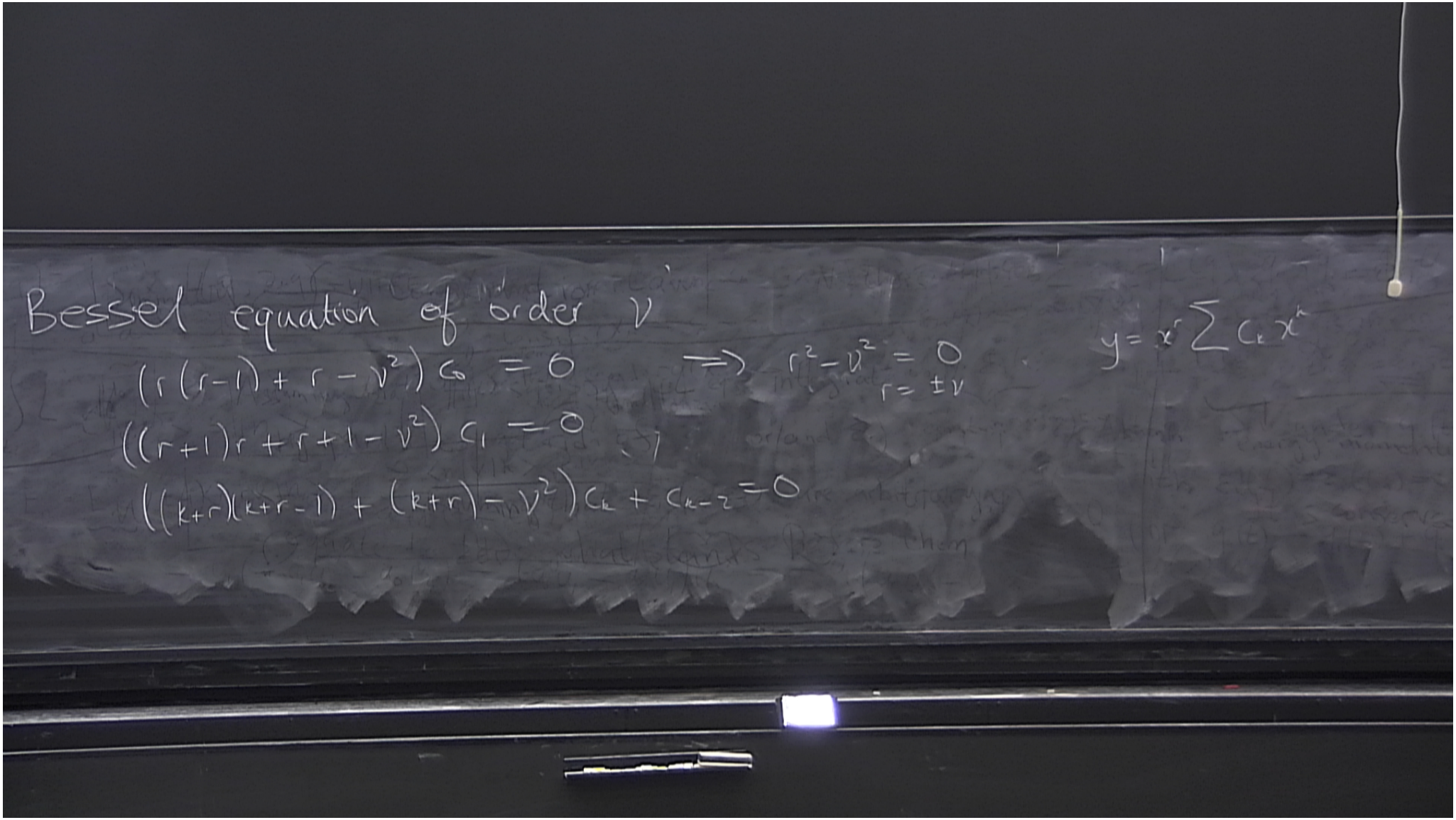
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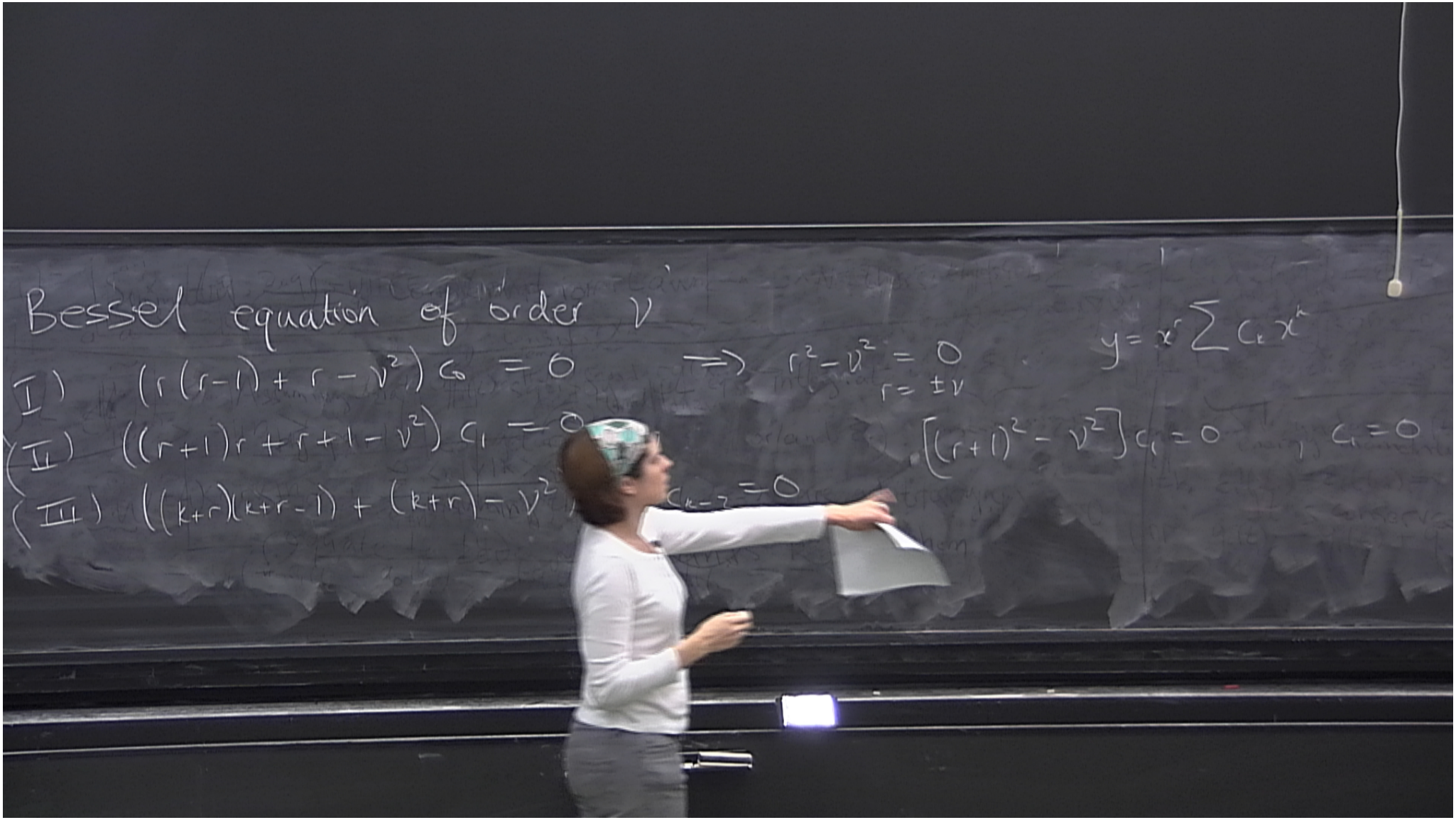
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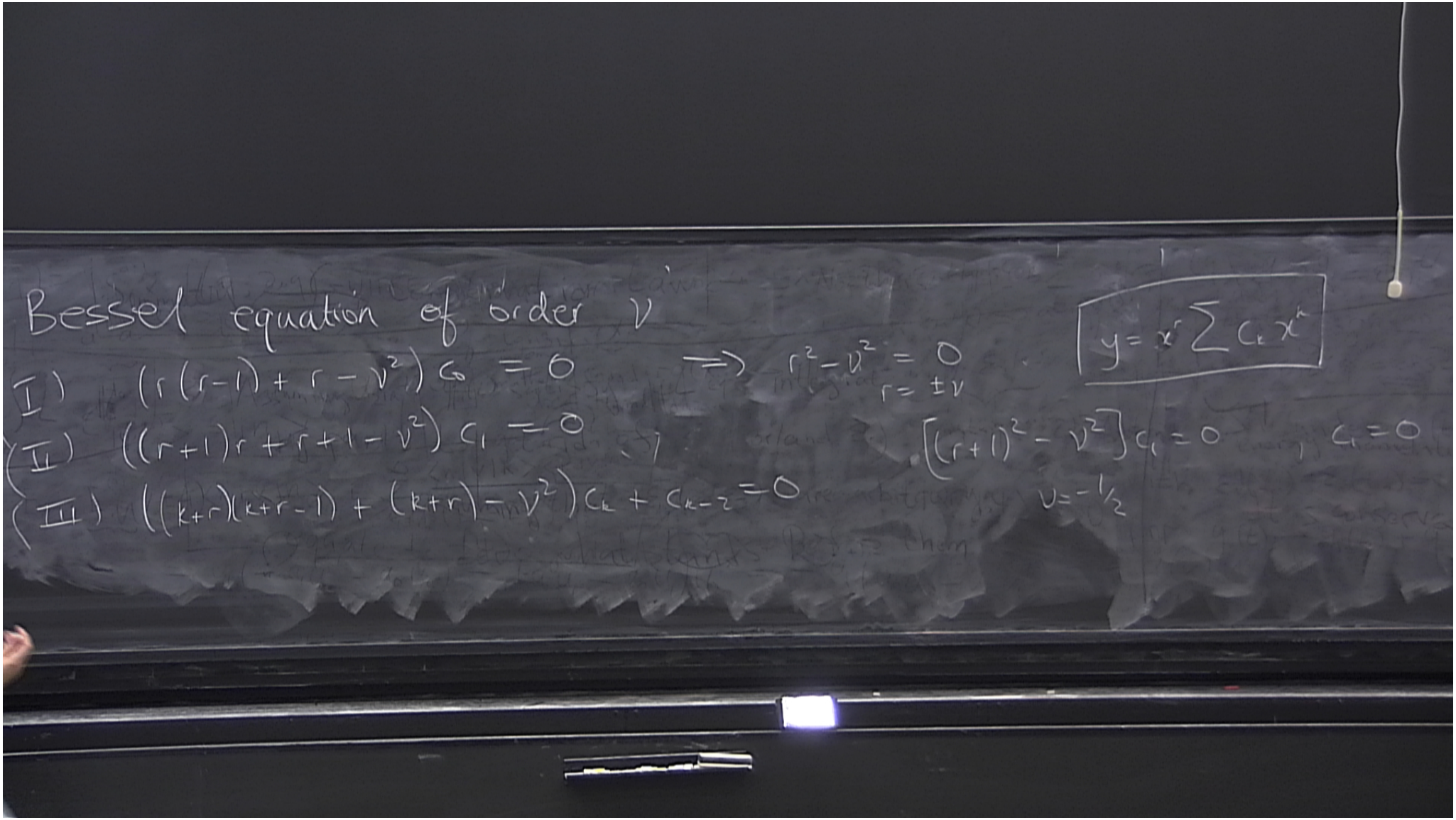
Abstract:


Bessel equation of order ν
$(r(r-1)+r-\nu^2)c_0=0$
$((r+1)r+r+1-\nu^2)c_1=0$
$((k+r)(k+r-1)+(k+r)-\nu^2)c_k+c_{k-2}=0$

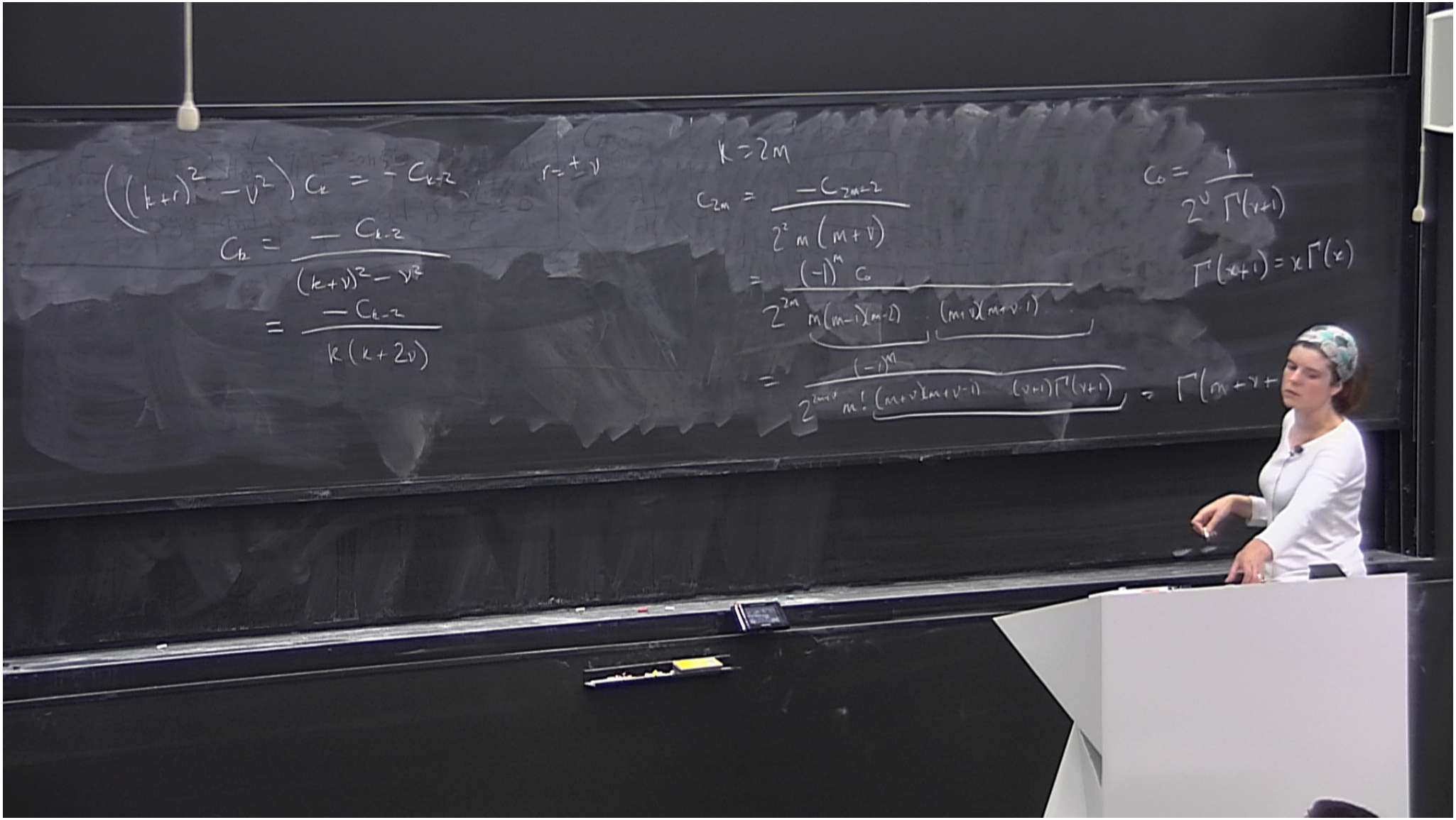

Bessel equation of order ν
$(r(r-1)+r-\nu^2)c_0 = 0 \Rightarrow r^2-\nu^2=0$
$y = x^r \sum c_k x^k$
$(r+1)r+r+1-\nu^2)c_1 = 0$
$(k+r-1)+(k+r)-\nu^2)c_k + c_{k-2} = 0$


Bessel equation of order ν
$(r(r-1)+r-\nu^2)c_0 = 0 \Rightarrow r^2-\nu^2=0$
$r=\pm\nu$
$y = x^r \sum c_k x^k$
$((r+1)r+r+1-\nu^2)c_1 = 0$
$((k+r)(k+r-1)+(k+r)-\nu^2)c_k + c_{k-2} = 0$

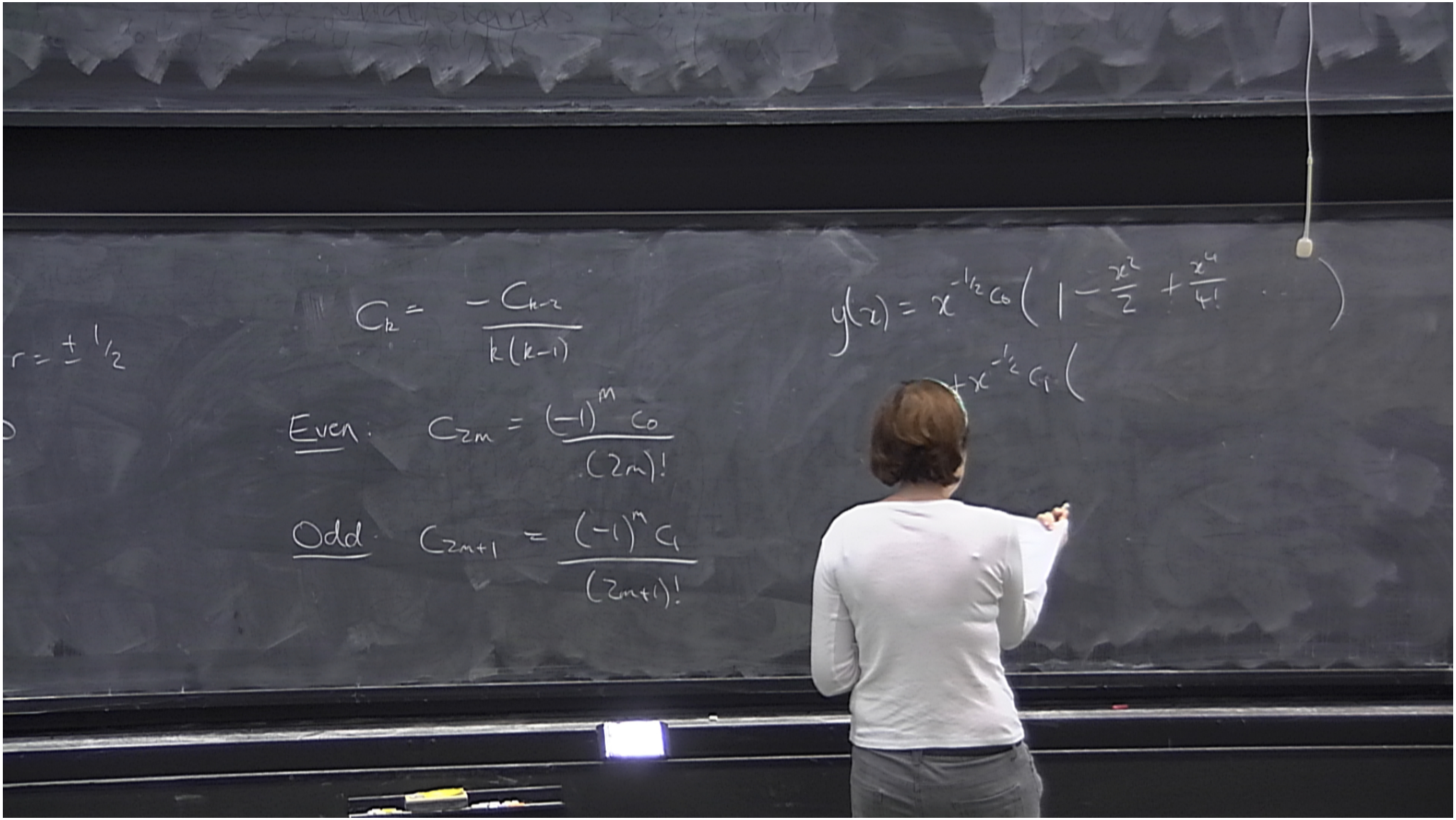

Bessel equation of order ν
I) (r(r-1) + r - ν²) c₀ = 0 ⇒ r² - ν² = 0
r = ±ν
y = x^r Σ c_k x^k
II) ((r+1)r + r + 1 - ν²) c₁ = 0
[(r+1)² - ν²] c₁ = 0 , c₁ = 0
III) ((k+r)(k+r-1) + (k+r) - ν²) c_{k-2} = 0

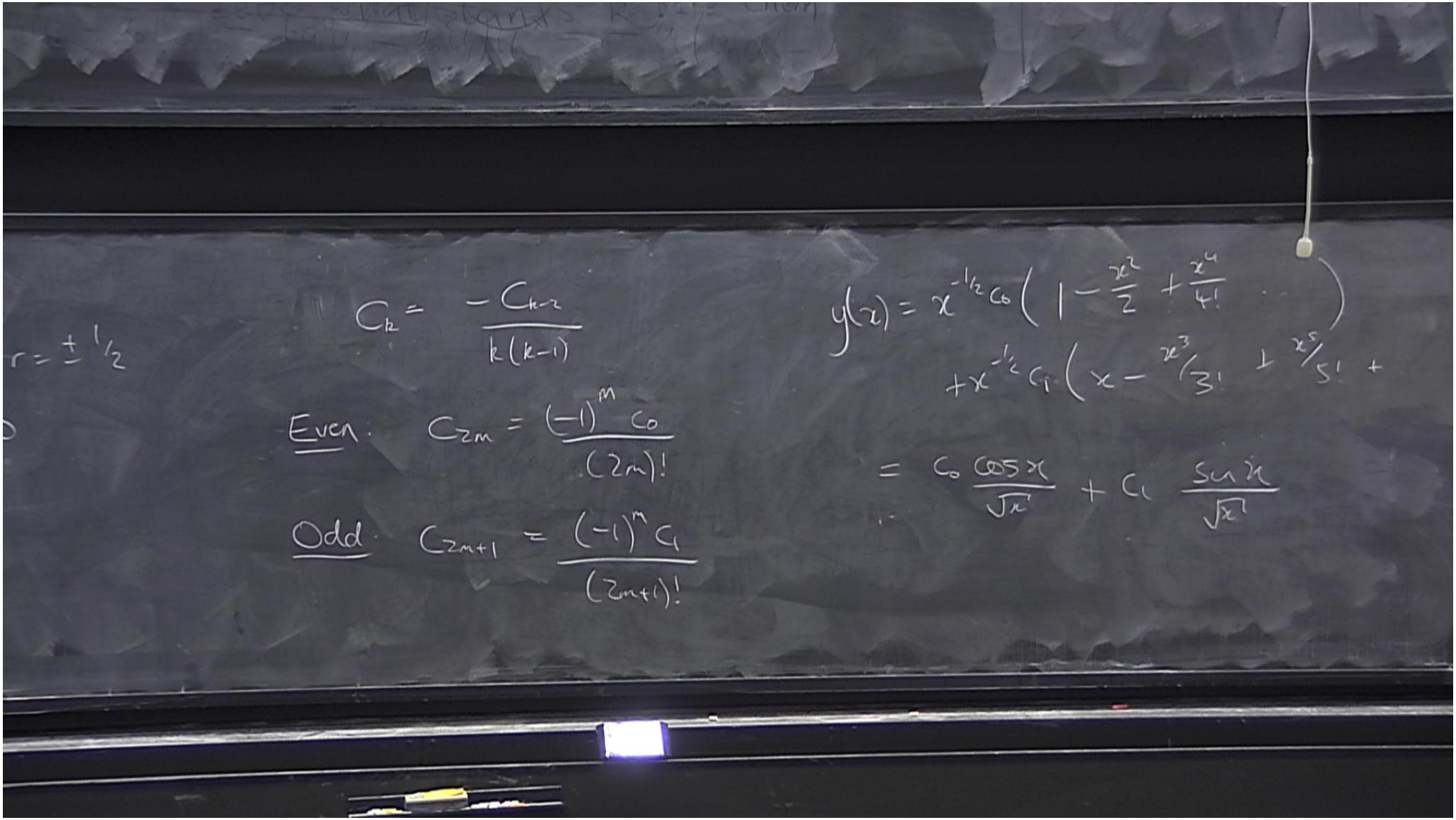

Bessel equation of order ν
I) $(r(r-1)+r-\nu^2)c_0=0 \Rightarrow r^2-\nu^2=0$, $r=\pm\nu$
$y=x^r\sum c_k x^k$
II) $((r+1)r+r+1-\nu^2)c_1=0$
$[(r+1)^2-\nu^2]c_1=0$, $c_1=0$
$\nu=-1/2$
III) $((k+r)(k+r-1)+(k+r)-\nu^2)c_k+c_{k-2}=0$




$\left((k+r)^2 - \nu^2\right) c_k = -c_{k-2}$
$r = \pm\nu$
$c_k = \frac{-c_{k-2}}{(k+\nu)^2 - \nu^2}$
$= \frac{-c_{k-2}}{k(k+2\nu)}$
$k = 2m$
$c_{2m} = \frac{-c_{2m-2}}{2^2 m(m+\nu)}$
$= \frac{(-1)^m c_0}{2^{2m} m(m-1)(m-2)\cdots (m+\nu)(m+\nu-1)\cdots}$
$= \frac{(-1)^m}{2^{2m} m!\,(m+\nu)(m+\nu-1)\cdots(\nu+1)\Gamma(\nu+1)}$
$c_0 = \frac{1}{2^\nu \Gamma(\nu+1)}$
$\Gamma(x+1) = x\Gamma(x)$




$r = \pm 1/2$
$C_k = \frac{-C_{k-2}}{k(k-1)}$
Even: $C_{2m} = \frac{(-1)^m C_0}{(2m)!}$
Odd: $C_{2m+1} = \frac{(-1)^m C_1}{(2m+1)!}$
$y(x) = x^{-1/2} C_0 \left(1 - \frac{x^2}{2} + \frac{x^4}{4!} \cdots\right)$
$+ x^{-1/2} C_1 ($


$r = \pm 1/2$
$C_k = \frac{-C_{k-2}}{k(k-1)}$
Even $C_{2m} = \frac{(-1)^m C_0}{(2m)!}$
Odd $C_{2m+1} = \frac{(-1)^m C_1}{(2m+1)!}$
$y(x) = x^{-1/2} C_0 \left(1 - \frac{x^2}{2} + \frac{x^4}{4!} \cdots\right)$
$+ x^{-1/2} C_1 \left(x - x^3/3! + x^5/5! + \right.$
$= C_0 \frac{\cos x}{\sqrt{x}} + C_1 \frac{\sin x}{\sqrt{x}}$

Odd $c_{2m+1} = \frac{(-1)^m c_1}{(2m+1)!}$

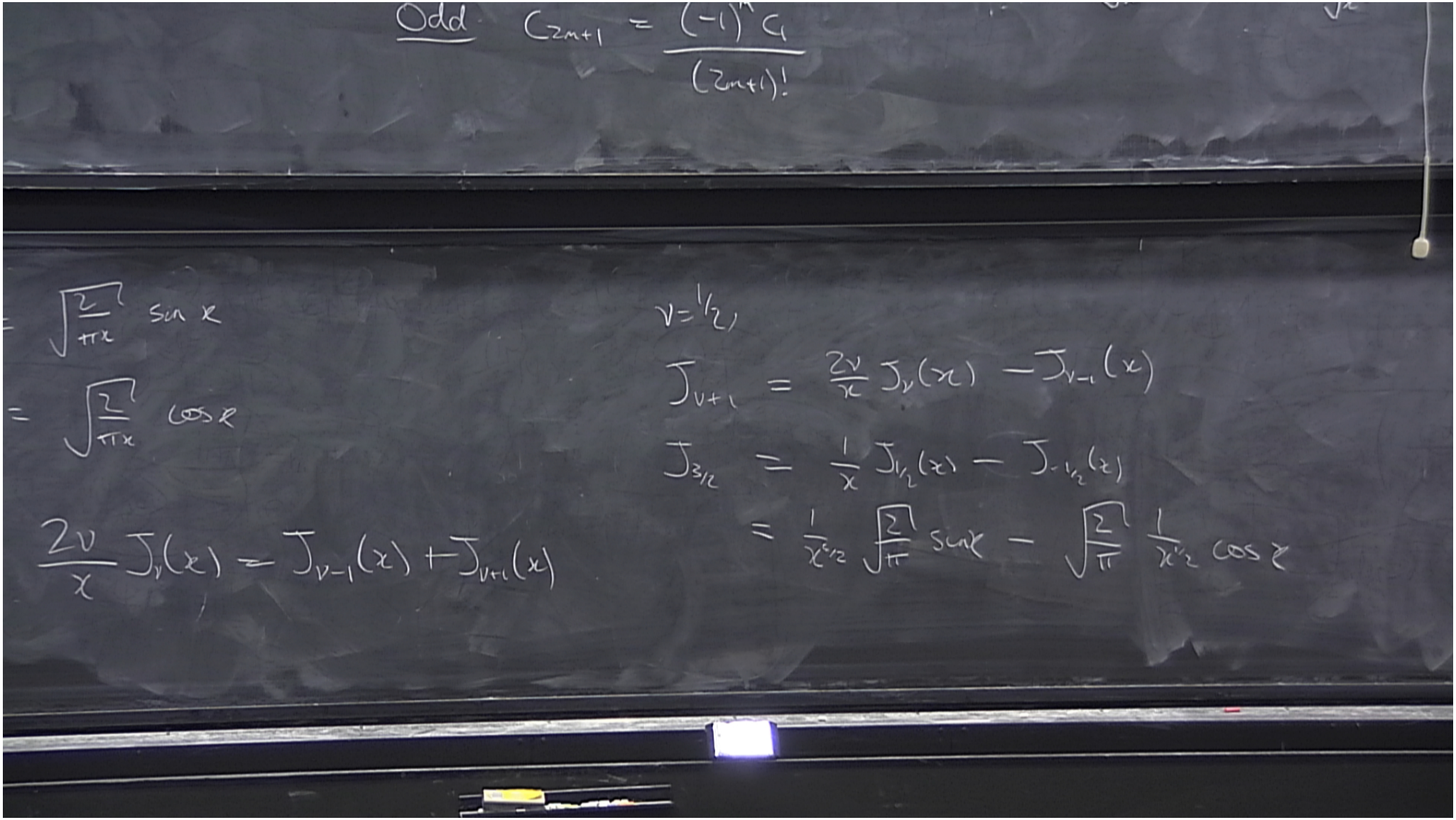
$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$

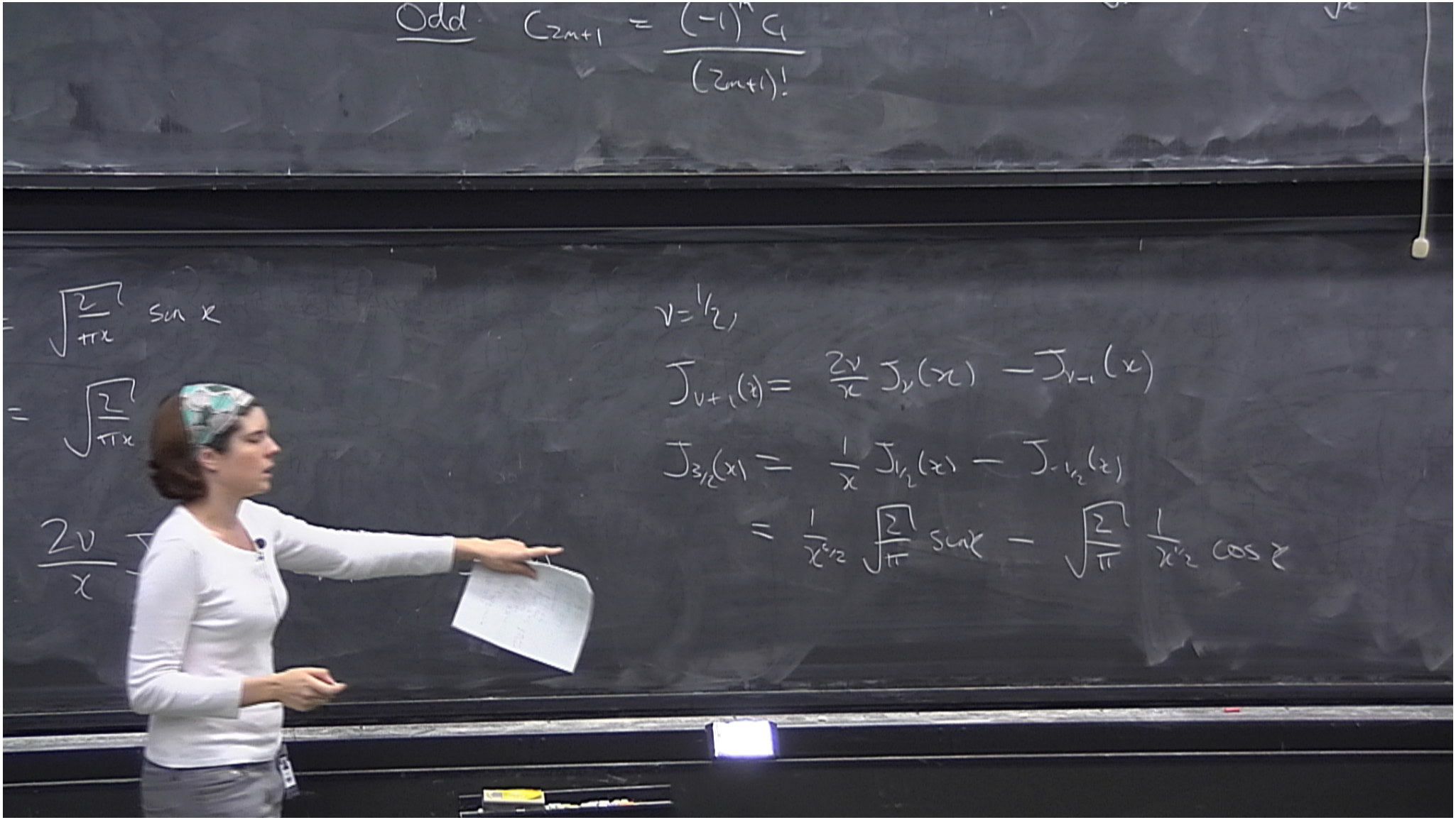
$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$

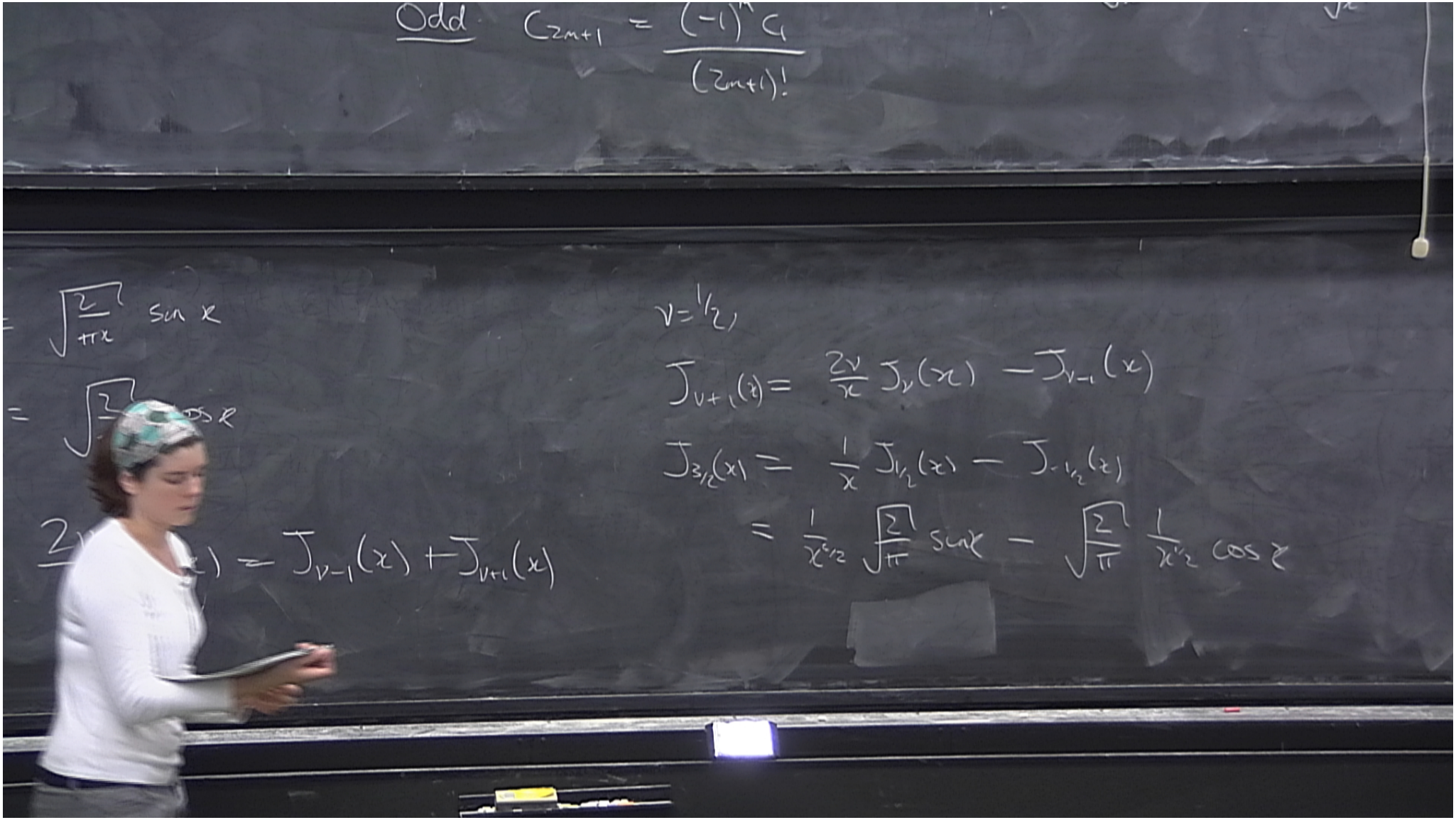
$\nu = 1/2,$

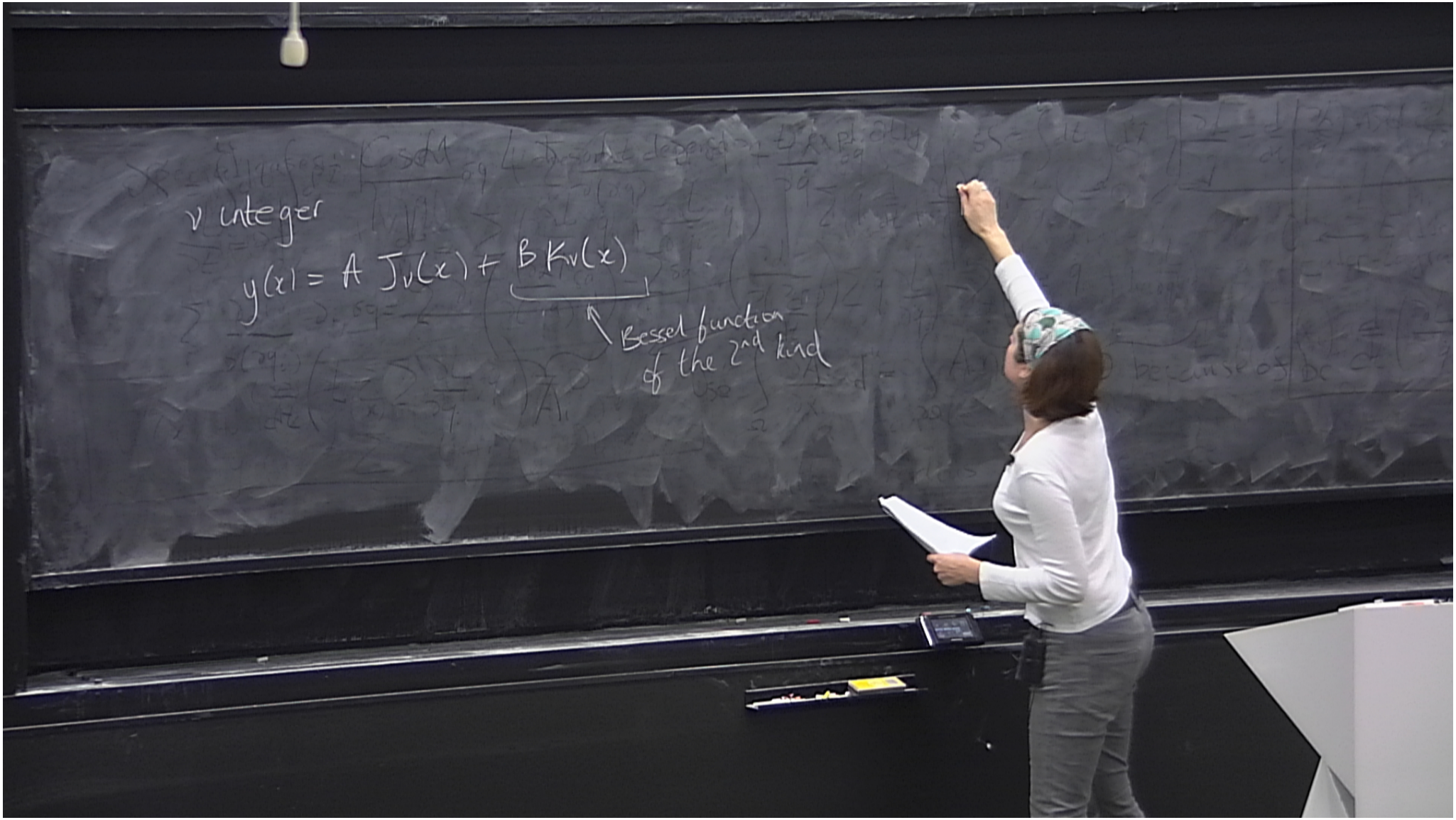
$J_{\nu+1} = \frac{2\nu}{x} J_\nu(x) - J_{\nu-1}(x)$

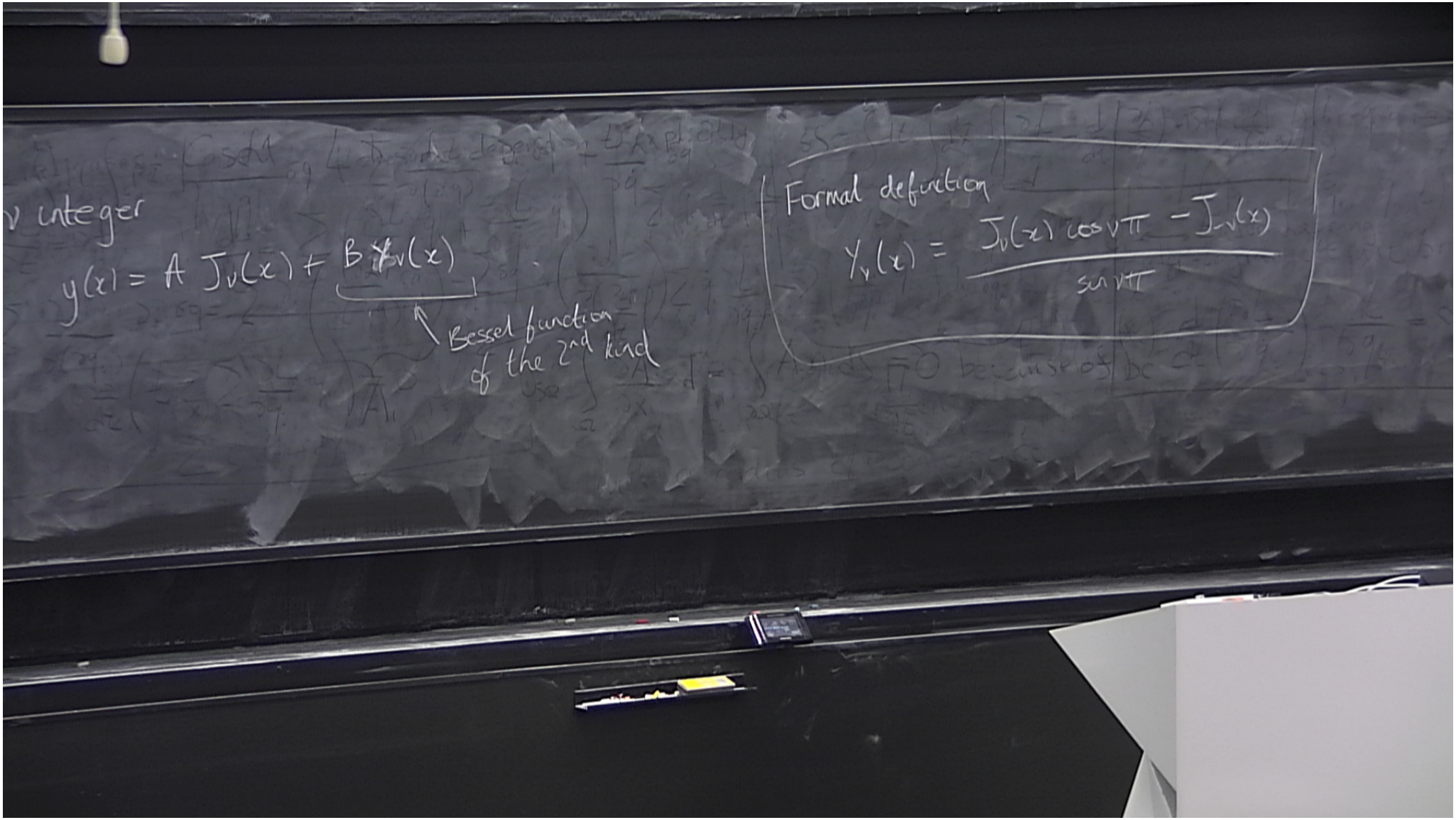
Recursion $\frac{2\nu}{x} J_\nu(x) = J_{\nu-1}(x) + J_{\nu+1}(x)$

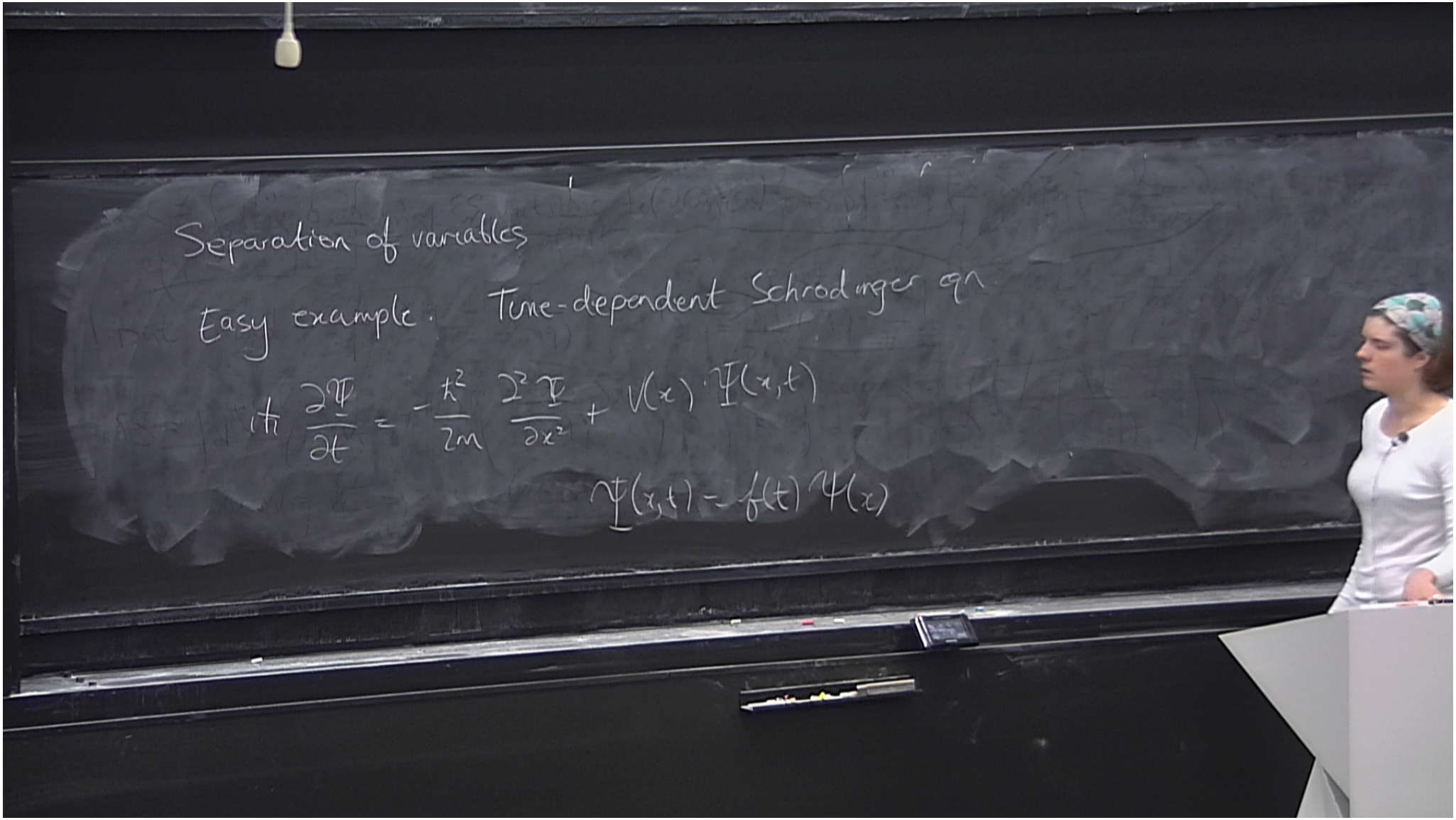

Odd $C_{2m+1} = \frac{(-1)^m C_1}{(2m+1)!}$
$= \sqrt{\frac{2}{\pi x}} \sin x$
$= \sqrt{\frac{2}{\pi x}} \cos x$
$\frac{2\nu}{x} J_\nu(x) = J_{\nu-1}(x) + J_{\nu+1}(x)$
$\nu = 1/2$
$J_{\nu+1} = \frac{2\nu}{x} J_\nu(x) - J_{\nu-1}(x)$
$J_{3/2} = \frac{1}{x} J_{1/2}(x) - J_{-1/2}(x)$
$= \frac{1}{x^{1/2}} \sqrt{\frac{2}{\pi}} \sin x - \sqrt{\frac{2}{\pi}} \frac{1}{x^{1/2}} \cos x$

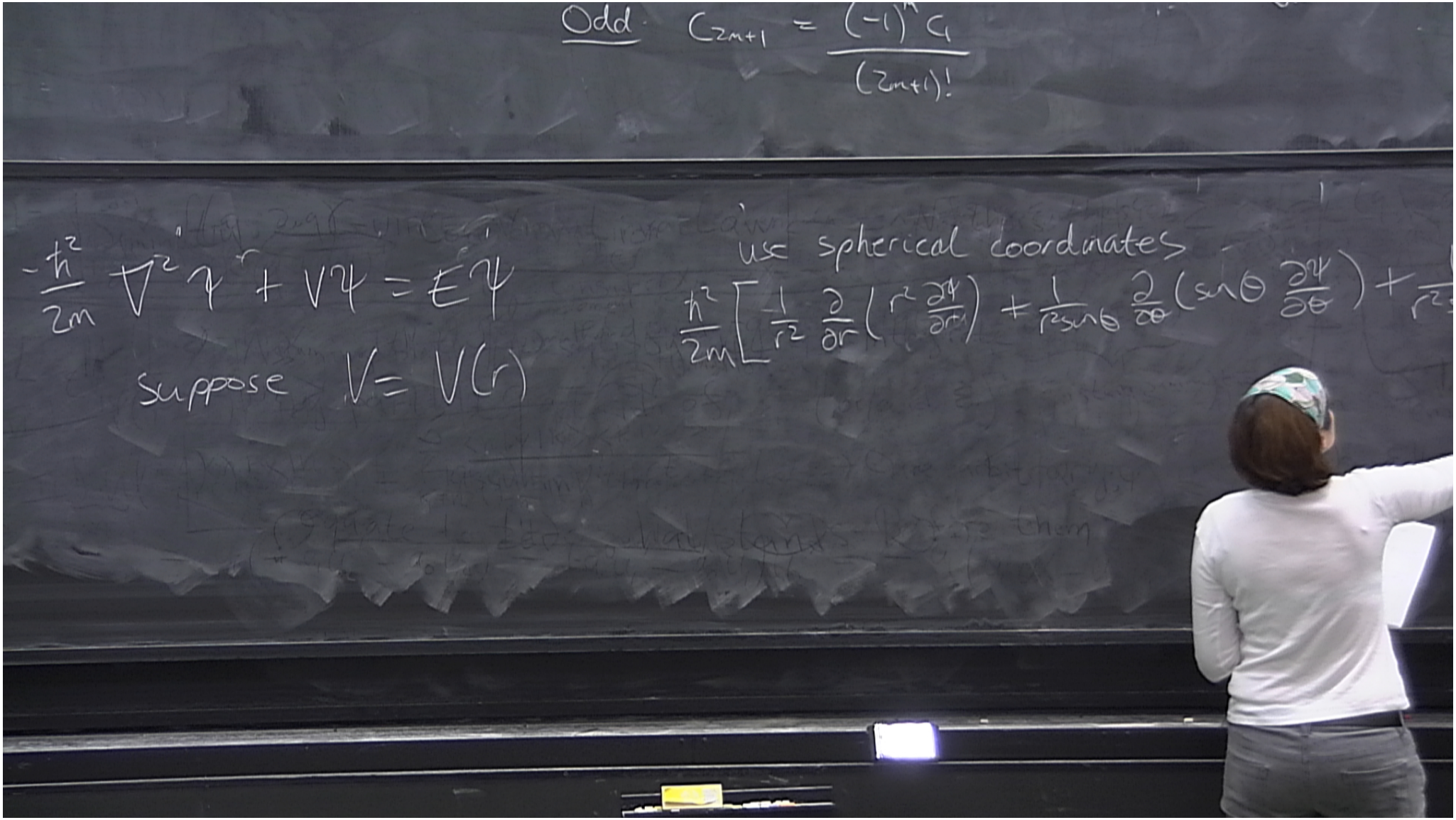

Odd $C_{2m+1} = \frac{(-1)^m C_1}{(2m+1)!}$
$\sqrt{\frac{2}{\pi x}} \sin x$
$\nu = 1/2$
$J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x) - J_{\nu-1}(x)$
$J_{3/2}(x) = \frac{1}{x} J_{1/2}(x) - J_{-1/2}(x)$
$= \frac{1}{x^{1/2}} \sqrt{\frac{2}{\pi}} \sin x - \sqrt{\frac{2}{\pi}} \frac{1}{x^{1/2}} \cos x$

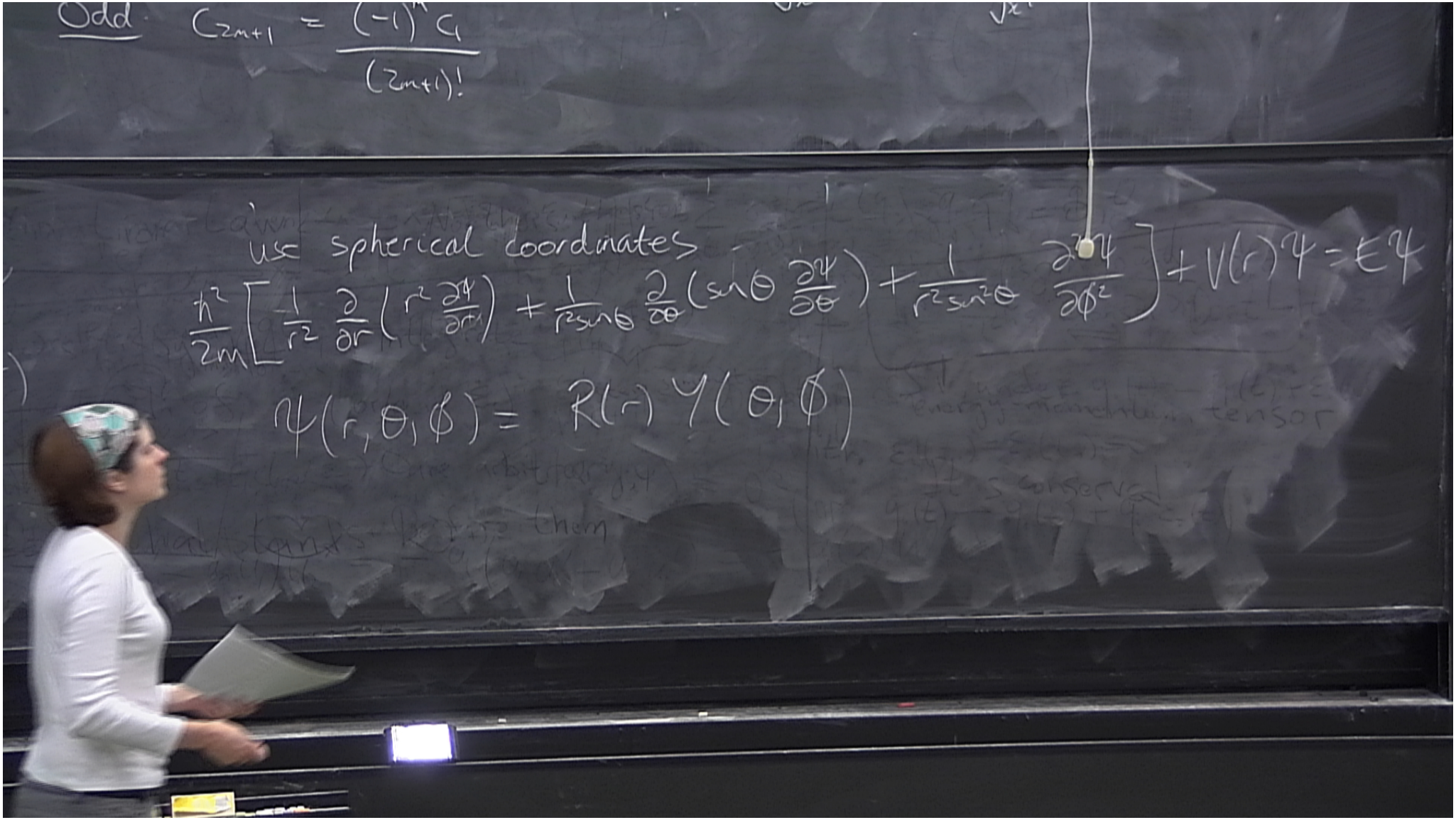

Odd $C_{2m+1} = \frac{(-1)^m C_1}{(2m+1)!}$
$= \sqrt{\frac{2}{\pi x}} \sin x$
$= \sqrt{\frac{2}{\pi x}} \cos x$
$\nu = 1/2$
$J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x) - J_{\nu-1}(x)$
$J_{3/2}(x) = \frac{1}{x} J_{1/2}(x) - J_{-1/2}(x)$
$= \frac{1}{x^{1/2}} \sqrt{\frac{2}{\pi}} \sin x - \sqrt{\frac{2}{\pi}} \frac{1}{x^{1/2}} \cos x$
$J_{\nu-1}(x) + J_{\nu+1}(x)$

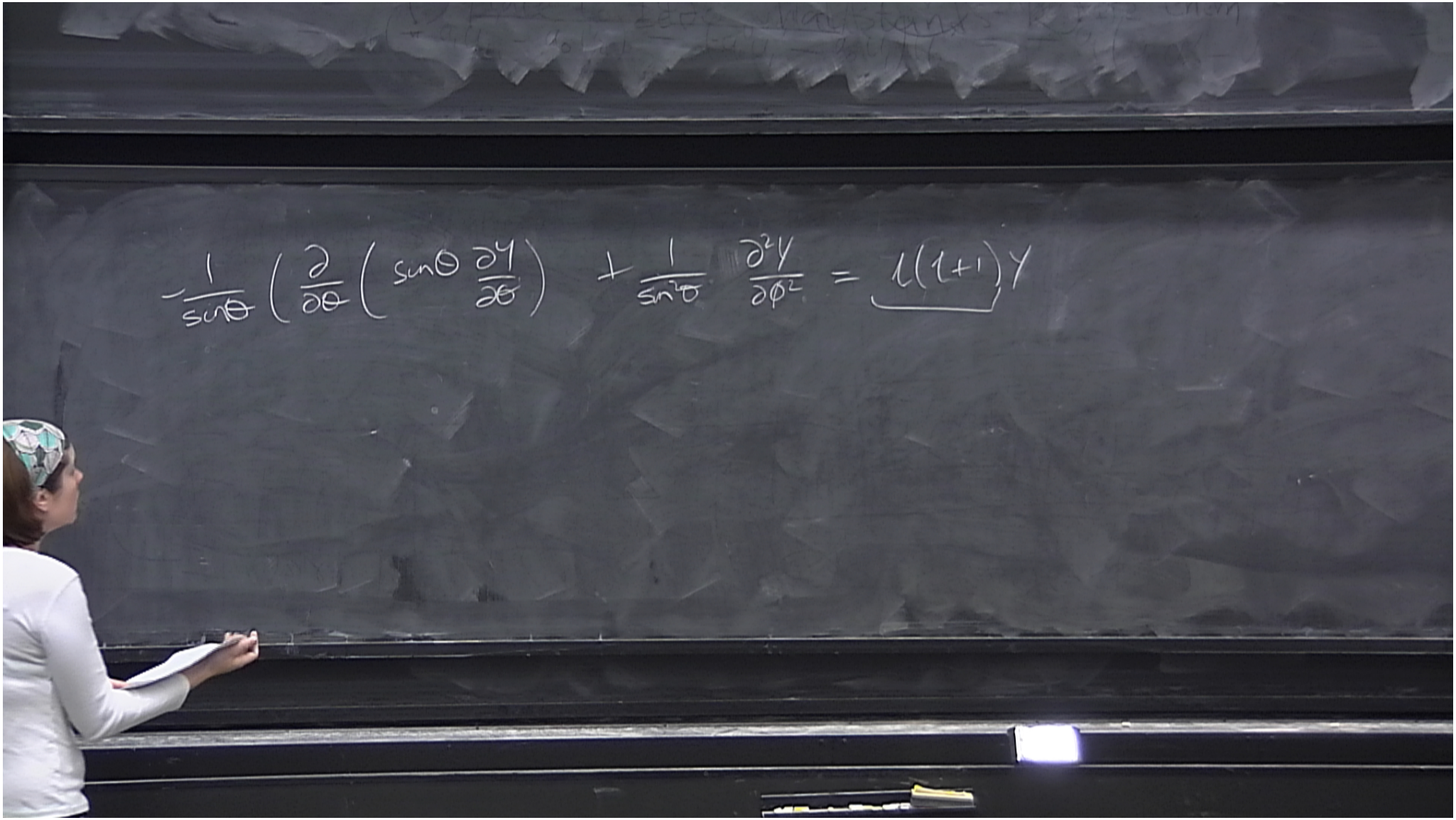

ν integer
$y(x) = A J_\nu(x) + B K_\nu(x)$
Bessel function of the 2nd kind


ν integer
$y(x) = A J_\nu(x) + B Y_\nu(x)$
Bessel function of the 2nd kind
Formal definition
$Y_\nu(x) = \frac{J_\nu(x)\cos\nu\pi - J_{-\nu}(x)}{\sin\nu\pi}$


Separation of variables
Easy example: Time-dependent Schrodinger eqn
$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V(x)\cdot\Psi(x,t)$
$\Psi(x,t) = f(t)\,\psi(x)$


Odd $C_{2m+1} = \frac{(-1)^m C_1}{(2m+1)!}$
$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi$
suppose $V = V(r)$
use spherical coordinates
$\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2}$

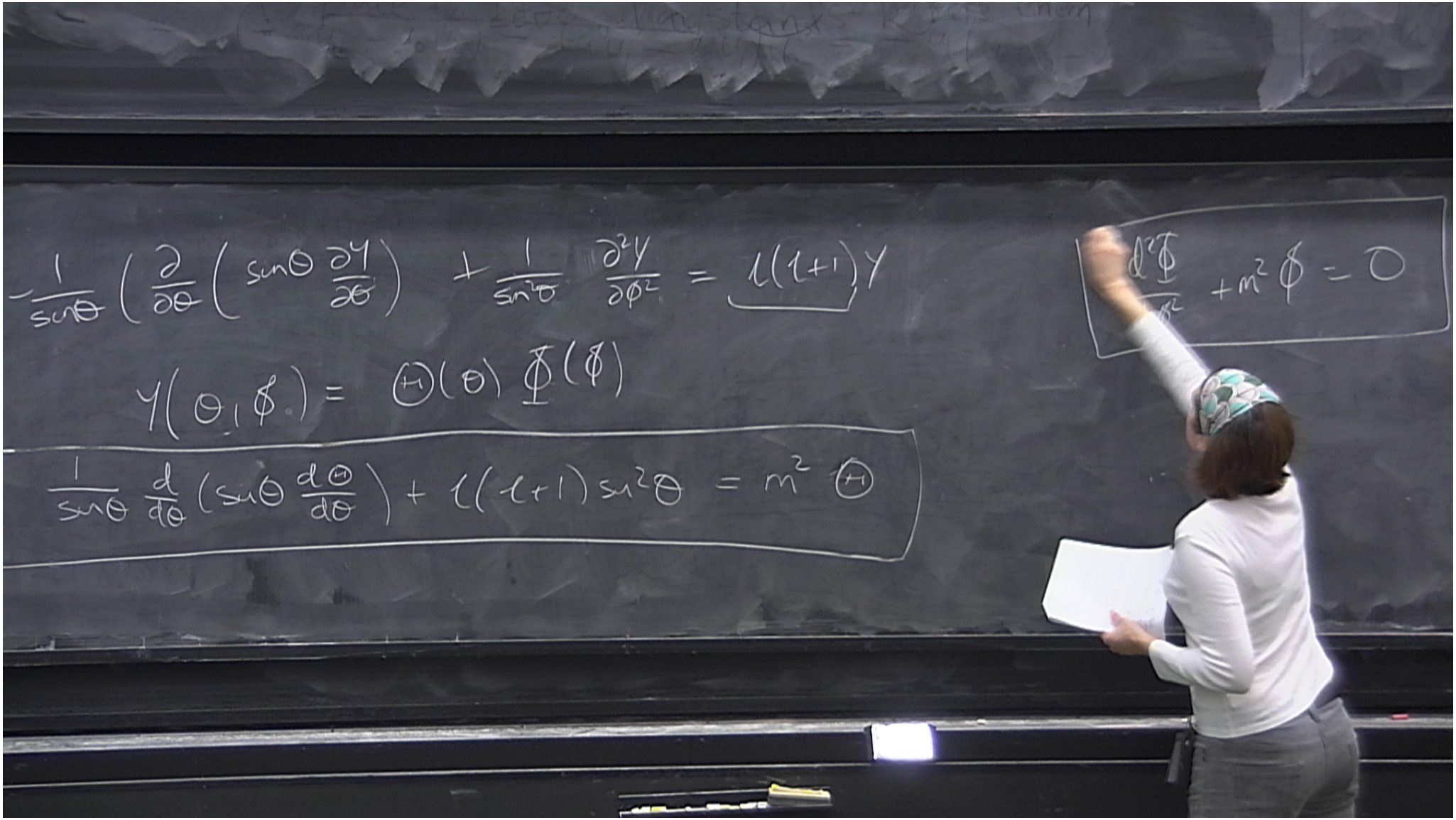

Odd $C_{2m+1} = \frac{(-1)^m C_1}{(2m+1)!}$
use spherical coordinates
$\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}\right] + V(r)\psi = E\psi$
$\psi(r,\theta,\phi) = R(r)\,Y(\theta,\phi)$


$-\frac{1}{\sin\theta}\left(\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial Y}{\partial\theta}\right)\right) + \frac{1}{\sin^2\theta}\,\frac{\partial^2 Y}{\partial\phi^2} = \underline{l(l+1)}\,Y$

$$-\frac{1}{\sin\theta}\left(\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial Y}{\partial\theta}\right)\right) + \frac{1}{\sin^2\theta}\,\frac{\partial^2 Y}{\partial\phi^2} = \underline{l(l+1)}\,Y$$

$$Y(\theta,\phi) = \Theta(\theta)\,\underline{\Phi}(\phi)$$

$$\frac{1}{\sin\theta}\,\frac{d}{d\theta}\left(\sin\theta\,\frac{d\Theta}{d\theta}\right) + l(l+1)\sin^2\theta = m^2\,\Theta$$


$-\frac{1}{\sin\theta}\left(\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial Y}{\partial\theta}\right)\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2} = \ell(\ell+1)Y$
$\frac{d^2\Phi}{d\phi^2} + m^2\Phi = 0$
$Y(\theta,\phi) = \Theta(\theta)\,\Phi(\phi)$
$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{d\Theta}{d\theta}\right) + \ell(\ell+1)\sin^2\theta = m^2\,\Theta$

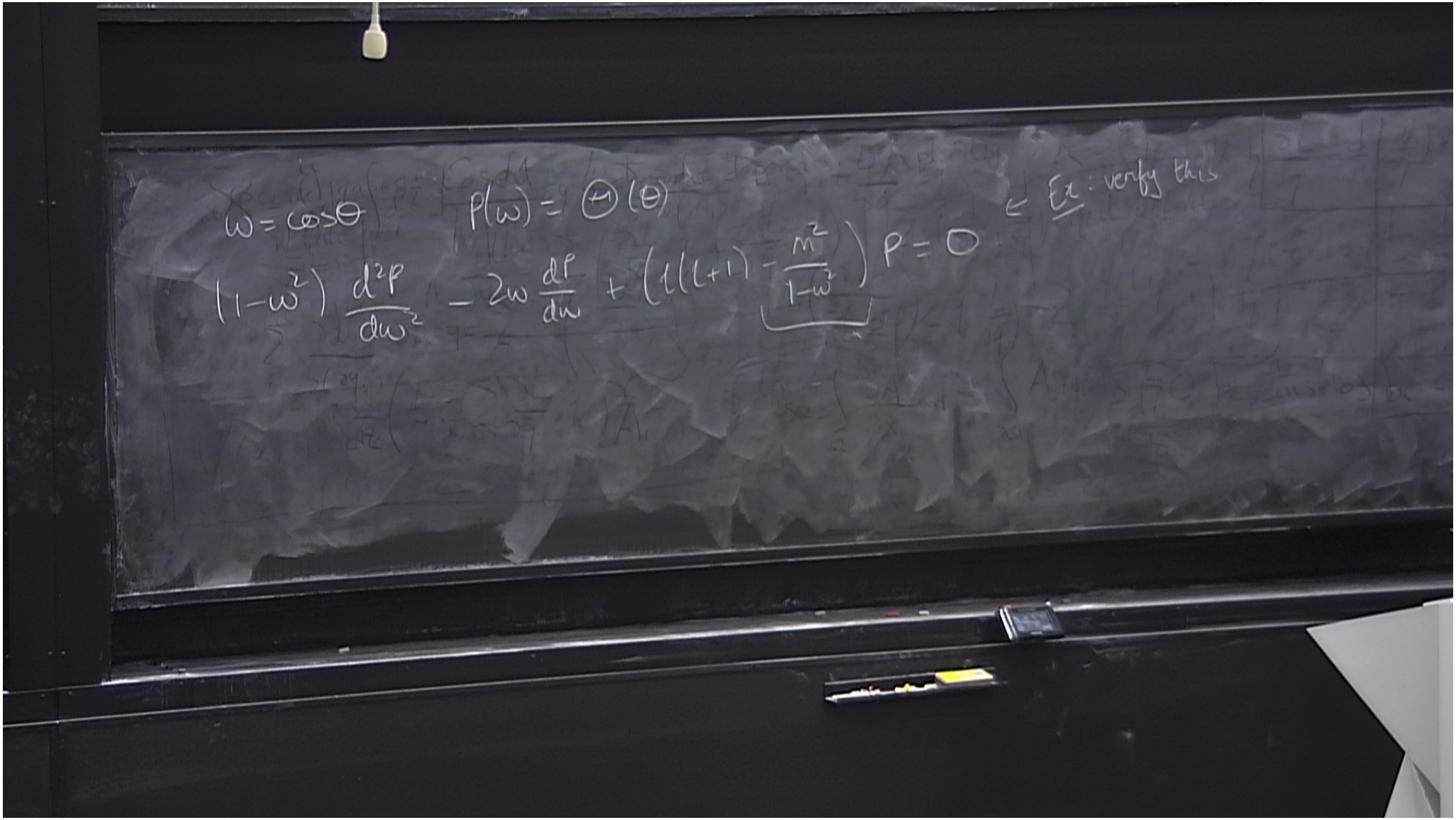
$$\left(\frac{\partial Y}{\partial \theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial \phi^2} = \underline{l(l+1)}\,Y$$

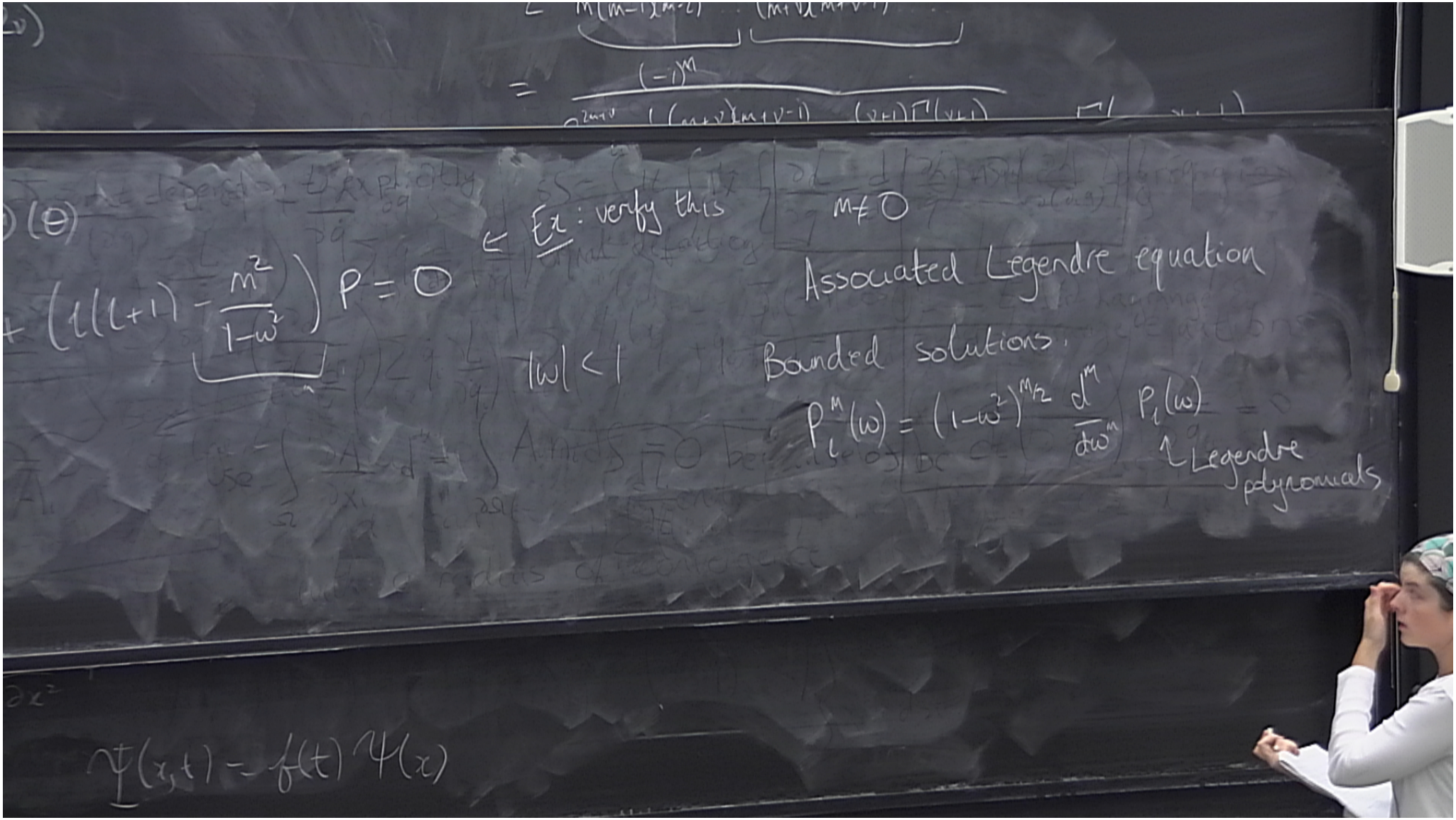
$$= \Theta(\theta)\,\Phi(\phi)$$

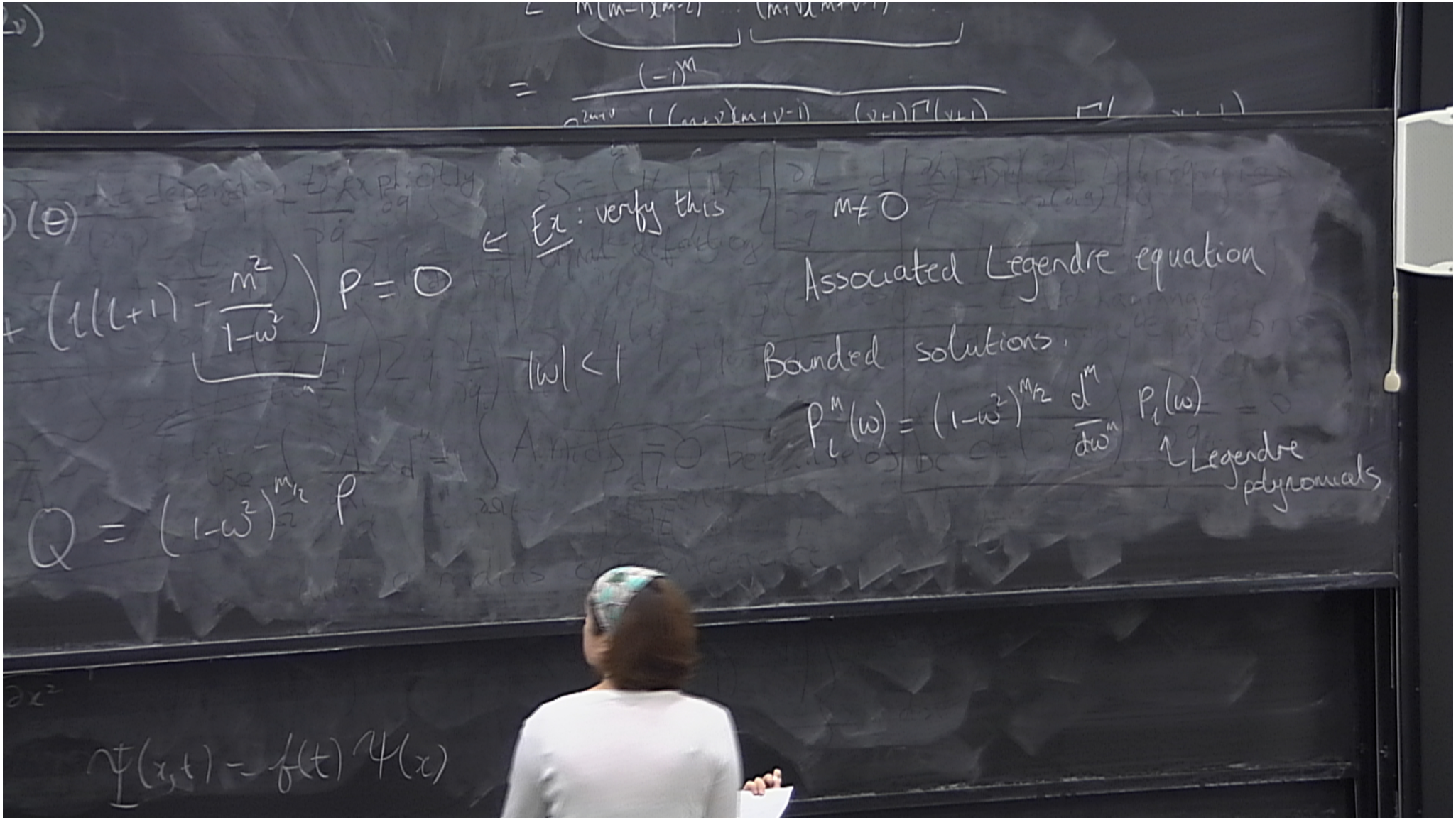
$$\left(\frac{\partial}{\partial \theta}\right) + l(l+1)\sin^2\theta = m^2\,\Theta$$

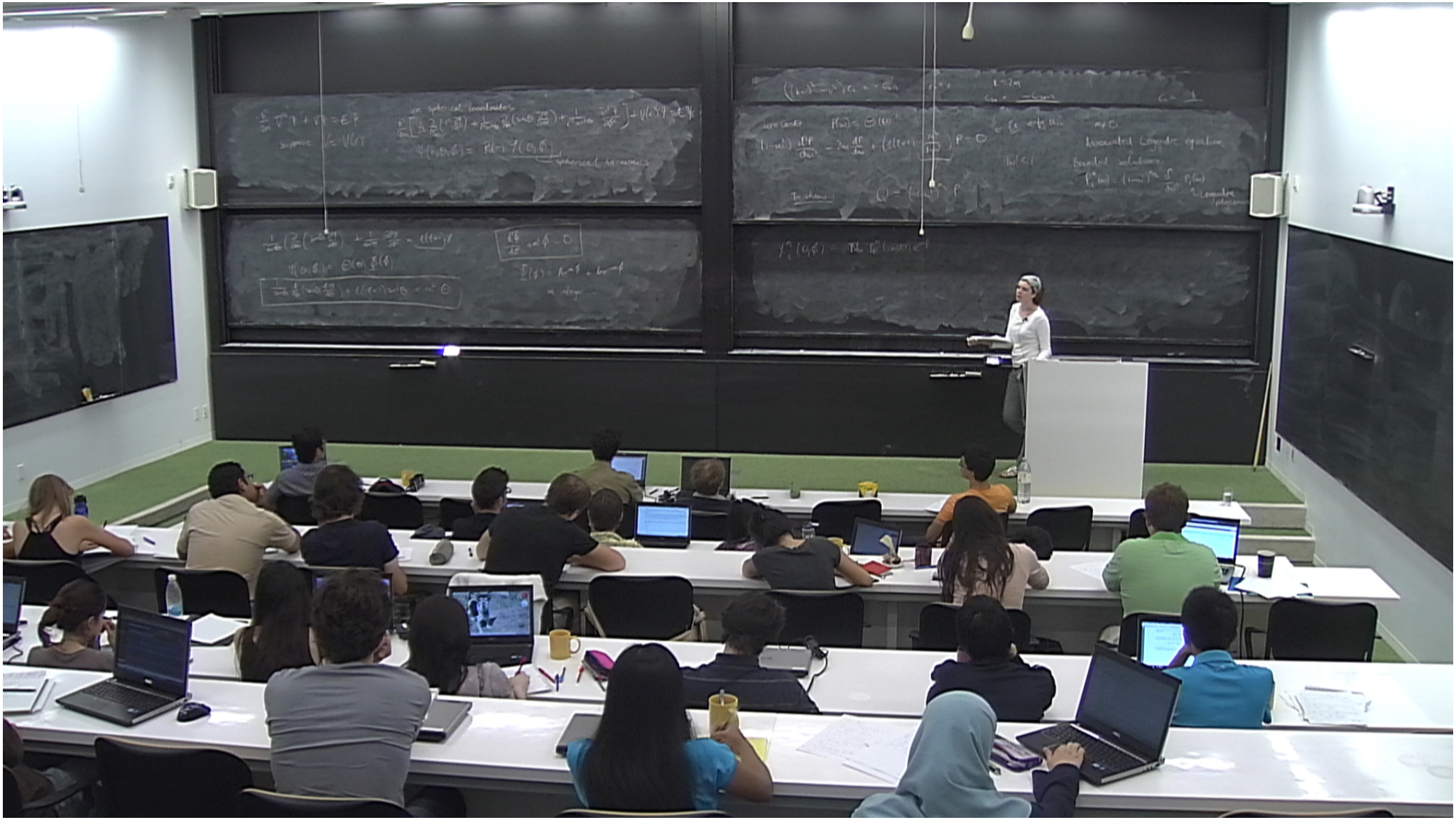
$$\frac{d^2\Phi}{d\phi^2} + m^2\Phi = 0$$

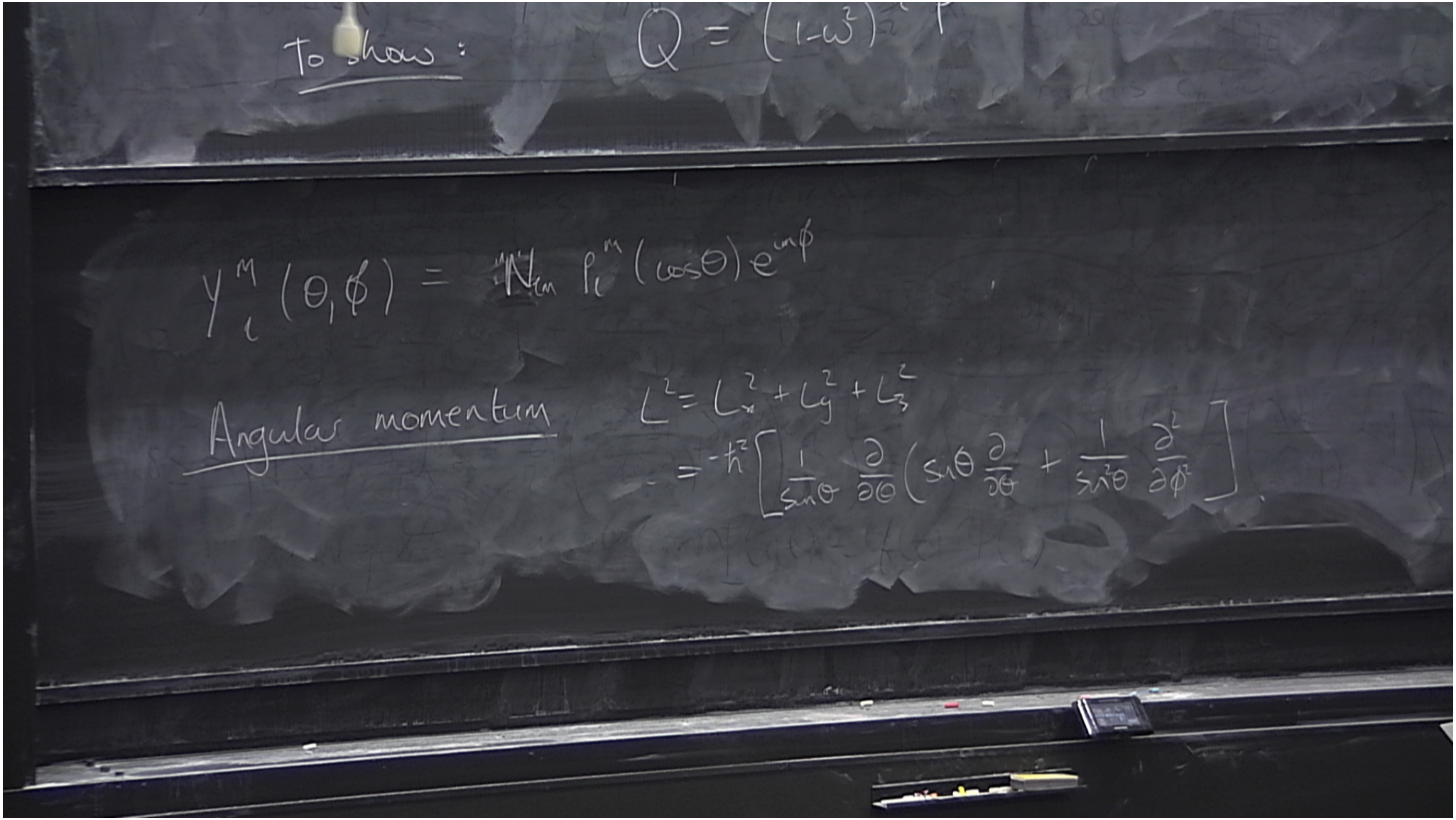
$$\Phi(\phi) = Ae^{im\phi} + Be^{-im\phi}$$

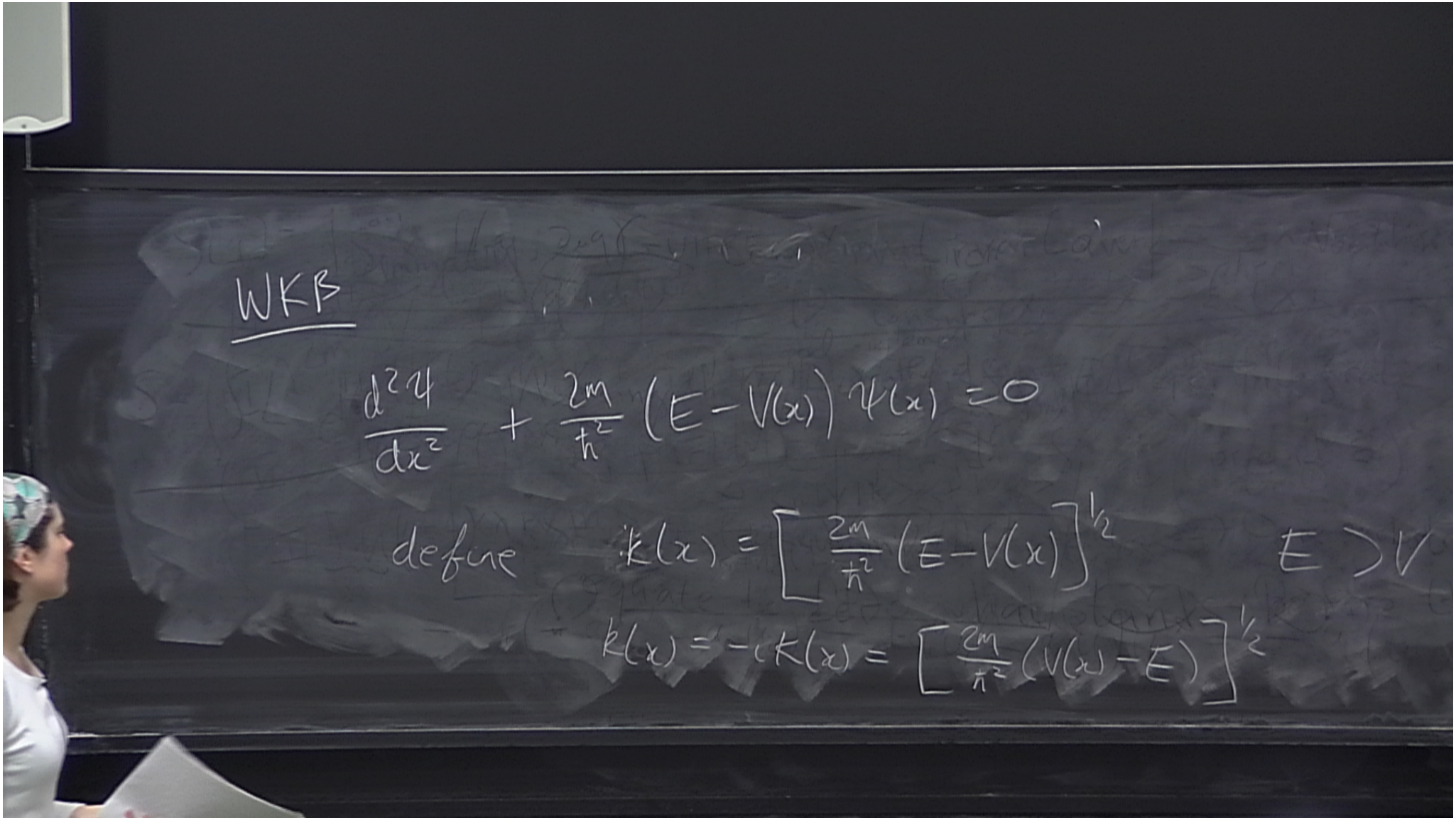

$w = \cos\theta$
$P(w) = \Theta(\theta)$
$(1-w^2)\frac{d^2P}{dw^2} - 2w\frac{dP}{dw} + \left(l(l+1) - \frac{m^2}{1-w^2}\right)P = 0$
← Ex: verify this

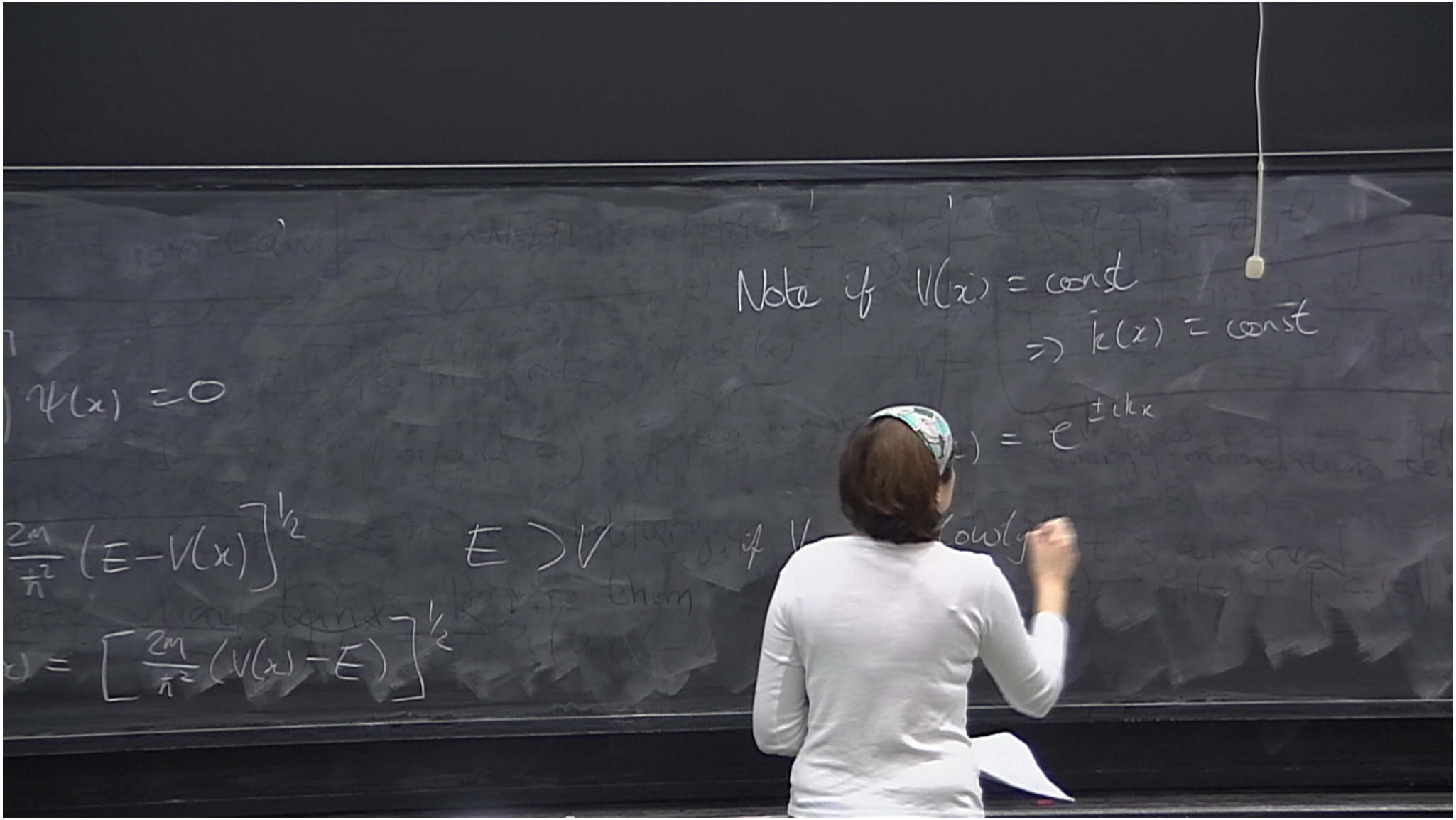

$\Theta(\theta)$
$\left(l(l+1) - \frac{m^2}{1-\omega^2}\right) P = 0$
Ex: verify this
$m \neq 0$
Associated Legendre equation
$|\omega| < 1$
Bounded solutions:
$P_l^m(\omega) = (1-\omega^2)^{m/2} \frac{d^m}{d\omega^m} P_l(\omega)$
Legendre polynomials
$\Psi(x,t) = f(t)\,\psi(x)$

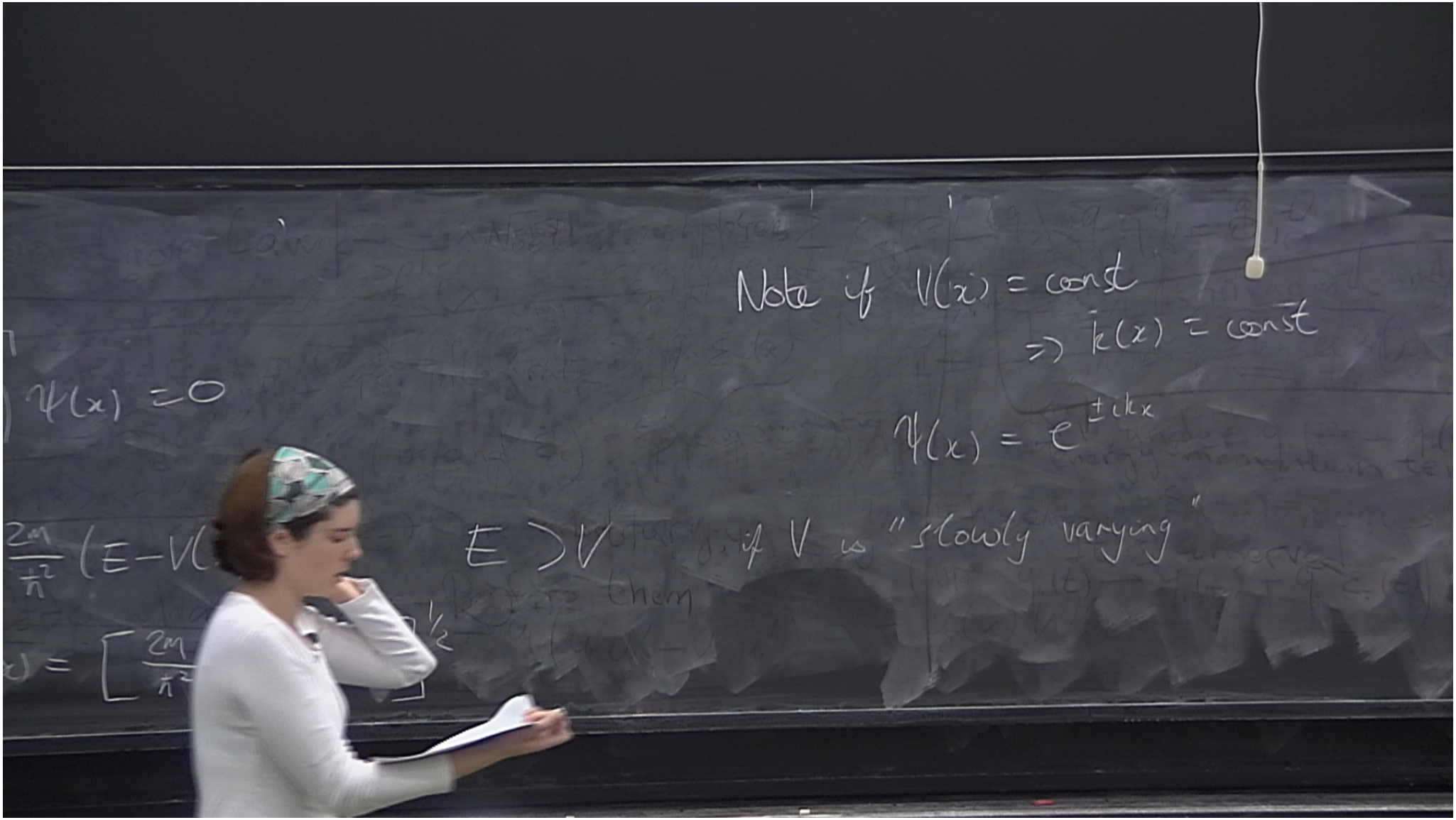

Θ
$\left(l(l+1) - \frac{m^2}{1-\omega^2}\right) P = 0$
Ex: verify this
$m \neq 0$
Associated Legendre equation
$|\omega| < 1$
Bounded solutions:
$P_l^m(\omega) = (1-\omega^2)^{m/2} \frac{d^m}{d\omega^m} P_l(\omega)$
Legendre polynomials
$Q = (1-\omega^2)^{m/2} P$
$\Psi(x,t) = f(t)\,\psi(x)$

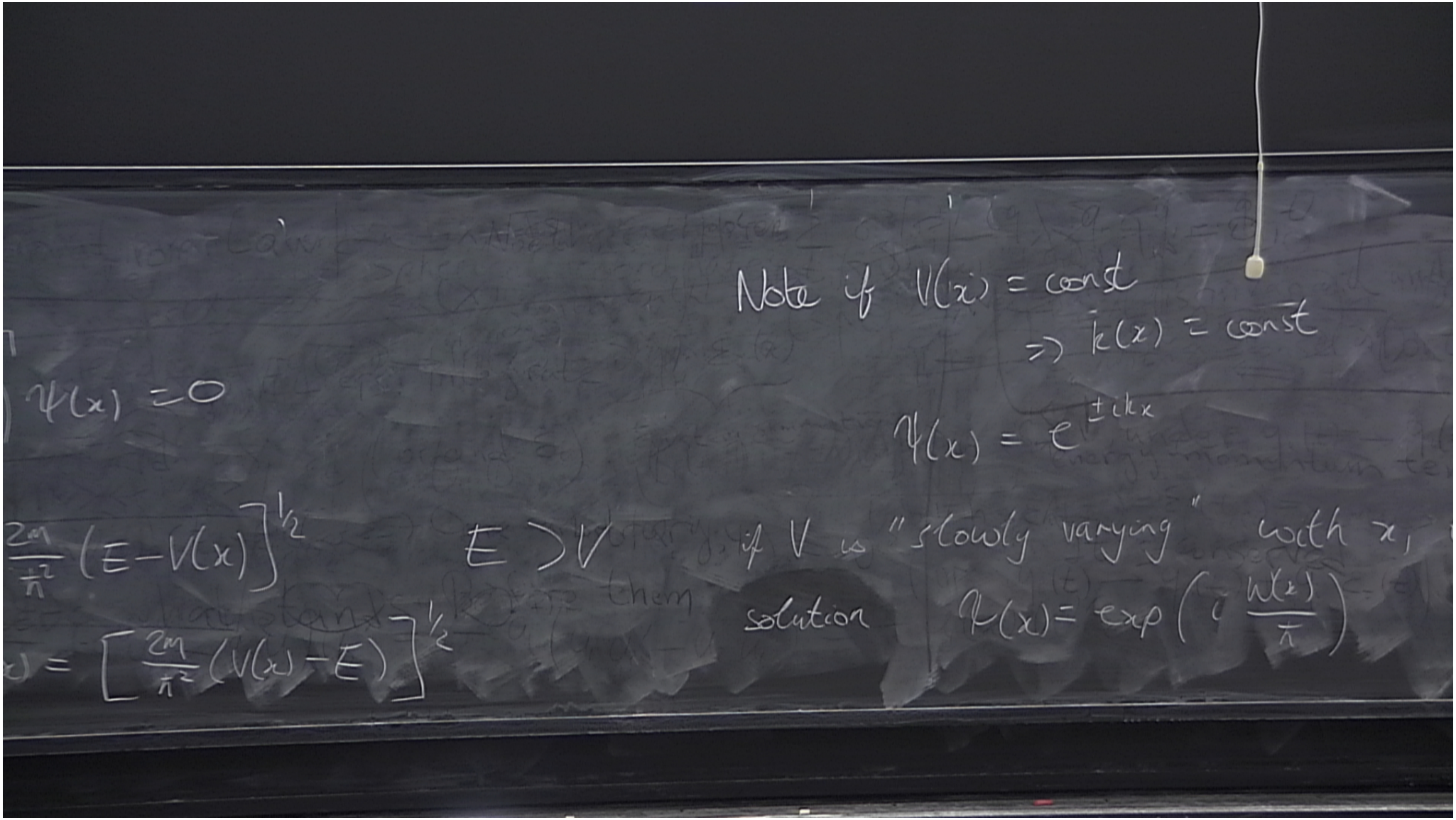


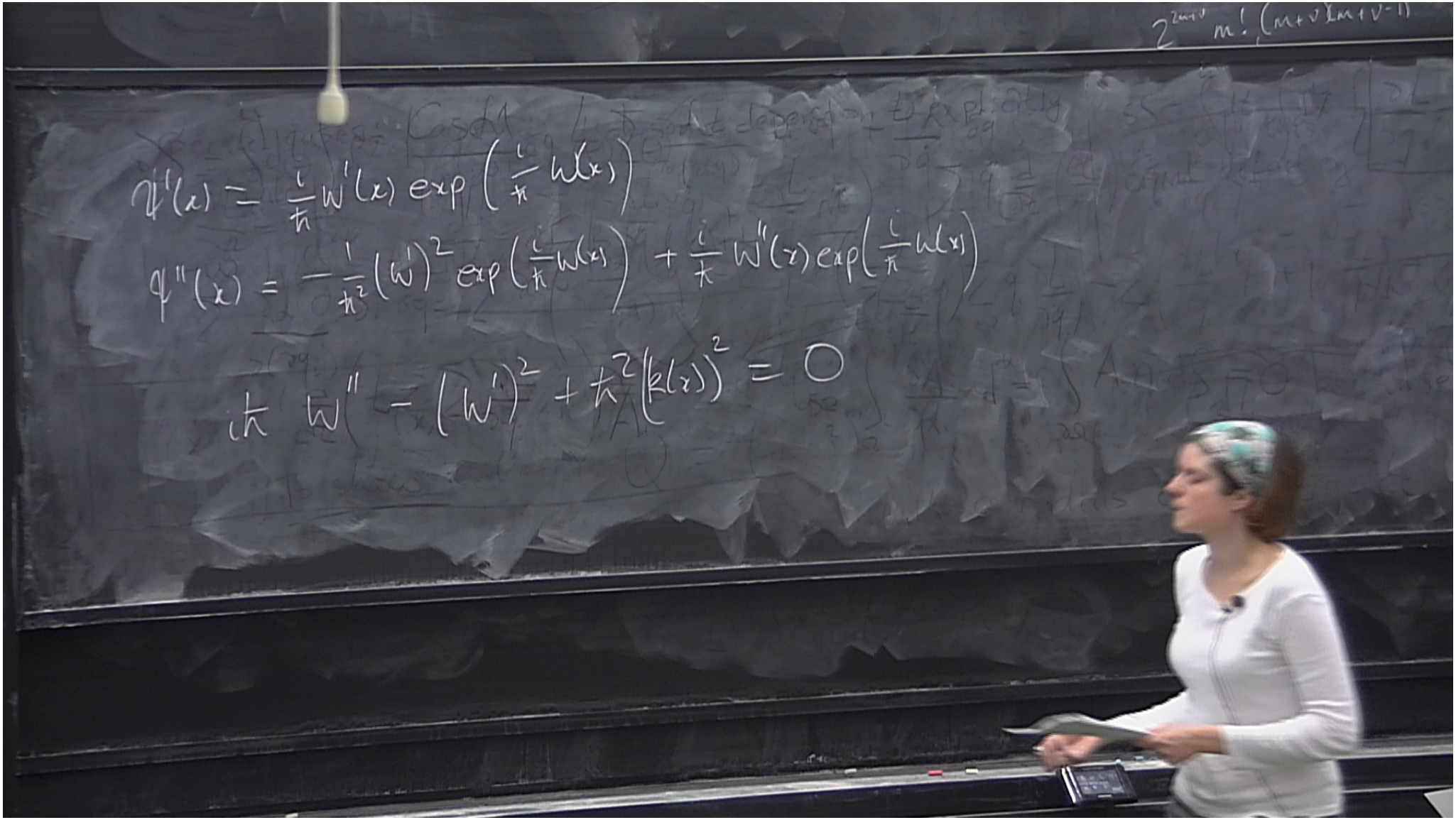

To show:
$Q = (1-\omega^2)$
$Y_\ell^m(\theta,\phi) = N_{\ell m} P_\ell^m(\cos\theta) e^{im\phi}$
Angular momentum
$L^2 = L_x^2 + L_y^2 + L_z^2$
$= -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right]$

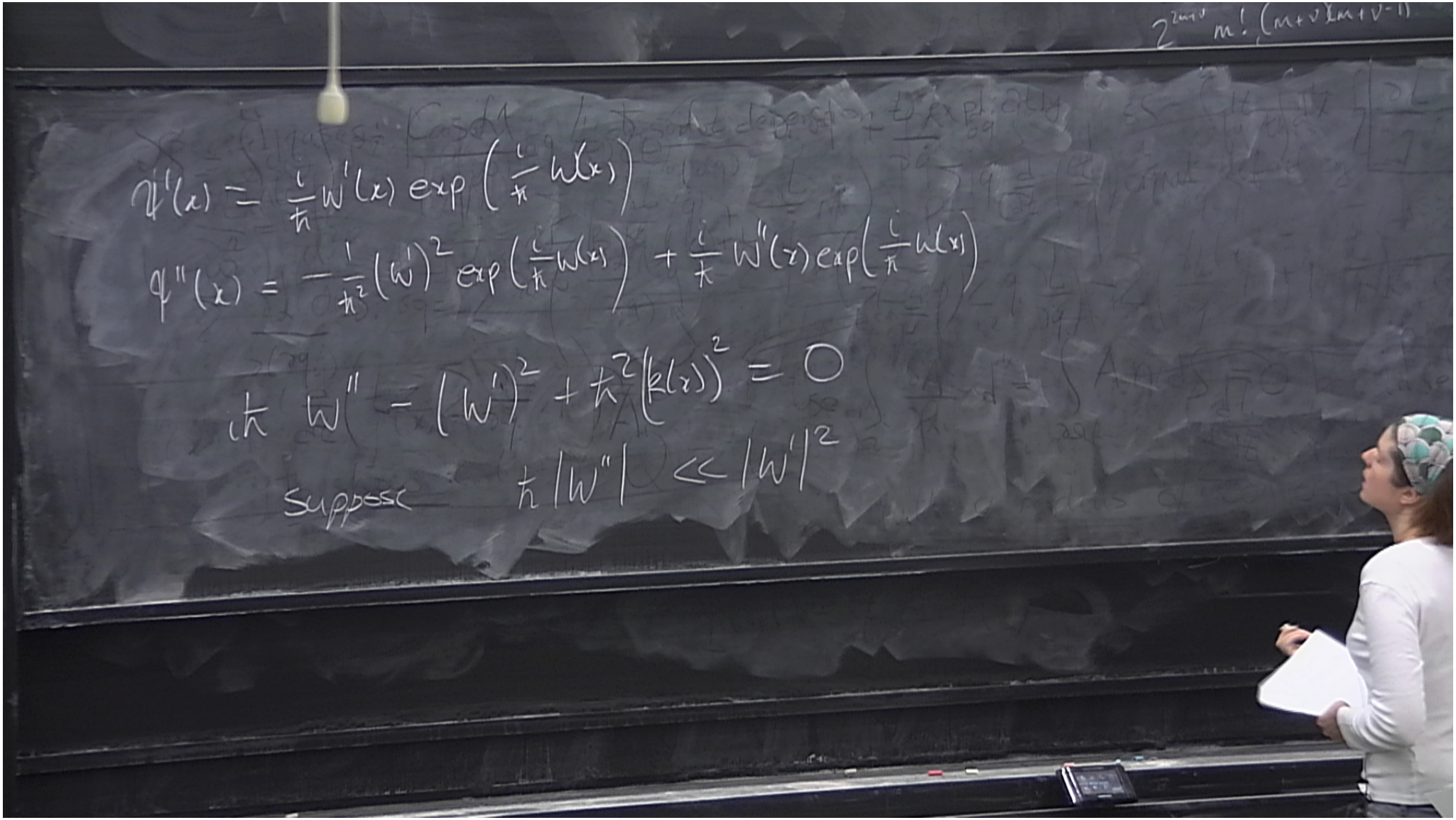

WKB
$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}(E - V(x))\psi(x) = 0$
define $k(x) = \left[\frac{2m}{\hbar^2}(E - V(x))\right]^{1/2}$ $E > V$
$k(x) = -iK(x) = \left[\frac{2m}{\hbar^2}(V(x) - E)\right]^{1/2}$

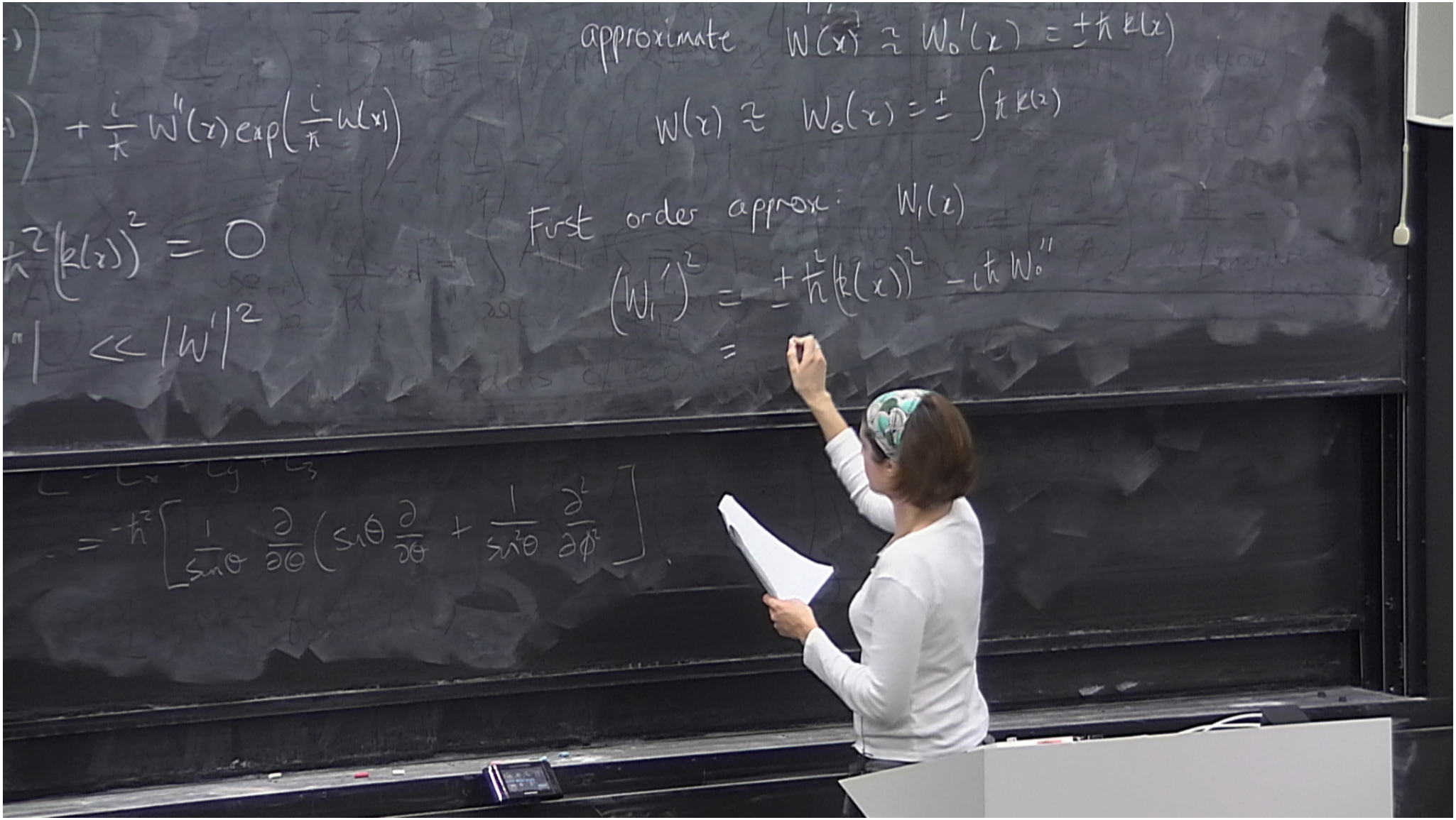

Note if V(x) = const
⇒ k(x) = const
ψ(x) = 0
E > V

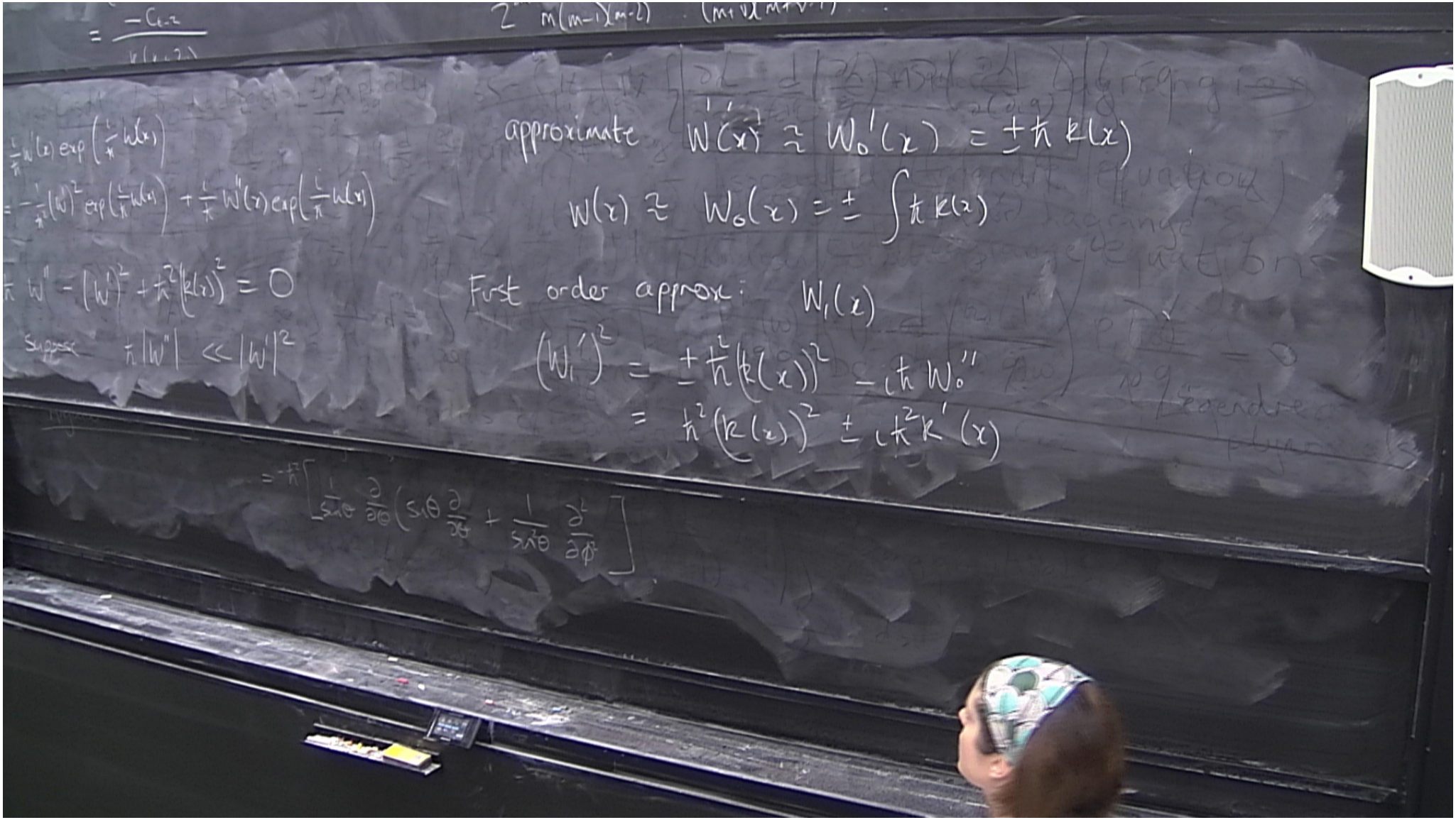

Note if V(x) = const
⇒ k(x) = const
ψ(x) = e^{±ikx}
E > V
if V is "slowly varying"

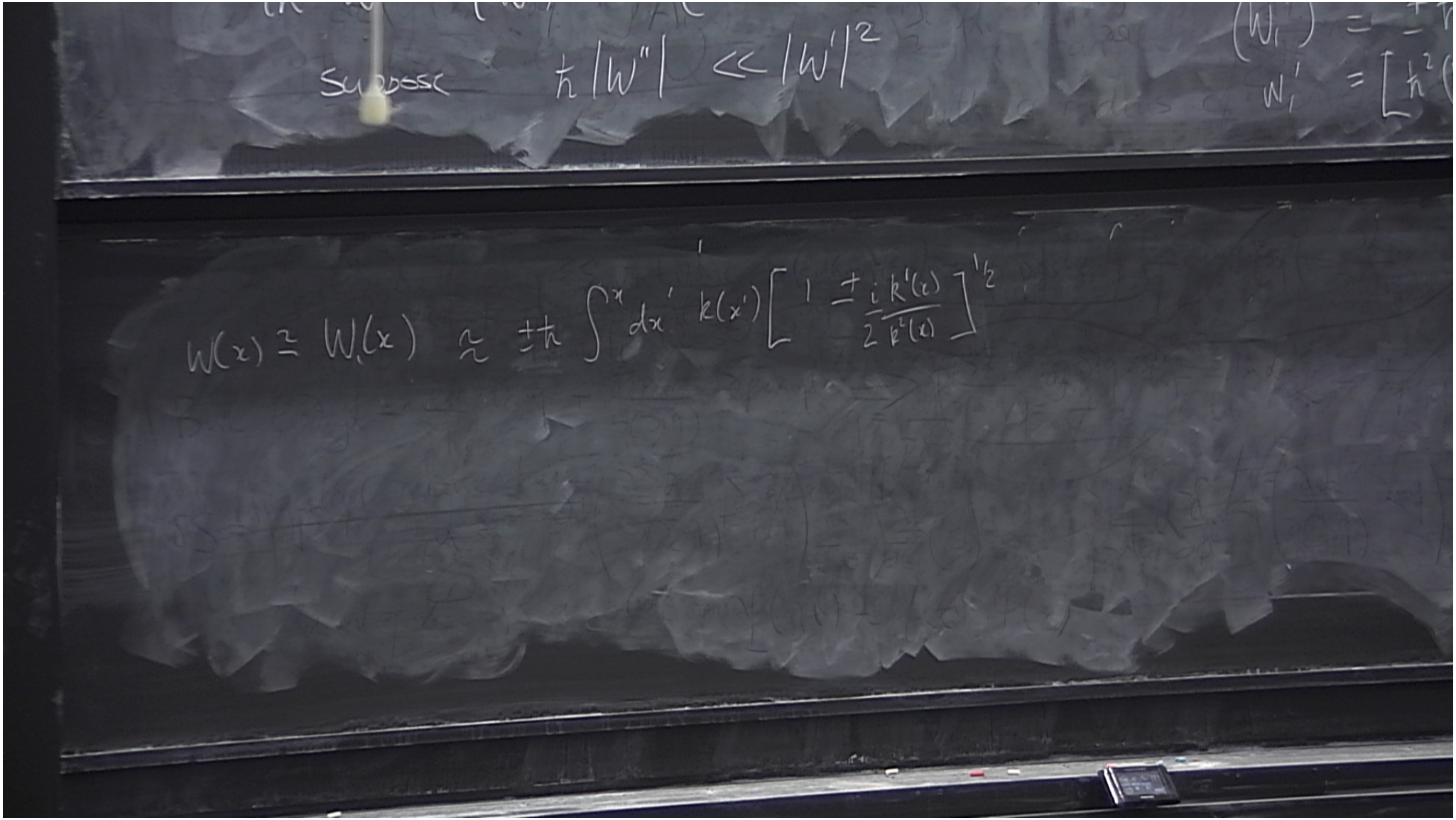

Note if V(x) = const
⇒ k(x) = const
ψ(x) = e^{±ikx}
ψ(x) = 0
E > V
if V is "slowly varying" with x,
solution
ψ(x) = exp(i ∫ k(x) dx)

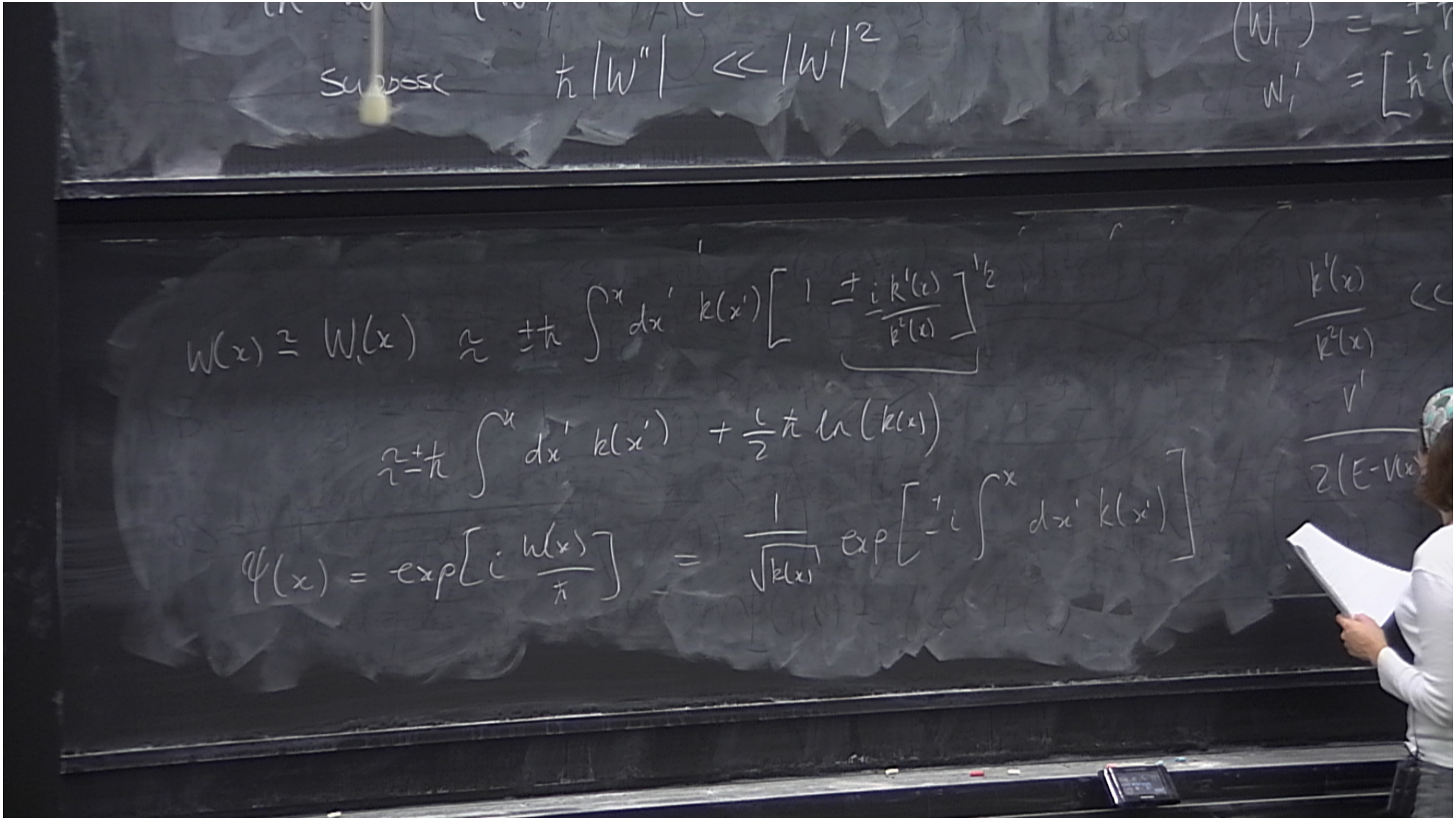

$\psi'(x) = \frac{i}{\hbar} W'(x) \exp\left(\frac{i}{\hbar} W(x)\right)$
$\psi''(x) = -\frac{1}{\hbar^2}(W')^2 \exp\left(\frac{i}{\hbar} W(x)\right) + \frac{i}{\hbar} W''(x) \exp\left(\frac{i}{\hbar} W(x)\right)$
$i\hbar\, W'' - (W')^2 + \hbar^2 (k(x))^2 = 0$

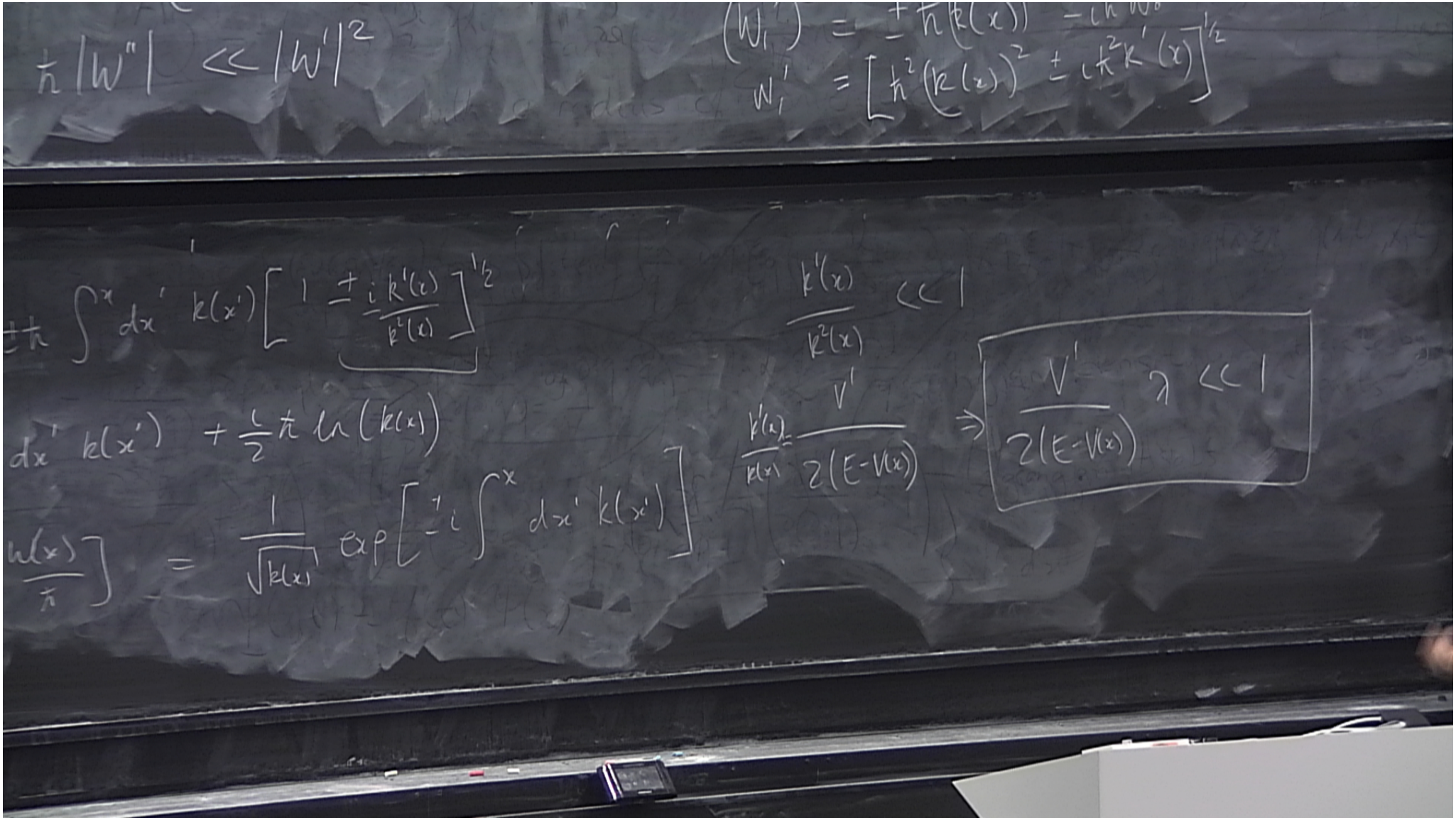

$2^{2m+\nu}\, m!\,(m+\nu)(m+\nu-1)$
$\psi'(x) = \frac{i}{\hbar} W'(x) \exp\left(\frac{i}{\hbar} W(x)\right)$
$\psi''(x) = -\frac{1}{\hbar^2}(W')^2 \exp\left(\frac{i}{\hbar} W(x)\right) + \frac{i}{\hbar} W''(x) \exp\left(\frac{i}{\hbar} W(x)\right)$
$i\hbar\, W'' - (W')^2 + \hbar^2 (k(x))^2 = 0$
suppose $\hbar |W''| \ll |W'|^2$

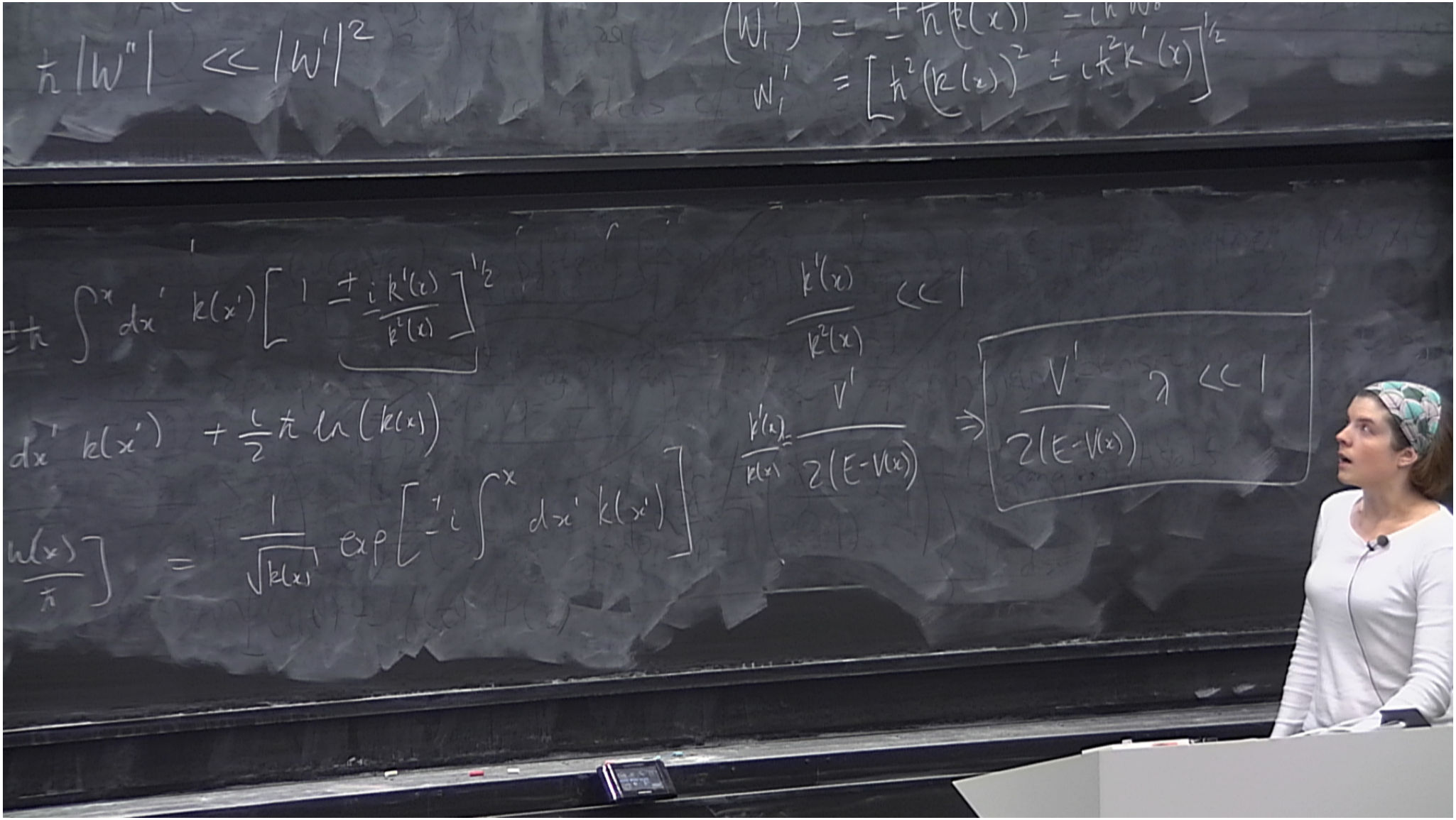

approximate
First order approx:


approximate $W'(x) \approx W_0'(x) = \pm \hbar k(x)$
$W(x) \approx W_0(x) = \pm \int \hbar k(x)$
First order approx: $W_1(x)$
$(W_1')^2 = \pm \hbar^2 (k(x))^2 - i\hbar W_0''$
$= \hbar^2 (k(x))^2 \pm i\hbar k'(x)$
$\frac{1}{\hbar} W(x) \exp\left(\frac{i}{\hbar} W(x)\right)$
$= -\frac{1}{\hbar^2}(W')^2 \exp\left(\frac{i}{\hbar} W(x)\right) + \frac{i}{\hbar} W'(x) \exp\left(\frac{i}{\hbar} W(x)\right)$
$\hbar W'' - (W')^2 + \hbar^2 (k(x))^2 = 0$
Suppose $\hbar |W''| \ll |W'|^2$


Suppose $\hbar|W''| \ll |W'|^2$
$W(x) \simeq W_1(x) \approx \pm\hbar \int^x dx' \, k(x') \left[1 \pm \frac{i k'(x)}{2 k^2(x)}\right]^{1/2}$


Suppose $\hbar|W''| \ll |W'|^2$
$W(x) \simeq W_1(x) \approx \pm\hbar \int^x dx' \, k(x') \left[1 \pm \frac{i k'(x)}{k^2(x)}\right]^{1/2}$
$\approx \pm\hbar \int^x dx' \, k(x') + \frac{1}{2}\hbar \ln(k(x))$
$\psi(x) = \exp\left[i \frac{W(x)}{\hbar}\right] = \frac{1}{\sqrt{k(x)}} \exp\left[\pm i \int^x dx' \, k(x')\right]$
$\frac{k'(x)}{k^2(x)} \ll$
$\frac{V'}{2(E-V(x)}$


$\hbar |W''| \ll |W'|^2$
$W_1' = \pm \hbar k(x)$
$W_1' = \left[\hbar^2 k(x)^2 \pm i\hbar k'(x)\right]^{1/2}$
$\pm \frac{i}{\hbar}\int^x dx'\, k(x')\left[1 \pm \frac{i k'(x)}{k^2(x)}\right]^{1/2}$
$\frac{k'(x)}{k^2(x)} \ll 1$
$\int^x dx'\, k(x') + \frac{i}{2}\hbar \ln(k(x))$
$\frac{k'(x)}{k(x)} = \frac{V'}{2(E-V(x))}$
$\Rightarrow \frac{V'}{2(E-V(x))}\lambda \ll 1$
$\frac{w(x)}{\hbar}] = \frac{1}{\sqrt{k(x)}} \exp\left[\pm i \int^x dx'\, k(x')\right]$


$\hbar |W''| \ll |W'|^2$
$W_1' = \left[\hbar^2 (k(x))^2 \pm i\hbar^2 k'(x)\right]^{1/2}$
$\pm \frac{i}{\hbar}\int^x dx'\, k(x')\left[1 \pm \frac{i k'(x)}{k^2(x)}\right]^{1/2}$
$\frac{k'(x)}{k^2(x)} \ll 1$
$\int dx'\, k(x') + \frac{i}{2}\hbar \ln(k(x))$
$\frac{k'(x)}{k(x)} = \frac{V'}{2(E-V(x))}$
$\Rightarrow \frac{V'}{2(E-V(x))}\lambda \ll 1$
$\frac{\psi(x)}{\pi}] = \frac{1}{\sqrt{k(x)}} \exp\left[\pm i \int^x dx'\, k(x')\right]$