

Title: Relativity - Lecture 11

Date: Sep 23, 2011 10:30 AM

URL: <http://pirsa.pi.local/11090035>

Abstract:

Journey into a black hole: from Schwarzschild to Kruskal

$$ds^2 = \left(1 - \frac{r_s}{r}\right) dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

$\frac{dr}{dt} = 0$ for $0 < r < r_s$ and $r > r_s$
What happens? and at $r=0$? Note that for $r < r_s$, r is the "time" coordinate.

Journey into a black hole: from Schwarzschild to

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

- solves $R_{\alpha\beta} = 0$ for $0 < r < r_s$ and $r > r_s$

what happens at r_s ? and at $r=0$? Note that for $r < r_s$, r

Radial light signals
(null geodesics)

$$r_s = \frac{2GM}{c^2}$$

$$g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = 0$$

Journe

$$ds^2 = -$$

- solves

what ha

Radial light

$$r_s = \frac{2GM}{c^2}$$

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what happens at r_s ? and at $r=0$?

Note that for $r < r_s$, r is the "time"!

Radial light signals
(null geodesics)

$$ds^2 = 0 \Rightarrow$$

$$d\theta = d\phi = 0$$

$$dt = \pm \frac{dr}{1 - \frac{r_s}{r}} = \pm dr \left(1 + \frac{r_s}{r - r_s}\right)$$

$$d\theta = d\phi = 0$$

Journey into a black hole: from Schwarzschild to Kruskal

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null geodesic

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Schwarzschild is
- more like

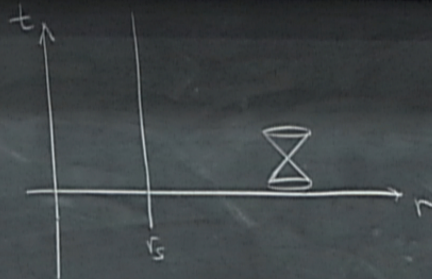
Radial light signals
(null geodesics)

$$d\theta = d\phi = 0$$

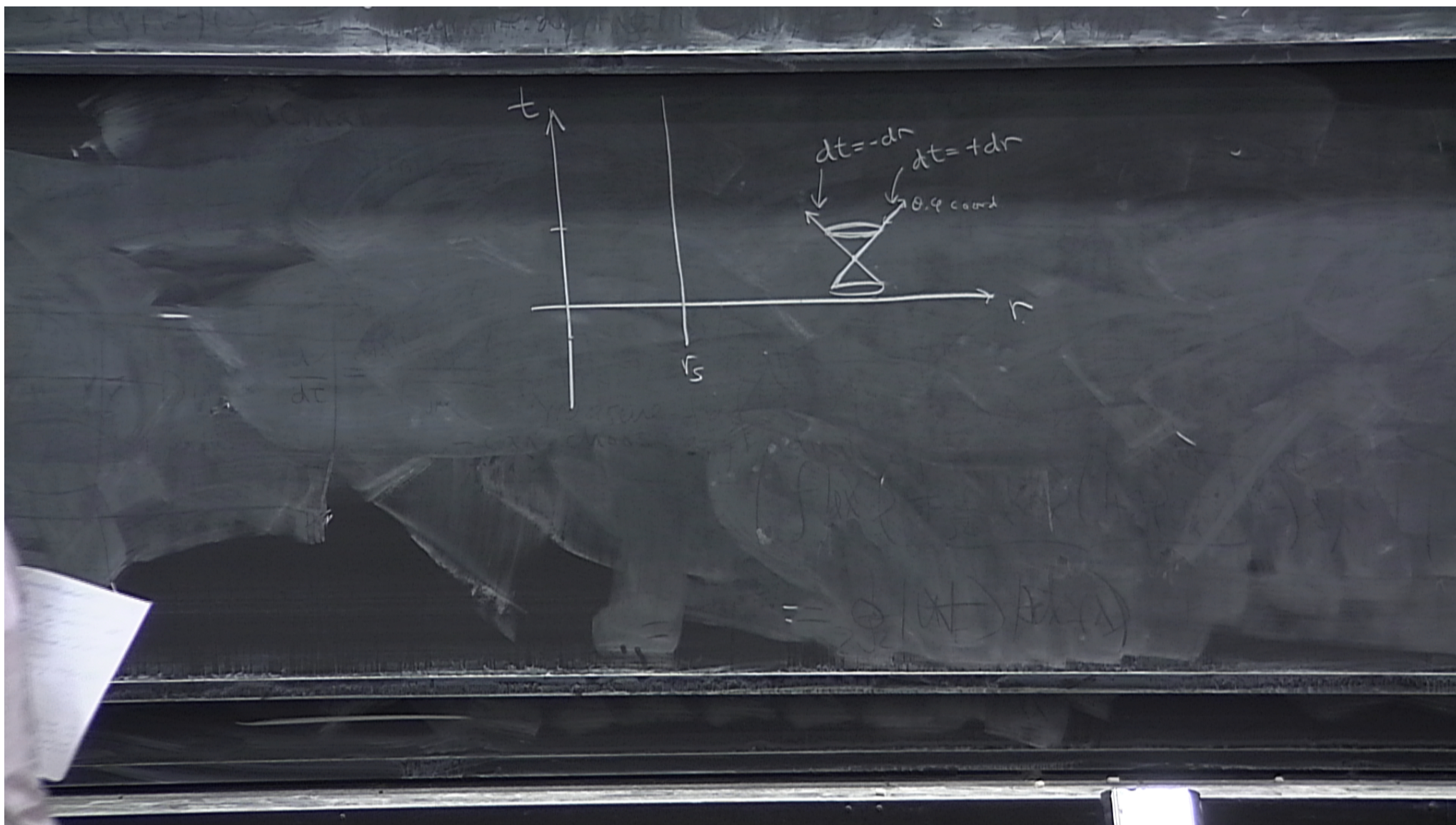
$$ds^2 = 0 \Rightarrow$$

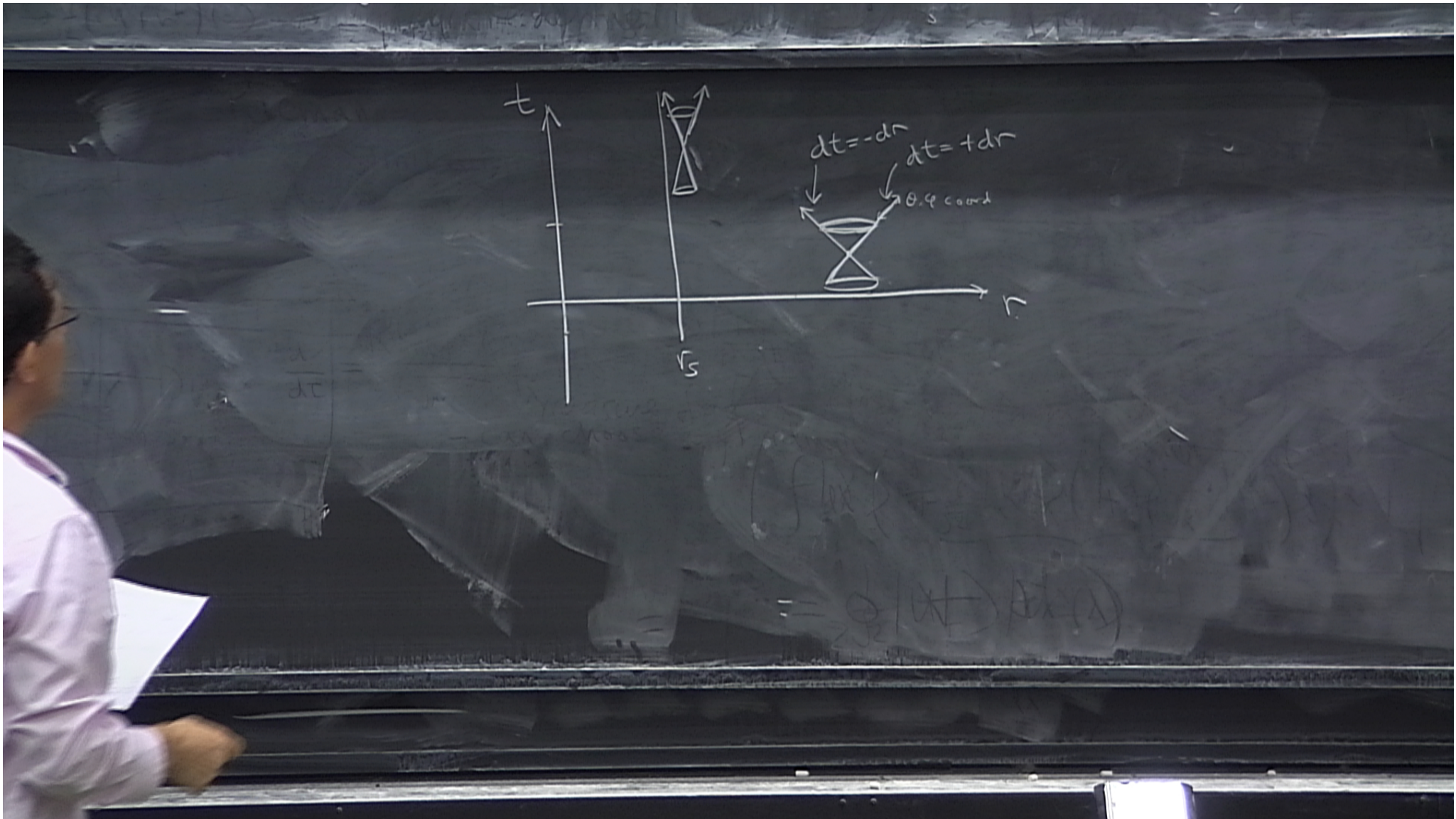
$$d\theta = d\phi = 0$$

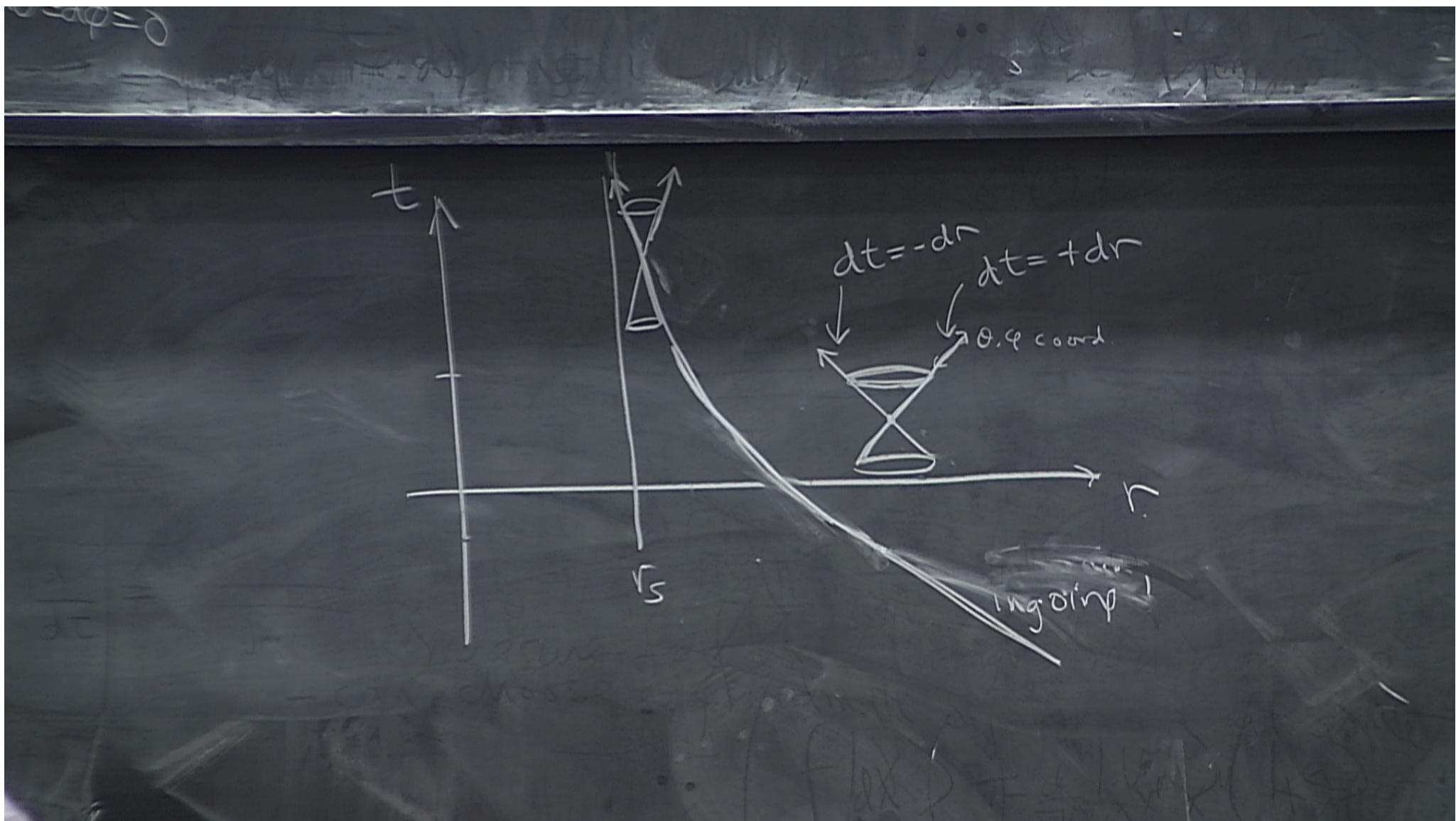
$$dt = \pm \frac{dr}{1 - \frac{r_s}{r}} = \pm dr \left(1 + \frac{r_s}{r - r_s}\right)$$

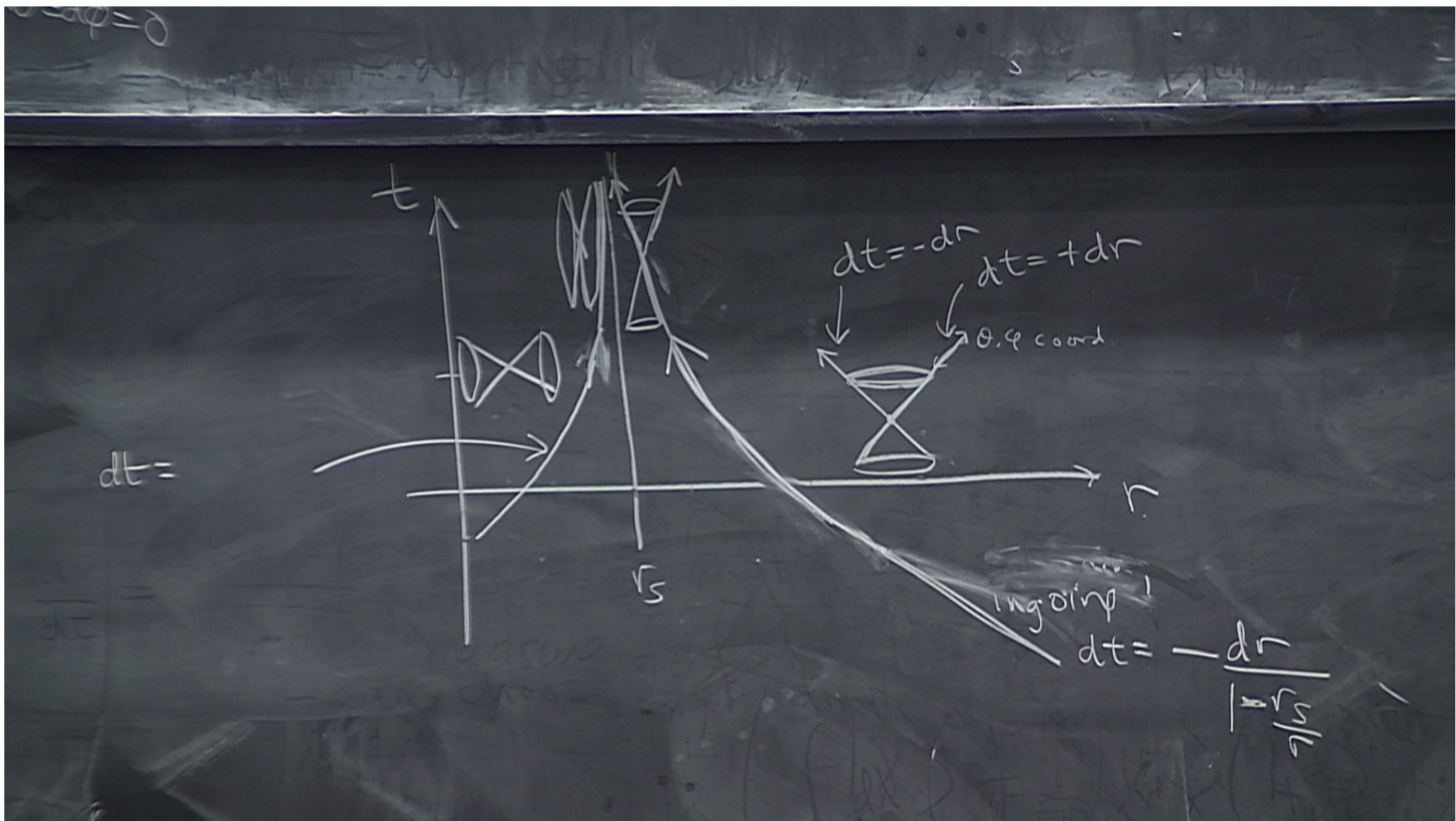


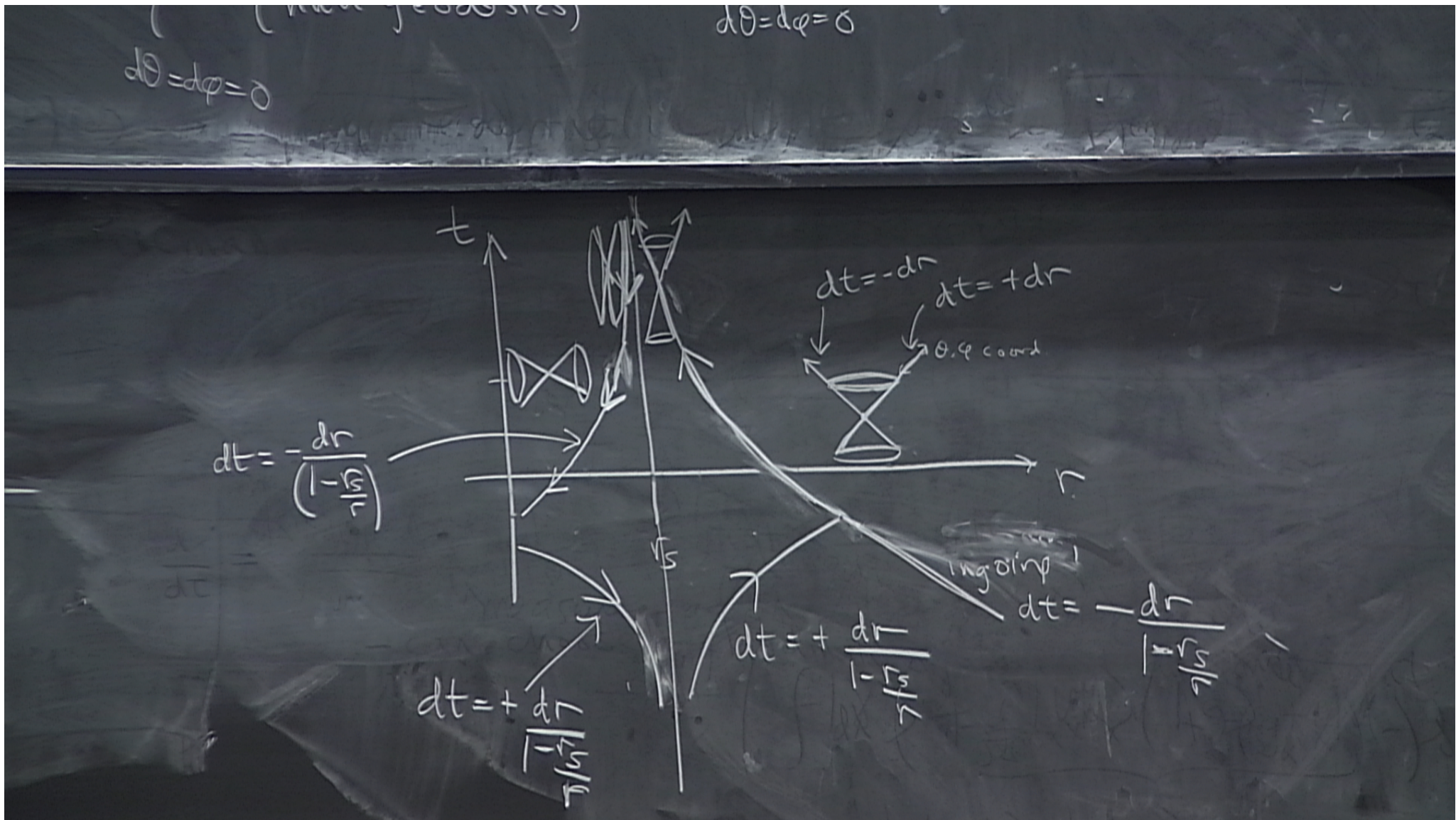












$$\Rightarrow t = \pm \left(r + r_s \ln \left(\frac{r}{r_s} - 1 \right) \right) + \text{const} \quad r > r_s$$

$$= \pm \left(r + r_s \ln \left(1 - \frac{r}{r_s} \right) \right) + \text{const} \quad r < r_s$$

ingoing null ray: choose - sign

$$t = -r - r_s \ln \left(\frac{r}{r_s} - 1 \right) \rightarrow \infty \text{ as } r \rightarrow r_s$$

$$\Rightarrow t = \pm \left(r + r_s \ln \left(\frac{r}{r_s} - 1 \right) \right) + \text{const} \quad r > r_s$$

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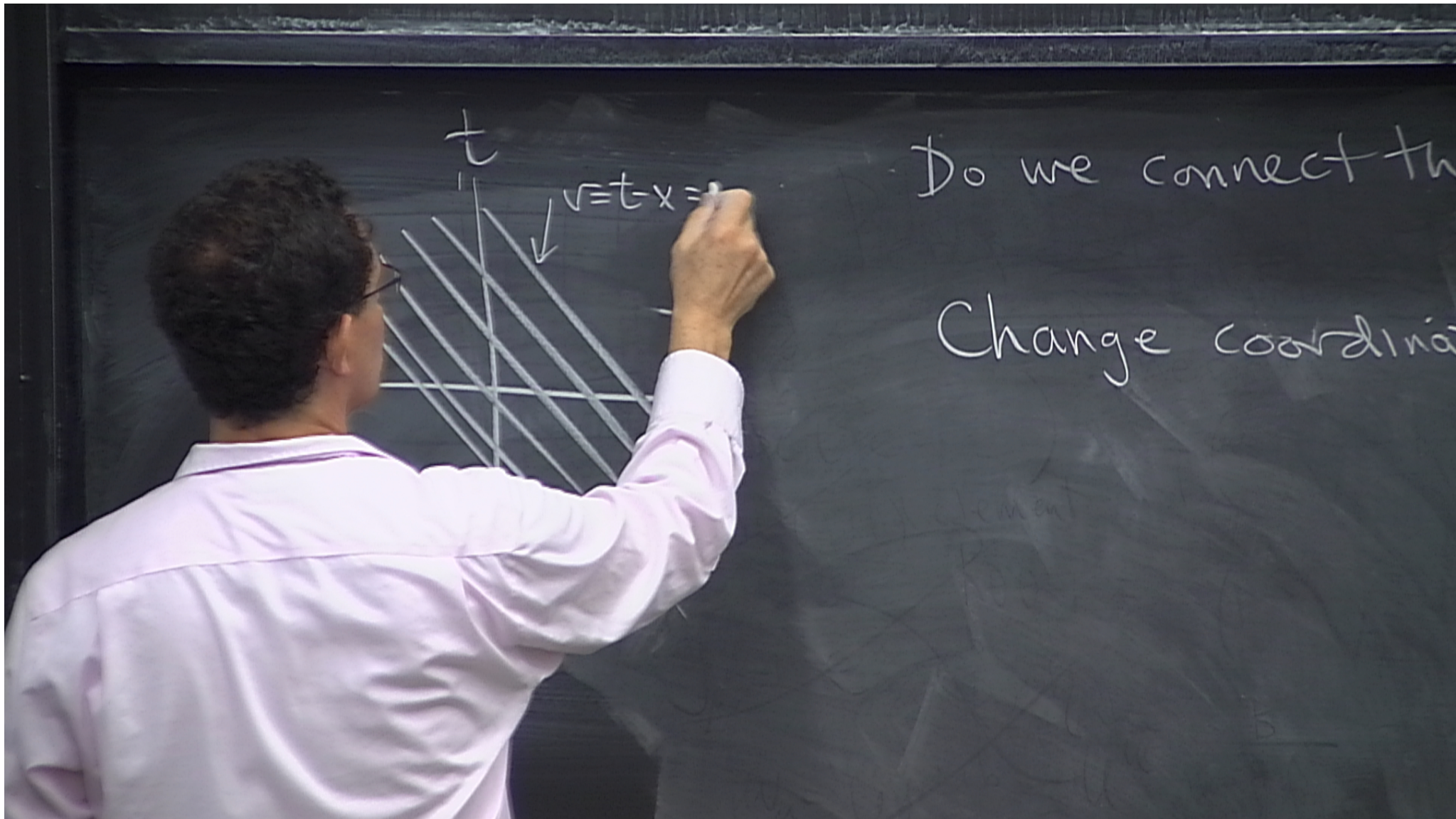
$$t = -r - r_s \ln \left(\frac{r}{r_s} - 1 \right) \rightarrow \infty \quad \text{as } r \rightarrow r_s \text{ from above}$$

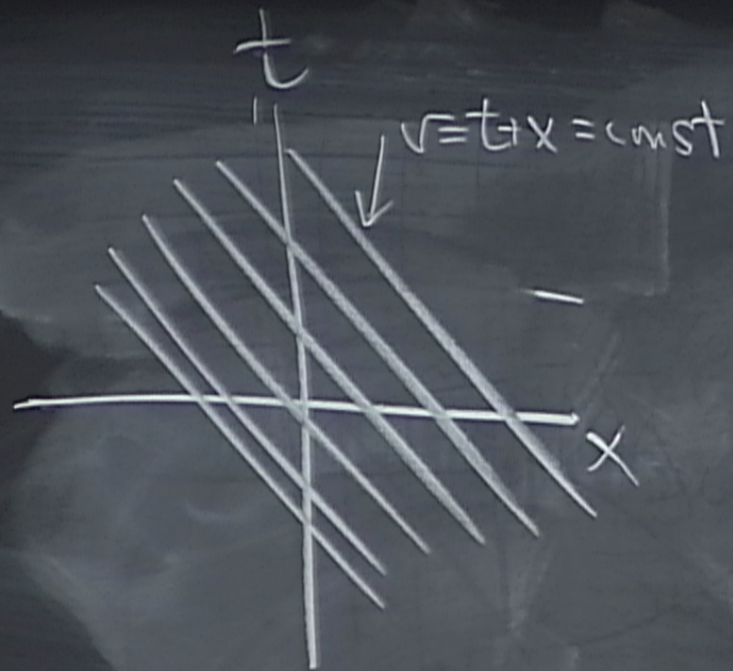
$$= -r - r_s \ln \left(1 - \frac{r}{r_s} \right) \rightarrow \infty \quad \text{as } r \rightarrow r_s \text{ from below}$$

Do we connect these null rays across $r=r_s$?

Change coordinates:
$$v = t + r + r_s \ln \left| \frac{r}{r_s} - 1 \right|$$

So that ingoing null rays obey $v = \text{const.}$





Do we connect the

Change coordinates

se null rays across $r=r_s$?

s:
$$v = t + r + r_s \ln \left| \frac{r}{r_s} - 1 \right|$$

So that ingoing

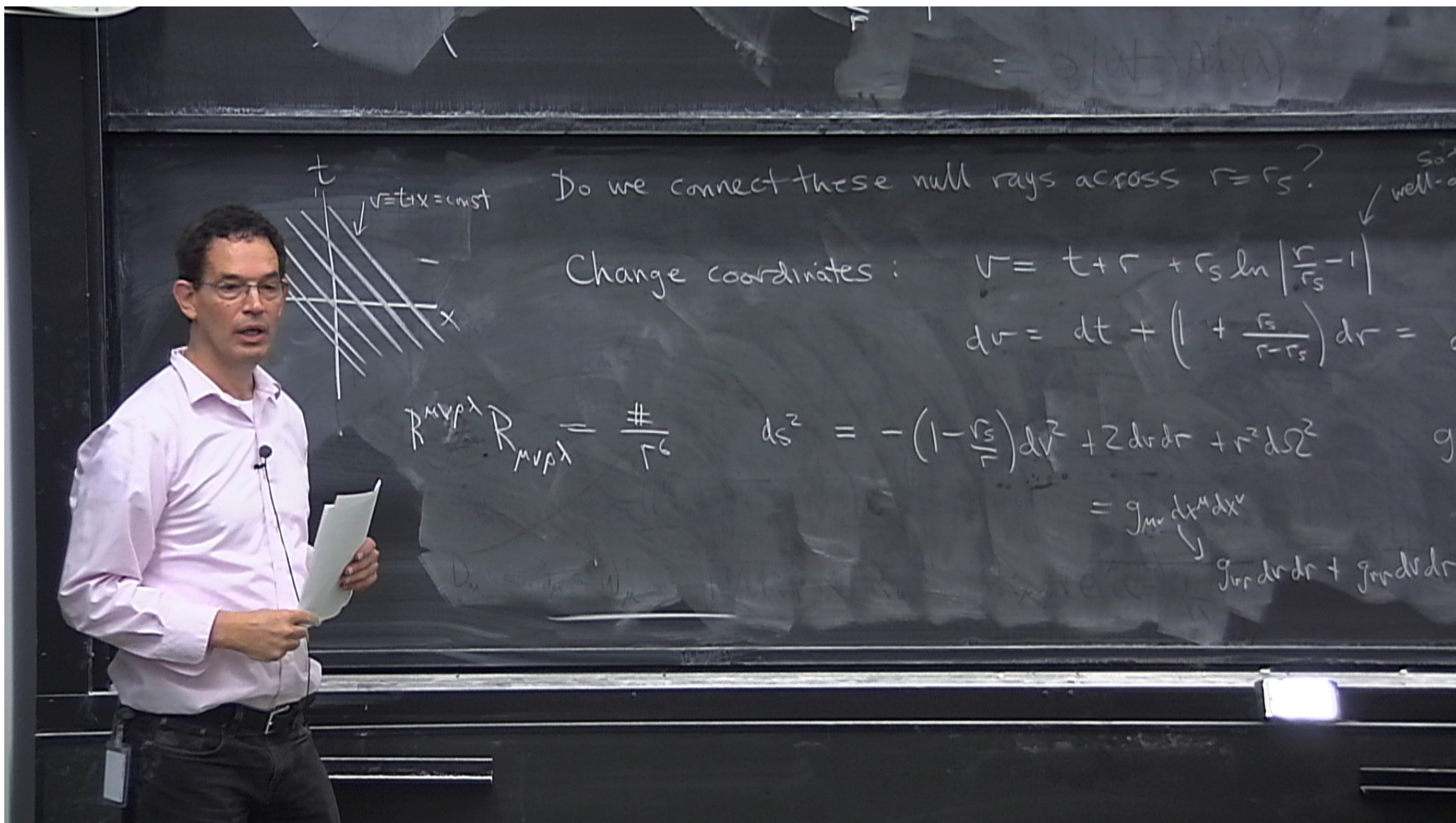
$$dv = dt + 1 + \frac{r_s}{r - r_s}$$

Do we connect these null rays across $r=r_s$?

So that
well-defined for $r > r_s$
or $r < r_s$

Change coordinates: $v = t + r + r_s \ln \left| \frac{r}{r_s} - 1 \right|$ So that ingoing null rays

$$dv = dt + \left(1 + \frac{r_s}{r - r_s} \right) dr = dt + \frac{dr}{1 - \frac{r_s}{r}}$$



Do we connect these null rays across $r=r_s$?

So that well-defined for $r > r_s$ or $r < r_s$

Change coordinates: $v = t + r + r_s \ln \left| \frac{r}{r_s} - 1 \right|$

So that ingoing null rays

$$dv = dt + \left(1 + \frac{r_s}{r-r_s} \right) dr = dt + \frac{dr}{1 - \frac{r_s}{r}}$$

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) dv^2 + 2 dr dv + r^2 d\Omega^2$$

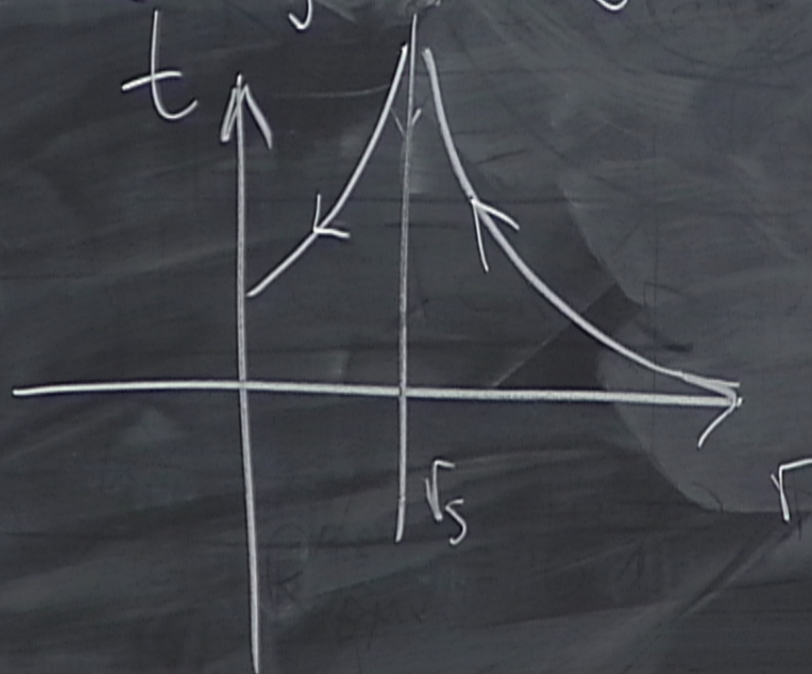
$$= g_{\mu\nu} dx^\mu dx^\nu$$

$$g_{rr} dr dr + g_{rd} dr dv$$

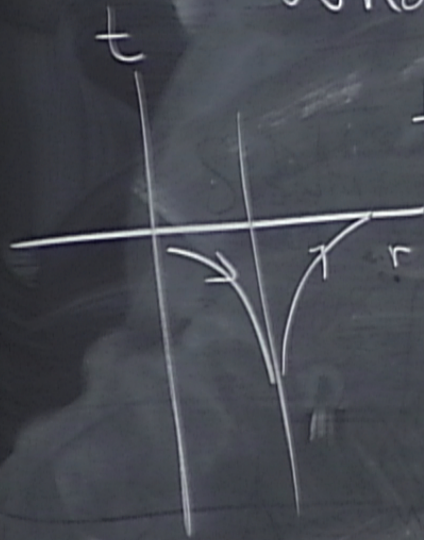
$$g_{\mu\nu} = \begin{pmatrix} -\left(1 - \frac{r_s}{r}\right) & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \theta \end{pmatrix}$$

$g_{\mu\nu}$ is not singular at $r=r_s$ in these coordinates $\Rightarrow$ Singularity of Schwarzschild metric at $r=r_s$ is only a coordinate singularity!

radial light rays are $v = \text{const}$
 $\Rightarrow$ choice



What about outgoing light rays?



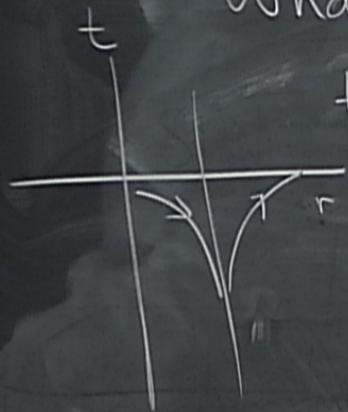
$$t = +\left(r + r_s \ln\left|\frac{r}{r_s} - 1\right|\right) + \text{const}$$

$$\text{define } u = t - \left(r + r_s \ln\left|\frac{r}{r_s} - 1\right|\right)$$

$$\Rightarrow du = dt + \frac{dr}{1 - \frac{r_s}{r}}$$

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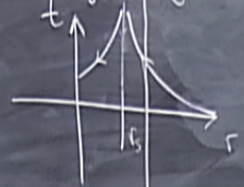
$$ds^2 = -\left(1 - \frac{r_s}{r}\right) du^2 - 2du dr + r^2 d\Omega^2$$

again, nonsingular

static for $r < r_s$!

radial light rays are $v = \text{const}$, $r = -\lambda$, $-\infty < \lambda < 0$

is only a coordinate singularity!



$\Rightarrow$ choice of (ingoing) coordinate r determines the direction of "time" r for $r < r_s$
 $\Rightarrow$ decreasing r is increasing time.
 (this is the physical case for gravitational collapse).

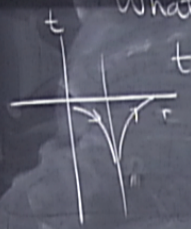
$r > r_s$

$r < r_s$

$r \rightarrow r_s$ from above

$r \rightarrow r_s$ from below

What about out-going light rays?



$$t = \left(r + r_s \ln \left| \frac{r}{r_s} - 1 \right| \right) + \text{const}$$

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$$\Rightarrow du = dt - \frac{dr}{1 - \frac{r_s}{r}}$$

$$ds^2 = - \left(1 - \frac{r_s}{r} \right) du^2 - 2 du dr + r^2 d\Omega^2 \quad \text{again, nonsingular at } r=r_s$$

- also valid

BUT r increases along this null, for $r < r_s$, but this is direction of increasing time!

- disagrees with dir*

$$r_s \sin \theta$$



define $u = t - \left(r + r_s \ln \left| \frac{r}{r_s} - 1 \right| \right)$

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resolution: the ingoing and outgoing null rays see different regions $r < r_s$.



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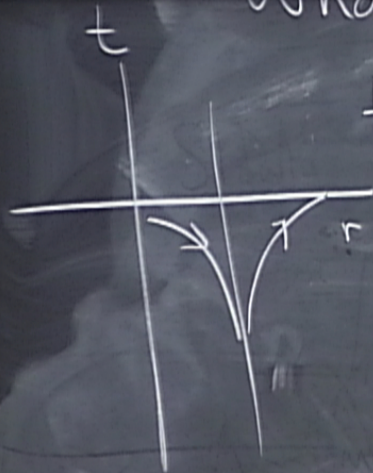
- disagrees with dr^u of increasing time along ingoing null ray!

resolution: the ingoing and outgoing null rays see different regions $r < r_s$.

if pick fixed direction of time in each inner space $\Rightarrow$ get two external spaces!

Kruskal: Phys Rev 119, 1743 (1960)

What about outgoing light rays?



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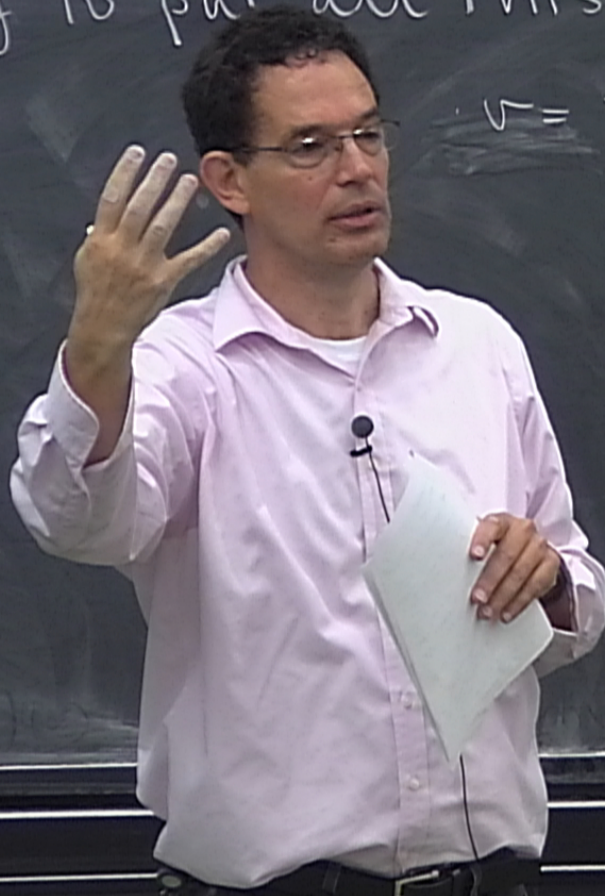
again, nonsingular at $r = r_s$

E-F outgoing coords

-also - lid

Try to put all this together:

$$r = t + r + r_s \ln\left(\frac{r}{r_s} - 1\right)$$



Kruskal: Phys Rev 119, 1743 (1960)

define for

$r > r_s$, we will continue new coord system to obtain entire spacetime

Try to put all this together:

$$v = t + r + r_s \ln\left(\frac{r}{r_s} - 1\right)$$

$$u = t - r - r_s \ln\left(\frac{r}{r_s} - 1\right)$$

$$\frac{dv}{du} = dt \pm \frac{dr}{1 - \frac{r_s}{r}}$$

define for
 $r > r_s$

Kruskal: Phys Rev 119, 174

define for

$r > r_s$, we will continue new coord system to ob

$$\Rightarrow ds^2 = -\left(1 - \frac{r_s}{r}\right) du dv + r^2 d\Omega^2$$

use new coord system to obtain entire spacetime

with
$$r + r_s \ln \left(\frac{r}{r_s} - 1 \right) = \frac{v - u}{2}$$

use new coord system to obtain entire spacetime

with $r + r_s \ln\left(\frac{r}{r_s} - 1\right) = \frac{v-u}{2} \Rightarrow r(u, v)$

Kruskal: Phys Rev 119, 17

define for

$r > r_s$, we will continue new coord system to

$$\Rightarrow ds^2 = -\left(1 - \frac{r_s}{r}\right) du dv + r^2 d\Omega^2, \text{ with } r + r_s \ln\left(\frac{r}{r_s} - 1\right) = v - u$$

$$\Rightarrow \frac{r}{r_s} - 1 = e^{-\frac{r}{r_s}} e^{\frac{v-u}{2r_s}}$$

Kruskal: Phys Rev 119, 1743 (1960)

define for

$r > r_s$, we will continue new coord system to obtain entire spacetime

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) du dv + r^2 d\Omega^2, \text{ with } r + r_s \ln\left(\frac{r}{r_s} - 1\right) = \frac{v-u}{2} \Rightarrow r(u, v)$$

$$\Rightarrow \frac{r}{r_s} - 1 = e^{-\frac{r}{r_s}} e^{\frac{v-u}{2r_s}}; \quad 1 - \frac{r_s}{r} = \frac{r_s}{r} e^{-\frac{r}{r_s}} e^{\frac{v-u}{2r_s}}$$

$$= -\frac{r_s}{r} e^{-\frac{r}{r_s}} e^{\frac{v-u}{2r_s}} du dv + r^2 d\Omega^2$$

now, set $U = r_s e^{-u/2r_s}$, $V = r_s e^{v/2r_s}$

$$ds^2 = \underbrace{\frac{4r_s}{r} e^{-r/r_s}}_{\text{completely regular at } r=r_s!} du dv + r^2 d\Omega^2$$

completely
regular at $r=r_s$!

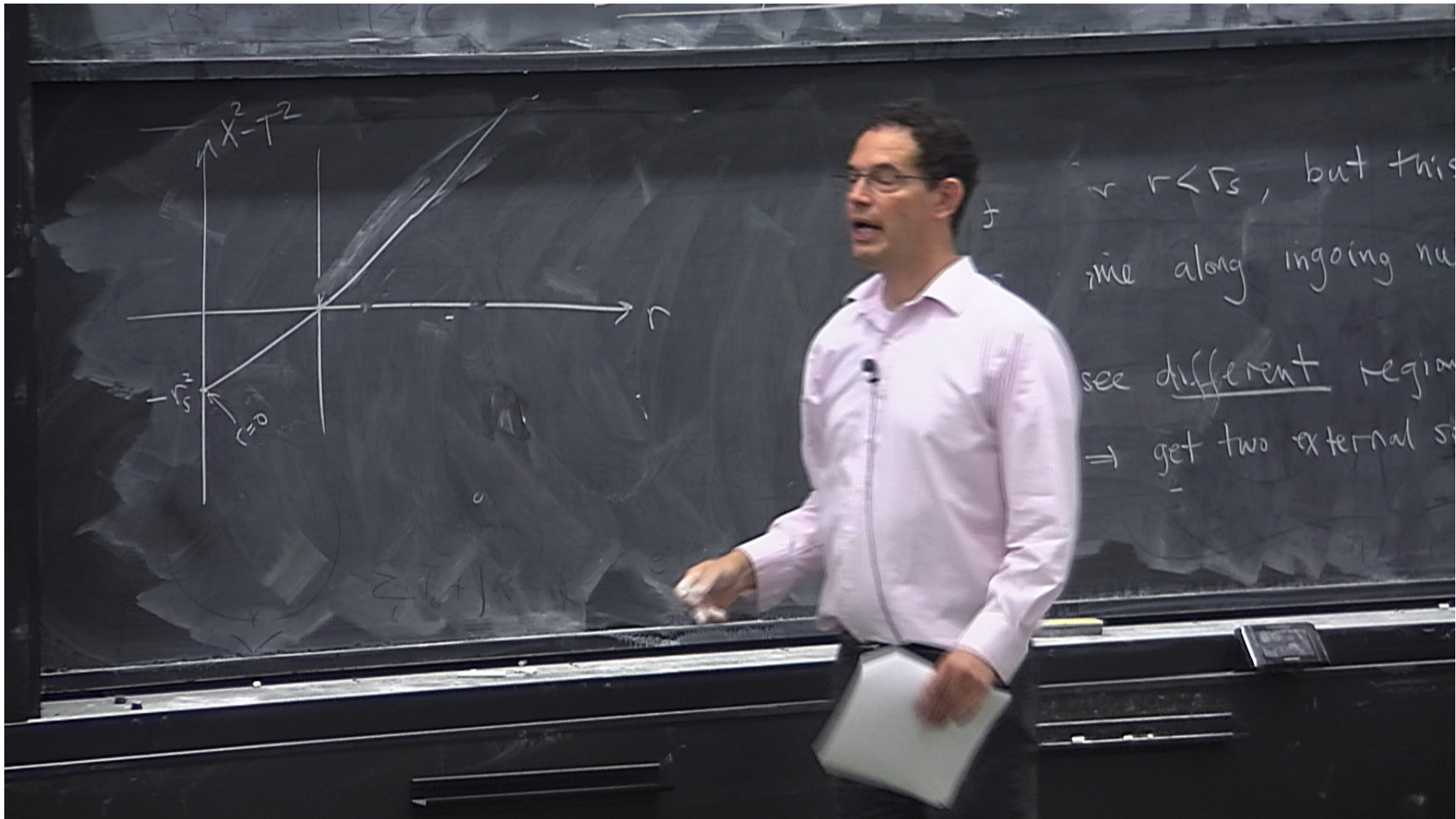
$$\Rightarrow ds^2 = -\frac{r_s}{r} e^{-\frac{r}{r_s}} e^{\frac{r-u}{2r_s}} du dr + r^2 d\Omega^2$$

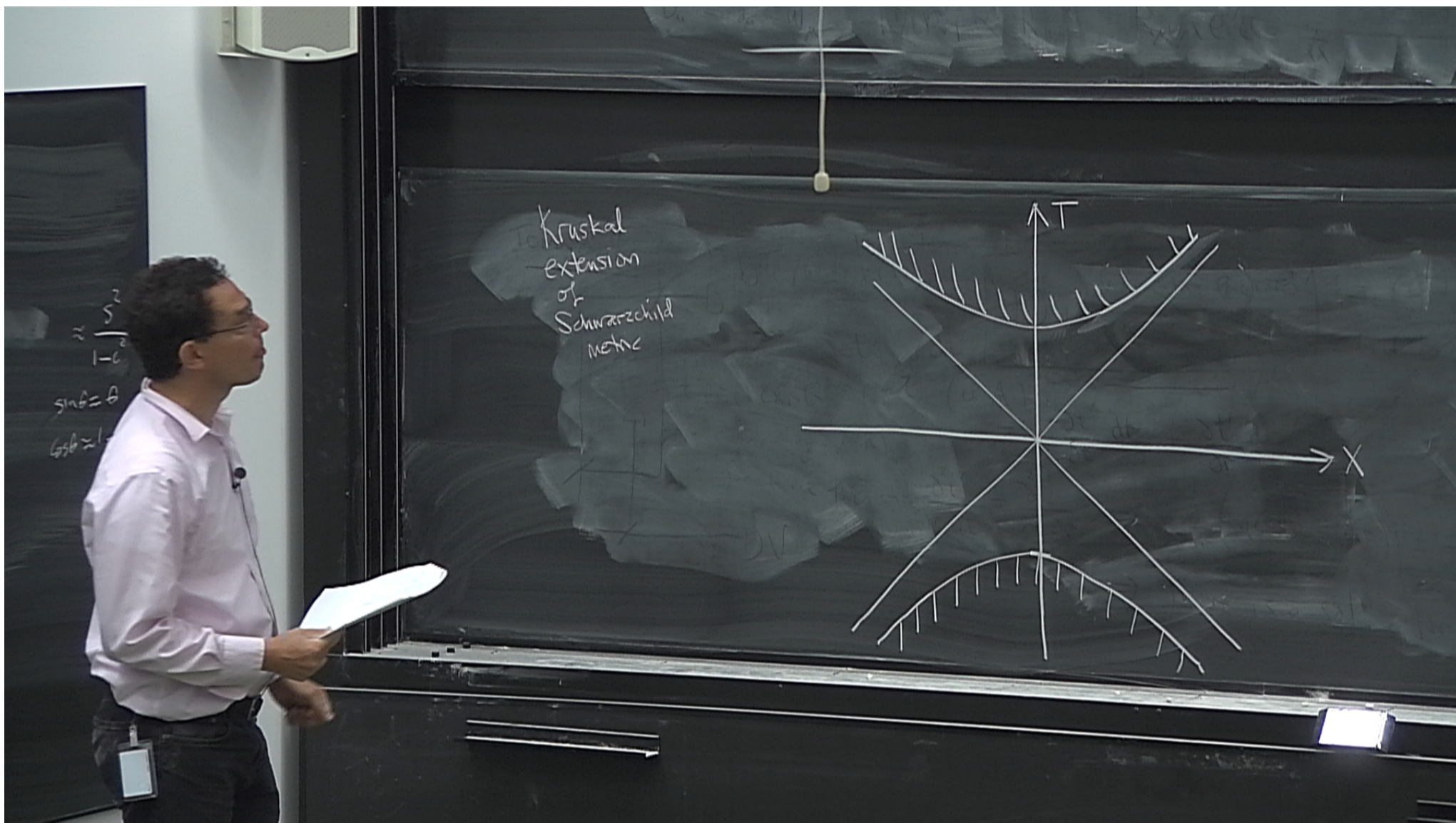
set $U = r_s e^{-\frac{u}{2r_s}}$, $V = r_s e^{\frac{r}{2r_s}}$ tricky! will extend U, V to negative values

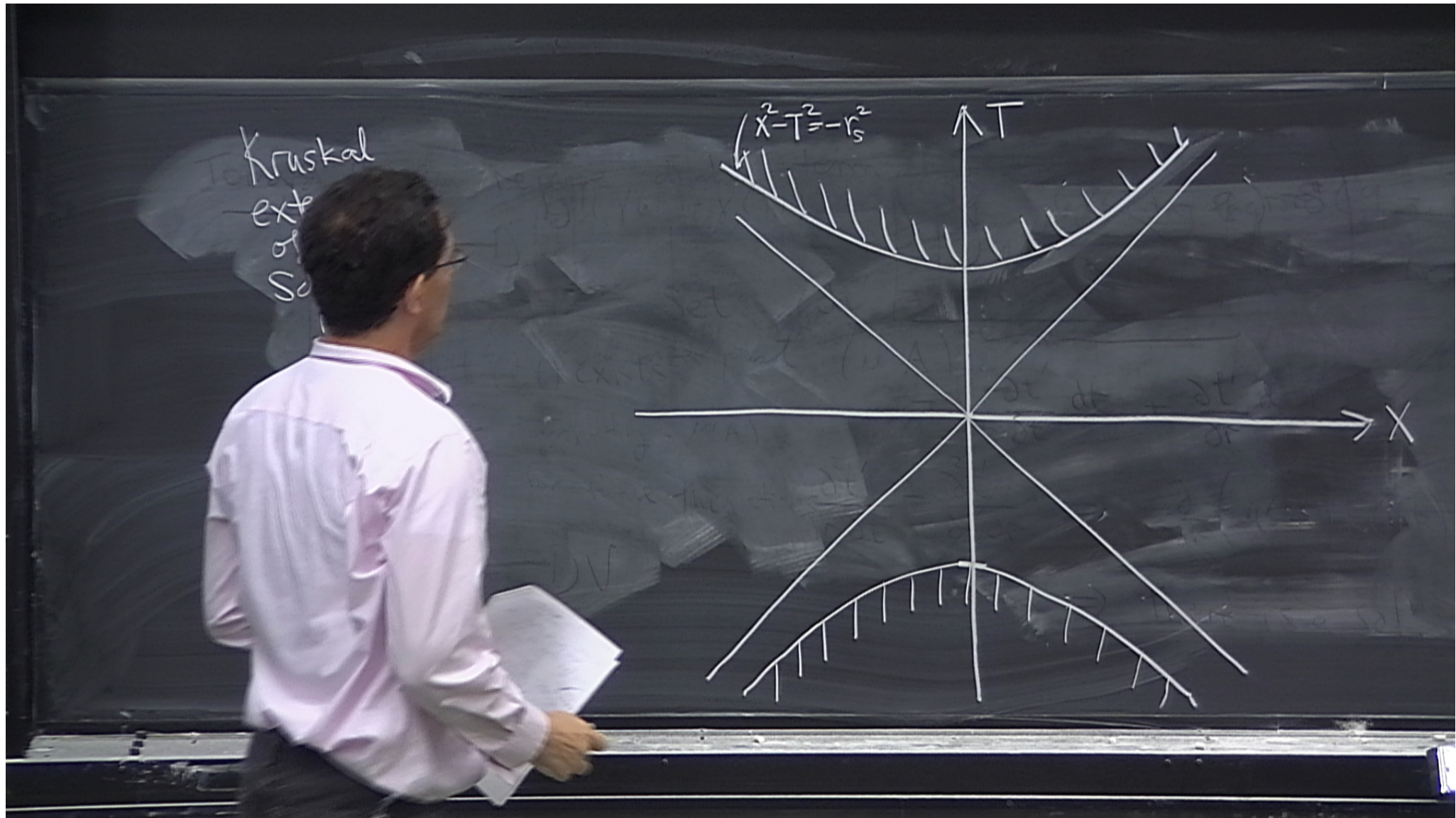
$$ds^2 = \underbrace{\frac{4r_s}{r} e^{-\frac{r}{r_s}}}_{\text{completely regular at } r=r_s!} du dv + r^2 d\Omega^2$$

where $X^2 - T^2 = UV = r_s^2 e^{\frac{v-u}{2r_s}}$

$$= r_s^2 e^{\frac{r}{r_s}} \left(\frac{r}{r_s} - 1 \right)$$

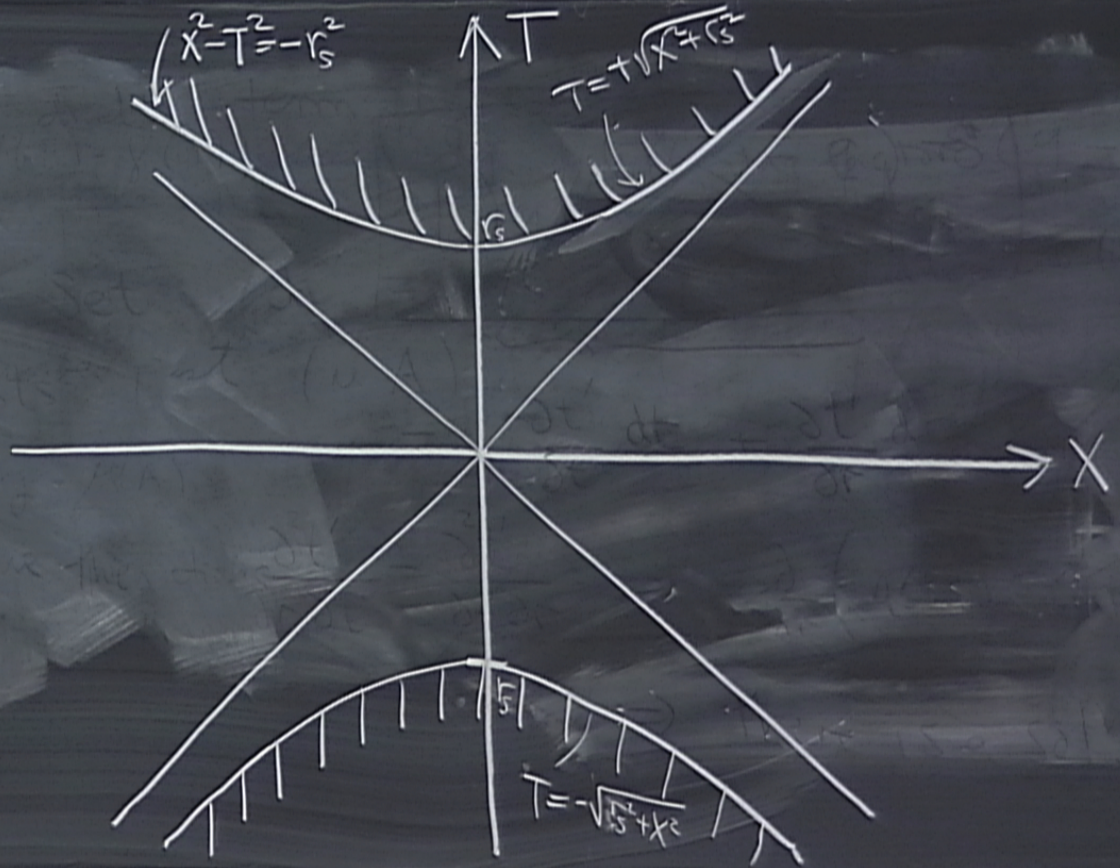






To Kruskal
extension
of
Schwarzschild
metric

$$T^2 = X^2 + r_s^2$$



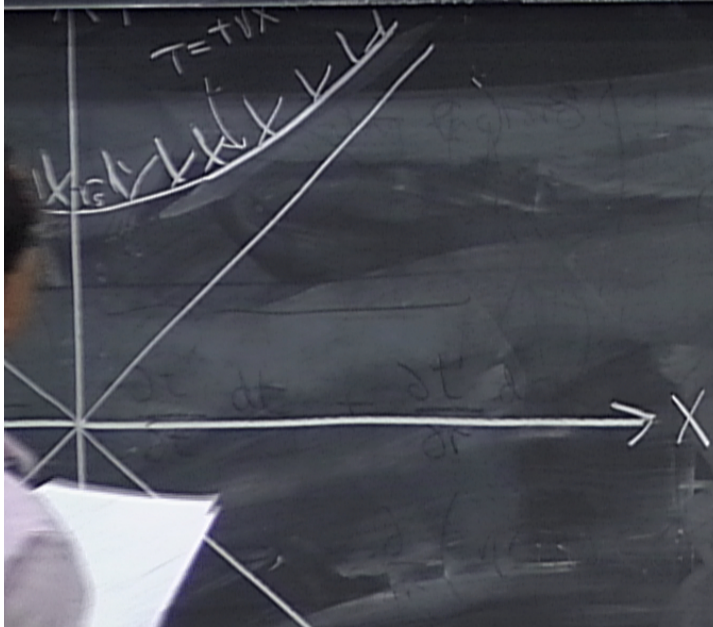
completely
regular at $r=r_s$!

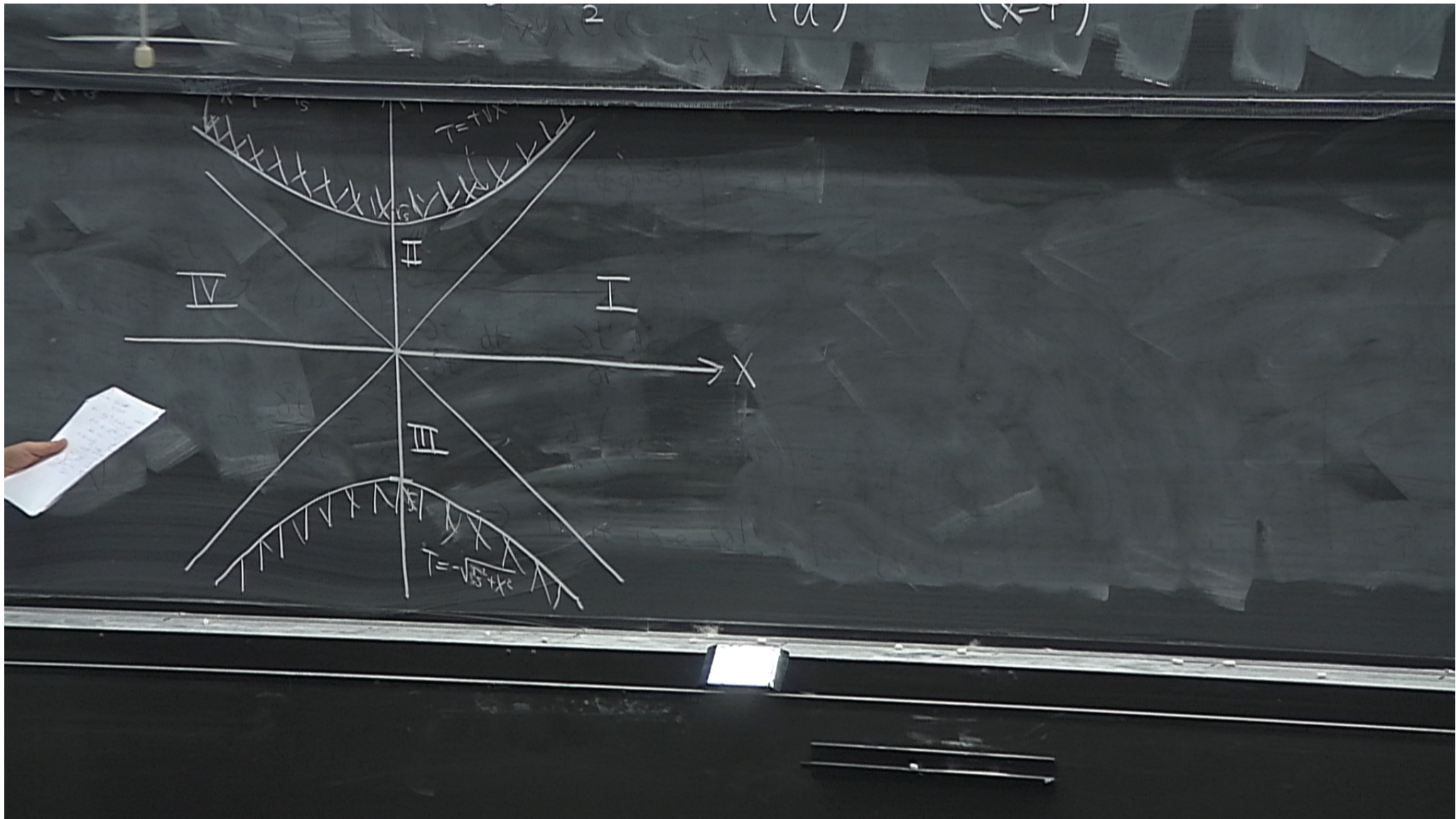
$$\Rightarrow ds^2 = 4\frac{r_s}{r} e^{-\frac{r}{r_s}} (-dT^2 + dX^2) + r^2 d\Omega^2$$

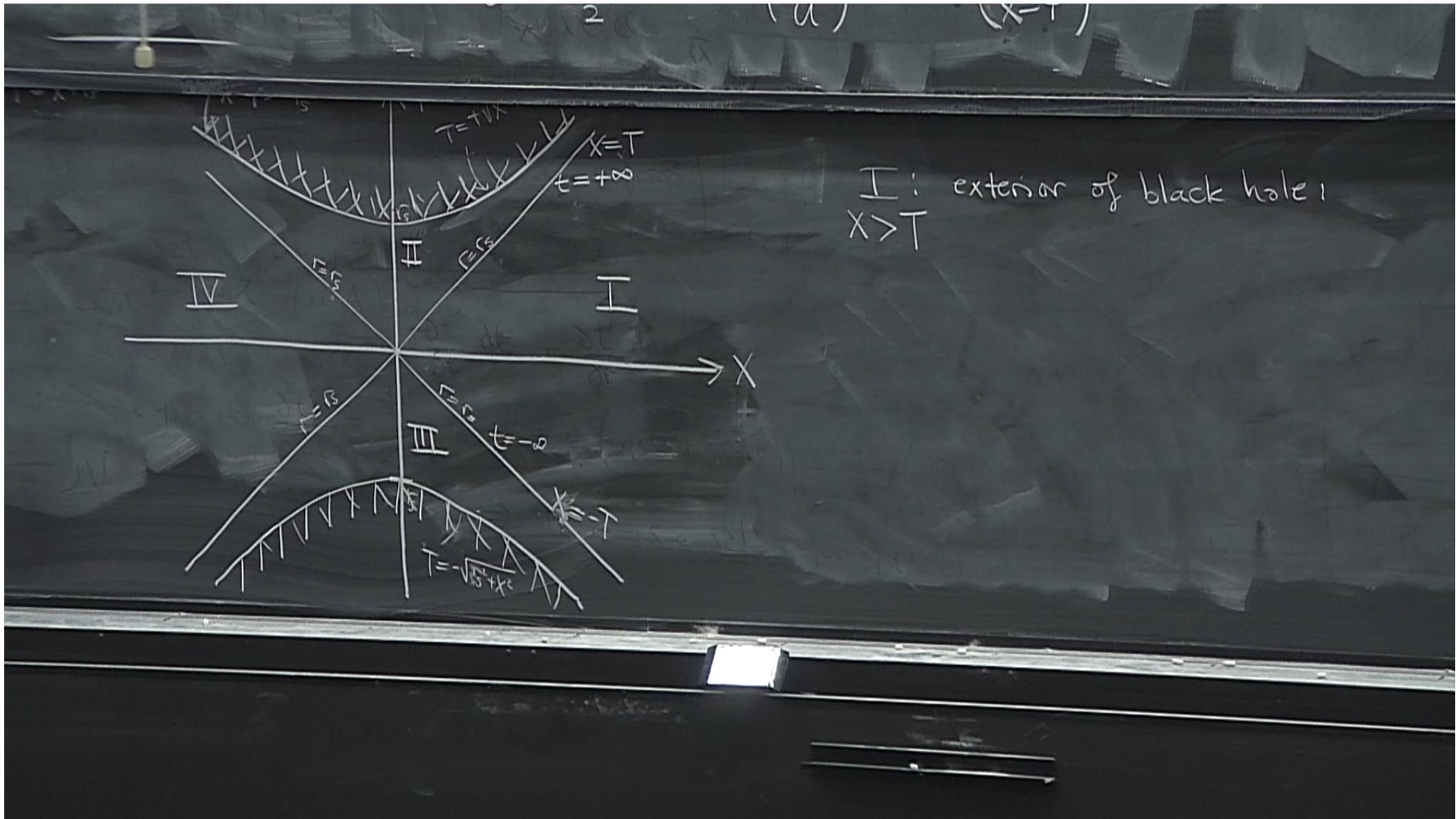
where $X^2 - T^2 =$

defines $r(X, T)$

$$t = \frac{u+v}{2} = r_s \ln\left(\frac{v}{u}\right)$$







completely regular at $r=r_s$!

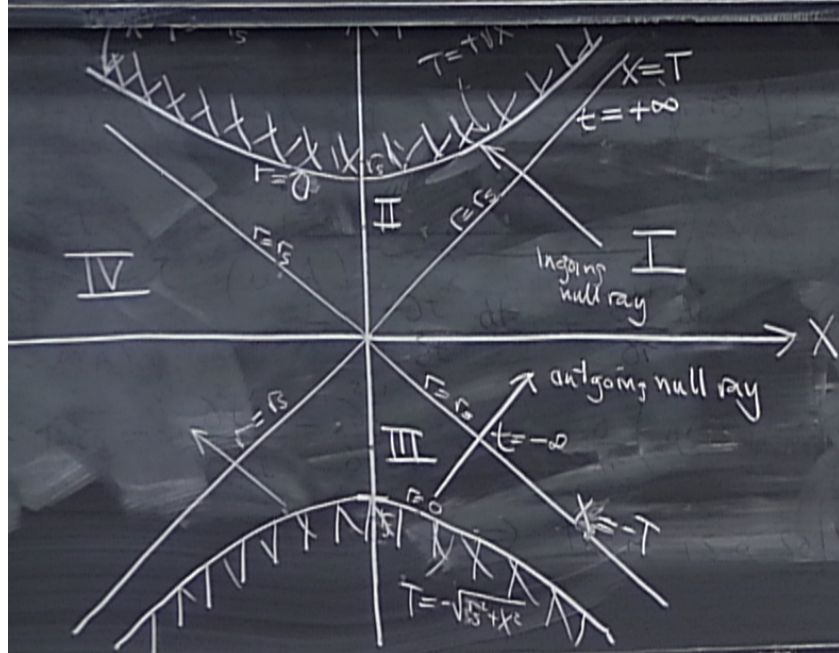
$$u = X - T$$

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$$t = \frac{u+v}{2} = r_s \ln\left(\frac{v}{u}\right) = r_s \ln\left(\frac{X+T}{X-T}\right)$$

where $X^2 - T^2 = uv = r_s^2 e^{\frac{r}{r_s}}$
 $= r_s^2 e^{\frac{r}{r_s}} \left(\frac{r}{r_s} - 1\right)$
 defines $r(X, T)$



I: exterior of black hole
 $X > T$

II and III are the two regions with $r < r_s$

IV is second copy of the exterior.

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↑
asymptotically
flat

$$ds^2 = \underbrace{\frac{4r_s}{r} e^{-\frac{r}{r_s}}}_{\text{completely regular at } r=r_s!} du dv + r^2 d\Omega^2$$

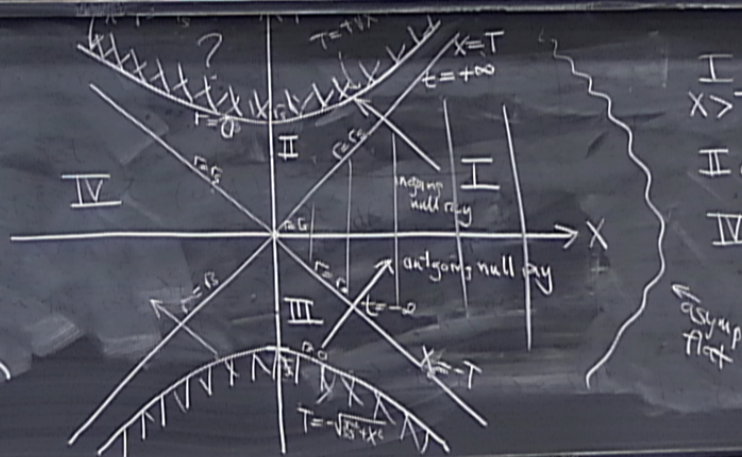
$$u = X - T$$

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where $X^2 - T^2 = UV = r_s^2 e^{\frac{v-u}{2r_s}}$
 $= r_s^2 e^{\frac{r}{r_s}} \left(\frac{r}{r_s} - 1\right)$
 defines $r(X, T)$



I: exterior of black hole (covered by $-\infty < t < \infty, r > r_s$, $X > T$)

II and III are the two regions with $r < r_s$

IV is second copy of the exterior.
 2 asymptotically flat regions

