

Title: Relativity - Lecture 9

Date: Sep 22, 2011 10:30 AM

URL: <http://pirsa.pi.local/11090034>

Abstract:

geometry energy

$$G_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta}$$

geometry

$$G_{\alpha\beta}$$

energy

$$= \frac{8\pi G}{c^4} T_{\alpha\beta}$$

Stress-energy tensor

$$T^{\alpha\beta} = T^{\beta\alpha}$$

: in SR

Stress-energy tensor

$$T^{\alpha\beta} = T^{\beta\alpha}$$

in SR

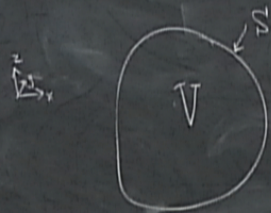
$$\text{ic } g = \eta$$

$$\partial_\alpha T^{\alpha\beta} = 0$$

conserved.

R/10

$$\overset{\text{geometry}}{G}_{\alpha\beta} = \frac{8\pi G}{c^4} \overset{\text{energy}}{T}_{\alpha\beta}$$



Stress-energy tensor

$$T^{\alpha\beta} = T^{\beta\alpha}$$

in SR $\partial_\alpha T^{\alpha\beta} = 0$ conserved

$$\partial_\alpha T^{\alpha 0} = -\partial_i T^{i0}, \quad \partial_\alpha T^{\alpha i} = -\partial_j T^{ji}$$

Stress-energy tensor

$$T^{\alpha\beta} = T^{\beta\alpha}$$

in SR $\partial_\alpha T^{\alpha\beta} = 0$ conserved.

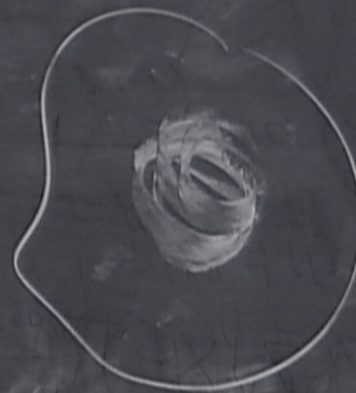
$$\partial_0 T^{00} = -\partial_i T^{i0}, \quad \partial_0 T^{0i} = -\partial_j T^{ji}$$

$$\frac{\partial}{\partial t} \int d^3x T^{00}(t, \mathbf{x}) = - \int d^3x \partial_i T^{i0}$$

divergence th^m

$$= - \int_S dS^i T^{i0}$$

R/10



Stress-energy tensor

$$T^{\alpha\beta} = T^{\beta\alpha}$$

in SR $\rightarrow i.e. g = \eta$

$$\partial_\alpha T^{\alpha\beta} = 0$$

conserved.

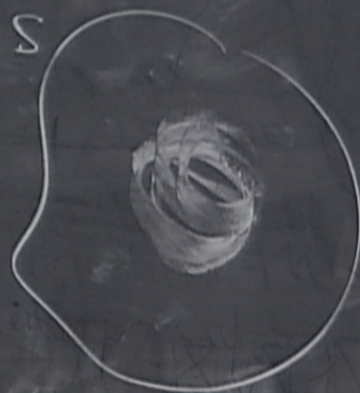
$$\partial_0 T^{00} = -\partial_i T^{i0}, \quad \partial_0 T^{0i} = -\partial_j T^{ji}$$

$$\frac{d}{d(ct)} \int d^3x T^{00}(t, \mathbf{x}) = - \int d^3x \partial_i T^{i0}$$

divergence th^m

$$= - \int_S dS^i T^{i0}$$

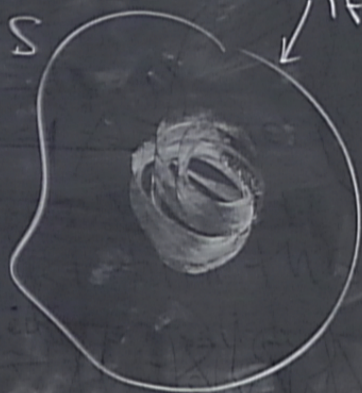
R/0



$$\rightarrow i\epsilon g^{\mu\nu}$$

$$\partial_\alpha T^{\alpha\beta} = 0 \quad \text{Conserved.}$$

$$-\partial_i T^{ji}$$



if $T^{0i} = 0$ on S , then $E \equiv \int d^3x T^{00}$ is conserved.

T^{0i} = energy flux

Similarly

$$\frac{d}{cdt} \int d^3x T^{0i}(t, \mathbf{x}) = - \int d^3x \partial_j T^{ji}$$

$$= - \int dS^j T^{ji}$$

$$P^i = \int_V d^3x T^{0i} \quad \text{is conserved if}$$

$$T^{ji} = 0 \text{ on } S'$$

$$P^i = \int_V d^3x T^{0i} \quad \text{is conserved if}$$

$T^{ji} = 0$ on S'

momentum

$$T^{0i} = T^{i0} \quad \text{is the momentum density.}$$

$$\overset{\uparrow}{\text{momentum}} \quad p^i = \int_V d^3x T^{0i} \quad \text{is conserved if}$$

$$T^{ji} = 0 \text{ on } S'$$

$$T^{0i} = T^{i0} \text{ is the momentum density.}$$

$$T^{ji} \equiv \text{stress tensor.}$$

Perfect fluid stress-energy tensor

fluid element

$T^{\alpha\beta}$

u^μ

Perfect fluid stress-energy tensor

fluid element

$$T^{\alpha\beta}$$

: in rest frame of fluid ① $u^\mu = (1, \underline{0})$

$$u^\mu$$

Perfect fluid stress-energy tensor

$$(g_{\mu\nu} = \eta_{\mu\nu})$$

fluid element

$$T^{\alpha\beta}$$

: in rest frame of fluid
(comoving)

$$\textcircled{1} u^\mu = (1, 0)$$

$$u^\mu$$

stress-energy tensor

$$(g_{\mu\nu} = \eta_{\mu\nu})$$

rest frame of fluid
(comoving)

$$\textcircled{1} \quad u^\mu = (1, \underline{0})$$

$$\textcircled{2} \quad T^{0i} = 0$$

$$\textcircled{3} \quad T^{ij} = p \delta^{ij}$$

isotropic

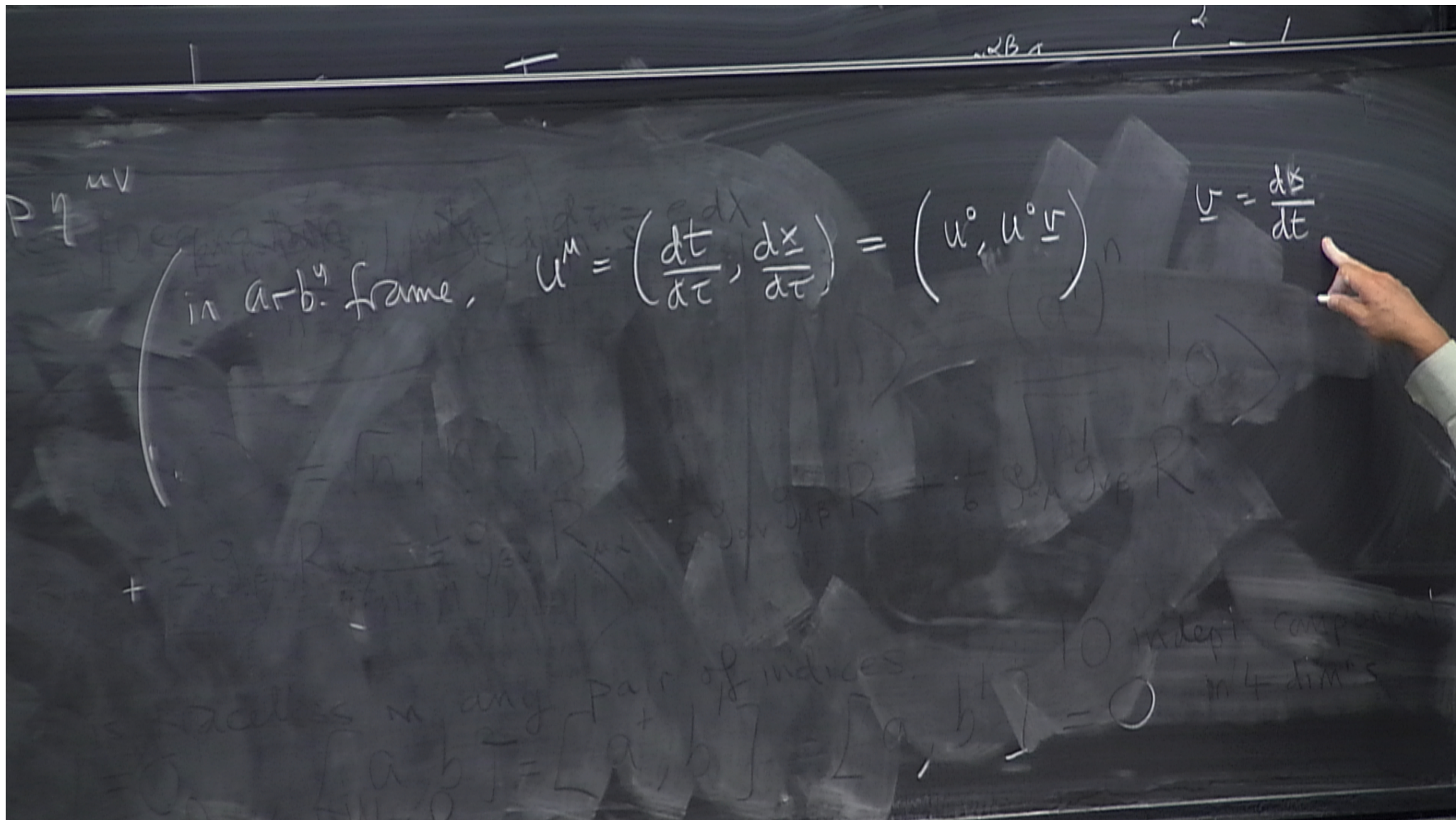
$$\textcircled{4} \quad T^{00} \equiv \rho$$

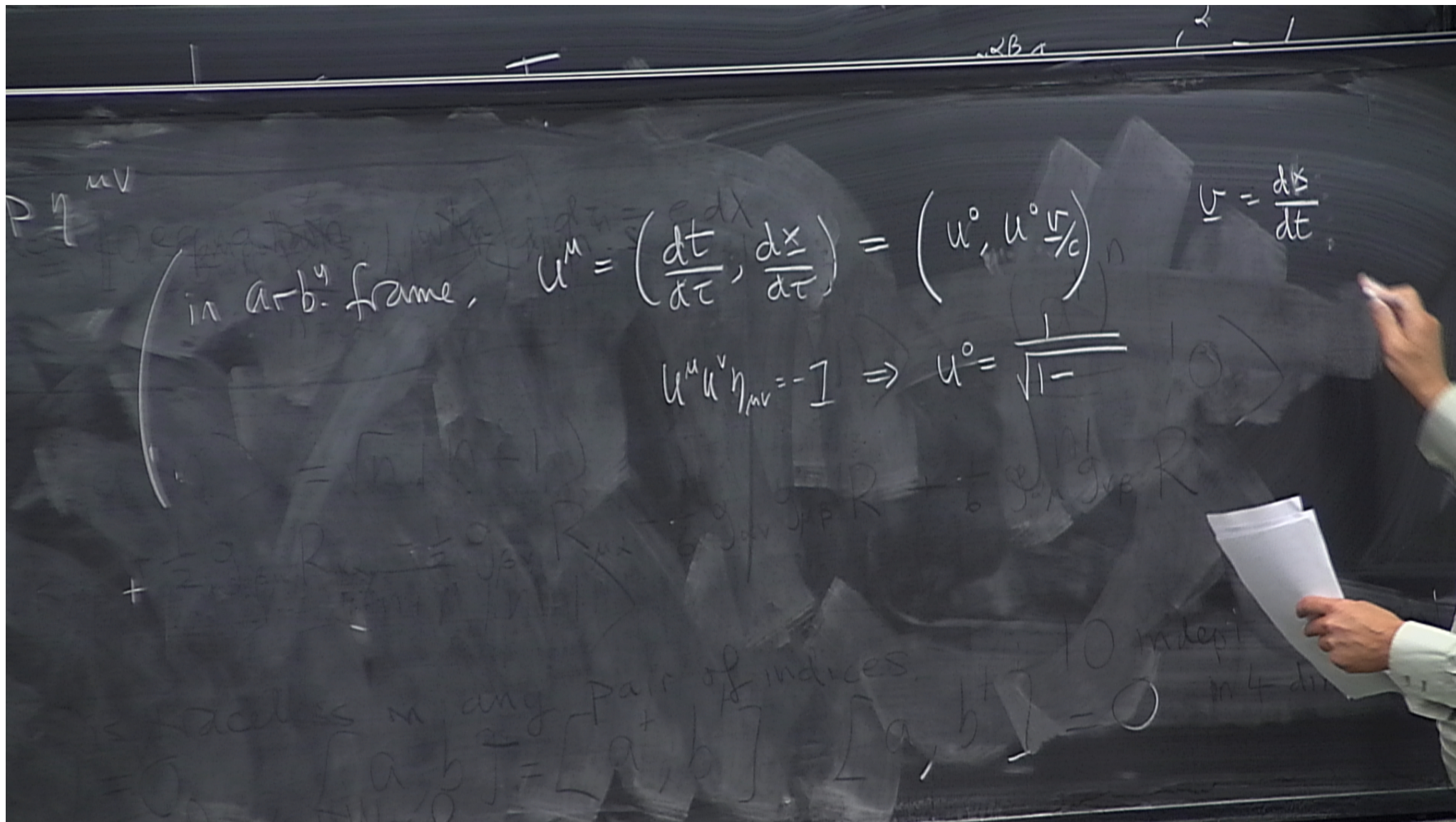
In an arbitrary frame, $T^{\mu\nu} = (p + \rho) u^\mu u^\nu + p \eta^{\mu\nu}$

check in comoving frame $u^\mu = (1, 0)$

$$T^{00} = (p + \rho) - p = \rho$$

$$T^{0i} =$$





$P^{\mu\nu}$

in a-b. frame, $u^{\mu} = \left(\frac{dt}{d\tau}, \frac{dx}{d\tau} \right) = \left(u^0, u^0 \frac{v}{c} \right)$ $\underline{v} = \frac{dx}{dt}$

$$u^{\mu} u^{\nu} \eta_{\mu\nu} = -1 \Rightarrow u^0 = \frac{1}{\sqrt{1 - v^2/c^2}} = \gamma$$

$$u^{\mu} = (\gamma, \gamma \underline{v})$$

10 indept components
in 4 dim's

antisym in any pair of indices.

$$[a, b] = [a, b] = [a, b] = 0$$

Conservation of $T^{\mu\nu}$

$$\partial_\mu T^{\mu\nu} = \partial_\mu ((\rho + p) u^\mu u^\nu) + \partial^\nu p$$

Conservation of $T^{\alpha\beta}$

$$\partial_\mu T^{\mu\nu} = \partial_\mu ((\rho + p) u^\mu u^\nu) + \partial^\nu p$$

$$V=0$$

$$\partial_0 ((\rho + p) u^0{}^2) + \partial_i ((\rho + p) u^0 u^i) - \partial_t p = 0$$

$$V=1$$

$$\partial_0 ((\rho + p) u^0{}^2 u^i) + \partial_j ((\rho + p) u^0{}^2 u^i u^j) + \partial_i p = 0$$

Conservation of $T^{\alpha\beta}$

$$\partial_\mu T^{\mu\nu} = \partial_\mu ((\rho + p) u^\mu u^\nu) + \partial^\nu p$$

$v=0$

$$\partial_0((\rho + p) u^0^2) + \partial_i((\rho + p) u^0 u^i) - \partial_t p = 0 \quad (1)$$

$v=1$

$$\partial_0((\rho + p) u^0^2 u^i) + \partial_j((\rho + p) u^0^2 u^i u^j) + \partial_i p = 0 \quad (2)$$

use (1) to remove $\partial_j((\rho + p) u^0^2 u^j)$

$$\Rightarrow \frac{\partial \underline{v}}{\partial t} + \underline{v}(\underline{\nabla} \cdot \underline{v}) = - \frac{(\underline{\nabla} p + \underline{v} \frac{\partial p}{\partial t})}{(\rho + p) u^0^2}$$

relativistic
Navier-Stokes

conservation of $T^{\mu\nu}$

$$\partial_\mu T^{\mu\nu} = \partial_\mu ((p+\rho) u^\mu u^\nu) + \partial^\nu p$$

$$V=0 \quad \partial_0 ((p+\rho) u^0^2) + \partial_i ((p+\rho) u^0 u^i) - \partial_t p = 0 \quad (1)$$

$$V=1 \quad \partial_0 ((p+\rho) u^i u^i) + \partial_j ((p+\rho) u^0 u^i u^j) + \partial_i p = 0 \quad (2)$$

use (1) to remove $\partial_j ((p+\rho) u^0 u^i u^j)$

$$\rho \left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \underline{\nabla} \underline{v} \right) \Rightarrow \frac{\partial \underline{v}}{\partial t} + (\underline{v} \cdot \underline{\nabla}) \underline{v} = - \frac{(\underline{\nabla} p + \underline{v} \cdot \frac{\partial \underline{p}}{\partial t})}{(p+\rho) u^0^2}$$

relativistic
Navier-Stokes

conservation of $T^{\mu\nu}$

$$\partial_\mu T^{\mu\nu} = \partial_\mu ((p+\rho) u^\mu u^\nu) + \partial^\nu p$$

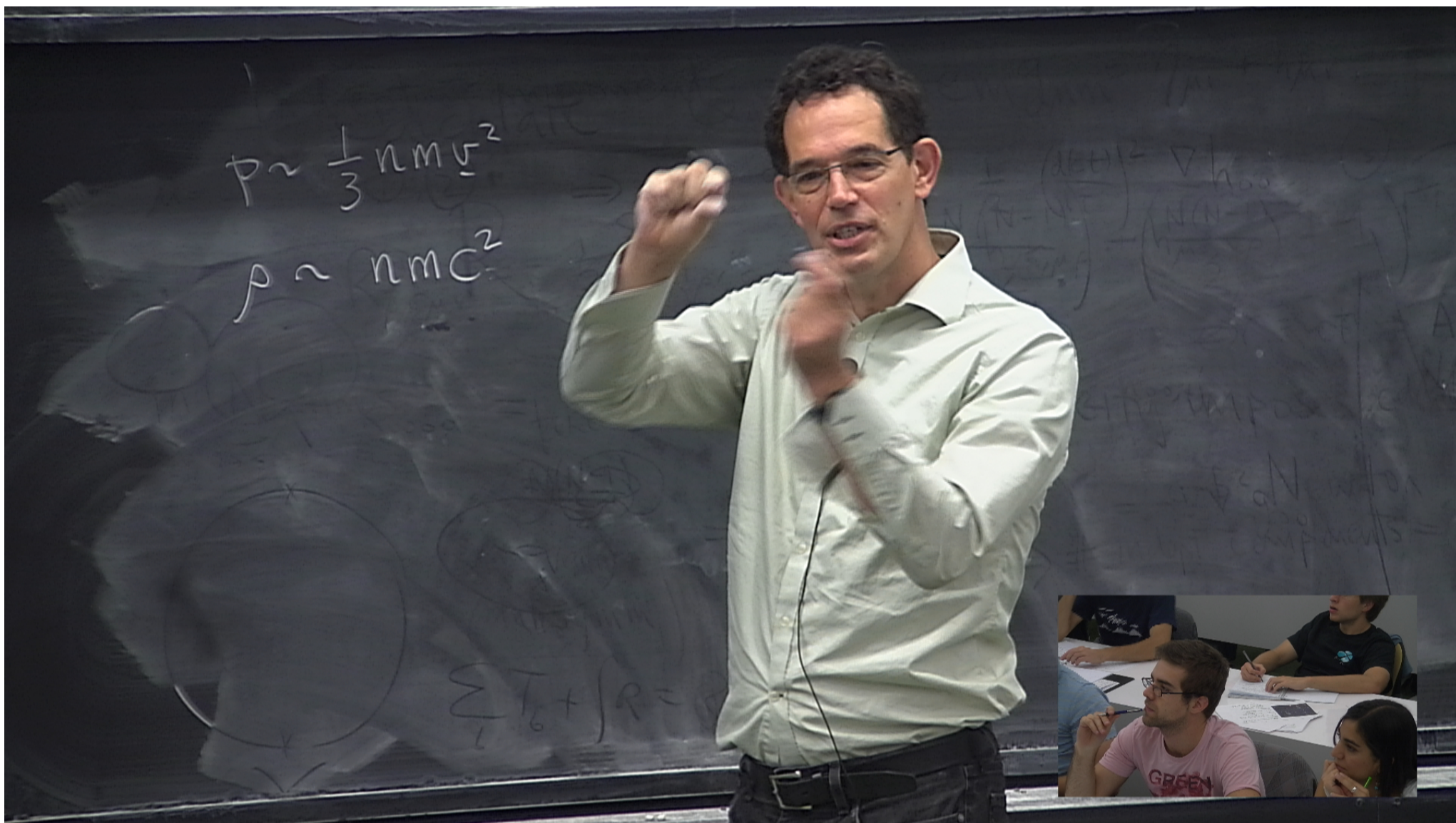
$$V=0 \quad \partial_0 ((p+\rho) u^0^2) + \partial_i ((p+\rho) u^0 u^i) - \partial_t p = 0 \quad (1)$$

$$V=1 \quad \partial_0 ((p+\rho) u^i u^i) + \partial_j ((p+\rho) u^0 u^i u^j) + \partial_i p = 0 \quad (2)$$

use (1) to remove $\partial_j ((p+\rho) u^0 u^i u^j)$

$$\rho \left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \underline{\nabla} \underline{v} \right) = - \underline{\nabla} p \quad \Rightarrow \quad \frac{\partial \underline{v}}{\partial t} + (\underline{v} \cdot \underline{\nabla}) \underline{v} = - \frac{(\underline{\nabla} p + \underline{v} \cdot \frac{\partial \underline{p}}{\partial t})}{(p+\rho) u^0^2}$$

relativistic
Navier-Stokes



Einstein - Hilbert Action

$$S_{\text{gravity}} = \frac{1}{16\pi G} \int \sqrt{-g} d^4x R$$

Einstein - Hilbert Action

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Einstein - Hilbert Action

$$S_{\text{gravity}}[g] = \frac{1}{16\pi G} \int \sqrt{-g} d^4x R$$

$$\delta S = S[g + \delta g] - S[g]$$

Einstein - Hilbert Action

$$S_{\text{gravity}}[g] = \frac{1}{16\pi G} \int \sqrt{-g} d^4x R$$

$$\delta S = S[g + \delta g] - S[g]$$

$$\delta S_{\text{grav}} = \int \sqrt{-g} \delta g_{\mu\nu} \left(-\frac{G^{\mu\nu}}{16\pi G} \right)$$

$$S_{\text{total}} = S_{\text{grav}} + S_{\text{matter}} \quad \text{by def'n} \quad \text{vars of matter fields}$$

$$\delta S_{\text{matter}} = \int \sqrt{-g} \left[\delta g_{\mu\nu} \left(\frac{T^{\mu\nu}}{2} \right) + \delta \phi_m \dots \right]$$

Einstein equations follow from $\delta S = 0 \quad \forall \delta g$.

$$\Rightarrow \bar{h}_{00} = -\frac{2\phi}{c^2} = K \left(\rho = \frac{1}{2} \rho \right) \quad \Rightarrow K = \frac{8\pi G}{c^4}$$

use ① to remove $\partial_j((\rho+p)u^j u^i)$

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p \Rightarrow \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = - \frac{(\nabla p + \mathbf{u} \cdot \frac{\partial \mathbf{p}}{\partial t})}{(\rho+p)u^2}$$

$p \ll \rho$ and $|u| \ll c$

relativistic Navier-Stokes (viscosity equals zero)

Einstein-Hilbert Action

$$S = \frac{1}{16\pi G} \int \sqrt{-g} d^4x R$$

δS

$$\delta g_{\mu\nu} \left(-\frac{G^{\mu\nu}}{16\pi G} \right)$$

$$\Rightarrow G^{\mu\nu} = 8\pi G T^{\mu\nu}$$

$$S_{\text{total}} = S_{\text{grav}} + S_{\text{matter}}$$

$$\delta S_{\text{matter}} = \int \sqrt{-g} \left[\delta g_{\mu\nu} \left(\frac{T^{\mu\nu}}{2} \right) + \delta \varphi_m \right]$$

vars of matter fields

Einstein equations follow from $\delta S = 0 \quad \forall \delta g$

$$S_{\text{total}} = S_{\text{grav}} + S_{\text{matter}} \quad \text{by def'n} \quad \swarrow \text{var'ns of matter fields}$$

$$\delta S_{\text{matter}} = \int \sqrt{-g} \left[\delta g_{\mu\nu} \left(\frac{T^{\mu\nu}}{2} \right) + \delta \varphi_m \dots \right]$$

Einstein equations follow from $\delta S = 0 \quad \forall \delta g$.

$$\Rightarrow G^{\mu\nu} = 8\pi G T^{\mu\nu}$$

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$$\delta S_{\text{matter}} = \int \sqrt{-g} \left[\delta g_{\mu\nu} \left(\frac{T^{\mu\nu}}{2} \right) + \delta \varphi_n \dots \right]$$

Einstein equations follow from $\delta S = 0 \quad \forall \delta g$.

$$\nabla_\alpha T^{\alpha\beta} = 0$$

$$\Rightarrow G^{\mu\nu} = 8\pi G T^{\mu\nu}$$

$$S_{\text{total}} = S_{\text{grav}} + S_{\text{matter}} \quad \text{by def'n}$$

$$\delta S_{\text{matter}} = \int \sqrt{-g} \left[\delta g_{\mu\nu} \left(\frac{T^{\mu\nu}}{2} \right) + \dots \right]$$

Einstein equations follow from

$$\Rightarrow G^{\mu\nu} = 8\pi G T^{\mu\nu}$$

various matter fields

symmetry
(Noether th.)
for Gen. Coord.
transf'n's

$$\nabla_\alpha T^{\alpha\beta} = 0$$

Solutions of Einstein's equations $\rightarrow$ black holes, cosmology.

Simplest case: spherical symmetry, vacuum solutions.
 $\Rightarrow T_{\mu\nu} = 0.$

s → black holes, cosmology.

$$G_{\alpha\beta} = 0$$

$$= R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R$$

$$g^{\alpha\beta} (R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R) = R - 2R = 0 \Rightarrow \underline{R=0}$$

$$\Rightarrow \boxed{R_{\alpha\beta} = 0}$$

$$g_{\mu\nu}(t, \underline{x})$$

rotations act via $\underline{x}' = \underline{O}\underline{x}$

$$\underline{O}^T \underline{O} = \underline{1}$$

$$g_{\mu\nu}(t, \underline{x})$$

rotations act via $\underline{x}' = \underline{O}\underline{x}$

$$\underline{O}^T \underline{O} = \underline{1}$$

ds^2 should depend only on rotational invariants

$$t, dt, r = |\underline{x}| = \sqrt{\underline{x}^2}, r dr = \underline{x} \cdot d\underline{x}$$

$$v(t, \underline{x})$$

ds^2 should depend on rotational invariants

$$t, dt$$

rotations act via $\underline{x}' = \underline{O}\underline{x}$
 $d\underline{x}' = \underline{O}d\underline{x}$

$$\underline{O}^T \underline{O} = \underline{1}$$

$$\underline{x} = r \begin{pmatrix} \sin\theta \cos\varphi \\ \sin\theta \sin\varphi \\ \cos\theta \end{pmatrix}$$

$$r = \sqrt{\underline{x}^2}, \quad r dr = \underline{x} \cdot d\underline{x}, \quad d\underline{x}^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

$$g_{\mu\nu}(t, \underline{x})$$

rotations act via $\underline{x}' = \underline{O}\underline{x}$
 $d\underline{x}' = \underline{O}d\underline{x}$

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$$\underline{x} = r \begin{pmatrix} \sin\theta \cos\varphi \\ \sin\theta \sin\varphi \\ \cos\theta \end{pmatrix}$$

$$t, dt, \quad r = |\underline{x}| = \sqrt{\underline{x}^2}, \quad r dr = \underline{x} \cdot d\underline{x}, \quad d\underline{x}^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

$$ds^2 = -C(r,t)dt^2 + D(r,t)dr^2 + 2E(r,t)drdt + F(r,t)r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

$$\underline{x} = r \begin{pmatrix} \sin \theta \cos \varphi \\ \sin \theta \sin \varphi \\ \cos \theta \end{pmatrix}$$

: most general rotational invariant
line element.

$d\varphi^2$)

Can remove dr/dt term by redefining t

$$\text{Set } dt' = \eta(r,t) [C(r,t) dt - E(r,t) dr]$$

Can remove $dr dt$ term by redefining t
↙ "integrating factor"

Set $dt' = \underbrace{\eta(r,t)}_{\partial t'}$ $[C(r,t) dt - E(r,t) dr]$

If $t'(r,t)$ exists

$$= \frac{\partial t'}{\partial t} dt + \frac{\partial t'}{\partial r} dr$$

Can remove $dr dt$ term by redefining t
↙ "integrating factor"

Set $dt' = \underbrace{\eta(r,t)}_{\text{integrating factor}} [C(r,t) dt - E(r,t) dr]$

If $t'(r,t)$ exists

$$= \frac{\partial t'}{\partial t} dt + \frac{\partial t'}{\partial r} dr$$

then $\frac{\partial^2 t'}{\partial r \partial t} = \frac{\partial^2 t'}{\partial t \partial r}$

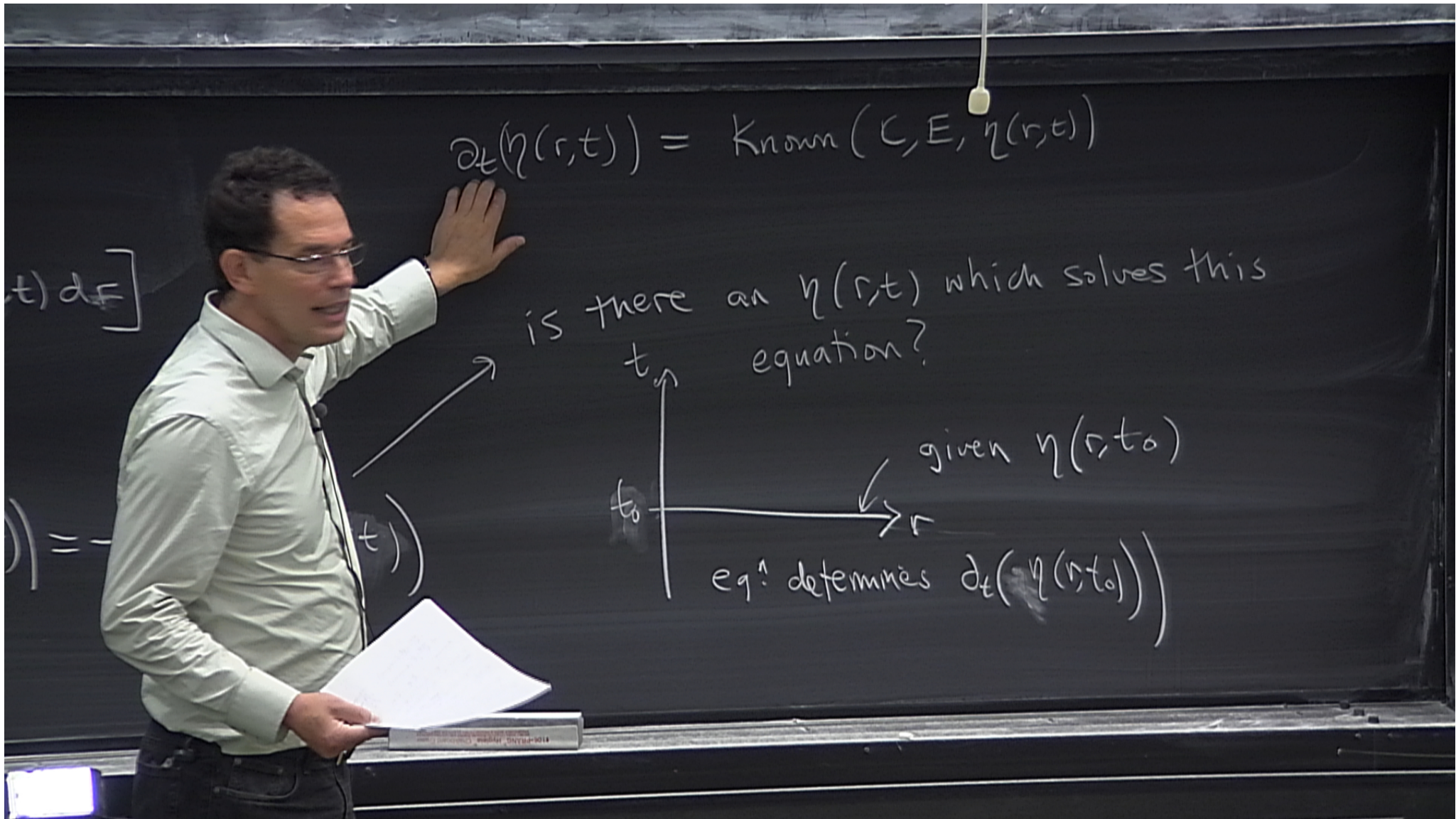
remove $dr dt$ term by redefining t
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Set $dt' = \underbrace{\eta(r,t)}_{\text{integrating factor}} [C(r,t) dt - E(r,t) dr]$

$t'(r,t)$ exists

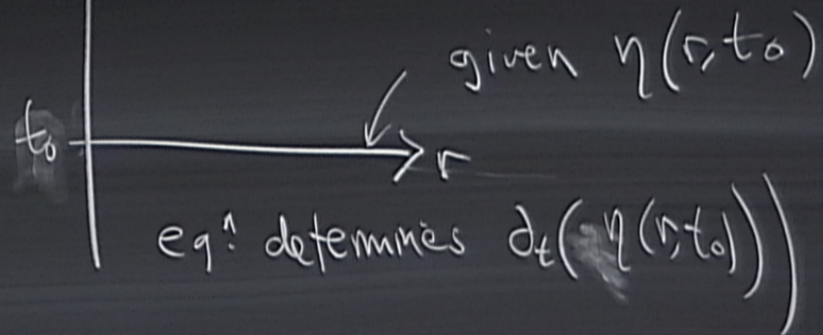
$$= \frac{\partial t'}{\partial t} dt + \frac{\partial t'}{\partial r} dr$$

then $\frac{\partial^2 t'}{\partial r \partial t} = \frac{\partial^2 t'}{\partial t \partial r} \Rightarrow \frac{\partial}{\partial r} (\eta(r,t) C(r,t)) = - \frac{\partial}{\partial t} (\eta(r,t) E(r,t))$



$$\partial_t(\eta(r,t)) = \text{Known}(C, E, \eta(r,t))$$

is there an $\eta(r,t)$ which solves this equation?



$$g_{\mu\nu}(t, \underline{x})$$

rotations act via $\underline{x}' = \underline{O}\underline{x}$
 $d\underline{x}' = \underline{O}d\underline{x}$

$$\underline{O}^T \underline{O} = \underline{1}$$

ds^2 should depend only on rotational invariants

$$\underline{x} = r \begin{pmatrix} \sin\theta \cos\phi \\ \sin\theta \sin\phi \\ \cos\theta \end{pmatrix}$$

$$t, dt, r = |\underline{x}| = \sqrt{\underline{x}^2}, r dr = \underline{x} \cdot d\underline{x}, d\underline{x}^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

$$ds^2 = -C(r,t)dt^2 + D(r,t)dr^2 + \underbrace{2E(r,t)drdt}_{\text{integrating factor}} + \underbrace{F(r,t)r^2}_{\text{most general rotational invariant line element}}(d\theta^2 + \sin^2\theta d\phi^2)$$

: most general rotational invariant line element.

Can remove $drdt$ term by redefining t
 ↓
 "integrating factor"

Set $dt' = \eta(r,t) [C(r,t)dt - E(r,t)dr]$
 If $t'(r,t)$ exists
 $= \frac{\partial t'}{\partial t} dt + \frac{\partial t'}{\partial r} dr$

then $\frac{\partial^2 t'}{\partial r \partial t} = \frac{\partial^2 t'}{\partial t \partial r} \Rightarrow \frac{\partial}{\partial r} (\eta(r,t) C(r,t)) = \frac{\partial}{\partial t} (\eta(r,t) E(r,t))$
 $\Rightarrow$ there is a soln

$$\partial_t(\eta(r,t)) = \text{Known}(C, E, \eta(r,t))$$

is there an $\eta(r,t)$ which solves this equation?

given $\eta(r, t_0)$
 e_1 determines $d_t(\eta(r, t_0))$

$$g_{\mu\nu}(t, \underline{x})$$

rotations act via $\underline{x}' = \underline{O}\underline{x}$
 $d\underline{x}' = \underline{O}d\underline{x}$

$$\underline{O}^T \underline{O} = \underline{1}$$

ds^2 should depend only on rotational invariants

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$$ds^2 = -C(r,t)dt^2 + D(r,t)dr^2 + \underbrace{2E(r,t)}_{r'^2} dr dt + \underbrace{F(r,t)}_{r'^2} r^2 (d\theta^2 + \sin^2\theta d\phi^2)$$

: most general rotational invariant line element.

$$r' = \sqrt{F} \rightarrow \text{solve for } r(r,t)$$

$$dr = () dr' + () dt$$

Can remove $dr dt$ term by redefining t
 "integrating factor"

$$\text{Set } dt' = \eta(r,t) [C(r,t) dt - E(r,t) dr]$$

If $t'(r,t)$ exists

$$= \frac{\partial t'}{\partial t} dt + \frac{\partial t'}{\partial r} dr$$

$$\text{then } \frac{\partial t'}{\partial r \partial t} = \frac{\partial t'}{\partial t \partial r} \Rightarrow \frac{\partial}{\partial r} (\eta(r,t) C(r,t)) = - \frac{\partial}{\partial t} (\eta(r,t) E(r,t))$$

$\Rightarrow$ there is a solⁿ for $\eta(r,t)$

$$\partial_t(\eta(r,t)) = \text{Known}(C, E, \eta(r,t))$$

is the t $\eta(r,t)$ which solves this

given $\eta(r,t_0)$

$$\rightarrow r \text{ times } d_t(\eta(r,t_0))$$

$\frac{\partial}{\partial t}(\rho + \mathbf{v} \cdot \nabla \rho) = -\nabla \cdot \mathbf{p}$
 $\Rightarrow \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = - \frac{(\nabla p + \frac{\mathbf{v} \cdot \nabla p}{c \partial t})}{(\rho + p) u^2}$

$p \ll \rho$ and $|\mathbf{v}| \ll c$

relativistic Navier-Stokes (viscosity equals zero)

$$ds^2 = - \frac{dt'^2}{\eta^2 c} + (D + \bar{C}^1 E^2) dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2)$$

$$\rho \left(\frac{dr}{dt} + \mathbf{u} \cdot \nabla \mathbf{r} \right) = -\nabla p \quad \Rightarrow \quad \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = - \frac{(\nabla p + \mathbf{r} \frac{\partial p}{\partial t})}{(\rho + p) u^2} \quad \text{Navier-Stokes}$$

$p \ll \rho$ and $|\mathbf{u}| \ll c$

$$ds^2 = - \frac{dt'^2}{\eta^2 C} + (D + \bar{C}^{-1} E^2) dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2)$$

drop primes

$$\equiv - B(r,t) dt^2 + A(r,t) dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2)$$